A finite cell approach for discretization of random fields

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Abstract: A new method for discretization of Gaussian random fields with only a small number of random variables in the representation is introduced. The method is based on the Karhunen-Loève (KL) expansion, which is optimal among series expansion methods with respect to the global mean square truncation error. The resulting integral eigenvalue problem in the KL-expansion is discretized using a finite cell (FC) approach; i.e. the domain of computation is extended beyond the physical domain up to the boundaries of an embedding domain with a primitive geometrical shape. Higher order polynomials are used as FC shape functions. The approach is useful for random fields defined on domains with complex geometries since it shifts the problem from the mesh generation to the integration of discontinuous functions defined over a fictitious domain. A suitable approach for numerical integration is described. The presented method is compared to the Expansion Optimal Linear Estimation (EOLE) method and to the finite element discretization of the KL-expansion with respect to the mean error variance and in terms of computational costs. On the one hand, the proposed approach shows an exponential rate of convergence in terms of the dimension of the matrix eigenvalue problem to solve for a fixed number of random variables. On the other hand, obtaining a solution for the random field approximation takes considerably longer than with the EOLE method. However, the generation of a realization of the random field representation with the finite cell approach is computationally more efficient than with EOLE.

1 Introduction

A stochastic analysis of structures in civil engineering often requires the modeling of input parameters that vary randomly in space (e.g. load distributions or material parameters). This type of uncertainty is modeled by means of ran-

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dom fields. A random field represents a random quantity at each point of a continuous domain, and, thus, consists of an infinite number of random variables. For computational purposes, the random field has to be expressed using a finite number of random variables. This step is referred to as random field discretization.

The efficiency of a random field discretization method depends on its ability to approximate the original random field accurately with a minimum number of random variables. Accuracy is to be defined with respect to a certain error measure such as the global mean square truncation error. It is advantageous to keep the number of random variables in the representation of the random field small, since it can have a considerable influence on the computational costs of a subsequent stochastic analysis. An example is finite element reliability analysis [2] where, for instance, a first-order reliability method (FORM) is employed to obtain an estimate of the failure probability of the investigated system. Another example is the spectral stochastic finite element method [4]. For this method, the size of the problem to solve is a function involving factorials of the input random variables and, thus, the problem size increases drastically with increasing number of random variables. An overview of random field discretization methods is given in [9].

The Karhunen-Loève (KL) expansion of random fields is optimal in the global mean square truncation error with respect to the number of random variables in the representation [5]. However, its analytical solution is available only for primitive geometries and for a few selected autocovariance functions. For complex-shaped geometries, a finite element based approach can be chosen to approximate the solution of the KL expansion. However, this requires a spatial decomposition of the domain.

The requirements to a good random field mesh are not the same as the requirements to a good mesh of the corresponding mechanical system (see [9]). Consequently, two different meshes might be necessary. However, working with different meshes is a handicap in writing efficient algorithms for post-processing the random field (e.g. evaluating the realization of the field at every finite element Gauss-point). A possible remedy is to use the elements in the FE mesh as a basis for the random field mesh, and to adapt the mesh by either refining individual elements or by coalescing different elements. This approach becomes impractical for two- or three-dimensional problems if the physical domain is of complex geometrical shape. This includes domains with curved boundaries, domains with holes, and porous media. Therefore, meshless approaches appear to be favorable on complex shaped domains.

The Expansion Optimal Linear Estimation (EOLE) method [6] does not require a mesh; the domain of the field is approximated by a number of points. Consequently, the shape of the physical domain is of minor importance, since the selection of points can be easily performed on a fictitious domain containing the actual physical domain, where all points outside of the physical domain are neglected. Another meshless approach [7] is to embed the physical domain in a larger domain of primitive geometrical shape. The KL expansion is then solved for the primitive domain, either analytically or numerically. However, the optimality of the KL expansion with respect to the mean square truncation error is lost in this approach since the expansion is solved on a domain that is larger than the actual physical domain.

The finite cell (FC) method [8] is a fictitious domain approach, developed as an extension of the finite element method. Following this approach, the physical domain is embedded in elements of primitive geometrical shape. Higher order shape functions are of crucial importance for the applicability of the method because they yield a fast rate of convergence [8]. The finite cell method shifts the problem of complex geometries from the mesh generation to the integration.

