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# Titel: Pricing of spread options on a bivariate jump market and stability to model risk


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Abstract. We study the pricing of spread options and we obtain a Margrabe type formula for a general bivariate jump diffusion model. Moreover we study the robustness of the price to model risk, in the sense that we consider two types of bivariate jump diffusions: one allowing for infinite activity small jumps and one not. In the second one an adequate continuous component describes the small variation of prices. We illustrate our computations by several examples.

## 1. Introduction

Recent considerations in finance have led to an increasing interest in multidimensional models with jumps taking into account the dependence between components (see for instance Cont and Tankov [9]). In this context one is interested in finding closed form formulas for the value of options written on such models. Eberlein et al. [15, 14] introduced the so-called duality principle. This principle aims at simplifying valuation problems that depend on several variables by associating them to the corresponding dual option pricing problem. Their approach allows in particular, to reduce an option pricing problem involving two random variables to an option pricing problem involving just one random variable under a dual measure.

In this paper we focus on spread options. These are options written on the difference of two underlying assets. As illustration, the holder of such a European call option receives at maturity $T$ a payoff given by

$$\max(S^{(1)}(T) - S^{(2)}(T) - K, 0),$$

where $S^{(1)}(t)$, $S^{(2)}(t)$, $0 \leq t \leq T$ are the asset prices and $K$ is the strike.

Specifically, we assume that the price process $(S^{(1)}(t), S^{(2)}(t))_{0\leq t\leq T}$ is a bivariate jump-diffusion and the strike $K = 0$. In this case these options are also called exchange options (see e.g. Margrabe [22]). We derive a Margrabe-type formula for the spread option price using the approach of Eberlein et al. [15] and specifically, we provide explicit formulas when the underlying price process is a two-dimensional stochastic volatility model and when the price process is a two-dimensional exponential Lévy model driven by a bivariate normal inverse Gaussian (NIG) distributed Lévy process. Both cases are popular models in the finance and energy literature (see e.g., Barndorff-Nielsen and Shephard [3] and Benth et al. [4]).

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Moreover, we study the robustness of the option price towards model risk, in the sense that we consider two types of jump-diffusion models: one allowing for infinite activity small jumps and one not. In the second case, an adequate continuous component models the small variations of prices. Our results show that the pricing of spread options is stable to such model risk.

The Margrabe formula is based on an appropriate change of measure which allows to move from pricing the spread option written on a bivariate process to pricing a European option written on a one-dimensional process. In Margrabe [22] spread options are evaluated assuming that the asset prices are geometric Brownian motions and using a measure change induced by treating one of the assets as numéraire. See also Carmona and Durrleman [10] for an overview about spread options in continuous models. In our case, we choose the price processes to be modeled by bivariate jump-diffusions since such models describe better some features of prices in financial and commodity markets. See, e.g. Cont and Tankov [9] and Benth et al. [4], for the use of models with jumps.

The pricing of financial derivatives is set in the non-arbitrage framework and technically is performed under a risk neutral pricing measure. Under this measure the discounted prices of the underlying primaries are martingales. However, in the context of energy derivatives, the underlying may be electricity which cannot be stored. Then the classical non-arbitrage arguments by hedging cannot be performed. Hence pricing of energy derivatives does not require that the pricing measure is a risk neutral measure as in classical financial markets (see e.g. Benth et al. [4] for more details on this). Thus in our computation we first value the option as the discounted expectation of the terminal payoff under the real-world measure, then we use the duality measure to derive a Margrabe type formula for the spread option price. This accommodates the pricing of energy derivatives on electricity. Following the approach by Eberlein et al. [15], we describe the density between the original and the dual measure by an Esscher change of measure (see Kallsen and Shiryaev [20]).

On other hand, when pricing derivatives in a financial market, the real-world measure must first be transformed into a risk neutral measure, and then a second appropriate measure change is performed to find the Margrabe formula. We recall that a market modeled by a jump-diffusion process is in general incomplete and there exist many equivalent martingale measures. A specific martingale measure is usually chosen according to some criterion minimizing the risk associated with (partial) hedging of the derivative. See Schweizer [26] for an overview about equivalent martingale measures in incomplete markets. In our examples we choose risk-neutral Esscher measures as they are structure preserving (see e.g., Gerber and Shiu [16]) and for consistency with the duality methods for spread options introduced above.

From the modeling point of view, one can approximate the small jumps of a jump-diffusion by a continuous martingale appropriately scaled. This was introduced by Asmussen and Rosinski [1] in the case of Lévy processes. Benth et al. [6, 8] studied convergence results of options written in one-dimensional jump-diffusion models. They also studied the robustness of the option prices after a change of measure where the measure depends on the model choice. From a simulation point of view, one may truncate off the small jumps and use an appropriate compound Poisson process to approximate a Lévy process. However, when the Lévy process has infinite variation, it was proved in Cohen and Rosinski [11] that adding a small-variance Brownian motion to a compound Poisson process improves the approximation in general.

The main contribution of this paper is to apply the Margrabe formula to prove the robustness of the spread option prices towards model risk. By approximating the small jumps by a two-dimensional Brownian motion appropriately scaled, we compute a rate for the convergence of the approximating spread option price to the underlying option price. This rate turns out to be proportional to the variance of the small jumps.

The paper is organised as follows: in Section 2 we define our market model and recall a Margrabe type formula for the spread option written on a bivariate jump-diffusion price dynamics. Moreover, we present several examples in which we give explicit formulas for spread option prices. In Section 3 we prove the robustness of the spread option prices and compute the convergence rate in the case the price process is driven by a bivariate Lévy process.

## 2. Pricing of spread options in a jump-diffusion framework

In this section we describe our model for the financial market and introduce necessary notation. Let $\mathbb{R}^2$ denote the 2-dimensional Euclidean space. The Euclidean scalar product between two vectors $u, v \in \mathbb{R}^2$ is denoted by $\langle u, v\rangle$ or $u^\star v$, where $.^\star$ denotes the transpose of a given vector. The Euclidean norm is denoted by $|u|$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, $(T > 0)$ satisfying the *usual conditions* (see Protter [23]). Consider a process $Y = Y(t)$, $0 \le t \le T$, with left continuous (càg) trajectories. We denote by $\mathcal{P}$ the $\sigma$-algebra on $\Omega \times [0, T]$ generated by all càg adapted processes $Y$ and by $\mathcal{M}_{loc}(\mathbb{P})$ the space of $\mathbb{P}$ local martingales $M$, with $M(0) = 0$. Heareafter we follow the notation of Jacod and Shiryaev [18]. We consider a 2-dimensional general semimartingale $H = (H^{(1)}(t), H^{(2)}(t))^\star$, $0 \le t \le T$, on the given probability space. Every semimartingale has *the canonical representation* (see II.2.34 in Jacod and Shiryaev [18]).

$$H(t) = H(0) + B(t) + H^c(t) + \int_0^t \int_{\mathbb{R}^2} h(z)\,\mathrm{d}(\mu-\nu) + \int_0^t \int_{\mathbb{R}^2} (z - h(z))\,\mathrm{d}\mu\,,$$

where

1. $H(0)$ is finite-valued and $\mathcal{F}_0$-measurable,
2. $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a truncation function, i.e. a bounded function with compact support that behaves like $h(z) = z$ in a neighborhood of zero. A canonical choice of $h$ is $h(z) = z\mathbf{1}_{\{|z|\le a\}}$, (for some $a > 0$),
3. $B = B(t)$, $0 \le t \le T$, is an $\mathbb{R}^2$-valued predictable process of bounded variation,
4. $H^c = H^c(t)$, $0 \le t \le T$, is the continuous martingale part of $H$, $H^c$ has the predictable quadratic characteristic $\langle H^c\rangle = C$, which is a predictable $\mathbb{R}^{2\times 2}$-valued process of bounded variation, whose values are nonnegative symmetric matrices,
5. $\nu = \nu(\omega; \mathrm{d}t, \mathrm{d}z)$ is a predictable random measure on $[0, T] \times \mathbb{R}^2$. It is the compensator of the random measure of jumps $\mu = \mu(\omega; \mathrm{d}t, \mathrm{d}z)$ of $H$.

The processes $B$, $C$, and the measure $\nu$ are called the *triplet of predictable characteristics* of the semimartingale $H$ associated to the truncation function $h$ with respect to the probability measure $\mathbb{P}$ and will be denoted by

$$\mathbb{T}(H|\mathbb{P}) = (B, C, \nu)\,.$$

In addition there exists an increasing predictable process $A$, predictable processes $b, c$, and a predictable kernel $F$ from $(\Omega \times [0,T], \mathcal{P})$ into $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ such that

$$B(t) = \int_0^t b(s)\,\mathrm{d}A(s), \qquad C(t) = \int_0^t c(s)\,\mathrm{d}A(s)\,,$$

$$\nu([0,t] \times E) = \int_0^t \int_E F(s, \mathrm{d}z)\,\mathrm{d}A(s)\,, \qquad E \in \mathcal{B}(\mathbb{R}^2)$$

(see Proposition II.2.9 in Jacod and Shiryaev [18]).

If the characteristics $(B, C, \nu)$ are *absolutely continuous*, which means $A(t) = t$, then the triplet $(b, c, F)$ is called the *differential characteristics* of the semimartingale $H$.

Notice that an $\mathbb{R}^2$-valued Lévy process $L$, is a semimartingale with $L(0) = 0$ which has a version with differential characteristics $(b, c, F)$ that does not depend on $(\omega, t)$ (see Prop. 2.3 in Kallsen [19]). That is, $\mathbb{T}(L|\mathbb{P}) = (B, C, \nu)$, where

$$B(\omega; t) = bt, \quad C(\omega; t) = ct, \quad \nu(\omega, \mathrm{d}t, \mathrm{d}z) = \mathrm{d}tF(\mathrm{d}z)\,.$$

2.1. **Description of the model.** We choose $H = (H^{(1)}, H^{(2)})^\star$ to be of the form

$$H(t) = H(0) + \int_0^t b(s)\,\mathrm{d}s + \int_0^t c^{\frac{1}{2}}(s)\,\mathrm{d}W(s) + \int_0^t \int_{\mathbb{R}^2} h(z)\,(\mu - \nu)\,(\mathrm{d}s, \mathrm{d}z)$$

$$+ \int_0^t \int_{\mathbb{R}^2} (z - h(z))\,\mu\,(\mathrm{d}s, \mathrm{d}z)\,, \tag{2.1}$$

where $H(0) = (0,0)^\star$, $b(t) = b(\omega; t) \in \mathbb{R}^2$, $c^{\frac{1}{2}}(t) = c^{\frac{1}{2}}(\omega; t) \in \mathbb{R}^{2\times 2}$ is given by

$$c^{\frac{1}{2}}(t) = \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{12}(t) & \sigma_{22}(t) \end{pmatrix}\,, \quad 0 \le t \le T\,, \tag{2.2}$$

$W = W(t)$, $0 \le t \le T$, is a standard 2-dimensional Brownian motion, and the jump-measure $\mu$ is a compound Poisson process with compensator $\nu([0,t] \times E) = t \int_E F(\mathrm{d}z)$, for $E \in \mathcal{B}(\mathbb{R}^2)$. The Lévy measure $F$ satisfies $\int_{\mathbb{R}^2} (|z|^2 \wedge 1) F(\mathrm{d}z) < \infty$. Notice that the triplet $\mathbb{T}(H|\mathbb{P})$ is absolutely continuous with differential characteristics $(b, c, F)$. We assume for $i = 1, 2$,

$$\int_0^T \left( |b_i(s)| + \sum_{j=1}^2 |\sigma_{ij}(s)|^2 \right) \mathrm{d}s < \infty\,, \quad \mathbb{P} - \text{a.s.} \tag{2.3}$$

The latter condition implies that the integrals in (2.1) are well defined.

We consider a price process given by the stochastic exponential of the semimartingale $H$ (see Shiryaev [27]). That is

$$\begin{aligned} S(t) &= S(0)\mathcal{E}(H)_t \\ &= S(0)\exp\left\{H(t) - \frac{1}{2}\langle H^c\rangle_t\right\}\prod_{0\le s\le t}(1+\Delta H(s))\,\mathrm{e}^{-\Delta H(s)}\,, \qquad 0\le t\le T. \end{aligned} \tag{2.4}$$

Here $\langle\cdot\rangle$ denotes the predictable quadratic variation and $S(0)\in\mathbb{R}^2_+$. To ensure positive prices we restrict our attention to semimartingales $H$ for which $\Delta H(s) > -1$ for all $s$. In this case there exists a semimartingale $X$ with $X(0) = 0$ such that

$$S(t) = S(0)\mathcal{E}(H)_t = S(0)\mathrm{e}^{X(t)}\,, \qquad 0\le t\le T.$$

$X$ is called the logarithmic transform of $H$ and it is explicitly given by

$$X = \log\mathcal{E}(H) = H - \frac{1}{2}\langle H^c\rangle + \sum_{0<s\le\cdot}\left[\log\left(1+\Delta H(s)\right) - \Delta H(s)\right]. \tag{2.5}$$

The latter may be written in the form

$$X = H - \frac{1}{2}\langle H^c\rangle + (\log(1+z) - z) * \mu\,,$$

where $\mu$ is the random measure of jumps of $H$ and $\xi * \mu$ denotes the stochastic integral with respect to $\mu$. Later we shall use $\xi * (\mu - \nu)$ to denote the stochastic integral with respect to the compensated random measure $\mu - \nu$. From (2.5), one can see that the characteristic triplet of $X$ is absolutely continuous, that is, $\mathbb{T}(X|\mathbb{P}) = (\widetilde{B}, \widetilde{C}, \widetilde{\nu})$, with

$$\begin{aligned} \widetilde{B}(t) &= \int_0^t \widetilde{b}(s)\,\mathrm{d}s\,, \qquad \widetilde{b} = b - \frac{c}{2} + (h(\log(1+z)) - h(z)) * F\,, \\ \widetilde{C}(t) &= \int_0^t \widetilde{c}(s)\,\mathrm{d}s\,, \qquad \widetilde{c} = c\,, \\ \widetilde{\nu}(\mathrm{d}t,\mathrm{d}z) &= \mathrm{d}t\,\widetilde{F}(\mathrm{d}z)\,, \qquad \mathbf{1}_A * \widetilde{F} = \mathbf{1}_A(\log(1+z)) * F\,, \quad A\in\mathcal{B}(\mathbb{R}^2\setminus\{0\})\,. \end{aligned} \tag{2.6}$$

The process $X$ can be expressed for $i = 1, 2$ as follows

$$\begin{aligned} dX^{(i)}(t) = &\left\{b_i(t) - \frac{1}{2}(\sigma_{i1}^2(t) + \sigma_{i2}^2(t)) + \int_{\mathbb{R}^2}\left[h_i\left(\ln(1+z)\right) - h_i(z)\right]F(\mathrm{d}z_1\,,\mathrm{d}z_2)\right\}\mathrm{d}t \\ &+ \sigma_{i1}(t)\,\mathrm{d}W^{(1)}(t) + \sigma_{i2}(t)\,\mathrm{d}W^{(2)}(t) + \int_{\mathbb{R}^2} h_i(z)(\mu^X - \widetilde{\nu})(\,\mathrm{d}t\,,\mathrm{d}z_1\,,\mathrm{d}z_2) \\ &+ \int_{\mathbb{R}^2}(z_i - h_i(z))\,\mu^X(\mathrm{d}t\,,\mathrm{d}z_1\,,\mathrm{d}z_2)\,, \qquad 0\le t\le T\,, \end{aligned} \tag{2.7}$$

$$X^{(i)}(0) = H^{(i)}(0) = 0\,,$$

where $\mu^X$ is the jump measure of $X$.

