Extremal behaviour of diffusion models in finance

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Abstract

We investigate the extremal behaviour of a diffusion (X_t) given by the SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t\,, \quad t > 0\,, \quad X_0 = x\,,$$

where W is standard Brownian motion, μ is the drift term and σ is the diffusion coefficient. Under some appropriate conditions on (X_t) we prove that the point process of ε–upcrossings converges in distribution to a homogeneous Poisson process. As examples we study the extremal behaviour of term structure models or asset price processes such as the Vasicek model, the Cox-Ingersoll-Ross model and the generalised hyperbolic diffusion. We also show how to construct a diffusion with pre-determined stationary density which captures any extremal behaviour. As an example we introduce a new model, the generalised inverse Gaussian diffusion.

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1 Introduction

Over the last decade a variety of stochastic models have been suggested as appropriate models for financial products. In a continuous time setting the dynamics of an interest rate or price process is often modelled as a diffusion process given by a stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t\,, \quad t > 0\,, \quad X_0 = x\,, \tag{1.1}$$

where W is standard Brownian motion, μ is the *drift term* and σ is the *diffusion coefficient or volatility*. Two standard models in finance are of the above form:

(i) The Black-Scholes model: (X_t) models the price process of an asset, here $\mu(x) = \mu x$ and the volatility $\sigma(x) = \sigma x$. The resulting model for the price process is geometric Brownian motion.

(ii) The Vasicek model: the process (X_t) models an interest rate, the drift term μ is linear and the volatility $\sigma > 0$ is some constant.

Both models can be considered in the framework of Gaussian models. It has been recognised for decades that financial data like interest rates and asset prices exhibit fluctuations which cannot be modelled by Gaussian processes or simple transformations as in the two standard models above.

There are two features, heavy-tailedness and the dependence structure, that require modelling for financial data. Various models have been suggested to capture these features. For a discussion of non-linear heavy-tailed models and further references we refer to Section 7.6 of Embrechts, Klüppelberg and Mikosch (1997). There are in principle two different approaches.

A first concept replaces the Gaussian driving process in the Black-Scholes or Vasicek model (or any other traditional model) by a process with heavy-tailed marginals as for instance a stable process, a Lévy process or a discrete time counterpart as an ARMA (autoregressive-moving average) process with heavy-tailed noise (see e.g. Barndorff-Nielsen (1995), Eberlein and Keller (1995), Klüppelberg and Mikosch (1996), Mittnik and Rachev (1997)).

The second concept sticks to Brownian motion as the driving dynamic of the process, but introduces a path-dependent, time-dependent or even stochastic volatility into the model. These models are commonly referred to as volatility models, and include diffusions given by the SDE (1.1). Hence this paper is about such models. Discrete time counterparts are for instance ARCH and GARCH models, which have been successfully applied in econometrics. The extremal behaviour of the ARCH(1) model has been studied in de Haan et al. (1989) and is an interesting complement to the present paper.

In this paper we study the extremal behaviour of diffusion processes defined by (1.1). The stationary distributions of the processes under investigation are well-known and one might expect that they influence the extremal behaviour of the process in some way. This is however not the case: for any pre-determined stationary distribution the process can exhibit quite different behaviour in its extremes.

Extremal behaviour of a stochastic process (X_t) is for instance manifested in the asymptotic behaviour of the maxima

$$M_t^X = \max_{0 \le s \le t} X_s\,, \quad t > 0\,. \tag{1.2}$$

The asymptotic distribution of M_t^X for $t \to \infty$ has been studied by various authors, see Davis (1982) for detailed references. Two monographs on this and related problems are by Leadbetter, Lindgren and Rootzén (1983) and Berman (1992). It is remarkable that running maxima and minima of (X_t) are asymptotically independent and have the same behaviour as the extremes of iid random variables. In this paper we restrict ourselves to the investigation of maxima, the mathematical treatment for minima being similar.

We furthermore investigate the point process of upcrossings (more precisely ε-upcrossings) of a high threshold u by (X_t). For fixed $\varepsilon > 0$ the process has an ε-upcrossing at t if it has remained below u on the interval $(t-\varepsilon, t)$ and is equal to u at t. Under weak conditions, the point process of ε-upcrossings, properly scaled in time and space, converges in distribution to a homogeneous Poisson process, i.e. it behaves again like iid random variables, coming however not from the stationary distribution of (X_t), but from the df F which also describes the maxima M_t^X (see Theorem 3.4).

The paper aims at applications in finance: for the various models under consideration we derive the distributional behaviour of M_t^X as $t \to \infty$, which together with Theorem 3.4 describes the extremal behaviour of the whole process (X_t). The results of this paper can be applied to study risk measures of financial products as for instance the value at risk or related quantile risk measures; see Embrechts, Klüppelberg and Mikosch (1997), Example 6.1.6. This work is currently under way and will be presented in another paper.

The course of this paper is as follows. In Section 2 we present the framework for the results to follow. We shall require certain properties on the speed measure and scale function of (X_t), which we explain and summarise in the usual conditions (2.5). They are assumed to hold throughout the paper.

In Section 3 we present some results on extreme value theory for diffusions. We show that, provided M_t^X has a weak non-degenerate limit, then, under weak conditions, also the point processes of ε-upcrossings converge to a homogeneous Poisson process (Theorem 3.4). Furthermore we derive the limit distribution of M_t^X (suitably normalised) under simple conditions on the drift term and the diffusion coefficient (Theorem 3.7). Finally we show how to construct a diffusion with pre-determined stationary density which captures any extremal behaviour (Theorem 3.8).

In Section 4 we apply these results in order to derive the extremal behaviour of such diffusions as the Vasicek model, the Cox-Ingersoll-Ross (CIR) model, including a generalised version, and the generalised hyperbolic diffusion. Depending on the choice of parameters the generalised CIR model allows for large fluctuations in the data. This is captured by the limit distribution of M_t^X and the intensity of the limit point process of ε-upcrossings.

We conclude in Section 5 the paper with a new model, the generalised inverse Gaussian diffusion, which is constructed with the pre-determined generalised inverse Gaussian stationary density and a pre-determined diffusion coefficient. If we choose the diffusion coefficient as in the CIR model we obtain a further generalisation of this important model. Whereas in Section 4 we mainly present results without explicit calculations, for this new model we derive certain quantities in detail.

We shall need some results on classical extreme value theory. For an introduction to extreme value theory we refer to Leadbetter, Lindgren and Rootzén (1983), Resnick (1987) or Embrechts, Klüppelberg and Mikosch (1997, Chapter 3).

2 The usual conditions

The diffusion (X_t) given by the SDE (1.1) has state space $(l,r) \subset \mathbb{R}$, where l, r can be $-\infty$ or $+\infty$. We only consider the case when the boundaries l and r are inaccessible and (X_t) is recurrent. We require furthermore that, for all $x \in (l,r)$, $\sigma^2(x) > 0$ and there exists some $\varepsilon > 0$ such that $\int_{x-\varepsilon}^{x+\varepsilon}(1+|\mu(t)|)/\sigma^2(t)dt < \infty$. These two conditions guarantee in particular that the SDE (1.1) has a weak solution which is unique in probability (see Karatzas and Shreve (1988), Chapter 5.5.C).

Associated with the diffusion is the scale function s and the speed measure m. The *scale function* is defined as

$$s(x) = \int_z^x \exp\left\{-2\int_z^y \frac{\mu(t)}{\sigma^2(t)}dt\right\} dy\,, \quad x \in (l,r)\,, \tag{2.1}$$

where z is any interior point of (l, r). Since the scale function is unique only up to a positive affine transformation (if $\widetilde{s}(x) = \alpha s(x) + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$, then $\widetilde{s}$ is a scale function if and only if s is), in a first order approximation, the choice of z is of no importance. The scale function s defines in the usual way a measure on (l, r), the so-called *scale measure*, which is absolutely continuous with Lebesgue density

$$s'(x) = \exp\left\{-2\int_z^x \frac{\mu(t)}{\sigma^2(t)}dt\right\}, \quad x \in (l, r)\,. \tag{2.2}$$

For the *speed measure* m we know that $m(I) > 0$ for every non-empty open subinterval I of the interior of (l, r). We only consider diffusions with finite speed measure m and denote its total mass by $|m| = m((l, r))$. The speed measure of model (1.1) is absolutely continuous with Lebesgue density

$$m'(x) = \frac{2}{\sigma^2(x)s'(x)}, \quad x \in (l, r)\,. \tag{2.3}$$

In this situation (X_t) is ergodic and its stationary distribution is absolutely continuous with Lebesgue density

$$h(x) = m'(x)/|m|\,. \tag{2.4}$$

Notice that the connection between stationary distribution, speed measure, scale function, drift term and diffusion coefficient (given by (2.1)-(2.4)) allows us to construct diffusions with arbitrary stationary distribution (see Examples 4.4 and the generalised inverse Gaussian diffusion of Section 5).

Since the process is recurrent and the boundaries l and r are inaccessible, we must have $s(u) \to \infty$ as $u \uparrow r$ and $s(u) \to -\infty$ as $u \downarrow l$. Conversely, if $s(u) \to \infty$ (resp. $-\infty$) as $u \uparrow r$ (resp. $u \downarrow l$), then l and r are inaccessible, and therefore (X_t) is recurrent.

For proofs of the above relations and further results on diffusions we refer to the monographs by Karlin and Taylor (1981), Karatzas and Shreve (1987), Rogers and Williams (1987), Revuz and Yor (1991) or any other advanced textbook on stochastic processes.

Throughout this paper, we assume that the diffusion process (X_t) defined in (1.1) satisfies *the usual conditions*:

$$\begin{aligned} s(r) = -s(l) &= \infty\,, \\ |m| &< \infty\,. \end{aligned} \tag{2.5}$$

3 Extremal behaviour of diffusions

The following formulation can be found in Davis (1982).

Proposition 3.1 *Let (X_t) satisfy the usual conditions (2.5). Then for any initial value $X_0 = y \in (l, r)$ and any $u_t \uparrow r$,*

$$\lim_{t\to\infty} |P^y(M_t^X \le u_t) - F^t(u_t)| = 0\,, \tag{3.1}$$

where F is a df, defined by

$$F(x) = e^{-1/(|m|s(x))} I_{(z,r)}(x)\,, \quad x \in \mathbb{R}\,, \tag{3.2}$$

for any $z \in (l, r)$. (I_A denotes the indicator function of A.) The function s and the quantity $|m|$ also depend on the choice of z. □

Various proofs of this result exist and we refer to Davis (1982) for further references. Davis' proof is based on a representation of such a diffusion as an Ornstein-Uhlenbeck process after a random time-change. Standard techniques for extremes of Gaussian processes apply leading to the above result.