In this work, a finite cell like approach is utilized to discretize the spatial domain of the random field and, thus, to approximate the solution of the Karhunen-Loève expansion numerically. The proposed method inherits the efficiency of the KL expansion if the error in the numerical integration is negligible and if the eigenmodes of the KL expansion can be approximated well by the chosen shape functions. The presented method is compared to the EOLE method and to the finite element discretization of the KL-expansion. The proposed approach shows an exponential rate of convergence with respect to the size of the matrix eigenvalue problem to solve. On the other hand, obtaining a solution for the random field approximation takes considerably longer than with the EOLE method. However, the generation of a realization of the random field representation with the finite cell approach is more efficient in terms of computational cost than with EOLE.

2 Discretization of random fields

A continuous random field $H(\mathbf{x},\theta)$ may be loosely defined as a random function that describes a random quantity at each point $\mathbf{x}\in\Omega$ of a continuous domain $\Omega\subset\mathbf{R}^d$, $d\in\mathbf{N}_{>0}$. $\theta\in\Theta$ is a coordinate in the sample space Θ, and (Θ,F,P) is a complete probability space. If the random quantity attached to

each point $\mathbf{x}$ is a random variable, the random field is said to be *univariate* or real-valued. If the random quantity is a random vector, the field is called *multivariate*. The dimension d of a random field is the dimension of its topological space Ω. One usually distinguishes between a *one-* and a *multidimensional* random field, the former one is also referred to as *random process*. The field is said to be Gaussian if the distribution of $(H(\mathbf{x}_1,\theta),\ldots,H(\mathbf{x}_n,\theta))$ is jointly Gaussian for any $(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in\Omega$ and any $n\in\mathbf{N}_{>0}$. It is completely defined by its mean function $\mu(\mathbf{x}):\Omega\to\mathbf{R}$ and auto-covariance function $Cov(\mathbf{x},\mathbf{x}'):\Omega\times\Omega\to\mathbf{R}$. In the following, we will restrict ourselves to continuous univariate multidimensional Gaussian random fields.

The approximation $\hat{H}(\cdot)$ of a continuous random field $H(\cdot)$ by a finite set of random variables $\{\chi_i(\theta), i=1,\ldots,M\}$ is referred to as *random field discretization*.

2.1 Error measures

Different error measures are available to quantify the error resulting from the discretization of a random field. For a given outcome θ, the *truncation error* $\varepsilon_H(\cdot)$ is defined at position $\mathbf{x}$ as the difference between the random field and its approximation:

$$\varepsilon_H(\mathbf{x},\theta) = H(\mathbf{x},\theta) - \hat{H}(\mathbf{x},\theta). \tag{1}$$

In the context of this work, we will assume that the mean function of the approximated random field can be modeled precisely, i.e. $\mathrm{E}\big(\varepsilon_H(\mathbf{x},\theta)\big)=0\ \forall\,\mathbf{x}\in\Omega$.

In general, the truncation error can only be evaluated if the exact representation of the random field is known explicitly. This is usually not the case. In the following, an error estimator is introduced which circumvents this problem. $\varepsilon_\sigma(\mathbf{x})$ is known as the *error variance* and has been commonly used in the literature; it is defined as:

$$\varepsilon_\sigma(\mathbf{x}) = \frac{\mathrm{Var}\big(H(\mathbf{x},\theta)-\hat{H}(\mathbf{x},\theta)\big)}{\mathrm{Var}\big(H(\mathbf{x},\theta)\big)} = \frac{\mathrm{Var}\big(H(\mathbf{x},\theta)-\hat{H}(\mathbf{x},\theta)\big)}{\sigma^2(\mathbf{x})}, \tag{2}$$

where $\sigma(\mathbf{x})$ is the standard deviation function of the random field $H(\mathbf{x},\theta)$.

Pointwise measures are of little use when making a quantitative assessment of the quality of the overall random field approximation. Therefore, the following global error norm $\bar{\varepsilon}_\sigma$, known as the *mean error variance*, is used here:

$$\bar{\varepsilon}_\sigma = \frac{\int_\Omega \varepsilon_\sigma(\mathbf{x})\,\mathrm{d}\,\mathbf{x}}{|\Omega|} \tag{3}$$

where $|\Omega| = \int_\Omega \mathrm{d}\,\mathbf{x}$. Besides the mean error variance, other global error measures haven been used in the literature. For example, in [6] the supremum norm of the error variance was used to compare different random field discretization methods. It has been noted in [10] that different global error measures might favor different discretization methods. In this work, we will only investigate convergence with respect to the mean error variance.

