Quantization-loss Reduction for 1-bit BOC Positioning

Friederike Wendler, Manuel Stein, Amine Mezghani, Josef A. Nossek
Institute for Circuit Theory and Signal Processing
Technische Universität München, Germany
{Stein, Nossek}@nws.ei.tum.de

BIOGRAPHY

Friederike Wendler (S'10) was born in Tübingen, Germany. She received the B.Sc. degree in engineering and information technology from Technische Universität München (TUM), Munich, Germany, in 2012. Currently, she is studying for a master at TUM and working at the Institute for Circuit Theory and Signal Processing (TUM). Her research interests focus on parameter estimation under coarse ADC. Since 2011 she is treasurer of the local IEEE student branch and head of the VDE student branch in Munich.

Manuel Stein (S'12) was born in Munich, Germany. He received the B.Sc. and Dipl.-Ing. degrees in engineering and information technology from Technische Universität München (TUM), Munich, Germany, in 2009 and 2010, respectively. After working as a software engineer for computer vision, image processing and aircraft engine control systems he joined the Institute for Circuit Theory and Signal Processing (TUM) in 2011 to work toward his doctorate degree. His research focuses on statistical signal processing for satellite-based navigation and synchronization.

Amine Mezghani (S'08) was born in Sfax, Tunisia. He received the "Diplome d'Ingénieur" degree from École Centrale de Paris, France, and the Dipl.-Ing. in electrical engineering from Technische Universität München (TUM), Munich, Germany, both in 2006. Since 2006, he has been working toward the doctorate degree at the Institute for Circuit Theory and Signal Processing (TUM). His research interests focus on the study on information theory and signal processing under low-precision ADCs.

Josef A. Nossek (S'72-M'74-SM'81-F'93) received the Dipl.-Ing. and Dr. techn. degrees in electrical engineering from Vienna University of Technology, Vienna, Austria, in 1974 and 1980, respectively. He joined Siemens AG, Munich, Germany, in 1974, where he was engaged in the design of both passive and active filters for communication systems. From 1987 to 1989, he was the Head of the Radio Systems Design Department, where he was instrumental in introducing high-speed VLSI signal processing into digital microwave radio. Since April 1989, he has been a Professor of circuit theory and design with the Technische Universität München (TUM), Munich, Germany. Josef A. Nossek was the President Elect, President, and Past President of the IEEE Circuits and Systems Society in 2001, 2002, and 2003, respectively. He was the President of Verband der Elektrotechnik, Elektronik und Informationstechnik e.V. (VDE) in 2007 and 2008. He was the recipient of the ITG Best Paper Award in 1988, the Mannesmann Mobilfunk (currently Vodafone) Innovations Award in 1998, and the Award for Excellence in Teaching from the Bavarian Ministry for Science, Research and Art in 1998. From the IEEE Circuits and Systems Society, he received the Golden Jubilee Medal for "Outstanding Contributions to the Society" in 1999 and the Education Award in 2008. He was the recipient of the "Bundesverdienstkreuz am Bande" in 2008. In 2011 he received the IEEE Guillemin-Cauer Best Paper Award for his paper "Toward a Circuit Theory of Communication".

ABSTRACT

Accurate time-delay measurement is essential for positioning with Global Navigation Satellite Systems (GNSS). For mobile receivers small cost, moderate complexity and low energy consumption are desirable. In contrast, Safety-of-Life (SoL) applications require fast processing in order to meet strict time constraints. Concerning these technical considerations, single bit analog-to-digital converters (ADC) are highly attractive as they are simple to build, require low energy and allow to realize high sampling rates and fast digital processing. On the other hand, such systems introduce a loss in the estimation accuracy. In this work we derive a mathematical model for 1-bit GNSS re-

ceivers, which takes into account the effects of the analog filters and the non-linear quantizer in the receiver front-end. Based on this refinement we derive analytic formulations for the receiver performance by resorting to the estimation theoretic tools which have become popular for the analysis of ideal receivers with infinite resolution. Finally, we verify the 1-bit quantization-loss for time-delay estimation systems, acting in low SNR scenarios, and investigate possible approaches to reduce this degradation by adjusting the analog front-end.

INTRODUCTION

Today, location-based services are an integral part of mobile devices, while time synchronization is essential for technical infrastructure like communication systems or networks which distribute electrical power or financial data. Therefore the reception of GNSS signals has emerged as a powerful technique in order to attain precise information about position in time and space. While high accuracy is desired in such applications, the architecture of the receiver is limited by technical constraints like complexity, chip-size and energy consumption.

Coarse analog-to-digital converters (ADC) have the advantage of simplicity and require low energy during system operation. In particular 1-bit quantizers allow very high sampling rates and can therefore be placed close to the antenna in order to realize a quasi all-digital implementation. Further, the signal processing chain is simplified significantly. For example the correlation of a reference signal with a 1-bit receive signal requires no multiplications. Especially in the case of Binary Offset Carrier (BOC) signals, which are used for Galileo and modernized GPS, this is advantageous. The auto-correlation function of a BOC signal has additional extrema close to the global maximum. This requires additional correlation operations in order to determine the time-delay precisely and track it reliably. Besides these attractive properties, 1-bit quantization introduces a loss in the effective receive SNR. Due to the non-linear operation, information about the signal at the input of the quantization device is lost.

The mathematical background to solve the problem of determining a channel parameter like the time-delay of a signal which has been disturbed by additive noise is provided by estimation theory. Estimation theory yields methods to derive estimators and to analytically determine their performance. When applying this theory to real technical GNSS systems, due to mathematical tractability, it is usually assumed that the sampled receive signal is available with infinite resolution and that the additive noise samples are white. Nevertheless, the assumption of signals with infinite resolution is not applicable for common digital GNSS receivers. These systems work with signals of finite and preferably coarse resolution due to the presence of ADC. In addition the assumption of white Gaussian noise after sampling with an ideal ADC of infinite resolution is only justified in very special cases. Therefore it is important to include all these aspects into the receiver model in order to provide the optimum design criteria and digital processing algorithms for future GNSS receivers.

Related Work

An early discussion on the problem of estimating unknown parameters based on quantized signals can be found in [1]. Channel parameter estimation based on single-bit quantization, under the assumption of uncorrelated noise, was treated in [2][3][4]. Also the problem of signal processing with low resolution when considering data transmission over noisy channels is discussed in different works. In [5] it has been shown that the well known reduction of low SNR channel capacity by factor $2/\pi$ ($-1.96\,\text{dB}$) due to 1-bit quantization holds also for the general MIMO case with uncorrelated noise. Paper [6] provides a survey on the quantization-loss of GNSS receivers, where the well known 1-bit loss of $-1.96\,\text{dB}$ has been verified. Contrary, in [7] the authors showed that this loss can be reduced for AWGN channels with low SNR by oversampling the analog receive signal. Additionally, in paper [8] the authors investigated the opportunity of improving the correlator output SNR under quantization by using high sampling frequencies and reducing the front-end bandwidth. In [9] it was recently shown that the capacity bound of MIMO channels under spatial noise correlation can be higher than under white noise for low SNR.