It is not difficult to show that Proposition 3.1 is true for arbitrary initial probability measure H. For the special choice of $H = m/|m|$ the diffusion (X_t) is stationary.

As a consequence of Proposition 3.1, the maxima M_t^X have, properly normalised, a non-degenerate limit distribution Q if and only if F belongs to the maximum domain of attraction of Q (we write MDA(Q)) for some extreme value distribution Q. In Proposition 3.1, any function $u_t \uparrow r$ is possible, but as usual in classical extreme value theory we restrict ourselves to positive affine functions, i.e.

$$u_t = a_t x + b_t\,. \tag{3.3}$$

The norming constants $a_t > 0$ and $b_t \in \mathbb{R}$ have to be chosen appropriately to ensure convergence to a non-degenerate limit.

The extremal behaviour (in particular the behaviour of the maximum) of an iid sequence with common df F is determined by the far end of the right tail $\overline{F} = 1 - F$. In our situation the asymptotic behaviour of the maxima M_t^X is determined by the tail of F as in (3.2): If $F \in \text{MDA}(Q)$ with norming constants $a_t > 0$ and $b_t \in \mathbb{R}$, then

$$a_t^{-1}\left(M_t^X - b_t\right) \xrightarrow{d} Q\,, \quad t \to \infty\,. \tag{3.4}$$

As already noted the scale and speed measure of a diffusion (X_t) are not unique. Different scale and speed measures (and therefore different z) lead to different df's F in Proposition 3.1. They are however all tail-equivalent.

Corollary 3.2 *Under the conditions of Proposition 3.1 the tail of the df F in (3.2) satisfies*

$$\overline{F}(x) \sim \left(|m| \int_z^x s'(y)dy \right)^{-1} \sim (|m|s(x))^{-1} , \quad x \uparrow r , \tag{3.5}$$

where $\sim$ means that the quotient of lhs and rhs converges to 1.

Proof. The representation of (3.5) follows immediately by Taylor expansion from (3.2) and the fact that $s(x) \to \infty$ as $x \uparrow r$.

We show that the rhs is for different z asymptotically equivalent, and thus independent of $z \in (l, r)$. Let $z_1, z_2 \in (l, r)$ and $z_1 \neq z_2$. Denote s_i, m_i' and $|m_i|$ the functions and constants corresponding to z_i for $i = 1, 2$. Then from (2.1) we obtain

$$s_1(x) = \alpha + \beta s_2(x) ,$$

where α and β are constants depending on z_1 and z_2. Furthermore, from (2.3) we obtain

$$m_1'(x) = \frac{2}{\sigma^2(x) s_1'(x)} = \frac{2}{\sigma^2(x)\beta s_2'(x)} = \frac{1}{\beta} m_2'(x) .$$

Hence $|m_1| = |m_2|/\beta$ and

$$\overline{F}(x) \sim (|m_1|s_1(x))^{-1} = \left(|m_2| (\frac{\alpha}{\beta} + s_2(x)) \right)^{-1} \sim (|m_2|s_2(x))^{-1} , \quad x \uparrow r ,$$

since $\lim_{x\uparrow r} s_2(x) = \infty$. $\square$

Proposition 3.1 reduces the asymptotic behaviour of the maximum of (X_t) to that of the maximum of iid random variables with df F having tail (3.5). It would be interesting to know more about the extremal behaviour of the corresponding diffusion (X_t) than just the behaviour of its maxima. From classical extreme value theory it is well-known that the point process of exceedances of an iid sequence of a level u_t, plotted at points i/t, converges to a homogeneous Poisson process for $u_t \uparrow r$ as $t \uparrow \infty$ in an appropriate way. Extremes of a continuous time stochastic process over a high threshold u_t typically occur on intervals and form excursions over this level. However, an analogous discrete skeleton which describes the behaviour of the extremes of a continuous time stochastic process is provided by a point process of the upcrossings (i.e. the events where excursions above a level begin). This is quite natural and upcrossings are well–defined if the sample paths of the corresponding process are regular (i.e. differentiable in the

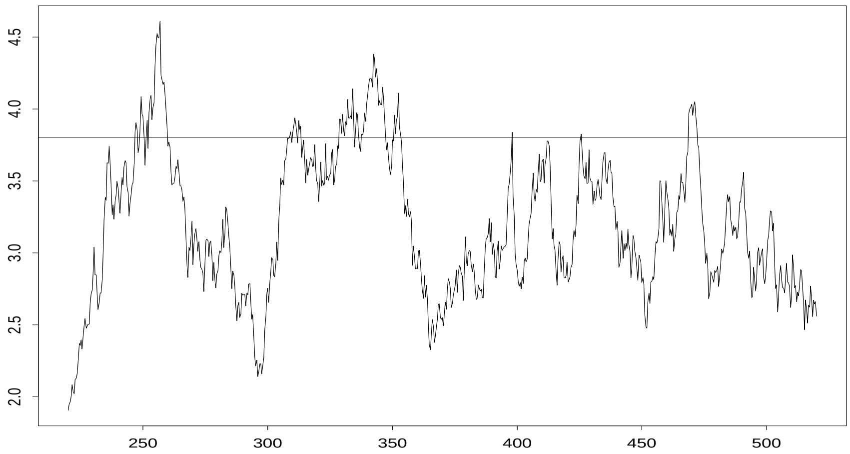

Figure 1: Sample path of a diffusion with threshold $u = 3.8$. For the values of $\varepsilon = 3.2,\ 1.2,\ 0.8,\ 0.4$ we get 6, 7, 10, 14 ε–upcrossings, respectively. The number of ε–upcrossings depends crucially on ε. The dependence only disappears in the limit.

L^2–sense). In cases with irregular sample paths there can be infinitely many upcrossings on a finite interval.

To avoid such problems special upcrossings, namely ε-uprossings, are considered. We use the definition given by Pickands (1969) for continuous processes. We also refer to Leadbetter, Lindgren and Rootzén (1983), Chapter 12, for more mathematical background.

Definition 3.3 *Let (X_t) be a diffusion satisfying the usual conditions (2.5). Take $\varepsilon > 0$.*
*(a) The process (X_t) is said to have an ε–*upcrossing of the level u at t_0 *if $X_t < u$ for $t \in (t_0-\epsilon, t_0)$ and $X_{t_0} = u$.*
(b) Let $N_{\varepsilon,u}(t)$ denote the number of ε-upcrossings of u by $(X_s)_{0\le s\le t}$. Then for any $t > 0$,

$$N_t^*(B) = N_{\varepsilon,u_t}(tB) = \#\{\varepsilon\text{-}upcrossings\ of\ u_t\ \ by\ (X_s)_{0\le s\le t} : \frac{s}{t} \in B\}$$

is the time–normalised point process of ε-upcrossings *on the Borel sets B of* $(0, 1]$. □

The point process (N_t^*) has a point at t_0 if $(X_s)_{0\le s\le t}$ has an ε–upcrossing at $t_0 t$. ε–upcrossings of a continuous time process correspond to exceedances of an iid sequence. It is well-known that for a sequence (X_t) of iid rvs, all with df F, the point processes (N_t^*) of exceedances converge to a homogeneous Poisson process with intensity τ, provided the u_t are appropriately chosen, namely such that

$$t\overline{F}(u_t) \to \tau \in (0, \infty)\,, \quad t \to \infty\,. \tag{3.6}$$

Recall from (3.4) that for the choice of $u_t = a_t x + b_t$:

$$P\left(M_t^X \le a_t x + b_t\right) = F^t(a_t x + b_t) \to Q(x) = e^{-\tau}\,, \quad x \in \mathbb{R}\,. \tag{3.7}$$

Taking logarithms in (3.7) shows that (3.6) is equivalent to (3.7). Convergence of the point processes of exceedances to a Poisson process also holds for more general sequences (X_t) if the dependence structure is nice enough to prevent clustering of the extremes in the limit. For diffusions (1.1) the dependence structure of the extremes is such that the point processes of ε-upcrossings converge to a homogeneous Poisson process, however, the intensity is not determined by the stationary df H, but by the df F from Proposition 3.1. This means that the ε-upcrossings of (X_t) are likely to behave as the exceedances of iid rvs with df F.

The extra condition (3.10) of the following theorem relates the scale function s and speed measure m of (X_t) to the corresponding quantities s_{ou} and m_{ou} of the standard Ornstein-Uhlenbeck process, defined by

$$s_{ou}(x) = \sqrt{2\pi}\int_0^x e^{t^2/2}dt \quad \text{and} \quad m'_{ou}(x) = 1/s'_{ou}(x)\,, \quad x \in \mathbb{R}\,. \tag{3.8}$$

Theorem 3.4 *Let (X_t) satisfy the usual conditions (2.5) and $u_t \uparrow r$ such that*

$$\lim_{t\to\infty} \frac{t}{|m|s(u_t)} = \tau \in (0,\infty)\,. \tag{3.9}$$

Assume there exists some positive constant c such that

$$\frac{m'_{ou}(s_{ou}^{-1}(s(z)))}{s'_{ou}(s_{ou}^{-1}(s(z)))}\frac{s'(z)}{m'(z)} \ge c\,, \quad \forall z \in (l,r)\,. \tag{3.10}$$

Then for all starting points $y \in (l,r)$ of (X_t) and $\varepsilon > 0$ the time-normalised point processes (N_t^*) of ε-upcrossings of the level u_t converge in distribution to N as $t \uparrow \infty$, where N is a homogeneous Poisson process with intensity τ on $(0,1]$.

Remark 3.5 (a) Notice from Corollary 3.2 that $t\overline{F}(u_t) \sim t/(|m|s(u_t))$. Hence, if $u_t = a_t x + b_t$ and $\tau = -\ln Q(x)$, then condition (3.9) guarantees that F belongs to some maximum domain of attraction.

(b) Pickands (1969) proved that the point processes of ε-upcrossings converge to a homogeneous Poisson process in the case when (X_t) is a Gaussian process. Notice that the assumptions of Theorem 3.4 are particular satisfied for the Ornstein-Uhlenbeck process with $c = 1$.