2.2 Karhunen-Loève expansion

The KL-expansion is a series expansion method for the representation of a random field. The expansion is based on the spectral decomposition of the covariance function of the field. It states that a random field can be represented exactly by the following expansion:

$$H(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{4}$$

where $\mu(\mathbf{x})$ is the mean function of the field, ξ_i are independent standard normal random variables, and λ_i, $\varphi_i(\mathbf{x})$ are the eigenvalues and eigenfunctions of the covariance kernel obtained from solving the integral eigenvalue problem:

$$\int_\Omega \varphi_i(\mathbf{x})\,\mathrm{Cov}(\mathbf{x},\mathbf{x}')\,\mathrm{d}\,\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x}) \tag{5}$$

The eigenfunctions are by definition orthonormal, i.e. $\int_\Omega \varphi_i(\mathbf{x})\,\varphi_j(\mathbf{x})\,\mathrm{d}\,\mathbf{x} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

2.2.1 Truncated Karhunen-Loève expansion

The truncated KL-expansion is obtained by arranging the eigenvalues and eigenfunctions in a descending series with respect to the magnitude of the eigenvalues, and truncating the ordered expansion after M terms. The truncated KL-expansion does no longer represent the random field $H(\mathbf{x})$ exactly, but provides an approximation $\tilde{H}(\mathbf{x})$ of the field. Hence, the truncated KL-expansion is a random field discretization method. The discretized random field is written as:

$$\tilde{H}(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i}\,\varphi_i(\mathbf{x})\,\xi_i \tag{6}$$

An important property of the truncated KL-expansion is that the global mean square error is minimized with respect to any other complete basis of $L^2(\Omega)$ [5].

2.2.2 Error variance

For the truncated KL-expansion, the error variance can be expressed as [9]:

$$\varepsilon_\sigma(\mathbf{x}) = 1 - \frac{\sum_{i=1}^{M} \lambda_i \varphi_i^2(\mathbf{x})}{\sigma^2(\mathbf{x})} \tag{7}$$

2.3 Finite element approximation of the KL-expansion

The KL-expansion involves solving the integral eigenvalue problem given in equation 5. Equation 5 can be solved analytically only for a few covariance functions and geometries (see [5]). Therefore, for general problems with arbitrary geometries and covariance functions, a numerical approach is necessary. This involves a spatial discretization of the integral eigenvalue problem. Obviously, this introduces yet another approximation to the representation of the random field. The obtained eigenvalues $\hat{\lambda}_i$ and eigenfunctions $\hat{\varphi}_i(\mathbf{x})$ are, therefore, approximations to the eigenvalues λ_i and eigenfunctions $\varphi_i(\mathbf{x})$ of the analytical solution of the KL-expansion. The approximation of the random field can be expressed as:

$$\hat{H}(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \sqrt{\hat{\lambda}_i}\,\hat{\varphi}_i(\mathbf{x})\,\xi_i \tag{8}$$

In the finite element approximation of the KL-expansion (in the following referred to as *FE-KL method*), the eigenfunctions are approximated as:

$$\hat{\varphi}_i(\mathbf{x}) = \sum_{n=1}^{N} d_n^i N_n(\mathbf{x}) = \mathbf{N}^{\mathrm{T}}(\mathbf{x})\,\mathbf{d}_i \tag{9}$$

where N is the number of shape functions, $N_n(\mathbf{x}) \in L^2(\Omega)$ are the global shape functions forming a basis in a chosen sub-space of the set of all Lebesgue square-integrable functions on Ω, and $d_n^i \in \mathbf{R}$ are the coordinates of the *i*th eigenfunction in the basis formed by all shape functions. $\mathbf{N}^{\mathrm{T}}(\mathbf{x})$ is a vector function of $\mathbf{x}$ with elements $N_n(\mathbf{x})$, and $\mathbf{d}_i$ is a vector containing the coeffi-

cients d_n^i. It is important to note that the eigenfunctions are by definition orthonormal and, therefore, the vectors $\mathbf{d}_i$ have to be scaled appropriately.