The semimartingale $X$ with triplet $\mathbb{T}(X|\mathbb{P}) = (\widetilde{B}, \widetilde{C}, \widetilde{\nu})$ can be associated to the Laplace cumulant process denoted by $K(\mathtt{u}) = K(\mathtt{u})_t\,,\ 0 \le t \le T$, defined for $\mathtt{u} \in \mathbb{R}^2$ by

$$K(\mathtt{u}) := \langle \mathtt{u}, \widetilde{B}\rangle + \frac{1}{2}\langle \mathtt{u}, \widetilde{C}\,\mathtt{u}\rangle + \left(\mathrm{e}^{\langle \mathtt{u}, \log(1+z)\rangle} - 1 - \langle \mathtt{u}, h(\log(1+z))\rangle\right) * \nu\,.$$

Moreover, we have that $K(\mathtt{u})_t = \int_0^t \kappa(\mathtt{u})_s \mathrm{d}s$ , where

$$\kappa(\mathtt{u}) = \langle \mathtt{u}, \widetilde{b}\rangle + \frac{1}{2}\langle \mathtt{u}, \widetilde{c}\,\mathtt{u}\rangle + \int_{\mathbb{R}^2} \left(\mathrm{e}^{\langle \mathtt{u}, \log(1+z)\rangle} - 1 - \langle \mathtt{u}, h(\log(1+z))\rangle\right)\, F(\mathrm{d}z)\,. \tag{2.8}$$

The Laplace cumulant process satisfies the following property

$$\frac{\mathrm{e}^{\mathtt{u}^\star X}}{\mathcal{E}(K(\mathtt{u}))} \in \mathcal{M}_{loc}(\mathbb{P})\,, \tag{2.9}$$

for all $\mathtt{u} \in \mathbb{R}^2$, assuming that $\mathcal{E}(K(\mathtt{u}))$ never vanishes (see Cor II.2.48 in Jacod and Shiryaev [18]). Moreover, $\widetilde{K}(\mathtt{u})$ denotes the logarithmic transform of the cumulant process $K(\mathtt{u})$, that is $\mathcal{E}(K(\mathtt{u})) = \exp(\widetilde{K}(\mathtt{u}))$.

Note that the cumulant of a Lévy process is continuous and deterministic. We denote it by $K_L$. Hence $\mathcal{E}(K_L(\mathtt{u})) = \mathrm{e}^{K_L(\mathtt{u})}$ and never vanishes. In addition, we have for all $t \in [0, T]$ and $\mathtt{u} \in \mathbb{R}^2$

$$\mathbb{E}[\mathrm{e}^{\mathtt{i}\langle \mathtt{u}, L(t)\rangle}] = \mathrm{e}^{K_L(-\mathtt{i}\mathtt{u})_t}\,.$$

2.2. **Margrabe formula.** In the sequel, we consider a spread option of European type written on the difference of two underlying assets whose values are driven by the jump-diffusion (2.4). Eberlein et al. [15] simplify the valuation problem of an option written on two random variables to a problem involving just one random variable exploiting a change of measure technique. We state the following result from Eberlein et al. [15] in which the density between the original and the dual measure is described by an Esscher transform.

**Theorem 2.1.** *Let $X$ be as in (2.5), (2.7), and $u, v \in \mathbb{R}^2$. Define the measure $\widetilde{\mathbb{P}}_v$ via the Radon-Nikodym derivative*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_v}{\mathrm{d}\mathbb{P}} = \exp\left\{v^\star X(T) - \widetilde{K}(v)_T\right\}\,,$$

*assuming that $\mathrm{e}^{v^\star X - \widetilde{K}(v)} \in \mathcal{M}(\mathbb{P})$. Then the process $X^u$ with $X^u := u^\star X$, is a one-dimensional semimartingale with absolutely continuous characteristic triplet (with respect to a truncation function $h$) $\mathbb{T}(X^u|\widetilde{\mathbb{P}}_v) = (B^u_v, C^u_v, \nu^u_v)$ having differential characteristics given by*

$$\begin{aligned} b^u_v &= u^\star\widetilde{b} + u^\star\widetilde{c}v + \left(h\left(u^\star \log(1+z)\right)\mathrm{e}^{v^\star \log(1+z)} - u^\star h\left(\log(1+z)\right)\right) * F\,,\\ c^u_v &= u^\star\widetilde{c}u\,,\\ F^u_v(E) &= \mathbf{1}_E\left(u^\star \log(1+z)\right)\mathrm{e}^{v^\star \log(1+z)} * F\,, \qquad E \in \mathcal{B}(\mathbb{R}\setminus\{0\})\,. \end{aligned} \tag{2.10}$$

The spread is defined by the difference of the two underlying asset prices $S^{(1)}(t)-S^{(2)}(t)$, $t \geq 0$. Thus, the payout function of a European call spread option with strike $K=0$ at maturity date $T$ (sometimes called a zero-exercise spread option) is given by

$$\max(S^{(1)}(T) - S^{(2)}(T), 0)\,. \tag{2.11}$$

This means that the buyer has the right to be paid at the maturity date $T$ the difference $S^{(1)}(T) - S^{(2)}(T)$ whenever it is positive and zero otherwise. We choose the risk-free instantaneous interest rate $r(t) = r(t,\omega)$ to be an $\mathcal{F}_t$-adapted stochastic process. Moreover we assume that it is Lebesgue integrable on any compact subset of the positive real line. We denote the value of the spread option by

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \mathbb{E}\left[\mathrm{e}^{-\int_0^T r(s)\mathrm{d}s} \max\left(S^{(1)}(T) - S^{(2)}(T), 0\right)\right].$$

In the following, we state a Margrabe type formula for a spread option written on the process (2.4). The result is taken from Eberlein et al. [15].

**Theorem 2.2.** *Assume that the asset price process evolves as in* (2.4)*. Then we can relate the value of a spread option and a put option via the following duality*

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = S^{(1)}(0)\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\mathrm{e}^{-\int_0^T (r(s)-a(s))\mathrm{d}s} \max(1 - S^u(T), 0)\right], \tag{2.12}$$

*where* $S^u := S^{(2)}/S^{(1)}$, $a(s) = b_1(s) + \int_{\mathbb{R}^2}\{z_1 - h_1(z)\}F(\mathrm{d}z)$, *the measure* $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_v$, *and the characteristics* $(B^u_v, C^u_v, \nu^u_v)$ *of* $X^u = \log\left(S^u S^{(1)}(0)/S^{(2)}(0)\right)$ *are given by Theorem 2.1 for* $v = (1,0)^\star$ *and* $u = (-1,1)^\star$.

Applying Theorem 2.1 with $v = (1,0)^\star$, we get the density for the dual measure associated with $X$ as $\mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P} = \exp\{Y(T)\}$, where

$$\begin{aligned} Y(t) = &-\frac{1}{2}\int_0^t \left(\sigma_{11}^2(s) + \sigma_{12}^2(s)\right)\,\mathrm{d}s + t\int_{\mathbb{R}^2} \left[h_1\left(\ln(1+z)\right) - z_1\right]\,F(\mathrm{d}z_1\,,\mathrm{d}z_2) \\ &+ \int_0^t \sigma_{11}(s)\,\mathrm{d}W^{(1)}(s) + \int_0^t \sigma_{12}(s)\,\mathrm{d}W^{(2)}(s) \\ &+ \int_0^t\int_{\mathbb{R}^2} h_1(z)\,(\mu^X - \widetilde{\nu})(\mathrm{d}s\,,\mathrm{d}z_1\,,\mathrm{d}z_2) \\ &+ \int_0^t\int_{\mathbb{R}^2} (z_1 - h_1(z))\;\mu^X(\mathrm{d}s\,,\mathrm{d}z_1\,,\mathrm{d}z_2)\,, \qquad 0 \leq t \leq T\,. \end{aligned} \tag{2.13}$$

Furthermore, the dynamics of $X^u = X^{(2)} - X^{(1)}$ under the measure $\widetilde{\mathbb{P}}$ are given by

$$\begin{aligned} X^u(t) = &\int_0^t \{b_2(s) - b_1(s)\}\, \mathrm{d}s - \frac{1}{2}\left\{\int_0^t (\sigma_{11}(s) - \sigma_{12}(s))^2\, \mathrm{d}s + \int_0^t (\sigma_{12}(s) - \sigma_{22}(s))^2\, \mathrm{d}s\right\} \\ &+ t\int_{\mathbb{R}^2} \left\{h\left(\log(\frac{1+z_2}{1+z_1})\right)(1+z_1) + h_1(z) - h_2(z)\right\} F(\mathrm{d}z_1\,, \mathrm{d}z_2) \\ &+ \int_0^t (\sigma_{12}(s) - \sigma_{11}(s))\, \mathrm{d}W^{(1)}_{\widetilde{\mathbb{P}}}(s) + \int_0^t (\sigma_{22}(s) - \sigma_{12}(s))\, \mathrm{d}W^{(2)}_{\widetilde{\mathbb{P}}}(s) \\ &+ \int_0^t \int_{\mathbb{R}} h(z)(\mu^u - \nu^u_v)(\mathrm{d}s\,, \mathrm{d}z) + \int_0^t \int_{\mathbb{R}} (z - h(z))\, \mu^u(\mathrm{d}s\,, \mathrm{d}z)\,, \qquad 0 \le t \le T\,, \end{aligned}$$

where $W_{\widetilde{\mathbb{P}}}$ is a two-dimensional Brownian motion under the dual measure $\widetilde{\mathbb{P}}$ and $\mu^u$ is the jump measure of $X^u$.

**Remark 2.3.** *In the calculations above we have applied $\mathbb{P}$ as pricing measure, without assuming that the underlying discounted price process $S$ is a martingale under $\mathbb{P}$. This means that we are not assuming that $\mathbb{P}$ is a risk neutral measure. In this case the Margrabe type formula* (2.12) *is suitable for markets in which the underlying is not tradable in the usual sense. This is the case, for example, of weather or electricity markets. See e.g. Benth et al.* [4] *for more details.*

*In the case we consider a financial market, then we work under a risk neutral measure, say $\mathbb{Q} \sim \mathbb{P}$, and the discounted price process $S$ will be a martingale under $\mathbb{Q}$ and thus the dynamics of $S$ under $\mathbb{Q}$ correspond to* (2.4) *with*

$$\begin{aligned} b_1(t) + t\int_{\mathbb{R}^2} \{z_1 - h_1(z)\}\, F(\mathrm{d}z) &= b_2(t) + t\int_{\mathbb{R}^2} \{z_2 - h_2(z)\}\, F(\mathrm{d}z) \\ &= r(t)\,, \quad \text{a.s. for all } t\,. \end{aligned} \tag{2.14}$$

*In that case we apply Theorem 2.2 with the dynamics of $S$ written under $\mathbb{Q}$ and the price of the spread option is given by the formula*

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = S^{(1)}(0)\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\max(1 - S^u(T), 0)\right], \tag{2.15}$$

*where $\mathrm{d}\widetilde{\mathbb{Q}}/\mathrm{d}\mathbb{Q} = \exp\{Y(T)\}$ and the process $Y$ is given by equation* (2.13)*. Note that the measure $\widetilde{\mathbb{Q}}$ with respect to the real world measure $\mathbb{P}$ can be defined by $\mathrm{d}\widetilde{\mathbb{Q}}/\mathrm{d}\mathbb{P} = \mathrm{d}\widetilde{\mathbb{Q}}/\mathrm{d}\mathbb{Q} \times \mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ as a composition of two measure changes. We develop these arguments further in Section 2.4 using the Esscher transform and we give an example where the logreturns follow a normal inverse Gaussian process in Section 2.4.*

2.3. **Application: Stochastic volatility model.** We price a zero-strike spread option in case of a two-dimensional stochastic volatility model for the price $S$. The dynamics of the volatility are given by a Barndorff-Nielsen and Shephard model (see Barndorff-Nielsen and Shephard [3]).

To this end, we assume that the price process $S = (S^{(1)}, S^{(2)})$ is already given under a risk neutral measure $\mathbb{Q} \sim \mathbb{P}$, see Remark 2.3. Moreover, we assume $S = S(0)\mathcal{E}(H)$, $S^{(i)}(0) = 1$, $i = 1, 2$, where the differential characteristics of $H$ are given by

$$b(t) = \begin{pmatrix} r(t) \\ r(t) \end{pmatrix}, \qquad c^{\frac{1}{2}}(t) = \begin{pmatrix} \sigma_1(t) & 0 \\ 0 & \sigma_2(t) \end{pmatrix}, \qquad F \equiv 0, \qquad 0 \le t \le T$$

and $H(0) = (0, 0)^\star$. The process $r$ is the risk-free instantaneous interest rate and the volatility is modeled by

$$\mathrm{d}\sigma_i^2(t) = -\lambda_i \sigma_i^2(t)\,\mathrm{d}t + \mathrm{d}L^{(i)}(t), \qquad \sigma_i^2(0) \ge 0,\ i = 1, 2, \qquad 0 \le t \le T, \tag{2.16}$$

where $\lambda_1$ and $\lambda_2$ are positive constants and $L = (L^{(1)}, L^{(2)})$ is a two-dimensional Lévy subordinator, that is, a two-dimensional Lévy process which is non decreasing in each of its coordinates. Note that the marginal Lévy processes $L^{(1)}$ and $L^{(2)}$ can be dependent. We suppose that the Lévy process $L$ has no drift and the Lévy measure has density $\rho(z_1, z_2)$ under $\mathbb{Q}$, so that the Laplace cumulant functions $K_{L^{(i)}}(\theta) := K_{L^{(i)}}(\theta)_1 = \log \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{\theta L^{(i)}(1)}]$, $\theta \in \mathbb{R}$, whenever they exist, take the form

$$K_{L^{(i)}}(\theta) = \int_{\mathbb{R}_+^2} (\mathrm{e}^{\theta z_i} - 1)\rho(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2, \qquad i = 1, 2.$$

Notice that $S = \mathrm{e}^X$, where

$$X^{(i)}(t) = \int_0^t (r(s) - \frac{1}{2}\sigma_i^2(s))\,\mathrm{d}s + \int_0^t \sigma_i(s)\,\mathrm{d}W^{(i)}(s), \qquad 0 \le t \le T.$$

We assume for simplicity that $W^{(1)}$ and $W^{(2)}$ are independent. Note that a subordinator has paths of finite variation since it is monotone increasing. It therefore has to be independent of $W^{(1)}$ and $W^{(2)}$, which are processes with paths of infinite variation.

