Technische Universität München

Department of Mathematics

Chair of Mathematical Statistics

Master's Thesis

Systemic risk in credit networks in the presence of a financial accelerator

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Submission Date: 29.10.2013

I assure the single handed composition of this master's thesis only supported by declared resources.

Garching,

Acknowledgment

I gratefully acknowledge Dr. Ben Craig of the Federal Reserve Bank of Cleveland for providing data on the German interbank market which found usage in the end of this work. Furthermore I thank Prof. Claudia Klüppelberg and Dr. Oliver Kley for their helpful advice and patience referring my working speed. Additionally, I mention my parents without whose support I would have failed to manage both my entire studies and this work.

Zusammenfassung

Systemisches Risiko in Kreditnetzwerken ist ein wachsendes Gebiet intensiver Forschung. Es wird der Aufsatz "Liaisons dangereuses: Increasing connectivity, risk sharing and systemic risk" von Battiston et al. (2012) rezensiert, der systemisches Risiko vom Grad der Diversifizierung in homogenen Kreditnetzwerken ableitet und dabei die Rolle von negativen Feedback-Schleifen im Hinblick auf die Konstitution der Akteure des Netzwerkes mit einbezieht. Zum einen erfolgt eine Zusammenfassung des Aufsatzes in mathematisch präziser Art und Weise und zum anderen werden die Resultate des Aufsatzes vorgestellt. Weiterhin wird mit Hilfe von Simulationen geprüft, ob diese Resultate bestätigt werden können, falls Annahmen, die aus analytischen Gründen eingeführt werden, unberücksichtigt bleiben. Abschließend wird der deutsche Interbankmarkte betrachtet, der einer Studie zu Folge eine "Core-Periphery" Struktur aufweisen soll. Hierbei wird ein Teil des zusammengefassten Aufsatzes verwendet, um eine Aussage über systemisches Risiko auf Kreditnetzwerken mit solch einer Struktur treffen zu können.

Abstract

Systemic risk analysis of credit networks is an emerging field of interest. In our work we will first review the paper "Liaisons dangereuses: Increasing connectivity, risk sharing and systemic risk" of Battiston et al. (2012), which deals with systemic risk induced by interdependencies, diversification and feedback mechanisms on credit networks. This paper shall be summarized in a mathematically rigorous manner. A subsequent discussion assesses the procedure of the paper and its results by resorting to some simulations. Particularly, we deal with the simplifications and assumptions used in the paper which allow on the one hand an analytical treatment but on the other hand possibly involve the risk of falsification. Finally, we consider in a practical application the German interbank market, which possesses, according to a research study, a core-periphery structure. For an examination we will partly refer to the methods of the summarized paper in order to figure out some conclusions on systemic risk within networks of such a structure.

Keywords: Systemic Risk, Credit Network, Interbank Market, Financial Contagion, Financial Accelerator, Core-Periphery Structure

Contents

1 Introduction

Credits are an essential component of the global economy. Many investment projects either of private or public nature may only be financed by the way of credit. On February 24th, 2009 U.S. President Obama (2009) has stated in his speech to the joint session of the Congress that "*... the flow of credit is the lifeblood of our economy.*" For guaranteeing this flow of credits an economy requires a working credit network. Such a network may contain all kinds of participants: private consumers, small and large companies, the government and last but not least the financial industry. Particularly banks execute the role of financial intermediaries which organize the credit exchange between the network's participants. Hereby, banks do not only organize the credit flow but also act as creditors and debtors on the credit markets themselves. For instance there is a large interbank market on which banks borrow and lend money among each other in order to run their businesses. Just this market has came under scrutiny during the financial crisis of the last decade. Though this financial crisis had first limited to a small segment of the financial industry it vastly spread to the remaining, global financial system and has resulted in a tenacious distrust between all agents on the interbank market. The following distortion of this market, which is for instance examined by the work of Heider et al. (2009) or Tintchev (2013), was not only a problem for the financial industry but also affected the credit flow of the remaining economy.

The crisis spread from a small segment of the financial industry to the entire, global banking system has originated a plenty of research activity about connectivity between banks and the consequences on systemic risk. For instance, Markose et al. (2012) headline their work by "Too Interconnected To Fail" following the title "Too Big To Fail" of Sorkin (2009). Their work deals with "*... [t]he dominance of a few big players in the chains of insurance and reinsurance for CDS credit risk mitigation in banks' assets ...*" and the effects of this dominance on systemic risk. Further examinations about banks' interconnectivity and systemic risk are for instance given by the pioneering work of Allen and Gale (2000), the work of Hurd and Gleeson (2011) or by Battiston et al. (2012).

In the following we are going to review the work of Battiston et al. (2012) whose core result is an alternative draft to the classical theory of risk diversification. They state that a too vast risk diversification on the interbank credit market, which implies a higher connectivity on this market, may increase systemic risk if negative feedback loops on the banks' solvency are additionally assumed to exist. We are going to summarize the setting of Battiston et al. (2012) in a mathematically rigorous manner and will present their results in detail. While the motivation of the setting is based on a sophisticated model containing various aspects of financial contagion, the analytical results of Battiston et al. (2012) rely on a simplified version of this model. Since it is not obviously clear, whether the core of the analytical results, namely systemic risk as a non-decreasing function of diversification in the banks' credit portfolios, are possibly falsified by these simplifications, we use a simulation procedure in order to confirm that the originally sophisticated model also provides systemic risk as a non-decreasing function of diversification.

While the examination of Battiston et al. (2012) is limited to an homogeneous interbank credit market, that means that every agent of the market has the same degree of diversification in her credit portfolio, we partially extend the procedure of Battiston et al. (2012) to an interbank market consisting of a core and a periphery. The last is a practically

Figure 1.1: The upper plot shows the number of failures and assistance transactions per quarter where periods of less or no defaults are partially excluded from the time line. The lower plots are the cumulative numbers corresponding to the three periods of noticeable default sequences. The data have been drawn from www.fdic.gov.

relevant structure for the interbank credit market as it is for instance identified by the works of Bech and Atalay (2008), Craig and von Peter (2010) or Langfield et al. (2012). In the end of this work we like to examine the fragility of such a network structure relying on observed data of the German interbank market.

1.1 Historical examples of distressed bank markets

In the forefront of the actual work we like to mention some historical examples on distressed bank markets in order to illustrate that systemic risk is a cyclical appearance. For giving an overview about the history of distressed interbank markets we refer to data of the Federal Deposit Insurance Corporation (FDIC). In Figure 1.1 we illustrate for the years 1934 - 2013 the counts of failures and assistance transactions[1] among commercial and saving banks which had been member of the FDIC system. Apparently, there are three periods of noticeable default activity:
The recent financial crisis has its origin in the mid of 2007 when the first financial institutions - predominantly on the US mortgage market as it is explained by Lang and Jagitani (2010) - came under pressure and partially defaulted. The following spread of the mortgage market's distortions to the remaining financial system has augmented the list of defaulted institutions by large, internationally well-respected institutions like AIG or Lehman Brothers. This in turn led to massive liquidity problems on the entire interbank credit market and subsequent failures in the financial industry. Figure 1.1 shows that after a duration of 5 years we still find ourselves in a noticeable period of bank defaults.
The second period in the 1980s and early 1990s is often tagged as *Savings and Loan Crisis* which is characterized by a massive number of failed savings and loan associations on the U.S. market. Ely (2008) sees the trigger of this misery in the Federal Reserve's reversal to a restrictive monetary policy in 1979 which increased interest rates and led to accumulated losses of 9 USD billion in the savings and loan industry for the following two years. These losses in an any case fragile savings and loan industry and a sequence of wrong policy decisions finally resulted in the high number of failed institutions during this time.
The observable defaults in the 1930s may be referred to the era of *The Great Depression*. Actually, for a complete look on this time we would have to extend the time line up to the late 1920s. As it is described in Galbraith (1988) The Great Depression's trigger dated from the *Black Friday* in 1929. Losses on banks' balance sheets, an a priori fragile banking industry and political disabilities amass in the aftermath a high number of failed banks in the period between 1929 and the Second World War.[2]

[1] Assistance transaction are any activities by the FDIC to prevent a complete failure of troubled banks. A more detailed definition can be found in FDIC (2003).

[2] The FDIC had been established in 1934. This is the reason why our observation period about failures

Interestingly, the intervening periods between these crises possess significantly fewer counts of defaults. For these periods we calculate an averaged count of defaults per quarter of 1.1. Obviously, this figure is essentially less than the observed counts of defaults in the described crises.
For the following review on the work of Battiston et al. (2012) we can keep in mind that distress on bank markets appear cyclically. The idea of Battiston et al. (2012) is to measure systemic risk on the interbank credit market as the intensity with which such cycles of instability occur. Hereby the intensity is derived from the first passage time of a process which ought to represent the system-wide financial constitution of the agents on the interbank market. This first passage time in turn is driven by the inserted financial accelerator on the one hand and by the diversification degree of the agents' credit portfolios on the other hand. As we will see the interdependence of both elements is the basis for the core result of Battiston et al. (2012) which states that systemic risk in the presence of a financial accelerator is a non-decreasing function with respect to diversification.

in the FDIC system illustrated in Figure 1.1 stops in 1934.

Figure 2.1: The figure illustrates an undirected graph on the left and a directed graph on the right. Both graphs have the same set of nodes, namely, $V = \{1, 2, 3, 4, 5\}$. While for the left graph the set of edges is $\{\{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 5\}\}$, the set of edges for the right one is $\{(1, 2), (1, 4), (4, 5), (5, 4), (5, 2)\}$.

2 Modeling an interbank credit mark

This section serves to build up the framework in which systemic risk is going to be examined. We strictly follow Battiston et al. (2012) by defining the interbank credit market as a graph whereas the graph's nodes and edges represent the network's agents and their bilateral credit relationships among each other. The discussion on credit networks by referring to the techniques of graphs is widely used. At this point we again mention as references the work of Hurd and Gleeson (2011) and Markose et al. (2012). A general introduction to the theory of graphs may be found in Boccaletti et al. (2006).
First we introduce some definitions which concatenate graphs and credit networks in general. Secondly we give a small example on contagion effects in credit networks. At the end of this section we state the assumptions which are used by Battiston et al. (2012) to restrict their examination on a certain kind of network structure.

2.1 Definitions

We give a general definition on graphs which is borrowed from Boccaletti et al. (2006).

Definition 2.1. *An undirected (directed) graph $G = (V, E)$ consists of two sets V and E such that $V \neq \emptyset$ and E is a set of unordered (ordered) pairs of the elements of V. While the elements of E are the edges of the graph, the elements of V are the graph's nodes.*

This definition emphasizes that we have to distinguish between two kinds of graphs, namely, directed and undirected ones. While in the latter it would not matter whether an edge between two nodes starts in the one or the other, the former exactly makes this difference. As an example see Figure 2.1 which graphically shows an undirected graph on the left and a directed graph on the right.
Battiston et al. (2012) demonstrate the interlinkage within credit networks by applying directed graphs, so, the next definitions are used to introduce ordinary expressions and notations exclusively for directed graphs.

Definition 2.2. *Let $i \in V$ be a node in the directed graph $G = (V, E)$. Then*

$$V_{in}(i) = \{j \in V : (j, i) \in E\}$$

is the in-neighborhood of $i \in V$ and analogously

$$V_{out}(i) = \{j \in V : (i, j) \in E\}$$

the out-neighborhood of $i \in V$. Additionally we define by

$$\begin{aligned} d_{in}(i) &= |\{j \in V : (j, i) \in E\}| \\ d_{out}(i) &= |\{j \in V : (i, j) \in E\}|, \end{aligned}$$

the in- and out-degree of $i \in V$ respectively, where $|S|$ denotes the cardinality of any set S.

The next definition presents a convenient way to express graphs in a compact form.

Definition 2.3. *Let $G = (V, E)$ be a directed graph with $|V| = N < \infty$. The $N \times N$ adjacency matrix $A = (a_{ij})_{i=1}^{N}$ of the graph G is defined by the entries*

$$a_{ij} = \begin{cases} 1, & \textit{if } (i,j) \in E \\ 0, & \textit{else} \end{cases}$$

for $i, j = 1, \ldots, N$.

The entries of the adjacency matrix immediately signal whether there is a link between two nodes and in which direction if so. Unless otherwise stated, we suppose for the adjacency matrix of any directed graph a zero diagonal. This constraint means an exclusion of self-loops in directed graphs and implies for all $i \in V$ that (i, i) may not be found in E. To extend the notion of a directed graph a third measure is going to be introduced by the next definition.

Definition 2.4. *A weighted, directed graph is a directed graph $G^w = (V, E, W)$ with additional set W, where for all $l \in E$ there is a unique weight $w_l \in W$.*

The weights $w_l \in W$ allow to distinguish and compare the various links within a graph. Thus, a weighted, directed graph may be considered as a subtle extension of the primarily introduced version of a graph and provides more information about the interaction between its nodes. Based on this we define the weighted adjacency matrix $A^w = (a^w_{ij})_{i,j=1}^{N}$ with entries

$$a^w_{ij} = \begin{cases} w_{(i,j)}, & \text{if } (i,j) \in E \\ 0, & \text{else.} \end{cases}$$

The adjacency matrix of a weighted, directed graph with entries

$$a_{ij} = \begin{cases} 1, & \text{if } a^w_{ij} \neq 0 \\ 0, & \text{else} \end{cases}$$

is the corresponding unweighted adjacency matrix, that is the matrix signaling the existing edges but disregarding the weights.

2.2 Contagion in credit networks

In various scientific disciplines network analysis on graphs is a popular method to investigate complex interaction and connectivity between network participants. The consideration of disease spread in the field of epidemiology has always been a popular application of network analysis. In this context, the pioneering work of Kermack and McKendrick (1927) carried out a theory which resulted in the extensively used SIR-models. Meanwhile, interaction in the world wide web is widely examined - for instance the spread of virtually viruses by Berger et al. (2005). The list of examples can be arbitrarily extended, however, many applications share the same interest: How is the entire network affected by an initial disruption anywhere in the network? If we now turn to financial credit markets, not only the last financial crisis revealed that disruptions in the credit market, which are first locally limited, may lead to market-wide distortions. Not without reasons economists like referring to the image of a virus spread within a population when they talk about the development of a crisis on the financial markets.[3] In literature the transition of financial difficulties from one agent to another agent is often tagged as *financial contagion*. In the following we exemplify how financial contagion may happen if two agents are linked by a bilateral credit relationship.

Example 1. *1. Given two agents A and B with initial balance sheets as follows*

Balance Sheet (1) of A

Cash	*5*	*Equity*	*10*
Other Assets	*35*	*Debt*	*30*

Balance Sheet (1) of B

Cash	*20*	*Equity*	*40*
Other Assets	*60*	*Debt*	*40*

2. Now, suppose that A is interested in realizing an investment project but has not got sufficient liquidity. Therefore she borrows cash from B. The balance sheets would change to

Balance Sheet (2) of A

Cash	*20*	*Equity*	*10*
Other Assets	*35*	*Debt*	*45*

Balance Sheet (2) of B

Cash	*5*	*Equity*	*40*
Other Assets	*75*	*Debt*	*40*

3. For any reason A slides into financial difficulties and is not able to reimburse the loan to B who then must depreciate her accounts receivable on A. Since depreciations are expenditures this directly results in a reduction of equity.

Balance Sheet (3) of B

Cash	*5*	*Equity*	*25*
Other Assets	*60*	*Debt*	*40*

We see that a primal weakness at A may immediately effect the financial strength of B subsequently. Obviously, the shape of these effects depend on the amount of credit and B's own initial financial situation, however, in worst case it might even develop to the point of B's bankruptcy. In this example B probably survived the case due to enough equity, but imagine that she had defaulted as well, then the creditors of B's debt had to conduct depreciations on their balance sheets, too. We can conclude that starting difficulties at one agent are under certain circumstances not only limited to bilateral business partners but may indeed spread by financial contagion to other parts of the credit network as well. The example has supposed a complete payment default of A. Let us now assume that A does not default in step 3, however, her credit-worthiness is seriously stricken. This might be reflected by rising spreads on CDS which insure creditors against the default of A. If and how this is reflected in B's balance sheet depends on the applied accounting standards. In the case of strict fair value accounting and if there were also an opportunity to revalue B's account receivables against A reliably, B would have to adjust them in her balance

[3]For instance, see Dodd and Mills (2008) who motivate their presentation on the last financial crisis by the example of a virus spread.

sheet downwards. Consequently, not only an actual default but also partial changes in the financial strength of a debtor may effect the creditor's strength immediately.[4]
By continuing along the contents of Battiston et al. (2012) we develop a model which is thought to examine financial contagion in interbank credit markets based on these kinds of examples.

2.3 The interbank credit market as a graph

By the next definition we will formulate an interbank credit market as weighted, directed graph.

Definition 2.5. *In the following, a directed, weighted graph $G^w = (V, E, W)$ matches a credit interbank market where the market's agents are represented by the set V. Each agent $i \in V$ manages a credit portfolio $E_i \subseteq E$, where for all $j \in V$ it is $(i,j) \in E_i$ if agent j is a debtor of agent i. It is agreed that for all $i \in V$*

$$\sum_{(i,j)\in E_i} w_{(i,j)} = 1$$

holds, where a weight $w_{(i,j)} \in W$ corresponds to $w_{(i,j)} \cdot 100$ % of the total credit amount agent i has issued to other agents in V.

To be in line with Battiston et al. (2012) we underlay the interbank market with the following specifying assumptions:

- A(1.1) The interbank credit market consists of N agents, i.e. $|V| = N$.
- A(1.2) Each agent acts as creditor and issues credits to $0 < k \leq N$ other agents from the same market.
- A(1.3) Each issued credit amounts to $\frac{1}{k} \cdot 100$ % of the agent's credit portfolio for the interbank market.

Assumption A(1.1) restricts our considerations to finite graphs where A(1.2) and A(1.3) include a simplification of the real world. Obviously in real world credit markets there will be agents with different credit portfolio sizes, however, A(1.2) and A(1.3) can be thought as an approximation to an homogeneous credit market with agents all of the same size and with comparable diversified credit portfolios. Particularly the market does not have any key player or a group of creditors which comparatively issue a higher number of credits than the remaining market. Hereby k is taken as measure for the present connectivity within the credit market and simultaneously as degree of diversification in the agents' credit portfolios.
If we use the technical expressions from graph theory and referring to Definition 2.2 we obtain on an interbank market fulfilling assumption A(1.2) for all $i \in V$

$$d_{out}(i) = k.$$

The next proposition characterizes the adjacency matrix of a graph representing an interbank credit market with assumptions A(1.1) - A(1.3).

[4]For a definition and close discussion about fair value accounting as a reason for financial contagion effects see Ryan (2008).

Proposition 2.1. *Any weighted directed graph $G^w = (V, E, W)$ which represents an interbank credit market fulfilling assumptions A(1.1) - A(1.3) has a weighted, row stochastic adjacency matrix $A^w = (a^w_{ij})^N_{i,j=1}$ and an unweighted adjacency matrix $A = (a_{ij})^N_{i,j=1}$ for which it holds that $A = kA^w$.*

Proof. By definition one knows that

$$\sum_{j=1}^{N} a_{ij} = deg_{out}(i)$$

for all $i \in V$. Assumption A(1.2) prescribes that $deg_{out}(i) = k$. Assumption A(1.3) in turn indicates that $w_{(i,j)} = \frac{1}{k}$ for all $(i, j) \in E$. So, we can conclude that

$$\sum_{j=1}^{N} a^w_{ij} = \sum_{j=1}^{N} \frac{1}{k} a_{ij} = \frac{1}{k} deg_{out}(i) = 1.$$

□

The proposition gives an idea how to represent a credit interbank market in matrix form which is going to be helpful for approaching some simulations on the suggested credit interbank market at a later point.

3 The risk of default in homogeneous networks

Up to now we have implemented a credit market as a graph while such a graph is first a static formation. Since we are interested in the agents' risk of default over time, we must extend the model by a dynamical component. This is going to be done by deriving a stochastic differential equation. We follow Battiston et al. (2012) by motivating a sophisticated, multivariate system of stochastic differential equations (SDEs) which includes a series of possible effects impacting the participants of a credit network. However, in order to receive explicit formulas for determining the individual risk of default for one agent the system of stochastic differential equations is reduced to a simple, univariate drift-diffusion model. In an added discussion we deal with the question whether the resulting univariate process still includes sufficient information in order to make inferences on an agent's default risk within the originally multivariate system of coupled SDEs.

3.1 Financial robustness in continuous time

Each agent i in our network is going to be endowed by a measure called financial robustness which shall quantify an agent's financial constitution. First, there is no concrete financial ratio such as, for instance, the ratio equity to total assets which matches the financial robustness directly. Battiston et al. (2012) rather perceive loosely the financial robustness as a measure for an agent's level of equity.
In their brief calibration exercise Battiston et al. (2012) use the normalized market capitalization of financial institutions, where normalized means that market capitalization is divided by its own maximum value on a given observation time line.
Another possible interpretation of the financial robustness might be just a kind of rating reflecting the public opinion about an agent's financial strength.

3.1.1 Specification of the financial robustness

The financial robustness of B in Example 1 would have been suffered during the three steps due to the default of her debtor A or in other words because the financial robustness of her debtor A has been dropped to zero and generated a loss on B's equity. This functional chain is taken by Battiston et al. (2012) as a motivation to specify the financial robustness of agent $i \in V$, which we denote in the following by $\rho_i^D(n)$ for discrete time steps $n = 0, 1, \dots$, in dependence of the financial robustness of agent i's debtors. The specification is enriched by three further assumptions on the credit market and its agents:

- A(1.4) The graph G^w which represents the credit network is fixed in $n = 0$ and then stays constant for $n > 0$, i.e. the composition of one agent's credit portfolio will not change over time.

- A(1.5) Next to the robustness of other agents the robustness of any agent $i \in V$ is additionally driven by an external effect which is going to be modeled by a Gaussian process $(\sigma\xi_i(n))_{n=0,1,\dots}$ where $\xi_i(n) \sim N(0, 1)$ iid and $\sigma > 0$ deterministic. The Gaussian processes are supposed to be independent for all $i \in V$.

- A(1.6) The agents apply fair value accounting to their credit portfolios and adjustments in their balance sheets occur permanently.

By A(1.5) we take into account that the financial robustness of an agent is not only subject to the developments in her credit portfolio alone but also to some random noise. We can suppose that an agent i will probably not only finance other banks' liabilities but also invest in other projects. Of course, the return on these projects will impact the financial robustness as well and if one assumes stochastic returns, it is a nearby to insert the projects' impact on the financial robustness over time as random noise.
Based on this and following Battiston et al. (2012) we specify the financial robustness ρ_i^D of agent i as

$$\rho_i^D(n+1) = \sum_{j \in V_{out}(i)} w_{(i,j)} \left(\rho_j^D(n) + \sigma\xi_j(n)\right) = \frac{1}{k} \sum_{j \in V_{out}(i)} \left(\rho_j^D(n) + \sigma\xi_j(n)\right). \tag{3.1}$$

for $n = 0, 1, \dots$. So, the financial robustness of agent i in $n+1$ arises from the weighted robustness of her debtors in n which is additively adjusted by the random component from (A5).
This equation implies that each and every minor change $\epsilon > 0$ in the robustness of a debtor is directly reflected in the robustness of the creditor with $w_{(i,j)} \cdot \epsilon$ one step later. Without assumption (A6) only actual losses would affect an agent's equity in her balance sheet and we would have a problem to argue for this equation. For instance, suppose an accounting standard which states, that a creditor must only depreciate account receivables on a credit if the debtor's default has indeed occurred, then the equation

$$\rho_i^\star(n+1) = \sum_{j \in V_{out}(i)} w_{(i,j)} \mathbb{1}_{(\rho_j^\star(n) > 0)} + \sigma\xi_i(n)$$

would have presumably been more convenient where $\mathbb{1}_M$ denotes the indicator function on the set M.[5]
By proceeding with Equation (3.1) we see that this equation may be rewritten in multivariate form as matrix-vector notation with column vectors $\rho^D(n) := \left(\rho_1^D(n), \dots, \rho_N^D(n)\right)^T, \xi(n) = \left(\xi_1(n), \dots, \xi_N(n)\right)^T$ and the network's weighted adjacency matrix A^w as

$$\rho^D(n+1) = A^w \rho^D(n) + \sigma A^w \xi(n). \tag{3.2}$$

This representation of the financial robustness, however, is not the final representation used by Battiston et al. (2012). For deriving analytical results the last equation will be further extended but also simplified as follows:

- Transformation of the last equation into continuous time
- Inserting the financial accelerator effect as a jump process
- Reduction to one dimension.

[5] There has not been a discussion on accounting standards and its implications on the model in Battiston et al. (2012), however, we think that for the specification of the financial robustness, as in Equation (3.1), assumption A(1.6) should be postulated.

3.1.2 Continuous transformation

The transformation of Equation (3.2) into continuous time takes place in order to rely for future analytical results on the mean first passage time of stochastic processes in continuous time. Actually, Battiston et al. (2012) derive a SDE in continuous time from Equation (3.2). We argue this step by relying on the Euler-Maruyama scheme which is defined in Mao (1997) as follows.

Definition 3.1. *Let $(\zeta(t))_{t\geq 0}$ be a $\mathbb{R}^N$-valued stochastic process in continuous time which satisfies the stochastic integral equation*

$$\zeta(t) = \zeta(0) + \int_0^t G(\zeta(s))ds + \int_0^t \Sigma(\zeta(s))dW(s), \tag{3.3}$$

where $G \in \mathcal{C}\left(\mathbb{R}^N;\mathbb{R}^N\right)$, $\Sigma \in \mathcal{C}\left(\mathbb{R}^N;\mathbb{R}^{N\times N}\right)$ and W is a N-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F})_{t\leq 0}$. Then, the Euler-Maruyama scheme $\zeta^m(t)$ defined on the interval $[0,T]$ for the equidistant partition $t_n = \frac{n}{m}T$, $0 \leq n \leq m$ denotes

$$\zeta^m(t) = \zeta^m(t_n) + G(\zeta^m(t_n))(t-t_n) + \Sigma(\zeta^n(t_n))(W(t) - W(t_n)), \tag{3.4}$$

for $t \in (t_n, t_{n+1}]$ and $\zeta^m(t_0) = \zeta(0)$.

The Euler-Maruyama scheme serves as an approximation procedure of SDEs whose solutions are not explicitly known. For numerical usage one predetermines the time horizon T and the discretization size $\frac{T}{m}$ in order to evaluate ζ^m at the sampling points $t = 0, \frac{T}{m}, \ldots, T$ by simulating the increments of the Brownian motion. The next theorem ensures strong convergence of the Euler-Maruyama scheme to the stochastic process ζ as $m \to \infty$ under some regularity conditions for G and Σ.

Theorem 3.1. *Assume a Euler-Maruyama scheme $\zeta^m(t)$ for $\zeta(t)$ being the unique solution of Equation (3.3) whereas for functions G and Σ there are two positive constants $\bar{C}$ and C such that for all $x, y \in \mathbb{R}^N$ the conditions*

$$\begin{aligned} \max\left(|G(x)-G(y)|^2, |\Sigma(x)-\Sigma(y)|^2\right) &\leq \bar{C}|x-y|^2 \\ \max\left(|G(x)|^2, |\Sigma(x)|^2\right) &\leq C(1+|x|^2) \end{aligned}$$

are fulfilled. Then there is a positive constant $C^\star$ depending on $\bar{C}$ and C so that

$$E\left[\sup_{0\leq t\leq T} |\zeta^m(t) - \zeta(t)|^2\right] \leq \frac{C^\star}{m}.$$

Proof. The proof and the formula on $C^\star$ may be found in Mao (1997). □

We use the last result to argue for the transformation of the process in Equation (3.2) into continuous time as follows:

1. At first $\rho^D(n)$ is subtracted on both sides of Equation (3.2) which yields

$$\rho^D(n+1) - \rho^D(n) = (A^w - I_{N\times N})\rho^D(n) + \sigma A^w \xi(n),$$

where $I_{N\times N}$ denotes the $N \times N$ identity matrix.

2. We rewrite the last equation as

$$\rho^D(n+1) = \rho^D(n) + (A^w - I_{N\times N})\rho^D(n)(n+1-n) + \sigma A^w \xi(n).$$

   Now, by setting $t_{n+1} = n+1, t_n = n, G(\rho^D(t_n)) = (A^w - I_{N\times N})\rho^D(t_n), \Sigma(\rho^D(t_n)) = \sigma A^w$ and by considering $\xi(n)$ as the increment of a Brownian motion over an interval of length 1, the last equation fits to the Euler-Maruyama scheme in Equation (3.4), while the discretization size here is 1.

3. If we reduce the discretization size of the last equation towards zero, we may argue that $\rho^D(n+1)$ converges in the sense of Theorem 3.1 to a stochastic process ρ in continuous time having the dynamics

$$d\rho(t) = (A^w - I_{N\times N})\rho(t)dt + \sigma A^w dW(t) \tag{3.5}$$

   where $W(t)$ denotes a N-dimensional Brownian motion.

For the last step we additionally have to check whether the functions G and Σ induced by step 2 fulfill the conditions of Theorem 3.1. Therefore, we argue that the linear functions G and Σ are Lipschitz-continuous due to their bounded derivatives. Hence, one can find positive constants $\bar{C}$ and C so that both conditions from Theorem 3.1 are fulfilled.
Before we proceed with the contains of Battiston et al. (2012) we insert a proposition which points out the explicit solution of Equation (3.5) denoting the dynamics of a multivariate Ornstein-Uhlenbeck process. The derivation of this result may be found in Gardiner (2004).

Proposition 3.1. *Let the SDE from Equation* (3.5) *be given with initial condition* $\rho(0) = I_N$*, where* $I_N := \left[1, \ldots, 1\right]^T \in \mathbb{R}^N$.

1. *The SDE's explicit solution is given by*

$$\rho(t) = \exp\left[t(A^w - I_{N\times N})\right] I_N + \int_0^t \sigma \exp\left[(t-s)(A^w - I_{N\times N})\right] A^w dW(s)$$

   with the matrix exponential $\exp[X] := \sum_{m=0}^{\infty} \frac{1}{m!} X^m$.

2. $E\left[\rho(t)\right] = \exp\left[t(A^w - I_{N\times N})\right] I_N$

3. *For any* $t, t' > 0$

$$Cov\left[\rho(t), \rho(t')\right] = \int_0^{\min(t,s)} \sigma^2 \exp\left[(t-s)(A^w - I_{N\times N})\right] A^w (A^w)^T \exp\left[(t'-s)(A^w - I_{N\times N})^T\right] ds$$

   where $Cov[x,y] := E[xy^T] - E[x]E[y]^T$ *denotes the covariance matrix for two* $\mathbb{R}^N$*-valued random vectors* x *and* y.

3.1.3 Financial Accelerator

The i-th row of Equation (3.5) contains the component

$$\left(\sum_{j \in V_{out}(i)} \frac{1}{k} \rho_j(t) - \rho_i(t) \right) dt,$$

which may be considered as a kind of mean-reverting element. More precisely, the financial robustness of agent i is permanently adjusted towards her debtors' averaged robustness. This is in line with the motivating Example 1 and the associated notes about fair value accounting. Battiston et al. (2012) indicate that single losses in the balance sheet which have at first only a one-time effect on an agent's robustness may be succeeded by further negative effects. In order to illustrate such a succession, we are going to extend Example 1 from Section 2.2 by a further step:

Example (cont.) 1. *4. Due to credit losses in step 3 B's rating is downgraded by several rating agencies. As a trader of derivatives and other securities B has to face increasing margin calls by her brokers. In order to satisfy all margin calls she has to sell a set of her assets under time pressure. In doing so she suffers losses since some of the assets are sold under book value. Thus, her equity is reduced once again.*[6]

The extension of the example shows that a creditor's equity does not have to be affected just by the non-payed credit but might also suffer subsequently further negative effects. There is a series of other examples which can depict such effects. For one further example, let us assume that B herself is debtor and has to renegotiate her credit line regularly. If B's creditors have been informed about B's losses in step 3 of Example 1, the new credit terms will probably turn out worse from B's perspective. This in turn effects her robustness negatively again and may eventually trigger a downward spiral on her robustness.
Battiston et al. (2012) name such succession effects *financial accelerator* and use a jump process to incorporate these effects into Equation (3.5). The set $\mathcal{T} = \{t_0, t_1, \dots\}$ is going to include the assumed deterministic jump times of this process. It is furthermore agreed that

$$t_n = t_{n-1} + s$$

for $n \geq 1$ where $s > 0$. In order to avoid any ambiguities, we additionally set $t_0 \geq d + s$ for $0 < d < s$. Actually, Battiston et al. (2012) have not payed attention to t_0, however, without our restriction we would have to consider the process of the financial robustness in the later proceeding for time t less than 0, which is not wanted, since the process is going to be started always in $t = 0$ and there is no consensus by Battiston et al. (2012) about how to handle the process for $t < 0$.
In practice financial accelerator effects presumably arise when an agent has to report her balance sheets which normally occurs in regular time intervals, hence, the deterministic jump times with a constant distant should not disturb.

[6] Suddenly occurring margin calls oftentimes cause irrevocable defaults of stricken agents in the financial industry. For a practical example one is advised to Abelow (2011), who declares, as the former COO of MF Global, the reasons of MF Global's bankruptcy to the US Bankruptcy Court. He argues in point 35 that increasing margin requirements against MF Global as a consequence of rating downgrades have contributed to MF Global's bankruptcy due to a hereby stricken liquidity.

In order to incorporate the jump process of the financial accelerator into the dynamics of the financial robustness, we apply a measure which is substantiated in the next lemma. For a general introduction and definitions about measure theory we refer to Elstrodt (2009).