The approximation of the integral eigenvalue problem defined in equation 5 by means of equation 9 introduces an error term, denoted $\varepsilon_N^i(\mathbf{x})$. The coefficients of the vectors $\mathbf{d}_i$ are selected such that the error term $\varepsilon_N^i(\mathbf{x})$ becomes orthogonal to the space spanned by the shape functions. A solution to this problem is given by the matrix eigenvalue problem:

$$\mathbf{B}\mathbf{d}_i = \hat{\lambda}_i \mathbf{M}\mathbf{d}_i \tag{10}$$

The coefficients B_{kn} of the matrix $\mathbf{B}$ are defined as:

$$B_{kn} = \int_{\mathbf{x}\in\Omega} N_k(\mathbf{x}) \int_{\mathbf{x}'\in\Omega} N_n(\mathbf{x}') \operatorname{Cov}(\mathbf{x},\mathbf{x}') \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} \tag{11}$$

The coefficients M_{kn} of the matrix $\mathbf{M}$ are defined as:

$$M_{kn} = \int_{\mathbf{x}\in\Omega} N_k(\mathbf{x})\, N_n(\mathbf{x}) \,\mathrm{d}\mathbf{x} \tag{12}$$

The error variance of the FE-KL approach can be expressed as [1]:

$$\varepsilon_\sigma(\mathbf{x}) = 1 + \frac{\sum_{i=1}^{M} \hat{\lambda}_i \hat{\varphi}_i^2(\mathbf{x}) - 2 \cdot \sum_{i=1}^{M} \hat{\varphi}_i^2(\mathbf{x}) \int_\Omega \operatorname{Cov}(\mathbf{x},\mathbf{x}') \hat{\varphi}_i(\mathbf{x}')\, d\mathbf{x}'}{\sigma^2(\mathbf{x})} \tag{13}$$

In case of a constant standard deviation σ within the domain of the field, the mean error variance reduces to (compare [1]):

$$\bar{\varepsilon}_\sigma = 1 - \frac{1}{\sigma^2 \cdot |\Omega|} \cdot \sum_{i=1}^{M} \hat{\lambda}_i \tag{14}$$

2.4 Finite cell approximation of the KL-expansion

The finite cell method [8] was developed as an extension to the finite element method for the solution of linear elasticity problems. Let $\Omega \subset \mathbf{R}^d$ be the domain of interest and $\Omega^* \subset \mathbf{R}^d$ a geometrically simpler domain with $\Omega \subseteq \Omega^*$. The geometrically simpler domain Ω^* is called *primitive domain*, and the original domain Ω is called *physical domain*. Furthermore, let the shape functions $N_n^*(\mathbf{x}) \in L^2(\Omega^*)$ form a basis of a subspace in $L^2(\Omega^*)$. We are searching a solution of the integral equation defined on Ω, and approximate it with functions defined on Ω^*.

The spatial decomposition of the problem is performed on the primitive domain Ω^* (this is illustrated in figure 1). Since Ω^* is by definition of primitive geometrical shape (e.g. a hyperrectangle), the meshing of the domain is a triv-

ial task. However, the region $\Omega^* \cap \overline{\Omega}$ is not part of the physical domain. In order to solve the original, i.e. physical, problem, the non-physical part of the extended domain Ω^* must not influence the solution. For this reason, we introduce the mapping $\alpha : \Omega^* \to \{0,1\}$ as:

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \forall\, \mathbf{x} \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

In order to solve the problem defined in equation 10 we have to assemble the matrices $\mathbf{M}$ and $\mathbf{B}$. The integral in equation 12 can be transformed to an integral over Ω^* as:

$$M_{kn} = \int_{\mathbf{x} \in \Omega^*} \alpha(\mathbf{x})\, N_k(\mathbf{x})\, N_n(\mathbf{x})\, \mathrm{d}\mathbf{x} \tag{16}$$

In a similar way, the integral in equation 11 can be written as:

$$B_{kn} = \int_{\mathbf{x} \in \Omega^*} \alpha(\mathbf{x})\, N_k(\mathbf{x}) \int_{\mathbf{x}' \in \Omega^*} \alpha(\mathbf{x}')\, N_n(\mathbf{x}')\, \mathrm{Cov}(\mathbf{x}, \mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \tag{17}$$

In the finite cell approach, the shape functions are defined locally on the cells Ω_e, see Figure 1. Higher order hierarchical shape functions based on the integrated Legendre polynomials [11] are used, compare [8]. Note that the integrals in equations 16 and 17 are smooth over the domain Ω but not even continuous over the domain Ω^*. Therefore, it is important to use appropriate numerical integration schemes in order to keep the integration error small.