The solution of (2.16) is given by

$$\sigma_i^2(t) = \mathrm{e}^{-\lambda_i t}\sigma_i^2(0) + \int_0^t \mathrm{e}^{-\lambda_i(t-s)}\,\mathrm{d}L^{(i)}(s), \qquad i = 1, 2, \qquad 0 \le t \le T.$$

We denote the integrated variance over the time period $[0, T]$ by $\sigma_i^{*2}(T) := \int_0^T \sigma_i^2(t)\,\mathrm{d}t$. A simple computation shows that

$$\sigma_i^{*2}(T) = \sigma_i^2(0)(1 - \mathrm{e}^{-\lambda_i T})\lambda_i^{-1} + \int_0^T (1 - \mathrm{e}^{-\lambda_i(T-u)})\lambda_i^{-1}\,\mathrm{d}L^{(i)}(u), \qquad i = 1, 2. \tag{2.17}$$

The risk neutral valuation of the spread option price is given by

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r(s)\,\mathrm{d}s} \max\left(S^{(1)}(T) - S^{(2)}(T), 0\right)\right].$$

In the context of Theorem 2.1, we define the measure $\widetilde{\mathbb{Q}}$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t} = \exp\{Y(t)\}, \qquad 0 \le t \le T, \tag{2.18}$$

where

$$Y(t) = \frac{1}{2}\int_0^t \sigma_1^2(s)\,\mathrm{d}s - \int_0^t \sigma_1(s)\,\mathrm{d}W^{(1)}(s)\,.$$

Notice that the processes $L^{(1)}$ and $L^{(2)}$ remain Lévy processes under the new measure $\widetilde{\mathbb{Q}}$. In fact $\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle}] = \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle}]$. To explain, we have

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle}] = \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle} \exp\left\{\frac{1}{2}\int_0^t \sigma_1^2(s)\,\mathrm{d}s - \int_0^t \sigma_1(s)\,\mathrm{d}W^{(1)}(s)\right\}\right].$$

Denote by $\mathcal{F}_T^L$ the $\sigma$-algebra generated by $L$ up to time $T$. Therefore conditioning on $\mathcal{F}_T^L$, we get

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle}] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle} \exp\left\{\frac{1}{2}\int_0^t \sigma_1^2(s)\,\mathrm{d}s - \int_0^t \sigma_1(s)\,\mathrm{d}W^{(1)}(s)\right\} \Big|\, \mathcal{F}_T^L\right]\right] \\ &= \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{\mathtt{i}\langle\theta, L(t)\rangle}].\end{aligned}$$

In particular the Laplace cumulant function is preserved: $\log \mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{\langle\theta, L(1)\rangle}] = K_L(\theta)$. We obtain the following Proposition for the price of the spread option.

**Proposition 2.4.** *Let $f(x) = \max((1-\mathrm{e}^x), 0)$. Denote by $\widehat{h}$ the Fourier transform of a given function $h$. Then for $R < 0$, the price of the spread option written on $S$ is given by*

$$\begin{aligned}&\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) \\ &\quad = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\mathtt{i}R - \mathtt{u}) \exp\left\{-\sigma_1^2(0) g_1(\mathtt{u}, 0) - \sigma_2^2(0) g_2(\mathtt{u}, 0)\right\} \\ &\qquad \times \exp\left(\int_0^T \{K_{L^{(1)}}(g_1(\mathtt{u}, s)) + K_{L^{(2)}}(g_2(\mathtt{u}, s))\}\mathrm{d}s\right) \mathrm{d}\mathtt{u},\end{aligned}$$

*where $g_1(\mathtt{u}, s) := -\frac{1}{2}(\mathtt{i}\mathtt{u} - R)(\mathtt{i}\mathtt{u} - R + 1)(1 - e^{-\lambda_1(T-s)})\lambda_1^{-1}$, $g_2(\mathtt{u}, s) := -\frac{1}{2}(\mathtt{i}\mathtt{u} - R)(\mathtt{i}\mathtt{u} - R + 1)(1 - e^{-\lambda_2(T-s)})\lambda_2^{-1}$, and $K_{L^{(1)}}$, $K_{L^{(2)}}$ are the Laplace cumulant functions of $L^{(1)}, L^{(2)}$, resp.*

Before stating the proof of this Proposition, let us recall a general result on pricing options by the Fourier transform: suppose the payoff at time $T$ of an option can be expressed by functions $f$ and $\phi$ such that

$$f(Y(T)) = \phi(\mathrm{e}^{Y(t)}, 0 \le t \le T)\,, \tag{2.19}$$

where $\phi$ is the option's payoff function (a call, say) and $S(t) = \mathrm{e}^{Y(t)} = S(0)\mathrm{e}^{X(t)}$, $0 \le t \le T$, is the price process of the underlying asset of the option. Then the following result holds (see Eberlein et al. [13] for this Theorem and its proof):

**Theorem 2.5.** *Let $Y$ be a jump-diffusion in $\mathbb{R}$ and $L^1_{bc}(\mathbb{R})$ denotes the space of bounded continuous functions in $L^1(\mathbb{R})$. Let $f$ be as in (2.19). Assume that for some $R \in \mathbb{R}$ we have*

(1) $\mathrm{e}^{-Rx} f(x) \in L^1_{bc}(\mathbb{R})$,
(2) $\mathtt{u} \mapsto \mathbb{E}[\mathrm{e}^{(R-\mathtt{iu})Y(T)}]\widehat{f}(\mathtt{i}R - \mathtt{u}) \in L^1(\mathbb{R})$,

*Then we have*

$$\mathbb{E}[f(Y(T))] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{E}[\mathrm{e}^{(R-\mathtt{iu})Y(T)}]\widehat{f}(\mathtt{i}R - \mathtt{u})\,\mathrm{d}\mathtt{u}\,.$$

*Proof of Proposition 2.4.* From Theorem 2.2 and Remark 2.3, the risk neutral formula for the spread option price is given by

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\max(1 - S^u(T), 0)\right].$$

Here $\widetilde{\mathbb{Q}}$ is defined in (2.18) and $S^u = \exp(X^u)$, where

$$X^u(t) = -\frac{1}{2}\int_0^t \left(-\sigma_1^2(s) + \sigma_2^2(s)\right)\,\mathrm{d}s - \int_0^t \sigma_1(s)\,\mathrm{d}W^{(1)}_{\widetilde{\mathbb{Q}}}(s) + \int_0^t \sigma_2(s)\,\mathrm{d}W^{(2)}_{\widetilde{\mathbb{Q}}}(s).$$

Notice that the option price takes the form $\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(X^u(T)]$, where $f(x) = \max(1 - \mathrm{e}^x, 0)$. Eberlein et al. [13] show that for $R < 0$, condition (1) holds true and $\widehat{f}(\mathtt{i}R - \cdot) \in L^1(\mathbb{R})$, for this particular choice of $f$. Thus to show (2), it is enough to check that $\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}]$ is bounded in $\mathtt{u}$ by a constant. We have

$$|\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}]| \le \mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{RX^u(T)}]\,.$$

To check that the latter is finite is similar to analogous computations that we will perform below, so we do not detail the argument here. Thus, applying Theorem 2.5, we get for $R < 0$,

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}]\widehat{f}(\mathtt{i}R - \mathtt{u})\,\mathrm{d}\mathtt{u}.$$

Therefore to compute the spread option price, we need to compute $\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}]$. To this end we see that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}] = \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\exp\left\{(R-\mathtt{iu})\left(-\frac{1}{2}\int_0^T(\sigma_1^2(s)+\sigma_2^2(s))\,\mathrm{d}s - \int_0^T \sigma_1(s)\,\mathrm{d}W^{(1)}_{\widetilde{\mathbb{Q}}}(s) + \int_0^T \sigma_2(s)\,\mathrm{d}W^{(2)}_{\widetilde{\mathbb{Q}}}(s)\right)\right\}\right].$$

Conditioning on $\mathcal{F}_T^L$, and recalling the expressions of $\sigma_1^{*2}(T)$ and $\sigma_2^{*2}(T)$ in (2.17), we get

$$\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}]\\
&= \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\exp\left\{\frac{1}{2}(\mathtt{iu}-R)(\sigma_1^{*2}(T)+\sigma_2^{*2}(T))\right\}\right.\\
&\qquad \left.\times\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\exp\left\{(\mathtt{iu}-R)\left(\int_0^T\sigma_1(s)\,\mathrm{d}W^{(1)}_{\widetilde{\mathbb{Q}}}(s) - \int_0^T\sigma_2(s)\,\mathrm{d}W^{(2)}_{\widetilde{\mathbb{Q}}}(s)\right)\right\}\,\middle|\,\mathcal{F}_T^L\right]\right]\\
&= \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\exp\left\{\frac{1}{2}(\mathtt{iu}-R)(\mathtt{iu}-R+1)\sigma_1^{*2}(T)+\frac{1}{2}(\mathtt{iu}-R)(\mathtt{iu}-R+1)\sigma_2^{*2}(T))\right\}\right].
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}]\\
&\quad= \exp\left\{\frac{1}{2}(\mathtt{iu}-R)(\mathtt{iu}-R+1)\sigma_1^2(0)(1-\mathrm{e}^{-\lambda_1 T})\lambda_1^{-1}\right.\\
&\qquad\qquad \left.+\frac{1}{2}(\mathtt{iu}-R)(\mathtt{iu}-R+1)\sigma_2^2(0)(1-\mathrm{e}^{-\lambda_2 T})\lambda_2^{-1}\right\}\\
&\qquad\qquad\qquad \times\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\exp\left\{\int_0^T g_1(\mathtt{u},s)\,\mathrm{d}L^{(1)}(s)+\int_0^T g_2(\mathtt{u},s)\,\mathrm{d}L^{(2)}(s)\right\}\right].
\end{aligned}$$

Using an extension of the key formula in Eberlein and Raible [12], it holds that

$$\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{iu})X^u(T)}] &= \exp\left\{-\sigma_1^2(0)g_1(\mathtt{u},0)-\sigma_2^2(0)g_2(\mathtt{u},0)\right\}\\
&\qquad\times\exp\left(\int_0^T\{K_{L^{(1)}}(g_1(\mathtt{u},s))+K_{L^{(2)}}(g_2(\mathtt{u},s))\}\,\mathrm{d}s\right)
\end{aligned}$$

and the result follows. □

Next we consider a spread option written on an asset price dynamics defined by an exponential bivariate Lévy process.

2.4. **Application: Exponential Lévy processes and Esscher transforms.** We suppose that the risk-free rate of return is given by a positive constant, that is $r(t) = r > 0$, for all $0 \le t \le T$. We consider a price process

$$S(t) = S(0)\mathrm{e}^{L(t)} \qquad 0 \le t \le T\,, \tag{2.20}$$

where $S(0) = (1,1)$ and $L$ is a bivariate Lévy process with differential characteristics $(b,c,F)$. We define a probability $\mathbb{P}_\theta$ by means of the Esscher transform of a parameter $\theta \in \mathbb{R}^2$ as follows (see Gerber and Shiu [16])

$$\frac{\mathrm{d}\mathbb{P}_\theta}{\mathrm{d}\mathbb{P}} = \exp\{\theta^\star L(T) - K_L(\theta)_T\}\,, \tag{2.21}$$

where $K_L(\theta)$ is the cumulant function of the Lévy process $L$. We first apply an Esscher transform with parameter $\theta$ such that the corresponding measure $\mathbb{P}_\theta$ is risk neutral for the price dynamics (see Gerber and Shiu [16]) and the spread option price $\mathbb{M}(S^{(1)}(T), S^{(2)}(T))$ can be written as an expectation under $\mathbb{P}_\theta$. Afterwards, we apply the Margrabe formula as in Theorem 2.2 and Remark 2.3 and we write $\mathbb{M}(S^{(1)}(T), S^{(2)}(T))$ as an expectation under the dual measure $\widetilde{\mathbb{P}}_\theta$. Furthermore, we explore the relations between the real world measure $\mathbb{P}$, the risk neutral measure $\mathbb{P}_\theta$ and the dual measure $\widetilde{\mathbb{P}}_\theta$ in terms of Esscher transforms. In fact, the dual measure can be specified with respect to $\mathbb{P}$ directly through a single Esscher transform with parameter $\theta + \mathbf{1}_1$, where $\mathbf{1}_1$ denotes the first unit vector as we see in the computations below.