Lemma 3.1. *Let $\mathcal{B}(\mathbb{R})$ denote the Borel-algebra over $\mathbb{R}$. Then, the function*

$$\pi_i\colon \mathcal{B}(\mathbb{R}) \to [0, +\infty], \qquad \pi_i(B) = \sum_{t_n \in \mathcal{T}} \mathbb{1}_B(t_n)$$

defines a measure on $\mathcal{B}(\mathbb{R})$ where $i \in V$.

Proof. Per definition of a measure we have to check

(i) $\pi_i(\emptyset) = 0$,

(ii) $\pi_i \geq 0$,

(iii) $\pi_i\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \pi_i(B_j)$ for any sequence $(B_j)_{j\geq 1}$ of disjoint sets in $\mathcal{B}(\mathbb{R})$.

Obviously (i) and (ii) are fulfilled. For (iii) we refer to a well known property of indicator functions, namely, for any sequence $(B_j)_{j\geq 1}$ of disjoint sets it holds

$$\mathbb{1}_B(s) = \sum_{j=1}^{\infty} \mathbb{1}_{B_j}(s),$$

where $B := \bigcup_{j=1}^{\infty} B_j$. Then, we can deduce that

$$\begin{aligned}
\pi_i(B) &= \sum_{t_n \in \mathcal{T}} \mathbb{1}_B(t_n) \\
&= \sum_{t_n \in \mathcal{T}} \sum_{j=1}^{\infty} \mathbb{1}_{B_j}(t_n) \\
&= \sum_{j=1}^{\infty} \sum_{t_n \in \mathcal{T}} \mathbb{1}_{B_j}(t_n) \\
&= \sum_{j=1}^{\infty} \pi_i(B_j).
\end{aligned}$$

The interchange of summations in step 3 is reasoned by Tonelli's theorem due to the non-negativity of the occurring summands. The final step is simply obtained by the definition of π_i. □

Remark: We can use measure π_i for Lebesgue integration, i.e. for any $\mathcal{B}(\mathbb{R})$-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ and any $B \in \mathcal{B}(\mathbb{R})$, we have

$$\int_B f(u) d\pi_i(u) = \sum_{t_n \in \mathcal{T} \cap B} f(t_n).$$

We now define the financial accelerator effect as a stochastic process and allude that our definition is the same as the one of Battiston et al. (2012), though inscribed by a slightly different way.

Definition 3.2. *We define the financial accelerator of agent i as a stochastic process $(h_i(t))_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, P)$ by setting*

$$h_i(t,\omega) = -\alpha \mathbb{1}_{C_i(t-d)}(\omega)\mathbb{1}_{\mathcal{T}}(t)$$

with $C_i(t-d)$ supposed to be in $\mathcal{F}$, $\mathcal{T} = \{t_0, t_1, \dots\}$ as above, $\alpha \geq 0$ and $0 < d < s$. The $N-$dimensional vector h denotes the multivariate financial accelerator, that is

$$h(t,\omega) = \big(h_1(t,\omega), \dots, h_N(t,\omega)\big)^T.$$

We rewrite the dynamics in Equation (3.5) by incorporating the financial accelerator effect as

$$d\rho(t) = (A^w - I_{N\times N})\rho(t)dt + \sigma A^w dW(t) + h(t)d\pi(t), \tag{3.6}$$

where $\pi(t) = \begin{pmatrix}\pi_1(t) & \dots & \pi_N(t)\end{pmatrix}^T$.
The financial robustness is now driven by two sources: On the one side the multivariate Ornstein-Uhlenbeck Process which contains permanent adjustment of the financial robustness due to one-time accounting effects resulting from the changes in the debtors' robustness or some random noise. On the other side there are now additionally periodical jumps denoting the financial accelerator effect. The jumps' arrivals are not clear yet since the set $C_i(t-d)$ has not been specified so far. Prior to their definition we allude that for any fixed $\omega \in \Omega$ the fibers of $h_i(t,\omega)$ are subsets of the countable set $\mathcal{T}$. Hence, $h_i(t,\omega)$ is $\mathcal{B}(\mathbb{R})$-measurable and the integral

$$\int_0^t h(s)d\pi(s)$$

is component-by-component defined with respect to Lebesgue-integration.
By following the procedure of Battiston et al. (2012) and recalling the extension of Example 1 as a motivation, we agree upon specifying the sets C_i as

$$C_i(t) := \begin{cases} \{\rho_i(t) - \rho_i(t-s) < -\epsilon\} & \text{if } t \geq t_0 - d \\ \emptyset, & \text{else} \end{cases} \tag{3.7}$$

with respect to the process ρ_i resulting from the dynamics in Equation (3.6). In the next lemma we show that C_i is $\mathcal{F}$-measurable.

Lemma 3.2. *For the process ρ with dynamics as in Equation* (3.6) *and the N-dimensional Brownian motion W defined on the probability space $(\Omega, \mathcal{F}, P)$ the sets $C_i(t)$ of Equation* (3.7) *are contained in $\mathcal{F}$ for all $t \geq 0$ and $i \in V$.*

Proof. As $\emptyset \in \mathcal{F}$ the claim is fulfilled for $t \in [0, t_0 - d)$. For $t = t_0 - d$ we have

$$\begin{aligned} C_i(t_0 - d) &= \{\rho_i(t_0 - d) - \rho_i(t_0 - d - s) < -\epsilon\} \\ &= \Bigg\{ \int_{t_0-d-s}^{t_0-d} \Bigg(\sum_{j\in V_{out}(i)} \frac{1}{k}\rho_j(u) - \rho_i(u) \Bigg) du + \int_{t_0-d-s}^{t_0-d} h_i(u)d\pi_i(u) \end{aligned}$$

$$+ \quad \frac{\sigma}{k} \sum_{j \in V_{out}(i)} (W_j(t_0 - d) - W_j(t_0 - d - s)) < -\epsilon \Big\}.$$

By definition of π_i it is

$$\int_{t_0-d-s}^{t_0-d} h_i(u)\pi_i(u) \quad = \quad 0$$

So, we can deduce that

$$\begin{aligned} C_i(t_0 - d) \quad &= \quad \Big\{ \int_{t_0-d-s}^{t_0-d} \left(\sum_{j \in V_{out}(i)} \frac{1}{k}\rho_j(u) - \rho_i(u) \right) du \\ &+ \quad \frac{\sigma}{k} \sum_{j \in V_{out}(i)} (W_j(t_0 - d) - W_j(t_0 - d - s)) < -\epsilon \Big\}, \end{aligned}$$

where the left side of the inequality within the brackets is according to Proposition 3.1 the increment of the i-th element of a multivariate Ornstein-Uhlenbeck process. Particularly, it is well defined on the probability space $(\Omega, \mathcal{F}, P)$. Hence, $C_i(t_0 - d) \in \mathcal{F}$ which we use as base clause for the following induction procedure.
Assume the induction hypothesis $C_i(t_n - d) \in \mathcal{F}$ and let $t \in (t_n - d, t_{n+1} - d]$. Then we have per definition

$$\begin{aligned} C_i(t) \quad &= \quad \Big\{ \int_{t-s}^{t} \left(\sum_{j \in V_{out}(i)} \frac{1}{k}\rho_j(u) - \rho_i(u) \right) du + \int_{t-s}^{t} h_i(u) d\pi_i(u) \\ &+ \quad \frac{\sigma}{k} \sum_{j \in V_{out}(i)} (W_j(t) - W_j(t - s)) < -\epsilon \Big\}, \end{aligned}$$

where

$$\begin{aligned} \int_{t-s}^{t} h_i(u) d\pi_i(u) \quad &= \quad \int_{t-s}^{t} -\alpha \mathbb{1}_{C_i(u-d)} \mathbb{1}_{\mathcal{T}}(u) d\pi_i(u) \\ &= \quad -\alpha \mathbb{1}_{C_i(t_n - d)}. \end{aligned}$$

Due to the induction hypothesis the last indicator function is $\mathcal{F}$-measurable. Hence, we conclude that the left side of the inequality in the set $C_i(t)$ is $\mathcal{F}$-measurable as well for any $t \in (t_n, t_{n+1}]$. Particularly, this holds true for $t = t_{n+1} - d$ so that we deduce the lemma's claim. □

For any $t_n \in \mathcal{T}$ the set $C_i(t_n - d)$ may be regarded as a control set for the financial accelerator affecting agent i. Only if an $\omega \in \Omega$ has been realized so that the increment of the realization of ρ_i undershoots $-\epsilon$ on the interval $[t_n - d - s, t_n - d]$, the financial accelerator effect will occur in t_n as a jump with size $-\alpha$. Since $t_{n-1} = t_n - s$ and $d < s$ the control of the increment on the interval $[t_n - d - s, t_n - d]$ incorporates a possible financial accelerator effect in t_{n-1} which is a justification of the name *financial accelerator*. We follow Battiston et al. (2012) by setting for the remaining considerations always

$$s \quad := \quad 1.$$

Equation (3.6) covers all the impacts on an agent's financial robustness mentioned in Example 1 and its extension. Due to its computational complexity Battiston et al. (2012), however, avoid working with these dynamics but reduce their complexity up to a univariate drift-diffusion SDE, which is then used for deriving analytical results on the financial robustness in the context of default risk. By following these steps it will admittedly not be clear whether all of the original motivation entirely covered by Equation (3.6) will be adequately regained in the later analytical results or whether some of the contains of the motivation will become lost at the cost of complexity reduction. Having derived analytical results on the reduced SDE, we will later proceed some simulations on the dynamics in Equation (3.6) and try to figure out whether the analytical results can still be confirmed.

3.1.4 Reduction to one dimension

In a first step of complexity reduction Battiston et al. (2012) pick out one row of Equation (3.6), that is

$$d\rho_i(t) = \left(\sum_{j \in V_{out}(i)} \frac{1}{k} \rho_j(t) - \rho_i(t) \right) dt + \frac{\sigma}{k} \sum_{j \in V_{out}(i)} dW_j(t) + h_i(t) d\pi_i(t).$$

At this point Battiston et al. (2012) decouple the row's dependency on the remaining system of SDEs by removing the dynamics' dependency on the adjacency matrix A^w. Therefore, they first neglect the mean-reverting term

$$\left(\sum_{j \in V_{out}(i)} \frac{1}{k} \rho_j(t) - \rho_i(t) \right) dt$$

and secondly compress the sum of the diffusion term by replacing

$$\frac{\sigma}{k} \sum_{j \in \in V_{out}(i)} dW_j(t)$$

with

$$\frac{\sigma}{\sqrt{k}} dW_i^\star.$$

Hereby $W_i^\star$ denotes a Brownian motion defined on the same probability space as the N-dimensional Brownian motion W. The argumentation for the last manipulation is based on nothing else than the well-known property that the sum of k independent $N(0,1)$-distributed random variables equals a $N(0,k)$-distributed random variable again.

3.1.5 Transformation of the financial accelerator

The originally multivariate dynamics in Equation (3.6) has thus reduced to

$$d\rho_i(t) = h_i(t) d\pi_i(t) + \frac{\sigma}{\sqrt{k}} dW_i^\star(t) \tag{3.8}$$

i.e. the reaming effects on the financial robustness are the financial accelerator and some random noise.
At this point we allude on the specification of the threshold ϵ in the control set $C_i(t)$ of Equation (3.7) which is determined by Battiston et al. (2012) as

$$\epsilon \quad := \quad \epsilon^\star \frac{\sigma}{\sqrt{k}}.$$

In Equation (3.8) the random noise impacting ρ_i is expected to have a magnitude of $\frac{\sigma}{\sqrt{k}}$ on an interval of length 1 by neglecting a possible financial accelerator effect. If one sets $\epsilon^\star = 1$ a financial accelerator effect will occur as soon as there is a downward motion on the process stronger than expected. According to Battiston et al. (2012) that is in line with practice. If an agent's performance, during a quarter for instance, has been worse than expected, her poor performance will be subsequently penalized by measures of various stakeholders like rating agencies, creditors, investors and so on which is then modeled by the jump process h_i.
A readjustment of $\epsilon^\star$ may be used to regulate the accelerator's sensitivity. A value less than 1 would induce a faster occurrence while a value greater than 1 causes the opposite. Nervous markets, which respond highly sensitively to negative information, would for instance suggest a value less than 1.
The term $h_i(t_n)$, however, still possesses analytical difficulties as it depends on past realizations of ρ_i. Battiston et al. (2012) avoid those difficulties by asking for the expected value of $h_i(t_n)$ at any $t_n \in \mathcal{T}$ in order to remove the randomness on the financial accelerator. We give the lemma which presents the expected value of the financial accelerator based on the simplifying dynamics of Equation (3.8).

Lemma 3.3. *Assume a process $\rho_i(t)$ with dynamics*

$$d\rho_i(t) = h_i(t)d\pi_i(t) + \frac{\sigma}{\sqrt{k}} dW_i^\star(t), \tag{3.9}$$

where h_i is defined by Definition 3.2 with $0 < d < s = 1$ and $\alpha > 0$. For $\mathcal{T} = \{t_0, t_1, \dots\}$ we agree upon $t_0 \geq 1 + d$, $t_n = t_{n-1} + 1$ and

$$\epsilon = \epsilon^\star \frac{\sigma}{\sqrt{k}}.$$

Then for t_0 it holds

$$E[h_i(t_0)] = -\alpha \Phi\left(-\epsilon^\star\right),$$

where Φ denotes the distribution function of a Gaussian distribution with zero mean and standard deviation 1. For $t_n \in \mathcal{T}$ with $n > 0$ we have

$$E[h_i(t_n)] = -\alpha \left(\Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) q(t_{n-1}) + \Phi\left(-\epsilon^\star\right) \Big(1 - q(t_{n-1})\Big) \right).$$

Proof. Note that for any $t_n \in \mathcal{T}$, $n \geq 0$

$$E[h_i(t_n)] \quad = \quad -\alpha E[\mathbb{1}_{C_i(t_n - d)}]$$

$$= \quad -\alpha P\left((\rho_i(t_n - d) - \rho_i(t_n - d - 1) < -\epsilon\right)$$

while we are going to set in the following for the sake of simplicity

$$q(t_n) := P\left((\rho_i(t_n - d) - \rho_i(t_n - d - 1) < -\epsilon\right). \tag{3.10}$$

If $n > 0$ we can deduce that

$$\begin{aligned}
\rho_i(t_n - d) - \rho_i(t_n - d - 1) &= \int_{t_n-d-1}^{t_n-d} h_i(u)d\pi_i(u) + \int_{t_n-d-1}^{t_n-d} \frac{\sigma}{\sqrt{k}} dW_i^\star(u) \\
&= h_i(t_{n-1}) + \frac{\sigma}{\sqrt{k}}\left(W_i^\star(t_n - d) - W_i^\star(t_n - d - 1)\right).
\end{aligned}$$

By using that $W_i^\star(t_n - d) - W_i^\star(t_n - d - 1) := \zeta_i^\star \sim N(0,1)$, we obtain

$$\begin{aligned}
q(t_n) &= P\left((\rho_i(t_n - d) - \rho_i(t_n - d - 1) < -\epsilon\right) \\
&= P\left(\zeta_i^\star < -\epsilon^\star - \frac{h_i(t_{n-1})\sqrt{k}}{\sigma}\right).
\end{aligned}$$

For $n = 0$ Equation (3.10) reduces to

$$q(t_0) = P\left(\zeta_i^\star < -\epsilon^\star\right) = \Phi\left(-\epsilon^\star\right)$$

since there is no financial accelerator effect at any $t < t_0$.
For the case $n > 0$ we refine $q(t_n)$ by applying the formula of total probability which is for instance stated in Klenke (2006). We get the recursive formula

$$\begin{aligned}
q(t_n) &= P\left(\zeta_i^\star < -\epsilon^\star - \frac{h_i(t_{n-1})\sqrt{k}}{\sigma}\Big| h_i(t_{n-1}) = -\alpha\right) P\Big(h_i(t_{n-1}) = -\alpha\Big) \\
&+ P\left(\zeta_i^\star < -\epsilon^\star - \frac{h_i(t_{n-1})\sqrt{k}}{\sigma}\Big| h_i(t_{n-1}) = 0\right) P\Big(h_i(t_{n-1}) = 0\Big) \\
&= P\left(\zeta_i^\star < -\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\Big| h_i(t_{n-1}) = -\alpha\right) q(t_{n-1}) \\
&+ P\Big(\zeta_i^\star < -\epsilon^\star \Big| h_i(t_{n-1}) = 0\Big)\Big(1 - q(t_{n-1})\Big).
\end{aligned}$$

Now, we use that $\zeta_i^\star$ is the increment of the Brownian motion $W_i^\star$ on $[t_n - d - 1, t_n - d]$ whereas $h(t_{n-1})$ is a function in $W_i^\star$ on the interval $[0, t_n - d - 1]$. Hence, we can assume stochastically independence between $\zeta_i^\star$ and $h_i(t_{n-1})$ so that the last equation may be written as

$$q(t_n) = \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) q(t_{n-1}) + \Phi\left(-\epsilon^\star\right)\Big(1 - q(t_{n-1})\Big).$$

□

Remark: The lemma's setting assumed $\alpha > 0$, i.e. a financial accelerator effect is available. In the case $\alpha = 0$, $h_i(t_n)$ is obviously 0 for all $t_n \in \mathcal{T}$, hence, we simply have $E[h_i(t_n)] = 0$.

Battiston et al. (2012) keeps on simplifying by arguing that $q(t_n)$ moves into a time independent steady state q as $n \to \infty$ which is substantiated in the next lemma.

Lemma 3.4. *The sequence $(q(t_n))_{n\geq 0}$ from Lemma 3.3 is monotonously increasing and there exists q such that*

$$q(t_n) \to q$$

as $n \to \infty$. Particularly, it holds that

$$q = \frac{\Phi\left(-\epsilon^\star\right)}{1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)}.$$

Proof. Consider function

$$f(x) = \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) x + \Phi\left(-\epsilon^\star\right)(1 - x)$$

which has the unique fixed point

$$x^\star = \frac{\Phi\left(-\epsilon^\star\right)}{1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)} \in [0,1].$$

Obviously $f(x) \in [0,1]$ for all $x \in [0,1]$. Moreover for any $x, y \in [0,1]$ we have

$$|f(x) - f(y)| = \left(\Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) - \Phi\left(-\epsilon^\star\right)\right)|x - y|$$

where

$$0 \leq L := \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) - \Phi\left(-\epsilon^\star\right) < 1$$

if $\alpha > 0$. Thus, function f is a contraction mapping on the bounded interval $[0,1]$ to which we can apply the Banach fixed point theorem. The theorem can be for instance found in Süli and Mayers (2003) and states - with respect to our case - that for $q(t_0) \in [0,1]$ the sequence $q(t_n) = f(q(t_{n-1}))$ converges to the unique fixed point of f, which we used to denote by $x^\star$. Hence, the convergence of $q(t_n)$ towards

$$q = \frac{\Phi\left(-\epsilon^\star\right)}{1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)}.$$

is shown. For the claim on monotonicity we use that for $\alpha > 0$

$$f'(x) = \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) - \Phi\left(-\epsilon^\star\right) > 0$$

and

$$\begin{aligned} q(t_1) &= \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) q(t_0) + \Phi\left(-\epsilon^\star\right)\left(1 - q(t_0)\right) \\ &= \Phi\left(-\epsilon^\star\right)\left(\Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) - \Phi\left(-\epsilon^\star\right) + 1\right) \\ &> \Phi\left(-\epsilon^\star\right) \\ &= q(t_0) \end{aligned}$$

The last one is taken as base clause for the induction step

$$q(t_{n+1}) = f(q(t_n)) \geq f(q(t_{n-1})) = q(t_n),$$

which finishes the proof. □

Battiston et al. (2012) use these results in order to replace in the dynamics

$$d\rho_i(t) = h_i(t)d\pi_i(t) + \frac{\sigma}{\sqrt{k}} dW_i^\star(t)$$

the original component of the financial accelerator, namely, $h_i(t)d\pi_i(t)$ by the continuous drift term $-\alpha q dt$, hence, they do not only remove any randomness from the financial accelerator but also distribute the jumps onto the whole continuous time. The dynamics for the financial robustness reads then

$$d\rho_i(t) = -\alpha q dt + \frac{\sigma}{\sqrt{k}} dW_i^\star(t). \tag{3.11}$$

The replacement of h_i involves the advantage that in further analytical research on the dynamics of the financial robustness one does not have to rely any more on the history of the robustness' process in order to execute the financial accelerator effect. Furthermore, by the usage of q instead of $q(t_n)$, any time dependency has been removed, though, this deduces an approximation error which is going to be discussed later.
In order to close the section about the financial accelerator, we point out the behavior of the introduced drift term in Equation (3.11) for increasing k which is going to be used for examining the later analytical results on varying diversification degree k.

Corollary 3.1. *Let q_k be given as in Lemma 3.3 for $\alpha > 0$, $\epsilon^\star > 0$ and $\sigma > 0$. Then, it holds that*

$$q_k = \frac{\Phi\left(-\epsilon^\star\right)}{1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)}$$

is monotonously increasing and converges towards 1 as $k \to \infty$.

Proof. Both claims hold since the distribution function Φ is monotonously increasing, continuous and $\Phi(x) \to 1$ as $x \to \infty$. □

Recall that the size of k denotes on the one side the network connectivity and on the other side the diversification degree in the agents' credit portfolios. The last result deduces that for an increasing diversification degree the financial accelerator's probability of occurring is increasing as well. Hence, the degree of diversification has not got a beneficial effect on the financial accelerator but rather a converse effect. The behavior for q, if k increases, may be explained by noting that on the financial accelerator's control set C_i the specified threshold

$$\epsilon = \epsilon^{\star} \frac{\sigma}{\sqrt{k}}$$

for triggering the financial accelerator effect is scaled down by $\sqrt{k}$, however, at the same time the magnitude of the financial accelerator effect $-\alpha$ remains constant for all k. So, once a financial accelerator effect with size $-\alpha$ has occurred in any interval, the probability of having an effect in the subsequent interval will be higher if k is large. This implies that agents with highly diversified credit portfolios are more sensitive to downward spiral - once such a spiral has been triggered. Battiston et al. (2012) point out that $q(t_0)$ is independent of k, hence, initially there is no higher probability for a financial accelerator if k is large. The deviation in the probability for an occurrence of the financial accelerator, if k differs, is not noticeable before t_n increases. Of course, by replacing $q(t_n)$ with q a priori, one considers already in the beginning a higher probability of having the financial accelerator, what we are going to examine in the later discussion.

Battiston et al. (2012) reason for an unscaled financial accelerator effect $-\alpha$ with respect to fixed transaction costs. One of their examples fits to the extension of our Example 1: The interest rates of agent B will be increased by B's creditors due to an increasing mistrust about the creditworthiness of B, who is financially stricken. However, the increasing costs of financing for B, which may be regarded as the financial accelerator effect, are independent of the diversification degree k in B's credit portfolio. This example illustrates that the financial accelerator always hits an agent in her position as a debtor whereas the diversification degree is a matter of the agent's role as a creditor, thus, by having this separation in mind, there are, in our point of view good reasons to keep the financial accelerator unscaled and particularly independent of k.

Furthermore, we also find the scaling in ϵ comprehensible. Assume a creditor who negotiates with agent B about the conditions of a credit. If the creditor is informed about a high diversification degree in the credit portfolio of B, she will first expect low fluctuations in the financial constitution of B and rewards this by offering low interest rates. Then, however, the creditor will also react more sensitive to small deviations compared to the case of low diversification degrees in which she would already have considered smaller deviations in a debtor's robustness a priori in her initially offered credit conditions.

The increase of the probability for the financial accelerator effect with respect to k is crucial for the observations in the later results. Consequently, if the financial accelerator as defined does not hold in practice, the results of Battiston et al. (2012) become worthless.

3.1.6 Mean first passage time

Having originally started with a multivariate and coupled system of stochastic differential equations, the financial robustness of an individual agents has been now reduced to a univariate stochastic differential equation with constant drift and common diffusion term. For further considerations the process ρ_i is additionally provided with a deterministic start value $x_0 > 0$ and a reflecting barrier $b_u \geq x_0$ in the sense that the process ρ_i may not pass the horizontal line at b_u but is immediately bounced back. The following definition is going to substantiate this, where we quote for the formal definition of a process with reflecting barrier the procedure of Asmussen et al. (1995).

Definition 3.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\hat{\rho}_i$ a stochastic process in continuous time on this probability space with dynamics*

$$d\hat{\rho}_i(t) \quad = \quad -\alpha q dt + \frac{\sigma}{\sqrt{k}} dW_i^{\star}(t), \tag{3.12}$$

and initial value

$$\hat{\rho}_i(0) \quad = \quad x_0.$$

The financial robustness ρ_i with reflecting barrier at $b_u \geq x_0$ is then defined by

$$\rho_i(t) = \hat{\rho}_i(t) + \min\left(\inf_{0 \leq s \leq t}(b_u - \hat{\rho}_i(s)), 0\right). \tag{3.13}$$

We are not sure but suppose that the reflecting barrier has been introduced by Battiston et al. (2012) in order to interpret ρ_i as a ratio ranging between 0 and 1. For a pure examination of ρ_i the reflecting barrier complicates things. Obviously, the process $\hat{\rho}_i$ is a simple process consisting of a scaled Brownian motion and a negative drift, however, that is not true any more for ρ_i since it is biased by the reflecting barrier.
We start the considerations on the analytical results about the process ρ_i by a further definition.

Definition 3.4. *Let ρ_i be the stochastic process of the financial robustness of an agent i. Then*

$$T_i(b_l) = inf\{t \geq 0 : \rho_i(t) = b_l\}.$$

denotes the time at which agent i's robustness reaches b_l the first time.

Battiston et al. (2012) are interested in calculating $E[T_i(b_l)]$ for a given absorbing barrier b_l, which may be handled by referring to the concept of *mean first passage times*. Actually, for ρ_i as it is defined, $E[T_i(b_l)]$ can be explicitly calculated. Therefore, see the next lemma which presents a formula on the calculation of $E[T_i(b_l)]$ in general.

Lemma 3.5. *Assume an univariate stochastic process η which is driven by the dynamics*

$$d\hat{\eta}(t) = c_1 dt + \sqrt{c_2} dW_t$$

where c_1 and c_2 are constants, $c_2 > 0$ and W_t is a Brownian motion. Let $\hat{\eta}(0) = x_0$ and suppose a reflecting barrier $b_u \geq x_0$ for η. Then for

$$T(b_l) = \inf\{t \geq 0 : \eta(t) = b_l\}$$

where $b_l < b_u$ is an absorbing barrier one obtains

$$E[T(b_l)] = 2\int_{b_l}^{x_0} \frac{1}{\Psi(y)} \int_y^{b_u} \frac{\Psi(z)}{c_2} dz dy$$

with

$$\Psi(x) = \exp\left(2\frac{c_1}{c_2}(x - b_l)du\right).$$

As it is stated in Battiston et al. (2012) the derivation of the formula for the mean first passage time $E[T(b_l)]$ can be found in Gardiner (2004) and demands for some previous knowledge about the Fokker-Planck Equation, hence, we omit the proof at this point. Actually, Gardiner (2004) gives the formula for processes that are described by general homogeneous SDEs, i.e. the coefficients c_1 and c_2 may depend under some regularity conditions on $\hat{\eta}(t)$. However, for our purpose the formula for constant coefficients is sufficient to calculate $E[T_i(b_l)]$. Therefore see the next theorem.

Theorem 3.2. *Let ρ_i be given by Definition 3.3 and assume an absorbing barrier b_l for this process, where $b_l < x_0 \leq b_u$. Then we have*

$$E[T_i(b_l)] = \begin{cases} \frac{\exp\left(2\alpha qk\sigma^{-2}(b_l - b_u)\right) - \exp\left(2\alpha qk\sigma^{-2}(x_0 - b_u)\right)}{2\alpha^2 q^2 k\sigma^{-2}} + \frac{x_0 - b_l}{\alpha q}, & \textit{for } \alpha > 0 \\ -k\sigma^{-2}\left((b_u - x_0)^2 - (b_u - b_l)^2\right), & \textit{for } \alpha = 0. \end{cases}$$

Proof. Having given the absorbing barrier $b_l < x_0$, we can apply Lemma 3.5 on the process from Definition 3.3 by identifying the constants c_1 and c_2 as

$$c_1 = -\alpha q, \quad c_2 = \frac{\sigma^2}{k}.$$

Hence, we must solve the integral

$$E[T_i(b_l)] = 2\int_{b_l}^{x_0} \exp\left(2\alpha qk\sigma^{-2}(y - b_l)\right) \left(\int_y^{b_u} k\sigma^{-2} \exp\left(2\alpha qk\sigma^{-2}(b_l - z)\right) dz\right) dy$$

which is for $\alpha > 0$

$$E[T_i(b_l)] = \frac{1}{2\alpha^2 q^2 k\sigma^{-2}} \left[\exp\left(2\alpha qk\sigma^{-2}(b_l - b_u)\right) - \exp\left(2\alpha qk\sigma^{-2}(x_0 - b_u)\right)\right] + \frac{x_0 - b_l}{\alpha q}.$$

For $\alpha = 0$ the integral simplifies to

$$\begin{aligned} E[T_i(b_l)] &= 2\int_{b_l}^{x_0} \int_y^{b_u} k\sigma^{-2} dz dy, \\ &= -k\sigma^{-2}\left((b_u - x_0)^2 - (b_u - b_l)^2\right). \end{aligned}$$

□

The next corollary adjusts the last theorem's formula to the specifications of x_0, b_l and b_u as they are used by Battiston et al. (2012).

Corollary 3.2. *For the process ρ_i from Definition 3.3 it is for $x_0 = b_u = 1$ and $b_l = 0$*

$$E[T_i(0)] = \begin{cases} \frac{\exp\left(-2\alpha q k \sigma^{-2}\right)-1}{2\alpha^2 q^2 k \sigma^{-2}} + \frac{1}{\alpha q}, & \textit{for } \alpha > 0 \\ k\sigma^{-2}, & \textit{for } \alpha = 0. \end{cases}$$

To sum up, by applying a series of simplification onto the original dynamic of the financial robustness from the beginning, the last result delivers an explicit formula to calculate the expected time of default for an agent if its robustness starts conservatively assumed in 1 and if one previously agreed upon that an agents' default is triggered in $T_i(0)$. The core of the research by Battiston et al. (2012) particularly consists of measuring $E[T_i(0)]$ for varying diversification degrees k. The subsequent section serves to give an overview about the research's results where we will refer to the notation

$$t^\star(k, \alpha, \sigma, \epsilon^\star) := E[T_i(0)].$$

Moreover, we allude to

$$f^\star(k, \alpha, \sigma, \epsilon^\star) := \frac{1}{t^\star(k, \alpha, \sigma, \epsilon^\star)} = \frac{1}{E[T_i(0)]}$$

which is the actual function being researched and visualized by Battiston et al. (2012). While $t^\star$ normally ranges across larger intervals for varying parameters, $f^\star$ is mostly limited to smaller intervals which features some optical advantages for visualization.[7] Battiston et al. (2012) use to refer to $f^\star$ as the probability of bankruptcy. Therefore, it is argued that "*[t]he probability [] that , at any given time t, an agent goes bankrupt is the expected frequency with which, over time, the robustness of the agent hits the bankruptcy threshold [b_l].*" Actually, we have some suspicion with respect to this view since it is not ensured that $f^\star$ is less than one. Thus. it does not appear to be convenient to rely on it as probability measure. We address this subject once again in the later discussion and give an alternative view on $f^\star$.
The next lemma discloses the behavior of $t^\star$ as $k \to \infty$.

Lemma 3.6. *For parameters $\epsilon^\star > 0$ and $\sigma > 0$ we have for $E[T_i(0)]$ as given by Corollary 3.2 the convergence*

$$t^\star(k, \alpha, \sigma, \epsilon^\star) := E[T_i(0)] \to \begin{cases} \frac{1}{\alpha} & \textit{for } \alpha > 0 \\ +\infty & \textit{for } \alpha = 0 \end{cases}$$

as $k \to \infty$.

Proof. The case $\alpha = 0$ is obvious. For the other case, $\alpha > 0$, we see that it suffices to show

$$\frac{\exp\left(-2\alpha q k \sigma^{-2}\right) - 1}{2\alpha^2 q^2 k \sigma^{-2}} \to 0$$

[7] For the sake of simplicity we are going to suppress the arguments of $t^\star$ and $f^\star$ respectively, unless their dependency on one or more arguments shall be highlighted. Mostly, all parameters will be fixed with the exception of k.

as $k \to \infty$ since we know from Corollary 3.1 that

$$q = q_k \to 1.$$

We apply l'Hôpital's rule such that we must compute the derivative of the numerator and denominator. For the former it is

$$\frac{d}{dk}(\exp\left(-2\alpha qk\sigma^{-2}\right) - 1) = -2\alpha\sigma^{-2}(q + kq')\exp\left(-2\alpha qk\sigma^{-2}\right)$$

where q' denotes the derivative of q with respect to k and is given by

$$\begin{aligned} q' &= \Phi(-\epsilon)\left(1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)\right)^{-2} \frac{\alpha}{2\sigma\sqrt{k}}\Phi'\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) \\ &= \frac{q\alpha}{\sqrt{8\sigma^2\pi k}}\left(1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)\right)^{-1} \exp\left(-\frac{1}{2}\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right)^2\right). \end{aligned}$$

So,

$$kq' = \underbrace{\frac{q\alpha\sqrt{k}}{\sqrt{8\sigma^2\pi}}}_{=\gamma_1}\underbrace{\left(1 - \Phi\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right) + \Phi\left(-\epsilon^\star\right)\right)^{-1}}_{=\gamma_2}\underbrace{\exp\left(-\frac{1}{2}\left(-\epsilon^\star + \frac{\alpha\sqrt{k}}{\sigma}\right)^2\right)}_{=\gamma_3}.$$

Obviously, $\gamma_1 \to \infty$, $\gamma_2 \to \Phi(-\epsilon^\star)^{-1}$ and $\gamma_3 \to 0$ as $k \to \infty$, however, the convergence of γ_3 dominates such that $kq' \to 0$. Overall, we have for the numerator

$$\frac{d}{dk}(\exp\left(-2\alpha qk\sigma^{-2}\right) - 1) \to 0.$$

The denominator's derivative is

$$\frac{d}{dk}(2\alpha^2 q^2 k\sigma^{-2}) = 2\alpha^2(q^2\sigma^{-2} + 2qkq'\sigma^{-2}),$$

where we can use the argumentation from above to conclude

$$\frac{d}{dk}(2\alpha^2 q^2 k\sigma^{-2}) \to 2\alpha\sigma^{-2}$$

as $k \to \infty$. Under consideration of l'Hôpital's rule the lemma's claim is shown. □

<u>**Remark**</u>: For the alternative expression $f^\star$ Lemma 3.6 allows to conclude for $\alpha \geq 0$

$$f^\star(k, \alpha, \sigma, \epsilon^\star) = \frac{1}{t^\star} \to \alpha$$

as $k \to \infty$.