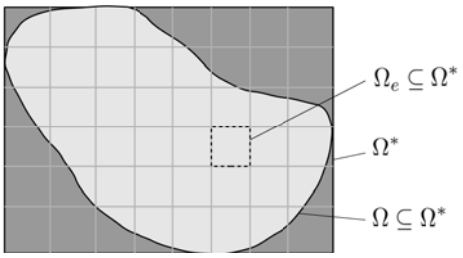


Figure 1. Notation for the finite cell approach.

A staggered Gaussian integration scheme is proposed. The principal idea of this integration scheme is illustrated in figure 2. For staggered Gaussian integration, a tree-based mesh refinement is used to mesh the domain of the finite cell for integration. For one-, two- and three-dimensional elements, a binary-, quad- and oct-tree is used, respectively. Gaussian integration is applied in the leaf elements of the tree. A tree-element is refined if it is cut by the boundary of the physical domain and if the element level is smaller than the maximum tree-depth. The element level of the finite cell itself is zero.

In the context of this work, the number of Gauss-points used on the respective levels of the tree is decreased with an increasing level. This is contrary to the approach presented in [8] and [3], where all sub-cells were integrated with a full number of Gauss-points. However, in the cut-cells the function to integrate is discontinuous and, therefore, cannot be approximated well using polynomials. Moreover, the influence area of the individual Gauss-points is not directly observable and not necessarily accumulated around the corresponding point.

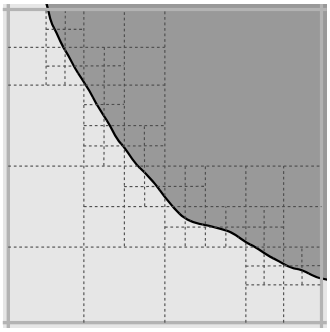


Figure 2. Staggered Gaussian integration: mesh for integration on a cut finite cell.

Assuming the integration error is small enough, $\varepsilon_\sigma(\mathbf{x})$ and $\bar{\varepsilon}_\sigma$ can be computed according to equation 13 and 14, respectively.

2.5 EOLE method

The EOLE method [6] is a series expansion method that is based on an optimal linear estimation using discrete points of the field and carries out a spectral decomposition of the covariance matrix $\mathbf{\Sigma}_{\chi\chi}$ corresponding to these points. The coefficients of the covariance matrix are defined as $\left(\mathbf{\Sigma}_{\chi\chi}\right)_{ij} = \mathrm{Cov}\left(\chi_i, \chi_j\right)$ with $i, j \in \{1,2,\ldots N\}$, where each χ_i is a random variable associated with a point $\mathbf{x}_i \in \Omega$.

The points $\mathbf{x}_i$ are used to discretize the domain Ω of the random field pointwise. Consequently, the domain is represented approximately by a finite number of points and no finite element mesh is required. The distribution of the points $\mathbf{x}_i$ has an influence on the random field approximation, especially if the field is approximated by a minimal number of points.

The random field representation in case of the EOLE method writes:

$$\hat{H}(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{M} \frac{\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{X}}(\mathbf{x})}{\sqrt{\omega_i}} \xi_i \tag{18}$$

where ω_i and $\mathbf{\Phi}_i^T$ are the M largest eigenvalues and their corresponding eigenvectors of the covariance matrix $\mathbf{\Sigma}_{\chi\chi}$, the ξ_i are independent standard normal random variables; $\mathbf{\Sigma}_{\chi\mathbf{X}}(\mathbf{x})$ is a vector function whose coefficients are defined as $\left(\mathbf{\Sigma}_{\chi\mathbf{X}}(\mathbf{x})\right)_j = \mathrm{Cov}\left(\mathbf{x}_j, \mathbf{x}\right)$ with $j \in \{1,2,\ldots,N\}$. The EOLE method minimizes the mean square error pointwise given values of the random field at the set of points $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$. For the EOLE method, the error variance can be expressed as [6]:

$$\varepsilon_\sigma(\mathbf{x}) = 1 - \frac{1}{\sigma^2(\mathbf{x})} \sum_{i=1}^{M} \frac{\left(\mathbf{\Phi}_i^T \mathbf{\Sigma}_{\chi\mathbf{X}}(\mathbf{x})\right)^2}{\omega_i} \tag{19}$$