Suppose there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^2} \mathrm{e}^{\langle x,z\rangle}\, F(\mathrm{d}z) < \infty\,, \qquad \text{for all } |x| \le c. \tag{2.22}$$

This condition ensures finite exponential moments for $L(1)$ up to order $c$. Thus the Radon-Nikodym derivative in (2.21) is well defined. We define the parameter $\theta$ such that, for $i = 1,2$, the discounted price process $\mathrm{e}^{-rt}S^{(i)}(t)$ is a martingale. We denote by $\mathbb{E}_\theta$ the expectation under the new measure $\mathbb{P}_\theta$. Hence

$$S^{(i)}(0) = 1 = \mathbb{E}_\theta[\mathrm{e}^{-rt}S^{(i)}(t)]\,,$$

which is equivalent to

$$\mathrm{e}^{rt} = \mathbb{E}_\theta[\mathrm{e}^{L^{(i)}(t)}] = \mathbb{E}_\mathbb{P}\left[\exp\{L^{(i)}(t) + \theta^\star L(t) - K_L(\theta)_t\}\right] = \exp\left(K_L(\mathbf{1}_i + \theta)_t - K_L(\theta)_t\right), \tag{2.23}$$

where $\mathbf{1}_i$ denotes the $i$th unit vector. The existence and uniqueness of the parameter $\theta$ which verifies (2.23) is proved in Gerber and Shiu [17]. By the risk neutral valuation rule, the price of the spread option is then given by

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \mathrm{e}^{-rT}\mathbb{E}_\theta\left[S^{(1)}(T)\max\left(1 - S^u(T), 0\right)\right].$$

Let $v = \mathbf{1}_1$. We define

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_\theta}{\mathrm{d}\mathbb{P}_\theta} = \exp\{v^\star L(T) - K_L^\theta(v)_T\}\,, \tag{2.24}$$

where $K_L^\theta(v)_T = \log \mathbb{E}_\theta[\mathrm{e}^{v^\star L(T)}]$. Using (2.23), we deduce $\mathrm{d}\widetilde{\mathbb{P}}_\theta/\mathrm{d}\mathbb{P}_\theta = \exp\{-rT + L^{(1)}(T)\}$. Furthermore, we get

$$\begin{aligned}\frac{\mathrm{d}\widetilde{\mathbb{P}}_\theta}{\mathrm{d}\mathbb{P}} &= \frac{\mathrm{d}\widetilde{\mathbb{P}}_\theta}{\mathrm{d}\mathbb{P}_\theta}\frac{\mathrm{d}\mathbb{P}_\theta}{\mathrm{d}\mathbb{P}} \\ &= \exp\{-rT + L^{(1)}(T)\}\exp\{\theta^\star L(T) - K_L(\theta)_T\} \\ &= \exp\left\{(\mathbf{1}_1+\theta)^\star L(T) - K_L(\mathbf{1}_1+\theta)_T\right\} \\ &=: \frac{\mathrm{d}\mathbb{P}_{\theta+\mathbf{1}_1}}{\mathrm{d}\mathbb{P}}\,.\end{aligned}$$

Thus $\widetilde{\mathbb{P}}_\theta$ corresponds to the measure $\mathbb{P}_{\theta+\mathbf{1}_1}$ defined through an Esscher transform with respect to the real world measure $\mathbb{P}$. Then applying Theorem 2.2 and Remark 2.3, it follows

$$\mathbb{M}(S^{(1)}(T), S^{(2)}(T)) = \mathbb{E}_{\theta+\mathbf{1}_1}\left[\max(1-S^u(T),0)\right], \tag{2.25}$$

where $S^u = \mathrm{e}^{L^u}$, $L^u := u^\star L$, and $u = (-1,1)^\star$. The differential characteristics of $L^u$ under the new measure $\mathbb{P}_{\theta+\mathbf{1}_1}$ are given by $(b^u_{\theta+\mathbf{1}_1}, c^u_{\theta+\mathbf{1}_1}, F^u_{\theta+\mathbf{1}_1})$, where (see Theorem 2.1)

$$\begin{aligned} b^u_{\theta+\mathbf{1}_1} &= u^\star b + u^\star c(\theta+\mathbf{1}_1) + \big(h(u^\star z)\mathrm{e}^{(\theta+\mathbf{1}_1)^\star z} - u^\star h(z)\big) * F\,,\\ c^u_{\theta+\mathbf{1}_1} &= u^\star c u\,,\\ F^u_{\theta+\mathbf{1}_1}(E) &= \mathbf{1}_E(u^\star z)\mathrm{e}^{(\theta+\mathbf{1}_1)^\star z} * F\,, \qquad E \in \mathcal{B}(\mathbb{R}\setminus\{0\})\,. \end{aligned} \tag{2.26}$$

Now we consider an example in which we apply the Esscher transform approach presented above for the evaluation of spread options.

*Example: Normal inverse Gaussian Lévy process.* Given the parameters of the distribution of a normal inverse Gaussian (NIG) Lévy process under the real world measure $\mathbb{P}$, one can derive parameters under a risk neutral measure $\mathbb{P}_\theta$ using the Esscher transform as in Benth and Henriksen [5]. The bivariate NIG distribution has parameters $\alpha > 0$, $\beta \in \mathbb{R}^2$, $\mu \in \mathbb{R}^2$, $\delta > 0$, and $\Delta \in \mathbb{R}^{2\times 2}$, where $\Delta$ is a positive definite matrix with determinant 1 (see Barndorff-Nielsen [2] and Rydberg [24] for more about bivariate NIG distributions). Let $L$ be a Lévy process such that $L(1) \sim NIG(\alpha,\beta,\mu,\delta,\Delta)$ under $\mathbb{P}$. In this case $L(t) \sim NIG(\alpha,\beta,t\mu,t\delta,\Delta)$, for all $t > 0$. The density function of $L(1)$ takes the form

$$\zeta(z) = \frac{\delta}{\sqrt{2}}\left(\frac{\alpha}{\pi q(z)}\right)^{\frac{3}{2}} \exp(p(z))\mathcal{K}_{\frac{3}{2}}(\alpha q(z))\,, \tag{2.27}$$

where $\mathcal{K}_{\frac{3}{2}}$ is the modified Bessel function of second kind of order $\frac{3}{2}$ and

$$\begin{aligned} p(z) &= \delta\sqrt{\alpha^2 - \beta^\star\Delta\beta} + \beta^\star(z-\mu),\\ q(z) &= \sqrt{\delta^2 + (z-\mu)^\star\Delta^{-1}(z-\mu)}. \end{aligned}$$

The parameters have the following interpretation: $\alpha$ corresponds to the tail heaviness of the marginals and $\delta$ is the scaling of the distribution. The centring is described by $\mu$ and

$\beta$ controls the skewness. The dependency structure between the marginals is modelled by $\Delta$. We define $\psi_L(s) := K_L(-\mathbb{i}s)_1 = \log \mathbb{E}[\mathrm{e}^{\mathbb{i}\langle s, L(1)\rangle}]$, $s \in \mathbb{R}^2$, to be the exponent of the characteristic function of $L(1)$. Then $\psi_L$ is given by

$$\psi_L(s) = \delta\sqrt{\alpha^2 - \beta^\star\Delta\beta} - \delta\sqrt{\alpha^2 - (\beta + \mathbb{i}s)^\star\Delta(\beta + \mathbb{i}s)} + \mathbb{i}s^\star\mu\,. \tag{2.28}$$

The price dynamics for the stocks are given by $S^{(1)}(t) = \exp\{L^{(1)}(t)\}$ and $S^{(2)}(t) = \exp\{L^{(2)}(t)\}$, $i = 1, 2$. Define a probability measure $\mathbb{P}_\theta \sim \mathbb{P}$, for $\theta \in \mathbb{R}^2$ through the Esscher transform as in (2.21). Calculating the characteristic function, it follows that under $\mathbb{P}_\theta$,

$$L(1) \sim \ NIG(\alpha, \beta + \theta, \mu, \delta, \Delta)\,.$$

We choose the parameter $\theta$ such that we have risk neutral dynamics. In this case the discounted price process is a $\mathbb{P}_\theta$ martingale, where discounting is done using the risk-free interest rate $r > 0$. Hence from (2.23), we have

$$\psi_L\left(\mathbb{i}(\mathbf{1}_i + \theta)\right) - \psi_L(\mathbb{i}\theta) = r\,, \quad i = 1, 2\,.$$

The latter condition is equivalent to the following system of equations,

$$\begin{aligned} r &= \mu_1 - \delta\sqrt{\alpha^2 - [\beta_1 + 1 + \theta_1, \beta_2 + \theta_2]\Delta\begin{bmatrix} \beta_1 + 1 + \theta_1 \\ \beta_2 + \theta_2 \end{bmatrix}} \\ &\quad + \delta\sqrt{\alpha^2 - [\beta_1 + \theta_1, \beta_2 + \theta_2]\Delta\begin{bmatrix} \beta_1 + \theta_1 \\ \beta_2 + \theta_2 \end{bmatrix}}, \\ r &= \mu_2 - \delta\sqrt{\alpha^2 - [\beta_1 + \theta_1, \beta_2 + 1 + \theta_2]\Delta\begin{bmatrix} \beta_1 + \theta_1 \\ \beta_2 + 1 + \theta_2 \end{bmatrix}} \\ &\quad + \delta\sqrt{\alpha^2 - [\beta_1 + \theta_1, \beta_2 + \theta_2]\Delta\begin{bmatrix} \beta_1 + \theta_1 \\ \beta_2 + \theta_2 \end{bmatrix}}\,. \end{aligned}$$

We define the measure $\mathbb{P}_{\theta + \mathbf{1}_1} = \widetilde{\mathbb{P}}_\theta$ as in (2.24). Notice that $\mathrm{d}\mathbb{P}_{\theta+\mathbf{1}_1}/\mathrm{d}\mathbb{P}_\theta = \mathrm{e}^{-rT + L^{(1)}(T)}$. It follows that under $\mathbb{P}_{\theta+\mathbf{1}_1}$

$$(L^{(1)}(1), L^{(2)}(1)) \sim \ NIG(\alpha, \beta_{\theta+\mathbf{1}_1}, \mu, \delta, \Delta)\,,$$

where $\beta_{\theta+\mathbf{1}_1} = \beta + \theta + \mathbf{1}_1$. We are interested in the characteristics of $S^u = \exp\{L^{(2)} - L^{(1)}\}$ under the measure $\mathbb{P}_{\theta+\mathbf{1}_1}$. We observe that

$$\psi^{\theta+\mathbf{1}_1}_{L^{(2)}-L^{(1)}}(s) := \ln \mathbb{E}_{\theta+\mathbf{1}_1}[\mathrm{e}^{\mathbb{i}s(L^{(2)}(1) - L^{(1)}(1))}] = \psi^{\theta+\mathbf{1}_1}_{(L^{(1)}, L^{(2)})}(-s, s)\,,$$

where $\psi_L^{\theta+\mathbf{1}_1}(s_1, s_2)$ is given by (2.28) with skewness $\beta_{\theta+\mathbf{1}_1}$. Thus we have

$$\psi^{\theta+\mathbf{1}_1}_{L^{(2)}-L^{(1)}}(s) = \widetilde{\delta}\sqrt{\widetilde{\alpha}^2 - \widetilde{\beta}_\theta^2} - \widetilde{\delta}\sqrt{\widetilde{\alpha}^2 - (\widetilde{\beta}_\theta + \mathbb{i}s)^2} + \mathbb{i}s\widetilde{\mu}\,, \tag{2.29}$$

where $\widetilde{\delta} = \delta\sqrt{z_1}$, $\widetilde{\alpha}^2 = \frac{1}{z_1}(\alpha^2 - \beta^\star_{\theta+\mathbf{1}_1}\Delta\beta_{\theta+\mathbf{1}_1} + \widetilde{\beta}_\theta^2)$, $\widetilde{\beta}_\theta = \frac{z_2}{2z_1}$, $\widetilde{\mu} = \mu_2 - \mu_1$, $z_1 = u^\star\Delta u$, $z_2 = u^\star\Delta\beta_{\theta+\mathbf{1}_1} + \beta^\star_{\theta+\mathbf{1}_1}\Delta u$, and $u = (-1, 1)^\star$. Hence $\psi^{\theta+\mathbf{1}_1}_{L^{(2)}-L^{(1)}}(s)$ is the exponent of a characteristic function of a NIG-distribution with parameters $\widetilde{\alpha}, \widetilde{\beta}_\theta, \widetilde{\mu}, \widetilde{\delta}$. We conclude that $L^{(2)} - L^{(1)}$ is a NIG Lévy process under $\mathbb{P}_{\theta+\mathbf{1}_1}$ and one can compute the price of the European spread option by means of Fourier transform. We can follow the same approach as in Proposition 2.4, however, with a different characteristic function for $(L^{(1)}, L^{(2)})$.

## 3. Robustness of spread options

In this section we consider two models to describe the dynamics of the price process. The first model is a jump-diffusion dynamics, where the driving Lévy process may be of infinite activity. In the second model we replace the small jumps by a continuous martingale which is chosen such that the variance is preserved with respect to the original model. The second model can be viewed as an approximation of the first one. The aim is to study the robustness of spread option prices towards the model choice of jumps. We refer to Benth et al. [6] Section 3.2 for a discussion on the relationship between the Brownian motion in the price dynamics and the Brownian motion of the approximating dynamics from the point of view of an efficient computation of option prices.

3.1. **Robustness of a bivariate Lévy process.** We first consider a pure-jump bivariate Lévy process $(L^{(1)}, L^{(2)})$ with differential characteristics $(0,0,F)$. From now on we choose the truncation function $h(z) = z\mathbf{1}_{\{|z|\le 1\}}$. We assume that the Lévy measure satisfies

$$\int_{\mathbb{R}^2} z_i^2 F(\mathrm{d}z_1, \mathrm{d}z_2) < \infty\,, \qquad i = 1,2\,.$$

We define $F_\varepsilon(\mathrm{d}z) = \mathbf{1}_{\{|z|>\varepsilon\}} F(\mathrm{d}z)$ and $\bar{F}_\varepsilon(\mathrm{d}z) = \mathbf{1}_{\{|z|\le\varepsilon\}} F(\mathrm{d}z)$. We can write $(L^{(1)}, L^{(2)})$ as the sum of the two following Lévy processes

$$\begin{pmatrix} L^{(1)}(t) \\ L^{(2)}(t) \end{pmatrix} = \begin{pmatrix} L^{(1)}_{1,\varepsilon}(t) \\ L^{(2)}_{1,\varepsilon}(t) \end{pmatrix} + \begin{pmatrix} L^{(1)}_{2,\varepsilon}(t) \\ L^{(2)}_{2,\varepsilon}(t) \end{pmatrix}, \qquad 0 \le t \le T\,, \tag{3.1}$$

where $(L^{(1)}_{1,\varepsilon}, L^{(2)}_{1,\varepsilon})$, $(L^{(1)}_{2,\varepsilon}, L^{(2)}_{2,\varepsilon})$ have the differential characteristics $(0,0,\bar{F}_\varepsilon)$, $(0,0,F_\varepsilon)$, resp. We allow the processes $L^{(1)}$ and $L^{(2)}$ to be dependent. Thus the processes $L^{(1)}_{1,\varepsilon}$ and $L^{(2)}_{1,\varepsilon}$ are dependent. The same holds true for $L^{(1)}_{2,\varepsilon}$ and $L^{(2)}_{2,\varepsilon}$. We denote the characteristic function of a bivariate Lévy process $L$ by

$$\Phi_{L(t)}(x) := \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle x, L(t)\rangle}], \qquad \forall x \in \mathbb{R}^2, \quad 0 \le t \le T\,.$$

It is easy to see that the characteristic function of the process $(L^{(1)}, L^{(2)})$ is given by

$$\Phi_{(L^{(1)}(t), L^{(2)}(t))}(x_1, x_2) \;=\; \Phi_{(L^{(1)}_{1,\varepsilon}(t), L^{(2)}_{1,\varepsilon}(t))}(x_1, x_2)\Phi_{(L^{(1)}_{2,\varepsilon}(t), L^{(2)}_{2,\varepsilon}(t))}(x_1, x_2)\,, \tag{3.2}$$

from which we conclude that the couples $(L^{(1)}_{1,\varepsilon}, L^{(2)}_{1,\varepsilon})$ and $(L^{(1)}_{2,\varepsilon}, L^{(2)}_{2,\varepsilon})$ are independent. The variance-covariance matrix of $(L^{(1)}_{1,\varepsilon}, L^{(2)}_{1,\varepsilon})$ is given by

$$\Sigma(\varepsilon) = \int_{\mathbb{R}^2} (z_1, z_2)(z_1, z_2)^\star\, \bar{F}_\varepsilon(\mathrm{d}z_1, \mathrm{d}z_2) = \begin{pmatrix} \sigma_{11}^2(\varepsilon) & \sigma_{12}(\varepsilon) \\ \sigma_{12}(\varepsilon) & \sigma_{22}^2(\varepsilon) \end{pmatrix}. \tag{3.3}$$

We approximate the Lévy process $L$ by a two-dimensional Lévy process $(L_\varepsilon^{(1)}, L_\varepsilon^{(2)})$ with differential characteristics $(0, c(\varepsilon), F_\varepsilon)$. We choose $c(\varepsilon) = \Sigma^{\frac{1}{2}}(\varepsilon)$. The approximating Lévy process is thus given by

$$\begin{pmatrix} L_\varepsilon^{(1)}(t) \\ L_\varepsilon^{(2)}(t) \end{pmatrix} = c(\varepsilon) \begin{pmatrix} W^{(1)}(t) \\ W^{(2)}(t) \end{pmatrix} + \begin{pmatrix} L_{2,\varepsilon}^{(1)}(t) \\ L_{2,\varepsilon}^{(2)}(t) \end{pmatrix}, \qquad 0 \le t \le T, \tag{3.4}$$

where $W^{(1)}$ and $W^{(2)}$ are two independent standard Brownian motions.