Contribution

To the best of our knowledge the performance of signal parameter estimation under 1-bit quantization and colored noise has not been considered in a systematic way. Using a refined expression of the Cramér-Rao lower bound (CRLB), which takes into account the non-linear effects of quantization, we validate the possibility of reducing the well known quantization-loss of $-1.96\,\text{dB}$ through the receiver front-end design. To this end, we provide a general framework for optimizing the front-end with respect to the Fisher information measure. Possible degrees of freedom in the front-end are the analog filter and the sampling rate. Consequently, the impact of the filter bandwidth, the filter shape and oversampling on time-delay estimation performance is shown for the case of a GNSS BOC receiver with a 1-bit ADC.

Outline

The paper is organized as follows. After introducing the notation, a signal model of the GNSS receive signal with a single satellite in an interference free scenario is given. For this model with infinite ADC resolution we review the derivation of the the maximum-likelihood estimator (MLE) for the time-delay as well as a lower bound for the variance of this estimator. Then the system model is refined in order to characterize the influence of sampling and filtering on the noise covariance and a linear model for the quantization device is introduced. For simulations we focus on Galileo BOC signals and show that the performance of time-delay estimation for quantized signals can be improved by optimizing the sampling rate and the analog filter.

Notation

Throughout this work scalars are denoted by lower case letters, whereas column vectors and matrices are written with lower case bold letters and upper case bold letters, respectively. In addition, the transpose of a matrix $\boldsymbol{A}$ is given by $\boldsymbol{A}^T$, while $\boldsymbol{A}^H$ represents the conjugate transpose (Hermitian) of a matrix. $\mathrm{diag}(\boldsymbol{a})$ yields a matrix with the elements of $\boldsymbol{a}$ on its diagonal and all other elements equal to zero, while $\mathrm{diag}(\boldsymbol{A})$ denotes a matrix with the diagonal elements of $\boldsymbol{A}$ on its diagonal and all other elements equal to zero. The n-th element of a vector $\boldsymbol{a} \in \mathbb{R}^{N\times 1}$ is denoted $\boldsymbol{a}_n$ with $n = 1, 2, \ldots, N$, while the entry in the n-th row and m-th column of a matrix $\boldsymbol{A} \in \mathbb{R}^{N\times M}$ is written $\boldsymbol{A}_{nm}$, where the indices always range from $n = 1, 2, \ldots, N$ and $m = 1, 2, \ldots, M$.

SYSTEM MODEL

Consider a coherent analog GNSS receive signal from one satellite with receive carrier-power $C \in \mathbb{R}$, time-delay $\tau \in \mathbb{R}$

$$\begin{aligned} y(t) &= \left(d(t) * \sqrt{C}\delta(t-\tau) + \eta'(t)\right) * h(t) * g(t) \\ &= \sqrt{C}s(t-\tau) + \eta(t), \end{aligned} \tag{1}$$

where

$$\delta(t-\tau) = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{else.} \end{cases} \tag{2}$$

The power spectral density (PSD) of the additive white Gaussian noise $\eta'(t) \in \mathbb{R}$ is $\Phi(\omega) = \frac{N_0}{2}$. The impulse responses $h(t)$ of an ideal low-pass filter with one-sided bandwidth B and $g(t)$ of an additional filter with arbitrary shape characterize the analog receive filter. The satellite transmit signal is assumed to be

$$d(t) = \sum_{m=-\infty}^{\infty} \boldsymbol{b}_{(\mathrm{mod}(m,M)+1)} p(t - mT_c) \tag{3}$$

where $\boldsymbol{b}_m \in \{-1, 1\}^{M\times 1}$ is a code sequence of length M with chip duration T_c and chip-rate $f_c = \frac{1}{T_c}$, while $p(t)$ is the transmit pulse shape. The structure of the receive signal

$$s(t) = d(t) * h(t) * g(t) \tag{4}$$

is known at the receiver. Contrary, τ is a deterministic time-delay parameter but unknown at the receiver. Collecting N samples of the receive signal at a rate of $f_s = \frac{1}{T_s}$ yields

$$\boldsymbol{y} = \sqrt{C}\boldsymbol{s}(\tau) + \boldsymbol{\eta}, \tag{5}$$

where for convenience N is restricted to be an even number and

$$\begin{aligned} \boldsymbol{s}(\tau) &= [s(-\tau), s(T_s-\tau), \ldots, s((N-1)T_s-\tau)]^T \in \mathbb{R}^{N\times 1}, \\ \boldsymbol{\eta} &= [\eta(0), \eta(T_s), \ldots, \eta((N-1)T_s)]^T \in \mathbb{R}^{N\times 1}, \\ \boldsymbol{y} &= [y(0), y(T_s), \ldots, y((N-1)T_s)]^T \in \mathbb{R}^{N\times 1}. \end{aligned} \tag{6}$$

The noise samples $\boldsymbol{\eta}$ are assumed to be additive Gaussian noise with zero mean and a covariance matrix

$$\mathrm{E}\left[\boldsymbol{\eta}\boldsymbol{\eta}^T\right] = \boldsymbol{R}. \tag{7}$$

The probability density function (pdf) of the random receive signal $\boldsymbol{y}$, parametrized by τ, is therefore

$$\begin{aligned} p_{\boldsymbol{y}}(\boldsymbol{y};\tau) = &\frac{1}{\sqrt{(2\pi)^N \det(\boldsymbol{R})}} \cdot \\ &\exp\left(-\frac{1}{2}\left(\boldsymbol{y} - \sqrt{C}\boldsymbol{s}(\tau)\right)^T \boldsymbol{R}^{-1}\left(\boldsymbol{y} - \sqrt{C}\boldsymbol{s}(\tau)\right)\right). \end{aligned} \tag{8}$$

PARAMETER ESTIMATION

The receiver determines its position by estimating the distance to four or more in-view satellites. If the receiver and the satellites have access to the same reference point in time, the distance can be calculated with the estimated signal propagation delay. In order to determine τ the receive signal is correlated with a delayed local replica of the transmit signal $\boldsymbol{s}(\tau)$.

