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***Abstract*—From an information theoretic point of view, multiple-input multiple-output (MIMO) broadcast channels are separable, i.e., the optimal (capacity achieving) strategy performs encoding and decoding separately on each of the parallel channels (e.g., carriers in multicarrier system). In recent publications, it was shown that this separability can be lost if a restriction to a certain class of transmit strategies (such as linear strategies or zero-forcing strategies) is imposed. In this paper, we categorize the recent results into three different kinds of inseparability, and we discuss the characteristic properties of each kind. Moreover, two new inseparability results that fit into these categories are presented: for the case of linear zero-forcing with time-sharing and for linear transceivers in systems with two users.**

***Index Terms*—linear transceivers, multiple-input multiple-output (MIMO), multiuser multicarrier systems, parallel broadcast channels, separable and inseparable channels.**

I. Introduction

The notion of inseparability first attained significant attention in the context of parallel interference channels, where it might happen that capacity can only be achieved by coding jointly across the parallel channels instead of separately on each channel [1]. In parallel MIMO broadcast channels, i.e., in the case of a single base station that transmits on multiple orthogonal carriers, the situation is different: in this setting, the capacity achieving strategy based on dirty paper coding (DPC) [2] can be applied separately on each carrier (e.g., [3]). Only the power allocation has to be optimized across all carriers.

However, DPC is not an adequate method for implementation in a real system since even approximate implementations as in [4] have prohibitive complexity due to the involved vector quantization operations. Also approximate implementations based on Tomlinson-Harashima precoding (THP) have drawbacks such as the shaping, power, and modulo loss (e.g., [5]). An adequate alternative for practical systems are MIMO techniques with linear transceivers, where all operations involving more than one data stream have to be linear while nonlinear operations (encoding, detection, . . .) are only applied to single data streams (e.g., [6]). On the other hand, the optimization of the transmit strategy becomes more involved in this case and additional aspects such as inseparability have to be considered.

As we could recently show in a series of publications [7]–[9], the question of inseparability becomes important also in broadcast channels if linear transceivers are employed. Let us consider a restriction of the allowed transmit strategies to a certain class of strategies, e.g., to linear transceivers or, even stricter, to linear transceivers with zero-forcing (complete interference cancellation), and let us assume that, among all strategies in this class that employ separate coding on each carrier, we know the optimal one. Then, there is some performance gap between this strategy and the capacity-achieving DPC strategy that also uses separate coding. However, among the strategies in the considered class that employ joint coding across carriers, there might be some strategies that perform closer to the DPC solution. Whenever this happens, i.e., whenever allowing joint coding makes the performance gap smaller, we say that a scenario is inseparable. As the possibility of inseparability can be shown by constructing appropriate examples, it often suffices to consider very simple special cases of MIMO broadcast channels, maybe even with a single antenna at some of the terminals (cf. e.g., [9]).

In each of the publications [7]–[9], inseparability was shown for different performance criteria and under different assumptions on the permitted transmit strategies. In [7], problems with quality of service constraints were considered in overloaded systems (i.e., with more users than degrees of freedom) under the assumption of linear transceivers without time-sharing (switching between several transmit strategies and considering average per-user rates as well as the average sum transmit power). Quality of service problems without time-sharing were also considered in [8], but without the restriction to overloaded systems. Therein, it was shown that inseparability can occur if a constraint to zero-forcing strategies is present. The most general inseparability result was shown in [9] for the class of strategies with linear transceivers without further constraints. Restricted to this class, parallel broadcast channels can be inseparable also for weighted sum rate problems or for quality of service problems with time-sharing.

Inseparability of MIMO broadcast channels with linear transceivers was shown in each of the abovementioned publications, but, as we have just discussed, the results can be considered as three different kinds of inseparability. In Section III to V, we present a systematic classification of these kinds of inseparability, and we discuss their characteristic properties.