In Asmussen and Rosinski [1] it is shown that the jumps smaller than $\varepsilon$ of a one-dimensional Lévy process behave (asymptotically) very similar in distribution to a Brownian motion scaled with the standard deviation of the small jumps. Cohen and Rosinski [11] extended this result to a multidimensional setting. Kohatsu-Higa and Tankov [21] showed that for many Lévy processes, replacing the small jumps by a scaled Brownian motion, gives better convergence rates for the related option prices compared with simply truncating the small jumps in the approximation. This provides a rationale for our choice of the matrix $c(\varepsilon)$. Indeed our aim is to keep unchanged the variance-covariance matrix of $(L_{1,\varepsilon}^{(1)}, L_{1,\varepsilon}^{(2)})$.

We first show that the coefficients of the matrix $c(\varepsilon)$ vanish when $\varepsilon$ goes to 0. We have

$$\mathrm{Var}\left( \begin{pmatrix} c_1(\varepsilon) & c_2(\varepsilon) \\ c_2(\varepsilon) & c_3(\varepsilon) \end{pmatrix} \begin{pmatrix} W^{(1)}(t) \\ W^{(2)}(t) \end{pmatrix} \right) = t\Sigma(\varepsilon). \tag{3.5}$$

Since $W^{(1)}$ and $W^{(2)}$ are two independent Brownian motions, (3.5) is equivalent to

$$\mathrm{Var}\left( \begin{pmatrix} c_1(\varepsilon) W^{(1)} \\ c_2(\varepsilon) W^{(1)} \end{pmatrix} \right) + \mathrm{Var}\left( \begin{pmatrix} c_2(\varepsilon) W^{(2)}(t) \\ c_3(\varepsilon) W^{(2)}(t) \end{pmatrix} \right) = t\Sigma(\varepsilon)$$

and we get the following set of equations for the coefficients $c_1(\varepsilon)$, $c_2(\varepsilon)$, and $c_3(\varepsilon)$

$$\begin{aligned} c_1^2(\varepsilon) + c_2^2(\varepsilon) &= \sigma_{11}^2(\varepsilon), \\ c_1(\varepsilon)c_2(\varepsilon) + c_2(\varepsilon)c_3(\varepsilon) &= \sigma_{12}(\varepsilon), \\ c_2^2(\varepsilon) + c_3^2(\varepsilon) &= \sigma_{22}^2(\varepsilon). \end{aligned} \tag{3.6}$$

We know that $\sigma_{11}^2(\varepsilon)$, $\sigma_{12}^2(\varepsilon)$, and $\sigma_{22}^2(\varepsilon)$ vanish when $\varepsilon$ goes to 0. Therefore $c_1(\varepsilon)$, $c_2(\varepsilon)$, and $c_3(\varepsilon)$ converge to 0 when $\varepsilon$ goes to 0. We use this fact to prove the following result.

**Proposition 3.1.** *Let the process $L, L_\varepsilon$ be as defined in* (3.1), (3.4), *resp. Then, for every* $0 \le t \le T$,

$$\lim_{\varepsilon \to 0} (L_\varepsilon^{(1)}(t), L_\varepsilon^{(2)}(t)) = (L^{(1)}(t), L^{(2)}(t)), \quad \mathbb{P} - a.s.$$

*In fact, the limit above also holds in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ with*

$$\begin{aligned} \mathbb{E}\left[|L_\varepsilon^{(1)}(t) - L^{(1)}(t)|\right] &\le (c_1(\varepsilon) + c_2(\varepsilon) + \sigma_{11}(\varepsilon))\sqrt{t}, \\ \mathbb{E}\left[|L_\varepsilon^{(2)}(t) - L^{(2)}(t)|\right] &\le (c_2(\varepsilon) + c_3(\varepsilon) + \sigma_{22}(\varepsilon))\sqrt{t}, \qquad 0 \le t \le T \end{aligned} \tag{3.7}$$

*Proof.* The $\mathbb{P}$-a.s. convergence follows directly from the proof of the Lévy-Kintchine formula (See Theorem 19.2 in Sato [25]). Concerning the $L^1$-convergence, we argue as follows. Applying the triangle and Cauchy-Schwarz inequalities, we get

$$\begin{aligned}\mathbb{E}\left[|L^{(1)}(t)-L^{(1)}_\varepsilon(t)|\right] &= \mathbb{E}\left[\left|c_1(\varepsilon)W^{(1)}(t)+c_2(\varepsilon)W^{(2)}(t)-\int_0^t\int_{|z|\le\varepsilon} z_1\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}z_1,\mathrm{d}z_2)\right|\right]\\ &\le c_1(\varepsilon)\mathbb{E}\left[|W^{(1)}(t)|\right]+c_2(\varepsilon)\mathbb{E}\left[|W^{(2)}(t)|\right]\\ &\quad+\mathbb{E}\left[\left|\int_0^t\int_{|z|\le\varepsilon} z_1\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}z_1,\mathrm{d}z_2)\right|\right]\\ &\le c_1(\varepsilon)\mathbb{E}\left[|W^{(1)}(t))|^2\right]^{\frac12}+c_2(\varepsilon)\mathbb{E}\left[|W^{(2)}(t))|^2\right]^{\frac12}\\ &\quad+\mathbb{E}\left[\left|\int_0^t\int_{|z|\le\varepsilon} z_1\,(\mu-\nu)(\mathrm{d}s,\mathrm{d}z_1,\mathrm{d}z_2)\right|^2\right]^{\frac12}\\ &\le (c_1(\varepsilon)+c_2(\varepsilon)+\sigma_{11}(\varepsilon))\sqrt{t}\,.\end{aligned}$$

We can prove (3.7) in the same manner and the result follows. □

3.2. **Robustness of the price process.** Recall the expression of $H$ in (2.1). We approximate $H$ by a process $H_\varepsilon$ which has differential characteristics under $\mathbb{P}$ given by $(b, c+\Sigma(\varepsilon), F_\varepsilon)$, where $\Sigma(\varepsilon)$ is as in (3.3) and $F_\varepsilon(\mathrm{d}z)=F(\mathrm{d}z)\mathbf{1}_{\{|z|>\varepsilon\}}$. That is

$$\begin{aligned}H_\varepsilon(t) &= \int_0^t b(s)\,\mathrm{d}s+\int_0^t c^{\frac12}(s)\,\mathrm{d}W(s)+\Sigma^{\frac12}(\varepsilon)\widetilde{W}(t)+\int_0^t\int_{\mathbb{R}^2} h(z)\,(\mu_\varepsilon-\nu_\varepsilon)\,(\mathrm{d}s,\mathrm{d}z)\\ &\quad+\int_0^t\int_{\mathbb{R}^2}(z-h(z))\,\mu_\varepsilon\,(\mathrm{d}s,\mathrm{d}z)\,,\qquad 0\le t\le T\,,\end{aligned}\tag{3.8}$$

where $\widetilde{W}$ is a two-dimensional standard Brownian motion independent of $W$, $\mu_\varepsilon$ is a Poisson random measure with compensator $\nu_\varepsilon([0,t]\times E)=t\int_E F_\varepsilon(\mathrm{d}z)$, $E\in\mathcal{B}(\mathbb{R}^2)$. We consider the approximating price process

$$S_\varepsilon(t)=S(0)\mathcal{E}(H_\varepsilon)_t\,,\qquad 0\le t\le T\,.\tag{3.9}$$

We know that it exists a semimartingale $X_\varepsilon$ such that

$$S_\varepsilon(t)=S(0)\mathrm{e}^{X_\varepsilon(t)}\,,\qquad 0\le t\le T\,.\tag{3.10}$$

The differential characteristics of $X_\varepsilon$ are given by

$$\widetilde{b}_\varepsilon=b-\frac12(c+\Sigma(\varepsilon))+(h(\log(1+z))-h(z))*F_\varepsilon\,,$$

$$\widetilde{c}_\varepsilon = c + \Sigma(\varepsilon)\,,$$

$$\mathbf{1}_A * \widetilde{F}_\varepsilon = \mathbf{1}_A(\log(1+z)) * F_\varepsilon\,, \quad A \in \mathcal{B}(\mathbb{R}^2 \setminus \{0\})\,. \tag{3.11}$$

We state the following lemma which shows the boundedness of $S$ and $S_\varepsilon$. The proof is similar to the proof of Lemma 3.2 in Benth et al. [6].

**Lemma 3.2.** *Let $S$, $S_\varepsilon$ be as in (2.4), (3.9), resp. Then we have*

$$\|S^{(i)}(t)\|_2^2, \|S_\varepsilon^{(i)}(t)\|_2^2 \le C(T)\,, \qquad \forall \quad 0 \le t \le T\,, \quad i = 1, 2\,,$$

*where $C(T)$ is a positive constant depending on $T$ and independent of $\varepsilon$.*

In the following proposition we prove the convergence of $S_\varepsilon^{(i)}$ to $S^{(i)}$ in $L^2(\Omega)$ with rate $\sigma_{ii}^2(\varepsilon)$, $i = 1, 2$.

**Proposition 3.3.** *Assume*

$$\sup_{0\le s\le T} |b_i(s)| < C_1, \qquad \sup_{0\le s\le T} |\sigma_{ij}(s)| < C_2, \quad i,j = 1,2\,, \quad \mathbb{P}-a.s.\,, \tag{3.12}$$

*where $C_1$ and $C_2$ are positive constants. Let $S$, $S_\varepsilon$ be the solutions of (2.4), (3.9), resp. Then we have, for every $0 \le t \le T < \infty$*

$$\|S^{(1)}(t) - S_\varepsilon^{(1)}(t)\|_2^2 \le A_1(T)\sigma_{11}^2(\varepsilon)\,,$$
$$\|S^{(2)}(t) - S_\varepsilon^{(2)}(t)\|_2^2 \le A_2(T)\sigma_{22}^2(\varepsilon)\,,$$

*and $A_1(T)$ and $A_2(T)$ are positive constants depending on $T$ and independent of $\varepsilon$.*

*Proof.* We prove the result for the process $S^{(1)}$. The proof for $S^{(2)}$ follows the same lines. From Protter [23], Chapter II, we know that we can write $S^{(1)}$ and $S_\varepsilon^{(1)}$ as follows

$$S^{(1)}(t) = S(0) + \int_0^t S^{(1)}(s-)\,\mathrm{d}H^{(1)}(s)\,,$$

$$S_\varepsilon^{(1)}(t) = S^{(1)}(0) + \int_0^t S_\varepsilon^{(1)}(s-)\,\mathrm{d}H_\varepsilon^{(1)}(s)\,, \qquad 0 \le t \le T\,.$$

Hence,

$$\begin{aligned} &S^{(1)}(t) - S_\varepsilon^{(1)}(t) \\ &\quad = \int_0^t \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\}\, b_1(s)\,\mathrm{d}s - c_1(\varepsilon)\int_0^t S_\varepsilon^{(1)}(s)\mathrm{d}\widetilde{W}^{(1)}(s) - c_2(\varepsilon)\int_0^t S_\varepsilon^{(1)}(s)\mathrm{d}\widetilde{W}^{(2)}(s) \\ &\qquad + \int_0^t \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\}\,\sigma_{11}(s)\,\mathrm{d}W^{(1)}(s) + \int_0^t \{S^{(1)}(s) - S_\varepsilon^{(1)}(s)\}\,\sigma_{12}(s)\,\mathrm{d}W^{(2)}(s) \end{aligned}$$

$$+\int_0^t \int_{|z|>\varepsilon} \{S^{(1)}(s-) - S^{(1)}_\varepsilon(s-)\} h_1(z)(\mu-\nu)(\mathrm{d}s,\mathrm{d}z)$$

$$+\int_0^t \int_{\mathbb{R}} \{S^{(1)}(s-) - S^{(1)}_\varepsilon(s-)\}(z_1 - h_1(z))\mu(\mathrm{d}s,\mathrm{d}z)$$

$$+\int_0^t \int_{|z|\le\varepsilon} S^{(1)}(s-) h_1(z)(\mu-\nu)(\mathrm{d}s,\mathrm{d}z)\,.$$

We denote $\sigma_{11}^2(\infty) = \int_{\mathbb{R}} z_1^2 F(\mathrm{d}z_1,\mathrm{d}z_2)$. Applying Hölder's inequality and the Itô isometry we get

$$\begin{aligned}
&\|S^{(1)}(t) - S^{(1)}_\varepsilon(t)\|_2^2 \\
&\quad\le T\mathbb{E}\left[\int_0^t \{S^{(1)}(s) - S^{(1)}_\varepsilon(s)\}^2\, b_1^2(s)\,\mathrm{d}s\right] + \{c_1^2(\varepsilon)+c_2^2(\varepsilon)\}\mathbb{E}\left[\int_0^t (S^{(1)}_\varepsilon(s))^2\,\mathrm{d}s\right] \\
&\qquad + T\mathbb{E}\left[\int_0^t \{S^{(1)}(s) - S^{(1)}_\varepsilon(s)\}^2\,(\sigma_{11}^2(s)+\sigma_{12}^2(s))\,\mathrm{d}s\right] + \sigma_{11}^2(\varepsilon)\mathbb{E}\left[\int_0^t (S^{(1)}(s))^2 \mathrm{d}s\right] \\
&\qquad + (2\sigma_{11}^2(\infty)+\sigma_{11}^2(\infty))\mathbb{E}\left[\int_0^t \{S^{(1)}(s) - S^{(1)}_\varepsilon(s)\}^2 \mathrm{d}s\right].
\end{aligned}$$