Maximum Likelihood Estimation

For the problem of determining an estimate $\hat{\tau}(\boldsymbol{y}) \in \mathbb{R}$, the maximum likelihood estimator (MLE) is efficient if N is large. The MLE decides for the value $\hat{\tau}(\boldsymbol{y})$ which maximizes the parametrized pdf $p_{\boldsymbol{y}}(\boldsymbol{y};\tau)$

$$\begin{aligned} \hat{\tau} &= \arg\max_{\tau} p_{\boldsymbol{y}}(\boldsymbol{y};\tau) \\ &= \arg\min_{\tau} \left(\boldsymbol{y} - \sqrt{C}\boldsymbol{s}(\tau)\right)^T \boldsymbol{R}^{-1}\left(\boldsymbol{y} - \sqrt{C}\boldsymbol{s}(\tau)\right) \\ &= \arg\min_{\tau} \boldsymbol{y}^T \boldsymbol{R}^{-1}\boldsymbol{s}(\tau), \end{aligned} \tag{9}$$

where it was assumed that $\boldsymbol{s}(\tau)^T\boldsymbol{R}^{-1}\boldsymbol{s}(\tau)$ is constant in τ for sufficiently large N, which is shown in (57).

Performance Bound

The variance of the estimator $\hat{\tau}(\boldsymbol{y})$ with $\boldsymbol{y}$ being distributed according to the parametrized pdf (8) is given by

$$\mathrm{var}(\hat{\tau}) = \mathrm{E}\left[\left(\hat{\tau}(\boldsymbol{y}) - \mathrm{E}\left[\hat{\tau}(\boldsymbol{y})\right]\right)^2\right], \tag{10}$$

where $\mathrm{E}[\bullet]$ is the expectation with respect to $p_{\boldsymbol{y}}(\boldsymbol{y};\tau)$. A lower bound for the achievable variance of an unbiased estimator, i.e., an estimator which asymptotically in N satisfies

$$\mathrm{E}\left[\hat{\tau}(\boldsymbol{y})\right] = \tau, \tag{11}$$

is given by the CRLB [10]

$$\mathrm{var}(\hat{\tau}) \geq \frac{1}{F(\tau)}, \tag{12}$$

where

$$F(\tau) = C\frac{\partial \boldsymbol{s}(\tau)^T}{\partial \tau}\boldsymbol{R}^{-1}\frac{\partial \boldsymbol{s}(\tau)}{\partial \tau} \tag{13}$$

is the Fisher information measure.

FILTERS AND SAMPLING RATE

When considering the problem of GNSS parameter estimation, often the assumption is made, that the noise $\eta(t)$ behind the analog filter $h(t) * g(t)$ is white and therefore $\boldsymbol{R}$ is a scaled identity matrix. This assumption neglects the noise correlations produced by the receiver front-end. To achieve optimum performance with the MLE, the correct noise correlation has to be used in (9). In the following the properties of $\eta(t)$ due to the filter bandwidth B, the filter shape of $g(t)$ and the sampling rate f_s are discussed and an exact expression for $\boldsymbol{R}$ is derived.

Noise characterization for filtered and sampled signals

The frequency response of the ideal low-pass filter $h(t)$ in the analog filter front-end of the receiver is

$$H(\omega) = \begin{cases} 1 & \text{if } -2\pi B \leq \omega \leq 2\pi B \\ 0 & \text{else,} \end{cases} \tag{14}$$

while $g(t)$ has an arbitrary frequency response $G(\omega)$. The covariance matrix $\boldsymbol{R}$ of the sampled noise $\boldsymbol{\eta}$ can be determined with the auto-correlation function of $\eta(t)$

$$r(t) = \int_{-\infty}^{\infty} \eta(t)\eta^*(\lambda - t)\mathrm{d}\lambda \tag{15}$$

which has a Fourier representation

$$R(\omega) = \frac{N_0}{2}|H(\omega)|^2|G(\omega)|^2. \tag{16}$$

Using the inverse Fourier transform, the auto-correlation function of the noise after filtering with $H(\omega)G(\omega)$ is given by

$$r(t) = \frac{N_0}{4\pi}\int_{-2\pi B}^{2\pi B} |G(\omega)|^2 e^{j\omega t} d\omega \tag{17}$$

and therefore the entries of the noise covariance matrix $\boldsymbol{R}$ are

$$\boldsymbol{R}_{ij} = r\left(T_s|i-j|\right). \tag{18}$$

Special case: Ideal low-pass filter

Considering the case $G(\omega) = 1$, the auto-correlation function of the noise, strictly band-limited to B, is

$$\begin{aligned} r(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\omega) e^{j\omega t} d\omega \\ &= \frac{N_0}{4\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega t} d\omega \\ &= \frac{N_0}{4\pi}\int_{-2\pi B}^{2\pi B} e^{j\omega t} d\omega \\ &= BN_0 \,\mathrm{sinc}\left(2Bt\right), \end{aligned} \tag{19}$$

where a normalized version of the classical *sinc*-function is used

$$\mathrm{sinc}\left(x\right) = \frac{\sin(\pi x)}{\pi x}. \tag{20}$$

Consequently, the entries of the noise covariance matrix $\boldsymbol{R}$ of the band-limited and sampled noise $\boldsymbol{\eta}$ are

$$\boldsymbol{R}_{ij} = BN_0 \,\mathrm{sinc}\left(2BT_s|i-j|\right). \tag{21}$$

Note, that the common assumption of uncorrelated noise with a covariance matrix that has vanishing off-diagonal entries is only correct if the signal is filtered with an ideal low-pass $H(\omega)$ and the sampling rate is exactly $f_s = 2B$.

QUANTIZATION

For digital signal processing the discrete-time but continuous-valued signal $\boldsymbol{y}$ is quantized by a non-linear function

$$\boldsymbol{r} = Q(\boldsymbol{y}). \tag{22}$$

Especially for coarse quantization this has a significant influence on the noise correlation. In order to achieve optimum estimation performance for quantized receive signals, the noise covariance matrix has to be adjusted with respect to the quantization operation. In the following the extreme case of 1-bit hard-limiting quantization is discussed where the quantization device $Q(\bullet)$ processes every signal component $\boldsymbol{y}_i$ individually

$$Q(\boldsymbol{y}_i) = \begin{cases} 1 & \text{if } \boldsymbol{y}_i \geq 0 \\ -1 & \text{if } \boldsymbol{y}_i < 0. \end{cases} \tag{23}$$