(c) Examples which satisfy condition (3.10) are the Vasicek model, the Cox-Ingersoll-Ross model or the generalised Cox-Ingersoll-Ross model for $\gamma \neq 1$. All these models are presented in Section 4. Nevertheless not every diffusion satisfies the assumptions in Theorem 3.4. Lemma 3.6 indicates that for the generalised inverse Gaussian diffusion with $\chi > 0, \psi > 0$ and $\gamma > 1.5$ or $\gamma < 0.5$ the assertion of Theorem 3.4 may not hold.

Proof. The proof invokes a random time change argument. An application of Theorem 12.4.2 of Leadbetter et al. (1983) shows that the theorem holds for the standard Ornstein-Uhlenbeck (O_t) process. Denote by

$$Z_t = s_{ou}(O_t)\,, \quad t \geq 0\,,$$

the Ornstein-Uhlenbeck process in natural scale. Now define

$$Y_t = s(X_t)\,, \quad t \geq 0\,,$$

which is again a diffusion process in natural scale. (Y_t) can then be considered as a random time change of the process (Z_t), i.e.

$$Y_t = Z_{\tau_t} \quad a.s. \tag{3.11}$$

The random time τ_t has a representation via the local time of the process Y. This is a consequence of the Dambis-Dubins-Schwarz Theorem (Revuz and Yor (1991), Theorem 1.6, p.173), Theorem 47.1 of Rogers and Williams (1987), p.277 and Exercise 1.27 of Revuz and Yor (1991), p.226. For $z \in (l, r)$ denote $L_t(z)$ the local time of $(Y_s)_{0 \leq s \leq t}$ in z. Then

$$\begin{aligned}
\tau_t &= \int_{-\infty}^{\infty} L_t(z) dm_{ou}(s_{ou}^{-1}(z)) \\
&= \int_{-\infty}^{\infty} L_t(z) \frac{m'_{ou}(s_{ou}^{-1}(z))}{s'_{ou}(s_{ou}^{-1}(z))} \frac{s'(s^{-1}(z))}{m'(s^{-1}(z))} dm(s^{-1}(z)) \\
&= \int_0^t \frac{m'_{ou}(s_{ou}^{-1}(Y_s))}{s'_{ou}(s_{ou}^{-1}(Y_s))} \frac{s'(s^{-1}(Y_s))}{m'(s^{-1}(Y_s))} ds \\
&= \int_0^t \frac{m'_{ou}(s_{ou}^{-1}(s(X_s)))}{s'_{ou}(s_{ou}^{-1}(s(X_s)))} \frac{s'(X_s)}{m'(X_s)} ds\,, \quad t \geq 0\,,
\end{aligned}$$

where we used the occupation time formula (cf. Revuz and Yor (1991), p.215). Notice also that τ_t is continuous and strictly increasing. Under condition (3.10) we obtain

$$\tau_t - \tau_{t-\varepsilon} \geq c\varepsilon\,, \quad t \geq 0\,. \tag{3.12}$$

Moreover, Itô and McKean (1974), p. 228 proved the following ergodic theorem

$$\frac{\tau_t}{t} \overset{\text{a.s.}}{\to} \frac{1}{|\widetilde{m}|} = \frac{1}{|m|}\,. \tag{3.13}$$

Wlog we assume $|m| = 1$ in the following.

According to Theorem 4.7 of Kallenberg (1983) it suffices to show for any $y \in (l, r)$

$$\lim_{t\to\infty} P^y(N^X_{\varepsilon,u_t}(tU) = 0) = P(N(U) = 0)\,, \tag{3.14}$$

where U is an arbitrary union of semi-open intervals, and

$$\limsup_{t\to\infty} E^y(N^X_{\varepsilon,u_t}(t(a,b])) \le E(N((a,b])) < \infty, \quad \text{for every } (a,b] \subset (0,1]\,. \tag{3.15}$$

By definition of the processes O, Z, X and Y, setting $v_t = s(u_t)$, $z = s(y)$, $w_t = s_{ou}^{-1}(v_t)$ and $x = s_{ou}^{-1}(z)$, we have for $k \ge 1$,

$$
\begin{aligned}
&P^y(N^X_{\varepsilon,u_t}(t(a,b]) \ge k)\\
&= P(\#\{\varepsilon\text{-upcrossings of } u_t \text{ by} X_\nu\,, \nu \in t(a,b]\} \ge k \mid X_0 = y)\\
&= P(\#\{\varepsilon\text{-upcrossings of } v_t \text{ by } Y_\nu\,, \nu \in t(a,b]\} \ge k \mid Y_0 = z)\\
&= P(\{\exists \nu_1,\dots,\nu_k \in t(a,b] : \forall i = 1,\dots,k\,, Y_\nu < v_t \, \forall \nu \in (\nu_i - \varepsilon, \nu_i) \text{ and } Y_{\nu_i} = v_t\} \mid Y_0 = z)\\
&= P(\{\exists \tau_{\nu_1},\dots,\tau_{\nu_k} \in (\tau_{ta},\tau_{tb}] : \forall i = 1,\dots,k\,, Z_u < v_t \, \forall u \in (\tau_{\nu_i-\varepsilon},\tau_{\nu_i}) \text{ and } Z_{\tau_{\nu_i}} = v_t\,\} \mid Z_0 = z)\\
&\le P(\{\exists \tau_{\nu_1},\dots,\tau_{\nu_k} \in (\tau_{ta},\tau_{tb}] : \forall i = 1,\dots,k\,, Z_u < v_t \;\; \forall u \in (\tau_{\nu_i} - c\varepsilon,\tau_{\nu_i}) \text{ and } Z_{\tau_{\nu_i}} = v_t\,\} \mid Z_0 = z)
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
&= P(\#\{c\varepsilon\text{-upcrossings of } v_t \text{ by } Z_u\,, u \in (\tau_{ta},\tau_{tb}]\} \ge k \mid Z_0 = z)\\
&= P(\#\{c\varepsilon\text{-upcrossings of } s_{ou}^{-1}(v_t) \text{ by } s_{ou}^{-1}(Z_u)\,, u \in (\tau_{ta},\tau_{tb}]\} \ge k \mid s_{ou}^{-1}(Z_0) = s_{ou}^{-1}(z))\\
&= P^x(N^O_{c\varepsilon,w_t}((\tau_{ta},\tau_{tb}]) \ge k)\,.
\end{aligned}
$$

The inequality is a consequence of (3.12). Notice, since all transformations are strictly monotone and continuous, when we start with $(a,b]$, then we get again an interval $(\tau_a,\tau_b]$. Furthermore, we know already that the theorem holds for the OU-process O. We show that for all $k \ge 0$,

$$\limsup_{t\to\infty} \; \left|P^x(N^O_{c\varepsilon,w_t}((\tau_{ta},\tau_{tb}]) \ge k) - P^x(N^O_{c\varepsilon,w_t}((ta,tb] \ge k)\right| = 0\,, \quad x \in \mathbb{R},$$

equivalently, for all $k \ge 0$,

$$\limsup_{t\to\infty} \; \left|P^x(N^O_{c\varepsilon,w_t}((\tau_{ta},\tau_{tb}]) \le k) - P^x(N^O_{c\varepsilon,w_t}((ta,tb] \le k)\right| = 0\,, \quad x \in \mathbb{R}, \tag{3.17}$$

For any $0 < \delta < 1$, define

$$A_t = \{|\tau_{ta} - ta| \le \delta ta\,, |\tau_{tb} - tb| \le \delta tb\}\,, \quad t \ge 0\,.$$

By the triangular inequality, the lhs of (3.17) is bounded by

$$\begin{aligned}
&\limsup_{t\to\infty} \left| P^x(N^O_{c\varepsilon,v_t}((\tau_{ta},\tau_{tb}]) \le k, A_t) - P^x(N^O_{c\varepsilon,w_t}((ta,tb]) \le k, A_t) \right| \\
&\quad +2\limsup_{t\to\infty} \{P^x(|\tau_{ta}-ta| \ge \delta ta) + P^x(|\tau_{tb}-tb| \ge \delta tb)\} \quad =: \quad I_1 + I_2\,.
\end{aligned}$$

Notice that I_2=0 by (3.13).

Again by the triangular inequality and the fact that $(\tau_{ta},\tau_{tb}] \subset ((1-\delta)ta,(1+\delta)tb]$ in A_t,

$$\begin{aligned}
I_1 &\le \limsup_{t\to\infty} \big(P^x(N^O_{c\varepsilon,w_t}((\tau_{ta},\tau_{tb}]) \le k, A_t) - P^x(N^O_{c\varepsilon,w_t}(((1-\delta)ta,(1+\delta)tb]) \le k, A_t) \big) \\
&\quad + \limsup_{t\to\infty} \big(P^x(N^O_{c\varepsilon,w_t}((ta,tb]) \le k, A_t) - P^x(N^O_{c\varepsilon,w_t}(((1-\delta)ta,(1+\delta)tb]) \le k, A_t) \big) \\
&=: \; J_1 + J_2\,.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
J_1 &\le \limsup_{t\to\infty} \big(P^x(N^O_{c\varepsilon,w_t}((\tau_{ta},\tau_{tb}]) \le k, A_t, N^O_{c\varepsilon,w_t}(((1-\delta)ta,(1+\delta)tb]) > k) \big) \\
&\le \limsup_{t\to\infty} \big(P^x(N^O_{c\varepsilon,w_t}(((1-\delta)ta,(1+\delta)ta]) > 0) + P^x(N^O_{c\varepsilon,w_t}(((1-\delta)tb,(1+\delta)tb]) > 0) \big) \\
&= \limsup_{t\to\infty} \big(P^H(N^*_t((0,2\delta a] > 0) + P^H(N^*_t((0,2\delta b] > 0) \big) \\
&= P(N((0,2\delta a] > 0) + P(N((0,2\delta b] > 0) \le 2(1-e^{-\tau 2\delta b})\,,
\end{aligned}$$

where H is the stationary distribution and N^*_t is the time-normalised point process of ε-upcrossings of the process O. We used that the Ornstein-Uhlenbeck process O has the strong Markov property and is ergodic, and that the result holds for O.