From (3.12) we know it exists a positive constant $C$ such that

$$\begin{aligned}
\|S^{(1)}(t) - S^{(1)}_\varepsilon(t)\|_2^2 \le C\int_0^t \|S^{(1)}(s) - S^{(1)}_\varepsilon(s)\|_2^2\,\mathrm{d}s + (c_1^2(\varepsilon)+c_2^2(\varepsilon))\int_0^t \|S^{(1)}_\varepsilon(s)\|_2^2\,\mathrm{d}s \\
+ \sigma_{11}^2(\varepsilon)\int_0^t \|S^{(1)}(s)\|_2^2\,\mathrm{d}s\,.
\end{aligned}$$

Observing that $c_1^2(\varepsilon)+c_2^2(\varepsilon) = \sigma_{11}^2(\varepsilon)$ (see (3.6)) and applying Lemma 3.2 and Gronwall's inequality, the result follows. □

3.3. **Robustness of the Margrabe formula: jump-diffusion.** We suppose that we work under a martingale measure $\mathbb{Q}$ and that the dynamics of the price process $S$, $S_\varepsilon$ are given under $\mathbb{Q}$ by (2.4), (3.9), resp. We assume $S(0) = (1,1)^\star$. Thus the parameters of $S$ verify (2.14) and the value of the spread option written in $S$ is given by (2.15). Let $X_\varepsilon$ be as in (3.10). Denote by $K_\varepsilon$ the cumulant process of $X_\varepsilon$. Define the measure $\widetilde{\mathbb{Q}}_{v,\varepsilon}$, for

$v \in \mathbb{R}^2$, via the Radon-Nikodym derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}_{v,\varepsilon}}{\mathrm{d}\mathbb{Q}} = \exp\{v^\star X_\varepsilon(T) - \widetilde{K}_\varepsilon(v)_T)\} \,,$$

where $\widetilde{K}_\varepsilon(v)$ denotes the logarithmic transform of the cumulant process $K_\varepsilon(v)$. Applying Theorem 2.2 and Remark (2.3), the value of the spread option written in $S_\varepsilon$ is given by

$$\mathbb{M}(S_\varepsilon^{(1)}(T), S_\varepsilon^{(2)}(T)) = \mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\left[\max(1 - S_\varepsilon^u(T), 0)\right] \,, \tag{3.13}$$

where $\widetilde{\mathbb{Q}}_\varepsilon = \widetilde{\mathbb{Q}}_{v,\varepsilon}$, for $v = (1,0)^\star$, $S_\varepsilon^u := S^{(2)}(0)\mathrm{e}^{X_\varepsilon^u}/S^{(1)}(0)$, $X_\varepsilon^u = uX_\varepsilon$, and $u = (-1,1)^\star$. From Theorem 2.1, the differential characteristics $(b_{v,\varepsilon}^u, c_{v,\varepsilon}^u, \nu_{v,\varepsilon}^u)$ of $X_\varepsilon^u$ are given by

$$\begin{aligned} b_{v,\varepsilon}^u &= u^\star \widetilde{b}_\varepsilon + u^\star(c + \Sigma(\varepsilon))v \\ &\qquad + \big(h\left(u^\star \log(1+z)\right)\mathrm{e}^{v^\star \log(1+z)} - u^\star h\left(\log(1+z)\right)\big) * F_\varepsilon \,, \\ c_{v,\varepsilon}^u &= u^\star(c + \Sigma(\varepsilon))u \,, \\ F_{v,\varepsilon}^u(E) &= \mathbf{1}_E\left(u^\star \log(1+z)\right)\mathrm{e}^{v^\star \log(1+z)} * F_\varepsilon \,, \qquad E \in \mathcal{B}(\mathbb{R}\setminus\{0\}) \,. \end{aligned} \tag{3.14}$$

We present the following technical lemma which is used in the forthcoming convergence result for spread option prices.

**Lemma 3.4.** *Let $X$, $X_\varepsilon$ be as in* (2.5), (3.10), *resp. Assume*

$$\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\left[\mathrm{e}^{|R| X_\varepsilon^u(t)}\right], \mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\left[\mathrm{e}^{|R| X^u(t)}\right] < \infty \,. \tag{3.15}$$

*Then for $t \in [0,T]$ and $\mathbb{u}, R \in \mathbb{R}$, we have*

$$\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\left[\mathrm{e}^{(R - \mathbb{i}\mathbb{u})X_\varepsilon^u(t)}\right] \le C(T,R) \,, \qquad \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{(R - \mathbb{i}\mathbb{u})X^u(t)}\right] \le \widetilde{C}(T,R) \,,$$

*where $C(T,R)$ and $\widetilde{C}(T,R)$ are positive constants depending on $T$ and $R$.*

*Proof.* From the characteristics of $X_\varepsilon^u$ in (3.14), we deduce

$$\begin{aligned} &\left|\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\left[\mathrm{e}^{(R - \mathbb{i}\mathbb{u})X_\varepsilon^u(t)}\right]\right| \\ &\quad \le \mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}\left[|\mathrm{e}^{R X_\varepsilon^u(t)}|\right] \\ &\quad \le \exp\Big\{ tR \int_{\mathbb{R}^2} \Big[h\Big(\log(\frac{1+z_2}{1+z_1})\Big)(1+z_1) + h_1(z) - h_2(z)\Big] F_\varepsilon(\mathrm{d}z_1\,, \mathrm{d}z_2) \\ &\qquad + t \int_{\mathbb{R}^2} \Big[\big(\frac{1+z_2}{1+z_1}\big)^R - 1 - Rh\Big(\log(\frac{1+z_2}{1+z_1})\Big)\Big](1+z_1) F_\varepsilon(\mathrm{d}z_1\,, \mathrm{d}z_2) \\ &\qquad + \frac{tR}{2}(R-1)\left[\sigma_{11}^2(\varepsilon) + \sigma_{22}^2(\varepsilon) - 2\sigma_{12}(\varepsilon)\right] \Big\} \\ &\qquad\ \mathbb{E}\Bigg[\exp\Bigg\{R\Bigg(\int_0^t (b_2(s) - b_1(s))\mathrm{d}s - \frac{1}{2}\int_0^t (\sigma_{11}(s) - \sigma_{12}(s))^2 \mathrm{d}s \end{aligned}$$

$$
-\frac{1}{2}\int_0^t(\sigma_{11}(s)-\sigma_{12}(s))^2\mathrm{d}s+\int_0^t(\sigma_{22}(s)-\sigma_{12}(s))^2\mathrm{d}W^{(1)}(s)
$$

$$
+\int_0^t(\sigma_{22}(s)-\sigma_{12}(s))^2\mathrm{d}W^{(2)}(s)\Bigg)\Bigg\}\Bigg]
$$

$$
\leq \exp\Big\{2T|R||R-1|
$$

$$
+T|R|\int_{\mathbb{R}^2}\Big|(1+z_1)\Big(\big(\frac{1+z_2}{1+z_1}\big)^R-1\Big)+h_1(z)-h_2(z)\Big|F(\mathrm{d}z_1\,,\mathrm{d}z_2)\Big\}
$$

$$
\mathbb{E}\left[\exp\left\{R\left(\int_0^t(b_2(s)-b_1(s))\mathrm{d}s-\frac{1}{2}\int_0^t(\sigma_{11}(s)-\sigma_{12}(s))^2\mathrm{d}s\right.\right.\right.
$$

$$
-\frac{1}{2}\int_0^t(\sigma_{11}(s)-\sigma_{12}(s))^2\mathrm{d}s+\int_0^t(\sigma_{22}(s)-\sigma_{12}(s))^2\mathrm{d}W^{(1)}(s)
$$

$$
+\int_0^t(\sigma_{22}(s)-\sigma_{12}(s))^2\mathrm{d}W^{(2)}(s)\Bigg)\Bigg\}\Bigg]\,,
$$

and the result follows for $X^u_\varepsilon$ since the latter expectation is finite. Moreover, the statement follows for $X^u$ by observing that $|\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathrm{e}^{(R-\mathtt{i}\mathtt{u})X^u(t)}\right]|\leq \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[|\mathrm{e}^{RX^u(t)}|\right]$. □

We conclude now with the following convergence result.

**Proposition 3.5.** *Let $S$, $S_\varepsilon$, be as in* (2.4), (3.9), *resp. Assume* (3.15) *holds. Then*

$$
\lim_{\varepsilon\longrightarrow 0}\mathbb{M}(S^{(1)}_\varepsilon(T),S^{(2)}_\varepsilon(T))=\mathbb{M}(S^{(1)}(T),S^{(2)}(T))\,.
$$

*Proof.* We first write the Margarbe formula in terms of Fourier techniques as in Theorem 2.5. Let $f(x)=\max(1-\mathrm{e}^x\,,0)$. We know that for $R<0$, condition (1) of Theorem 2.5 holds true and $\widehat{f}(\mathtt{i}R-\cdot)\in L^1(\mathbb{R})$ (see Eberlein et al. [13]). Moreover from Lemma 3.4, we know that $\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[\mathrm{e}^{(R-\mathtt{i}\mathtt{u})X^u_\varepsilon(T)}]$ is bounded uniformly in $\mathtt{u}$ by a constant. Thus we have, for $R<0$

$$
\mathbb{M}(S^{(1)}_\varepsilon(T),S^{(2)}_\varepsilon(T))=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(R-\mathtt{i}\mathtt{u})\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[\mathrm{e}^{(R-\mathtt{i}\mathtt{u})X^u_\varepsilon(T)}]\,\mathrm{d}\mathtt{u}\,. \tag{3.16}
$$

Applying Lemma 3.4 again we know that $\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[\mathrm{e}^{(R-\mathtt{i}\mathtt{u})X^u_\varepsilon(T)}]$ is bounded uniformly in $\varepsilon$. Next, applying the dominated convergence theorem we can take the limit inside the integral in (3.16) to obtain

$$
\lim_{\varepsilon\longrightarrow 0}\mathbb{M}(S^{(1)}_\varepsilon(T),S^{(2)}_\varepsilon(T))=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(R-\mathtt{i}\mathtt{u})\mathbb{E}_{\widetilde{\mathbb{Q}}}[\mathrm{e}^{(R-\mathtt{i}\mathtt{u})X^u(T)}]\mathrm{d}\mathtt{u}\,.
$$

Hence, the result follows easily from Lemma 3.4.

□

The convergence in Prop. 3.5 can be quantified with a rate in terms of $\varepsilon$, which is reported in the next proposition:

**Proposition 3.6.** *Let $S$, $S_\varepsilon$, be as in* (2.4), (3.9), *resp. Assume* (3.15) *holds for $R = 2$. Then*

$$|\mathbb{M}(S_\varepsilon^{(1)}(T), S_\varepsilon^{(2)}(T)) - \mathbb{M}(S^{(1)}(T), S^{(2)}(T))| \leq C(T)\xi(\varepsilon)\,,$$

*where $C(T)$ is a positive constant depending on $T$ and*

$$\begin{aligned}\xi(\varepsilon) &= \sigma_{11}^2(\varepsilon) + \sigma_{22}^2(\varepsilon) + |\sigma_{12}(\varepsilon)| + |c_1(\varepsilon)| + |c_2(\varepsilon)| + |c_3(\varepsilon)| \\ &\quad + \Big| \int\limits_{|z|\leq\varepsilon} \Big\{ \log(\frac{1+z_2}{1+z_1})(1+z_1) + z_1 - z_2 \Big\} F(\mathrm{d}z) \Big| \\ &\quad + \int\limits_{|z|\leq\varepsilon} \Big\{ \log(\frac{1+z_2}{1+z_1})(1+z_1) \Big\}^2 F(\mathrm{d}z)\,.\end{aligned}$$

*Proof.* Recall the function $f(x) = \max(1 - \mathrm{e}^x, 0)$. Note that we can write

$$\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[f(X_\varepsilon^u(T))] - \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(X^u(T))] = \mathbb{E}[f(X_\varepsilon^u(T)) - f(X^u(T))]\,,$$

where the characteristics of $X^u$, $X_\varepsilon^u$ are given under $\widetilde{\mathbb{Q}}$, $\widetilde{\mathbb{Q}}_\varepsilon$ by (2.10), (3.14), resp. The function $f(x)$ is not differentiable in 0. However since the expectation is null in a negligible set, we can apply the mean value theorem to $f(x)$ and we know there exists a random variable $Z_\varepsilon(t) \in L_{X_\varepsilon^u(t), X^u(t)}$ , i.e. the line connecting $X_\varepsilon^u(t), X^u(t)$, such that

$$\begin{aligned}&|\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[f(X_\varepsilon^u(T))] - \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(X^u(T))]| \\ &\quad \leq \mathbb{E}[\mathrm{e}^{Z_\varepsilon(T)}|X_\varepsilon^u(T) - X^u(T)|] \\ &\quad \leq \mathbb{E}[\mathrm{e}^{2Z_\varepsilon(T)}]^{\frac{1}{2}}\mathbb{E}[|X_\varepsilon^u(T) - X^u(T)|^2]^{\frac{1}{2}}\,.\end{aligned}$$