Figure 3.1: For both plots it has been $\epsilon^\star = 1$ and $\sigma = 0.25$. For the sake of simulations in Plot 2 we applied an Euler-Maruyama scheme with discretization size of 0.01 where due to the reflecting barrier the simulation algorithm has been adapted to Algorithm A of Asmussen et al. (1995).

3.1.7 Results on default intensity

Supposing the classical investment theory based on Markowitz (1952), one generally expects a beneficial effect of risk diversification on the default risk of one agent. In the setting of Battiston et al. (2012) one might think for k becoming large the value $f^\star$ is decreasing. If we look at Plot 1 in Figure 3.1, we see that this is not the case in general. For all α a beneficial effect of increasing diversification is indeed initially noticeable, however, after having reached a minimum, $f^\star$ is rising again for positive values of α which is the contrary behavior of what we have originally expected.
The observation of a regrowing $f^\star$ appears maybe surprising, however, it is nothing else than an interaction between drift and diffusion term in the dynamics of ρ_i if k changes. Recall that the robustness' driving dynamics has been given in Definition 3.3 by

$$d\hat{\rho}_i(t) \quad = \quad -\alpha q dt + \frac{\sigma}{\sqrt{k}} dW_i^\star(t).$$

Corollary 3.1 points out that the drift term proceeds from above to $-\alpha$ as $k \to \infty$. Simultaneously, the impact of the diffusion term disappears for increasing k as $\sigma/\sqrt{k}$ is a zero sequence, so for k large we approximately have the degenerated dynamics

$$d\hat{\rho}_i(t) \quad = \quad -\alpha dt.$$

These opposing trends of drift and diffusion and their effects on $f^\star$ are illustrative exemplified in Plot 2 of Figure 3.1. The plot contains simulated paths on ρ_i of Definition 3.3 for the case $\alpha = 0.055$ and different values of k. For $k = 1$ the path is characterized by a large fluctuations due to the relatively large coefficient in the diffusion term and a fast hitting time at 0. For $k = 15$ the fluctuation of the path already becomes less and the process stays above 0 for a longer time. In the case of $k = 80$ and $k = 150$ hardly any fluctuation is noticeable, however, the time of defaults becomes earlier again and we almost exclusively observe a negative, deterministic drift.
It must be mentioned that Plot 2 of Figure 3.1 only contains the outcome of one specific simulation on ρ_i for different values of k and does not allow for a general statement about the behavior of ρ_i with respect to a change in k, however, the illustration points out the opposing change of random and deterministic part in ρ_i and its ostensible effect on the default times of an agent. The next section more generally examines the reasons for this effect and the occurrence of a minimum for $f^\star$ if k increases.

3.1.8 Identifying two drifts

We'd like to reason the appearing minimum for $f^\star$ by pointing out the various forces on the process' path and their changing power with respect to k. By definition it is assumed that the process ρ_i is affected by a reflecting barrier at 1. This reflecting barrier is one of the forces which pushes the process' paths downwards. One may claim that the reflecting barrier implies a further negative drift on the process ρ_i additionally to the already available negative drift of the financial accelerator, which is the other force pushing the process downwards. We illustrate this in Plot 1 of Figure 3.2 which contains for different diversification degrees k the simulation of process ρ_i, but different from Plot 2 in Figure 3.1 we set $\alpha = 0$ in order to isolate the drift implied by the reflecting barrier. The regression lines laid trough the simulations' interpolation points suggest that the implied drift is absolutely decreasing for increasing k. We state a lemma which may be taken as an explanation for this observation.

Lemma 3.7. *Let $\hat{\rho}_i$ be a stochastic process with dynamics*

$$d\hat{\rho}_i(t) = \frac{\sigma}{\sqrt{k}} dW_i^\star(t)$$

and initial value $\hat{\rho}_i(0) = 1$. Then, we have for any $\delta > 0$ and fixed $t \geq 0$

$$P\left(\inf_{0 \leq s \leq t}(1 - \hat{\rho}_i(s)) < -\delta\right) \to 0$$

as $k \to \infty$.

Proof. We deduce that

$$\begin{aligned} P\left(\inf_{0 \leq s \leq t}(1 - \hat{\rho}_i(s)) < -\delta\right) &= P\left(\inf_{0 \leq s \leq t}(-\frac{\sigma}{\sqrt{k}} W_i^\star(s)) < -\delta\right) \\ &= P\left(\sup_{0 \leq s \leq t} W_i^\star(s) > \frac{\sqrt{k}\delta}{\sigma}\right) \end{aligned}$$

and apply to the last expression the reflection principle of a Brownian motion, which may be found in Klenke (2006). Hence, we have

$$\begin{aligned} P\left(\inf_{0 \leq s \leq t}(1 - \hat{\rho}_i(s)) < -\delta\right) &= P\left(1 - P\left(W_i^\star(t) \leq \frac{\sqrt{k}\delta}{\sigma}\right)\right) \\ &= 2\left(1 - \Phi\left(\frac{\sqrt{k}\delta}{\sigma\sqrt{t}}\right)\right), \end{aligned}$$

where $\Phi \to 1$ as $k \to \infty$ which finishes the proof. □

Recall that in Definition 3.3 we used to consider the reflecting barrier by the term

$$\min\left(\inf_{0 \leq s \leq t}(b_u - \hat{\rho}_i(s)), 0\right)$$

which reduces for the initial value $\hat{\rho}_i(0) = 1$ and the reflecting barrier $b_u = 1$ to

$$\inf_{0 \leq s \leq t}(1 - \hat{\rho}_i(s)).$$

Figure 3.2: For all plots it has been set $\epsilon^\star = 1$ and $\sigma = 0.25$. For the sake of simulations in Plot 1 and 2 we applied an Euler-Maruyama scheme with a discretization size of 0.01 where due to the reflecting barrier the simulation algorithm has been adapted to Algorithm A of Asmussen et al. (1995). The dashed lines in Plot 1 are the corresponding regression lines while the dashed vertical lines in Plot 3 signalize the positions of $\hat{k}^\star_\alpha$ and $k^\star_\alpha$. In order to distinguish the lines of same color, consider that in th plot it is always $\hat{k}^\star_\alpha > k^\star_\alpha$.

If we use this term to measure the influence of the reflecting barrier on the process ρ_i, the last lemma's result suggests that the barrier's impact disappears for a proceeding diversification degree k.
Of course, for fixed k, the slope of the regression lines in Plot 1 of Figure 3.2 may vary for each simulation due to the randomness of the occurring Brownian motion which is comparable to the case in Plot 2 of Figure 3.1. Hence, for $k = 1, \dots, 50$ we simulate for varying k the process' paths 500 times and calculated the averaged slopes of the regression lines from the observed simulation outcome, i.e.

$$b(k) = \frac{\sum_{j=1}^{500} b_j(k)}{500},$$

where $b_j(k)$ denotes the slope of the regression determined by the j-th simulation loop. The results for $b(k)$ may be found in Plot 2 of Figure 3.2 which confirms the declining effect of the reflecting barrier. The plot is added by

$$b(k)^{+,-} = b(k) \pm \hat{\sigma}(b(k))$$

where $\sigma(b(k))$ denotes the estimated standard deviation in the sample $b_1(k), \dots, b_{500}(k)$. Actually, we do not have to refer to an estimator for the implied drift, but can directly state it in a closed form, namely,

$$\beta(k) := -f^\star(k, 0, \sigma, \epsilon^\star) = -\frac{\sigma^2}{k}$$

which is according to Corollary 3.2 the negative inverse of the expected mean first passage time $E[T_i(0)]$ depending on k if $\alpha = 0$.
In order to understand this claim, draw a line in a coordinate system from $(0, 1)$ to $(E[T_i(0)], 0)$, where $(0, 1)$ corresponds to the start point of ρ_i and $(E[T_i(0)], 0)$ to the expected exit time. The line's slope obviously equals $\beta = -1/E[T_i(0)]$, hence, by filtering out any random fluctuations, the paths of ρ_i would then imply a drift of β. As a verification we can refer to the red line in Plot 2 which corresponds to $\beta(k)$ and properly fits the line determined by the estimates $b(k)$.
Our actual interest, the minimum of $f^\star$, becomes comprehensible if we superpose both forces on the robustness' paths, namely $\beta(k)$, the drift implied by the reflecting barrier, and the financial accelerator drift $-\alpha q(k)$ for $\alpha > 0$. Plot 3 of Figure 3.2 illustrates

$$\mu(k) = \beta(k) - \alpha q(k)$$

for different $\alpha > 0$. We see that first $\mu(k)$ absolutely decreases which may be referred to the beneficial effect of risk diversification, however, for later k the sum $\mu(k)$ starts

to absolutely increase as the changes for the summand $-\alpha q(k)$ begin to dominate the oppositional changes of the summand $\beta(k)$ if a k of certain size has exceeded.
One may summarize this core result of Battiston et al. (2012) by stating that the marginal utility of risk diversification will be overlaid by a stronger increasing financial accelerator effect if the agent diversifies her credit portfolio too much.
It must be noted that $-\mu(k)$ is not identical with $f^\star(k)$ if $\alpha > 0$. We see for instance in Plot of Figure 3.2 that the maximum of $\mu(k)$ does not perfectly fit the minimum of $f^\star(k)$. Actually, for our cases of α, if $k^\star_\alpha$ denotes the minimum of $f^\star(k)$ and $\hat{k}^\star_\alpha$ is the maximum of $\mu(k)$, it holds $\hat{k}^\star_\alpha > k^\star_\alpha$. This can be explained by considering that $\beta(k)$ is the implied drift assuming that $\alpha = 0$, however, the drift implied by the reflecting barrier will be actually lower than $\beta(k)$ if α becomes larger than zero. This may also be seen by the inequality

$$\begin{aligned} P\left(\inf_{0\leq s\leq t}(1-\hat{\rho}_i(s)) < -\delta\right) &= P\left(\inf_{0\leq s\leq t}(\alpha q s - \frac{\sigma}{\sqrt{k}}W_i^\star(s)) < -\delta\right) \\ &\leq P\left(\inf_{0\leq s\leq t}(-\frac{\sigma}{\sqrt{k}}W_i^\star(s)) < -\delta\right) \end{aligned}$$

for a process $\hat{\rho}_i$ as in Definition 3.3 with $\alpha > 0$. The left hand side measures the implied drift for this process, whereas the right hand side of the inequality implicitly measures the implied drift for a process with $\alpha = 0$. So, the drift implied by the reflecting barrier for the case $\alpha > 0$ is actually less than $\beta(k)$, and the resulting overestimate in $\mu(k)$ pretends a longer ongoing positive diversification effect which explains the differences between $-\mu(k)$ and $f^\star(k)$.

3.2 Discussion on the financial robustness and its results

Having presented the results of Battiston et al. (2012) about the process ρ_i and its behavior with respect to the diversification degree k, we now comment some points which we came across by presenting the way of how to get these results. First we clarify our interpretation of the value $f^\star$ which will also explain why we choose the name "Results on default intensity" for Section 3.1.7. Then, we are going to reconsider the usage of the steady state q instead of $q(t_n)$ for the probability of having a financial accelerator effect. This example, however, will also illustrate that it is difficult to consider all made simplifications and assumptions individually since one simplification may have effects on the examination of another simplification. For a final validation we want to simulate the originally motivated and more sophisticated process from Equation (3.6) and look whether the results in Section 3.1.7, which have been deduced from the reduced process ρ_i of Definition 3.3, may be regained.

3.2.1 Intensity of a renewal process

As in Battiston et al. (2012) we used to visualize $f^\star$ rather than $t^\star$, however, different from Battiston et al. (2012) we avoid referring to $f^\star$ as a default probability due to the commented reasons in Section 3.1.6. Instead of that we offer another way to interpret $f^\star$. Hereby, we assume that the development of the financial robustness between its initial value 1 and 0 denoting the agent's default is an arbitrarily often repeatable experiment.

In practice, this might mean that if agent j failed at t she will be replaced by a new agent $j+1$ with robustness of 1 while this new agent in turn will then be inductively replaced at her default and so on.

Definition 3.5. *Let (Z_i) be an iid sequence of a.s. positive random variables. Then the random walk*

$$S_0 = 0, \qquad S_n = Z_1 + \ldots Z_n, \quad n \geq 1$$

is said to be a renewal sequence and the counting process

$$N(t) = \#\{i \geq 1 : S_i \leq t\} \quad t \geq 0$$

is the corresponding renewal (counting) process.

The last definition being borrowed by Mikosch (2004) is popularly applied to insurance mathematics where Z_i denotes the inter-arrival times between the occurrence of claims. For our purpose we do not consider Z_i as inter-arrival time of insurance claims but as the time which is required by the robustness of the i-th agent in our experiment from above to reach 0. Frankly speaking, our loss occurrence is the default of an agent. In Definition 3.5 we set

$$T_j(0) = Z_j$$

where $T_j(0)$ is as defined in Definition 3.4. Hereby, subscript j stands for the j-th agent in our experiment. Obviously $T_j(0)$ is a.s. positive and to be in line with Definition 3.4 we additionally assume that the $T_j(0)$ are identical and independent across all j.
Hence, S_n from Definition 3.4 becomes the accumulated inter-arrival times of defaults up to the n-th agent. Correspondingly $N(t)$ is the number of defaults up to t.
Then, the following theorem bares how we can interpret $f^\star$.

Theorem 3.3. *If $E[T_1(0)] = \lambda^{-1}$ is finite, we have the convergence*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \quad a.s.$$

Proof. The theorem and proof may be found in Mikosch (2004) and is named as *Strong law of large numbers for the renewal process.* □

By pointing out that

$$f^\star = \frac{1}{E[T_1(0)]} = \lambda$$

we can conclude that

$$\lim_{t\to\infty} \frac{N(t)}{t} = f^\star$$

almost surely. Hence, $f^\star$ corresponds approximately to the number of defaults per time unit and is alternatively called the *intensity* of the renewal process $N(t)$ whereas for our purposes we are going to call it the intensity of defaults.

Figure 3.3: The red line denotes the theoretical value $E[T_i(0)]$ computable via Corollary 3.2. Both black lines rely on the simulations' outcome where the stars and points respectively on the lines signalize the interpolation points k, at which the simulations have been performed. For the sake of simulations we applied an Euler-Maruyama scheme with a discretization size of 0.01, where due to the reflecting barrier the simulation algorithm has been adapted to Algorithm A of Asmussen et al. (1995).

In insurance mathematics it is often assumed that

$$T_i(0) \sim \exp(f^\star)$$

where the restriction on an exponential distribution allows to deduce further properties on the renewal process. If such an assumption hold for $T_i(0)$, we would have the well known property of an exponential distribution, namely,

$$E[T_i(0)] = (f^\star)^{-1} = Var[T_i(0)]^{1/2},$$

In Figure 3.3 we plotted the exit times averaged over 500 simulations of the process ρ_i for different k and $\alpha = 0.075$. This is denoted by $\hat{T}_i(0)$, while $\sigma(\hat{T}_i(0))$ is the estimated standard deviation on the observed sample. Apparently, $\sigma(\hat{T}_i(0))$ is lower than it would have to be in the case of an exponential distribution. Based on this, we would claim that an exponential distribution on $T_i(0)$ were not appropriate.

3.2.2 Usage of the steady state

We have described in Section 3.1.4 the replacement of the stochastic jump process $h_i(t)$ by the deterministic term $-\alpha q$. Besides we have alluded that there might be a bias due to the usage of the steady state q instead of the actual probability for the occurrence of the financial accelerator in $t_n \in \mathcal{T}$, namely $q(t_n)$.
According to Lemma 3.4, $q(t_n)$ converges towards q bottom up, thus, there will be an overestimated financial accelerator effect by referring to q. Plot 1 in Figure 3.4 shows the distance between $q(t_0)$ and q for varying k and α. Since on the one hand q is monotonously increasing towards 1 in k, which has been shown by Corollary 3.4, but on the other hand

$$q(t_0) = \Phi(-\epsilon^\star)$$

for all k, it is not surprising that the distance between q and $q(t_0)$ is increasing in k. Furthermore, Plot 1 suggests that a higher value for α implies a larger distance.
In order to point out the accumulated size of approximation error between q and $q(t_n)$ over time we define for $\delta > 0$

$$n^\star = \inf\{n \in \mathbb{N}_0 : q - q(t_n) < \delta\},$$

i.e. the iteration step at which the sequence of $q(t_n)$ has reached a sufficiently small distance to q. Then

$$S(n^\star) = \sum_{j=0}^{n^\star} (q - q(t_j))$$

Figure 3.4: Plot 1 illustrates the absolute distance between $q(t_0)$ and q. Plot 2 contains the values $n^\star$ and $S(n^\star)$. Hereby, the former is denoted by the step functions and the latter is represented by the dashed lines. The color code is as follows: green denotes $\alpha = 0.1$, red denotes $\alpha = 0.05$ and blue stands for $\alpha = 0.03$. For the calculations of $n^\star$ we have set $\delta = 10^{-4}$, the remaining parameters have been $\sigma = 0.25$ and $\epsilon^\star = 1$.

is the accumulated approximation error up to $t_{n^\star}$. Both values obviously coincide in Plot 2 of Figure 3.4 with the distance between $q(t_0)$ and q and also possess higher values when α is large.

For measuring the accumulated overestimate on the financial accelerator effect up to $t_{n^\star}$, $S(n^\star)$ must additionally be scaled by α. For the specification $\alpha = 0.05$ and $k = 50$ we for instance have

$$\alpha S(n^\star) = 0.016$$

at $n^\star = 12$ if $\delta = 10^{-4}$. On the other hand it is

$$\alpha S(n^\star) = 0.34$$

at $n^\star = 43$ for the setting $\alpha = 0.1, k = 50$ and $\delta = 10^{-4}$. While in the first case the overestimation is low, the latter appears large if one has in mind that ρ_i only ranges between 0 and 1.

By a simulation procedure we'd like to figure out the effects of the overestimated financial accelerator probability on $E[T_i(0)]$ for the case $\alpha = 0.1$. Our idea hereby is to simulate the jump-diffusion process

$$d\hat{\rho}_i^\star(t) = h_i(t)d\pi_i(t) + \frac{\sigma}{\sqrt{k}}dW_i^\star(t)$$

from Equation (3.8). By additionally considering a reflecting barrier at 1 we denote in the following the reflected process of this dynamics by $\rho_i^\star$. In 500 simulation runs we will then estimate the mean first passage time of this process where we are going to denote this in the following by $\hat{T}_i(0)$ and its estimated standard deviation correspondingly by $\sigma(\hat{T}_i(0))$. The simulation is undertaken for different k and finally compared to the results of $E[T_i(0)]$ which is the expected mean first passage time for the process ρ_i from Definition 3.3. Particularly, this process relies on the continuous drift $-\alpha q$ where q has been found to overestimate the probability of having a financial accelerator effect. Due to our examination from above we would hypothesize that

$$\hat{T}_i(0) > E[T_i(0)],$$

i.e. the process driven by $\hat{\rho}^\star$ is expected to stay longer above 0 than ρ_i due to the supposed overestimated financial accelerator effect in q.

We see in Figure 3.5 that our hypothesis is only fulfilled for k larger than 20. We can explain this unexpected observation by alluding that the calculations on q in Section 3.1.5 have been proceeded by disregarding the reflecting barrier which is in turn incorporated into the simulation of $\rho_i^\star$ and the calculations of $E[T_i(0)]$. A reflecting barrier obviously

Figure 3.5: The plot on the left hand side compares $E[T_i(0)]$, which is known as closed formula in Corollary 3.2, against $\hat{T}_i(0)$. For the simulations we have applied an Euler-Maruyama scheme with discretization size 0.01. We used to incorporate the jump process h_i into the simulation as suggested by Battiston et al. (2012) in Appendix B. The reflection barrier has been considered by adjusting the Euler-Maruyama scheme as in Algorithm A of Asmussen et al. (1995). The parameters have been $\alpha = 0.1, \sigma = 0.25, \epsilon^\star = 1$. On the right hand side we compare the probability of having a financial accelerator effect suggested by q against the probability which is deduced by the simulations.

favors the occurrence of the financial accelerator effect, hence, by having calculated the probability of its occurrence without considering the reflecting barrier, $q(t_n)$ turns out to be too small. Particularly, this will hold for k small, since in Section 3.1.8 we have pointed out that the reflecting barrier influences ρ_i mostly for k small. Due to Figure 3.5 even the steady state q underestimates the occurrence of the financial accelerator effect for $k < 20$. Apparently for $k > 20$ the calculation of $q(t_n)$ in Lemma 3.3 serves as better approximation since the reflecting barrier more and more looses its influence on the process' path so that the usage of the steady state begins to incorporate an overestimation. Hence, for $k > 20$ our hypothesis is eventually fulfilled.

This explanation is substantiated in Plot 2 of Figure 3.5, where we compare q, the theoretical steady state probability of the financial accelerator, against the probability of observing jumps in the simulations. We hereby count for each simulation run the occurrence of the jumps h_i whereas we eventually divided this number by the possible number of jump times in each simulation run up to the time at which the simulated path of $\rho^\star$ has reached 0. For theses observations $\hat{h}_i$ denotes the average over all simulation runs and correspondingly $\sigma(\hat{h}_i)$ is the estimated standard deviation.

Apparently the probability of having a jump is indeed higher than it is estimated by q for k being small. On the other side, while for proceeding k the curve of $\hat{h}_i$ seems to slab to a constant level q is still proceeding towards 1.

The observation of differing exit times between simulation and theoretical value may be analogously made in Figure B1 of Battiston et al. (2012) which proceed the same simulation, however, for $\alpha = 0.03$. Since for this size the overestimate of q on $q(t_n)$ is less significant as we can see in Figure 3.4, the spread between the curves of $\hat{T}_i(0)$ and $E[T_i(0)]$ is less noticeable in their Figure for k becoming large.

To sum up, the examination has shown that the simplifying procedure to replace the jump process of the financial accelerator by a continuous drift term with a time-independent coefficient can bias the results on the default intensity, however, it is not possible to uniquely determine in which direction. While for lower values of k one must suppose that $E[T_i(0)]$ is too high, we have for larger values of k possibly the contrary. However, the concrete characteristic of this discrepancy is a matter of parameter values.

Apparently the assessment of $E[T_i(0)]$ as reliable measure for the mean first passage time of the originally multivariate system of SDEs in Equation (3.6) is challenging if one considers every step of simplification individually. The problem with analyzing the usage of q turns out to be the different settings between its calculation in Section 3.1.5, in which a reflecting barrier has not yet been assumed, and its eventual application with respect to the reflected process ρ_i which assumes a reflecting barrier. In order to cope with this challenge we will draw a comparison of the mean first passage time in a simulation procedure between Equation (3.6), that is the originally multivariate SDE system before simplifications, and the finally univariate process ρ_i.

3.2.3 Simulation of the full dynamics

If we refer in the following to the multivariate process from Equation (3.6), we are going to use the notation $\hat{\rho}^\star$ and for its components $\hat{\rho}_i^\star$. Recall that the dynamics of Equation (3.6) contain all elements of the original model before one started to reduce the dynamics to the simpler, univariate process ρ_i from Definition 3.3 in order to get analytical results. Our intention with this final simulation is to figure out whether the analytical results for this simplified process are still in line with the original model in spite of the different occurring simplifications. We cannot expect an exact fitting, however, we are more interested in the general behavior of the default intensity with respect to the diversification degree.

For the simulation of

$$d\hat{\rho}^\star(t) = (A^w - I_{N\times N})\hat{\rho}(t)dt + \sigma A^w dW(t) + h(t)d\pi_i(t)$$

a reflecting barrier at 1 is considered for every component where we denote in the following the reflected multivariate process by $\rho^\star$ and its reflected components by $\rho_i^\star$.

The simulations procedure furthermore requires to take the weighted adjacency matrix A^w into account. We conform with the mean field approach of Battiston et al. (2012) by not relying on one specific adjacency matrix A^w, but by regenerating a new adjacency matrix in the beginning of each simulation loop. For generating the adjacency matrices we keep in mind that assumptions A(1.1) - A(1.3) must be fulfilled which is going to be treated in the next section. Prior to that still a comment on the specification of N. Craig and von Peter (2010) allude in their study about the German interbank credit market to a size of about 2.000 banks consisting of a periphery and a core where the latter is built up of 40-50 banks. Battiston et al. (2012) mention that their assumption of an homogeneous network most likely fits the core of an overall credit market. By combining both, we restrict ourself to $N = 50$ which would similarly correspond to the core of the German interbank credit market. The diversification degree will then possibly range between $k = 1$, which is almost no diversification, and $k = 49$, which is the largest possible diversification degree.

3.2.4 Simulating an homogeneous network structure

For generating the weighted adjacency matrix we apply an algorithm which regards assumptions A(1.1) - A(1.3) by incorporating N and k as input parameters.

Algorithm 3.1: Generating a weighted adjacency matrix which represents an homogeneous credit network

Data: Parameter N and k
Result: Adjacency matrix A^w with weights
Initialize a $N \times N$ matrix A^w with zero entries;
for $i = 1$ *to* N **do**
  Select randomly k numbers from the set $\{1, \dots N\} \setminus \{i\}$ and save them in *ind*;
  $A^w[i, ind] = \frac{1}{k}$;
end

The random selection of k numbers in this algorithm intends to determine the k debtors of agent i, whereas it is additionally controlled that the diagonal entries stay 0. We realize this in R Core Team (2013) by relying on the function `sample()` which draws randomly k numbers from a predefined set without replacement and without any preference on the choice of the numbers. Hence, for any agent $i \in V$ the algorithm's random selection of her credit portfolio does not favor any composition. More precisely, if $\mathcal{V}_{out}(i)$ denotes all subsets of $\{1, \dots, N\} \setminus \{i\}$ with cardinality k, we have

$$P(V_{out}(i) = \tilde{V}) = \frac{1}{\binom{N-1}{k}} = \frac{(N-1-k)! \cdot k!}{(N-1)!}$$

for all $\tilde{V} \in \mathcal{V}_{out}(i)$. Furthermore, the credit portfolios of all agents are chosen independently in the algorithm which yields

$$P\left(V_{out}(1) = \tilde{V}_1, \dots, V_{out}(N) = \tilde{V}_N\right) = \left(\frac{1}{\binom{N-1}{k}}\right)^N = \left(\frac{(N-1-k)! \cdot k!}{(N-1)!}\right)^N$$

for all $\tilde{V}_1 \times \dots \times \tilde{V}_N \in \mathcal{V}_{out}(1) \times \dots \times \mathcal{V}_{out}(N)$. Since the overall network is fully determined by the agents' credit portfolios, we conclude that the algorithm generates any possible network fulfilling assumptions A(1.1) - A(1.3) with the same probability.

While the number of debtors for each agent is predetermined on k, the number of creditors is random and depends on the concrete realization of the adjacency matrix. We state in the next lemma the distribution of $d_{in}(i)$ as it occurs in Algorithm 3.1. This result will allow to conclude that the expected value of $d_{in}(i)$ will agree with the deterministic value of $d_{out}(i)$.

Lemma 3.8. *Having given a weighted adjacency matrix A^w generated by Algorithm 3.1 and let V be the set of agents. Then, it is*

$$d_{in}(i) \sim Bin\left(N-1, \frac{k}{N-1}\right)$$

for all $i \in V$.

Proof. If A denotes the unweighted version of the weighted adjacency matrix A^w generated by Algorithm 3.1, the entries of the i-th column in A will be independent Bernoulli distributed random variables with success probability

$$P\left(a_{ji}=1\right)=\frac{\binom{N-2}{k-1}}{\binom{N-1}{k}}$$

for all $j \in V, j \neq i$. For $j = i$ the success probability is 0 by definition. We deduce that

$$\frac{\binom{N-2}{k-1}}{\binom{N-1}{k}}=\frac{(N-2)!\cdot(N-1-k)!\cdot k!}{(N-1)!\cdot(N-2-k+1)!\cdot(k-1)!}=\frac{k}{N-1}.$$

Now, let $d \in \{0, \ldots, N-1\}$. Since the matrix entries a_{ji} are independent for all $j \in V \setminus \{i\}$ it is

$$\begin{aligned} P\left(d_{in}(i)=d\right) &= P\left(|V_{in}(i)|=d\right) \\ &= P\left(|\{j \in V : a_{ji}=1\}|=d\right) \\ &= \binom{N-1}{d}\left(\frac{k}{N-1}\right)^{d}\left(1-\frac{k}{N-1}\right)^{N-1-d} \end{aligned}$$

which finishes the proof. □

In directed graphs the condition

$$\sum_{i\in V} deg_{in}(i)=\sum_{i\in V} deg_{out}(i).$$

must always hold, hence, we conclude for the convolution on the random variables $d_{in}(i)$ of the last lemma that

$$\sum_{i\in V} deg_{in}(i)=Nk \quad a.s.$$

This fact is used in order to prove the next lemma.

Lemma 3.9. *For $0 < k < N-1$ the in-degrees $d_{in}(1), \ldots, d_{in}(N)$ of a graph represented by a weighted adjacency matrix A^w from Algorithm 3.1 are not independently distributed.*

Proof. Assumed that $d_{in}(1), \ldots, d_{in}(N)$ were independently distributed, then we know from probability theory - for an argumentation see for instance Example 3.4 (ii) in Klenke (2006) - that

$$D:=\sum_{i\in V} d_{in}(i)\sim Bin\left(N(N-1),\frac{k}{N-1}\right). \tag{$\star$}$$

By assumption it is $0 < \frac{k}{N-1} < 1$ so that $(\star)$ would yield

$$P(D=l)>0$$

for $0 \leq l \leq N(N-1)$. Particularly, this holds for $l = 0$ which is a contradiction to

$$\sum_{i \in V} deg_{in}(i) = Nk \quad a.s.$$

□

While in Algorithm 3.1 the choice of the adjacency matrix's rows occur independently the last result suggests that this is not true for the matrix's columns.
In the following we will examine the mean first passage times of the simulated financial robustness with respect to k but disregard the occurring in-degrees which cannot be controlled by our algorithm. That is in line with the approach of Battiston et al. (2012) who exclusively focus on the agents' out-degrees in the context of diversification. Nevertheless, we allude that a given structure with respect to the in-degrees might also be interesting to examine. For instance, assume that there is a small subset of agents which are highly active as debtors meaning that they borrow more money than other agents. Some distortion on the financial robustness of these agents will probably effect the network's stability in an other way than a distortion on agents which are less active as debtors. We will later develop an algorithm to generate networks with an heterogeneous core-periphery structure in which we will find an asymmetrical distribution of the credit exposure and try to point out the effect of this structure on the network's stability.

3.2.5 The results of the simulation

For the simulation of $\rho^\star$ we use the multivariate version of the Euler-Maruyama scheme with discretization size 0.01. Hereby, the reflecting barrier for each component has been incorporated by following Algorithm A of Asmussen et al. (1995). In the following $\hat{T}_1(0)$ will denote the estimated mean first passage time of $\rho_1^\star$ in the simulation procedure and $\sigma(\hat{T}_1(0))$ is the estimated standard deviation. Since the network is renewed in each simulation run, the position in the network of agent 1 permanently changes and the value of $\hat{T}_1(0)$ may be thus considered as the mean first passage time for any agent in any homogeneous network.
Analogously to Section 3.2.2 we again measure by $\hat{h}_1$ the observed relative frequency of the financial accelerator effects for the first component of the simulated vector $\rho^\star$ and compare theses values to q.
Before referring to the results we must allude a further subtle detail in our procedure which has not been elaborately addressed by Battiston et al. (2012). While we are interested in the default time of agent 1, probably there have already been some defaults of other agents. In the univariate considerations of Battiston et al. (2012) there is no need to think about handling such a situation. For our purposes we agree upon that the robustnesses of the defaulted agents are retained at 0 for the remaining duration of the simulation loop. In particular, there will be no recovery of any agent in the network.
Secondly, we must lead a discussion how to handle the adjacency matrix. Assumed that agent i defaults it would be advisable to set the i-th column of the adjacency matrix to zero in order to prohibit that the stochastic noise arising from the i-th component of the Brownian motion may still impact the robustness of other agents after i has failed. However, then, the stochastic impact on an agent's financial robustness would run dry if

Figure 3.6: The figure compares the simulated mean first passage times at 0 between a component of the full sytem of SDEs from Equation (3.6) and the univariate process of Definition 3.3. The simulation has been based on a multivariate extension of Algorithm A of Asmussen et al. (1995) with discretization size 0.01. The incorporation of the financial accelerator for each component has been executed as in the univariate case. Further parameters have been $\sigma = 0.25$ and $\epsilon^\star = 1$.

Figure 3.7: The figure compares the steady-state probability q of having a financial accelerator effect with the probability deduced by the simulations.

her debtors subsequently default. In practice, this would then mean, that random noise impacting an agent's robustness is going to be reduced for every newly defaulted debtor. We decide not to change the entries of the adjacency matrix during the simulation loops in order to ensure that the stochastic impact on the financial robustness is constantly a sum of k independent Brownian motions. This also ensures a better comparability to the univariate process ρ_i which is continuously affected by a constant diffusion term. Actually we tried out different alternatives to handle this and observe that the outcome of these alternatives predominantly deviate for small k if the credit portfolios just consist of a few debtors so that any individual default counts relatively much.

