Hybridized Discontinuous Galerkin Discretizations for Flow Problems

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Introduction

Discontinuous Galerkin methods have several properties that make them attractive for the simulation of fluid flow:

- Mimic physical directionality in transport problems: Fluxes into and out of the cells balanced (generalization of finite volumes to high order)
- Work well also for convection-dominated problems, as opposed to continuous FEM which need stabilization
- Can easily couple non-conforming grids together
- Stable approximation with standard polynomial spaces

However, their cost is typically higher than continuous FEM or finite volumes (more degrees of freedom, wider stencils). Hybridized discontinuous Galerkin (HDG) methods try to mitigate this cost disadvantage by reducing the final linear problem to degrees of freedom on element faces.

HDG for the steady convection–diffusion equation

For a given convection velocity $\mathbf{c}$ and diffusivity κ, solve for

$$\nabla \cdot (\mathbf{c}u) - \nabla \cdot (\kappa \nabla u) = f$$

Write the equation as a system

$$\left.\begin{aligned} \mathbf{q} + \kappa \nabla u &= 0 \\ \nabla \cdot (\mathbf{c}u + \mathbf{q}) &= f \end{aligned}\right\} \text{ in } \Omega; \qquad \begin{aligned} u &= g_D && \text{on } \Gamma_D \text{ (Dirichlet)}, \\ (\mathbf{q} + \mathbf{c}u) \cdot \mathbf{n} &= g_N && \text{on } \Gamma_N \text{ (Neumann)}. \end{aligned}$$

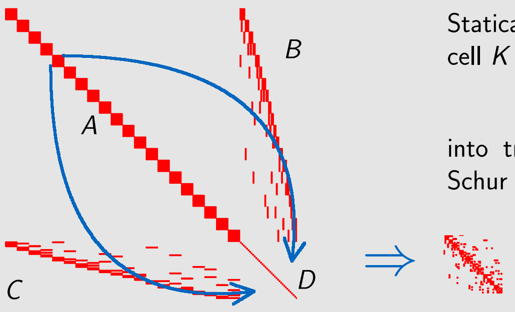
Weak HDG form solves for the discontinuous element variables u and $\mathbf{q}$ and the discontinuous trace variable $\widehat{u}$ [1]:

$$\begin{aligned} &\left(\mathbf{w}, \kappa^{-1}\mathbf{q}\right)_{\mathcal{T}_h} - (\nabla \cdot \mathbf{w}, u)_{\mathcal{T}_h} + \langle \mathbf{w} \cdot \mathbf{n}, \widehat{u} \rangle_{\partial \mathcal{T}_h} = 0 && \forall \mathbf{w} \in \mathbf{V}_h^d \\ &-(v, \mathbf{c}u + \mathbf{q})_{\mathcal{T}_h} + \langle v, (\mathbf{c}\widehat{u} + \mathbf{q}) \cdot \mathbf{n} + \tau(u - \widehat{u}) \rangle_{\partial \mathcal{T}_h} = (v, f)_{\mathcal{T}_h} && \forall v \in V_h \\ &\langle \mu, (\mathbf{c}\widehat{u} + \mathbf{q}) \cdot \mathbf{n} + \tau(u - \widehat{u}) \rangle_{\partial \mathcal{T}_h} = \langle \mu, g_N \rangle_{\Gamma_N} && \forall \mu \in M_h \end{aligned}$$

Concept of hybridizable discontinuous Galerkin schemes:
Use the trace $\widehat{u}$ as a new variable, solved alongside with u and $\mathbf{q}$ [2].