Bussgang decomposition for quantized signals

In [9] it was shown that using the Bussgang theorem [11] the output of a non-linear device, such as a 1-bit quantizer, can be approximated by a linear signal component and an uncorrelated distortion $\boldsymbol{e}$

$$\boldsymbol{r} = \boldsymbol{Q}\boldsymbol{y} + \boldsymbol{e}. \tag{24}$$

By minimizing the mean squared error

$$\mathrm{E}\left[||\boldsymbol{e}||^2\right] = \mathrm{E}\left[||\boldsymbol{r} - \boldsymbol{Q}\boldsymbol{y}||^2\right] \tag{25}$$

the matrix

$$\begin{aligned}\boldsymbol{Q} &= \arg\min_{\boldsymbol{Q}} \mathrm{E}\left[||\boldsymbol{r} - \boldsymbol{Q}\boldsymbol{y}||^2\right] \\ &= \arg\min_{\boldsymbol{Q}} \mathrm{tr}\left(\boldsymbol{R}_{\boldsymbol{rr}} - \boldsymbol{R}_{\boldsymbol{ry}}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{yr}} + \boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{yy}}\boldsymbol{Q}^T\right) \\ &= \arg\min_{\boldsymbol{Q}} \Lambda(\boldsymbol{Q}) \end{aligned} \tag{26}$$

can be obtained. A solution to (26) has to fulfill

$$\frac{\partial \Lambda(\boldsymbol{Q})}{\partial \boldsymbol{Q}} = -2\boldsymbol{R}_{\boldsymbol{ry}} + 2\boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{yy}} = \boldsymbol{0}, \tag{27}$$

which yields

$$\boldsymbol{Q} = \boldsymbol{R}_{\boldsymbol{ry}}\boldsymbol{R}_{\boldsymbol{yy}}^{-1}, \tag{28}$$

where $\boldsymbol{R}_{\boldsymbol{ry}}$ denotes the cross-correlation matrix of the input signal of the quantizer $\boldsymbol{y}$ with the output of the quantizer $\boldsymbol{r}$ and $\boldsymbol{R}_{\boldsymbol{yy}}$ denotes the auto-correlation matrix of the signal $\boldsymbol{y}$. For a 1-bit hard-limiter the cross-correlation matrix is given by [12, p. 307]

$$\boldsymbol{R}_{\boldsymbol{ry}} = \sqrt{\frac{2}{\pi}}\, \mathrm{diag}(\boldsymbol{R}_{\boldsymbol{yy}})^{-\frac{1}{2}}\boldsymbol{R}_{\boldsymbol{yy}} \tag{29}$$

such that

$$\boldsymbol{Q} = \sqrt{\frac{2}{\pi}}\, \mathrm{diag}(\boldsymbol{R}_{\boldsymbol{yy}})^{-\frac{1}{2}}. \tag{30}$$

Note, that due to the orthogonality principle

$$\mathrm{E}\left[\boldsymbol{e}\boldsymbol{y}^T\right] = \boldsymbol{0}. \tag{31}$$

Effective noise model for the quantized signal

For the GNSS signal model (5), the output of the quantizer is given by

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{Q}\boldsymbol{y} + \boldsymbol{e} \\ &= \sqrt{C}\boldsymbol{Q}\boldsymbol{s}(\tau) + \boldsymbol{Q}\boldsymbol{\eta} + \boldsymbol{e} \\ &= \sqrt{C}\boldsymbol{s}_Q(\tau) + \boldsymbol{\eta}_Q. \end{aligned} \tag{32}$$

The covariance matrix of the error $\boldsymbol{e}$

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{ee}} &= \boldsymbol{R}_{\boldsymbol{rr}} - \boldsymbol{R}_{\boldsymbol{ry}}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{yr}} + \boldsymbol{Q}\boldsymbol{R}_{\boldsymbol{yy}}\boldsymbol{Q}^T \\ &= \boldsymbol{R}_{\boldsymbol{rr}} - \boldsymbol{R}_{\boldsymbol{ry}}\boldsymbol{R}_{\boldsymbol{yy}}^{-1}\boldsymbol{R}_{\boldsymbol{yr}} \end{aligned} \tag{33}$$

allows to characterize $\boldsymbol{\eta}_Q$ by the covariance matrix

$$\begin{aligned}\boldsymbol{R}_Q &= \mathrm{E}\left[(\boldsymbol{Q}\boldsymbol{\eta} + \boldsymbol{e})(\boldsymbol{Q}\boldsymbol{\eta} + \boldsymbol{e})^T\right] \\ &= \boldsymbol{Q}\boldsymbol{R}\boldsymbol{Q}^T + \boldsymbol{R}_{\boldsymbol{ee}} \\ &= \boldsymbol{Q}\boldsymbol{R}\boldsymbol{Q}^T + \boldsymbol{R}_{\boldsymbol{rr}} - \boldsymbol{R}_{\boldsymbol{ry}}\boldsymbol{R}_{\boldsymbol{yy}}^{-1}\boldsymbol{R}_{\boldsymbol{yr}}, \end{aligned} \tag{34}$$

where it was used that in the low SNR regime

$$\begin{aligned}\mathrm{E}\left[\boldsymbol{\eta}\boldsymbol{e}^T\right] &= \mathrm{E}_{\boldsymbol{y}}\left[\mathrm{E}\left[\boldsymbol{e}\boldsymbol{\eta}^T|\boldsymbol{y}\right]\right] \\ &\approx \mathrm{E}_{\boldsymbol{y}}\left[\mathrm{E}\left[\boldsymbol{e}\boldsymbol{y}^T|\boldsymbol{y}\right]\right] \\ &= \mathrm{E}\left[\boldsymbol{e}\boldsymbol{y}^T\right] \\ &= \boldsymbol{0}. \end{aligned} \tag{35}$$

Using the *arcsine law* [12, p. 307] allows to specify the auto-correlation matrix $\boldsymbol{R}_{\boldsymbol{rr}}$

$$\boldsymbol{R}_{\boldsymbol{rr}} = \frac{2}{\pi}\arcsin\left(\mathrm{diag}\left(\boldsymbol{R}_{\boldsymbol{yy}}\right)^{-\frac{1}{2}}\boldsymbol{R}_{\boldsymbol{yy}}\,\mathrm{diag}\left(\boldsymbol{R}_{\boldsymbol{yy}}\right)^{-\frac{1}{2}}\right) \tag{36}$$

for a 1-bit hard-limiting quantizer. Inserting (30) and (36) into (34)