Applying Lemma 3.4, we know that $\mathbb{E}[\mathrm{e}^{2Z_\varepsilon(T)}] \leq C(T)$. Moreover writing $X^u(T)$ and $X_\varepsilon^u(T)$ explicitly in terms of their characteristics, we get

$$\begin{aligned}&|\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[f(X_\varepsilon^u(T))] - \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(X^u(T))]| \\ &\leq C(T)\mathbb{E}\Big[\Big| - T(-\frac{1}{2}\sigma_{11}^2(\varepsilon) - \frac{1}{2}\sigma_{22}^2(\varepsilon) + \sigma_{12}(\varepsilon)) - (c_1(\varepsilon) - c_2(\varepsilon))\widetilde{W}^{(1)}(T) \\ &\quad - (c_3(\varepsilon) - c_2(\varepsilon))\widetilde{W}^{(2)}(T) + \int\limits_0^T \int\limits_{|z|\leq\varepsilon} z(\mu^u - \nu^u)(\mathrm{d}s, \mathrm{d}z) \\ &\quad + T \int\limits_{|z|\leq\varepsilon} \Big\{ \log(\frac{1+z_2}{1+z_1})(1+z_1) + (z_1 - z_2) \Big\} F(\mathrm{d}z) \Big|^2\Big]^{\frac{1}{2}}\end{aligned}$$

$$\leq C(T)\Big(|-\frac{1}{2}\sigma_{11}^2(\varepsilon)-\frac{1}{2}\sigma_{22}^2(\varepsilon)+\sigma_{12}(\varepsilon)|+|c_1(\varepsilon)-c_2(\varepsilon)|\mathbb{E}[|\widetilde{W}^{(1)}(T)|^2]^{\frac{1}{2}}$$

$$+|c_3(\varepsilon)-c_2(\varepsilon)|\mathbb{E}[|\widetilde{W}^{(2)}(T)|^2]^{\frac{1}{2}}+\mathbb{E}\Big[\Big|\int_0^T\int_{|z|\leq\varepsilon} z(\mu^u-\nu^u)(\mathrm{d}s,\mathrm{d}z)\Big|^2\Big]^{\frac{1}{2}}$$

$$+\Big|\int_{|z|\leq\varepsilon}\Big\{\log(\frac{1+z_2}{1+z_1})(1+z_1)+z_1-z_2\Big\}F(\mathrm{d}z)\Big|\Big)$$

$$\leq C(T)\Big(\sigma_{11}^2(\varepsilon)+\sigma_{22}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|+|c_1(\varepsilon)|+|c_2(\varepsilon)|+|c_3(\varepsilon)|$$

$$+\Big|\int_{|z|\leq\varepsilon}\Big\{\log(\frac{1+z_2}{1+z_1})(1+z_1)+z_1-z_2\Big\}F(\mathrm{d}z)\Big|$$

$$+\int_{|z|\leq\varepsilon}\Big\{\log(\frac{1+z_2}{1+z_1})(1+z_1)\Big\}^2F(\mathrm{d}z)\Big)$$

and the result follows. □

Notice that a Taylor series expansion around $(0,0)$ of $z\mapsto\log(\frac{1+z_2}{1+z_1})$, shows that

$$\log(\frac{1+z_2}{1+z_1})(1+z_1)+z_1-z_2\approx C_1\left(\sigma_{11}^2(\varepsilon)+\sigma_{22}^2(\varepsilon)+\sigma_{12}(\varepsilon)\right)$$

$$\Big\{\log(\frac{1+z_2}{1+z_1})(1+z_1)\Big\}^2\approx C_2\left(\sigma_{11}^2(\varepsilon)+\sigma_{22}^2(\varepsilon)+\sigma_{12}(\varepsilon)\right).$$

Thus $\xi(\varepsilon)\approx C\left(\sigma_{11}^2(\varepsilon)+\sigma_{22}^2(\varepsilon)+\sigma_{12}(\varepsilon)+|c_1(\varepsilon)|+|c_2(\varepsilon)|+|c_3(\varepsilon)|\right)$, where $C_1$, $C_2$, and $C$ are positive constants.

So far we have considered the approximation of a general price process after having performed a measure change. That is we approximated the jump-diffusion dynamics specified under the martingale measure and then studied the robustness of the Margrabe formula. In the following subsection we consider a more specific price processes, namely a two-dimensional exponential Lévy process. For this model, we approximate the small jumps under the real world measure before defining the martingale measure for the *approximating* process through an Esscher transform. For this we can apply the duality measure to compute the Margrabe formula, which is then used to prove the rate of convergence. We refer to Benth et al. [8] for robustness studies of options under different martingale measures in the univariate case.

3.4. **Robustness of the Margrabe formula: geometric Lévy process.** Let $S$ be as in (2.20). We approximate $L$ by a Lévy process $L_\varepsilon$ that has differential characteristics $(b,c+\Sigma(\varepsilon),F_\varepsilon)$, where $\Sigma(\varepsilon)$ is given by (3.3) and $F_\varepsilon(\mathrm{d}z)=F(\mathrm{d}z)\mathbf{1}_{\{|z|>\varepsilon\}}$. We consider the price process

(3.17) $$S_\varepsilon(t)=\mathrm{e}^{L_\varepsilon(t)},\qquad 0\leq t\leq T.$$

For simplicity, we omit $L$ in the notation of the cumulant function of the Lévy process. Thus we denote $K = K_L$ and $K_\varepsilon = K_{L_\varepsilon}$. Let $\theta^\varepsilon \in \mathbb{R}^2$, for $0 < \varepsilon < 1$ be such that

$$e^{rt} = \exp\left(K_\varepsilon(\mathbf{1}_i + \theta^\varepsilon)_t - K_\varepsilon(\theta^\varepsilon)_t\right), \qquad 0 \le t \le T, \tag{3.18}$$

where $r$ is the risk-free rate of return. Benth et al. [8] proved the existence and uniqueness of the parameter $\theta^\varepsilon$ when $L_\varepsilon$ is a one-dimensional Lévy process. With similar computations one can prove the existence and uniqueness of the parameter $\theta^\varepsilon$ in our case. We define a new measure $\mathbb{P}_{\theta^\varepsilon + \mathbf{1}_1}$ as follows

$$\frac{\mathrm{d}\mathbb{P}_{\theta^\varepsilon+\mathbf{1}_1}}{\mathrm{d}\mathbb{P}} := \exp\left\{(\mathbf{1}_1 + \theta^\varepsilon)^\star L_\varepsilon(T) - K_\varepsilon(\mathbf{1}_1 + \theta^\varepsilon)_T\right\}. \tag{3.19}$$

Applying Theorem 2.2 and Remark 2.3, we get the price of the spread option written on $S_\varepsilon$ as follows

$$\mathbb{M}(S_\varepsilon^{(1)}(T), S_\varepsilon^{(2)}(T)) = \mathbb{E}_{\theta^\varepsilon+\mathbf{1}_1}\left[\max(1 - S_\varepsilon^u(T), 0)\right], \tag{3.20}$$

where $S_\varepsilon^u = \mathrm{e}^{L_\varepsilon^u}$, $L_\varepsilon^u := u^\star L_\varepsilon$, and $u = (-1,1)^\star$. The differential characteristics of $L_\varepsilon^u$ under the new measure $\mathbb{P}_{\theta^\varepsilon+\mathbf{1}_1}$ are given by $(b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon), c^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon), F^u_{\theta^\varepsilon+\mathbf{1}_1})$, where (see Theorem 2.1)

$$\begin{aligned} b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) &= u^\star b + u^\star(c + \Sigma(\varepsilon))(\theta^\varepsilon + \mathbf{1}_1) + \left(h(u^\star z)\mathrm{e}^{(\theta^\varepsilon+\mathbf{1}_1)^\star z} - u^\star h(z)\right) * F_\varepsilon, \\ c^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) &= u^\star(c + \Sigma(\varepsilon))u, \\ F^u_{\theta^\varepsilon+\mathbf{1}_1}(E) &= \mathbf{1}_E(u^\star z)\mathrm{e}^{(\theta^\varepsilon+\mathbf{1}_1)^\star z} * F_\varepsilon, \qquad E \in \mathcal{B}(\mathbb{R}\setminus\{0\}). \end{aligned} \tag{3.21}$$

Before we prove the robustness of the Margrabe formula, we need the following lemma in which we prove the convergence of the Esscher parameter $\theta_\varepsilon$ to $\theta$ when $\varepsilon$ goes to 0. Notice that the same type of result was proved in a paper by Benth et al. [8] in the case of a one-dimensional Lévy process. We present the proof for the sake of completeness.

**Lemma 3.7.** *Let $\theta$, $\theta^\varepsilon$ be as in (2.23), (3.18), resp. Then $|\theta_i^\varepsilon|$, $i = 1,2$ is bounded uniformly in $\varepsilon \in (0,1)$. Moreover, we have*

$$|\theta_i^\varepsilon - \theta_i| \le C_\theta(\sigma_{11}^2(\varepsilon) + |\sigma_{12}(\varepsilon)|), \qquad i = 1,2, \tag{3.22}$$

*where $C_\theta$ is a positive constant depending on $\theta$ and $\sigma_{ij}(\varepsilon)$, $i,j = 1,2$, are the components of the matrix $\Sigma(\varepsilon)$ defined in (3.3).*

*Proof.* We introduce the function $G_\varepsilon(x)$, $x \in \mathbb{R}$ as follows

$$\begin{aligned} G_\varepsilon(x) &= \sigma_{11}^2(\varepsilon)x + \sigma_{12}(\varepsilon)\theta_2 + (\sigma_{11}^2 + \sigma_{12}^2)x + (\sigma_{11}\sigma_{12} + \sigma_{12}\sigma_{22})\theta_2 \\ &\quad + \int_{\mathbb{R}^2} \mathrm{e}^{xz_1+\theta_2 z_2}(\mathrm{e}^{z_1} - 1 - h_1(z))F_\varepsilon(\mathrm{d}z) + \int_{\mathbb{R}^2} h_1(z)(\mathrm{e}^{xz_1+\theta_2 z_2} - 1)F_\varepsilon(\mathrm{d}z). \end{aligned}$$

By the dominated convergence theorem, we deduce that $G_\varepsilon(x)$ is differentiable and the derivative is given by

$$G'_\varepsilon(x) = \sigma_{11}^2(\varepsilon) + (\sigma_{11}^2 + \sigma_{12}^2) + \int_{\mathbb{R}^2} z_1 \mathrm{e}^{xz_1+\theta_2 z_2}(\mathrm{e}^{z_1} - 1)F_\varepsilon(\mathrm{d}z) > 0.$$

It follows that $G_\varepsilon(x)$ is a strictly increasing function. Thus the inverse $G_\varepsilon^{-1}(x)$ exists and its derivative is given by $(G_\varepsilon^{-1}(x))' = 1/G'_\varepsilon(x)$. From (2.23) and (3.18), we deduce that $\theta$ and $\theta_\varepsilon$ verify the following two equations

$$\begin{aligned} G_\varepsilon(\theta_1^\varepsilon) &= r - b_1 - \frac{1}{2}\sigma_{11}^2(\varepsilon) - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)\,, \\ G_\varepsilon(\theta_1) &= r - b_1 - \frac{1}{2}\sigma_{11}^2(\varepsilon) - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) + \sigma_{11}^2(\varepsilon)\theta_1 + \sigma_{12}(\varepsilon)\theta_2 \\ &\quad - \int\limits_{|z|\le\varepsilon} \mathrm{e}^{\theta_1 z_1 + \theta_2 z_2}(\mathrm{e}^{z_1} - 1 - z_1)F(\mathrm{d}z) - \int\limits_{|z|\le\varepsilon} z_1(\mathrm{e}^{\theta_1 z_1 + \theta_2 z_2} - 1)F(\mathrm{d}z)\,. \end{aligned}$$

Hence

$$\begin{aligned} |\theta_1 - \theta_1^\varepsilon| = \Big| & G_\varepsilon^{-1}\Big(r - b_1 - \frac{1}{2}\sigma_{11}^2(\varepsilon) - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) + \sigma_{11}^2(\varepsilon)\theta_1 + \sigma_{12}(\varepsilon)\theta_2 \\ & - \int\limits_{|z|\le\varepsilon} \mathrm{e}^{\theta_1 z_1 + \theta_2 z_2}(\mathrm{e}^{z_1} - 1 - z_1)F(\mathrm{d}z) - \int\limits_{|z|\le\varepsilon} z_1(\mathrm{e}^{\theta_1 z_1 + \theta_2 z_2} - 1)F(\mathrm{d}z)\Big) \\ & - G_\varepsilon^{-1}\Big(r - b_1 - \frac{1}{2}\sigma_{11}^2(\varepsilon) - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)\Big)\Big|\,. \end{aligned}$$

Let $\sigma_{11} > 0$ or $\sigma_{12} > 0$. In this case $(G_\varepsilon^{-1}(x))' \le 1/(\sigma_{11}^2 + \sigma_{12}^2)$. Then applying the mean value theorem to the function $G_\varepsilon^{-1}(x)$, we get

$$\begin{aligned} |\theta_1 - \theta_1^\varepsilon| \le \frac{1}{\sigma_{11}^2 + \sigma_{12}^2}\Big| & \sigma_{11}^2(\varepsilon)(\theta_1 + \frac{1}{2}) + \sigma_{12}(\varepsilon)\theta_2 \\ & + \int\limits_{|z|\le\varepsilon} \{\mathrm{e}^{(\theta_1+1)z_1 + \theta_2 z_2} - 1 - z_1(\theta_1 + 1)\}F(\mathrm{d}z) \\ & - \int\limits_{|z|\le\varepsilon} \{\mathrm{e}^{\theta_1 z_1 + \theta_2 z_2} - 1 - z_1\theta_1\}F(\mathrm{d}z)\Big|\,. \end{aligned}$$

Applying the mean value theorem to the function $\theta_1 \mapsto \mathrm{e}^{\theta_1 z_1 + \theta_2 z_2} - 1 - z_1\theta_1$, we know there exists $a_1 \in L_{\theta_1, \theta+1}$ such that

$$|\theta_1 - \theta_1^\varepsilon| \le \frac{1}{\sigma_{11}^2 + \sigma_{12}^2}\Big(\sigma_{11}^2(\varepsilon)|\theta_1 + \frac{1}{2}| + |\sigma_{12}(\varepsilon)||\theta_2| + \int\limits_{|z|\le\varepsilon} |z_1||\mathrm{e}^{a_1 z_1 + \theta_2 z_2} - 1|F(\mathrm{d}z)\Big)\,.$$

Let $U$ be an open set in $\mathbb{R}^2$. Suppose $a_1$ and $\theta_2$ are two points of $U$ such that $U$ contains the line segment from $(0,0)$ to $(a_1, \theta_2)$. Applying the mean value theorem to the function $(a_1, \theta_2) \mapsto \mathrm{e}^{a_1 z_1 + \theta_2 z_2} - 1$, we know there exists $\tilde{a}_1, a_2$ such that

$$|\theta_1 - \theta_1^\varepsilon| \le \frac{1}{\sigma_{11}^2 + \sigma_{12}^2}\Big(\sigma_{11}^2(\varepsilon)|\theta_1 + \frac{1}{2}| + |\sigma_{12}(\varepsilon)||\theta_2|$$

$$+ \int\limits_{|z|\le\varepsilon} |z_1| \mathrm{e}^{\tilde{a}_1 z_1 + a_2 z_2} (|z_1||\theta_1+1| + |z_2||\theta_2|) F(\mathrm{d}z)\Big)$$

$$\le C(\theta)\left(\sigma_{11}^2(\varepsilon) + |\sigma_{12}(\varepsilon)|\right) ,$$

which proves the result for $\theta_1$ in the case $\sigma_{11} > 0$ or $\sigma_{12} > 0$. In the case $\sigma_{11} = \sigma_{12} = 0$, we derive the result using the fact that $(G_\varepsilon^{-1}(x))' \le 1/\int_{|z|>1} z_1 \mathrm{e}^{\theta_1 z_1 + \theta_2 z_2}(\mathrm{e}^{z_1} - 1) F(\mathrm{d}z)$. Similar computations lead to the result for $\theta_2$. □

In the following proposition we show that the drift of the Lévy process $L_\varepsilon^u$ converges to the drift of the Lévy process $L^u$.