Similar considerations yield the same upper bound for J_2 and hence the lhs of (3.17) is bounded by $4(1-e^{-\tau 2\delta b})$. Letting $\delta \downarrow 0$ we have proved (3.17) for all $k \ge 0$, which yields together with (3.16) and $x = s^{-1}_{ou}(s(y))$,

$$\begin{aligned}
\limsup_{t\to\infty} E^y(N^X_{\varepsilon,u_t}(t(a,b])) &= \limsup_{t\to\infty} \sum_{k=1}^{\infty} P^y(N^X_{\varepsilon,u_t}(t(a,b]) \ge k) \\
&\le \sum_{k=1}^{\infty} \limsup_{t\to\infty} P^x(N^O_{c\varepsilon,w_t}(t(a,b]) \ge k) \\
&= \sum_{k=1}^{\infty} P(N((a,b]) \ge k) \;=\; E(N((a,b]))\,,
\end{aligned}$$

and therefore (3.15) holds. Now we check (3.14):

Wlog choose an arbitrary U of the form $U = \bigcup_{i=1}^d (a_i,b_i]$ with disjoint intervals and $a_1 \le a_2 \le \ldots \le a_d$. Then, by definition of the ε-upcrossings,

$$\lim_{t\to\infty} P^y(N^X_{\varepsilon,u_t}(tU) = 0) \quad = \quad \lim_{t\to\infty} P^y(\{N^X_{\varepsilon,u_t}(tU) = 0\} \cap \bigcap_{i=1}^d \{M^X_{[ta_i,ta_i+\varepsilon]} < u_t\})$$

$$
\begin{aligned}
& + \lim_{t\to\infty} P^y(\{N^X_{\varepsilon,u_t}(tU)=0\} \cap \bigcup_{i=1}^d \{M^X_{[ta_i,ta_i+\varepsilon]} \geq u_t\}) \\
= \quad & \lim_{t\to\infty} P^y(\bigcap_{i=1}^d \{M^X_{t[a_i,b_i]} < u_t\}) \\
& + \lim_{t\to\infty} P^y(\{N^X_{\varepsilon,u_t}(tU)=0\} \cap \bigcup_{i=1}^d \{M^X_{[ta_i,ta_i+\varepsilon]} \geq u_t\}) \\
=: \quad & K_1 + K_2 \,.
\end{aligned}
$$

We show by induction that the rhs equals $P(\bigcap_{i=1}^d \{N((a_i,b_i]) = 0)$. Because of Proposition 3.1 and the fact that $K_2 = 0$ (see below) this is true for $d = 1$. Now we may assume that

$$
I_{\{\bigcap_{i=1}^{d-1}\{M^X_{t[a_i,b_i]}<u_t\}\}} \quad \stackrel{d}{\to} \quad I_{\{\bigcap_{i=1}^{d-1}\{N((a_i,b_i])=0\}\}} \,, \quad t\to\infty \,, \tag{3.18}
$$

and by the Markov property,

$$
P(M^X_{t[a_d,b_d]} < u_t | X_{ta_d}) \quad \stackrel{P}{\to} \quad e^{-\tau(b_d-a_d)} \quad t\to\infty \,. \tag{3.19}
$$

By Slutzki's theorem, the product of the lhss of (3.18) and (3.19) converges in distribution to the product of their rhss. Applying Theorem 5.2 of Billingsley (1968) we obtain

$$
\begin{aligned}
K_1 &= \lim_{t\to\infty} E^y(1_{\{\bigcap_{i=1}^{d-1}\{M^X_{t[a_i,b_i]}<u_t\}\}} P(M^X_{t[a_d,b_d]} < u_t | X_{ta_d})) \\
&= E(1_{\{\bigcap_{i=1}^{d-1}\{N((a_i,b_i])=0\}\}} e^{-\tau(b_d-a_d)}) \\
&= P(\bigcap_{i=1}^{d-1} \{N((a_i,b_i])=0\}) e^{-\tau(b_d-a_d)} \\
&= P(\bigcap_{i=1}^{d} \{N((a_i,b_i])=0\}) = P(N(U)=0) \,.
\end{aligned}
$$

In the last step we used that a homogeneous Poisson process has independent increments. It remains to show $K_2 = 0$. With the same notation as before we have

$$
\begin{aligned}
K_2 &\leq \lim_{t\to\infty} P^y(\bigcup_{i=1}^d \{M^X_{[ta_i,ta_i+\varepsilon]} \geq u_t\}) \\
&\leq \sum_{i=1}^d \lim_{t\to\infty} P^y(M^X_{[ta_i,ta_i+\varepsilon]} \geq u_t) \\
&\leq \sum_{i=1}^d \lim_{t\to\infty} P^z(M^Z_{[\tau_{ta_i},\tau_{ta_i+\varepsilon}]} \geq v_t) \\
&\leq \sum_{i=1}^d \lim_{t\to\infty} P^z(M^Z_{[\tau_{ta_i},\tau_{ta_i+\varepsilon}]} \geq v_t, |\tau_{ta_i} - ta_i| < \delta ta_i, |\tau_{ta_i+\varepsilon} - ta_i| < \delta ta_i) \\
&\quad + \sum_{i=1}^d \lim_{t\to\infty} (P^z(|\tau_{ta_i} - ta_i| \geq \delta ta_i) + P^z(|\tau_{ta_i+\varepsilon} - ta_i| \geq \delta ta_i)) \,.
\end{aligned}
$$

Because of (3.13), the second and third term vanish. Again by ergodicy and Proposition 3.1, $K_2 \le \sum_{i=1}^d (1 - e^{-2\tau\delta a_i})$. Letting $\delta \downarrow 0$, $K_2 = 0$ and we proved

$$\lim_{t\to\infty} P^y(N^X_{\varepsilon,u_t}(tU) = 0) = P(N(U) = 0)\,. \tag{3.20}$$

and hence (3.14). □

Theorem 3.4 describes the asymptotic behaviour of the number of ε-upcrossings of a suitably increasing level. In particular, on average there are τ ε-upcrossings of u_t by $(X_s)_{0\le s\le t}$ for large t. Notice furthermore, that we get a Poisson process in the limit which is independent of the choice of $\varepsilon > 0$. A visualisation of the Poisson approximation of Theorem 3.4 is shown in Figure 8 for the generalised inverse Gaussian diffusion.

The next lemma provides a simple sufficient condition, only on scale function and speed measure of (X_t), for (3.10). Notice that by positivity and continuity, (3.10) holds automatically on compact intervals. It remains to check this condition for z in a neighbourhood of r and l.

Lemma 3.6 *Let (X_t) satisfy the usual conditions (2.5). Assume furthermore that (3.9) holds and that there exist $c_1, c_2 \in (0,\infty]$ such that*

$$\frac{1}{4\ln(|s(z)|)s(z)}\left(\frac{s''(z)}{s'(z)m'(z)} - \frac{m''(z)}{(m'(z))^2}\right) \quad\to\quad c_1 \textit{ or } c_2 \tag{3.21}$$

according as $z \uparrow r$ or $z \downarrow l$, then the assertion of Theorem 3.4 holds.

Proof. By l'Hospital,

$$s_{ou}(x) \sim g(x) = \sqrt{2\pi}e^{x^2/2}/x\,, \quad x\to\infty\,, \tag{3.22}$$

and s_{ou} and g are unbounded and non-decreasing for all x large enough. Moreover, s_{ou} and g are inversely asymptotic, i.e. for all $\lambda > 1$, there exists some $x_0(\lambda)$ such that

$$s_{ou}(x/\lambda) \le g(x) \le s_{ou}(\lambda x), \quad \forall x \ge x_0(\lambda)\,.$$

This implies by Exercise 14 of Bingham, Goldie and Teugels (1987), Section 3.13, that $s_{ou}^{-1}(x) \sim g^{-1}(x) \sim \sqrt{2\ln x}$ as $x\to\infty$. Thus, by l'Hospital,

$$\begin{aligned}
\frac{m'_{ou}(s_{ou}^{-1}(s(z)))}{s'_{ou}(s_{ou}^{-1}(s(z)))}\frac{s'(z)}{m'(z)} &\sim \frac{s''(z)/m'(z) - s'(z)m''(z)/(m'(z))^2}{2s''_{ou}(s_{ou}^{-1}(s(z)))s'(z)}\\
&\sim \frac{1}{2(s_{ou}^{-1}(s(z)))^2 s(z)}\left(\frac{s''(z)}{s'(z)m'(z)} - \frac{m''(z)}{(m'(z))^2}\right)\\
&\sim \frac{1}{4\ln(|s(z)|)s(z)}\left(\frac{s''(z)}{s'(z)m'(z)} - \frac{m''(z)}{(m'(z))^2}\right), \quad z\uparrow r \text{ or } z\downarrow l\,.
\end{aligned}$$

The second line is a consequence of (3.22) for $x = s_{ou}^{-1}(s(z))$ which tends to $\pm\infty$ as $z \downarrow l$ or $z \uparrow r$. In the last line we have used that $s_{ou}^{-1}(x) \sim \pm\sqrt{2\ln|x|}$ as $x \to \pm\infty$. $\square$

In the following situations we work out conditions on μ and σ such that the tail behaviour of F can easily be described. We apply these results to the examples in Sections 4 and 5.

Theorem 3.7 *Assume that the usual conditions hold.*
(a) Assume that $\mu \equiv 0$. Then $(l,r) = (-\infty, \infty)$ and

$$\overline{F}(x) \sim \left(\int_{-\infty}^{\infty} (2/\sigma^2(t))dt \right)^{-1} x^{-1}, \quad x \to \infty.$$

(b) Assume that $r = \infty$ and $-\infty < \rho = \int_z^r \mu(t)/\sigma^2(t)dt < \infty$ for some $z \in (l, \infty)$. Then

$$\overline{F}(x) \sim e^{2\rho}|m|^{-1}x^{-1}, \quad x \to \infty. \tag{3.23}$$

(c) Let μ and σ^2 be differentiable functions on (x_0, r) for some $x_0 < r$ such that

$$\lim_{x\uparrow r} \frac{d}{dx}\left\{ \frac{\sigma^2(x)}{\mu(x)} \right\} = 0 \qquad \text{and} \qquad \lim_{x\uparrow r} \frac{\sigma^2(x)}{\mu(x)} \exp\left\{ -2\int_z^x \frac{\mu(t)}{\sigma^2(t)}dt \right\} = -\infty. \tag{3.24}$$

Then

$$\overline{F}(x) \sim |\mu(x)|h(x), \quad x \uparrow r, \tag{3.25}$$

where h is the stationary density of (X_t).