$$\begin{aligned}\boldsymbol{R}_Q =& \frac{2}{\pi}\,\mathrm{diag}(\boldsymbol{R}_{\boldsymbol{yy}})^{-\frac{1}{2}}\boldsymbol{R}\,\mathrm{diag}(\boldsymbol{R}_{\boldsymbol{yy}})^{-\frac{1}{2}} \\ &- \frac{2}{\pi}\,\mathrm{diag}(\boldsymbol{R}_{\boldsymbol{yy}})^{-\frac{1}{2}}\boldsymbol{R}_{\boldsymbol{yy}}\,\mathrm{diag}(\boldsymbol{R}_{\boldsymbol{yy}})^{-\frac{1}{2}} \\ &+ \frac{2}{\pi}\arcsin\left(\mathrm{diag}\left(\boldsymbol{R}_{\boldsymbol{yy}}\right)^{-\frac{1}{2}}\boldsymbol{R}_{\boldsymbol{yy}}\,\mathrm{diag}\left(\boldsymbol{R}_{\boldsymbol{yy}}\right)^{-\frac{1}{2}}\right) \end{aligned} \tag{37}$$

gives a full characterization of the gain-noise model for the 1-bit hard-limiter. Note, that the covariance matrix of $\boldsymbol{\eta}_Q$ only depends on the covariance matrices of $\boldsymbol{y}$ and $\boldsymbol{\eta}$. For signals in the low SNR regime $\boldsymbol{R}_{yy} = C\boldsymbol{R}_{ss} + \boldsymbol{R}$ is dominated by $\boldsymbol{R}$, which allows to approximate

$$\boldsymbol{R}_Q \approx \frac{2}{\pi}\arcsin\left(\mathrm{diag}\left(\boldsymbol{R}\right)^{-\frac{1}{2}}\boldsymbol{R}\,\mathrm{diag}\left(\boldsymbol{R}\right)^{-\frac{1}{2}}\right), \tag{38}$$

by $\boldsymbol{R}_{\boldsymbol{yy}} \approx \boldsymbol{R}$. In [13] it was shown that under a fixed second moment additive Gaussian noise minimizes the Fisher information. Therefore a Gaussian model in general yields an equivalent system, which is pessimistic with respect to the parameter estimation performance.

Considering the Fisher information measure for a quantized GNSS receive signal

$$F(\tau) = C\frac{\partial \boldsymbol{s}(\tau)^T}{\partial \tau}\boldsymbol{Q}^T\boldsymbol{R}_Q^{-1}\boldsymbol{Q}\frac{\partial \boldsymbol{s}(\tau)}{\partial \tau} \tag{39}$$

it is useful to define an effective noise covariance matrix

$$\boldsymbol{R}'_Q = \boldsymbol{Q}^{-1}\boldsymbol{R}_Q(\boldsymbol{Q}^T)^{-1} \tag{40}$$

which includes the effects of quantization on the noise correlation as well as on the signal.

FISHER INFORMATION - FREQUENCY DOMAIN

Due to the inversion of $\boldsymbol{R}$, the calculation of the Fisher information $F(\tau)$ in the time domain (39) is costly if N becomes large. In the following, an expression for the Fisher information in the frequency domain is derived, which allows a simple calculation for a large but finite number of samples.

As the GNSS signal $s(t)$ is periodic with periodic time T_0 it can be represented by an infinite sum of complex sinusoids

$$s(t) = \sum_{k=-\infty}^{\infty} \tilde{s}_k e^{jk\omega_0 t}, \tag{41}$$

with $\omega_0 = \frac{2\pi}{T_0}$ and

$$\tilde{s}_k = \frac{1}{T_0} S\left(k\omega_0\right), \tag{42}$$

where

$$S(\omega) = \int_0^{T_0} s(t) e^{-j\omega t} \mathrm{d}t. \tag{43}$$

As $s(t)$ is band-limited, the sum (41) can be replaced by a finite counterpart

$$s\left(t\right) = \sum_{k=-\frac{K}{2}}^{\frac{K}{2}-1} \tilde{s}_k e^{jk\omega_0 t}, \tag{44}$$

with

$$K \geq \frac{4\pi B}{\omega_0} = 2BT_0. \tag{45}$$

Sampling the band-limited signal at a rate $f_s \geq 2B$, each element of the sampled transmit signal can be written

$$s\left(nT_S\right) = \sum_{k=-\frac{K}{2}}^{\frac{K}{2}-1} \tilde{s}_k e^{jk\omega_0 nT_S}. \tag{46}$$

For $N = K$ and $\frac{T_s}{T_c} = \frac{M}{N}$ the transmit signal is given by

$$\boldsymbol{s} = \boldsymbol{W}\tilde{\boldsymbol{s}}, \tag{47}$$

where $\boldsymbol{W}$ is the well known Inverse Discrete Fourier Transformation (IDFT) matrix, which for convenience is here defined in a modified form

$$\boldsymbol{W}_{kn} = \frac{1}{\sqrt{N}} e^{j2\pi \frac{\left(-\frac{N}{2}-1+k\right)\left(-\frac{N}{2}-1+n\right)}{N}} \in \mathbb{C}^{N\times N}, \tag{48}$$

and

$$\tilde{\boldsymbol{s}} = [\tilde{s}_{-\frac{N}{2}}, \tilde{s}_{-\frac{N}{2}+1}, \dots, \tilde{s}_{\frac{N}{2}-1}]^T \in \mathbb{C}^N. \tag{49}$$

The time shifted signal

$$s\left(nT_S - \tau\right) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{s}_k e^{jk\omega_0 (nT_S - \tau)} \tag{50}$$

in vector notation is given accordingly by

$$\boldsymbol{s}(\tau) = \boldsymbol{W}\boldsymbol{T}(\tau)\tilde{\boldsymbol{s}}, \tag{51}$$

where $\boldsymbol{T}(\tau) \in \mathbb{C}^{N\times N}$ is a diagonal matrix with

$$\boldsymbol{T}_{kk}(\tau) = e^{-j\left(-\frac{N}{2}-1+k\right)\omega_0 \tau}. \tag{52}$$

For large N the temporal noise covariance matrix $\boldsymbol{R}$ is approximately circulant and therefore can be replaced by

$$\boldsymbol{R} \approx \boldsymbol{W}\boldsymbol{\Omega}_R \boldsymbol{W}^{-1}, \tag{53}$$

where $\boldsymbol{\Omega}_R \in \mathbb{C}^{N\times N}$ is a diagonal matrix with

$$\boldsymbol{\Omega}_{R,kk} = \frac{1}{T_0} \Phi\left(\left(-\frac{N}{2} - 1 + k\right)\omega_0\right) \tag{54}$$

and $\Phi(\omega)$ is the PSD of the noise. Using (51) and (53)

$$\begin{aligned} \boldsymbol{s}\left(\tau\right)^T \boldsymbol{R}^{-1} \boldsymbol{s}\left(\tau\right) &= \left(\boldsymbol{W}\boldsymbol{T}(\tau)\tilde{\boldsymbol{s}}\right)^H \boldsymbol{W}\boldsymbol{\Omega}_R^{-1}\boldsymbol{W}^{-1}\boldsymbol{W}\boldsymbol{T}(\tau)\tilde{\boldsymbol{s}} \\ &= \tilde{\boldsymbol{s}}^H \boldsymbol{T}(\tau)^H \boldsymbol{\Omega}_R^{-1} \boldsymbol{T}(\tau)\tilde{\boldsymbol{s}} \\ &= \tilde{\boldsymbol{s}}^H \boldsymbol{\Omega}_R^{-1} \tilde{\boldsymbol{s}}, \end{aligned} \tag{55}$$

where we used

$$\begin{aligned} \boldsymbol{W}^H &= \boldsymbol{W}^{-1} \\ \boldsymbol{T}^H &= \boldsymbol{T}^{-1}. \end{aligned} \tag{56}$$