Moreover, we answer a question that was left open for future research in [8]. Since the inseparability of broadcast channels with linear zero-forcing strategies was only shown for the case of quality of service problems without time-sharing in [8], it was not clear if inseparability in the zero-forcing case can also occur without time-sharing or for weighted sum rate problems. In Section VI-A, we answer this question in the positive, by constructing an example that demonstrates this inseparability. This new separability result can be classified by means of the systematics introduced in this paper.

Finally, based on the classification, we identify another scenario that has not yet been considered under the point of view of inseparability, and we again provide a corresponding example to demonstrate the inseparability in Section VI-B.

II. Types of Linear Transmit Strategies

Consider a set of C parallel MIMO broadcast channels with M transmit antennas, K users, and N_k receive antennas for user k, where the channel matrix of user k on carrier c is given by $\boldsymbol{H}_k^{(c),\mathrm{H}} \in \mathbb{C}^{N_k \times M}$, and the additive circularly symmetric complex Gaussian noise is characterized by $\boldsymbol{\eta}_k^{(c)} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{C}_k^{(c)})$. Transmission with linear transceivers can be described by

$$\hat{\boldsymbol{x}}_k = \boldsymbol{G}_k^{\mathrm{H}} \underbrace{\begin{bmatrix} \boldsymbol{H}_k^{(1),\mathrm{H}} & & \\ & \ddots & \\ & & \boldsymbol{H}_k^{(C),\mathrm{H}} \end{bmatrix}}_{\boldsymbol{H}_k^{\mathrm{H}}} \sum_{j=1}^{K} \boldsymbol{B}_j \boldsymbol{x}_j + \underbrace{\begin{bmatrix} \boldsymbol{\eta}_k^{(1)} \\ \vdots \\ \boldsymbol{\eta}_k^{(C)} \end{bmatrix}}_{\boldsymbol{\eta}_k} \quad (1)$$

where $S_k \leq \min\{N_k C, MC\}$ streams of i.i.d. Gaussian data symbols are intended for user k, i.e., $\boldsymbol{x}_k = [x_{k,1}, \ldots, x_{k,S_k}] \sim \mathcal{CN}(\boldsymbol{0}, \mathbf{I}_{S_k})$. The matrices $\boldsymbol{B}_k \in \mathbb{C}^{MC \times S_k}$ and $\boldsymbol{G}_k^{\mathrm{H}} \in \mathbb{C}^{S_k \times N_k C}$ are the transmit filters (beamforming matrices) and the receive filters (linear equalizers), respectively.

A. Separate and Joint Transmission

If the transmit filter matrices $\boldsymbol{B}_k$ have arbitrary structure, a transmit symbol might be transmitted over several carriers so that the transmit strategy is a *joint strategy* (or carrier-cooperative strategy, according to the nomenclature in [10]). In this case, to find the optimal receive filters for given transmit filters, an arbitrary structure should also be allowed for the receive filters $\boldsymbol{G}_k^{\mathrm{H}}$. On the other hand, if the transmit filters can be decomposed as

$$\boldsymbol{B}_k = \mathrm{blockdiag}\left(\boldsymbol{B}_k^{(1)}, \ldots, \boldsymbol{B}_k^{(C)}\right) \quad (2)$$

matching the block-diagonal structure of the channel matrices $\boldsymbol{H}_k^{\mathrm{H}}$, we have a *separate strategy* (or carrier-noncooperative strategy [10]), and the corresponding receive filters

$$\boldsymbol{G}_k^{\mathrm{H}} = \mathrm{blockdiag}\left(\boldsymbol{G}_k^{(1),\mathrm{H}}, \ldots, \boldsymbol{G}_k^{(C),\mathrm{H}}\right) \quad (3)$$

match the block structure as well.