Let us now turn to the results which are illustrated in Figure 3.6 and 3.7 respectively. First we clarify in the former one the diverging behavior between $E[T_i(0)]$ and $\hat{T}_1(0)$ for $\alpha = 0$. We know from Lemma 3.6 that $\alpha = 0$ implies in $E[T_i(0)] \to \infty$ as $k \to \infty$. In our simulation we restrict the time horizon to $T = 300$. For large k, however, the paths have commonly not reached 0 before $T = 300$. In this case we set the mean first passage time to 300, which explains the given plot. Certainly, its explanatory power is bounded, though it at least indicates that the true value of $\hat{T}_1(0)$ would also monotonously increase as $E[T_i(0)]$ for $\alpha = 0$ if the time horizon is enlarged. Since the simulation of $\rho^\star$ turned out to take a long time we avoid doing this. In the cases of positive α the simulated paths have always arrived in 0 prior to the technical time limit of $T = 300$ so that there is no risk of falsification. Apparently for these cases, the mean first passage time $E[T_i(0)]$ runs underneath the simulated passage time $\hat{T}_1$ for small k whereas for k becoming large we observe an approach between both lines. That is an interesting result since in Section 3.2.2 we made the contrary observation, that was $E[T_i(0)] > \hat{T}_i(0)$ for small k. We reasoned this by an underestimation of the financial accelerator affect in q if k is small which has been confirmed by Plot 2 in Figure 3.5. Figure 3.7, however, suggests that the financial accelerator effect in the simulations of this section occurs first with smaller probability than it is proposed by q but then rises steeper and intersect the curve of q at an intermediate level of k. Hence, we have for the simulation of the full dynamic a mirror-inverted observation compared to the univariate simulation of the jump-diffusion SDE in Section 3.2.2.

In Equation (3.6) we trace the mean reversion term as main distinctive feature to the simulated process in Section 3.2.2. Hence, we attribute the oppositional observations between Plot 2 in Figure 3.5 and Figure 3.7 to the additional consideration of this term. Apparently, it has at first a restraining effect on the decrease of the paths so that the financial accelerator effect less frequently occurs, but then expands the negative effect of risk diversification on the financial robustness for increasing k. Hereby note, that the mean reversion term will arrange a transfer of the financial accelerator from the debtors to their creditors which is completely blended out in the univariate case. Hence, in the multivariate case an agent's robustness is not only affected by her own financial accelerator but also by the one of her debtors. So, we explain the steeper increase of $\hat{h}_1$ compared to q by network contagion which becomes recognizable if one refers to the whole system of coupled SDEs rather than just looking at the univariate, decoupled process ρ_i.

3.3 Resume on the continuous time process

Our discussion aimed to figure out whether the simplified, univariate process ρ_i from Definition 3.3 may still reproduce the results of the sophisticated, multivariate system from Equation (3.6) which includes the whole motivation of the model construction. Admittedly, one cannot expect that the mean first passage times of the univariate process ρ_i perfectly fit the mean first passage times of the components from the multivariate process. In order to receive an analytically tractable form the costs of complexity reduction lead to a discrepancy between both values as it is observable in Figure 3.6. However, the qualitative core result of Battiston et al. (2012), namely, the negative effect on the default intensity for a too extensive diversification of the credit portfolios is accurately described by the behavior of the univariate process with respect to a changing k. By the additional contagion effects through the mean-reverting term, which has been detected in the last section, the core result of Battiston et al. (2012) actually based on the univariate process becomes even more distinctive in the multivariate case.
In the following the formulas of the mean first passage time for the univariate process ρ_i from Definition 3.3 enables to extend the default intensity of one agent to a general statement about systemic risk. Due to the last section we may suppose that the core of these results can be qualitatively transferred to the multivariate case as well, though one must keep in mind that the missing contagion effects in the univariate case will bias the results quantitatively.

4 The cascade mechanism in homogeneous networks

Section 3 has derived and examined a dynamics which reflects the financial constitution of an agent in continuous time. After all, this section has only presented a model which allows to investigate the dynamics and default frequency from an individual perspective but does not manifest the risk of having a system-wide distortion. The further idea of Battiston et al. (2012) is to examine the stability of an interbank market fulfilling assumptions A(1.1) - A(1.3) in a hypothesized period of system-wide distortion.
Battiston et al. (2012) argue that in times of distress the financial robustness displays another behavior than in times of a relaxed interbank market. While they are considering the dynamics from Definition 3.3 as a description of the financial robustness in calm times, they argue for a different development in times of distress which is evocative of a domino-effect or cascade mechanism. We are going to present the setting of this adapted cascade mechanism for the financial robustness in this section. Its analysis rests upon simplifying assumptions which incorporate an approximation error. However, as we will see in the end, the simplification still allow to figure out a mean robustness where the network becomes fragile for a total collapse.

4.1 Setting of the cascade mechanism in homogeneous networks

Analogously to Section 3 we again proceed on an interbank credit market which fulfills assumptions A(1.1) - A(1.3). The interbank market is still represented as a graph $G^w = (V, E, W)$. We also claim assumption A(1.4) in the following, that is, once the network has been initialized it will not change over time.

4.1.1 Financial robustness in the cascade mechanism

The core of Section 3.1 consists of deriving a further dynamics for the financial robustness, however, different from the last section the development of the financial robustness is now considered in a situation of fast and serial occurring defaults. For a motivation we refer to the introductory Section 1.1, which named three periods of significantly higher number of defaults than otherwise. It might be helpful to regard the time between these periods as the continuous time line on which ρ_i is developing in the long run as defined in Definition 3.3 while, what follows is the description of the robustness' dynamics in one of these periods with frequent defaults. Therefore, we introduce a new definition on the financial robustness. In order to guard against confusion let us first assume that we are going to deal with a new process completely disconnected from the last section.

Definition 4.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $i \in V$. Then the financial robustness of agent i in the cascade mechanism is a $\mathbb{R}$-valued sequence $(\tilde{\rho}_i(\tau))_{\tau \in \mathbb{N}_0}$ where its initial value $\tilde{\rho}_i(0)$ is a random variable defined on the given probability space with distribution function F_i. For $\tau > 0$ and $a > 0$ we set*

$$\tilde{\rho}_i(\tau + 1) = \tilde{\rho}_i(0) - a \sum_{j \in V_{out}(i)} w_{(i,j)} \mathbb{1}_{(\tilde{\rho}_j(\tau) \leq 0)}. \tag{4.1}$$

We may depict for $\tau > 0$ and initial random vector $\tilde{\rho}(0)$ the last definition in matrix-vector notation, if $\tilde{\rho}(\tau) := \big(\tilde{\rho}_1(\tau), \dots, \tilde{\rho}_N(\tau)\big)^T$ and $z(\tau) = \big(\mathbb{1}_{(\tilde{\rho}_1(\tau) \leq 0)}, \dots, \mathbb{1}_{(\tilde{\rho}_N(\tau) \leq 0)}\big)^T$

respectively, namely, by

$$\tilde{\rho}(\tau+1) = \tilde{\rho}(0) - aA^w z(\tau), \tag{4.2}$$

where A^w is the adjacency matrix of graph G^w. In the following we will assume that the elements of $\tilde{\rho}(0)$ are independently and identically distributed. This is substantiated by assumption A(1.7).

- A(1.7) The initial robustnesses $\tilde{\rho}_1(0), \ldots, \tilde{\rho}_N(0)$ are supposed to be identically and independently distributed with distribution function F.

4.1.2 Intention of the cascade mechanism

After having redefined the financial robustness technically let us discuss the intention of Battiston et al. (2012) to use this new definition. The mechanism driving the financial robustness in Definition 4.1 is thought to describe the process of successive defaults in a credit network which is suffering a system-wide distortion. Hereby $\tau = 0$ denotes the starting point of this mechanism, i.e. it may be considered as the origin of the distortion while the reason for its trigger is not matter of considerations. It is just assumed that in $\tau = 0$ the values of $\tilde{\rho}_i(0)$ are randomly drawn with some resulting fraction of defaults exclusively depending on the assumed initial distribution. The step $\tau = 0$ is followed by discrete time steps $\tau > 0$ on which further defaults might be induced due to the given setting in Definition 4.1. Hereby the variable $w_{(i,j)}$ is still the relative part of the credit portfolio, agent i has issued to agent j. So, if j, being the debtor of agent i, has defaulted during the cascade in one of the steps $0, \ldots, \tau$, the robustness of agent i from the beginning of the distortion is reduced in $\tau+1$ by the loss due to j's default. By relying on assumption A(1.3) we can replace $w_{(i,j)}$ with $\frac{1}{k}$ for all $(i,j) \in E$ in the following.
By assumption A(1.7) it is assumed that the initial defaults occur independently. Actually, a validation of this assumption requires information about interdependencies between agents prior to a cascade, however, this setting only deals with available correlation, once a phase of distress has already triggered.
The newly introduced variable a shall indicate to which extent the robustness of i is affected by j's default. Let us suppose two different interbank credit markets, where one market is endowed with a more advanced insolvency law than the other. One might expect that creditors on the market with advanced insolvency law would probably not have to suffer a complete debt default but may expect a partial compensation of their receivables. Consequently, one would choose a lower parameter a there, compared to the market without advanced insolvency regulations. In Battiston et al. (2012) it is by default $a = 1$ which we are going to handle for our purpose similarly.
If the initial financial robustness of agent i is low, it is likely that in any step τ agent i will default herself due to defaults of her debtors in previous steps. However, the default of agent i may probably trigger the defaults of her own creditors and so on. What follows is a sequence of failures, which is called a *bankruptcy cascade* by Battiston et al. (2012).
Equation (4.2) suggests that the only criteria deciding about an agent's default in any time step is on the one side the initial distribution of the agent's robustness and on the other side her position in the network, hence, it implies that the realization of $\tilde{\rho}$ in 0 already determines whether an agent will default or survive in $\tau \geq 0$.

4.1.3 Assumptions for analytical results

For the given cascade mechanism Battiston et al. (2012) target on the value

$$
\begin{aligned}
s(\tau) &:= \frac{E\left[\sum_{i\in V} \mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)}\right]}{N} \\
&= \frac{\sum_{i\in V} P(\mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)} = 1)}{N}.
\end{aligned} \tag{4.3}
$$

By relying on the cascade mechanism in Definition 4.1 we can argue that the variables $\mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)}$ are monotonously increasing in τ. Since $s(\tau)$ is furthermore bounded by 1 we can deduce by the Bolzano-Weierstrass theorem that there is a $s \in [0,1]$ so that $s(\tau) \to s$ as $\tau \to \infty$.
For an explicit calculation of $s(\tau)$ one might start with

$$
\begin{aligned}
P(\mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)} = 1) &= P\left(\tilde{\rho}_i(0) \leq \frac{1}{k} \sum_{j\in V_{out}(i)} \mathbb{1}_{(\tilde{\rho}_j(\tau-1)\geq 0)}\right) \\
&= \sum_{u=0}^{k} P\left(\tilde{\rho}_i(0) \leq \frac{u}{k} \middle| \sum_{j\in V_{out}(i)} \mathbb{1}_{(\tilde{\rho}_j(\tau-1)\geq 0)} = u\right) \\
&\quad P\left(\sum_{j\in V_{out}(i)} \mathbb{1}_{(\tilde{\rho}_j(\tau-1)\geq 0)} = u\right)
\end{aligned}
$$

where the second equality is obtained by the formula of the total probability. The possible complexity of an arbitrary network discloses analytical difficulties for the expression in the last equation. Particularly, the distribution of the indicator functions' convolution is presumably not known.
If we wanted to determine the possible outcomes of $\mathbb{1}_{(\tilde{\rho}_j(\tau-1)\leq 0)}$, we would have to consider $\mathbb{1}_{(\tilde{\rho}_{j_1}(\tau-2)\leq 0)}$ for all $j_1 \in V_{out}(j)$, however, in order to determine $\mathbb{1}_{(\tilde{\rho}_{j_1}(\tau-2)\leq 0)}$ we must then in turn consider $\mathbb{1}_{(\tilde{\rho}_{j_2}(\tau-3)\leq 0)}$ for all $j_2 \in V_{out}(j_1)$ and so on. Thus, for proceeding τ and increasing complexity of the network an explicit calculation of $s(\tau)$ and its limit s becomes intractable. Battiston et al. (2012) resolve this by pragmatically endowing the variables $\tilde{\rho}_i(0)$ and $\mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)}$ of Definition 4.1 with relaxing assumptions as follows.

- A(1.8) For all $i \in V$ and $\tau > 0$, it is assumed that

  $$P(\mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)} = 1) = p(\tau),$$

  where $p(\tau) \in [0,1]$. Additionally it is supposed that $\mathbb{1}_{(\tilde{\rho}_i(\tau)\leq 0)}$ are stochastically independent for all $i \in V$.

If we apply this assumption on agent i's credit portfolio it turns out that the distribution on the number of defaulted debtors in $\tau \geq 0$, namely

$$d_i(\tau) := \sum_{j\in V_{out}(i)} \mathbb{1}_{(\tilde{\rho}_j(\tau)\leq 0)}$$

will be implicitly binomial. More precisely, $d_i(\tau) \sim Bin(k, p(\tau))$ since every agent in the network is supposed to have k debtors in her portfolio which are now supposed to fail independently with probability $p(\tau)$. Based on this we rewrite the financial robustness of Definition 4.1 as

$$\tilde{\rho}_i(\tau+1) = \tilde{\rho}_i(0) - \frac{1}{k} d_i(\tau). \tag{4.4}$$

Furthermore assumption A(1.8) implies together with assumption A(1.7)

$$\begin{aligned} \frac{E\left[\sum_{i\in V} \mathbb{1}_{(\tilde{\rho}_i(\tau)\le 0)}\right]}{N} &= \frac{\sum_{i\in V} P(\mathbb{1}_{(\tilde{\rho}_i(\tau)\le 0)} = 1)}{N} \\ &= \begin{cases} F(0), & \text{if } \tau = 0 \\ p(\tau), & \text{if } \tau > 0. \end{cases} \end{aligned}$$

Battiston et al. (2012) rely on a further assumption which affects the correlation between the initial state of the financial robustness and the subsequent defaults.

- A(1.9) $\tilde{\rho}_i(0)$ and $d_j(\tau)$ are supposed to be independent for all $i, j \in V, i \neq j$ and $\tau > 0$.

Note that A(1.7) and A(1.8) coincide in $\tau = 0$, however, for $\tau > 0$ assumption A(1.8) does not hold in the setting of Definition 4.1. Also assumption A(1.9) is actually not correct. We refer to Section 4.2 which points out in a manageable network of small size that correlation among agents in the cascade mechanism is indeed present, which is not surprising since networks are actually characterized by correlation. Admittedly, there is the question about the usage of introducing assumptions on the variables of Definition 4.1 which must be taken as falsification of the given setting. We will discuss this question later, however, first we'd like to derive the results of Battiston et al. (2012), which becomes possible due to accessing assumptions A(1.8) and A(1.9). For a classification we will use the expression *cascade model* if we rely on the cascade mechanism from Definition 4.1 together with the introduced relaxing assumptions A(1.8) and A(1.9).

4.1.4 Analytical results for the cascade model

We state the theorem, being drawn from Battiston et al. (2012), which manifests a recursive formula for the sequence

$$\tilde{s}(\tau) = \frac{E\left[\sum_{i\in V} \mathbb{1}_{(\tilde{\rho}_i(\tau)\le 0)}\right]}{N}. \tag{4.5}$$

Note that we use $\tilde{s}(\tau)$ for the result of the cascade model, while $s(\tau)$ still denotes as in Equation (4.3) the fraction of defaults in the cascade mechanism without relying on the falsifying assumptions A(1.8) and A(1.9).

Theorem 4.1. *Let G^w be a graph that fulfills assumptions A(1.1) - A(1.3). For $\tau \geq 0$, agent i's financial robustness $\tilde{\rho}_i(\tau)$ is defined by Definition 4.1. Furthermore, it is assumed that A(1.7) - A(1.9) hold. Then, for $\tilde{s}(\tau)$ as defined in Equation* (4.5) *we have the following results:*

(i) $\tilde{s}(0) = F(0)$,

(ii) $\tilde{s}(\tau+1) = \sum_{j=0}^{k} \binom{k}{j} \tilde{s}(\tau)^j \left(1-\tilde{s}(\tau)\right)^{k-j} F\left(a\frac{j}{k}\right)$ *for* $\tau \geq 0$,

(iii) the sequence $(\tilde{s}(\tau))$ *converges to a value* $\tilde{s} \in [0,1]$.

Proof. It has already been showed above that (i) is implied by the assumptions A(1.7) and A(1.8).
For claim (ii) we remind likewise from above

$$\tilde{s}(\tau+1) = p(\tau+1) = P(\rho_1(\tau+1) \leq 0).$$

We use the redefining Equation (4.4) and see that

$$P\left(\tilde{\rho}_1(\tau+1) \leq 0\right) \;=\; P\left(\tilde{\rho}_1(0) \leq \frac{a}{k} d_1(\tau)\right).$$

The formula of the total probability yields for the last expression

$$P\left(\tilde{\rho}_1(0) \leq \frac{a}{k} d_1(\tau)\right) = \sum_{j=0}^{k} P\Big(\tilde{\rho}_1(0) \leq \frac{a}{k} d_1(\tau) \Big| d_1(\tau) = j\Big) P\Big(d_1(\tau) = j\Big).$$

With assumption A(1.9) it is then

$$P\left(\tilde{\rho}_1(0) \leq \frac{a}{k} d_1(\tau)\right) = \sum_{j=0}^{k} P\Big(\tilde{\rho}_1(0) \leq a\frac{j}{k}\Big) P\Big(d_1(\tau) = j\Big),$$

which directly results into the second claim if we apply the binomial distribution implied by A(1.8) on $P\Big(d_1(\tau) = j\Big)$, i.e. to sum up we have shown for $\tau \geq 0$

$$\begin{aligned} \tilde{s}(\tau+1) &= \sum_{j=0}^{k} P\Big(\tilde{\rho}_1(0) \leq a\frac{j}{k}\Big) P\Big(d_1(\tau) = j\Big) \\ &= \sum_{j=0}^{k} \binom{k}{j} \tilde{s}(\tau)^j \left(1-\tilde{s}(\tau)\right)^{k-j} F\left(a\frac{j}{k}\right). \end{aligned}$$

In order to prove claim (iii) we have to find a value $\tilde{s} \in [0,1]$ such as $\tilde{s}(\tau) \to \tilde{s}$ for $\tau \to \infty$. From its definition we directly see that $\tilde{s}(\tau) \in [0,1]$. Furthermore, we can argue that $(\tilde{s}(\tau))$ is a monotonously increasing sequence. Therefore, we consider the last equation as function in $\tilde{s}(\tau)$, namely

$$G(x) = \sum_{j=0}^{k} \binom{k}{j} x^j \left(1-x\right)^{k-j} F\left(a\frac{j}{k}\right).$$

The function's derivative is given by

$$G'(x) \;=\; \sum_{j=1}^{k} \binom{k}{j} j x^{j-1}(1-x)^{k-j} F\left(a\frac{j}{k}\right) - \sum_{j=0}^{k-1} \binom{k}{j} (k-j) x^j (1-x)^{k-j-1} F\left(a\frac{j}{k}\right)$$

$$= \sum_{j=0}^{k-1} \left[\binom{k}{j+1} (j+1) F\left(a\frac{j+1}{k}\right) - \binom{k}{j} (k-j) F\left(a\frac{j}{k}\right) \right] x^j (1-x)^{k-j-1}$$

where the last step has resulted from an index shift. Furthermore

$$\begin{aligned} \binom{k}{j+1}(j+1) &= \frac{k! \cdot (j+1)}{(k-j-1)! \cdot (j+1)!} \\ &= \frac{k! \cdot (k-j)}{(k-j)! \cdot j!} \\ &= \binom{k}{j}(k-j), \end{aligned}$$

hence, we have

$$G'(x) = \sum_{j=0}^{k-1} \left[F\left(a\frac{j+1}{k}\right) - F\left(a\frac{j}{k}\right) \right] \binom{k}{j} (k-j) x^j (1-x)^{k-j-1}.$$

Due to the monotonicity of distribution functions we can conclude that $G'(x) \geq 0$ for all $x \in [0,1]$.
We can then reason by induction that the sequence $\tilde{s}(\tau)$ is monotonously increasing on $[0,1]$. As base clause for this induction we show that $\tilde{s}(0) \leq \tilde{s}(1)$ holds.

$$\begin{aligned} \tilde{s}(1) &= G(\tilde{s}(0)) \\ &= \sum_{j=0}^{k} \binom{k}{j} \tilde{s}(0)^j \left(1-\tilde{s}(0)\right)^{k-j} F\left(a\frac{j}{k}\right) \\ &\geq \sum_{j=0}^{k} \binom{k}{j} \tilde{s}(0)^j \left(1-\tilde{s}(0)\right)^{k-j} F(0) \\ &= F(0) \\ &= \tilde{s}(0) \end{aligned}$$

The induction hypothesis $\tilde{s}(\tau) \geq \tilde{s}(\tau-1)$ and the monotonicity of G yield

$$\tilde{s}(\tau+1) = G(\tilde{s}(\tau)) \geq G(\tilde{s}(\tau-1)) = \tilde{s}(\tau),$$

which shows that the sequence $\tilde{s}(\tau)$ itself is monotonously increasing.
Now, by having a bounded and monotonously increasing sequence on $[0,1]$, we can apply the Bolzano-Weierstrass theorem and conclude that there must be a value $\tilde{s} \in [0,1]$ so that the third claim holds. □

The last theorem is going to be extended by the following corollary.

Corollary 4.1. *The limit $\tilde{s} \in [0,1]$ of the last theorem is a fixed point of function G.*

Proof.

$$G(\tilde{s}) = G(\lim_{\tau\to\infty} \tilde{s}(\tau)) = \lim_{\tau\to\infty} G\left(\tilde{s}(\tau)\right) = \lim_{\tau\to\infty} \tilde{s}(\tau+1) = \tilde{s},$$

where the second step holds due to the continuity of G.

□

Note that the existence of a fixed point for G cannot be proved by the Banach fixed point theorem. Though G is Lipschitz continuous on the compact set $[0,1]$, it is not ensured that G is a contraction mapping, since its derivative does not have to be less than 1 on the whole set $[0,1]$. Thus, we must not conclude that G has a unique fixed point on $[0,1]$, which is also confirmed by the later visualization of G in Section 4.1.6.
Besides we allude that claim (i) of the last theorem may also be derived without relying on the relaxing assumptions A(1.8) and A(1.9) of the cascade model but just on A(1.7). The same is true for claim (ii) if $\tau = 0$, hence, we conclude $\tilde{s}(0) = s(0)$ and $\tilde{s}(1) = s(1)$.

4.1.5 Remark on the analytical results

We have to advise of a difference between the results here and the results of Battiston et al. (2012). By using the notations from here, Battiston et al. (2012) derive the recursive sequence

$$\begin{aligned}
\tilde{s}(0) &= F(0) \\
\tilde{s}(\tau+1) &= \max\left\{\tilde{s}(0), \sum_{j=1}^{k} \binom{k}{j} \tilde{s}(\tau)^j \left(1-\tilde{s}(\tau)\right)^{k-j} F\left(a\frac{j}{k}\right)\right\}
\end{aligned}$$

in the cascade model. Their argumentation hereto has been the same up to the conclusion

$$\tilde{s}(\tau+1) = P\left(\tilde{\rho}_1(0) \le \frac{a}{k} d_1(\tau)\right)$$

in our proof above. Then, different from here, they do not refer to the formula of total probability, but argue that the possible events for $d_1(\tau)$ are $1, 2, \ldots, k$. From their argumentation it is not clear why they left out the event $d_1(\tau) = 0$.
If we had assumed

$$P\left(d_1(\tau) = 0\right) = 0$$

in our argumentation, we would have received the same results, however, in our point of view, this would not be feasible. By neglecting the corresponding summand for $j = 0$ in our calculation of $\tilde{s}(\tau+1)$, the fraction of defaulted agents in $\tau+1$ would not include the part of defaulted agents, which - though their debtors are still solvent in τ - have already defaulted in 0. One may suppose that such a case is only of little account in cascade scenarios, nevertheless, it originates the problem, that in general

$$\sum_{j=1}^{k} \binom{k}{j} \tilde{s}(\tau)^j \left(1-\tilde{s}(\tau)\right)^{k-j} F\left(a\frac{j}{k}\right)$$

does not have to increase monotonously. For instance, if we assume $\tilde{\rho}_1(0) \sim N(0.2, 0.0036)$ and choose $k = 10$, the recursion

$$\begin{aligned}
\tilde{s}(0) &= F(0) \\
\tilde{s}(\tau+1) &= \sum_{j=1}^{k} \binom{k}{j} \tilde{s}(\tau)^j \left(1-\tilde{s}(\tau)\right)^{k-j} F\left(a\frac{j}{k}\right)
\end{aligned}$$

would yield the following results for $\tau = 0, \ldots, 5$:

$\tilde{s}(0)$	$\tilde{s}(1)$	$\tilde{s}(2)$	$\tilde{s}(3)$	$\tilde{s}(4)$	$\tilde{s}(5)$
$4.29 \cdot 10^{-4}$	$2.08 \cdot 10^{-4}$	$1.00 \cdot 10^{-4}$	$4.82 \cdot 10^{-5}$	$2.31 \cdot 10^{-5}$	$1.10 \cdot 10^{-5}$

Battiston et al. (2012) indicate that recoveries during a cascade ought to be excluded, i.e. the cascade size must not decrease in time. That is the reason, why they used to apply the maximum function as above which eventually results in an increasing and converging sequence. In the following we will always use the version of $\tilde{s}$ as it has been derived by us in Theorem 4.1.
As a further remark we want to state a lemma which discloses some monotonicity property for the cascade size with respect to a variation in the mean robustness m if one assumes a normally distribution for $\tilde{\rho}_i(0)$ in our derived formula for $\tilde{s}(\tau)$.

Lemma 4.1. *Let $\tilde{\rho}_i^1(0)$ and $\tilde{\rho}_i^2(0)$ be the initial financial robustnesses adapted to the cascade sequences $(\tilde{s}_1(\tau))$ and $(\tilde{s}_2(\tau))$ respectively from Theorem 4.1 whereas it is assumed that $\tilde{s}_1(\tau) \to \tilde{s}_1$, $\tilde{s}_2(\tau) \to \tilde{s}_2$ and*

$$\tilde{\rho}_i^1(0) \sim N\left(m_1, \frac{\sigma^2}{k}\right)$$
$$\tilde{\rho}_i^1(0) \sim N\left(m_2, \frac{\sigma^2}{k}\right)$$

for $m_1 \geq m_2$. Then we have for all $\tau \geq 0$

$$\tilde{s}_1(\tau) \leq \tilde{s}_2(\tau) \text{ and } \tilde{s}_1 \leq \tilde{s}_2.$$

Proof. We prove the lemma by induction. First we substantiate the base clause, that is, we show $\tilde{s}_1(0) \leq \tilde{s}_2(0)$. Recall from Theorem 4.1 that

$$\begin{aligned}
\tilde{s}_1(0) &= P(\tilde{\rho}_i^1(0) \leq 0) = \Phi\left(\frac{-m_1\sqrt{k}}{\sigma}\right) \\
\tilde{s}_2(0) &= P(\tilde{\rho}_i^2(0) \leq 0) = \Phi\left(\frac{-m_2\sqrt{k}}{\sigma}\right),
\end{aligned}$$

where the second equality is due to the assumed distribution. Since distribution functions are monotonously increasing the assumption $m_1 \geq m_2$ yields

$$\tilde{s}_1(0) \leq \tilde{s}_2(0).$$

For the induction step we use the induction hypothesis $\tilde{s}_1(\tau) \leq \tilde{s}_2(\tau)$ and conclude

$$\begin{aligned}
\tilde{s}_1(\tau+1) &= \sum_{j=1}^{k} \binom{k}{j} \tilde{s}_1(\tau)^j \left(1 - \tilde{s}_1(\tau)\right)^{k-j} \Phi\left(\frac{aj/\sqrt{k} - m_1\sqrt{k}}{\sigma}\right) \\
&\leq \sum_{j=1}^{k} \binom{k}{j} \tilde{s}_2(\tau)^j \left(1 - \tilde{s}_2(\tau)\right)^{k-j} \Phi\left(\frac{aj/\sqrt{k} - m_1\sqrt{k}}{\sigma}\right) \\
&\leq \sum_{j=1}^{k} \binom{k}{j} \tilde{s}_2(\tau)^j \left(1 - \tilde{s}_2(\tau)\right)^{k-j} \Phi\left(\frac{aj/\sqrt{k} - m_2\sqrt{k}}{\sigma}\right)
\end{aligned}$$

$$= \tilde{s}_2(\tau+1).$$

While the first inequality comes from the monotonicity of G, which has been shown in the proof of Theorem 4.1, the second inequality is again the monotonicity of the distribution function Φ. The final claim, that is, $\tilde{s}_1 \leq \tilde{s}_2$, holds, since we know from analysis that for general sequences $b_1(n) \to b_1$ and $b_2(n) \to b_2$ with $b_1(n) \leq b_2(n)$, one can deduce that $b_1 \leq b_2$ holds as well. □

A further remark may give some support to analyze the following visualization of the analytical results from Theorem 4.1. Recall that the underlying function of $\tilde{s}(\tau)$ in the cascade model is given by

$$G(x) = \sum_{j=0}^{k} \binom{k}{j} x^j (1-x)^{k-j} F(j\frac{a}{k})$$

for $x \in [0,1]$ and F being the cumulative distribution function of the initial robustness $\tilde{\rho}_i(0)$. Apparently, $G(x)$ may also be regarded as the weighted sum of function F evaluated at the supporting points $0, \frac{a}{k}, 2\frac{a}{k}, \ldots, 1$. The $k+1$ weights are defined by

$$g_j := \binom{k}{j} x^j (1-x)^{k-j},\ j = 0, \ldots, k$$

and as g_j equals the probability mass of having j successes in a binomial experiment with k trials and success probability x, the weights obviously sum up to 1. Actually, for $a = 1$ this type of weighted sum is well known as *Bernstein polynomial* which is generally defined in the next definition.

Definition 4.2. *The Bernstein polynomial $B_k(f, x)$ is given by*

$$B_k(f,x) = \sum_{j=0}^{k} \binom{k}{j} x^j (1-x)^{k-j} f(\frac{j}{k})$$

for any function f on $[0,1]$, $x \in [0,1]$ and $k \in \mathbb{N}$.

Based on this general definition we can consider $G(x)$ as the Bernstein polynomial with respect to function F - assumed that parameter a is set to 1. While Bernstein polynomials possess a series of useful properties as for instance described by Philips (2003) we just state the probably most important one in the next theorem.

Theorem 4.2. *For a given function $f \in \mathcal{C}(0,1)$ and any $\delta > 0$, there exists $K \in \mathbb{N}$ such as*

$$|f(x) - B_k(f,x)| < \delta$$

for all $x \in [0,1]$ and $k \geq K$.

Proof. Let x be an arbitrary point in $[0,1]$. Note that we can rewrite the Bernstein polynomial as the expectation

$$B_k(f,x) = E\left[f\Big(\frac{S_k(x)}{k}\Big)\right]$$

for a random variable $S_k(x) \sim Bin(k,x)$, which is binomial distributed with k trials and success probability x. The idea of the proof is to show first that

$$\left|f(x) - E\left[f\Big(\frac{S_k(x)}{k}\Big)\right]\right| \to 0$$

as $k \to \infty$ in order to conclude pointwise convergence for the Bernstein polynomial in x. By Dini's Theorem we may then immediately follow that the continuous function $B_k(f,x)$ converges uniformly on the compact set $[0,1]$ towards f and the theorem's claim would have been proved.

We consider the random variable $S_k(x)$ as the convolution of k independent Bernoulli random variables $\xi_j(x)$ with success probability x, i.e.

$$\frac{1}{k}S_k(x) = \frac{1}{k}\sum_{j=1}^{k}\xi_j(x).$$

Hence, we may apply the weak law of large numbers to $\frac{1}{k}S_k(x)$ and conclude that

$$\frac{1}{k}S_k(x) \xrightarrow{P} E\left[\xi_1(x)\right] = x$$

where $\xrightarrow{P}$ denotes convergence in probability. Klenke (2006) states that convergence in probability implies weak convergence, that means, per definition of weak convergence we deduce

$$E\left[g\Big(\frac{S_k(x)}{k}\Big)\right] \to E[g(x)]$$

for all $g \in \mathcal{C}(0,1)$ and $k \to \infty$. Since, $f \in \mathcal{C}(0,1)$ by assumption, the theorem is proved. $\square$

Theorem 4.2 states that Bernstein polynomials uniformly converge to a sufficiently smooth function f which is a useful result in numerical approximation theory. For our purpose it enables the insight that - assumed that the distribution function F is continuous on the interval $[0,1]$ - function G approximates function F so that characteristics of F are regained in function G. One example is already stated in Lemma 4.1. So, for plots of G in the following section it might be useful to keep in mind that a change in F will effect function G in a similar manner.

Figure 4.1: For all plots the distribution of $\tilde{\rho}(0)$ has been $N(m, \frac{\sigma^2}{k})$ with fixed $\sigma = 0.25$. The parameter k is set to 10 for Plot 1, 2 and 4 - in Plot 3 k is correspondingly varying. Function G in Plot 2 is in the same way defined as in proof for Theorem 4.1. The dashed line in the same plot represents the line through the origin. The line in Plot 3 signalizes the values of the critical robustness for different values of k. τ^* in Plot 4 is the smallest time step τ for which $1 - \tilde{s}(\tau) < 10^{-6}$.