Therefore,

$$\frac{\partial \boldsymbol{s}\left(\tau\right)^T \boldsymbol{R}^{-1} \boldsymbol{s}\left(\tau\right)}{\partial \tau} = 0, \tag{57}$$

showing that the assumption in (9) is justified. Moreover, one element of the sampled derivative $\frac{\partial \boldsymbol{s}(\tau)}{\partial \tau}$ is given by

$$\frac{\partial s\left(nT_S - \tau\right)}{\partial \tau} = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} -jk\omega_0 \tilde{s}_k e^{jk\omega_0(nT_S - \tau)}. \tag{58}$$

Using the modified IDFT matrix (48), the derivative vector

$$\frac{\partial \boldsymbol{s}(\tau)}{\partial \tau} = \boldsymbol{W}\boldsymbol{T}(\tau)\boldsymbol{\Omega}_0 \tilde{\boldsymbol{s}}, \tag{59}$$

with the diagonal matrix $\boldsymbol{\Omega}_0 \in \mathbb{C}^{N\times N}$

$$\boldsymbol{\Omega}_{0,kk} = -j\omega_0 \left(-\frac{N}{2} - 1 + k\right), \tag{60}$$

allows to write the Fisher information of the time-delay

$$\begin{aligned} F(\tau) &= C \frac{\partial \boldsymbol{s}\left(\tau\right)^T}{\partial \tau} \boldsymbol{R}^{-1} \frac{\partial \boldsymbol{s}\left(\tau\right)}{\partial \tau} \\ &= C \tilde{\boldsymbol{s}}^H \boldsymbol{\Omega}_0^H \boldsymbol{\Omega}_R^{-1} \boldsymbol{\Omega}_0 \tilde{\boldsymbol{s}} \\ &= \frac{C}{T_0} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} (k\omega_0)^2 \frac{|S(k\omega_0)|^2}{\Phi(k\omega_0)} = F. \end{aligned} \tag{61}$$

Note that by (61) the Fisher information can be calculated in the frequency domain by evaluating a finite sum which is independent of τ. For $T_0 \to \infty$, (61) becomes a Riemann sum and converges to

$$\bar{F} = \frac{C}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2 |S(\omega)|^2}{\Phi(\omega)} \mathrm{d}\omega. \quad (62)$$

The spectrum of the signal is

$$S(\omega) = B(\omega)P(\omega)H(\omega)G(\omega) \quad (63)$$

with $P(\omega)$ being the spectrum of the transmit pulse $p(t)$ and $B(\omega)$ being the spectrum of the binary sequence $\boldsymbol{b}$ which for sufficiently large M is equal to

$$B(\omega) \approx \frac{T_0}{T_c}. \quad (64)$$

Therefore the Fisher information (61) can be approximated

$$F \approx \frac{T_0}{T_c} \frac{C}{2\pi} \int_{-2\pi B}^{2\pi B} \frac{\omega^2 |G(\omega)|^2 |P(\omega)|^2}{\Phi(\omega)} \mathrm{d}\omega. \quad (65)$$

ANALOG FRONT-END OPTIMIZATION

In the following a GNSS receiver with a 1-bit quantizer is optimized. Using the expression for the Fisher information in the frequency domain (61) and the refined expression for the noise correlation (73), it is demonstrated that the analog receiver front-end has an impact on the estimation performance of the time-delay. This is used to find optimum front-end parameters in order to achieve a performance gain in comparison to an unoptimized 1-bit GNSS receiver.

GNSS system

To visualize the possible performance improvements we consider a Galileo E1 OS receiver, for which $T_c = 977.52\,\mathrm{ns}$ ($f_c = \frac{1}{T_c} = 1.023\,\mathrm{MHz}$). The code sequence $\boldsymbol{b}$ has length $M = 4092$ and the spectrum of the BOC transmit pulse $p(t)$ is [14]

$$P(\omega) = -j2 \frac{\sin\left(\frac{\omega}{2f_c}\right)}{\omega} \tan\left(\frac{\omega}{4f_c}\right). \quad (66)$$

Reference systems

For illustration, the 1-bit receiver is compared to a reference receiver with infinite resolution. For an unquantized, band-limited signal with one-sided bandwidth $B = 2f_c$ which is sampled at Nyquist rate $f_s = 4f_c$, the Fisher information with respect to τ is

$$F_\infty = \frac{2C}{T_0 N_0} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} |k\omega_0|^2 |S(k\omega_0)|^2. \quad (67)$$

As an appropriate measure for the performance of a quantized system we employ the ratio of its Fisher information F_q over the Fisher information F_∞ of an ideal system with infinite resolution

$$\chi = \frac{F_q}{F_\infty}. \quad (68)$$

For a standard receiver with a 1-bit hard limiting quantizer and a receive signal with one-sided bandwidth $B = 2f_c$ which is sampled at Nyquist rate $f_s = 4f_c$ the Fisher information with respect to τ is

$$F_{q,\mathrm{ref}} = \frac{2}{\pi} \frac{2C}{T_0 N_0} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} |k\omega_0|^2 |S(k\omega_0)|^2 \quad (69)$$

and therefore the performance loss of this receiver is $\chi_{\mathrm{ref}} = \frac{F_{q,\mathrm{ref}}}{F_\infty} = 0.636$ ($-1.96\,\mathrm{dB}$) in comparison to the receiver with infinite resolution [15].