Proof. We first prove (b). By l'Hospital and (2.2),

$$\lim_{x\to\infty} \frac{s(x)}{x} = \lim_{x\to\infty} s'(x) \to e^{-2\rho}, \quad x \to \infty.$$

This implies that $s(x) \sim e^{-2\rho}x$ as $x \to \infty$. Now Corollary 3.2 applies and (3.23) follows by l'Hospital.
(a) Immediately from (2.2) we have $s'(x) = 1$ for all $x \in (l,r)$. Hence by (2.1) $s(x) = x - z$ for $z \in (l,r)$. Since $\lim_{x\uparrow r} s(x) = \infty$ and $\lim_{x\downarrow l} s(x) = -\infty$, we must have $l = -\infty$ and $r = \infty$. Then part (b) applies with $\rho = 0$ and $|m| = \int_{-\infty}^{\infty}(2/\sigma^2(t))dt$.
(c) Notice that s' is an exponential function, hence

$$s''(x) = -2s'(x)\frac{\mu(x)}{\sigma^2(x)}, \quad x \in (l,r).$$

Then by l'Hospital (which can be applied because of (3.24)),

$$\lim_{x\uparrow r} \frac{2\int_z^x s'(y)dy}{-s'(x)\sigma^2(x)/\mu(x)} = \lim_{x\uparrow r} \frac{2s'(x)}{-s'(x)o(1) - s''(x)\sigma^2(x)/\mu(x)} = 1, \quad x \uparrow r.$$

Inserting this in (3.5) yields $\overline{F}(x) \sim -2\mu(x)/(|m|s'(x)\sigma^2(x))$ as $x \uparrow r$, and the result follows from (2.3) and (2.4). $\square$

From equations (2.1)-(2.4) it is clear that (X_t) is also uniquely determined by its stationary density $h(x)$ and the diffusion coefficient $\sigma(x)$. They determine the drift term which is for differentiable volatility σ

$$\mu(x) = \frac{\sigma^2(x)}{2}\frac{d}{dx}\ln(\sigma^2(x)h(x)), \quad x \in (l,r). \tag{3.26}$$

Theorem 3.8 *Assume that the usual conditions hold with $r = \infty$. Let h be the stationary density, h positive on (x_0,∞) for some $x_0 > 0$.*
(a) If $\sigma^2(x) \sim x^{1-\delta}\ell(x)/h(x)$ as $x \to \infty$ for some $\delta > 0$, where ℓ is a slowly varying function such that $1/\ell$ is locally bounded. Then

$$\overline{F}(x) \sim \frac{\delta}{2}x^{-\delta}\ell(x), \quad x \to \infty.$$

(b) If $\sigma^2(x) \sim cx^{\delta-1}e^{-\alpha x^\beta}/h(x)$ as $x \to \infty$ for $\alpha,\beta,c > 0, \delta \in \mathbb{R}$, then

$$\overline{F}(x) \sim \frac{c\alpha\beta}{2}x^{\delta+\beta-2}e^{-\alpha x^\beta}, \quad x \to \infty. \tag{3.27}$$

Proof. (a) By (2.3) and (2.4) $s'(x) \sim 2x^{-(1-\delta)}/(|m|\ell(x))$ as $x \to \infty$. Hence s' is regularly varying with index $\delta - 1$ and is locally bounded. From Corollary 3.2 it follows with Karamata's theorem (Theorem 1.5.11 of Bingham, Goldie and Teugels (1987)) that

$$\overline{F}(x) \sim \frac{\delta}{2}x^{-\delta}\ell(x), \quad x \to \infty.$$

(b) By (2.3) and (2.4) we obtain $s'(x) \sim 2x^{-(\delta-1)}e^{\alpha x^\beta}/(c|m|)$ as $x \to \infty$. Then by l'Hospital

$$s(x) \sim \frac{2}{|m|c\alpha\beta}x^{-\delta-\beta+2}e^{\alpha x^\beta}, \quad x \to \infty.$$

giving (3.27) by Corollary 3.2. $\square$

Notice that this result provides a method to construct diffusions with any arbitrary stationary density (with right endpoint $r = \infty$) and any extremal behaviour.

4 Extremes of stochastic models in finance

Diffusion processes given by the SDE (1.1)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t > 0,$$

with properties as described in Section 2 are common models in finance; see e.g. Lamberton and Lapeyre (1991), Duffie (1992), Merton (1994) or Baxter and Rennie (1996). Examples 4.1, 4.2, and 4.3 are standard models for the term structure of interest rates; diffusions as Example 4.4 have been successfully fitted to share prices (Küchler et al. (1994), Eberlein and Keller (1995)).

The state space (l, r) and the range of parameters of all models below is such that $\lim_{x \uparrow r} s(x) = \infty$ and $\lim_{x \downarrow l} s(x) = -\infty$, hence the boundaries are inaccessible. This can easily be checked by standard calculations and (2.1). Furthermore, the speed measure m is finite for all models, the processes are ergodic with stationary distribution which is absolutely continuous with density h given by (2.4). Hence all these models satisfy the usual conditions (2.5).

Once F is determined for any of these models, classical extreme value theory takes over. Recall that there are three extreme value dfs (up to affine transformations). Since all the examples we treat in Section 4 are diffusions with state space unbounded above, we only consider the Fréchet df and the Gumbel df given by

$$\begin{aligned} \Phi_\alpha(x) &= \exp\{-x^{-\alpha}\} I_{(0,\infty)}(x) \,, \quad \alpha > 0 \,, \\ \Lambda(x) &= \exp\{-e^{-x}\} \,, \quad x \in \mathbb{R} \,. \end{aligned}$$

If $F \in \mathrm{MDA}(\Phi_\alpha)$, then the norming constants a_t and b_t can be chosen such that

$$\overline{F}(a_t) \sim t^{-1} \quad \text{and} \quad b_t = 0 \,. \tag{4.1}$$

If $F \in \mathrm{MDA}(\Lambda)$, then the norming constants a_t and b_t can be chosen such that

$$\overline{F}(b_t) = t^{-1} \quad \text{and} \quad a_t \sim a(b_t) \,, \tag{4.2}$$

where a is the so-called auxiliary function; see e.g. Theorem 3.3.26 of Embrechts et al. (1997). Calculating the norming constants explicitly is then a standard, though often tedious task. For b_t a Taylor expansion leads to the necessary accurracy required by the convergence to types theorem. We refer to the monographs by Leadbetter, Lindgren and Rootzén (1983), Resnick (1987) or Embrechts et al. (1997) for some tutorial examples.

Then (3.4) implies that

$$\frac{M_t^X}{a_t} \xrightarrow{d} \Phi_\alpha \quad \text{if} \quad F \in \mathrm{MDA}(\Phi_\alpha) \tag{4.3}$$

and

$$\frac{M_t^X - b_t}{a_t} \xrightarrow{d} \Lambda \quad \text{and} \quad \frac{M_t^X}{b_t} \xrightarrow{P} 1 \quad \text{if} \quad F \in \mathrm{MDA}(\Lambda) \,. \tag{4.4}$$

Furthermore, all the models in this section except the generalised Cox-Ingersoll-Ross model $\gamma = 1$ satisfy condition (3.21) of Lemma 3.6, hence the Poisson approximation of the ε–upcrossings is also explicitly given for $u_t = a_t x + b_t$ and $\tau = -\ln Q(x)$, where Q is either Φ_α or Λ.

Figures 2, 3, 4, 5, 6 and 7 show simulated sample paths (of length $t = 1000$ respectively 25 000) of the different models. The solid line indicates those norming constants which describe the increase of M_t^X for large t, i.e. in MDA(Φ_α) we plot a_t (see (4.3)) and in MDA(Λ) we plot b_t (see (4.4)).

Figure 9 shows the empirical df, the empirical density and the QQ-plot (based on 350 simulated maxima, each taken from a sample path with $t = 25\,000$) of the normalised maxima of the generalised inverse Gaussian model for certain parameter values together with the corresponding limit df and density. The models were simulated by means of the Milstein Scheme (strong Taylor approximation of convergence order 1) and we refer to Kloeden and Platen (1992) for details.

Example 4.1 (The Vasicek model)
In this model the drift term is $\mu(x) = c - dx$ for $c \in \mathbb{R}$, $d > 0$ and the diffusion coefficient $\sigma(x) \equiv \sigma > 0$. The solution of the SDE (1.1) with $X_0 = x$ is given by

$$X_t = \frac{c}{d} + (x - \frac{c}{d})e^{-dt} + \sigma \int_0^t e^{-d(t-s)} dW_s \,, \quad t \geq 0 \,.$$

(X_t) has state space $\mathbb{R}$, mean value function

$$EX_t = \frac{c}{d} + (x - \frac{c}{d})e^{-dt} \to \frac{c}{d} \quad \text{and} \quad \mathrm{var} X_t = \frac{\sigma^2}{2d}\left(1 - e^{-2dt}\right) \to \frac{\sigma^2}{2d} \,, \quad t \to \infty \,.$$

It is well-known and easy to calculate from (2.2)-(2.4) that (X_t) has a normal stationary distribution, more precisely, it is $N(\frac{c}{d}, \frac{\sigma^2}{2d})$, where $N(a, b)$ denotes the normal distribution with mean a and variance b.