4.1.6 Visualization of the analytical results

Analogously to Battiston et al. (2012) we discuss the results of Theorem 4.1 by visualizing the sequence $\tilde{s}(\tau)$. The following plots have been created by resorting to the version of the sequence as it has been derived by us in Theorem 4.1, i.e. by taking the case $d_1(\tau) = 0$ into account. Indeed, we see that the visualized results are structurally the same as in Battiston et al. (2012) who use the slightly different formula for $\tilde{s}(\tau)$. To ensure comparability we assume like Battiston et al. (2012) a normal distribution on the initial financial robustness, i.e.

$$\tilde{\rho}_i(0) \sim N\left(m, \frac{\sigma^2}{k}\right)$$

for all $i \in V$, where a diversification effect on the robustness' variance for an increasing number of debtors, if parameter σ is assumed to be fixed, is additionally regarded.
The result of Lemma 4.1 has formalized as intuitively supposed, that a lower mean robustness of $\tilde{\rho}_i(0)$ implies a higher final cascade size of defaulted agents in the network. While this fact is indeed recognizable in the following plots, we see in the first illustration that the cascade size is not necessarily a continuous function in m but may possesses a jump at a certain mean robustness. Plot 1 of Figure 4.1 illustrates the cascade size at step $\tau = 600$, fixed diversification degree $k = 10$ and varying mean robustness m on the interval $[0.215, 0.22]$. The discretization size for the given interval has been chosen to be 10^{-5}. Apparently, the observed cascade sizes separate the robustness interval into two parts, where for $m \leq 0.21904$ the cascade sizes in $\tau = 600$ are approximately 1 and for $m \geq 0.21905$ the cascade sizes drop to low values in the field of 0. Hence, the plot suggests that the cascade size is not continuous in m but rather jumps from almost no defaults to a total collapse of the system at a mean robustness $m^\star$ which we'd like to call *critical robustness* in the following.
Plot 2 on the right hand side can be taken as insight for the observation in the previous plot. We here draw function G(x) on the interval $[0, 1]$ for three different values of parameter m. The dashed line represents the line through the origin and shall help to detect fixed points. The upper blue line is $G(x)$ for a mean robustness of $m = 0.025$ which has no fixed point up to a value close to 1. The lower red line corresponds to $G(x)$ with $m = 0.3$. Though it is not visible in this plot, $G(0) > 0$ holds. What we see, is, that the red line crosses the dashed line at a value $x_1^\star \approx 0.18$ which is not the fixed point to which $\tilde{s}(\tau)$ converges. Actually, there is a further intersection point $x_2^\star$, where $0 < x_2^\star < x_1^\star$ and for which we eventually have $\tilde{s}(\tau) \to x_2^\star$. If we use Figure 4.2 to zoom into Plot 2 we may reason the convergence of $\tilde{s}(\tau)$ into x_2 by alluding that the Banach fixed point theorem holds locally for G on the interval $[0, x_2]$. Finally the black line corresponds to $G(x)$ with $m = 0.21905$, which is due to Plot 1 close to the critical robustness. Note that a reduction

Figure 4.2: The figure is used to zoom into Plot 2 of Figure 4.1.

in m shifts the curve of G towards left so that the first intersection point of G with the dashed line wanders bottom up to the point at which the black lines is almost tangent to the dashed line. For these wandering intersection points the Banach fixed point theorem always locally hold and we can always register a convergence to these points. However, if m is less than 0.21905 the curve of G is shifted too far so that there is no intersection point any more with the dashed line - at least on the range of the x-axis which is noticeable in Figure 4.2. In this situation the first intersection point would be close to 1 implying a cascade of large size. To sum up, the critical robustness $m^\star$ is the mean robustness for which the graph of G switches from intersecting to bothering the line through the origin. Plot 3 in Figure 4.1, which can be likewise found in Battiston et al. (2012), illustrates the dependency of the critical robustness on diversification degree k. We used to approximate these values in the range between $k = 5$ and $k = 90$ by an algorithm which is based on the idea of generating nested intervals. For the algorithm we have to initially set $\tau^\star$ sufficiently large, *tol* sufficiently small and $\tilde{s}^\star$ as the cascade size of interest. Furthermore, we must define initial values m_1, m_2 such as $\tilde{s}_{m_1}(\tau^\star) > \tilde{s}^\star$ and $\tilde{s}_{m_2}(\tau^\star) < \tilde{s}^\star$. Having given these input parameters the algorithm works as follows.

Algorithm 4.1: An algorithm for approaching the critical robustness.

Data: m_1, m_2, $\tilde{s}^\star$, *tol*, $\tau^\star$
Result: Critical robustness m
$m = \frac{m_1+m_2}{2}$;
Calculate $\tilde{s}(\tau^\star)$ w.r.t. mean robustness m;
while $m_2 - m_1 > tol$ *and* $\tilde{s}(\tau^\star) \neq \tilde{s}^\star$ **do**
    $m = \frac{m_1+m_2}{2}$;
    Calculate $\tilde{s}(\tau^\star)$ w.r.t. mean robustness m;
    if $\tilde{s}(\tau^\star) > \tilde{s}^\star$ **then**
        $m_1 = m$;
    else
        if $\tilde{s}(\tau^\star) < \tilde{s}^\star$ **then**
            $m_2 = m$;
        else
            STOP
        end
    end
end

The algorithm's intention is laid out to approximate the robustness m at which $\tilde{s}(\tau^\star)$ equals $\tilde{s}^\star$ and works as the cascade size increases if m decreases due to Lemma 4.1. If function G does not possess a stable fixed point in $\tilde{s}^\star$ for any value of m, the algorithm will approximate

$$m^\star = \inf\{m : \tilde{s}_m(\tau^\star) < \tilde{s}^\star\}.$$

For our purposes we set $\tilde{s} = 0.5$ and have only received approximations $\hat{m}^\star$ for which the

cascade size in $\tau^\star$ is less than 0.05 whereas minor reductions of the found values $\hat{m}^\star$ effect jumps to cascade sizes larger than 0.9. We allude that it is not excluded in general that there might be intermediate cascade sizes on the interval $[0, 1]$, which eventually happens for a certain setting of parameters m and σ. Actually, such cases would particularly occur if one chooses a standard deviation σ exceeding the assumed mean m whereas this has not been considered by Battiston et al. (2012).[8] If we talk about large or small cascade size in the following, we will always assume that the size of cascades is separated by the critical robustness $m^\star$ as in Plot 1 of Figure 4.1.
We are not informed how Battiston et al. (2012) found $m^\star$, and we could not figure out any analytical way to determine $m^\star$ exactly for the assumed normal distribution on $\tilde{\rho}_i(0)$, however, Algorithm 4.1 has turned out to work reliably.
In addition to the contents of Battiston et al. (2012) we analyze the duration of convergence for fixed k and σ but varying robustness m below the critical robustness, i.e. how fast does $\tilde{s}(\tau)$ approach towards 1. Plot 4 in Figure 4.1 points out that for robustness m close to the critical robustness the duration exponentially increases, whereas for smaller m the duration remains a long time on a low level. Figure 4.2 suggests that the continuous derivative of $G(x_2)$ converge towards 1 if m goes to $m^\star$ from above. The Banach fixed point theorem in turn allows to deduce that a higher value of the derivative in the fixed point slows the speed of convergence of the fixed point iteration which explains the increasing duration in Plot 4.

4.2 Discussion on the cascade model

Recall that the original motivation for the relaxing assumptions A(1.8) and A(1.9) has been to make an inference on $s(\tau)$ of Equation (4.3), however, it is not clear at all whether the analytical result on $\tilde{s}$ in Theorem 4.1 can be effectively taken as a good reference for s. Battiston et al. (2012) avoid doing a close discussion on this but state about A(1.8) that "*[t]his assumption is strictly valid only when there are few defaults in the system. However, when there are instead many defaults, the result shows that eventually the cascade involves the whole network. Thus, the presence of correlation becomes irrelevant and the result is still correct.*". In our following discussion we will at first illustrate by relying on a small network that assumptions A(1.8) and A(1.9) does not hold generally in the cascade mechanism of Definition 4.1 and then point out in a succeeding simulation procedure that $\tilde{s}$ cannot serve commonly as reliable inference on s, however, we will eventually conclude that the cascade model has its right to exist in the procedure of Battiston et al. (2012) anyway.

4.2.1 The cascade mechanism on a small network

For the construction of a counter-example with respect to assumptions A(1.8) and A(1.9) we rely on a network of small size. More precisely, we will refer to a network with $N = 3$ and $k = 2$. This network, illustrated on the left in Figure 4.3, contains only three agents 1,

[8] Recall that function G approximates the distribution function F as Bernstein polynomial, hence, for having a stable fixed point of G close to 0.5, $F(0)$ would have to be large, but at the same time the derivative of F must not to be too large so that a stable fixed point is available at 0.5. This in turns requires a large σ in order to stretch F.

Figure 4.3: The left hand side illustrates the network of consideration with $N = 3$ and $k = 2$. The upper plot on the right hand side reports the correlation coefficient as defined in Equation (4.6). The plot below shows $F(0)$ for different values of m and $\sigma = 0.25$ fixed.

2 and 3, and all three agents interchange bilaterally credits. In order to fulfill assumptions A(1.1) - A(1.3) entirely it is furthermore assumed that the weights of the issued credits equal 0.5.

Review assumption A(1.8)

We first assume $\tilde{\rho}_i(0) \sim N(m, \sigma^2)$ for $i = 1, 2$ and 3, where it is additionally assumed, in order to match assumption A(1.7), that $\tilde{\rho}_i(0)$ are independently distributed In the following we will denote by F the distribution function of $\tilde{\rho}_i(0)$.
Recall that assumption A(1.8) has been introduced in Section 4.1.3 in order to elude an involved analytical treatment of the network structure. The given network, however, is manageable enough to calculate, for instance, the correlation coefficient between $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ and $\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}$ explicitly by directly relying on the setting of Definition 4.1, while a correlation coefficient unequal 0 will refute assumption A(1.8). The correlation coefficient between $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ and $\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}$ reads

$$Corr[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}, \mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}] = \frac{Cov[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}, \mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}]}{\sqrt{Var[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}]}\sqrt{Var[\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}]}}. \tag{4.6}$$

For the given network its calculation is based on the identities

$$\begin{aligned} E[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}] &= P(\tilde{\rho}_2(1) \leq 0, \tilde{\rho}_3(1) \leq 0), \\ E[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}] &= E[\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}] = P(\tilde{\rho}_3(1) \leq 0) \end{aligned}$$

which can be combinatorially computed by keeping in mind that due to Equation (4.1)

$$\begin{aligned} \tilde{\rho}_2(1) &= \tilde{\rho}_2(0) - \frac{1}{2}\mathbb{1}_{(\tilde{\rho}_1(1)\leq 0)} - \frac{1}{2}\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)} \\ \tilde{\rho}_3(1) &= \tilde{\rho}_3(0) - \frac{1}{2}\mathbb{1}_{(\tilde{\rho}_1(1)\leq 0)} - \frac{1}{2}\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}. \end{aligned}$$

It is

$$\begin{aligned} E[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}] &= F(0)^2 + F(0)(F(0.5) - F(0))^2 + 2F(0)^2(F(1) - F(0)) \\ &\quad +2F(0)(1 - F(0))(F(0.5) - F(0)) \\ E[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}] &= F(0)(1 + F(0)(F(1) - F(0)) + 2(F(0.5) - F(0))(1 - F(0))) \end{aligned}$$

where a corresponding combination of these terms under consideration of the identity

$$E[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}] = E[\mathbb{1}^2_{(\tilde{\rho}_2(1)\leq 0)}]$$

yields the correlation coefficient of $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ and $\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}$.
For fixed $\sigma = 0.25$, but varying mean robustness m in the range of $[-1.5, 1.5]$, we draw

the correlation coefficient in Figure 4.3 on the right hand side. Additionally, the figure contains below the development of

$$F(0) = P(\tilde{\rho}_1(0) \leq 0)$$

with respect to m. IF $F(0)$ is high, we will presumably observe independent defaults for all agents 1, 2 and 3 already in $\tau = 0$. In this case any linkage between the agents becomes redundant for possible contagion effect and we observe just a low correlation in $\tau = 1$. On the other side, if $F(0)$ is low, we will have hardly any defaults in $\tau = 0$ and both agents will be still solvent in $\tau = 1$ without risk of infecting each other. Again, just a slight correlation is recognizable. One may take this as a kind of illustration for the statement of Battiston et al. (2012).
However, for an intermediate level of $F(0)$ a correlation between $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ and $\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}$ is clearly noticeable. The occurring correlation may be explained by now having an indifferent probability for a default or no default in $\tau = 0$. Typically, the fraction of defaults and no defaults will be balanced in $\tau = 0$, then, however, the linkages between the agents will influence $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ and $\mathbb{1}_{(\tilde{\rho}_3(1)\leq 0)}$ since the default of one agent will affect the status of the other agent in $\tau = 1$ which is then displayed by the noticeable correlation.

Review assumption A(1.9)

For reviewing assumption A(1.9) we again rely on the small network of Figure 4.3 fulfilling assumptions A(1.1) - A(1.3). But therefore we assume for the sake of simplicity a Bernoulli distribution for $\tilde{\rho}_i(0)$, that is

$$\begin{aligned} P(\tilde{\rho}_i(0) = 1) &= 1 - p \\ P(\tilde{\rho}_i(0) = 0) &= p, \end{aligned}$$

where $i = 1, 2, 3$. This yields

$$\begin{aligned} E[\tilde{\rho}_1(0)\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}] &= E[\tilde{\rho}_1(0)\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}|\tilde{\rho}_1(0) = 1]P(\tilde{\rho}_1(0) = 1) \\ &\quad + E[\tilde{\rho}_1(0)\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}|\tilde{\rho}_1(0) = 0]P(\tilde{\rho}_1(0) = 0) \\ &= E[\tilde{\rho}_1(0)\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}|\tilde{\rho}_1(0) = 1](1 - p) \\ &= P\left(\rho_2(0) \leq \frac{1}{2}\mathbb{1}_{(\tilde{\rho}_3(0)\leq 0)}\right)(1 - p) \\ &= p(1 - p). \end{aligned}$$

Furthermore we deduce that

$$E[\tilde{\rho}_1(0)] = 1 - p$$

and

$$\begin{aligned} E[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}] &= P\left(\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)} = 1\right) \\ &= p + (1 - p)p^2. \end{aligned}$$

We compose this and obtain the covariance

$$Cov\left[\tilde{\rho}_1(0), \mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}\right] = p(1 - p) - (1 - p)(p + (1 - p)p^2)$$

Figure 4.4: The figure illustrates the correlation between $\tilde{\rho}_1(0)$ and $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ for different values of the initial default probability p.

$$= \quad -(1-p)^2p^2.$$

By

$$\begin{aligned} Var\left[\tilde{\rho}_1(0)\right] &= p(1-p) \\ Var\left[\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}\right] &= p+(1-p)p^2-(p^2+(1-p)p^2)^2 \end{aligned}$$

we can deduce the correlation coefficient of $\tilde{\rho}_1(0)$ and $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$, which is plotted in Figure 4.4 for p ranging between 0 and 1. The correlation in Figure 4.4 tends to 0 if the default probability becomes large or small, which somehow matches the statement of Battiston et al. (2012) again, however, for intermediate values of the initial default probability the correlation becomes more distinctive. Actually, we observe a negative correlation between $\tilde{\rho}_1(0)$ and $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ meaning that high a probability for $\tilde{\rho}_1(0)$ being 1 decreases the probability of having 1 for $\mathbb{1}_{(\tilde{\rho}_2(1)\leq 0)}$ as well and vice versa.

4.2.2 Simulation on a small network

We'd like to proceed a simulation on the small network of Figure 4.3 in order to compare s, that is the expected fraction of defaults in the cascade mechanism of Definition 4.1 against the outcome of $\tilde{s}$ based on the formula of Theorem 4.1. For the simulation we construct the network's weighted adjacency matrix

$$A^w = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

generate the random variables $\tilde{\rho}_1(0)$, $\tilde{\rho}_2(0)$ and $\tilde{\rho}_3(0)$ a hundred times for different m and apply Equation (4.1) for receiving the number of defaulted agents. In this procedure it is assumed $\tilde{\rho}_i(0) \sim N(m, \sigma^2)$ for all $i = 1, 2, 3$, where σ is set to 0.25 constantly for all runs. Our particular impetus is to range the mean robustness m so that the three cases high, low and indifferent default probability in $\tau = 0$ are covered.
The following table lists how often we have observed which count of defaults in percentage of 100 runs for each selected mean robustness. In the last line we report $\tilde{s}$ which is the limit for the sequences $\tilde{s}(\tau)$ as suggested by the cascade model.

m	-1	-0.5	-0.25	0	0.25	0.5	0.75	1
Simulation								
3 Defaults	100%	100%	100%	88%	39%	3%	0%	0%
2 Defaults	0%	0%	0%	0%	0%	0%	0%	0%
1 Default	0%	0%	0%	0%	0%	1%	0%	0%
0 Defaults	0%	0%	0%	12%	61%	96%	100%	100%
Cascade Model $\tilde{s}$	1	1	1	1	0.998	0.5	0.002	$3.3 \cdot 10^{-5}$

We see that in both cases, low and high default probability in $\tau = 0$, the cascade model and the simulations estimate likewise the same fraction of defaults, namely, 0% and 100% respectively. The range between $m = 0$ and $m = 0.5$ appears more interesting. While the expected fraction of defaults is prognosticated to be close to 1 in the cascade model, the simulations rather suggest that the size of the cascades alternates between total collapse and no defaults. Obviously, the cascade model only works reliable for the border area of m, however, if we choose m so that we become an intermediate probability for defaults in $\tau = 0$, the cascade model tends to overestimate the expected cascade size.
By relying on our review of assumption A(1.8) and A(1.9), it is traceable that $\tilde{s}$ will not serve as accurate measure for the expected defaults in the cascade mechanism if m is chosen such that the initial default probability is balanced.
The overestimation appears first as surprising, since one would presumably expect an underestimation of the default probability due to the neglected correlation for $\tau > 0$. Note that for the given cascade mechanism without the relaxing assumptions A(1.8) and A(1.9) we have on a general network

$$\mu_1(\tau) := P\left(\sum_{i \in V} \mathbb{1}_{(\tilde{\rho}_i(\tau) \leq 0)} = 0\right) = (1 - F(0))^N$$

for all $\tau \geq 0$. However, in the cascade model, it is due to A(1.8)

$$\mu_2(\tau) := P\left(\sum_{i \in V} \mathbb{1}_{(\tilde{\rho}_i(\tau) \leq 0)} = 0\right) = (1 - p(\tau))^N,$$

i.e. the probability of having no defaults in the cascade model is depending on τ. Moreover, since $p(\tau) = \tilde{s}(\tau)$ and $\tilde{s}(\tau)$ is increasing in the cascade model, the probability of having no defaults is decreasing with growing τ. Thus, the cascade model cumulatively underestimates the event of having no defaults in the network over time. The error, which is induced by this underestimation, depends on the initial state of $\mu_2(0)$, which coincides with $\mu_1(0)$. If it is already low in the beginning, only less underestimation can be incorporated. On the other side, if the probability is close to 1, $\tilde{s}(\tau)$ will already terminate in an early stage and again just less or no underestimation flows into the recursive calculation of $\tilde{s}(\tau)$. However, if $\mu_2(0)$ is not too large, but sufficiently large in order to trigger a convergence of $\tilde{s}(\tau)$ towards 1, the calculation of $\tilde{s}(\tau)$ will incorporate in every step τ the wrongly decreasing probability $\mu_2(\tau)$. Hereby the error $\tilde{s}$ becomes the larger the longer the convergence of $\tilde{s}(\tau)$ takes. We refer to Figure 4.5, which includes μ_1, the time to convergence $\tau^\star$ with respect to machine accuracy for $\tilde{s}(\tau) \to \tilde{s}$ and the error between $\tilde{s}$ and s, which is going to be ascertained by

$$\kappa = \tilde{s} - \hat{s},$$

whereas these values are depending on m, and $\hat{s}$ is the observed average for the simulations on s from above. We observe that the peak of κ is reached when μ_1 bends down from 1 which is simultaneously the point where the convergence of $\tilde{s}(\tau)$ towards 1 is triggered the first time and requires the most iterations. Apparently, the calculations of the cascade model respond sensitively as soon as the probability of having no defaults in the network slightly leaves a level of almost 1. The error κ will not reach 0 before the probability of

Figure 4.5: The figure contains the values μ_1, κ, and $\tau^\star$ determined on the setting of the small network illustrated in Figure 4.3 and the assumptions $\tilde{\rho}_i(0) \sim N(m, \sigma^2)$, $\sigma = 0.25$.

having no defaults in the network undershoots a certain level close to 0 again.
Admittedly, the last examination has been only executed on a very specific network setting. We are going to undertake in the next section the comparison between simulation and cascade model on larger networks and will see that the problem of overestimation may be analogously found on these networks as well.

4.2.3 Cascade model vs. simulation on large networks

As in the last section on the specific, small network we now simulate the cascade mechanism on larger networks. For the generation of a general network which satisfies the assumptions of an homogeneous network we rely on Algorithm 3.1 in Section 3.2.4. Having generated this network with the corresponding $N \times N$ adjacency matrix A^w, we simulate the vector $\tilde{\rho}(0)$ of dimension N and then iteratively apply Equation (4.2). In each iteration step we can read out the vector $z(\tau)$ which indicates the fraction of defaulted agents in the cascade mechanism at τ. With the same argumentation as in Section 3.2.3 we restrict ourself to a network with size $N = 50$. Additionally, we assume for each component of the vector $\tilde{\rho}(0)$,

$$\tilde{\rho}_i(0) \sim N\left(m, \frac{\sigma^2}{k}\right) \quad i = 1, \ldots, N,$$

with $\sigma = 0.25$ and k fixed but varying values for m. In the following we denote by $\hat{s}_j(\tau)$ the fraction of defaults at iteration step τ in the j-th run of the simulation. We repeat the simulations for all m 300 times and consider the average

$$\hat{s}(\tau) = \frac{\sum_{j=1}^{300} \hat{s}_j(\tau)}{300}.$$

To be in line with the mean field approach of Battiston et al. (2012) the network's adjacency matrix is always renewed for every simulation run $j = 1, \ldots, 300$. Eventually, we compare $\hat{s}(\tau)$ to the cascade size $\tilde{s}(\tau)$ proposed by the cascade model and its iterative formula in Theorem 4.1.
The comparison is illustrated by the contour plots in Figure 4.6, where we focus ourselves on the cases $k = 5$ and $k = 35$. The contour plots consist of a y-axis for τ ranging from 0 to 50 and an x-axis for the mean robustness m. The colors signalize the size of $\tilde{s}(\tau)$ and $\hat{s}(\tau)$ respectively for the various values of m and τ. The observations between both cases $k = 5$ and $k = 35$ do not differ and may be analogously made for other degrees of diversification. Apparently, the sizes of the cascades in the cascade model are strictly separated between values close to 1 and 0, which is in line with the contents of Section 4.1.6. We see that the position of the critical robustness, which is highlighted by the horizontal lines, is indeed a boundary between two regimes in the cascade model. This coincides with the observation in Plot 1 of Figure 4.1 and we can also observe a long-lasting convergence of $\tilde{s}(\tau)$, as already seen in Plot 4 of Figure 4.1, if m proceeds towards the critical robustness

Figure 4.6: The figure illustrates the size of a cascade suggested by the formula of the cascade model and simulations respectively with respect to the discrete time unit τ on the y-axis and the mean robustness on the x-axis. For the code of the plots we refer to McGilliard et al. (2010).

Figure 4.7: The histogram on the left hand side illustrates the initial distribution of the cascade sizes for 300 simulations in the case $k = 5$ and $m = 0.225$. On the right hand side we illustrate the distribution of the final cascade size for the same set of simulations.

from left. Eventually, the both contour plots for the cascade model are just a confirmation of the observations in Section 4.1.6 by a different form of visualization.
For our purpose it is more interesting to draw a comparison with the contour plots of the simulations' outcome. For the boundary area on the interval of the mean robustness we detect a consensus between cascade model and simulations, however, for a range to the left of the critical robustness the contour plots display a discrepancy between model and simulations. While $\tilde{s}(\tau)$ is already close to 1, $\hat{s}(\tau)$ has just intermediate values.
The interpretation of $\hat{s}(\tau)$ must be taken carefully. We remind of the simulation exercise on the small network in the last section which has suggested that the final cascade size alternates in the simulations for some range of m between total collapse and no defaults. The same holds true for the consideration of larger networks. A value of 0.5 for $\hat{s}(\tau)$ does not necessarily mean that the outcomes of $\hat{s}_i(\tau)$ have been 0.5, but it means that in 50% of the cases the network has completely failed while in the other 50% no or just a few agents have defaulted.
We refer to Figure 4.7 which confirms on the right hand side that in $\tau = 50$ the simulations' outcome for $k = 5$ and $m = 0.225$ can be indeed grouped into both cases: total collapse and no or almost no defaults respectively. By bringing together this plot with the histogram on the left hand side which illustrates the distribution of the cascade sizes $\hat{s}_j(0)$ we conclude that a total collapse - with some exceptions - always occurs in the case $k = 5$ and $m = 0.225$ as soon as there is at least one defaulted agent in $\tau = 0$.
The cascade model always results in a cascade size of 1 for m less than the critical robustness as its recursive calculation procedure responds sensitive as soon as the probability of having no defaults in the network starts to decrease as explained in Section 4.2.2. This is also confirmed by the horizontal lines denoting the critical mean robustness at which we apparently observe in the simulations the first occurrence of subsequent defaults in $\tau > 0$ due to some initial defaults in $\tau = 0$.

4.3 Resume on the cascade model

The discussion has pointed out that the cascade model and its application of a binomial distribution under the recursive usage of the default probability and assumed independence between the initial robustness and the subsequent default counts - are not subtle enough to determine the expected size of a cascade reliable. It suggests a total collapse of the system in cases for which the simulations still generate outcomes without any noticeable loss.

Hence, an isolated application of the cascade model can result in misleading conclusions. Nevertheless, it has its right to exist in the procedure of Battiston et al. (2012). For the further procedure we will see that an accurate size of a cascade is actually not required but for the quantification of systemic risk it is just necessary to identify the mean robustness at which a network suffers a switch between two regimes: While in the first regime the network is stable enough so that individual defaults of some agents are absorbed without any system-wide consequence in the second regime the network becomes instable and is potentially exposed to a cascade mechanism which results in a total collapse of the system. The contour plots of the simulations in Figure 4.6 indeed show that the critical robustness, having been approximated by referring to the formula from the cascade model, indicates the mean robustness at which the cascade sizes of the simulations start to become slightly positive. Hence, the critical robustness resulting from the cascade model may be taken as an identifier of a point at which the network passes over into an instable state.
Independent of the used assumptions for getting the results in Theorem 4.1 the cascade model appears generally as naive way to model contagion effects between agents in the situation of a distressed system. Due to Equation (4.1) the cascade mechanism is based on depreciations in the agents' balance sheet if a debtor has defaulted. Certainly, the agents' robustness will be influenced by further impacts in such a situation. One possible impact might be interventions by policy maker and central banks. Their measures may stabilize the network's agents during a cascade and can prevent a total collapse of the system. Admittedly, such impacts are suppressed by the cascade model. The reasoning of Adrian and Shin (2008), for instance, may be considered as an argument against the convenience of the cascade model. So, if one wants to explicitly research the cascade mechanism in credit networks, we recommend to refer to a more complex model, however, for the procedure of Battiston et al. (2012), which is rather interested in the point at which a cascade mechanism with serious consequences for the networks can be possibly triggered, and in order to examine networks stability in general the assumed cascade mechanism may be a first attempt to get some insights how network participants effect each other in times of distressed credit markets.

5 Systemic risk in homogeneous networks

This section explains the finalizing step in the procedure of Battiston et al. (2012) which is ought to quantify the intensity of systemic risk. Up to now we have become acquainted with risk of default from an individual perspective in Section 3 while Section 4 has talked about a mechanism which considers defaults in a system-wide context. In their procedure Battiston et al. (2012) understand systemic risk as the possibility that all individuals of a network are so far weakened at the same time so that the system is exposed to a cascade mechanism with serious consequences. In our point of view this might be seen in line with the following qualitative concept of systemic risk from Kaufman and Scott (2003) which points out the boundary between individual default risk and the risk of a system wide distortion:

> Systemic risk refers to the risk or probability of breakdowns in an entire system, as opposed to breakdowns in individual parts or components, and is evidenced by comovements (correlation) among most or all of the parts.

The following section points out how Battiston et al. (2012) concatenate the concept of the individual financial robustness from Section 3 with the cascade mechanism in Section 4. The basic idea is to suppose that on the continuous time line every agent of the network starts in $t = 0$ with maximal robustness 1 and it is then measured how long it takes for the paths to reach a level at which the system-wide robustness is sufficiently low so that a cascade of large size can be potentially triggered.

5.1 Evolution of the initial distribution

Battiston et al. (2012) state in the beginning of their final section that "... *robustness is not a static quantity and that the distribution of robustness that is likely to be realized at the beginning of a cascade evolves in time[]*", i.e. it becomes somehow necessary to consider the evolution of the initial distribution in the cascade mechanism over time. In order to come up to this statement Battiston et al. (2012) refer to a two-stage process for which they assume that the initial distribution of the cascade mechanism is driven by the process ρ_i which is the financial robustness in continuous time from Definition 3.3. Actually, they synchronize the robustness $\tilde{\rho}_i(0)$ from the cascade mechanism with the distribution of ρ_i by agreeing upon

$$\tilde{\rho}_i(0) = \tilde{\rho}_i(0|t) \overset{d}{=} \rho_i(t), \tag{5.1}$$

where $\overset{d}{=}$ denotes that the left and right hand side are identical in distribution. Hence, the initial distribution of the financial robustness from the cascade mechanism is assumed to be dependent on the continuous time unit t which allows for considering its changing characteristic in the pre-cascade phase while during a cascade the financial robustness $\tilde{\rho}_i$ develops unchanged on the cascade mechanism's time scale τ. At this point Battiston et al. (2012) argue for the different time scales by alluding that the robustness evolution in a cascade-free period is rather a long-term development compared to the cascade period which "... *occurs at a time scale (e.g. days) which is much faster than the evolution of robustness* ..." in the cascade-free period.

The agreement in Equation (5.1) is based on the idea that the defaults occurring on the continuous time line correspond to the initial fraction of defaulted agents in the beginning of a cascade. Therefore we calculate by $v(t)$ the fraction of defaulted agents in t on an homogeneous network with N agents all having the same continuous time process ρ_i from Definition 3.3 if one supposes no recovery for a defaulted agent between 0 and t. It is

$$\begin{aligned} v(t) &= \frac{1}{N} E[\sum_{i \in V} \mathbb{1}_{(\rho_i(t) \leq 0)}] \\ &= \frac{1}{N} \sum_{i \in V} P(\rho_i(t) \leq 0) \\ &= P(\rho_i(t) \leq 0). \end{aligned}$$

So by agreeing upon Equation (5.1) we obtain

$$v(t) = s(0|t),$$

where $s(0|t)$ denotes the expected, initial fraction of defaults for a cascade starting on the continuous time line in t. Correspondingly we agree upon that $s(\tau|t)$ will be the fraction of defaults in step $\tau \geq 0$ within the cascade mechanism if the cascade is triggered in t for the assumed initial distribution $\tilde{\rho}_i(0|t)$.
In their given setting Battiston et al. (2012) deal with the question how long it takes in average for $m(t)$, which denotes the mean of the continuous time process ρ_i, to reach the critical robustness $m^\star$ from Section 4 so that $\tilde{s}(\tau|t)$ will converge to a value close to 1 as $\tau \to \infty$. In the following this time is defined by

$$Q_i(m^\star) = \inf\{t : m(t) = m^\star\}. \tag{5.2}$$

Obviously, for this procedure it is necessary to know the distribution of $\rho_i(t)$ for a fixed $t \in [0, +\infty)$. Recall hereto Definition 3.3, which states that ρ_i is driven by an underlying process $\hat{\rho}_i$ with dynamics

$$d\hat{\rho}_i = -\alpha q dt + \frac{\sigma}{\sqrt{k}} dW_i^\star(t),$$

and initial condition $\hat{\rho}_i(0) = 1$. By rewriting this SDE in integral form we obtain

$$\hat{\rho}_i(t) = 1 - \alpha q t + \frac{\sigma}{\sqrt{k}} W_i^\star(t),$$

that is

$$\hat{\rho}_i(t) \sim N\left(1 - \alpha q t, t\frac{\sigma^2}{k}\right)$$

however, this does not hold for $\rho_i(t)$ since it is additionally biased by the reflecting barrier at 1 and the absorbing barrier at 0. Nevertheless, Battiston et al. (2012) rely on a normality assumption and use for their purpose the assumption that

$$\rho_i(t) \sim N\left(m(t), \frac{\sigma^2}{k}\right). \tag{S1}$$

Figure 5.1: Plot 1 explains the trigger of a cascade by the decline of the mean robustness $m(t)$ of ρ_i in time. The dashed horizontal line denotes the point at which $m(t)$ reaches $m^\star$ which position on the y-axis is in turn highlighted by the horizontal line. Plot 2 zooms into Plot 1 in order to figure out the behavior of $\tilde{s}(0|t)$ and $\tilde{s}$ near $Q_i(m^\star)$. We set $\tilde{s}(10^3|t) = \tilde{s}$. $\tilde{s}$ does not entirely increase since the simulated $m(t)$ which is used as input for the calculation of $\tilde{s}$ is not monotonously decreasing but possesses slight up and down variation which is not noticeable in the plot.

holds approximately true, while $m(t)$ is not further specified. Furthermore, they assume for the sake of simplification, that

$$Q_i(m^\star) \approx E[T_i(m^\star)], \tag{S2}$$

where the right hand side of the last equation denotes the mean first passage time of process ρ_i at $m^\star$, which is obviously not in line with the actual definition of Equation (5.2).