3 Numerical convergence study

The convergence behavior of the proposed finite cell approach with respect to the mean error variance is investigated by means of a numerical example. The random field is modeled for a squared domain with a circular hole in its center. The length of a side of the square is four and the diameter of the circular hole is two, as shown in figure 3. The Gaussian random field has a constant mean value and standard deviation of $30 \cdot 10^3$ and $6 \cdot 10^3$, respectively.

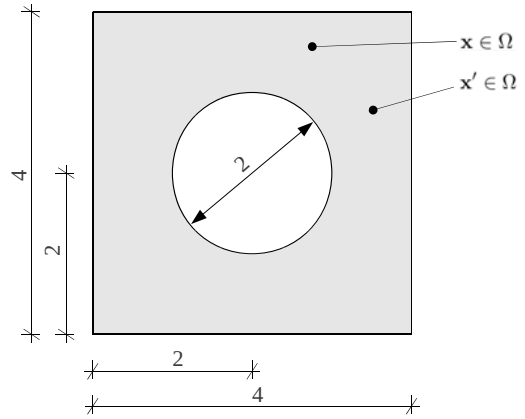


Figure 3. Domain used for the numerical convergence study.

Three different types of correlation coefficient functions are considered:

$$\text{Type A}: \quad \rho(\mathbf{x}, \mathbf{x}') = \exp\left(-\left(\frac{|\mathbf{x} - \mathbf{x}'|}{\delta_A} \right)^2 \right) \tag{20}$$

$$\text{Type B}: \quad \rho(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{|\mathbf{x} - \mathbf{x}'|}{\delta_B} \right) \tag{21}$$

$$\text{Type C}: \quad \rho(\mathbf{x},\mathbf{x}') = \frac{1}{1+\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\delta_C}\right)^{1.2}} \tag{22}$$

The correlation lengths δ_A, δ_B and δ_C used in the numerical study were chosen such that the reference mean error variances $\bar{\varepsilon}_{\sigma,\text{ref}}$ are close to ten percent for 100 random variables in the expansion. This reference value is the error from the truncation of the KL-expansion and was calculated with a uniform 10x10 finite cell mesh and a maximum polynomial order of the shape functions of ten. The so obtained reference value was verified with a uniform 14x14 finite cell mesh and a maximum polynomial order of eight. The applied correlation lengths and their corresponding reference mean error variances are $\delta_A = 0.3325$, $\delta_B = 1.08$, $\delta_C = 0.725$, $\bar{\varepsilon}_{\sigma,\text{ref,A}} = 0.099781$, $\bar{\varepsilon}_{\sigma,\text{ref,B}} = 0.099853$ and $\bar{\varepsilon}_{\sigma,\text{ref,C}} = 0.09953$. The correlation coefficient functions corresponding to the chosen correlation lengths are depicted in figure 4.