The assumptions of Theorem 3.7(c) are satisfied giving

$$\overline{F}(x) \quad \sim \quad d\frac{(x - c/d)^2}{\sigma^2/2d}\, \overline{H}(x) \,,$$

where $\overline{H}(x)$ is the tail of the stationary normal df; hence F has heavier tail than H. It can be shown that the rhs is the tail of a so-called von Mises function (see e.g. Example 3.3.23 of Embrechts et al. (1997)), hence $F \in \mathrm{MDA}(\Lambda)$ with norming constants

$$a_t = \frac{\sigma}{2\sqrt{d \ln t}} \quad \text{and} \quad b_t = \frac{\sigma}{\sqrt{d}}\sqrt{\ln t} + \frac{c}{d} + \frac{\sigma}{4\sqrt{d}} \frac{\ln \ln t + \ln(\sigma^2 d / 2\pi)}{\sqrt{\ln t}} \,.$$

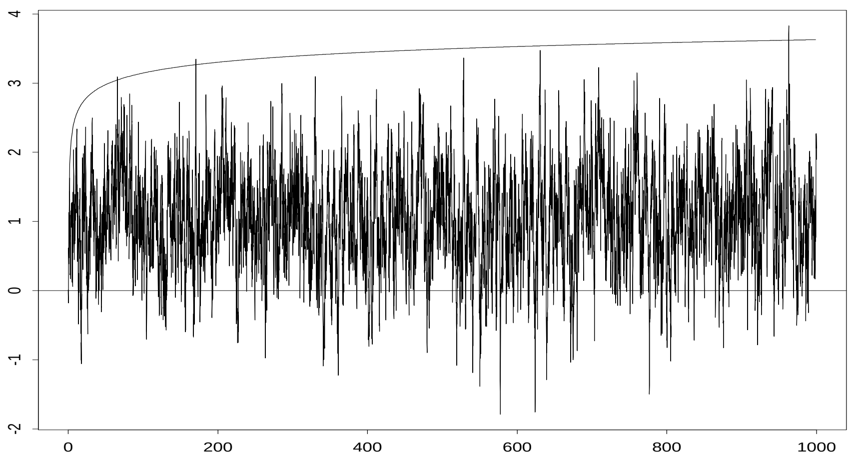

Figure 2: Simulated sample path of the Vasicek model (with parameters $c = d = \sigma = 1$) and corresponding normalising constants b_t.

Example 4.2 (The Cox-Ingersoll-Ross model)

In this model (X_t) satisfies the SDE (1.1) with $\mu(x) = c - dx$ for $d > 0$, $\sigma(x) = \sigma\sqrt{x}$ for $\sigma > 0$ and $2c \geq \sigma^2$. It has state space $(0, \infty)$, mean value function

$$EX_t = \frac{c}{d} + \left(x - \frac{c}{d}\right) e^{-dt} \to \frac{c}{d}, \quad t \to \infty$$

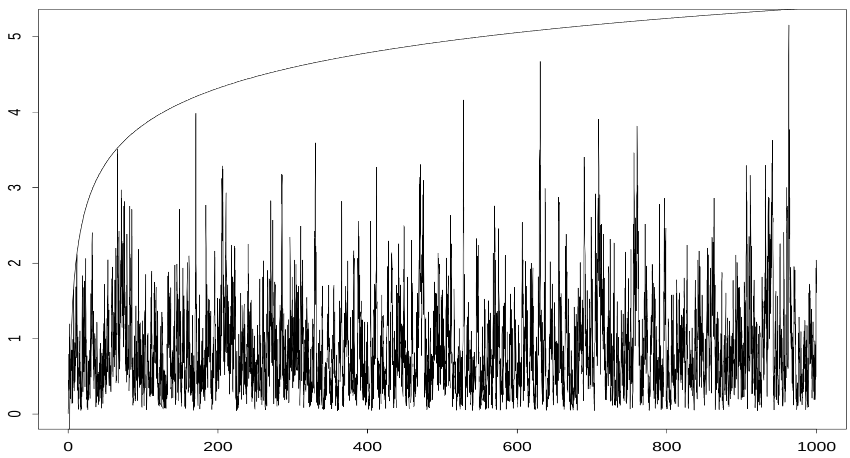

Figure 3: Simulated sample path of the Cox-Ingersoll-Ross model (with parameters $c = d = \sigma = 1$) and the corresponding norming constants b_t.

and

$$\mathrm{var} X_t = \frac{c\sigma^2}{2d^2}\left(1-\left(1+\left(x-\frac{c}{d}\right)\frac{2d}{c}\right)e^{-2dt}+\left(x-\frac{c}{d}\right)\frac{2d}{c}e^{-3dt}\right)\to\frac{c\sigma^2}{2d^2}\,,\quad t\to\infty\,,$$

where $X_0 = x$. From (2.2)-(2.4) we obtain that the stationary distribution H is $\Gamma(\frac{2c}{\sigma^2},\frac{2d}{\sigma^2})$.

Theorem 3.7(c) applies giving

$$\overline{F}(x)\sim\frac{2cd}{\sigma^2}\overline{G}(x)\,,\quad x\to\infty\,.$$

where $\overline{G}(x)$ is the tail of the $\Gamma(\frac{2c}{\sigma^2}+1,\frac{2d}{\sigma^2})$ distribution. Notice that $\overline{F}(x)\sim Ax\overline{H}(x)$ for some $A>0$. It is well-known (see e.g. Resnick (1987), p. 72-73) that the gamma distributions are in MDA(Λ) and the norming constants for F are

$$a_t=\sigma^2/(2d)\quad\text{and}\quad b_t=\frac{\sigma^2}{2d}\left(\ln t+\frac{2c}{\sigma^2}\ln\ln t+\ln\left(\frac{d}{\Gamma(2c/\sigma^2)}\right)\right)\,. \tag{4.5}$$

Notice that for $\sigma^2 << c$ the constant $\Gamma(2c/\sigma^2)$ is very large and consequently b_t may become negative for small t. In extreme cases b_t becomes positive only for very large t.

Example 4.3 (Generalised Cox-Ingersoll-Ross model)
In this model the drift term is given by $\mu(x)=c-dx$ and the diffusion coefficient has the form $\sigma(x)=\sigma x^\gamma$ for $\gamma\in[\frac{1}{2},\infty)$. For $\gamma<\frac{1}{2}$ we have $|m|=\infty$ and hence by Theorem 7 of Mandl (1968), p.90, the process is not ergodic. For $\gamma\geq\frac{1}{2}$ the process is ergodic with state space $(0,\infty)$.

We distinguish the following four cases:

$$\begin{array}{lcllll}
\gamma=1/2 & : & 2c\geq\sigma^2\,, & d>0 & & \text{(see Example 4.2)}\\
1/2<\gamma<1 & : & c>0\,, & d\geq 0 & & \\
\gamma=1 & : & c>0\,, & d>-\sigma^2/2 & & \\
\gamma>1 & : & c>0\,,\ d\in\mathbb{R} & \text{or} & c=0\,,\ d<0\,. &
\end{array} \tag{4.6}$$

For $\frac{1}{2}\leq\gamma\leq 1$ the mean value function of (X_t) is given by

$$EX_t=\begin{cases}\frac{c}{d}+\left(x-\frac{c}{d}\right)e^{-dt} & \to\frac{c}{d} \quad \text{if}\quad d>0\\ \frac{c}{d}+\left(x-\frac{c}{d}\right)e^{-dt} & \to\infty \quad \text{if}\quad d<0\\ x+ct & \to\infty \quad \text{if}\quad d=0\end{cases} \tag{4.7}$$

as $t\to\infty$ where $X_0=x$. This indicates already that for certain parameter values the model can capture large fluctuations in data, which will reflect also in the behaviour of the maxima.

- $\frac{1}{2} < \gamma < 1$

The stationary density, which can be calculated by (2.2)-(2.4), is

$$h(x) = \frac{2}{A\sigma^2} x^{-2\gamma} \exp\left\{ -\frac{2}{\sigma^2} \left(\frac{c}{2\gamma - 1} x^{-(2\gamma-1)} + \frac{d}{2-2\gamma} x^{2-2\gamma} \right) \right\}, \quad x > 0,$$

where

$$A = \frac{2}{\sigma^2} \int_0^\infty t^{-2\gamma} \exp\left\{ -\frac{2}{\sigma^2} \left(\frac{c}{2\gamma - 1} t^{-(2\gamma-1)} + \frac{d}{2-2\gamma} t^{2-2\gamma} \right) \right\} dt.$$

The assumptions of Theorem 3.7(c) are satisfied and hence

$$\overline{F}(x) \sim dx h(x) \sim \frac{d}{A} \frac{2}{\sigma^2} x^{-2\gamma+1} \exp\left\{ -\frac{d}{\sigma^2(1-\gamma)} x^{2(1-\gamma)} \right\}.$$

Notice that $\overline{F}(x) \sim B x^{2(1-\gamma)} \overline{H}(x)$ for some $B > 0$. The rhs is the tail of a von Mises function and hence in MDA(Λ) with norming constants

$$a_t = \frac{\sigma^2}{2d} \left(\frac{\sigma^2(1-\gamma)}{d} \ln t \right)^{\frac{2\gamma-1}{2-2\gamma}}$$

$$b_t = \left(\frac{\sigma^2(1-\gamma)}{d} \ln t \right)^{\frac{1}{2-2\gamma}} \left(1 - \frac{2\gamma - 1}{(2-2\gamma)^2} \frac{\ln\left(\frac{\sigma^2(1-\gamma)}{d} \ln t \right)}{\ln t} \right) + a_t \ln\left(\frac{2d}{A\sigma^2} \right).$$

Notice that a_t is continuous in the point $\gamma = 1/2$, i.e. a_t as above converges to $\sigma^2/(2d)$ as $\gamma \downarrow 1/2$, which is the same as (4.5). The first term of the norming constants b_t coincides with the first term of (4.5). The behaviour of a_t and b_t as $\gamma \uparrow 1$ is much more dramatic. It indicates already that at $\gamma = 1$ there must be a qualitative change in the extremal behaviour. This is confirmed below.