Prior to that we'd like to use Figure 5.1 in order to recap the idea of a triggered cascade by the decrease of $m(t)$ for the specific case $k = 10$ and $\alpha = 0.075$. Plot 1 partitions the continuous time interval in a pre and post cascade phase. The first phase lasts as long as $m(t)$ is larger than $m^\star$ which position on the y-axis is highlighted by the horizontal line. Hereby $m(t)$ has been drawn from a simulation on the continuous time process ρ_i. Additionally we have plotted as a blue line the value of $\tilde{s}(0|t)$ from the cascade model based on the assumption (S1) and by the red line the final cascade size $\tilde{s}$ as it is suggested by the cascade model for the start value $\tilde{s}(0|t)$. We see that exactly in $Q_i(m^\star)$ - where we here did not use assumption (S2) but the actual definition of $Q_i(m^\star)$ from Equation (5.2) - $\tilde{s}$ explodes, which can be taken as the trigger of a cascade with serious consequences for the network.

5.2 Results on systemic risk

By the simplifying assumption (S2) we get a formula for the time at which ρ_i hits $m^\star$. Therefore, we recall Theorem 3.2, which endows us with the formula

$$E[T_i(m^\star)] = \begin{cases} \frac{\exp\left(-2\alpha q k(1-m^\star)\sigma^{-2}\right)-1}{2\alpha^2 q^2 k \sigma^{-2}} + \frac{1-m^\star}{\alpha q}, & \text{for } \alpha > 0 \\ k\sigma^{-2}(1-m^\star)^2, & \text{for } \alpha = 0. \end{cases} \tag{5.3}$$

Hence, by neglecting that $E[T_i(m^\star)]$ is not measuring the time when the mean of ρ_i reaches $m^\star$, but the expected time when the path of ρ_i reaches $m^\star$ the first time, the last formula measures in dependence of the assumed financial accelerator α and diversification degree k the intensity with which a network becomes fragile for system-wide distortion. This is going to be summarized in the next section.

5.2.1 Visualizing systemic risk

Recall Section 4.1.6 where we used to calculate approximations on $m^\star$ by Algorithm 4.1 under the assumption

$$\tilde{\rho}_i(0) \sim N\left(m, \frac{\sigma^2}{k}\right).$$

Figure 5.2: The figure illustrates $f_s^\star$ for different α. The points attached to the lines signalize the interpolation points at which $f_s^\star$ has been evaluated.

Since assumption (S1) and the agreement in Equation (5.1) imply that

$$\tilde{\rho}_i(0|t) \sim N\left(m(t), \frac{\sigma^2}{k}\right)$$

we can use the calculations on $m^\star$ from Section 4.1.6 for determining $E[T_i(m^\star)]$. In the following we will refer to the notation $\hat{m}^\star$ in order to point out that the calculations on $E[T_i(m^\star)]$ happen here just with an approximative value of $m^\star$ as we do not know a closed formula on $m^\star$. In line with Section 3.1.7 we again follow Battiston et al. (2012) and visualize the inverse of Equation (5.3), that is

$$\begin{aligned} f_s^\star(k,\alpha,\sigma,\epsilon^\star,\hat{m}^\star(k)) &:= \frac{1}{E[T_i(\hat{m}^\star(k))]} \\ &= \begin{cases} \frac{\alpha q}{\sigma^2(2\alpha q k)^{-1}(\exp(-2\alpha q k(1-\hat{m}^\star(k))\sigma^{-2})-1)+1-\hat{m}^\star(k)}, & \text{for } \alpha > 0 \\ \frac{\sigma^2}{k(1-\hat{m}^\star(k))^2}, & \text{for } \alpha = 0 \end{cases} \end{aligned}$$

and which can be taken as the intensity of observing network fragility. Hereby, $q = q(k,\alpha,\sigma,\epsilon^\star)$ remains unchanged in the same way as in Section 3.1. Battiston et al. (2012) particularly focus on the development of $f_s^\star$ with respect to k whereas the other parameters are supposed to be fixed - exceptional $\hat{m}^\star$ which is depending on k.
Figure 5.2 illustrates the systemic risk with respect to the network's diversification degree k for various assumed magnitudes of the financial accelerator effect. This visualization is similar to Plot 1 in Figure 3.1, and the available lines possess the similar progressions. This is not surprising as in both cases we used to plot the inverse of mean first passage times just for differing absorbing barriers. So, the observations in Figure 5.2 may be analogously reasoned as we did it for Plot 1 in Figure 3.1. We repeat the core of our reasoning there:

The marginal utility of risk diversification will be overlaid by a stronger increasing financial accelerator effect if the agent diversifies her credit portfolio too much.

Applied to systemic risk in the sense of Battiston et al. (2012) Figure 5.2 allows for the conclusion that portfolio diversification in the presence of a financial accelerator may - other than maybe suspected - increase the intensity of having large cascades.
This statement is slightly limited by the observation in Figure 5.2 - noticeable at least for $\alpha = 0.075$ - that $f_s^\star$ decreases again after having passed a certain diversification degree. The same holds true for the other $\alpha > 0$, however, it is hardly or not viewable in this illustration. So, other than $f^\star$ in Plot 1 of Figure 3.1, $f_s^\star$ is not converging towards αq bottom-up but from above, which changes the statement about systemic risk in the presence of a financial accelerator only less, since the maximum for $f_s^\star$ is not reached before $k = 100$ and the decrease in $f^\star$ occurs later only slowly at least for the range of α, which we used to consider. Furthermore, Battiston et al. (2012) mention in their final calibration exercise an estimated diversification degree of 17 ± 13 in the core of the

global interbank credit market. Besides they refer to the work of Soramki et al. (2007) who figure out a core of 25 institutions on the U.S. interbank market, where the average degree for the overall markets has been estimated to be 11 ± 8 there. Craig and von Peter (2010) examine the German interbank market consisting of about 2000 institutions with an average diversification degree of about 11 across the whole network. Hence, a diversification degree ranging between 0 and 100 seems to be realistic.

5.3 Discussion on the final results

The final step has not entirely been in accordance to the original idea of synchronizing ρ_i and $\tilde{\rho}_i$ but relied on the approximations (S1) and (S2). Obviously, both approximations do not hold in general and only serve as simplification respectively. We are going to check whether both simplifying assumptions are nearby or whether the results of Battiston et al. (2012) must be rethought due to a lack of accuracy. Hereby we restrict our examination to the case $\alpha = 0.075$ - the procedure may be analogously applied to other values of α.

5.3.1 Review assumption (S1)

The approximative assumption (S1)

$$\rho_i(t) \sim N\left(m(t), \frac{\sigma^2}{k}\right)$$

may be considered as the link between the model of the financial robustness in Section 3.1 and the financial robustness from the cascade model in Section 4.1, since by the agreement $\tilde{\rho}_i(0|t) \stackrel{d}{=} \rho_i(t)$, the use of $N(m(t), \sigma^2/k)$ for $\tilde{\rho}_i(0|t)$ in the cascade model is implied. However, if there is a large discrepancy between (S1) and the actual distribution of ρ_i, the calculations of $\hat{m}^\star$ which then would have been undertaken under the wrong premise that

$$\tilde{\rho}_i(0) = \tilde{\rho}_i(0|t) \sim N\left(m, \frac{\sigma^2}{k}\right)$$

holds, might lead to wrong conclusions on systemic risk. We are going to figure out whether the distribution of $\rho_i(t^\star)$ is close to $N\left(m^\star, \sigma^2/k\right)$ where $t^\star = E[T_i(m^\star)]$. Before we start with our examination procedure we like to allude that assumption (S1) may be slightly weakened by recalling that the distribution function F for $\tilde{\rho}_i(0)$ only matters for positive values within the cascade model.[9] Thus, it suffices to compare $\rho_i(t^\star)$ to a variable defined as

$$\zeta = \max(0, \zeta^\star), \qquad \zeta^\star \sim N\left(m^\star, \frac{\sigma^2}{k}\right).$$

For both $\rho_i(t^\star)$ and ζ we simulate 1000 realizations and compare these realizations by visualizing the empirical distribution functions for different diversification degrees k. For the visualization of the empirical distribution function we have referred to the function

[9] Recall the formula on G in Section 4.1.4. G relies on the distribution function of $\tilde{\rho}_i$, that was F, only for arguments on the interval $[0, 1]$.

Figure 5.3: The figure compares the empirical distribution functions $\hat{F}(\zeta), \hat{F}(\zeta_{adj})$ and $\hat{F}(\rho_i(t^\star))$ for various k on the intervall $[0,1]$. The black line is covering $F(\rho_i(t^\star))$, the blue line denotes $\hat{F}(\zeta)$ and the red line represents $\hat{F}(\zeta_{adj})$. In the simulations on $\rho_i(t^\star)$ it has been taken $\alpha = 0.075$. For the simulation of $\rho_i(t^\star)$ we used to simulate the path of ρ_i on the interval $[0, t^\star]$ a thousand times by relying on the Euler-Maruyama scheme with discretization size 0.01. For incorporating the reflecting barrier the simulation has been adapted to Algorithm A of Asmussen et al. (1995).

`ecdf()`, which is part of the "stats" package in R Core Team (2013). In Figure 5.3 we observe that for different diversification degrees k the empirical distribution function $\hat{F}(\zeta)$ for the simulations of ζ clearly deviates from the empirical distribution function of the simulations of $\rho_i(t^\star)$ which is denoted by $\hat{F}(\rho_i(t^\star))$. Particularly, the plots allow to conclude a massive discrepancy between the variance of $\rho_i(t^\star)$ and $\zeta^\star$.
Recall that the underlying process of ρ_i is $\hat{\rho}_i$, and we know from above that

$$\hat{\rho}_i(t^\star) \sim N\left(1 - \alpha q t^\star, t^\star \frac{\sigma^2}{k}\right)$$

holds true. So, in our point of view the choice of

$$\zeta_{adj} = \max(0, \zeta^\star_{adj}), \qquad \zeta^\star_{adj} \sim N\left(m^\star, t^\star \frac{\sigma^2}{k}\right)$$

appears more accurate as an approximation on $\rho_i(t^\star)$. And indeed, the empirical functions of $\hat{F}(\zeta_{adj})$ fits $\hat{F}(\rho_i(t^\star))$ better than it has been the case for $\hat{F}(\zeta)$.[10] By looking at the red and black lines in Figure 5.3 it is conspicuous that for larger k the approximation error of (S1) becomes smaller. The reason is given by the decreasing coefficient in the diffusion term of $\hat{\rho}_i$ as k becomes large so that - as reasoned in Section 3.1.8 and illustrated by Lemma 3.7 - the process ρ_i is less influenced by the reflecting barrier and thus approaches towards a Brownian motion with drift. Eventually, (S1) may disturb if diversification degrees are small but becomes acceptable for k being sufficiently large.

5.3.2 Review assumption (S2)

Assumption (S2)

$$Q_i(m^\star) \approx E[T_i(m^\star)].$$

will not hold in general since for the process ρ_i with negative drift and upper reflecting barrier we would expect that the mean first passage time of hitting $m^\star$ occurs earlier than the time at which the mean of ρ_i reaches $m^\star$.

[10] For our purpose the simulations on ζ_{adj} have not been obtained by a completely new simulation run but by referring to the simulations on $\zeta^\star$ and adjust them in the following manner:

$$\zeta^\star_{adj} = (\zeta^\star - m^\star)\sqrt{t^\star} + m^\star.$$

Thus, we ensure that the observed deviation between $F(\rho_i(t^\star))$ and $\hat{F}(\zeta)$ is not a matter of simulations but indeed subject to a wrongly assumed time-invariant variance.

Figure 5.4: The figure compares for the case $\alpha = 0.075$ the estimate on $Q_i(\hat{m}^\star)$ with $E[T_i(\hat{m}^\star)]$ which value is known from Equation (5.3). Additionally we illustrate by the red line the values of $\tilde{Q}_i(\hat{m}^\star)$. The points on the lines denote the interpolation points k where values have been calculated.

We want to check the quality of this approximation by drawing $Q_i(\hat{m}^\star)$ via simulations and compare it to $E[T_i(\hat{m}^\star)]$ whose value is available to us as explicit formula. In our considerations we restrict ourself to the case $\alpha = 0.075$ - the remaining cases may be examined analogously.
We estimate $Q_i(m^\star)$ by simulating ρ_i with an applied discretization size of $d = 0.01$ on the Euler-Maruyama scheme, where due to the reflecting barrier the simulation algorithm has been adapted to Algorithm A of Asmussen et al. (1995). For each interpolation point $\hat{t}$ in the discretization scheme $\mathcal{D} = \{0, 0.01, 0.02, \dots\}$ we calculate the paths' empirical mean over $M = 1000$ simulations so that we have estimates $\hat{m}(\hat{t})$ for $E[\rho_i(\hat{t})] = m(\hat{t})$ if $\hat{t} \in \mathcal{D}$. The value

$$\hat{Q}_i(\hat{m}^\star) = \inf_{\hat{t} \in \mathcal{D}} \{\hat{t} : \hat{m}(\hat{t}) \leq \hat{m}^\star\}$$

is then our estimate on $Q_i(m^\star)$ and eventually compared to $E[T_i(\hat{m}^\star)]$.
The results are illustrated in Figure 5.4. Not surprisingly, the curve of $E[T_i(\hat{m}^\star)]$ runs permanently below the curve of $\hat{Q}_i(\hat{m}^\star)$, which makes sense as reasoned above. By considering the inverse of both curves in Figure 5.4, the intensity $f_s^\star = 1/E[T_i(\hat{m}^\star)]$ must be actually shifted downwards towards $1/\hat{Q}_i(\hat{m}^\star)$ - particularly for $k < 20$, however, the results of Battiston et al. (2012), that systemic risk is increasing again for increasing diversification degree holds true anyway.
The advantage of referring to $1/E[T_i(\hat{m}^\star)]$ instead of $1/Q_i(\hat{m}^\star)$ is that the former one endows us with a closed formula for systemic risk while the latter one must be identified via simulations.
To complete this review we give an alternative approximative attempt, namely

$$m(t) = 1 - \alpha q t$$

which would be the the mean of the process ρ_i disregarding any barriers. This is simple enough to make an inference on $Q_i(\hat{m}^\star)$ by solving just for t in

$$1 - \alpha q t = m^\star.$$

The results to this calculation denoted by $\tilde{Q}_i(\hat{m}^\star)$ is plotted in Figure 5.4 as red line. For small k the deviation to the black line arises from the reflecting barrier while for larger k we see that $\tilde{Q}_i(\hat{m}^\star)$ approaches towards the simulated version of $Q_i(\hat{m}^\star)$. Eventually, this is a repetition of the results in Figure 5.3, which suggests for k large enough that ρ_i is less effected by the reflecting barrier and close to a normal distribution.

5.3.3 Additional remark on assumptions (S1) and (S2)

We altogether conclude that the procedure of concatenating the continuous financial robustness with the financial robustness from the cascade model under the two assumptions

(S1) and (S2) bias the results, however, if one exclusively wants to cave out the core results of Battiston et al. (2012), that is the principle of systemic risk as a non-decreasing function in k without desiring an accurate quantification of systemic risk the pragmatical choice of (S1) and S(2) suffices for this purpose.
Nevertheless it is nearby to give some thought on assumptions (S1) and (S2), and to figure out whether there are some analytical instruments which allow to abstain from these biasing simplifications. The difficulty hereby is to deal with the reflecting and absorbing barrier of ρ_i. There are concepts, for instance, applied in the valuation of exotic options, which display some properties about processes with barriers and which might be used to avoid the biasing simplifications (S1) and (S2). However, most of the concepts, which we found, disregard either the reflecting or the absorbing barrier but do not consider both at the same time. As an example we refer to a formula which indicates the mean of

$$\eta_i(t) := \inf_{0 \leq s \leq t} (1 - \hat{\rho}_i(s)). \tag{5.4}$$

By recalling the composition of the process ρ_i in Definition 3.3, we can then consider for instance

$$\tilde{m}(t) = 1 - \alpha q t + E[\eta_i(t)].$$

as the mean of the process ρ_i without considering an absorption barrier. The formula for the mean of $\eta_i(t)$ is stated by the next lemma.

Lemma 5.1. *The mean of the process $\eta_i(t)$ as defined in Equation (5.4) is given by*

$$E[\eta_i(t)] \quad = \quad \frac{\alpha q t}{2} + \frac{\sigma^2}{k \alpha q} \left[\left(2\Phi(\sqrt{2}w) - 1 \right) \left(\frac{1}{2} + w^2 \right) + \frac{w e^{-w^2}}{\sqrt{\pi}} \right]$$

where

$$w := -\frac{\alpha q \sqrt{tk}}{\sqrt{2}\sigma}.$$

Proof. The calculation follows the procedure of Magdoni-Ismail et al. (2004) and starts with the consideration of

$$S(t) := \sup_{0 \leq s \leq t} x(s), \qquad I(t) := \inf_{0 \leq s \leq t} x(s).$$

Hereby $x(s)$ denotes a common Brownian motion W with drift, that is

$$x(s) = \mu s + \sigma_x W(s)$$

with $\mu \in \mathcal{R}, \sigma > 0$ and $x(0) = 0$. As, for instance, proved by Dominé (1996) the density of the absorption time

$$\lambda := \inf_{s \geq 0} \{x(s) = b\}$$

at an absorbing barrier $b > 0$ turns out to be the one of an inverse Gaussian distribution, namely

$$f_\lambda(u) = \frac{b}{(2\pi\sigma_x^2 u^3)^{1/2}} \exp\left(-\frac{(b-\mu u)^2}{2\sigma_x^2 u}\right).$$

We use that

$$1 - F_{S(t)}(b) = P\left(S(t) \geq b\right) = P\left(\lambda \leq t\right),$$

where $F_{S(t)}$ denotes the distribution function of $S(t)$. Hence,

$$\begin{aligned} E[S(t)] &= \int_0^{+\infty} 1 - F_{S(t)}(b) db \\ &= \int_0^{+\infty} P\left(\lambda \leq t\right) db \\ &= \int_0^{+\infty} \int_0^t f_\lambda(u) du db. \end{aligned}$$

Solving the integrals yields as described by Magdoni-Ismail et al. (2004)

$$E[S(t)] = \frac{\mu t}{2} + \frac{\sigma_x^2}{\mu}\left[\left(2\Phi(\sqrt{2}w_\mu) - 1\right)\left(\frac{1}{2} + w_\mu^2\right) + \frac{w_\mu e^{-w_\mu^2}}{\sqrt{\pi}}\right],$$

where $w_\mu = \frac{\mu\sqrt{t}}{\sqrt{2}\sigma_x}$. For

$$x^-(s) = -\mu s + \sigma W(s) \qquad S^-(t) := \sup_{t\in[0,t]} x^-(s)$$

we have the identity $E[I(t)] = -E[S^-(t)]$. Now, recall that $\hat{\rho}_i$ is assumed to start in 1, hence,

$$1 - \hat{\rho}_i(s) \stackrel{d}{=} x(s)$$

with $\mu = \alpha q$ and $\sigma_x = \frac{\sigma}{\sqrt{k}}$. Thus,

$$\begin{aligned} E[\eta(t)] &= E[\inf_{0\leq s\leq t}(1 - \hat{\rho}_i(s))] \\ &= \frac{\alpha q t}{2} + \frac{\sigma^2}{k\alpha q}\left[\left(2\Phi(\sqrt{2}w) - 1\right)\left(\frac{1}{2} + w^2\right) + \frac{w e^{-w^2}}{\sqrt{\pi}}\right] \end{aligned}$$

with $w = -\frac{\alpha q\sqrt{tk}}{\sqrt{2}\sigma}$.

$\square$

In Figure 5.5 we compare $\tilde{m}(t)$ with $m(t)$, where the latter denotes the mean of the process ρ_i under the assumption that $\alpha = 0.075$, $\sigma = 0.25$ and $k = 20$. $m(t)$ has been determined by a simulation run with both a reflecting barrier at 1 and an absorbing barrier at 0. Apparently, both curves coincide in the beginning but diverge for proceeding t which is

Figure 5.5: The figure illustrates the curves of $m(t)$ being the simulated mean of the process ρ_i and $\tilde{m}(t)$ resulting from Lemma 5.1. For the simulations we applied an Euler-Maruyama scheme with discretization size of 0.01, where due to the reflecting barrier the simulation algorithm has been adapted to Algorithm A of Asmussen et al. (1995).

due to the stated problem of the disregarded absorbing barrier in the calculations of $\tilde{m}(t)$. Nevertheless, for more accurate results with respect to systemic risk it might be worth to replace (S2) by the usage of $\tilde{m}(t)$ as long as $Q_i(m^*)$ is sufficiently small. Anyway, deriving results on the distribution of $\rho_i(t)$ in order to entirely eliminate (S1) and (S2) would require more efforts. Literature for further examination might be given by Dominé (1996), who states the first passage density of a process with a reflecting and an absorbing barrier which could be possibly used to derive the accurate expectation of ρ_i in the manner of Lemma 5.1.
At this point we content ourselves that, though (S1) and (S2) are not perfect, the core result of Battiston et al. (2012) with respect to systemic risk may be yet confirmed - particularly if the diversification degree k is large enough.

5.3.4 Resume

As a resume of the last discussion we suggest to modify the cascade model if it is linked to the continuous time process ρ_i, namely, by stating that for the initial distribution $\tilde{\rho}_i(0|t)$ it must be rather assumed

$$\tilde{\rho}_i(0|t) \sim N\left(m(t), t\frac{\sigma^2}{k}\right). \tag{5.5}$$

We adjust the variance term by incorporating the time unit t and take herewith into account that the variance of $\rho_i(t)$ expands over time. Of course this assumption is still only approximative correct since $\rho_i(t) \sim N(m(t), t\sigma^2/k)$ does not hold in general, but with respect to the visualization of the empirical distribution functions in the previous section it appears more accurate than the previously assumed $\tilde{\rho}_i(0|t) \sim N(m(t), \sigma^2/k)$. Due to this adjustment the results on systemic risk in Figure 5.2 must be revalued since $\hat{m}^\star$ has been originally calculated under the wrong variance term - disregarding the time unit. We refit $f_s^\star$ by now assuming Equation (5.5) which also requires in turn a readjustment of Algorithm 4.1 in order to find the critical robustness $m^\star$.

Figure 5.6: Plot 1 compares the version of $\hat{m}^\star$ before variance adjustment and its recalculated version after variance adjustment denoted by $\hat{m}^\star(adj)$. Plot 2 points out the change in systemic risk after having considered the adjustment on the variance term for the case $\alpha = 0.075$. The blue line is systemic risk $f_s^\star$ before adjustment, the black line denotes systemic risk after adjustment. The points on the lines signalize the interpolation points k where values have been calculated.

Algorithm 5.1: Adjustment of Algorithm 4.1 with respect to the time-dependent variance

---

```
Data: m_1, m_2, s, tol, τ*
Result: m
m = (m_1 + m_2)/2;
Calculate t_m = E[T_i(m)] ;
Calculate s(τ*) w.r.t. mean robustness m and variance t_m σ²/k;
while m_2 − m_1 > tol and s(τ*) ≠ s do
    m = (m_1 + m_2)/2;
    Calculate t_m = E[T_i(m)] ;
    Calculate s(τ*) w.r.t. mean robustness m and variance t_m σ²/k;
    if s(τ*) > s then
        m_1 = m;
    else
        if s(τ*) < s then
            m_2 = m;
        else
            STOP
        end
    end
end
```

The algorithm has been extended by calculating additionally $E[T_i(m)]$ which is then incorporated into the variance term. This is done by referring to the assumption (S1) and sure enough just approximatively correct.

The recalculation of $f_s^\star$, which is denoted by $f_s^\star(adj)$ and relies on the newly approximated critical robustness $\hat{m}^\star(adj)$, may be found in Figure 5.6. We observe that systemic risk is under the new assumption throughout higher than before. One can consider the increase of the systemic risk as the part of uncertainty resulting from time uncertainty which has not been considered before in the examination of Battiston et al. (2012). However, we also allude once again that the structure of systemic risk with respect to k does not change its face.

6 The cascade mechanism in heterogeneous networks

We remind that the examination of systemic risk by Battiston et al. (2012) has taken place on networks with an homogeneous network structure. Assumptions A(1.2) and A(1.3) state that the networks' agents are alike in their lending behavior and it is not possible to find any agent or a group of agents whose position in the credit market significantly differs from the position of other agents. However, there is evidence that interbank credit markets possess rather an heterogeneous structure than the assumed homogeneous structure of Battiston et al. (2012). We refer to the work of Langfield et al. (2012) who detect a core-periphery structure for the UK interbank system. Bech and Atalay (2008) in turn point out an heterogeneous structure for the U.S. funds market and Craig and von Peter (2010) see evidence for a core-periphery structure on the German interbank market. Their common conclusion is that the interbank market contains a core of a few but large banks who act as intermediaries for the remaining smaller banks, which account for the majority of the interbank market. For instance Langfield et al. (2012) write that ... *[s]maller domestic banks [...] are predominantly exposed to the large UK banks. This implies that the large UK banks act as intermediaries to cross-border transactions.*" Craig and von Peter (2010) describe the banks on the core as *money center banks* which provide services to the periphery in their role as market maker "... *including government securities, FX, derivatives, and offshore markets.*" Battiston et al. (2012) themselves allude that their procedure is rather a method for examining the core of a credit network which most likely fulfills the assumptions A(1.2) and A(1.3).
In this last section we first present some results of Craig and von Peter (2010) about the core-periphery structure on the German interbank market and do a brief data analysis. Afterwards, we'd like to point out the misleading effect of assuming homogeneity on a credit network though an heterogeneous structure is present. For measuring network stability we refer to the idea of the cascade mechanism from Section 4 and compare the magnitude of such an effect on a credit network under assumed homogeneity and heterogeneity respectively. In order to consider heterogeneous networks we will extend the setting of Battiston et al. (2012) so that groups of varying diversification degrees may be considered which is applied to observed data about connectivity in the German interbank market. Eventually, for simulating heterogeneous networks we will suggest a straightforward extension of Algorithm 3.1.

6.1 Core-periphery structure on the German interbank market

The work of Craig and von Peter (2010) aims to find statistically significance for a core-periphery structure on the German interbank market. First they reason theoretically for a tiered structure of the interbank market by arguing the existence of two sorts of banks forming the interbank market: top-tier banks and lower-tier banks. In a purely tiered network top-tier banks can potentially lend and borrow to any bank on the network while lower-tier banks exclusively interact with top-tier banks but not with banks from the own tier. Craig and von Peter (2010) formalize the tiered structure of a network as a model by relying on a blockmodel technique for the network's adjacency matrix A as

$$A = \begin{pmatrix} CC & CP \\ PC & PP \end{pmatrix}. \tag{6.1}$$

Hereby, the adjacency matrix indicates - as in the procedure of Battiston et al. (2012)- the bilateral credit relationships between the agents of the network by entries of ones and zeros. While the block CC lists the relationships among the top-tier banks, PP provides the informations about the relationships between the banks of the lower-tier. Correspondingly, the blocks PC and CP cover the exchange of credits between both tiers. Craig and von Peter (2010) require the blocks to fulfill the following properties in their model :

- P(1) CC is a matrix of ones exceptional the zero diagonal,
- P(2) PP is a matrix of zeros,
- P(3) CP is row regular, that is, each row is covered by at least one 1,
- P(4) PC is column regular, that is, each column is covered by at least one 1.

While due to P(1) all top-tier banks are linked among each other, P(2) excludes any direct credit relationships between banks of the lower-tier. On the other side P(3) and P(4) ensure the intermediary function of the top-tier banks, so that the linkage between banks of the lower-tier may be found as an indirect linkage via the top-tier banks. We allude that due to the given properties a lower-tier bank does not have to act necessarily as a creditor on the interbank market.
A network with properties P(1) - P(4) serves only as an illustrative model of a perfectly tiered interbank market, however, one cannot expect that the set of agents on a real world interbank credit network may be partitioned under consideration of the available linkages into a top-tier and lower-tier so that P(1) - P(4) are entirely fulfilled. Craig and von Peter (2010) rather take this strict structure as a benchmark for measuring whether an observed real world network approaches such a perfectly tiered structure. As a measure for the distance between the theoretical perfectly tiered network and the real world network they refer to an error score which involves penalties for every deviation of the real world network from the properties P(1) - P(4).
It is not obviously clear on networks of large size which set of banks should determine the set of top-tier banks so that the distance to a perfectly tiered network is minimized. In order to find this optimal set Craig and von Peter (2010) apply an algorithm which first initializes an arbitrary choice of top-tier banks and then subsequently adjusts this set towards the set of top-tier banks which eventually minimizes the error score. The partition of the network into top-tier and lower-tier found by the algorithm does not have to fulfill properties P(1) and P(2), however it is shown that properties P(3) and P(4) indeed hold.
For a final validation Craig and von Peter (2010) design a test procedure which aims to assess whether the distance of the real world network to a perfectly tiered structure is significantly small. Hereby, they generate random networks, which are not supposed to have a tiered structure, and compare the distance between a perfectly tiered network and theses random networks to the distance observed for the real world network. In the context of this test procedure they indeed point out significance for a tiered structure on the German interbank market. In their analysis they also conclude that this structure is resilient over time and they found low probabilities for a bank switching from one tier to the other tier. Commonly, Craig and von Peter (2010) call the found top-tier of the

German interbank market as the core of the market, while the lower-tier is ought to form the periphery.

6.1.1 Data analysis

Craig and von Peter (2010) rely on a data set consisting the quarterly observed linkage within the German interbank market for a sample period which starts in the first quarter of 1999 and goes up to the fourth quarter in 2007. We have been endowed with the counts of out and in-degrees in the blocks CP, PC and PP in the second quarter 2003 which allows for giving an overview about connectivity in core and periphery respectively. We introduce a definition which extends Definition 2.5 in order to allow for distinguishing between a possible core and periphery on a general interbank market.

Definition 6.1. *For a directed, weighted graph which represents according to Definition 2.5 a credit interbank market we additionally define the set V^C and V^P where both sets partition the set V, and $|V^C| := N^C, |V^P| := N^P$. We agree upon the following notation for the partition of the agents' credit portfolios:*

$$V_{out}(i) = \begin{cases} V_{out}^{CC}(i) \cup V_{out}^{CP}(i) = \{j \in V^C : (i,j) \in E\} \cup \{j \in V^P : (i,j) \in E\}, & \text{if } i \in V^C \\ V_{out}^{PP}(i) \cup V_{out}^{PC}(i) = \{j \in V^P : (i,j) \in E\} \cup \{j \in V^C : (i,j) \in E\}, & \text{if } i \in V^P. \end{cases}$$

The out-degrees are given by

$$d_{out}(i) = \begin{cases} d_{out}^{CC}(i) + d_{out}^{CP}(i) = |\{j \in V^C : (i,j) \in E\}| + |\{j \in V^P : (i,j) \in E\}|, & \text{if } i \in V^C \\ d_{out}^{PP}(i) + d_{out}^{PC}(i) = |\{j \in V^P : (i,j) \in E\}| + |\{j \in V^C : (i,j) \in E\}|, & \text{if } i \in V^P. \end{cases}$$

The in-degrees are analogously composed by

$$d_{in}(i) = \begin{cases} d_{in}^{CC}(i) + d_{in}^{CP}(i) = |\{j \in V^C : (j,i) \in E\}| + |\{j \in V^P : (j,i) \in E\}|, & \text{if } i \in V^C \\ d_{in}^{PP}(i) + d_{in}^{PC}(i) = |\{j \in V^P : (j,i) \in E\}| + |\{j \in V^C : (j,i) \in E\}|, & \text{if } i \in V^P. \end{cases}$$

The data which we have at hand are the vectors O_{CP}, O_{PC}, O_{PP}, I_{PC}, I_{CP} and I_{PP} where

$$O_{z\hat{z}} = \begin{pmatrix} d_{out}^{z\hat{z}}(1) \\ \vdots \\ d_{out}^{z\hat{z}}(N^z) \end{pmatrix} \text{ and } I_{z\hat{z}} = \begin{pmatrix} d_{in}^{z\hat{z}}(1) \\ \vdots \\ d_{in}^{z\hat{z}}(N^z) \end{pmatrix} \tag{6.2}$$

for $z, \hat{z} \in \{C, P\}$ contain the observed out- and in-degrees for the blocks CP, PC and PP by recalling the adjacency matrix in Equation (6.1).

The examination of the second quarter 2003 has displayed a core set V^C with cardinality $N^C = 45$ and a periphery of size $N^P = 1,757$. Craig and von Peter (2010) point out that over the sample period the adjacency matrices for the overall network have been rather sparse and only 0.61% of all cells have been 1. If we take the overall network size $N = 1,802$ in the second quarter 2003 and excluding the possibilities of self-loops in the network, we will thus get an averaged out-degree of

$$\hat{d}_{out} = 0.0061 \frac{N^2 - N}{N} \approx 11. \tag{6.3}$$

At the same time Craig and von Peter (2010) allude that for the block CC, which denotes the linkage just among core banks, 66% of the possible cells had an entry of 1. We thus conclude an averaged out-degree for the linkage among core banks of

$$\hat{d}_{out}^{CC} = 0.66\frac{(N^C)^2 - N^C}{N^C} \approx 29. \tag{6.4}$$

By taking into account that core banks will possibly issue credits into the periphery as well, the averaged diversification degree for core banks is noticeably higher than for banks from the periphery. We can confirm this by relying on the data at hand which suggest that[11]

$$\begin{pmatrix} \hat{d}_{out}^{CC} & \hat{d}_{out}^{CP} \\ \hat{d}_{out}^{PC} & \hat{d}_{out}^{PP} \end{pmatrix} = \begin{pmatrix} 29.00 & 114.11 \\ 6.56 & 0.98 \end{pmatrix}. \tag{6.5}$$

Consequently, we have

$$\begin{aligned} \hat{d}_{out}^{C} &:= \hat{d}_{out}^{CC} + \hat{d}_{out}^{CP} = 143.11, \\ \hat{d}_{out}^{P} &:= \hat{d}_{out}^{PP} + \hat{d}_{out}^{PC} = 7.54 \end{aligned}$$

which points out the difference between the out-degrees in the core and in the periphery respectively. Also note that we have

$$\begin{aligned} \hat{d}_{out}^{CC} &> \hat{d}_{out}^{PC} \\ \hat{d}_{out}^{CP} &> \hat{d}_{out}^{PP}, \end{aligned}$$

which illustrates that in the mean a creditor from V^C will always establish on the same set of potential debtors more bilateral relationships than an agent from V^P. Due to the stable structure of the network over time we may suppose that this does not only hold for the observed quarter but also in other periods.