B. Zero-Forcing Beamforming

In parts of this paper, zero-forcing strategies are considered. Zero-forcing requires complete suppression of the inter-stream interference, i.e., an estimate $\hat{x}_{k,s}$ of the sth stream of user k may not contain interference of any other data stream. In the context of linear transceivers, the term *zero-forcing beamforming* is often used to stress that the interference suppression is achieved only by means of the linear filters, i.e.,

$$\boldsymbol{g}_{k,s}^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{b}_{\ell,t} = 0 \quad \forall k, s, \ell, t : (\ell, t) \neq (k, s) \quad (4)$$

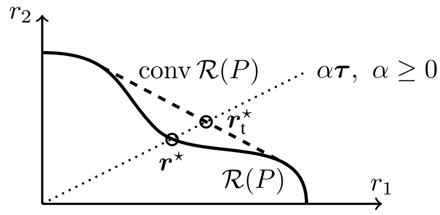

Fig. 1. Rate balancing solution $\boldsymbol{r}^\star$ without time-sharing and $\boldsymbol{r}_{\mathrm{t}}^\star$ with time-sharing.

where $\boldsymbol{g}_{k,s}^{\mathrm{H}}$ is the sth row of $\boldsymbol{G}_k^{\mathrm{H}}$ and $\boldsymbol{b}_{\ell,t}$ is the tth column of $\boldsymbol{B}_\ell$. In this paper, the term *zero-forcing* always refers to zero-forcing beamforming according to the definition in (4).

C. Time-Sharing

The term time-sharing refers to applying different transmit strategies one after another in order to achieve some average data rate with some average transmit power. Otherwise, if time-sharing is not allowed, one particular strategy has to be chosen and such an averaging is not possible. Both concepts, transmission with and without time-sharing, are worth being considered since they both have their advantages and disadvantages. In particular, transmission with time-sharing can lead to higher system performance, while strategies without time-sharing have a smaller signaling overhead.

Let r_k be the Shannon rate of user k, and let $\mathcal{R}(P)$ denote the rate region without time-sharing, i.e., the set of all rate vectors $\boldsymbol{r} = [r_1, \ldots, r_K]^{\mathrm{T}}$ achievable with sum transmit power P. Since time-sharing can be interpreted as creating convex combinations of points in the rate region $\mathcal{R}(P)$, allowing time-sharing corresponds to taking the convex hull $\mathrm{conv}\,\mathcal{R}(P)$.

If we consider a weighted sum rate maximization

$$\max_{\boldsymbol{r} \in \mathcal{R}(P)} \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{r} \quad (5)$$

for a given transmit power P, where $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_K]^{\mathrm{T}}$ is a vector of weights, transmitting with or without time-sharing does not make a difference since taking the convex hull of $\mathcal{R}(P)$ cannot increase the weighted sum rate [11, Corollary A5.9]. However, for problems with quality of service constraints, such as the so-called rate balancing problem

$$\max_{\boldsymbol{r} \in \mathcal{R}(P)} \sum_{k=1}^{K} r_k \quad \text{s.t.} \quad \frac{r_1}{\tau_1} = \frac{r_k}{\tau_k}\ \forall k \quad (6)$$

for a given transmit power P and a given vector of relative rate targets $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_K]^{\mathrm{T}}$, replacing $\mathcal{R}(P)$ by its convex hull can lead to an improved optimal value due to [11, Proposition A5.10]. A visualization can be found in Fig. 1. The same is true for the power minimization problem

$$\min_{P \geq 0} P \quad \text{s.t.} \quad \boldsymbol{\rho} \in \mathcal{R}(P) \quad (7)$$

for a given vector of rate requirements $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_K]^{\mathrm{T}}$ (e.g., [12]).

The solution to a quality of service problem with time-sharing consists of an optimal convex combination of several solutions of weighted sum rate problems (with or without time-sharing), cf., e.g., [12]. Therefore, for inseparability studies, considering quality of service problems with time-sharing is equivalent to considering weighted sum rate problems.

On the other hand, quality of service problems without time-sharing have qualitatively different properties than the weighted sum rate maximization. Moreover, for quality of service problems without time-sharing, we have to distinguish the case of an overloaded system with more users than degrees of freedom and a system where the transmitter has at least as many degrees of freedom as the system has users.