Implementation aspects

Aspect 1: Static condensation



Statically condense local $u, \mathbf{q}$ block matrix A_K on cell K in

$$\begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$$

into transformed global stiffness matrix M by Schur complement

$$M = \sum_{K=1}^{n\_cells} (D_K - C_K A_K^{-1} B_K)$$

HDG linear system: only solve trace system $M\Lambda = F$

Aspect 2: Superconvergent postprocessing

HDG produces a solution that is more accurate than standard FEM solutions:

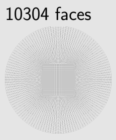
- u converges with rate $p+1$ for p-th order polynomials
- $\mathbf{q}$ converges with rate $p+1$

Main ingredient for postprocessing: If gradients $\mathbf{q}$ converge with rate $p+1$, can reconstruct a solution u^* that converges with rate $p+2$. Post-processing can include physically desired features, e.g. exactly divergence-free solutions for incompressible flow.

HDG trace system with Legendre basis: Computational efficiency

2D

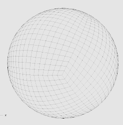
5120 elements
10304 faces



	matrix size (dofs) FEM	matrix size (dofs) HDG	matrix nonzeros FEM	matrix nonzeros HDG
$p=1$	5 185	20 608	0.021m	0.18m
$p=2$	20 609	30 912	0.32m	0.64m
$p=3$	46 273	41 216	1.1m	1.1m
$p=4$	82 177	51 520	2.9m	1.8m
$p=5$	128 321	61 824	6.2m	2.6m

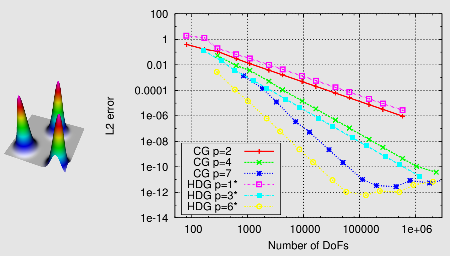
3D

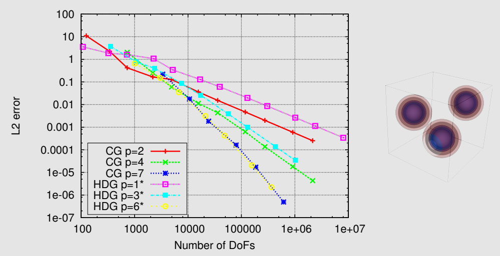
28672 elements
86784 faces



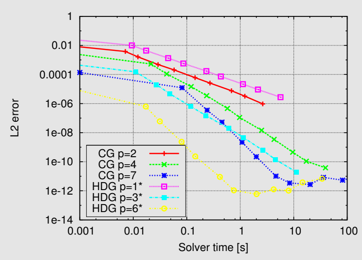
	matrix size (dofs) CG	matrix size (dofs) HDG	matrix nonzeros CG	matrix nonzeros HDG
$p=1$	29 521	260 352	0.74m	6.1m
$p=2$	232 609	520 704	14m	34m
$p=3$	781 297	867 840	94m	93m
$p=4$	1 847 617	1 301 760	390m	210m
$p=5$	3 603 601	1 822 464	1200m	410m

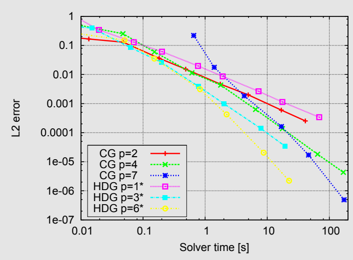
HDG is involves more work per element for lower orders compared to usual finite elements (CG), but is very competitive for higher orders $p \geq 3$, as pointed out also in [3]. With post-processing, HDG at degree p gives similar results as CG at degree $p+1$:





CG and HDG solver time: Use Trilinos ML algebraic multigrid preconditioner within GMRES iterative solver for diffusion-dominated problem, takes 20–40 iterations:

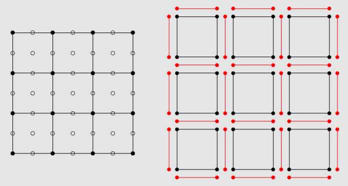




HDG solution representation and solutions

HDG solution space:

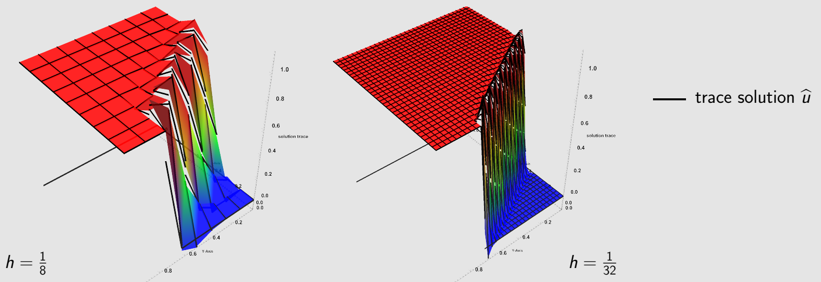
continuous FEM
$\mathcal{Q}_1$: 16 dofs
$\mathcal{Q}_2$: 49 dofs



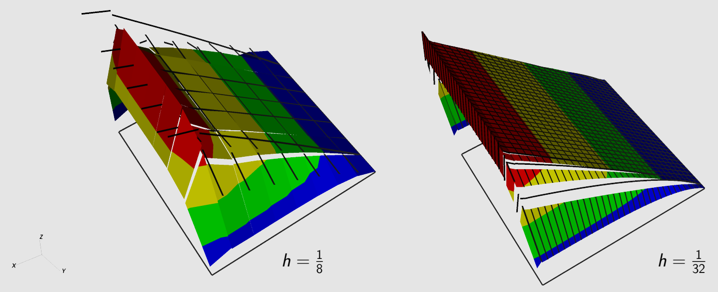
HDG, linears
12 interior faces, 12 boundary faces
u: $9 \times 4 = 36$ dofs
$\mathbf{q}$: 72 dofs
$\widehat{u}$: $24 \times 2 = 48$ dofs

HDG solutions are of good quality for difficult convection-dominated problems without additional stabilization

Problem 1: $\Omega = [0,1]^2$, $\kappa = 10^{-6}$, $\mathbf{c} = \frac{1}{2}\left(1, -\sqrt{3}\right)$, $f = 0$
Dirichlet conditions: $u = 0$ on $\{x = 1\}, \{y = 0\}, \{x = 0 \wedge y \leq 0.7\}$
$u = 1$ on $\{y = 1\}$ and $\{x = 0 \wedge y > 0.7\}$



Problem 2: $\Omega = [0,1]^2$, $\kappa = 10^{-6}$, $\mathbf{c} = (1, 0)$, $f = 1$, $u = 0$ on $\partial\Omega$



HDG for the incompressible Navier–Stokes equations

Consider the time-dependent incompressible Navier–Stokes equations in 2D/3D:

$$\begin{aligned} &\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \right) - \nabla \cdot (2\mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)) + \nabla p = \mathbf{f} \\ &\nabla \cdot \mathbf{u} = 0 \end{aligned}$$

HDG formulation [4]: Find $\mathbf{u}$, $\mathbf{L}$, p, and $\widehat{\mathbf{u}}$ such that

$$\begin{aligned} &(\mathbf{G}, \mathbf{L})_{\mathcal{T}_h} + (\nabla \cdot \mathbf{G}, \mathbf{u})_{\mathcal{T}_h} - \langle \widehat{\mathbf{u}}, \mathbf{G} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 && \forall \mathbf{G} \in \mathbf{V}_h^{d \times d} \\ &\left(\mathbf{v}, \rho \frac{\partial \mathbf{u}}{\partial t}\right)_{\mathcal{T}_h} + \left(\nabla \mathbf{v}, \mu(\mathbf{L} + \mathbf{L}^T) - p\mathbf{I} - \rho \mathbf{u} \otimes \mathbf{u}\right)_{\mathcal{T}_h} + \\ &\qquad \left\langle \mathbf{v}, (-\mu(\mathbf{L} + \mathbf{L}^T) + p\mathbf{I} + \rho \widehat{\mathbf{u}} \otimes \widehat{\mathbf{u}}) \cdot \mathbf{n} + \mathbf{s}_h(\mathbf{u}, \widehat{\mathbf{u}}) \right\rangle_{\partial \mathcal{T}_h} = (\mathbf{v}, \mathbf{f})_{\mathcal{T}_h} && \forall \mathbf{v} \in \mathbf{V}_h^d \\ &-(\nabla q, \mathbf{u})_{\mathcal{T}_h} + \langle q, \widehat{\mathbf{u}} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 && \forall q \in V_h \\ &\left\langle \boldsymbol{\mu}, (-\mu(\mathbf{L} + \mathbf{L}^T) + p\mathbf{I} + \rho \widehat{\mathbf{u}} \otimes \widehat{\mathbf{u}}) \cdot \mathbf{n} + \mathbf{s}_h(\mathbf{u}, \widehat{\mathbf{u}}) \right\rangle_{\partial \mathcal{T}_h} = 0 && \forall \widehat{\mathbf{u}} \in \mathbf{M}_h \end{aligned}$$