The data at hand additionally provide the absolute credit exposures in billion EUR which are given by

$$\begin{pmatrix} X^{CC} & X^{CP} \\ X^{PC} & X^{PP} \end{pmatrix} := \begin{pmatrix} 321.2 & 442.0 \\ 147.8 & 17.0 \end{pmatrix}. \tag{6.6}$$

For the sake of clarification, $X^{CC} = 321.2$ means hereby, for instance, that in the observation period core banks had outstanding credit receivables worth 321.2 billion EUR to other banks in the core. If we connect the absolute credit exposure to the the size of core and periphery, we may deduce by

$$\begin{pmatrix} \hat{x}^{CC} & \hat{x}^{CP} \\ \hat{x}^{PC} & \hat{x}^{PP} \end{pmatrix} = \begin{pmatrix} X^{CC}/N^C & X^{CP}/N^C \\ X^{PC}/N^P & X^{PP}/N^P \end{pmatrix} = \begin{pmatrix} 7.1378 & 9.8222 \\ 0.0841 & 0.0097 \end{pmatrix} \tag{6.7}$$

an estimator for the total credit exposure per agent in each block and additionally by

$$\begin{pmatrix} \hat{x}_b^{CC} & \hat{x}_b^{CP} \\ \hat{x}_b^{PC} & \hat{x}_b^{PP} \end{pmatrix} = \begin{pmatrix} \hat{X}^{CC}/(N^C\hat{d}_{out}^{CC}) & X^{CP}/(N^C\hat{d}_{out}^{CP}) \\ X^{PC}/(N^P\hat{d}_{out}^{PC}) & X^{PP}/(N^P\hat{d}_{out}^{PP}) \end{pmatrix} = \begin{pmatrix} 0.2461 & 0.0861 \\ 0.0128 & 0.0099 \end{pmatrix} \tag{6.8}$$

an estimator for each bilateral credit exposure in each block. These figures suggest that core and periphery banks do not only differ with respect to the counts of issued credits but also with respect to credit exposure in absolute values.

[11] As stated above the data at hand list the out and in-degrees in the blocks PP, PC, and CP. These data suffice to calculate the averaged out-degrees for the three blocks. The block CC is missing in the data set due to reasons of confidentiality, so, the number of $\hat{d}_{out}^{CC}$ is taken from Equation (6.4).

6.2 Setting of the cascade mechanism in heterogeneous networks

In order to analyze the German interbank market with the procedure of Battiston et al. (2012) one opportunity would be to neglect the periphery and to exclusively examine the core by assuming homogeneity in the sense of assumptions A(1.2) and A(1.3). Then it would be nearby to set $k = 29$. For a consideration of the overall network a first attempt would be to set $k = 11$ by following the result in Equation (6.3). However, as stated above this would disregard that we can find by the core a group of creditors who are more active and interconnected than the group of periphery banks. It is not clear whether there is a difference on the fragility of the network if one refers to the core-periphery structure in stead of assuming an homogeneous structure. We are going to extend the procedure of Battiston et al. (2012) which will be flexible enough to consider two groups with differing diversification degrees as it occurs in the core-periphery structure and measure network stability by relying on the cascade mechanism of Section 4.

6.2.1 Redefining the interbank market as a graph

For the extension we avail ourselves of Definition 6.1 which has revised the originally homogeneous interbank market from Definition 2.5 with respect to a partition of the interbank market agents into sets V^C and V^P respectively. Our idea is to consider agents from the same set as homogeneous, however, out-degrees may vary between agents from different sets.

- A(2.1) The interbank credit market consists of $0 < N < \infty$ agents and it is $|V^C| = N^C$ and $|V^P| = N^P$ respectively.
- A(2.2) We agree upon that

$$\begin{aligned}
d_{out}^{CC}(i) &= k_{CC}, \quad 0 \leq k_{CC} \leq N^C - 1, \quad \forall i \in V^C, \\
d_{out}^{CP}(i) &= k_{CP}, \quad 0 \leq k_{CP} \leq N^P, \quad \forall i \in V^C, \\
d_{out}^{PP}(i) &= k_{PP}, \quad 0 \leq k_{PP} \leq N^P - 1, \quad \forall i \in V^P, \\
d_{out}^{PC}(i) &= k_{PC}, \quad 0 \leq k_{PC} \leq N^C, \quad \forall i \in V^P.
\end{aligned}$$

- A(2.3) Each agent acts as a creditor, i.e. it must be

$$\begin{aligned}
\max\{k_{CC}, k_{CP}\} &> 0 \\
\max\{k_{PP}, k_{PC}\} &> 0.
\end{aligned}$$

So far, we have adjusted assumptions A(1.1) and A(1.2) by adding some more out-degrees so that we can now differ between agents from two groups with differing diversification. Note that these assumptions imply that we still consider agents from the same groups as homogeneous with respect to their out-degrees, hence, we still arrange a mean field approach, however, this approach now allows for two groups with different means.
Recall that assumption A(1.3) has determined the weight of every issued credit with respect to the overall credit portfolio of an agent. In the mean field approach of Battiston

et al. (2012) it has been assumed that these weights are uniformly $\frac{1}{k}$ for every credit issued to the network. For our extension a naive way would be to set by default

$$w_{(i,j)} = \begin{cases} \frac{1}{k_{CC}+k_{CP}}, & \text{if, } i \in V^C \\ \frac{1}{k_{PP}+k_{PC}}, & \text{if, } i \in V^P, \end{cases} \tag{6.9}$$

for all $j \in V$. Let us refer to the example of the German interbank market in order to validate this assumption. By relying on the rounded values in Equation (6.5) we choose as a specification of assumption A(2.2)

$$\begin{pmatrix} k_{CC} & k_{CP} \\ k_{PC} & k_{PP} \end{pmatrix} = \begin{pmatrix} 29 & 114 \\ 7 & 1 \end{pmatrix}.$$

The agreement in Equation (6.9) would then yield that any credit issued by a core bank would compose $\frac{1}{143} = 0.7\%$ of her credit portfolio while we have for a periphery bank the percentage $\frac{1}{8} = 12.5\%$. This means, for instance, in the case of a creditor from the core that the default of a debtor from the periphery would imply the same loss to the credit portfolio as the default of a debtor from the core. However, the figures in Equation (6.7) and (6.8) rather suggest that credits to the core and periphery differ with respect to the part in an agent's credit portfolio. We denote by $\hat{w}_{z\hat{z}}$ the estimated weight on the German interbank market of a credit in the credit portfolio of an agent $i \in V^z$, issued to a debtor $j \in V^{\hat{z}}$ where $z, \hat{z} \in \{C, P\}$ and calculate

$$\begin{pmatrix} w_{CC} & w_{CP} \\ w_{PC} & w_{PP} \end{pmatrix} = \begin{pmatrix} \frac{\hat{x}^{CC}}{\hat{x}^{CC}+\hat{x}^{CP}} \frac{1}{k_{CC}} & \frac{\hat{x}^{CP}}{\hat{x}^{CC}+\hat{x}^{CP}} \frac{1}{k_{CP}} \\ \frac{\hat{x}^{PC}}{\hat{x}^{PP}+\hat{x}^{PC}} \frac{1}{k_{PC}} & \frac{\hat{x}^{PP}}{\hat{x}^{PP}+\hat{x}^{PC}} \frac{1}{k_{PP}} \end{pmatrix} = \begin{pmatrix} 1.45\% & 0.51\% \\ 12.81\% & 10.32\% \end{pmatrix}.$$

These figures suggest that credits to the core should sustain more weights in the agents' credit portfolios. Hence, we agree upon the following assumption.

- A(2.4) The percentage of a credit issued to agent j on the credit portfolio of i is given by the weight

$$w_{(i,j)} = \begin{cases} w_{CC} = e_{CC}\frac{1}{k_{CC}}, & \text{if, } i, j \in V^C \\ w_{CP} = (1 - e_{CC})\frac{1}{k_{CP}}, & \text{if, } i \in V^C, j \in V^P \\ w_{PP} = e_{PP}\frac{1}{k_{PP}}, & \text{if, } i, j \in V^P \\ w_{PC} = (1 - e_{PP})\frac{1}{k_{PC}}, & \text{if, } i \in V^C, j \in V^P \end{cases}$$

if k_{CC}, k_{CP}, k_{PC} and $k_{PP} > 0$, where

$$e_{CC} := \frac{x^{CC}}{x^{CC} + x^{CP}}$$
$$e_{PP} := \frac{x^{PP}}{x^{PP} + x^{PC}},$$

denotes the percentage of the credit portfolio which is exposed to agents from the same group. If $k_{z\hat{z}} = 0$ for $z, \hat{z} \in \{C, P\}$, we set $w_{z\hat{z}} = 0$.

Note that by agreeing upon assumptions A(2.1) - A(2.4), the requirements of a perfectly tiered structure of the interbank market as given by properties P(1) - P(4) are not met. While P(1), P(2) and P(3) can be possibly fulfilled by determining the out-degrees correspondingly, we can never guarantee the validity of P(4) since this property is a matter of in-degrees which are not addressed in assumptions A(2.1) - A(2.4). However, we will later state a probability of having property P(4) fulfilled in a network resulting from A(2.1) - A(2.4).
For approaching towards a perfectly tiered structure of the network as suggested by Craig and von Peter (2010), it might be constructive to further assume $k_{PP} = 0$. In this case the name periphery would become adequate for the set V^P. In the following, we will suppose that connectivity in V^P is rather low but will not fully restrict ourselves to $k_{PP} = 0$ in order to maintain the option to consider the German interbank market for which the data analysis in Section 6.1.1 suggests that some connectivity among periphery banks - though it looks low - is present.

6.2.2 Adjusted cascade mechanism

The idea of this section is to conform the cascade model from Section 4 to the refined setting of a network with groups of different out-degrees and weights. We start this section by readjusting the cascade mechanism of Definition 4.1 for our purposes and with respect to a network fulfilling assumptions A(2.1) - A(2.4). Therefore we rely on the notations

$$\begin{aligned} V_{out}^{CC}(i) &:= \{j \in V^C : (i,j) \in E\} \\ V_{out}^{CP}(i) &:= \{j \in V^P : (i,j) \in E\} \end{aligned}$$

for $i \in V^C$ and equivalently

$$\begin{aligned} V_{out}^{PP}(i) &:= \{j \in V^P : (i,j) \in E\} \\ V_{out}^{PC}(i) &:= \{j \in V^C : (i,j) \in E\} \end{aligned}$$

for $i \in V^P$. The adjusted definition of the cascade mechanism is given as follows.

Definition 6.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $i \in V^C, j \in V^P$. Then the financial robustness of agent i and j respectively are $\mathbb{R}$-valued sequences $(\tilde{\rho}_i^C(\tau))_{\tau \in \mathbb{N}_0}$ and $(\tilde{\rho}_j^P(\tau))_{\tau \in \mathbb{N}_0}$ where the initial values $\tilde{\rho}_i^C(0)$ and $\tilde{\rho}_j^P(0)$ are random variables defined on the given probability space with given distribution functions F_i^C and F_j^P. For $\tau > 0$ we set*

$$\begin{aligned} \tilde{\rho}_i^C(\tau+1) &= \tilde{\rho}_i^C(0) - a_C \sum_{q \in V_{out}^{CC}(i)} w_{CC} \mathbb{1}_{(\tilde{\rho}_q^C(\tau) \leq 0)} - a_C \sum_{q \in V_{out}^{CP}(i)} w_{CP} \mathbb{1}_{(\tilde{\rho}_q^P(\tau) \leq 0)} \\ \tilde{\rho}_j^P(\tau+1) &= \tilde{\rho}_j^P(0) - a_P \sum_{q \in V_{out}^{PC}(j)} w_{PC} \mathbb{1}_{(\tilde{\rho}_q^C(\tau) \leq 0)} - a_P \sum_{q \in V_{out}^{PP}(j)} w_{PP} \mathbb{1}_{(\tilde{\rho}_q^P(\tau) \leq 0)}, \end{aligned}$$

for $a_P, a_C > 0$.

This definition extends Definition 4.1 by incorporating a second sequence, thus, we may distinguish between the financial robustness for agents from V^C and V^P during a cascade. The parameters a_P and a_C allow to assume different effects of a defaulted debtor on the financial robustness of an agent from V^C and V^P respectively, which gives some more flexibility to the model, however, in our later application we will always set $a_C = a_P = 1$. The adjustment on the cascade mechanism with respect to the new, heterogeneous structure of the network targets on constructing a two dimensional sequence $(s_C(\tau), s_P(\tau))_{\tau=0,1,\ldots}$, where the first component denotes the expected cascade size on the set V^C in step τ and the second component contains the expected cascade size on the set V^P, i.e.

$$s_C(\tau) := \frac{E[\sum_{i\in V^C} \mathbb{1}_{(\tilde{\rho}_i^C(\tau)\leq 0)}]}{N^C}, \tag{6.10}$$

$$s_P(\tau) := \frac{E[\sum_{i\in V^P} \mathbb{1}_{(\tilde{\rho}_i^P(\tau)\leq 0)}]}{N^P}. \tag{6.11}$$

Obviously, the expected fraction of defaults in the overall network is then given by

$$s(\tau) = \frac{N^C}{N^C + N^P} s_C(\tau) + \frac{N^P}{N^C + N^P} s_P(\tau).$$

Similar to assumption A(1.7) it is assumed that the initial state of the financial robustness is independently and identically distributed for all agents from the same group.

- A(2.5) The initial robustnesses $\tilde{\rho}_1^C(0), \ldots, \tilde{\rho}_{N^C}^C(0), \tilde{\rho}_{N^C+1}^P(0), \ldots, \tilde{\rho}_{N^C+N^P}^P(0)$ are supposed to be stochastically independent with distribution function F^C for all $i \in V^C$ and F^P for all $i \in V^P$.

In Section 4.1.3 we have pointed out the difficulty to compute expressions as in Equation (6.10) and (6.11) respectively. The results of Battiston et al. (2012) for the univariate case stated by Theorem 4.1 may be analogously derived in the two dimensional case as well. Therefore we introduce the following relaxing assumptions.

- A(2.6) Let $z \in \{C, P\}$. For an agent $i \in V^z$ we assume

  $$P(\mathbb{1}_{\tilde{\rho}_i^z(\tau)\leq 0)} = 1) = p_z(\tau),$$

  where $p_z(\tau) \in [0, 1]$. Additionally, it is supposed that $\mathbb{1}_{(\tilde{\rho}_i^z(\tau)\leq 0)}$ are stochastically independent for all $z \in \{C, P\}$ and $i \in V^z$.

- A(2.7) For $i \in V^z$, where $z \in \{C, P\}$ it is supposed that $\tilde{\rho}_i^z(0)$ and $\mathbb{1}_{(\tilde{\rho}_i^{\hat{z}}(\tau)\leq 0)}$ are independent for all $\tau > 0$ and $\hat{z} \in \{C, P\}$.

Having the discussion about assumptions A(1.8) and A(1.9) in mind, we allude that A(2.6) and A(2.7) likewise do not hold in general for the setting of Definition 6.2. Nevertheless, we rely on these assumptions in order to derive the analogon for Theorem 4.1. Therefore we use the notations

$$\tilde{s}_C(\tau) := \frac{E[\sum_{i\in V^C} \mathbb{1}_{(\tilde{\rho}_i^C(\tau)\leq 0)}]}{N^C} \tag{6.12}$$

$$\tilde{s}_P(\tau) := \frac{E[\sum_{i\in V^P} \mathbb{1}_{(\tilde{\rho}_i^P(\tau)\leq 0)}]}{N^P} \tag{6.13}$$

in order to point out that the following results for Equations (6.10) and (6.11) are based on the falsifying assumptions A(1.8) and A(1.9). Furthermore, we are going to use the notations

$$
\begin{aligned}
d_i^{CC}(\tau) &= \sum_{q \in V_{out}^{CC}(i)} \mathbb{1}_{(\tilde{\rho}_q^C(\tau) \leq 0)} \\
d_i^{CP}(\tau) &= \sum_{q \in V_{out}^{CP}(i)} \mathbb{1}_{(\tilde{\rho}_q^P(\tau) \leq 0)},
\end{aligned}
$$

if $i \in V^C$ and equivalently for $j \in V^P$ it is agreed upon

$$
\begin{aligned}
d_i^{PP}(\tau) &= \sum_{q \in V_{out}^{PP}(i)} \mathbb{1}_{(\tilde{\rho}_q^P(\tau) \leq 0)} \\
d_i^{PC}(\tau) &= \sum_{q \in V_{out}^{PC}(i)} \mathbb{1}_{(\tilde{\rho}_q^C(\tau) \leq 0)}.
\end{aligned}
$$

Theorem 6.1. *Let G^w be a graph that fulfills assumptions A(2.1) - A(2.4). For $\tau \geq 0$ the financial robustnesses $\tilde{\rho}_i^C(\tau)$ and $\tilde{\rho}_i^P(\tau)$ are defined by Definition 6.2 and A(2.5) - A(2.7) are assumed to hold true. F^C and F^P denote the distribution functions of $\tilde{\rho}_i^C(0)$ and $\tilde{\rho}_i^P(0)$ respectively. Then, for $\big(\tilde{s}_C(\tau), \tilde{s}_P(\tau)\big)$ as defined in Equations* (6.12) *and* (6.13) *we have the following results:*

(i) $\big(\tilde{s}_C(0), \tilde{s}_P(0)\big) = \big(F^C(0), F^P(0)\big)$,

(ii) $\tilde{s}_z(\tau+1) = \sum_{x=0}^{k_{zz}} \sum_{y=0}^{k_{z\hat{z}}} \Bigg[F^z\left(a_z w_{zz} x + a_z w_{z\hat{z}} y\right) \binom{k_{zz}}{x} \tilde{s}_z(\tau)^x \Big(1 - \tilde{s}_z(\tau)\Big)^{k_{zz}-x}$
$\binom{k_{z\hat{z}}}{y} \tilde{s}_{\hat{z}}(\tau)^y \Big(1 - \tilde{s}_{\hat{z}}(\tau)\Big)^{k_{z\hat{z}}-y} \Bigg]$, *for* $\tau \geq 0$ *and* $z \in \{C, P\}$, $\hat{z} \in \{C, P\} \setminus \{z\}$,

(iii) the sequence $\Big(\big(\tilde{s}_C(\tau), \tilde{s}_P(\tau)\big)\Big)_{\tau=0,1,\ldots}$ *converges to a value* $\big(\tilde{s}_C, \tilde{s}_P\big) \in [0,1]^2$.

Proof. For claim (i) we use the definition of $\tilde{s}_C(\tau)$ in Equation (6.12) and assumption A(2.5) that $\tilde{\rho}_i^C(0)$ is identically distributed for all $i \in V^C$. Hence, we obtain for $\tau = 0$

$$
\begin{aligned}
\tilde{s}_C(0) &= \frac{E[\sum_{i \in V^C} \mathbb{1}_{(\tilde{\rho}_i^C(0) \leq 0)}]}{N^C} \\
&= \frac{\sum_{i \in V^C} P(\tilde{\rho}_i^C(0) \leq 0)}{N^C} \\
&= P(\tilde{\rho}_1^C(0) \leq 0) \\
&= F^C(0),
\end{aligned}
$$

where we have supposed without loss of generality that $1 \in V^C$. The calculations on the claim $\tilde{s}_P(0) = F^P(0)$ work similar.
For claim (ii) we again rely exclusively on $\tilde{s}_C(\tau)$ and allude that the argumentation for $\tilde{s}_P(\tau)$ is the same again. Per definition we have for $\tilde{s}_C(\tau+1)$ and $\tau \geq 0$,

$$
\tilde{s}_C(\tau+1) = \frac{E[\sum_{i \in V^C} \mathbb{1}_{(\tilde{\rho}_i^C(\tau+1) \leq 0)}]}{N^C}
$$

$$
\begin{aligned}
&= \frac{\sum_{i \in V^C} P(\mathbb{1}_{(\tilde{\rho}_i^C(\tau+1) \le 0)} = 1)}{N^C} \\
&= P(\mathbb{1}_{(\tilde{\rho}_1^C(\tau+1) \le 0)} = 1),
\end{aligned}
$$

since $\mathbb{1}_{(\tilde{\rho}_i^C(\tau+1) \le 0)}$ are supposed to be identically distributed for all $i \in V^C$. With respect to Definition 6.2 the last one may be equivalently written as

$$
P\left(\tilde{\rho}_1^C(0) \le a_C w_{CC} \sum_{q \in V_{out}^{CC}(1)} \mathbb{1}_{(\tilde{\rho}_q^C(\tau) \le 0)} + a_C w_{CP} \sum_{q \in V_{out}^{CP}(1)} \mathbb{1}_{(\tilde{\rho}_q^P(\tau) \le 0)}\right),
$$

which we rewrite more convenient as

$$
P\left(\tilde{\rho}_1^C(0) \le a_C w_{CC} d_1^{CC}(\tau) + a_C w_{CP} d_1^{CP}(\tau)\right).
$$

We apply the formula of total probability and obtain

$$
\begin{aligned}
\tilde{s}_C(\tau+1) = \sum_{x=0}^{k_{CC}} \sum_{y=0}^{k_{CP}} \Bigg[& P\left(\tilde{\rho}_1^C(0) \le a_C w_{CC} d_1^{CC}(\tau) + a_C w_{CP} d_1^{CP}(\tau) \middle| d_1^{CC}(\tau) = x, d_1^{CP}(\tau) = y\right) \\
& P\left(d_1^{CC}(\tau) = x, d_1^{CP}(\tau) = y\right) \Bigg].
\end{aligned}
$$

Due to A(2.6) and A(2.7) the last one becomes

$$
\begin{aligned}
\tilde{s}_C(\tau+1) = \sum_{x=0}^{k_{CC}} \sum_{y=0}^{k_{CP}} \Bigg[& F^C\left(a_C w_{CC} x + a_C w_{CP} y\right) \binom{k_{CC}}{x} \tilde{s}_C(\tau)^x \Big(1 - \tilde{s}_C(\tau)\Big)^{k_{CC}-x} \\
& \binom{k_{CP}}{y} \tilde{s}_P(\tau)^y \Big(1 - \tilde{s}_P(\tau)\Big)^{k_{CP}-y} \Bigg]
\end{aligned}
$$

which is the claim of (ii).
For claim (iii) we first allude that $\big(\tilde{s}_C(\tau), \tilde{s}_P(\tau)\big) \in [0,1]^2$ per definition. Secondly, we show as an induction base for the following procedure that $\tilde{s}_C(1) \ge \tilde{s}_C(0)$:

$$
\begin{aligned}
\tilde{s}_C(1) &= \sum_{x=0}^{k_{CC}} \sum_{y=0}^{k_{CP}} \Bigg[F^C\left(a_C w_{CC} x + a_C w_{CP} y\right) \binom{k_{CC}}{x} \tilde{s}_C(0)^x \Big(1 - \tilde{s}_C(0)\Big)^{k_{CC}-x} \\
&\quad \binom{k_{CP}}{y} \tilde{s}_P(0)^y \Big(1 - \tilde{s}_P(0)\Big)^{k_{CP}-y} \Bigg] \\
&\ge F^C(0) \sum_{x=0}^{k_{CC}} \binom{k_{CC}}{x} \tilde{s}_C(0)^x \Big(1 - \tilde{s}_C(0)\Big)^{k_{CC}-x} \sum_{y=0}^{k_{CP}} \binom{k_{CP}}{y} \tilde{s}_P(0)^y \Big(1 - \tilde{s}_P(0)\Big)^{k_{CP}-y} \\
&= F^C(0) \\
&= \tilde{s}_C(0)
\end{aligned}
$$

In the same way one may show that $\tilde{s}_P(1) \ge \tilde{s}_P(0)$ is true. For the further argumentation we insert a function $H_z \colon \mathbb{R}^2 \to \mathbb{R}$,

$$
H_z(u_1, u_2) = \sum_{x=0}^{k_{zz}} \sum_{y=0}^{k_{z\hat{z}}} \left[F^z\left(a_z w_{zz} x + a_z w_{z\hat{z}} y\right) \binom{k_{zz}}{x} u_1^x \Big(1 - u_1\Big)^{k_{zz}-x} \binom{k_{z\hat{z}}}{y} u_2^y \Big(1 - u_2\Big)^{k_{z\hat{z}}-y} \right]
$$

for $z \in \{C, P\}$ and $\hat{z} \in \{C, P\} \setminus \{z\}$. Obviously, it is

$$H_z\left(\tilde{s}_z(\tau), \tilde{s}_{\hat{z}}(\tau)\right) = \tilde{s}_z(\tau + 1).$$

By recalling the argumentation for determining the derivative of function G in the proof on Theorem 4.1 we deduce for the partial derivative of H

$$\frac{\partial H_z}{\partial u_1} = \sum_{x=1}^{k_{zz}} \sum_{y=0}^{k_{z\hat{z}}} \Bigg\{ \Big[F^z\left(a_z w_{zz}(x+1) + a_z w_{z\hat{z}} y\right) - F^z\left(a_z w_{zz} x + a_z w_{z\hat{z}} y\right) \Big] \\ \binom{k_{zz}}{x} (k_{zz} - x) u_1^x \Big(1 - u_1\Big)^{k_{zz}-x-1} \binom{k_{z\hat{z}}}{y} u_2^y \Big(1 - u_2\Big)^{k_{z\hat{z}}-y} \Bigg\}$$

which is greater or equal zero for $\big(u_1, u_2\big) \in [0,1]^2$. Due to symmetry we also have

$$\frac{\partial H_z}{\partial u_2} \geq 0.$$

Hence, if ∇H_z denotes the gradient of H_z it is

$$\nabla H_z(u_1, u_2) \geq 0$$

for all $\big(u_1, u_2\big) \in [0,1]^2$.
Due to the mean value theorem we then know for $\big(\tilde{s}_z(\tau), \tilde{s}_{\hat{z}}(\tau)\big)$ and $\big(\tilde{s}_z(\tau-1), \tilde{s}_{\hat{z}}(\tau-1)\big) \in (0,1)^2$ that there must be $\lambda \in (0,1)$ with

$$\big(\tilde{s}_1, \tilde{s}_2\big) = \lambda\left(\tilde{s}_z(\tau), \tilde{s}_{\hat{z}}(\tau)\right) + (1-\lambda)\left(\tilde{s}_z(\tau-1), \tilde{s}_{\hat{z}}(\tau-1)\right) \in (0,1)^2$$

so that

$$H_z(\tilde{s}_z(\tau), \tilde{s}_{\hat{z}}(\tau)) - H_z(\tilde{s}_{\hat{z}}(\tau-1), \tilde{s}_z(\tau-1)) = \nabla H_z(\tilde{s}_1, \tilde{s}_2)^T \begin{pmatrix} \tilde{s}_z(\tau) - \tilde{s}_z(\tau-1) \\ \tilde{s}_{\hat{z}}(\tau) - \tilde{s}_{\hat{z}}(\tau-1) \end{pmatrix}$$

holds true, where the last one is greater or equal zero by recalling the stated value of the gradient and the present induction hypothesis $\tilde{s}_z(\tau) \geq \tilde{s}_z(\tau-1)$ and $\tilde{s}_{\hat{z}}(\tau) \geq \tilde{s}_{\hat{z}}(\tau-1)$. Thus, the last yields

$$\tilde{s}_z(\tau+1) = H_z(\tilde{s}_C(\tau), \tilde{s}_P(\tau)) \geq H_z(\tilde{s}_C(\tau-1), \tilde{s}_P(\tau-1)) = \tilde{s}_z(\tau)$$

For the marginal conditions $s_z(\tau) = 0, s_{\hat{z}}(\tau) = 0, s_z(\tau-1) = 1$ or $s_{\hat{z}}(\tau-1) = 1$ function H_z reduces to an univariate function in one variable so that we can examine monotonicity by relying on function G in Theorem 4.1. For the other possible marginal conditions the monotonicity is shown by the definition of H_z as it has been executed for deriving the induction base which is skipped here.
To sum up, we have a sequence $\Big(\big(\tilde{s}_C(\tau), \tilde{s}_P(\tau)\big)\Big)_{\tau\geq 0}$ which components monotonously increase on the bounded set $[0,1]^2$, hence, by applying the Bolzano-Weierstarss theorem, we find that there must be a point $\big(\tilde{s}_C, \tilde{s}_P\big) \in [0,1]^2$ so that the claim

$$\big(\tilde{s}_C(\tau), \tilde{s}_P(\tau)\big) \to \big(\tilde{s}_C, \tilde{s}_P\big)$$

holds for $\tau \to \infty$. □

6.2.3 Simulating an heterogeneous network structure

Analogously to Section 3.2.4 we want to simulate an adjacency matrix of an interbank market being in line with assumptions A(2.1) - A(2.4). Therefore, we rely on the block-model technique of Craig and von Peter (2010) and define the weighted adjacency matrix A^w as follows.

$$A^w = \begin{pmatrix} A^w_{CC} & A^w_{CP} \\ A^w_{PC} & A^w_{PP} \end{pmatrix},$$

where A^w_{CC} denotes a $N^C \times N^C$ matrix, A^w_{CP} is $N^C \times N^P$, A^w_{PC} is $N^P \times N^C$ and A^w_{PP} is a $N^P \times N^P$ matrix all having entries of zeros or the corresponding weights for each block as stated in assumption A(2.4). By being endowed with the out-degrees k_{CC}, k_{CP}, k_{PC}, k_{PP} and the exposure percentages e_{CC} and e_{PP}, we generate A^w by extending Algorithm 3.1 as follows.

Algorithm 6.1: Generating a weighted adjacency matrix which represents a heterogeneous credit network.

Data: Parameter N^C, N^P, k_{CC}, k_{CP}, k_{PC}, k_{PP}, e_{CC} and e_{PP}
Result: Adjacency matrix A^w with weights
Initialize a $(N^C + N^P) \times (N^C + N^P)$ matrix A^w with zero entries;
Initialize block matrices $A^w_{CC}, A^w_{CP}, A^w_{PC}$ and A^w_{PP} with zero entries and dimensions as above;
for i = *1 to* N^C **do**
  Select randomly k_{CC} numbers from the set $\{1, \dots N^C\} \setminus \{i\}$ and save them in *indC*;
  Select randomly k_{CP} numbers from the set $\{1, \dots N^P\}$ and save them in *indP*;
  if $k_{CC} > 0$ **then**
    $A^w_{CC}[i, indC] = e_{CC} \frac{1}{k_{CC}}$;
  end
  if $k_{CP} > 0$ **then**
    $A^w_{CP}[i, indP] = (1 - e_{CC}) \frac{1}{k_{CP}}$;
  end
end
for i = *1 to* N^P **do**
  Select randomly k_{PP} numbers from the set $\{1, \dots N^P\} \setminus \{i\}$ and save them in *indP*;
  Select randomly k_{PC} numbers from the set $\{1, \dots N^C\}$ and save them in *indC*;
  if $k_{PP} > 0$ **then**
    $A^w_{PP}[i, indP] = e_{PP} \frac{1}{k_{PP}}$;
  end
  if $k_{PC} > 0$ **then**
    $A^w_{PC}[i, indC] = (1 - e_{PP}) \frac{1}{k_{PC}}$;
  end
end
Compose adjacency matrix A^w by the blocks $A^w_{CC}, A^w_{CP}, A^w_{PC}$ and A^w_{PP}.

Figure 6.1: The figure illustrates Network 1,2 and 3. The blue nodes compose the set V^C.

For the random selection of the debtors we suggest - as in Algorithm 3.1 - the usage of the function `sample()` from R Core Team (2013).
In Figure 6.1 we illustrate three alternative networks which would be in line with assumptions A(2.1) - A(2.4), that is, they possess two groups with agents of different connectivity. The following table reports the chosen out-degrees for the networks:

	k_{CC}	k_{CP}	k_{PP}	k_{PC}
Network 1	3	1	1	1
Network 2	3	4	4	3
Network 3	4	2	0	1

Additionally, we have chosen $N^C = 5$ and $N^P = 10$. Note, that the out-degrees in network 2 are also in line with assumption A(1.2) of homogeneous networks, namely

$$d_{out}(i) = 7$$

for all $i \in V$. However, the block of the set V^C is denser than the block of V^P. In particular, with respect to the procedure in Algorithm 6.1 and recalling the proof of Lemma 3.8 we have

$$P\left(j \in V_{out}(i), j, i \in V^C\right) = \frac{3}{4} > \frac{4}{9} = P\left(j \in V_{out}(i), j, i \in V^P\right),$$

that means, for any pair of agents from the same group there is a higher probability for an interlinkage within the set V^C than in V^P. Nevertheless, network 2 in Figure 6.1 does not give the impression of a tiered structure - particularly a periphery cannot be identified.
For network 3 we have chosen the out-degrees so that properties P(1) - P(3) of a perfectly tiered network in the context of Craig and von Peter (2010) are fulfilled. Furthermore, we set in network 3 the out-degree k_{PP} to 0. Consequently, we can eventually recognize a factual core-periphery structure with the core formed by the blue nodes.
The probability for receiving a network from Algorithm 6.1, which additionally fulfills property P(4), can be explicitly determined. Of course, for $k_{PC} = N^C$ this probability becomes 1 and for $k_{PC} = 0$ we have probability 0. If $0 < k_{PC} < N^C$ we will obtain the probability

$$\begin{aligned}
P\left(d_{in}^{CP}(i) > 0 \ \forall i \in V^C\right) &= P\left(\bigcap_{i \in V^C} \left\{d_{in}^{CP}(i) > 0\right\}\right) \\
&= 1 - P\left(\bigcup_{i \in V^C} \left\{d_{in}^{CP}(i) = 0\right\}\right) \\
&= 1 - \sum_{z=k_{PC}}^{N^C-1} P\left(\left|\left\{i \in V^C : d_{in}^{CP}(i) = 0\right\}\right| = N^C - z\right)
\end{aligned}$$

$$= 1 - \sum_{z=k_{PC}}^{N^C-1} \binom{N^C}{N^C - z} \left(\frac{\binom{z}{k_{PC}}}{\binom{N^C}{k_{PC}}} \right)^{N^P},$$

where for the meaning of $d_{in}^{CP}(i)$ we recall Definition 6.1. This probability turns out to be close to 1 for network 2 and 0.40 for network 1 and 3 respectively.
Finally, we extend Lemma 3.8 with respect to the variables $d_{in}^{CC}(i), d_{in}^{CP}(i), d_{in}^{PC}(i)$ and d_{in}^{PP} from Definition 6.1.