III. Overloaded Systems without Time-Sharing

In [7], the quality of service (QoS) feasibility was studied for multicarrier MIMO broadcast channels with linear transceivers. Without interference, arbitrary QoS requirements can be fulfilled as long as enough transmit power is spent. Therefore, this kind of study is only relevant for overloaded system without time-sharing. If time-sharing is allowed, arbitrary QoS requirements can be fulfilled, e.g., by assigning each time slot exclusively to one user [7]. The same is true if the number of degrees of freedom at the transmitter, i.e., the product MC of the number of transmit antennas and the number of carriers, is at least as high as the number of users. In this case, interference-free transmission is possible without time-sharing by means of linear zero-forcing beamforming, and arbitrary QoS requirements can be fulfilled.

However, in an overloaded system, i.e., if the number of users K is larger than the number of degrees of freedom MC, a given set of QoS requirements might be impossible to fulfill without time-sharing—even with arbitrarily high transmit power (e.g., [7], [13], [14]). We restrict our considerations to overloaded systems with so-called *regular channels* [14] with

$$\operatorname{Rank}\left[\boldsymbol{H}_{\mathcal{K}}^{(c)}\right] \geq \min(|\mathcal{K}|, M) \quad \forall \mathcal{K} \subseteq \{1, \ldots, K\} \tag{8}$$

on each carrier c, where $\boldsymbol{H}_{\mathcal{K}}^{(c)} \in \mathbb{C}^{M \times \sum_{k \in \mathcal{K}} N_k}$ comprises all matrices $\boldsymbol{H}_k^{(c)}$ with $k \in \mathcal{K}$ in a block row. This condition is fulfilled almost surely if the channels are drawn from a general continuous distribution.

With joint transmission, the system can be treated as an equivalent single-carrier broadcast channel with MC transmit antennas and $N_k C$ receive antennas, and rate requirements $\rho_1, \ldots, \rho_K$ are feasible if they fulfill [7], [14]

$$\sum_{k=1}^{K} (1 - 2^{-\rho_k}) < MC. \tag{9}$$

With separate transmission, additional conditions have to be fulfilled. In this case, it must be possible to find per-carrier rates $r_k^{(c)}$ for each user such that $\sum_{c=1}^{C} r_k^{(c)} = \rho_k$ and such that the feasibility condition is fulfilled on each carrier [7]:

$$\sum_{k=1}^{K} (1 - 2^{-r_k^{(c)}}) < M. \tag{10}$$

Summing up (10) over all carriers and taking into consideration that $(1 - 2^{-r_k^{(c)}})$ is concave in $r_k^{(c)}$, we see that the constraints in (10) are stricter than the one in (9) unless the required rate of each user is achieved with a single data stream, i.e., $r_k^{(c_k)} = \rho_k$ for some c_k and all k. However, in case of an unsuitable combination of the individual requirement sizes, there might be no solution with single-stream transmission (bin packing problem, cf. [7]). Therefore, as discussed in detail in [7], we can find quality of service requirements $\rho_1, \ldots, \rho_K$ for which it is impossible to fulfill (10), even though (9) is satisfied. In these cases, the scenario is inseparable.

This first kind of inseparability of parallel MIMO broadcast channels with linear transceivers is the most specific one as it only occurs for quality of service problems without time-sharing in overloaded system. The characteristic property of this kind of inseparability is that the question whether a setting is inseparable does not depend on the channel realization (at least as long as we restrict ourselves to the practically relevant case of regular channels). Instead, the only decisive factors are the system dimensions and the values of the minimum rate requirements. The effect leading to the first kind of inseparability is that despite of being feasible for joint transmission, the rate requirements might form unsuitable combinations that can not be allocated to the various carriers in a feasible manner [7], [9].