Receiver with optimized radio front-end

For a receive low-pass filter $H(\omega)$ with bandwidth

$$B = 2f_c\rho \quad \rho \in [0, 1], \quad (70)$$

sampling rate

$$f_s = 4f_c\mu \quad \mu \geq 1, \quad (71)$$

and an arbitrary filter with frequency response $G(\omega)$ the Fisher information is given by

$$\begin{aligned} F_q(\tau) &= \frac{C}{T_0} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{|k\omega_0|^2 |S(k\omega_0)|^2}{\Phi_q(k\omega_0; \boldsymbol{\nu})} \\ &= \frac{C}{T_0} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{|k\omega_0|^2 |D(k\omega_0)|^2 |H(k\omega_0)|^2 |G(k\omega_0)|^2}{\Phi_q(k\omega_0; \rho, \mu, \boldsymbol{g})}, \end{aligned} \quad (72)$$

where $\boldsymbol{\nu} = [\rho, \mu, \boldsymbol{g}]^T$ is the vector of all parameters influencing the noise correlation and $\boldsymbol{g}$ is the vector of all parameters specifying the filter $G(\omega)$. The spectrum of the quantized noise is given by

$$\Phi_q(\omega; \rho, \mu, \boldsymbol{g}) = T_s \sum_{n=-\infty}^{n=\infty} r(0; \rho, \mu, \boldsymbol{g}) \arcsin\left(\frac{r(nT_S; \rho, \mu, \boldsymbol{g})}{r(0; \rho, \mu, \boldsymbol{g})}\right) e^{-j\omega n T_s}. \quad (73)$$

Note, that it is not possible to apply the *arcsine law* directly in the frequency domain and therefore $\Phi_q(\omega; \rho, \mu, \boldsymbol{g})$ has to be calculated with the help of a Fourier series (73).

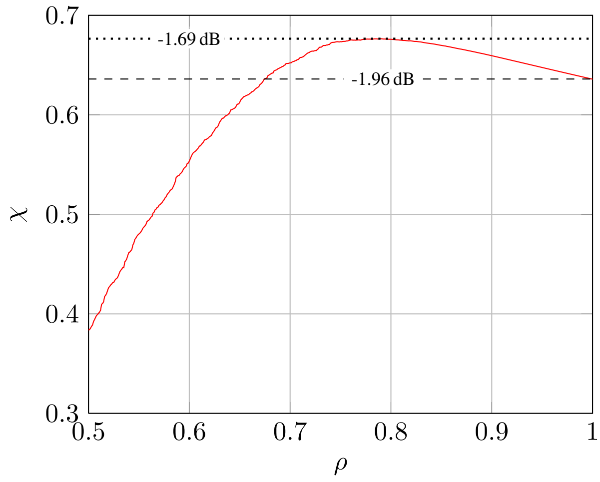

Fig. 1 Optimization of bandwidth

Bandwidth optimization

Fig. 1 shows the relative Fisher information χ for different bandwidth factors ρ, a fixed $\mu = 1$ and $G(\omega) = 1$. By using a filter $H(\omega)$ with $\rho < 1$ the noise of the analog signal becomes correlated. However, the 1-bit quantizer reduces this correlation due to the characteristic of the *arcsine* and distributes the noise power to higher frequencies which do not contain information on the transmit signal. For $\rho^* = 0.79$ the performance reaches its maximum (-1.7 dB).

Sampling rate optimization

Fig. 2 shows the relative Fisher information χ for different sampling factors μ under $\rho = 1$ and ρ^*, while no additional filter-shaping is applied, i.e. $G(\omega) = 1$. Contrary to the receiver with infinite resolution, oversampling increases the estimation performance for quantized signals, as the noise power can spread into higher frequencies which contain no signal information. For oversampling the performance increases significantly by doubling the sampling rate to $\mu = 2$. Increasing the sampling rate further just gives marginal additional improvements. For $\mu \to \infty$ the possible gain is approximately 1 dB. While increasing μ the optimum bandwidth factor approaches $\rho^* = 1$ and an optimization of the filter $H(\omega)$ becomes less important.

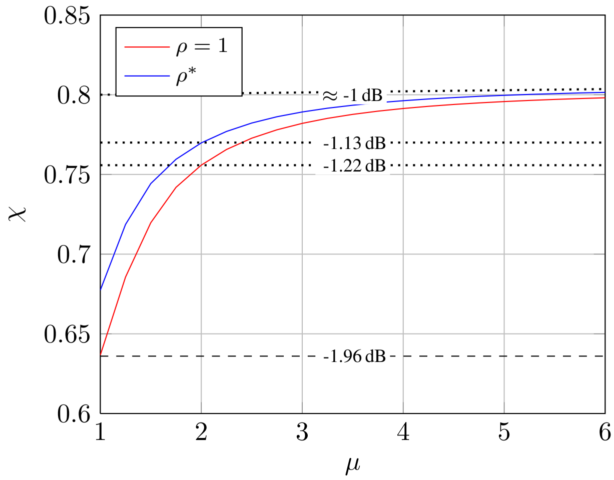

Fig. 2 Oversampling

High-pass filter optimization

For $\mu = 1$, Fig. 3 shows the relative Fisher information χ for a concatenation of $H(\omega)$ with an additional first-order high-pass filter $G(\omega)$ with frequency response

$$G(\omega) = \frac{\frac{j\omega}{\alpha}}{1 + \frac{j\omega}{\alpha}}, \tag{74}$$

for different α and a fixed bandwidth factor ρ and $\rho^* = 0.79$. Apparently, there is an optimum $\alpha^* \approx 0.5 \cdot 2\pi f_c$.

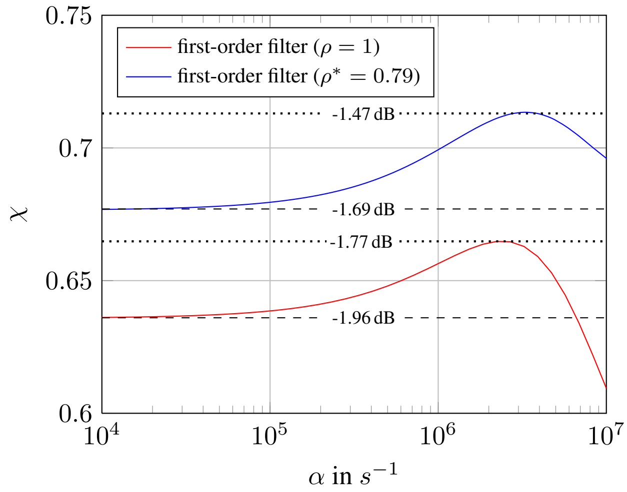

Fig. 3 Optimization of a high-pass filter

For α^* and ρ^* the possible gain in comparison to the unoptimized 1-bit GNSS receive system is close to 0.5 dB. The high-pass filter suppresses the center part of the signal spectrum, where also the spectrum of the BOC signal has a minimum and therefore has less impact on the estimation accuracy. Through 1-bit quantization the noise power spreads also into these free low frequencies.