Lemma 6.1. *Having given a weighted adjacency matrix A^w generated by Algorithm 6.1. Then, it is*

$$\begin{aligned}
d_{in}^{CC}(i) &\sim Bin(N^C - 1, \frac{k_{CC}}{N^C - 1}) \qquad \forall i \in V^C, \\
d_{in}^{CP}(i) &\sim Bin(N^P, \frac{k_{PC}}{N^C}) \qquad \forall i \in V^C, \\
d_{in}^{PC}(i) &\sim Bin(N^C, \frac{k_{CP}}{N^P}) \qquad \forall i \in V^P, \\
d_{in}^{PP}(i) &\sim Bin(N^P - 1, \frac{k_{PP}}{N^P - 1}) \qquad \forall i \in V^P.
\end{aligned}$$

Proof. The proof works as the proof for Lemma 3.8 whereas, for the off-diagonal blocks A_{CP}^w and A_{PC}^w we do not have to regard cells which require zero entries. □

As in the aftermath to Lemma 3.8 we allude that Lemma 6.1 states an agent's in-degrees unconditioned on the in-degrees of some other agents from the same block. If we are informed about the realized in-degrees for some agent, the distribution for the in-degree of another agent from the same block will change since we do not have independence for the in-degrees of the same block. However, the in-degrees for different blocks are - by the construction of Algorithm 6.1 - supposed to be independent.

6.3 Application: German interbank market

Section 6.2.2 has extended the concept of Section 4 in order to consider the cascade mechanism within a credit network, which can be partitioned into two groups of different out-degrees and exposure to the interbank market as assumed by Section 6.2.1. We are going to use this extension in order to figure out, whether network stability is affected by an heterogeneous structure of its agents. Hereby we are going to assess this stability by measuring the expected cascade size for networks with heterogeneous structure in comparison to the size on homogeneous networks. This examination can be motivated from two different points of view: On the one hand we see whether network stability changes if the exposure of the credit network is not distributed homogeneously over each agent in the network but rather heterogeneously as it is the case on the German interbank market. On the other hand we can draw conclusions what happens if one examines a network with a lack of information on its structure. While we are endowed by a widely detailed list of links within the German interbank market, such data are normally not available for public use. If at all, one knows the averaged degree of diversification for the overall network, that is according to Equation (6.3) on the German interbank market k = 11. From the data at hand we know that due to the given core-periphery structure this is just a rough estimation. Moreover, the credit exposure is apparently concentrated on a few agents and we can find a group of debtors who will account to more weight in the portfolios of the network's creditors. However, as mentioned, this information are commonly not provided to the public and due to the lack of information one might be tempted to refer to the procedure of Battiston et al. (2012) and assume $k = 11$ as a diversification degree for all agents on the network and uniformly $\frac{1}{k}$ as a weight for the issued credits. By relying on the data of the German interbank market, our approach in the next section will be based on this problem and tries to figure out, whether the homogeneity assumption generates cascade sizes which deviate noticeably from the cascade sizes in an heterogeneous structure.

6.3.1 Starting point of the application

We now present the setting on which we are going to undertake our application. The idea is to create for the given German interbank market three different settings for which we calculate the final cascade size. For all settings we will examine the German interbank market, that is we consider a network where its overall size is given by $N = 1802$. In the following we specify the three different settings in which we make different assumptions with respect to weights and out-degrees on the network. The settings may be taken as the result of having different stages of knowledge about the structure of the German interbank market starting with almost no information in setting one.

Setting 1

In setting 1 we assume for the out-degrees

$$k = 11$$

for all $i \in V$ which is according to Equation (6.3) the averaged out-degree for the overall German interbank market. Furthermore we agree upon that this setting fulfills assumption A(1.3), i.e. every issued credit composes uniformly $\frac{1}{11} \cdot 100\%$ of the agents' credit portfolios. The total credit exposure on the interbank market, that is according to Equation (6.6) $X = 928$ billion EUR, is supposed to be uniformly distributed on every existing credit, hence, each credit amounts to

$$x_{b,1} = \frac{X}{kN} = \frac{928}{11 \cdot 1802} = 0.0468 \tag{6.14}$$

billion EUR.

This setting can be taken as the one which comes into consideration if one has just limited information for the overall network like total exposure or the overall average for the out-degrees but not any information about a possible partition of the networks into groups with different out-degrees or varying credit exposure. This setting would be in line with assumptions A(1.1) - A(1.3) for the procedure of Battiston et al. (2012)

Setting 2

This setting relies on the idea of Section 6.2.1 and corresponds to assumptions A(2.1) - A(2.4). Compared to Setting 1 we now have some more information about the interbank market. Particularly, we are provided by the information that the adjacency matrix of the network may be partitioned into a core V^C and a periphery V^P where the sizes of these groups are given by $N^C = 45$ and $N^P = 1757$. In setting 2 we assume too now know the averaged out-degrees for the four different blocks which are given in Equation (6.5). Correspondingly, we set

$$\begin{pmatrix} k_{CC} & k_{CP} \\ k_{PC} & k_{PP} \end{pmatrix} = \begin{pmatrix} 29 & 114 \\ 7 & 1 \end{pmatrix}.$$

Furthermore we are informed about the credit exposure in each block. The figures can be found in Equation (6.6) and read

$$\begin{pmatrix} X^{CC} & X^{CP} \\ X^{PC} & X^{PP} \end{pmatrix} = \begin{pmatrix} 321.2 & 442.0 \\ 147.8 & 17.0 \end{pmatrix}.$$

For the weights of the credit links we take the results of the calculations in Section 6.2.1, that is

$$\begin{pmatrix} w_{CC} & w_{CP} \\ w_{PC} & w_{PP} \end{pmatrix} = \begin{pmatrix} 1.45\% & 0.51\% \\ 12.81\% & 10.32\% \end{pmatrix}.$$

The absolute values for each credit in the different blocks are assumed to be given in billion EUR by[12]

$$\begin{pmatrix} x_{b,2}^{CC} & x_{b,2}^{CP} \\ x_{b,2}^{PC} & x_{b,2}^{PP} \end{pmatrix} = \begin{pmatrix} X^{CC}/(N^C k^{CC}) & X^{CP}/(N^C k^{CP}) \\ X^{PC}/(N^P k^{PC}) & X^{PP}/(N^P k^{PP}) \end{pmatrix} = \begin{pmatrix} 0.2461 & 0.0862 \\ 0.0120 & 0.0097 \end{pmatrix}. \tag{6.15}$$

Setting 3

Setting 3 may be considered as the situation with the most filtrated information. We rely on the vectors O_{CP}, O_{PC} and O_{PP}, which have been defined in Equation (6.2) and which contain the observed out-degrees in the blocks CP, PC and PP. Thus, we do not have to rely on averages but can construct an adjacency matrix with out-degrees as actually observed on the German interbank market. However, the data are somewhat restricted: First we have not been endowed by the vector O_{CC}, i.e. for the block CC we do not know the out-degrees individually but can only refer to the average of 29 for this block. We pragmatically create O_{CC} by setting $d_{out}^{CC}(i) = 29$ for all $i \in V^C$. Secondly, we must allude that the vectors O_{PC} and O_{PP} are not ordered in the sense that the i-th element in both vectors possesses the out-degrees for one and the same agent. That means if we choose the first element of O_{PC} as the out-degree of the first periphery agent in the block PC, we do not know for sure the out-degree of this agent on the block PP and can only guess the adequate out-degree in vector O_{PP}. In order to construct an adjacency matrix we agree upon that the elements of O_{PC} and O_{PP} are given in a descending order meaning that the

[12]The figures slightly deviate from the figures in Equation (6.8) since we here use rounded out-degrees in the numerator.

first periphery agent is assumed to be simultaneously the largest creditor on both blocks PC and PP.

For the distribution of the credit exposure we can only make inferences as in the previous settings. We assume that the credit exposure in each block is uniformly distributed to the sum of the available links, that is

$$\begin{pmatrix} x_{b,3}^{CC} & x_{b,3}^{CP} \\ x_{b,3}^{PC} & x_{b,3}^{PP} \end{pmatrix} = \begin{pmatrix} X^{CC} / \sum_{i \in V^C} d_{out}^{CC}(i) & X^{CP} / \sum_{i \in V^C} d_{out}^{CP}(i) \\ X^{PC} / \sum_{i \in V^P} d_{out}^{PC}(i) & X^{PP} / \sum_{i \in V^P} d_{out}^{PP}(i) \end{pmatrix}.$$

The portfolio weights of each issued credit may vary from agent to agent though, for one and the same agent the weights only differ between the blocks. We define

$$\begin{aligned} w_{CC}(i) &= \frac{x_{b,3}^{CC}}{d_{out}^{CC}(i) x_{b,3}^{CC} + d_{out}^{CP}(i) x_{b,3}^{CP}}, \ w_{CP}(i) = \frac{x_{b,3}^{CP}}{d_{out}^{CC}(i) x_{b,3}^{CC} + d_{out}^{CP}(i) x_{b,3}^{CP}}, \ \text{if } i \in V^C \\ w_{PP}(i) &= \frac{x_{b,3}^{PP}}{d_{out}^{PP}(i) x_{b,3}^{PP} + d_{out}^{PC}(i) x_{b,3}^{PC}}, \ w_{PC}(i) = \frac{x_{b,3}^{PC}}{d_{out}^{PP}(i) x_{b,3}^{PP} + d_{out}^{PC}(i) x_{b,3}^{PC}}, \ \text{if } i \in V^C. \end{aligned}$$

There are agents $i \in V^P$ for which it is $d_{out}^{PP}(i) = d_{out}^{PC}(i) = 0$. These agents do not act as creditor on the interbank market and in this case we nearby set 0 for $w_{PP}(i)$ and $w_{PC}(i)$. While we have already settled by Algorithms 3.1 and 6.1 opportunities to simulate the networks of setting 1 and 2, we now give an extension which allows to generate a network being in line with setting 3. The principle is the same as in the two previous algorithms, i.e. for each agent the algorithm generates a random composition of the agent's credit portfolio, whereas all agents of the same block have the same probability to become debtor. The extension may be eventually found in the input parameters which are now the individual out-degrees of the vectors O_{CC}, O_{CP}, O_{PC}, O_{PP} and the weights as defined above.

Algorithm 6.2: Generating a weighted adjacency matrix which represents the network of the German interbank market.

Data: Parameter N^C, N^P, O_{CC}, O_{CP}, O_{PC}, O_{PP} and weights
Result: Weighted adjacency matrix A^w
Initialize a $(N^C + N^P) \times (N^C + N^P)$ matrix A^w with zero entries;
Initialize block matrices $A^w_{CC}, A^w_{CP}, A^w_{PC}$ and A^w_{PP} with zero entries and dimensions as above;
for i = *1 to* N^C **do**
  Select randomly $O_{CC}[i]$ numbers from the set $\{1, \ldots N^C\} \setminus \{i\}$ and save them in $indC$;
  Select randomly $O_{CP}[i]$ numbers from the set $\{1, \ldots N^P\}$ and save them in $indP$;
  $A^w_{CC}[i, indC] = w_{CC}(i)$;
  $A^w_{CP}[i, indP] = w_{CP}(i)$;
end
for i = *1 to* N^P **do**
  Select randomly $O_{PP}[i]$ numbers from the set $\{1, \ldots N^P\} \setminus \{i\}$ and save them in $indP$;
  Select randomly $O_{PC}[i]$ numbers from the set $\{1, \ldots N^C\}$ and save them in $indC$;
  $A^w_{PP}[i, indP] = w_{PP}(i)$;
  $A^w_{PC}[i, indC] = w_{PC}(i)$;
end
Compose adjacency matrix A^w by the blocks $A^w_{CC}, A^w_{CP}, A^w_{PC}$ and A^w_{PP}.

6.3.2 Procedure of the application

Our application arranges the consideration of the expected cascade size for the three different settings. Recall Section 4.2.3 where we used to compare the cascade size from the formula of the cascade model with the outcome of simulating the cascade mechanism. Due to both Theorems 4.1 and 6.1 we are endowed by a formula - based on falsifying assumptions - for the expected cascade size within the settings 1 and 2. The corresponding results for the expected fraction of defaults from these formulas are denoted by $\tilde{s}_1$ and $\tilde{s}_2$ in the following. Such a formula is not available for setting 3. Next to the expected cascade sizes from the Theorems 4.1 and 6.1 we will undertake simulations as in Section 4.2.3 for all three settings.
In order to examine the cascade mechanism we have to determine for both formula and simulation the distribution of the initial financial robustness. Hereby we agree upon that in all three settings it is assumed that

$$\tilde{\rho}_i(0) \sim N(m, \sigma^2) \tag{6.16}$$

for all $i \in V$ identically and independently distributed. Particularly, we do not distinguish between core and periphery in the settings 2 and 3. It might be arguable to assume the same mean and variance for the financial robustness on the core and periphery. For instance, one could argue for a higher variance on the core due to the riskier business

models of the larger core banks. However, that is just speculation and as we do not know it will be better to keep σ constant.
An identical mean for the core and periphery becomes reasonable, if take for the initial value of the financial robustness in the cascade mechanism, the ratio of a bank's equity to the receivables on the interbank market. We may then argue that such a ratio is determined by regulation authorities which do not distinguish between banks from the core and periphery but do apply the same rules for all banks on the interbank market. Consequently, the expected robustness in the beginning of a cascade mechanism should not differ. Due to the data of Deutsche Bundesbank[13] this ratio has been about 21% for the overall German banking industry in the second quarter 2003. In the context of our cascade mechanism this ratio means that in average 21% of the credit portfolio must default before an agent fails herself.
We are going to simulate a cascade for each setting 50 times while in each run the adjacency matrix is newly generated by the corresponding algorithms. The random vector $\tilde{\rho}(0)$ is generated once in each run and then used as an input vector for the cascade simulations of each setting, i.e. the deviating results of the cascade size are just due to network structure.
The cascade sizes are calculated and simulated respectively for fixed $\sigma = 0.1$ but a variation of the mean robustness m on the interval $[0.15, 0.3]$. In the following $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_3$ denote the observed fraction of defaults in the whole network averaged over all simulation runs in setting 1, 2 and 3. The iteration of the cascade mechanism has been limited to $\tau = 15$, however, the simulations of the three settings have already terminated for smaller τ, hence, $\tau = 15$ indeed indicates the final value of the cascade size.

6.3.3 Results of the application

The outcome of the application is illustrated in Figure 6.2. Hereby the fat continuous lines represent $\hat{s}_1$, $\hat{s}_2$, $\hat{s}_3$. We observe that all three lines leave a zero level at approximately $m = 0.25$ and approach towards 1 if the mean robustness becomes smaller. However, in setting 1 this occurs noticeably faster. While $\hat{s}_1$ has already reached 1 in $m = 0.225$, the values of $\hat{s}_2$ and $\hat{s}_3$ still possess intermediate values. We allude that the values of $\hat{s}_2$ and $\hat{s}_3$ - analogously to the simulation's outcome in Section 4.2.3 - may be taken as the part of large cascades, which have been observed in the 50 simulation runs for each m. Hence, there is a range of the mean robustness m for which the simulation suggests that in setting 1 we always have to expect a total collapse of the system while in setting 2 and 3 we still find a balanced chance for a cascade of small and large size. The thin dashed lines denote $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_3$ adjusted up- and downwards by their observed standard deviations. This suggests more variation in the simulations of setting 2 and 3.
The fat dashed lines represent the values of $\tilde{s}_1$ and $\tilde{s}_2$ in $\tau = 15$. We see that the time lag between setting 1 and 2 in the simulations may be regained by comparing the cascade model in both settings. Apparently $\tilde{s}_2$ requires a lower mean robustness to jump from small to large values, however, its discrepancy with respect to $\hat{s}_2$ suggest that it does not indicate the expected fraction of defaults reliably.

[13]Data has been retrieved from www.bundesbank.de, ESCB time series: Aggregated balance sheet of euro area monetary financial institutions position 2.5 dived by position 1.1.1

Figure 6.2: The figure illustrates the expected fraction of defaults in the network of setting 1 - 3. The fat continuous lines represent the outcome of the simulation. The thin dashed lines report the observed standard deviations in the simulations and the fat dashed lines correspond to the expected fraction of defaults suggested by the formulas of Theorem 4.1 and 6.1 respectively. The thin continuous blue line denotes $r(m)$ which is the probability of having at least one default on the core in $\tau = 0$.

Interestingly, we can observe a simultaneous development of $\hat{s}_2$ and $\hat{s}_3$. Recall that in each simulation run the randomly generated vector $\tilde{\rho}(0)$ has been the same for setting 2 and 3, hence, the observed distance between both lines points out the approximation error in the different settings. The simultaneity between $\hat{s}_2$ and $\hat{s}_3$ allow the conclusion that $\hat{s}_2$ might serve as a credible alternative to make inferences on network stability without relying on the actually observed out-degrees but just by taking into account the fragmentation of the network. A slight constraint about this must be stated by alluding that the block CC has been treated equivalently in setting 2 and 3 by assuming the same out-degree for all core agents on this block. In the next section we point out that the core mainly determines the network stability, hence, assumed that we had in setting 3 the actually observed out-degrees of block CC, the fit between $\hat{s}_2$ and $\hat{s}_3$ might be effected negatively. Nevertheless we will use the common trend of $\hat{s}_2$ and $\hat{s}_3$ in the next section by comparing setting 1 and 2 which are more simple to examine and then conclude that the outcome of this comparison would also hold approximately if one compares setting 1 with setting 3.

6.3.4 Asymmetrical distribution of debt

For reasoning the less frequent observation of large cascades in setting 2 compared to setting 1 we recall Equation (6.6), which shows the fragmentation of the credit exposure in the core-periphery structure. Note that the core consists only $\frac{N^C}{N} = 2.5\%$ of all network agents, however, this small fraction of agents holds

$$\frac{X^{CC} + X^{PC}}{X} = 47.8\%$$

of the total interbank market debt. This fact is not considered in the homogeneous network of setting 1. We are going to substantiate this in the following theorems, which disclose the distribution of debt on the interbank market of setting 1 and 2.

Theorem 6.2. *Assume a credit network with the properties of setting 1 and generated by Algorithm 3.1, then the distribution function of*

$$D_1(i) := x_{b,1} d_{in}(i)$$

is given by

$$P(D_1(i) \leq a) = \binom{N-1}{\lfloor a/x_{b,1} \rfloor} \left(\frac{k}{N-1}\right)^{\lfloor a/x_{b,1} \rfloor} \left(1 - \frac{k}{N-1}\right)^{N-1-\lfloor a/x_{b,1} \rfloor}$$

for all $i \in V$ and $a \in [0, +\infty)$. Hereby $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. For $a \geq 0$ we have

$$\begin{aligned} P(D_1(i) \leq a) &= P(x_{b,1} d_{in}(i) \leq a) \\ &= P(d_{in}(i) \leq a/x_{b,1}) \\ &= \binom{N-1}{\lfloor a/x_{b,1} \rfloor} \left(\frac{k}{N-1}\right)^{\lfloor a/x_{b,1} \rfloor} \left(1 - \frac{k}{N-1}\right)^{N-1-\lfloor a/x_{b,1} \rfloor}. \end{aligned}$$

The usage of the binomial distribution in the last step is due to the result of Lemma 3.8. □

While Lemma 6.2 states the distribution of debt per agent on setting 1, the next theorem is the analogon for the debt per agent on a network of setting 2. For the sake of simplicity we are going to set in the following

$$P(R \leq a) := B(a, M, p), \; P(R = a) := b(a, M, p)$$

for $R \sim Bin(M, p)$.

Theorem 6.3. *Assume a credit network with the properties of setting 2 and generated by Algorithm 6.1, then the distribution function of*

$$D_2(i) := \begin{cases} x_{b,2}^{CC} d_{in}^{CC}(i) + x_{b,2}^{PC} d_{in}^{CP}(i), & \text{if, } i \in V^C \\ x_{b,2}^{PP} d_{in}^{PP}(i) + x_{b,2}^{CP} d_{in}^{PC}(i), & \text{if, } i \in V^P \end{cases}$$

for a randomly chosen agent $i \in V$, where it is assumed that

$$p := P(i \in V^C) = \frac{N^C}{N^C + N^P},$$

is given by

$$\begin{aligned} P(D_2(i) \leq a) &= p \sum_{k=0}^{N^P} B\left(\lfloor (a - x_{b,2}^{PC} k)/x_{b,2}^{CC} \rfloor, N^C - 1, \frac{k_{CC}}{N^C - 1}\right) b\left(k, N^P, \frac{k_{PC}}{N^P}\right) \\ &+ (1-p) \sum_{k=0}^{N^C} B\left(\lfloor (a - x_{b,2}^{CP} k)/x_{b,2}^{PP} \rfloor, N^P - 1, \frac{k_{PP}}{N^P - 1}\right) b\left(k, N^C, \frac{k_{CP}}{N^P}\right) \end{aligned}$$

for $a \in [0, \infty)$.

Proof. For $a \geq 0$ it is

$$\begin{aligned} P(D_2(i) \leq a) &= P(D_2(i) \leq a | i \in V^C) p + P(D_2(i) \leq a | i \in V^P)(1-p) \\ &= pP\left(x_{b,2}^{CC} d_{in}^{CC}(i) + x_{b,2}^{PC} d_{in}^{CP}(i) \leq a\right) + (1-p) P\left(x_{b,2}^{PP} d_{in}^{PP}(i) + x_{b,2}^{CP} d_{in}^{PC}(i) \leq a\right) \\ &= p \sum_{k=0}^{N^P} P\left(d_{in}^{CC}(i) \leq (a - x_{b,2}^{PC} k)/(x_{b,2}^{CC})\right) P\left(d_{in}^{CP}(i) = k\right) \\ &+ (1-p) \sum_{k=0}^{N^C} P\left(d_{in}^{PP}(i) \leq (a - x_{b,2}^{CP} k)/(x_{b,2}^{PP})\right) P\left(d_{in}^{PC}(i) = k\right). \end{aligned}$$

The occurring distribution functions for the final claim are provided by Lemma 6.1. □

For $a < 0$ we have for D_1 and D_2 a zero probability since negative debt is due to the given settings 1 and 2 not designated. We give a final proposition which states that the means of D_1 and D_2 coincides.

Proposition 6.1. *Having given D_1 and D_2 as in Theorem 6.2 and 6.3 respectively, it is*

$$E[D_2(i)] = E[D_1(i)] = x_{b,1}k.$$

Proof. The mean of D_1 is given by

$$\begin{aligned} E[D_1(i)] &= x_{b,1}E[d_{in}(i)] \\ &= x_{b,1}k. \end{aligned}$$

for all $i \in V$ where the distribution of $d_{in}(i)$ is given by Lemma 3.8. For $D_2(i)$ we deduce

$$\begin{aligned} E[D_2(i)] &= pE[D_2(i)|i \in V^C] + (1-p)E[D_2(i)|i \in V^P] \\ &= pE[x_{b,2}^{CC}d_{in}^{CC}(i) + x_{b,2}^{PC}d_{in}^{CP}(i)] + (1-p)E[x_{b,2}^{PP}d_{in}^{PP}(i) + x_{b,2}^{CP}d_{in}^{PC}(i)] \\ &= p\left(x_{b,2}^{CC}k_{CC} + x_{b,2}^{PC}\frac{N^P}{N^C}k_{PC}\right) + (1-p)\left(x_{b,2}^{PP}k_{PP} + x_{b,2}^{CP}\frac{N^C}{N^P}k_{CP}\right), \end{aligned}$$

where the third equality is deduced by Lemma 6.1. Recall from Equation (6.15) that

$$\begin{pmatrix} x_{b,2}^{CC} & x_{b,2}^{CP} \\ x_{b,2}^{PC} & x_{b,2}^{PP} \end{pmatrix} = \begin{pmatrix} X^{CC}/(N^C k^{CC}) & X^{CP}/(N^C k^{CP}) \\ X^{PC}/(N^P k^{PC}) & X^{PP}/(N^P k^{PP}) \end{pmatrix}$$

This applied to $E[D_2(i)]$ and noting that $p = \frac{N^C}{N^C+N^P}$ yields

$$\begin{aligned} E[D_2(i)] &= p\left(\frac{X^{CC}}{N^C} + \frac{X^{PC}}{N^C}\right) + (1-p)\left(\frac{X^{PP}}{N^P} + \frac{X^{CP}}{N^P}\right) \\ &= \frac{X^{CC} + X^{PC} + X^{PP} + X^{CP}}{N^C + N^P} \\ &= \frac{X}{N} \\ &= x_{b,1}k \end{aligned}$$

where the last equality results from the agreement of Equation (6.14). □

Note that in our simulation procedure of Section 6.3.2 we used to assume that $\tilde{\rho}_i(0)$ are identically distributed for all $i \in V$ meaning that a possible default in $\tau = 0$ occurs with the same probability for all $i \in V$. Theorems 6.2 and 6.3 endow us with the probability about the credit value which must be depreciated by the creditors of a randomly defaulted agent. Apparently the distribution of D_2, which is illustrated by the blue line in Figure 6.3, possesses an asymmetrical structure which is due to the remark from the beginning of this section, namely, most of the debt is concentrated on the small group of core agents while the majority of the agents - located in the periphery - only displays small amounts of debt. On the contrary, in setting 1 the homogeneous structure on the asset side is

Figure 6.3: The plot on the left hand side compares the distribution function of D_1 and D_2 from Theorem 6.2 and 6.3. The horizontal line denotes the position of the mean value of both distributions as derived in Proposition 6.1. On the right hand side we see the illustration of the final application assuming different variance on core and periphery respectively. The fat continuous lines belong to the scenario of no variance on the periphery, while the fat dashed lines denote the case of no variance on the core. The thin dashed lines indicate the observed standard deviation during the simulations.

regained on the debt side in the sense of a symmetrical distribution of the debt around the expected value.
Hence, if the defaulted agents in the beginning are chosen uniformly, there is a larger probability in setting 2 to have debtors which owe less money to the interbank market, hence, the accumulated depreciations in the agents' balance sheets will be lower in $\tau = 0$ and larger cascades are less frequently observed compared to the case of setting 1.
The possibility of a large cascade is dominated by the state of the core in $\tau = 0$. A detailed insight into the behavior of the cascade for individual simulation has shown that a default of a core bank in $\tau = 0$ is not always but commonly the trigger for a large cascade. This is also confirmed by the probability of having at least one default on the core in $\tau = 0$ which is due to the assumed distribution in Equation (6.16) given by

$$r(m) := P\left(|\{i \in V^C : \mathbb{1}_{(\tilde{\rho}_i(0) \leq 0)} = 1\}| \geq 1\right) = 1 - \left(1 - \Phi\left(-\frac{m}{\sigma}\right)\right)^{N^C}.$$

Obviously, this probability grows in Figure 6.2 simultaneously with the expected cascade size in setting 2.
The fragmentation of the credit exposure is of course also given in setting 3, however, the distribution of an agent's debt size may not be computed in such a straightforward way as for setting 1 and 2. Actually, for a computation one must rely on the Poisson binomial distribution in order to cope with the varying out-degrees in setting 3, which we skip at this point. Alternatively, we state, as setting 2 can be taken as an approximation of setting 3, that the asymmetrical distribution of debt will be presumably regained on setting 3 as well. The lower expected cascade size of setting 3 in comparison to setting 2, highlighted in Figure 6.2, suggests that we must suppose in setting 3 an even more asymmetrical distribution of debt than in setting 2.

6.3.5 The core as the key driver of systemic risk

A final application reconfirms the importance of the core in the context of network stability. We repeat the simulation for setting 2 and 3, however, this time we assume in a first attempt a zero variance for the periphery, i.e. we set

$$\tilde{\rho}_i(0) \sim N(m, \sigma^2), \forall i \in V^C, \qquad \tilde{\rho}_i(0) = m, \forall i \in V^P$$

and in a second attempt we inversely assume

$$\tilde{\rho}_i(0) = m, \forall i \in V^C, \qquad \tilde{\rho}_i(0) \sim N(m, \sigma^2), \forall i \in V^P.$$

As in Section 6.3.2 we again rely on $\sigma = 0.1$ and 50 simulation runs for each m - this time from the interval $[0.125, 0.3]$. Moreover, the same generated outcome of $\tilde{\rho}_i(0)$ is again used for both settings 2 and 3 simultaneously.

Figure 6.3 shows that for large cascades it requires a lower value for the system-wide mean if we assume no variance on the core in comparison to the inverse case. This somehow confirms what one would have probably expected, namely, that most of the systemic risk emanates from the core on which both the majority of receivables and debt of the interbank market are concentrated.

7 Conclusion

Our work has conducted a detailed review of quantifying systemic risk in the sense of Battiston et al. (2012). In Sections 2 - 5 we undertook a reformulation of the setting used by Battiston et al. (2012) in a more rigorous manner. Particularly, the derivation of the univariate process in Section 3, which later served as the starting point for the quantification of systemic risk, has been disassembled stepwise so that we were able to point out the various chances of falsification in the pragmatical procedure of Battiston et al. (2012). We recall, for instance, the transformation of the financial accelerator, originally modeled as a jump process, into a deterministic negative drift or the entirely negligence of the mean-reverting component in order to decouple the multivariate dynamics in Equation (3.6), which has been aimed at reducing complexity for an analytical examination of the financial robustness over time. Simulations in Section 3.2 suggest that the derived mean first passage time for the simple univariate process does not exactly match the mean first passage times of the financial robustness prior to complexity reduction, hence, the univariate process' usage for quantifying systemic risk will involve a forwarded bias. Figure 3.6, for instance, provides the information that this bias due to simplification is predominantly viewable vertically, which means, that deviation of the mean first passage time occurs for a fixed k. However, if one is exclusively interested in the behavior of the mean first passage times for different diversification degrees k, the results on the univariate process can be taken as a good indication. Similarly, in Section 5 we used to point out the approximative simplifications (S1) and (S2) for completing the procedure of Battiston et al. (2012). Again, we observed that this pragmatical approach distorts the final results about systemic risk, however, the shape of the final core result, that is systemic risk as non-decreasing function of connectivity and diversification respectively, is unaffected.

We generally conclude for the work of Battiston et al. (2012) that it yields the principle of systemic risk if one assumes that network contagion is accurately described by the sophisticated multivariate dynamics in Equation (5) and if an interbank market may be indeed approximatively described by assumptions A(1.2) and A(1.3). In this context the most crucial point in the work of Battiston et al. (2012) is in our point of view less the pragmatical approach but rather the specification of the financial accelerator. As described in Section 6 the final outcome about systemic risk basically relies on the agreement of scaling the sensitivity of the financial accelerator with respect to the degree of diversification but not the magnitude. If this is not in line with practice, the conclusion of Battiston et al. (2012) about systemic risk in dependence of diversification must be rethought.

In Section 6.3 we examine the effect of different structures of the interbank market with respect to its stability and the chance of system-wide distortions. The core of this section has been the finding that exposure on the interbank market is asymmetrical distributed between core and periphery. Our extension of the setting of Battiston et al. (2012) to a known heterogeneous structure has allowed to refer to the cascade mechanism in order to examine network stability, though, we did not develop a setting for examining systemic risk in the comprehensive sense of Battiston et al. (2012). Particularly, we blended out, different from the idea of Battiston et al. (2012), the financial robustness during the precascade phase, and thus, we do not supply the opportunity to incorporate the financial accelerator effect into an heterogeneous structure. However, Section 6.3.5 suggests that the overall network stability and the size of a cascade is predominantly depending on the core's state. Based on this one might be advised to restrict the examination of systemic risk exclusively to the core, i.e. an inclusion of the periphery into the setting of Battiston et al. (2012) appears redundant. Nevertheless, it might be necessary to analyze what happens to the mean field approach reflected by assumptions A(1.2) and A(1.3) if the core itself possesses strongly varying diversification.

Admittedly, one must also challenge the attempt to research systemic risk on interbank markets in the whole manner of Battiston et al. (2012). Even the sophisticated dynamics in Equation (3.6) will probably not contain all information which are necessary to entirely describe the behavior of the financial robustness over time. Furthermore, the cascade mechanism in Section 4 is the most simple way to describe contagion during periods of distressed interbank markets, whereas it does not consider further elements such as asymmetrical information between agents in such a period as exemplified by Bullard et al. (2009) or the accelerating impact of falling asset prices if markets are distorted, which is discussed by Adrian and Shin (2008). Further advancements in the setting of Battiston et al. (2012) are conceivable in order to incorporate such effects, however, it always remains on the one hand the question whether a full description of systemic risk for the complex interdependence between agents on the interbank market is indeed fully covered and on the other hand the problem of arising complexity for the analytical treatment which probably makes a pragmatical avoidance necessary again.

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