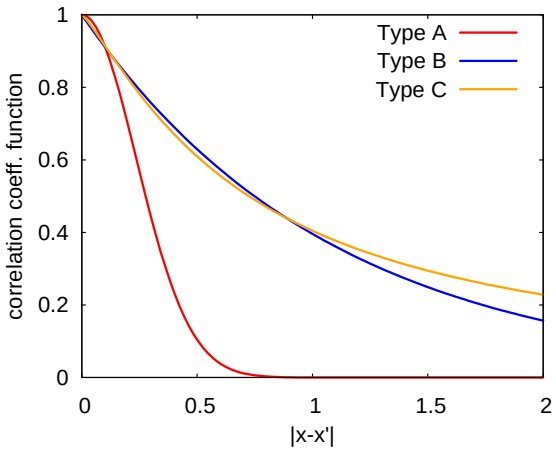


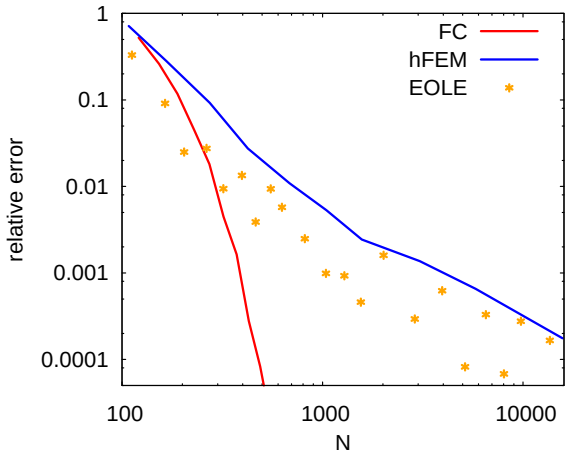
Figure 4. Plot of the investigated correlation coefficient functions.

For the convergence study, the following relative error is defined:

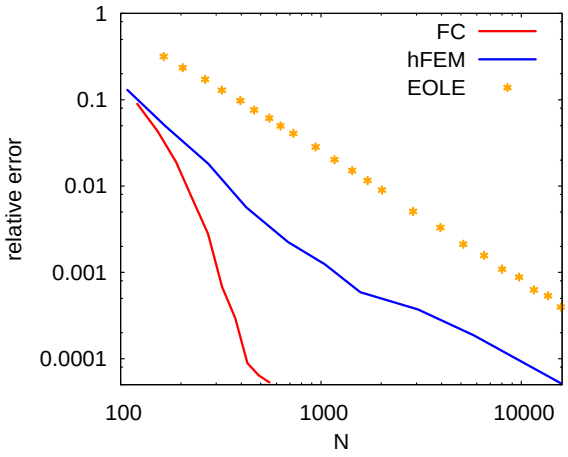
$$\varepsilon_{\text{rel,N}} = \frac{\left|\bar{\varepsilon}_{\sigma,\text{N}} - \bar{\varepsilon}_{\sigma,\text{ref}}\right|}{\bar{\varepsilon}_{\sigma,\text{ref}}} \tag{23}$$

where $\bar{\varepsilon}_{\sigma,\text{N}}$ is the mean error variance for a given size N of the matrix eigenvalue problem to solve.

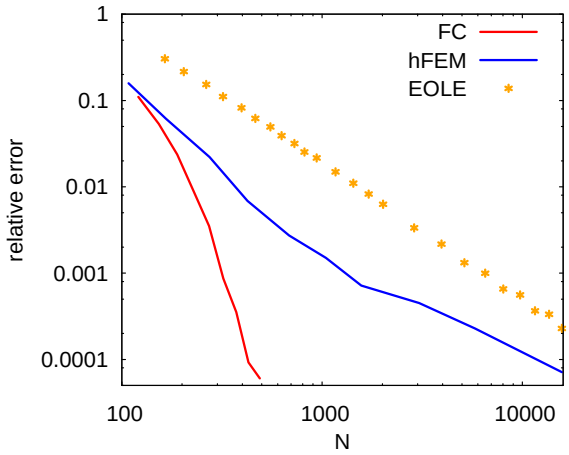
The errors obtained by the finite cell approximation of the KL-expansion (FC), the finite element approximation of the KL-expansion using linear shape functions (hFEM), and the EOLE method are shown in figure 5. For the FC-approach, a uniform 2x2 finite cell mesh is used. The size N of the matrix eigenvalue problem to solve is increased by increasing the maximum poly-



(a) convergence (Type A)



(b) convergence (Type B)



(c) convergence (Type C)

Figure 5. Convergence in the relative error w.r.t. the size of the size of the problem.