IV. General Systems without Time-Sharing

In our recent work [8], another inseparability result for quality of service problems was presented. Therein, the case of zero-forcing beamforming was considered with the assumption that time-sharing is not allowed. This setting obviously differs from the one in the preceding section since zero-forcing is not possible in an overloaded system. Therefore, the result that is summarized below shows that inseparability can also happen in general systems that are not overloaded. We will see later that the result extends to the case without zero-forcing.

The inseparable example in [8] is a two-user single-antenna system with two carriers. In this system, the number of users $K = 2$ is equal to the degrees of freedom at the transmitter $MC = 2$ so that the system is not overloaded, and zero-forcing is possible. It was shown in [8] that inseparability under the assumption of zero-forcing without time-sharing can occur in case of so-called *spectrally similar* channels that fulfill

$$\frac{\left|H_k^{(c)}\right|^2}{C_k^{(c)}} \geq \frac{\left|H_k^{(d)}\right|^2}{C_k^{(d)}} \;\forall k \in \{1, 2\} \tag{11}$$

for some carrier c and $d \neq c$, where the noise covariance matrices $\boldsymbol{C}_k^{(c)}$ and channel matrices $\boldsymbol{H}_k^{(c),\mathrm{H}}$ are reduced to scalars $C_k^{(c)}$ and $H_k^{(c),*}$ due to the single-antenna assumption. The question whether or not the setting is separable then depends on the particular channel realization and on the value of ρ_1 and ρ_2. For the case that the channels do not fulfill (11), the setting was proven to be separable no matter what values of ρ_1 and ρ_2 are chosen.

Unlike the first kind of inseparability, this second kind does not only depend on the rate requirements, but also on the channel realization. Now, the main effect leading to inseparability is that several users might compete for a carrier on which they all have good channel conditions. Note that this effect is specific to quality of service problems: from a (weighted) sum rate perspective, there is no competition between users since only the (weighted) sum of the rates is important. Moreover, this effect does not occur with time-sharing, which enables the users to use the good carrier one after the other. Therefore, this kind of inseparability is specific to quality of service problems without time-sharing. However, the zero-forcing assumption is not a prerequisite to obtain this kind of inseparability as will be shown in Section VI-B.

V. Systems with Time-Sharing

The most general inseparability result for parallel MIMO broadcast channel with linear transceivers, which is not limited to quality of service problems without time-sharing, was shown in [9].

Parallel broadcast channels with $C = 2$ carriers, $M = 2$ transmit antennas, $K = 3$ users, and $N_k = 1$ receive antenna $\forall k$ with noise variances $C_k^{(c)} = 1 \ \ \forall k, \ \forall c$ and channels

$$\boldsymbol{H}_1^{\mathrm{H}} = \begin{bmatrix} \boldsymbol{h}_1^{(1),\mathrm{H}} & \\ & \boldsymbol{h}_1^{(2),\mathrm{H}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{12}$$

$$\boldsymbol{H}_2^{\mathrm{H}} = \begin{bmatrix} \boldsymbol{h}_2^{(1),\mathrm{H}} & \\ & \boldsymbol{h}_2^{(2),\mathrm{H}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{13}$$

$$\boldsymbol{H}_3^{\mathrm{H}} = \begin{bmatrix} \boldsymbol{h}_3^{(1),\mathrm{H}} & \\ & \boldsymbol{h}_3^{(2),\mathrm{H}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\mathrm{j}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-\mathrm{j}}{\sqrt{2}} \end{bmatrix} \tag{14}$$

was considered in [9]. It was shown that the globally optimal strategy for separate linear processing requires a transmit power $P_{\text{sep}} \approx 3.5684$ to fulfill the rate targets $\rho_k = 1 \ \forall k$. Then, a joint linear strategy that only needs $P_{\text{joint}} \approx 3.1174$ was presented, which shows inseparability.

Due to the relations discussed in Section II-C, this kind of inseparability not only holds for quality of service problems without time-sharing, but also for weighted sum rate maximization.[1] In fact, this kind of inseparability can occur for various objective functions that take their optimal values on the Pareto boundary of the rate region [9]. This makes it significantly different from the first two kinds of inseparability.