ML RECEIVER

The previous results show that it is possible to improve the Fisher information with respect to a time-delay parameter and therefore to reduce the estimation error. By simulations of the MLE (9) it is verified that the optimization of the front-end based on a theoretic information measure also leads to an improvement for 1-bit GNSS systems that can be realized in practice. The root-mean-square error

(RMSE) of the estimator in meter

$$\text{RMSE}(\tau) \hat{=} \sqrt{\text{E}\left[(\hat{\tau} - \tau)^2\right]} \cdot v_p, \qquad (75)$$

where v_p is the propagation velocity, is used as a figure-of-merit. The RMSE corresponds to the standard deviation $\sqrt{\text{var}(\hat{\tau})}$ which was bounded through the CRLB. Simulations are carried out for the Galileo satellite number 3 and a fixed time-delay of $\tau = 49\,\text{ns}$, while coherent signal reception is assumed. In Fig. 4 the CRLB and the RMSE for the two reference systems are shown. For C/N_0 between

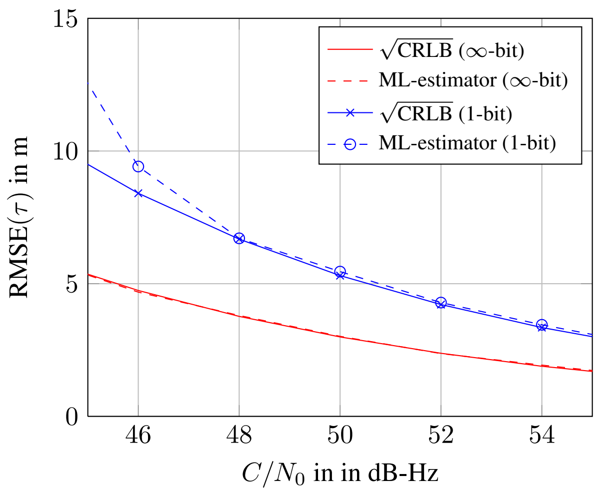

Fig. 4 RMSE(τ) for the reference receiver

50dB − Hz and 54dB − Hz the MLE achieves the lower bound. Therefore simulations are carried out in this range. In Fig. 5 CRLB and RMSE are shown for $\rho^* = 0.79$ as well as for oversampling with $\mu = 2$. It can be seen that the gain of $-0.26\,\text{dB}$ and $-0.77\,\text{dB}$, respectively, is achievable also for a real estimator.

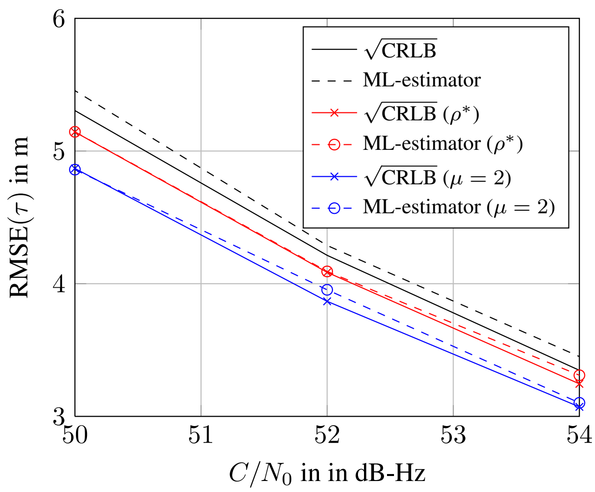

Fig. 5 RMSE(τ) for the optimized receivers (1-bit)

CONCLUSION

Coarse quantization is advantageous for the analog-to-digital conversion and signal processing complexity. Here it was shown that it is possible to significantly reduce the well known $\frac{2}{\pi}(-1.96\,\text{dB})$ loss due to 1-bit quantization. Using the fact that this performance loss is not valid in the low SNR regime if noise correlation is present, an analog GNSS receiver front-end can be designed with respect to the Fisher information in order to attain a significant performance gain. Note, that this gain is achievable without any additional effort, once the optimum analog filter parameters are determined.

ACKNOWLEDGMENTS

The financial support of the Federal Ministry of Economics and Technology, Germany, through grant 50NA1110 is kindly acknowledged.

REFERENCES

[1] R. E. Curry. *Estimation And Control With Quantized Measurements*. M.I.T. Press, 1970.

[2] T. M. Lok and V. K. W. Wei. Channel Estimation with Quantized Observations. *IEEE International Symposium on Information Theory*, page 333, August 1998.

[3] M. Ivrlač and J.A. Nossek. On MIMO Channel Estimation with Single-Bit Signal-Quantization. *ITG Workshop on Smart Antennas*, February 2007.

[4] A. Mezghani, F. Antreich, and J. A. Nossek. Multiple Parameter Estimation with Quantized Channel Output. *International ITG Workshop on Smart Antennas*, pages 143–150, February 2012.

[5] A. Mezghani and Josef A. Nossek. Analysis of 1-bit Output Noncoherent Fading Channels in the Low SNR Regime. *IEEE International Symposium on Information Theory (ISIT)*, June 2009.

[6] J. W. Betz and N. R. Shnidman. Receiver Processing Losses with Bandlimiting and One-Bit Sampling. In *Proceedings of the 20th International Technical Meeting of the Satellite Division of The Institute of Navigation (ION GNSS 2007)*, TX, pages 1244–1256, Fort Worth, September 2007.

[7] T. Koch and A. Lapidoth. Increased Capacity per Unit-Cost by Oversampling. *http://arxiv.org/abs/1008.5393*, August 2010.

[8] J.T. Curran, D. Borio, C. C. Murphy, and G. Lachapelle. Reducing Front-End Bandwidth may Improve Digital GNSS Receiver Performance. *IEEE Transactions on Signal Processing, Vol. 58, No. 4*, April 2010.

[9] A. Mezghani and J. A. Nossek. Capacity Lower Bound of MIMO Channels with Output Quantization and Correlated Noise. *ISIT*, 2012.

[10] S. M. Kay. *Fundamentals of Statistical Signal Processing, Volume 1: Estimation Theory*. Prentice Hall PTR, 1993.

[11] J. J. Bussgang. Cross-correlation function of amplitude-distorted Gaussian signals. *Res. Lab. Elec., Mas. Inst. Technol.*, Tech. Rep. 216, March 1952.

[12] A. Papoulis. *Probability, Random Variables, and Stochastic Processes Third Edition*. Mcgraw-Hill College, 1991.

[13] M. Stein and A. Mezghani. On information measures and worst-case additive noise. Technical report, Institute for Circuit Theory and Signal Processing, 2012.

[14] F. Antreich. *Array Processing and Signal Design for Timing Synchronization*. Dissertation, Technische Universität München, 2011.

[15] J. H. Van Vleck and D. Middleton. The spectrum of clipped noise. *Proceedings of the IEEE , vol.54, no.1*, pages 2– 19, January 1966.