**Proposition 3.8.** *Let $b^u_{\theta+\mathbf{1}_1}$ , $b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon)$ be as defined in* (2.26)*,* (3.21)*, resp. Then we have*

$$|b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) - b^u_{\theta+\mathbf{1}_1}| \le C_\theta(\sigma_{11}^2(\varepsilon) + \sigma_{22}^2(\varepsilon) + |\sigma_{12}(\varepsilon)|) ,$$

*where $C_\theta$ is a positive constant depending on $\theta$ and $\sigma_{ij}(\varepsilon)$, $i,j = 1,2$, are the components of the matrix $\Sigma(\varepsilon)$ defined in* (3.3)*.*

*Proof.* First, define

$$F_\theta(\mathrm{d}z) := \mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2} F(\mathrm{d}z) , \qquad F_{\theta^\varepsilon}(\mathrm{d}z) := \mathrm{e}^{(\theta_1^\varepsilon+1)z_1+\theta_2^\varepsilon z_2} F(\mathrm{d}z) ,$$

where $\theta, \theta^\varepsilon$ are as in (2.23), (3.18), resp. Let $U$ be an open set in $\mathbb{R}^2$. Suppose that $\theta^\varepsilon + \mathbf{1}_1$ and $\theta + \mathbf{1}_1$ are two points of $U$ such that $U$ contains the line segment $L$ from $\theta^\varepsilon + \mathbf{1}_1$ to $\theta + \mathbf{1}_1$. Then applying the mean value theorem to the function $\theta \mapsto \mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2}$, we have

$$|\mathrm{e}^{(\theta_1^\varepsilon+1)z_1+\theta_2^\varepsilon z_2} - \mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2}| \le \mathrm{e}^{(a_1+1)z_1+a_2 z_2}\left(|z_1||\theta_1^\varepsilon - \theta_1| + |z_2||\theta_2^\varepsilon - \theta_2|\right) ,$$

where $(a_1, a_2) \in L$. Applying Lemma 3.7, we get

$$|F_{\theta^\varepsilon}(\mathrm{d}z) - F_\theta(\mathrm{d}z)| \le \mathrm{e}^{(a_1+1)z_1+a_2 z_2}\left(|z_1| + |z_2|\right) C_\theta(\sigma_{11}^2(\varepsilon) + |\sigma_{12}(\varepsilon)|) F(\mathrm{d}z) , \tag{3.23}$$

where $C_\theta$ is a positive constant depending on $\theta$.

Now, let $U'$ be another open set in $\mathbb{R}^2$. Suppose that $\theta + \mathbf{1}_1$ and $(-1,0)^\star$ are two points of $U'$ such that $U'$ contains the line segment $L'$ from $(-1,0)^\star$ to $\theta + \mathbf{1}_1$. Then applying the mean value theorem to the function $\theta \mapsto \mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2}$, we get

$$|\mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2} - 1| \le \mathrm{e}^{(a_1'+1)z_1+a_2' z_2}\left(|z_1||\theta_1+2| + |z_2||\theta_2|\right) , \tag{3.24}$$

where $(a_1', a_2') \in L'$.

To prove the statement of the proposition, we compute

$$\begin{aligned}
&|b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) - b^u_{\theta+\mathbf{1}_1}| \\
&\quad \le |(\sigma_{11} - \sigma_{12})^2 + (\sigma_{11}\sigma_{12} - \sigma_{12}\sigma_{22})||\theta_1^\varepsilon - \theta_1| \\
&\qquad + |(\sigma_{11} - \sigma_{12})^2 + (\sigma_{11}\sigma_{12} - \sigma_{12}\sigma_{22})||\theta_2^\varepsilon - \theta_2| \\
&\qquad + [\sigma_{11}^2(\varepsilon) - \sigma_{12}(\varepsilon)]|\theta_1^\varepsilon - \theta_1| + [\sigma_{22}^2(\varepsilon) - \sigma_{12}(\varepsilon)]|\theta_2^\varepsilon - \theta_2|| \\
&\qquad + \Big| \int\limits_{\mathbb{R}^2} \{h(z_2 - z_1)\mathrm{e}^{(\theta_1^\varepsilon+1)z_1+\theta_2^\varepsilon z_2} + h_2(z) - h_1(z)\} F_\varepsilon(\mathrm{d}z)
\end{aligned}$$

$$
\begin{aligned}
&- \int\limits_{\mathbb{R}^2} \{h(z_2-z_1)\mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2} - (h_2(z)+h_1(z))\}F(\mathrm{d}z)\Big| \\
&\le C_\theta(\sigma_{11}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|) + \int\limits_{|z|\le\varepsilon} |h(z_2-z_1)\mathrm{e}^{(\theta_1+1)z_1+\theta_2 z_2} - (h_2(z)+h_1(z))|F(\mathrm{d}z) \\
&\quad + \int\limits_{|z|>\varepsilon} |h(z_2-z_1)||F_{\theta^\varepsilon}(\mathrm{d}z) - F_\theta(\mathrm{d}z)|\,,
\end{aligned}
$$

where in the latter we used Lemma 3.7. Observing that $h(z_2-z_2)=(z_2-z_2)\mathbf{1}_{\{|z_2-z_2|\le 1\}}$ and $\{|z_2-z_2|\le 1\}\cap\{|z|\le\varepsilon\}=\{|z|\le\varepsilon\}$ and applying (3.23) and (3.24), we get

$$
\begin{aligned}
&|b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) - b^u_{\theta+\mathbf{1}_1}| \\
&\quad \le C_\theta(\sigma_{11}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|) + \widetilde{C}_\theta \int\limits_{|z|\le\varepsilon} (|z_2|^2+|z_1|^2)\mathrm{e}^{(a_1'+1)z_1+a_2' z_2}F(\mathrm{d}z) \\
&\quad\quad + \widehat{C}_\theta(\sigma_{11}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|) \int\limits_{|z|>\varepsilon} (|z_2|^2+|z_1|^2)\mathbf{1}_{\{|z_2-z_1|\le 1\}}\mathrm{e}^{(a_1+1)z_1+a_2 z_2}F(\mathrm{d}z)\,,
\end{aligned}
$$

where $\widetilde{C}_\theta$ and $\widehat{C}_\theta$ are two positive constants depending on $\theta$. Moreover, since $(a_1',a_2')\in L'$, then $\mathrm{e}^{(a_1'+1)z_1+a_2'z_2}\le \mathrm{e}^{C_\theta'(|z_1|+|z_2|)}\le \mathrm{e}^{2C_\theta'}$, for $|z|\le\varepsilon$. Finally a Taylor series expansion of $z\mapsto \mathrm{e}^{(a_1+1)z_1+a_2z_2}$ around the point $(0,0)$, shows that the function $(|z_2|^2+|z_1|^2)\mathrm{e}^{(a_1+1)z_1+a_2z_2}$ is integrable in $\mathbb{R}^2$ with respect to $F(\mathrm{d}z)$. Hence

$$
\begin{aligned}
&|b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) - b^u_{\theta+\mathbf{1}_1}| \\
&\quad \le C_\theta(\sigma_{11}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|) + \widetilde{C}_\theta(\sigma_{11}^2(\varepsilon)+\sigma_{22}^2(\varepsilon)) \\
&\quad\quad + \widehat{C}_\theta(\sigma_{11}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|) \int\limits_{\mathbb{R}} (|z_2|^2+|z_1|^2)\mathbf{1}_{\{|z_2-z_1|\le 1\}}\mathrm{e}^{(a_1+1)z_1+a_2 z_2}F(\mathrm{d}z)
\end{aligned}
$$

and we proved the statement. □

In the following proposition we show the robustness of the Margrabe formula and we compute a convergence rate.

**Proposition 3.9.** *Let $S$, $S_\varepsilon$ be as in* (2.20), (3.17), *resp. Then it holds that*

$$\lim_{\varepsilon\longrightarrow 0} \mathbb{M}(S_\varepsilon^{(1)}(T), S_\varepsilon^{(2)}(T)) = \mathbb{M}(S^{(1)}(T), S^{(2)}(T))\,.$$

*Moreover*

$$|\mathbb{M}(S_\varepsilon^{(1)}(T), S_\varepsilon^{(2)}(T)) - \mathbb{M}(S^{(1)}(T), S^{(2)}(T))| \le C(\theta,T)\xi(\varepsilon)$$

*where $\xi(\varepsilon)=\sigma_{11}^2(\varepsilon)+\sigma_{22}^2(\varepsilon)+|\sigma_{12}(\varepsilon)|+|c_1(\varepsilon)|+|c_2(\varepsilon)|+|c_3(\varepsilon)|$ and $C(\theta,T)$ is a positive constant depending on $\theta$ and $T$.*

*Proof.* We follow the same steps as in the proof of Prop. 3.6. We apply the mean value theorem to $f(x) = \max(1 - \mathrm{e}^x, 0)$. Hence we know there exists a random variable $\widetilde{L}_\varepsilon(t) \in L_{L^u_\varepsilon(t), L^u(t)}$ i.e., the line connecting $L^u_\varepsilon(t)$ and $L^u(t)$ such that

$$\begin{aligned}
&|\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[f(L^u_\varepsilon(T))] - \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(L^u(T))]| \\
&\qquad \leq \mathbb{E}[\mathrm{e}^{2\widetilde{L}_\varepsilon(T)}]^{\frac{1}{2}} \mathbb{E}[|L^u_\varepsilon(T) - L^u(T)|] \,.
\end{aligned}$$

From the differential characteristics of $L^u$, $L^u_\varepsilon$ in (2.26), (3.21), resp., we get

$$\begin{aligned}
&|\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[f(L^u_\varepsilon(T))] - \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(L^u(T))]| \\
&\quad \leq C(T)\Big\{|b^u_{\theta^\varepsilon+\mathbf{1}_1}(\varepsilon) - b^u_{\theta+\mathbf{1}_1}| + \Big(|-\frac{1}{2}\sigma^2_{11}(\varepsilon) - \frac{1}{2}\sigma^2_{22}(\varepsilon) + \sigma_{12}(\varepsilon)|\Big) \\
&\qquad\qquad + |c_1(\varepsilon) - c_2(\varepsilon)|\mathbb{E}[|\widetilde{W}^{(1)}(T)|^2]^{\frac{1}{2}} + |c_3(\varepsilon) - c_2(\varepsilon)|\mathbb{E}[|\widetilde{W}^{(2)}(T)|^2]^{\frac{1}{2}} \\
&\qquad\qquad + \mathbb{E}\Big[\Big|\int_0^T \int_{|z|\leq\varepsilon} z(\mu^u_\theta - \nu^u_\theta)(\mathrm{d}s, \mathrm{d}z)\Big|^2\Big]^{\frac{1}{2}}\Big\} \,,
\end{aligned}$$

where $\mu^u_\theta$ is the jump measure of $L^u$. Hence from Prop. 3.8, we get

$$\begin{aligned}
&|\mathbb{E}_{\widetilde{\mathbb{Q}}_\varepsilon}[f(L^u_\varepsilon(T))] - \mathbb{E}_{\widetilde{\mathbb{Q}}}[f(L^u(T))]| \\
&\quad \leq C(\theta, T)\Big(\sigma^2_{11}(\varepsilon) + \sigma^2_{22}(\varepsilon) + |\sigma_{12}(\varepsilon)| + |c_1(\varepsilon)| + |c_2(\varepsilon)| + |c_3(\varepsilon)| \\
&\qquad + \Big|\int_0^T \int_{\mathbb{R}} (z_2 - z_1)\mathrm{e}^{(\theta_1+1)z_1 + \theta_2 z_2} F(\mathrm{d}z)\Big|\Big) \,.
\end{aligned}$$

Applying the mean value theorem, we can show that

$$|\int_0^T \int_{\mathbb{R}} (z_2 - z_1)\mathrm{e}^{(\theta_1+1)z_1 + \theta_2 z_2} F(\mathrm{d}z)\Big| \leq C_\theta(\sigma^2_{11}(\varepsilon) + \sigma^2_{22}(\varepsilon))$$

and the result follows. □

*Example: Normal inverse Gaussian Lévy process.* We consider the case where $(L^{(1)}, L^{(2)})$ follows a bivariate normal inverse Gaussian Lévy process as in Subsect. 2.4. For this case we investigate the behavior of $\sigma^2_{11}(\varepsilon)$ and $\sigma^2_{22}(\varepsilon)$, the entries of the variance-covariance matrix (3.3) of the small jumps. Using Proposition 8.9 in Sato [25] we know that the Lévy density of $L(1)$ is given by

$$g(z) = \frac{\delta}{\sqrt{2}}\left(\frac{\alpha}{\pi\sqrt{z'\Delta^{-1}z}}\right)^{\frac{3}{2}} \exp{(\beta z)} \mathcal{K}_{\frac{3}{2}}(\alpha\sqrt{z'\Delta^{-1}z}) \,.$$

Using the fact that for $z \to 0$, $\mathcal{K}_{\frac{3}{2}}(z) \sim z^{-\frac{3}{2}}$, we deduce

$$g(z) \sim (z'\Delta^{-1}z)^{-3/2} \,.$$

It follows

$$\sigma_{11}^2(\varepsilon), \sigma_{22}^2(\varepsilon) \sim \varepsilon \ln(\varepsilon) + \varepsilon \,,$$
$$\sigma_{12}^2(\varepsilon) \sim \varepsilon \,,$$

giving the speed of convergence in terms of the truncation level $\varepsilon$. With L'Hôpital's rule it is easy to see that $\varepsilon \ln(\varepsilon)$ vanishes when $\varepsilon$ goes to 0.

## 4. Conclusion

In this paper we considered spread options written on a two-dimensional jump-diffusion model. We applied the duality principle introduced by Eberlein et al. [14, 15] to express the spread option price in terms of the price of a put option under the dual measure. We applied this approach to derive an explicit formula for the value of a spread option in the case of a bivariate geometric Brownian motion with stochastic volatility following the Barndorff-Nielsen and Shephard model. In addition, we also considered a two-dimensional geometric Lévy process for the dynamics of the price process. We first applied the Esscher transform to derive a risk neutral measure. Then, we applied the duality principle to derive the Margrabe formula.

As for the study of robustness, we approximated the small jumps in the Lévy process by a two-dimensional Brownian motion scaled with the variance-covariance matrix of the small jumps. We first considered the approximation under the martingale measure and we proved the robustness of the Margrabe formula. Next, in the specific case of a geometric Lévy process, we considered the approximation under the real world measure. In this situation the Esscher transform as well as the duality measure become dependent on the approximation. For this choice of dynamics, we proved the robustness of the Margarbe formula and we computed the convergence rates.

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