As the third kind is not limited to quality of service problems, the rate targets do not seem to play the major role for the question of inseparability. However, their exact influence as well as the influence of the weights $\mu_1, \ldots, \mu_K$ in case of a weighted sum rate maximization is not yet studied. According to the study in [15], this kind of inseparability can occur in the high-SNR regime as well as in the low-SNR regime. As discussed in detail in [15], the important factor for deciding about separability or inseparability of this kind is the particular channel realization. However, unlike for the second kind presented above, inseparability is not caused by spectrally similar channels, where the channel quality strongly depends on the carrier index. Instead, symmetry among the carriers—and also among the users—seems to be conducive to this kind of inseparability [15]. Just like for the second kind of inseparability, absence or presence of zero-forcing constraints is not crucial for this kind of inseparability (cf. next section).

[1] An example with sum rate maximization can be found in [15].

VI. New Results on Inseparability

In this section, we present two new examples of inseparability in order to answer questions that have been left open for future research in previous publications. The presented examples are classified by means of the three categories introduced above, which helps to better understand both the new results and the classification scheme.

A. Zero-Forcing with Time-Sharing

In [8], the question was left open whether inseparability can occur in parallel MIMO broadcast channels with zero-forcing constraints also in the case where time-sharing is allowed. It was stated, that the example system from [8] cannot be used to show inseparability for this case.

The reason for this is that the result shown in [8] belongs to the second kind of inseparability while showing inseparability for zero-forcing with time-sharing would belong to the third category. As discussed above, the factors causing inseparability are fundamentally different for these two categories.

To obtain the desired inseparability result, we therefore propose to reconsider the example system from [9], which is reproduced in (12) to (14), since this scenario was already used to obtain an inseparability result of the third kind—albeit for the case without zero-forcing constraints. The channel vectors in this scenario have unit norm and symmetric angular separations $\theta_{k,j}^{(c)} = \arccos |\boldsymbol{h}_k^{(c),\mathrm{H}} \boldsymbol{h}_j^{(c)}| = 45^\circ$ for $k \neq j$.

As adding zero-forcing constraints cannot improve the optimal value of the objective function, the necessary power to fulfill $\rho_k = 1$ for all k with separate zero-forcing is $P_{\text{sep,ZF}} \geq P_{\text{sep}} \approx 3.5684$ (cf. Section V).[2]

To construct a joint zero-forcing strategy, we decide for $S_k = 1 \ \forall k$ and use receive filter vectors

$$\boldsymbol{t}_k^{\mathrm{H}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mathrm{e}^{\mathrm{j}\,\varphi_k} \end{bmatrix} \tag{15}$$

with $\varphi_1 = 0$, $\varphi_2 = -\frac{5\pi}{6}$, and $\varphi_3 = \frac{5\pi}{6}$ (similar as in [9]), yielding effective vector channels $\tilde{\boldsymbol{h}}_k^{\mathrm{H}} = \boldsymbol{t}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}}$. This particular choice yields high angular separations, i.e., $\cos \theta_{k,j} = |\tilde{\boldsymbol{h}}_k^{\mathrm{H}} \tilde{\boldsymbol{h}}_j|$ is given (as in [9]) by

$$\left| \tilde{\boldsymbol{h}}_k^{\mathrm{H}} \tilde{\boldsymbol{h}}_j \right| = \begin{cases} 1 & \text{if } k = j, \\ \frac{1}{\sqrt{2}} \cos\left(\frac{5\pi}{12}\right) \approx 0.1830 & \text{if } k \neq j \end{cases} \tag{16}$$

for all $k, j \in \{1, 2, 3\}$. The rates achievable with zero-forcing are then given by $r_k = \log(1 + p_k \gamma_k)$ where the channel gains γ_k are computed from (e.g., [16])

$$\gamma_k = \left[\left(\boldsymbol{H} \boldsymbol{H}^{\mathrm{H}} \right)^{-1} \right]_{k,k}^{-1} \quad \text{with} \quad \boldsymbol{H} = \begin{bmatrix} \tilde{\boldsymbol{h}}_1^{\mathrm{H}} \\ \tilde{\boldsymbol{h}}_2^{\mathrm{H}} \\ \tilde{\boldsymbol{h}}_3^{\mathrm{H}} \end{bmatrix}. \tag{17}$$