- $\gamma = 1$

In this case the solution of the SDE (1.1) with $X_0 = x$ is explicitly given by

$$X_t = e^{-(d+\frac{\sigma^2}{2})t + \sigma W_t} \left(x + c \int_0^t e^{(d+\frac{\sigma^2}{2})s - \sigma W_s} ds \right), \quad t \geq 0.$$

We obtain from (2.2)–(2.4) that the stationary density is inverse gamma:

$$h(x) = \left(\frac{\sigma^2}{2c} \right)^{-\frac{2d}{\sigma^2}-1} \left(\Gamma\left(\frac{2d}{\sigma^2} + 1 \right) \right)^{-1} x^{-2d/\sigma^2 - 2} \exp\left\{ -\frac{2c}{\sigma^2} x^{-1} \right\}, \quad x > 0.$$

Notice that h is regularly varying with index $-2d/\sigma^2 - 2$ and hence by Karamata's theorem (Theorem 1.5.11 of Bingham, Goldie and Teugels (1987)) the tail $\overline{H}$ of the stationary distribution

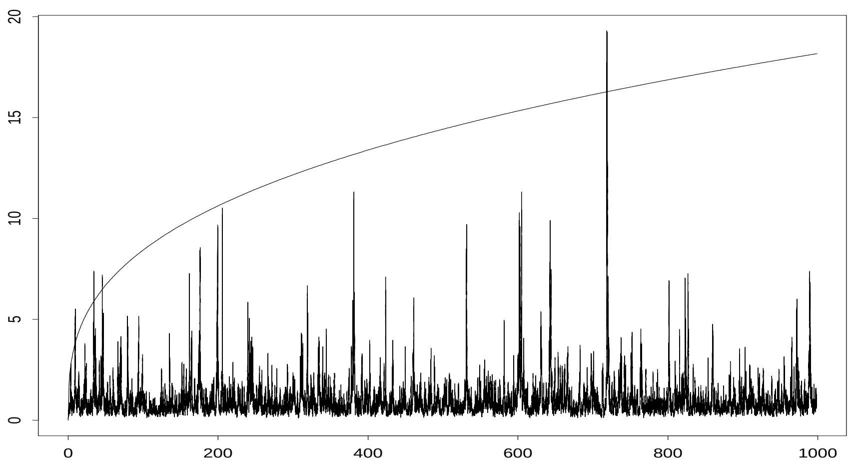

Figure 4: Simulated sample path of the generalised Cox-Ingersoll-Ross model for $\gamma = 1$ (with parameters $c = d = \sigma = 1$) and the corresponding norming constants b_t.

is also regularly varying. This implies that certain moments are infinite:

$$\lim_{t\to\infty} EX_t^\delta = \begin{cases} \left(\frac{2c}{\sigma^2}\right)^\delta \frac{\Gamma\left(\frac{2d}{\sigma^2}+1-\delta\right)}{\Gamma\left(\frac{2d}{\sigma^2}+1\right)} & \text{if } \delta < \frac{2d}{\sigma^2}+1\,, \\ \infty & \text{if } \delta \geq \frac{2d}{\sigma^2}+1\,. \end{cases}$$

In particular,

$$\lim_{t\to\infty} \mathrm{var} X_t = \begin{cases} \frac{2c^2}{d(2d-\sigma^2)} < \infty & \text{if } \frac{2d}{\sigma^2} > 1\,, \\ \infty & \text{if } -1 < \frac{2d}{\sigma^2} \leq 1\,. \end{cases}$$

For the tail of F we obtain by Theorem 3.7(c)

$$\begin{aligned} \overline{F}(x) &\sim \frac{\sigma^2}{2}\left(\frac{\sigma^2}{2c}\right)^{-\frac{2d}{\sigma^2}-1}\left(\Gamma\left(\frac{2d}{\sigma^2}+1\right)\right)^{-1}\left(\frac{2d}{\sigma^2}+1\right) x^{-2d/\sigma^2-1}\exp\left\{-\frac{2c}{\sigma^2}x^{-1}\right\} \\ &\sim \frac{\sigma^2}{2}\left(\frac{\sigma^2}{2c}\right)^{-\frac{2d}{\sigma^2}-1}\left(\Gamma\left(\frac{2d}{\sigma^2}+1\right)\right)^{-1}\left(\frac{2d}{\sigma^2}+1\right) x^{-2d/\sigma^2-1}\,, \quad x\to\infty\,. \end{aligned}$$

Here $\overline{F}(x) \sim B\overline{H}(x)$ for some $B > 0$. Hence $\overline{F}$ is regularly varying, equivalently, $F \in \mathrm{MDA}(\Phi_{1+2d/\sigma^2})$, with norming constants a_t chosen according to (4.1) as

$$a_t \sim \left(\frac{\sigma^2}{2}\left(\frac{\sigma^2}{2c}\right)^{-\frac{2d}{\sigma^2}-1}\left(\Gamma\left(\frac{2d}{\sigma^2}+1\right)\right)^{-1}\left(\frac{2d}{\sigma^2}+1\right) t\right)^{1/(1+2d/\sigma^2)} \quad \text{and} \quad b_t = 0\,.$$

Notice that $a_t \sim C\,t^{1/(1+2d/\sigma^2)}$ for a constant C, i.e. a_t is a decreasing function of d/σ^2 in t. Hence the maxima M_t^X are likely to increase slower, when d/σ^2 gets larger. In particular,

$$M_t^X / \left(Ct^{1/(1+2d/\sigma^2)}\right) \quad \xrightarrow{d} \quad \Phi_{1+2d/\sigma^2}\,, \quad t\to\infty\,.$$

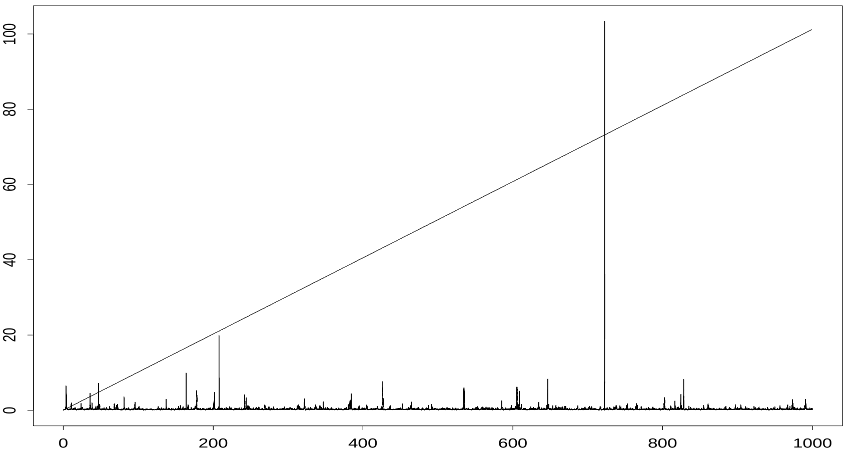

Figure 5: Simulated sample path of the generalised Cox-Ingersoll-Ross model for $\gamma = 1.5$ (with parameters $c = d = \sigma = 1$) and the corresponding norming constants a_t. In this case $A = 9.878$.

- $\gamma > 1$

Notice first that h is of the same form as in the case $\frac{1}{2} < \gamma < 1$, in particular $\overline{H}$ is regularly varying with index $-2\gamma + 1 \in (-\infty, -1)$. We apply Theorem 3.7(b) (alternatively Theorem 3.8(a)) and obtain

$$\overline{F}(x) \sim \frac{e^{2\rho}}{|m|} x^{-1} = (Ax)^{-1}, \quad x \to \infty,$$

where

$$\rho = \frac{1}{\sigma^2}\left(\frac{c}{2\gamma - 1} + \frac{d}{2 - 2\gamma}\right).$$

In this case the tail of the stationary distribution H has again a lighter tail than F. Hence $F \in \mathrm{MDA}(\Phi_1)$ with norming constants $a_t \sim t/A$. Notice that the order of increase of a_t is always linear. The constant A decides about the slope. We obtain in particular

$$AM_t^X/t \quad \stackrel{d}{\to} \quad \Phi_1 .$$

For $\gamma = 3/2$ it is possible to calculate A explicitly. We obtain

$$A = \frac{1}{c}\left(1 + 2\sqrt{\frac{d^2\pi}{\sigma^2 c}} e^{d^2/(c\sigma^2)} \Phi\left(\sqrt{\frac{2d^2}{c\sigma^2}}\right)\right),$$

where Φ denotes the standard normal df. Notice that (if we ignore the factor $1/c$ for the moment) A is increasing in the quotient $d^2/(c\sigma^2)$.

Example 4.4 (Generalised hyperbolic diffusion)

Diffusions with given stationary distribution have been considered as appropriate models for asset prices. It is often assumed that the price process follows the SDE (1.1) with drift term zero; i.e.

$$dX_t = \sigma(X_t)dW_t\,, \quad t > 0\,.$$

Choose

$$\sigma^2(x) = \sigma^2/h(x)\,,$$

where h is an arbitrary density, then (X_t) has exactly this stationary density h. These diffusion models and certain extensions have been investigated as alternatives to Gaussian processes for asset prices, see Küchler et al. (1994), Bibby and Sørensen (1995), Eberlein and Keller (1995), Barndorff-Nielsen (1995) and Rydberg (1996).

Here Theorem 3.8(a) applies, yielding

$$2M_t^X/(\sigma^2 t) \quad \stackrel{d}{\to} \quad \Phi_1\,, \quad t \to \infty\,,$$

regardless of their stationary distribution. $F \in \mathrm{MDA}(\Phi_1)$ means that the maximum of the process is likely to behave as the maximum of iid rvs with distribution tail $\overline{F}(x) \sim \frac{\sigma^2}{2}x^{-1}$, so the process is likely to show much more extreme fluctuation than one expects from its stationary distribution.

5 Generalised inverse Gaussian diffusion

In Example 4.4 we have seen how an ergodic diffusion with drift term $\mu \equiv 0$ and arbitrary stationary distribution can be constructed. This construction has the drawback that all these diffusions show the same behaviour in their maxima M_t^X represented by $\overline{F}(x) \sim Ax^{-1}$ for some $A > 0$. Guided by Theorem 3.8 we choose another method of construction. We choose a density $h(x)$ and a diffusion coefficient $\sigma(x)$. By equation (3.26) this defines a drift term μ, giving an SDE (1.1).

We shall present this method by introducing a new class of diffusions with generalised inverse Gaussian stationary distribution and state space $(0,\infty)$. Its stationary distribution has (like the generalised hyperbolic distribution) tails with asymptotic behaviour reaching from exponential to regularly varying. Moreover, this model can be viewed as a further generalisation of the Cox-Ross-Ingersoll model (Example 4.2). It also includes Example 4.3 for $\gamma = 1$.

The density of the generalised inverse Gaussian distribution is given by

$$h(x) = \frac{(\chi/\psi)^{\lambda/2}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} x^{\lambda-1} \exp\left\{-\frac{1}{2}(\chi x + \psi x^{-1})\right\}, \quad x > 0,$$

where K_λ is the modified Bessel function of the third kind and index λ. The following parameter sets are possible

$$\begin{aligned}
&\{\chi > 0, \quad \psi > 0, \quad \lambda \in \mathbb{R}\} \\
&\{\chi = 0, \quad \psi > 0, \quad \lambda < 0\} \\
&\{\chi > 0, \quad \psi = 0, \quad \lambda > 0\}.
\end{aligned}$$

Notice that the norming constant simplifies for $\chi = 0$ and $\psi = 0$.

Now we consider the special case of $\sigma(x) = \sigma x^\gamma$ for $\sigma > 0$ and $\gamma \geq 0$. For the sake of comparison we choose the diffusion coefficient to be the same as in Examples 4.2 and 4.3.