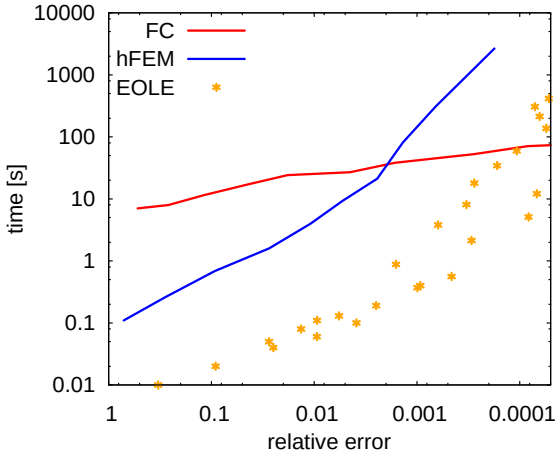


Figure 6. Time needed to converge to a certain relative error. (Type A)

nomial order of the shape functions. The maximum polynomial order in each coordinate direction is the same. For the hFEM-approach, the actual physical domain is meshed using four node quadrilateral elements. The problem size N is increased by refining the mesh. In case of the EOLE-method, the problem size N is equivalent to the total number of points used to discretize the field. The points were distributed uniformly over the domain.

The plots (a), (b) and (c) in figure 5 show the relative error defined in equation 23 for an increasing size N of the matrix eigenvalue problem to solve. The FC-approach shows an exponential rate of convergence for all three types of correlation coefficient functions. The convergence rate of the hFEM-method and the EOLE method is approximately linear in the log-log plots. The EOLE-method converges faster than the hFEM-method for the correlation coefficient function of type A. For the correlation coefficient functions of type B and C, the hFEM-method converges faster than the EOLE-method.

Figure 6 shows the time needed for the methods to converge to a certain relative error for the correlation coefficient function of type A. To obtain a reasonably well converged solution, the FC-approach needs considerably more time than the hFEM-method and the EOLE-method. For this particular correlation coefficient function, the EOLE-method solves the problem around one order of magnitude faster than the hFEM-method.

In a next study, the time required to evaluate a realization of the random field at a given position $\mathbf{x}$ is analyzed. This is of importance when the random field is used as input to finite element reliability analysis, because a realization of the field has to be evaluated at every finite element Gauss-point. In case of the hFEM-approach, the time needed to evaluate a realization of the random field at one position $\mathbf{x}$ does not depend on the mesh, because the number of shape functions per element remains constant. Consequently, it remains constant with increasing N. This time is denoted t_{hFEM} in the following. On the other hand, the time needed to obtain a realization depends in case of the FC-approach on the maximum polynomial degree of the shape functions, and for the EOLE-method on the number of points used to discretize the domain.

In the log-log plot depicted in figure 7, the time needed to obtain a realization of the random field is weighted by t_{hFEM} and plotted in terms of the relative error defined in equation 23. A correlation coefficient function of type A was employed to generate the plot. It is shown that a realization of the random field can be computed several times faster with the FC-approach than with the EOLE-method.

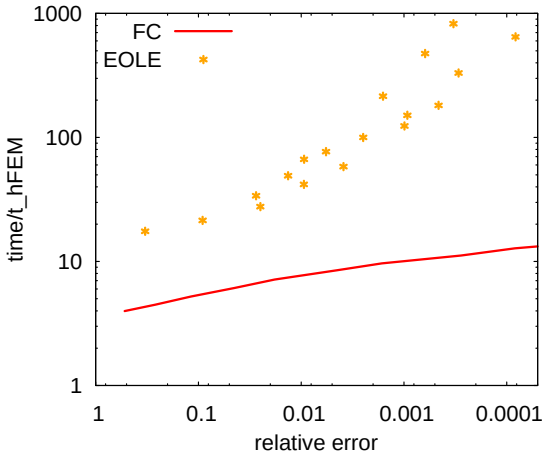


Figure 7. Time needed to compute a realization of the random field. Comparison between FC-approach and EOLE-method. Time is given relative to the time needed with the hFEM-method.

4 Summary and Conclusion

The proposed FC-approach exhibits an exponential rate of convergence with respect to the mean error variance. However, it is relatively expensive to compute a random field approximation. This effect will be even more severe for three-dimensional problems. On the other hand, compared to the EOLE-method, the proposed approach is computationally very efficient in obtaining a random field realization. This is advantageous, if many realizations of the random field have to be generated.

Compared to the hFEM method, the proposed approach is computationally more expensive in obtaining a random field realization. Therefore, for domains which are meshed with a linear finite element mesh that is fine enough to represent the correlation structure of the random field reasonably well, the hFEM-method is to be preferred. However, the FC-approach is useful for problems that do not require a mesh on the physical domain, e.g. meshless approaches or FC methods.

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