Navier–Stokes solution procedure

- Implicit time integration
  - In each time step, solve a nonlinear equation with Newton iteration
    - Assembly: condense local matrix A_K for $\mathbf{L}, \mathbf{u}, p$ into a trace matrix
    - Solve trace system
    - Reconstruct local solution $\mathbf{L}, \mathbf{u}, p$

Characterization of trace system

Local linearized Navier–Stokes system on element K is a Dirichlet problem—need to also fix the pressure average that couples the pressure between the elements,

$$p = (p - \bar{p}) + \psi,$$

where $\bar{p} = \int_K p \, d\mathbf{x}$ is the average of the pressure on the element K and ψ a the average element pressure that couples to other elements. This gives the linear system

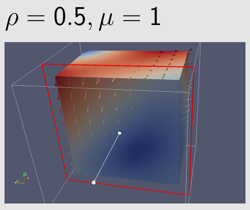
$$\begin{pmatrix} K & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \delta\Lambda \\ \delta\Psi \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{size}(\delta\Lambda) &= d \times n_{\text{faces}} \times \dim(\mathcal{P}_p(\text{face})) && \text{(trace velocity } \widehat{\mathbf{u}}) \\ \text{size}(\delta\Psi) &= n_{\text{elements}} \times \dim(\mathcal{P}_0(K)) = n_{\text{elements}} && \text{(average pressure } \psi) \end{aligned}$$

As for the convection–diffusion equation, this system is larger than similar CG systems for $p = \{1, 2\}$, but competitive for $p \geq 3$.

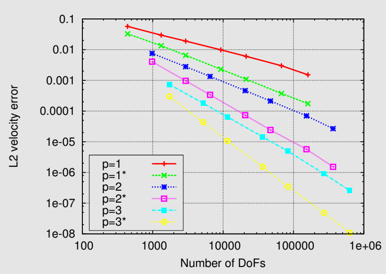
3D Beltrami flow:

Consider relative velocity error at time $t = 1$ for $\rho = 0.5, \mu = 1$



arrows:
velocity field $\mathbf{u}$
colored:
pressure p

Convergence rate velocity: $p+1$
Convergence rate post-processed velocity: $p+2$



References

[1] N.C. Nguyen, J. Peraire, B. Cockburn: An implicit high-order hybridizable discontinuous Galerkin method for linear convection–diffusion problems, *J. Comput. Phys.* 228 (2009): 3232–3254

[2] B. Cockburn, J. Gopalakrishnan, R. Lazarov: Unified hybridization of discontinuous Galkerin, mixed, and continuous Galerkin methods for second order elliptic problems, *SIAM J. Numer. Anal.* 47 (2009): 1319–1365

[3] R.M. Kirby, S.J. Sherwin, B. Cockburn: To CG or to HDG: A comparative study, *J. Sci. Comput.* 51 (2012): 183–212

[4] N.C. Nguyen, J. Peraire, B. Cockburn: An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier–Stokes equations, *J. Comput. Phys.* 230 (2011): 1147–1170