[2] Using the algorithm from [12], it can be shown numerically that $P_{\text{sep,ZF}} \approx 3.6569$, which can be achieved by time-sharing between single-user transmission, is optimal.

Note that the elements of $\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}$ only depend on the inner products $\tilde{\boldsymbol{h}}_i^{\mathrm{H}}\tilde{\boldsymbol{h}}_j$ between the effective channels.

We obtain $\gamma_k \approx 0.9180$ for all k, and the required transmit power is given by

$$P_{\text{joint,ZF}} = \sum_{k=1}^{K} \frac{2^{\rho_k} - 1}{\gamma_k} \approx 3.2679. \tag{18}$$

This proves that broadcast channels with linear zero-forcing are not always separable even if time-sharing is allowed. As discussed above, this is an inseparability of the third kind, i.e., it can also happen for other objective functions such as (weighted) sum rate maximization.

Similar as in the case without zero-forcing studied in [9], the loss in performance resulting from small angular separation of the channels in the case of separate transmission ($\theta_{k,j}^{(c)} = 45°$ versus $\theta_{k,j} \approx 79.5°$ in the joint case) cannot be compensated by the fact that two subchannels are available and by the use of optimal time-sharing.

B. Two Users without Zero-Forcing

Having seen that inseparability of the third kind can happen with and without zero-forcing constraints (Section VI-A), the question arises whether this is also true for the second kind, which has only been studied with zero-forcing constraints so far. Therefore, we reconsider the example system from [8] (cf. Section IV) with $H_k^{(1)} = 1\ \forall k$, $H_k^{(2)} = 0.1\ \forall k$, and $C_k^{(c)} = 1\ \forall k, \forall c$, but this time without zero-forcing constraints.

We consider a power minimization with $\rho_1 = \rho_2 = 1$. The globally optimal separate linear strategy without time-sharing can be found numerically up to an arbitrarily small error tolerance by means of the algorithm proposed in [17]. The optimal power is $P_{\text{sep}} \approx 27.2843$, which can be achieved by choosing for the two users the per-carrier rates $r_1^{(1)} = 1$, $r_1^{(2)} = 0$, $r_2^{(1)} \approx 0.8092$, $r_2^{(2)} \approx 0.1908$.

For the joint strategy, we use the receive filters

$$\boldsymbol{t}_k^{\mathrm{H}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & (-1)^k \end{bmatrix} \tag{19}$$

yielding effective vector channels $\tilde{\boldsymbol{h}}_k^{\mathrm{H}} = \boldsymbol{t}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}$. Applying the power minimization method for vector broadcast channels presented in [13], we get the transmit power $P_{\text{joint}} = 20 < P_{\text{sep}}$. This shows that inseparability of the second kind can also happen without zero-forcing constraints.

In [15], we have asked the question whether inseparability without zero-forcing constraints can also occur in a system with only two users. Using the classification proposed in this paper, we can answer this question in part and pose the remaining question more precisely: in this section, it was shown that inseparability of the second kind can occur with two users without zero-forcing, but we still do not know whether inseparability of the third kind can happen in a system with two users.

VII. Discussion

The categorization into three kinds of inseparability presented in this paper shows that the inseparability of MIMO broadcast channels with linear transceivers is a multifaceted topic that merits further study in order to be understood better. We hope that the systematic classification proposed in this paper helps to better understand existing as well as future results on the matter and assists in identifying separability-related questions that need to be studied. For instance, the systematic approach facilitated finding appropriate examples for the new inseparability results presented in Section VI.

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