By equation (3.26), (2.2) and (2.3),

$$\begin{aligned}
\mu(x) &= \frac{1}{4}\sigma^2 x^{2\gamma-2}\left(\psi + 2(2\gamma + \lambda - 1)x - \chi x^2\right), \\
s'(x) &= \exp\left\{-\frac{1}{2}(\chi + \psi)\right\} x^{-(2\gamma+\lambda-1)} \exp\left\{\frac{1}{2}(\chi x + \psi x^{-1})\right\} \text{ and} \\
m'(x) &= \frac{2}{\sigma^2}\exp\left\{\frac{1}{2}(\chi + \psi)\right\} x^{\lambda-1} \exp\left\{-\frac{1}{2}(\chi x + \psi x^{-1})\right\}.
\end{aligned}$$

As formulated in Section 2 we require the process to be recurrent and to have inaccessible boundaries 0 and ∞, i.e.

$$\int_1^\infty x^{1-\lambda-2\gamma} \exp\left\{\frac{\chi}{2}x\right\} dx = \infty \quad \text{and} \quad \int_0^1 x^{1-\lambda-2\gamma} \exp\left\{\frac{\psi}{2}x^{-1}\right\} dx = \infty.$$

This puts further restrictions on the parameter space and we consider

$$\begin{aligned}
&\{\chi > 0, \quad \psi > 0, \quad \lambda \in \mathbb{R}\} \\
&\{\chi = 0, \quad \psi > 0, \quad \lambda < 0 \quad \text{and} \quad \lambda \leq 2(1-\gamma)\} \\
&\{\chi > 0, \quad \psi = 0, \quad \lambda > 0 \quad \text{and} \quad \lambda \geq 2(1-\gamma)\}.
\end{aligned} \tag{5.1}$$

The SDE (1.1) with μ and σ as above with this restricted parameter space has a unique solution (X_t). This can be shown for instance by an application of Theorem 5.13 of Karatzas and Shreve (1987) to $\ln(X_t)$. We call this solution (X_t) *generalised inverse Gaussian diffusion* (GIG diffusion). For all parameters in (5.1), $|m| < \infty$ and hence by Theorem 7 of Mandl (1968), p. 90, (X_t) is ergodic.

Comparison of the drift terms show that the GIG diffusion for $\gamma = 1/2$ and $\psi = 0$ (which implies $\chi > 0$ and $\lambda \geq 1$) is just the CIR model with parameters $c = \sigma^2\lambda/2$ and $d = \sigma^2\chi/4$. If we choose $\gamma = 1$ and $\chi = 0$ (which implies $\psi > 0$ and $\lambda < 0$), then we obtain the generalised CIR model with parameters $c = \sigma^2\psi/4$ and $d = -\sigma^2(1+\lambda)/2$.

For the study of the extremal behaviour of (X_t) we distinguish three different cases:

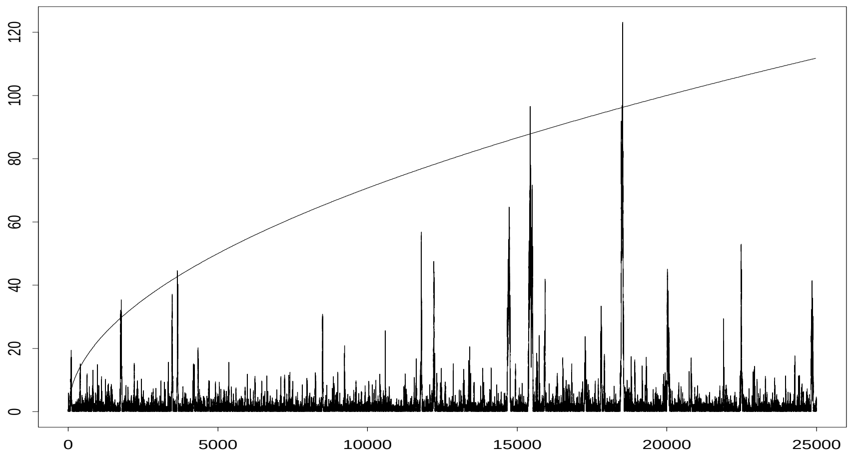

Figure 6: Simulated sample path of the GIG model (with parameters $\gamma = 0.5$, $\chi = 0$, $\psi = \sigma = 1$, $\lambda = -1$) and the corresponding norming constants a_t.

- $\chi = 0$

Then $\psi > 0$ and $\lambda < 0$ and $\lambda < 2(1-\gamma)$. The stationary density is inverse gamma of the form

$$h(x) = \frac{(\psi/2)^{-\lambda}}{\Gamma(-\lambda)} x^{\lambda-1} \exp\left\{-(\psi/2)x^{-1}\right\} \quad \sim \quad \frac{(\psi/2)^{-\lambda}}{\Gamma(-\lambda)} x^{\lambda-1}, \quad x \to \infty,$$

i.e. it is regularly varying. Hence

$$\overline{F}(x) \sim \frac{\sigma^2(\psi/2)^{-\lambda}(2-2\gamma-\lambda)}{2\Gamma(-\lambda)} x^{-(2-2\gamma-\lambda)}, \quad x \to \infty.$$

Thus $F \in \mathrm{MDA}(\Phi_{2-2\gamma-\lambda})$ with norming constants chosen according to (4.1) as

$$a_t \sim \left(\frac{\sigma^2(\psi/2)^{-\lambda}(2-2\gamma-\lambda)}{2\Gamma(-\lambda)} t\right)^{1/(2-2\gamma-\lambda)} \quad \text{and} \quad b_t = 0.$$

By Karamata's theorem (Theorem 1.5.11 of Bingham, Goldie and Teugels (1987))

$$\overline{H}(x) \sim \frac{(\psi/2)^{-\lambda}}{\Gamma(-\lambda)(-\lambda)} x^{\lambda}, \quad x \to \infty,$$

giving

$$\overline{H}(x) \sim \frac{2}{\sigma^2(-\lambda)(2-2\gamma-\lambda)} x^{2-2\gamma}\overline{F}(x), \quad x \to \infty.$$

Hence, depending on the choice of γ, the tail $\overline{H}$ of the stationary distribution can be heavier or lighter than or of the same order as $\overline{F}$.

- $\psi = 0$

Then $\chi > 0$ and $\lambda > 0$ and $\lambda \geq 2(1-\gamma)$. The stationary density simplifies

$$h(x) = \frac{(\chi/2)^\lambda}{\Gamma(\lambda)} x^{\lambda-1} \exp\{-(\chi/2)x\}\ , \quad x > 0\,,$$

which is a $\Gamma(\lambda, \chi/2)$ density. Now Theorem 3.8(b) applies giving

$$\overline{F}(x) \sim \frac{\sigma^2}{2}\left(\frac{\chi}{2}\right)^2 x^{2\gamma}\overline{H}(x)\,, \quad x \to \infty\,,$$

Then, as in the Cox-Ingersoll-Ross model, F is of gamma-type and hence $F \in \mathrm{MDA}(\Lambda)$ with norming constants chosen according to (4.2)

$$a_t = 2/\chi \quad \text{and} \quad b_t = \frac{2}{\chi}\left(\ln t + (2\gamma + \lambda - 1)\ln\ln t + \ln\left(\frac{\sigma^2(\chi/2)^{2-2\gamma}}{2\Gamma(\lambda)}\right)\right)\,.$$

Since $\gamma > 0$ this implies that F has a heavier tail than the stationary df H. Hence the extremal behaviour of (X_t) shows larger fluctuations than an iid family of random variables with df H is likely to show.

- $\psi > 0\,, \chi > 0$

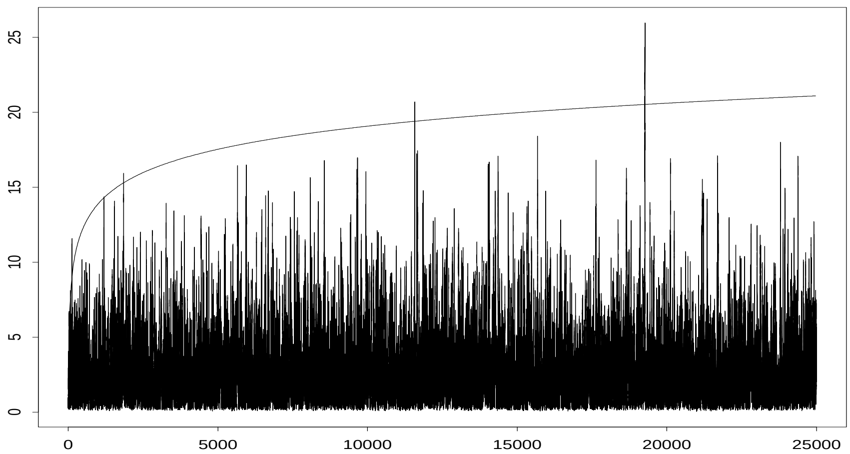

Figure 7: Simulated sample path of the GIG model (with parameters $\gamma = 0.5$, $\chi = \psi = \sigma = \lambda = 1$) and the corresponding norming constants b_t.

Then λ is arbitrary in $\mathbb{R}$. Theorem 3.8(b) applies giving

$$\overline{F}(x) \sim \frac{\sigma^2}{2}\left(\frac{\chi}{2}\right)^2 x^{2\gamma}\overline{H}(x)\,, \quad x \to \infty\,.$$

By (4.2) we obtain the norming constants

$$a_t = 2/\chi \quad \text{and} \quad b_t = \frac{2}{\chi}\left(\ln t + (2\gamma+\lambda-1)\ln\ln t + \ln\left(\frac{\sigma^2(\chi/2)^{2-2\gamma-\lambda}}{4K_\lambda(\sqrt{\chi\psi})}\left(\frac{\chi}{\psi}\right)^{\lambda/2}\right)\right).$$

The remark at the end of the case $\psi = 0$ applies.

Finally we investigate the assumptions in Theorem 3.4 for this case in detail. First notice that $s'(x) \to \infty$ for $x \downarrow 0$ or $x \uparrow \infty$. Thus by l'Hospital

$$\frac{s'(x)}{s(x)} \sim \frac{\chi}{2}, \quad x \uparrow \infty \quad \text{and} \quad \frac{s'(x)}{s(x)} \sim -\frac{\psi}{2}x^{-2}, \quad x \downarrow 0.$$

By Lemma 3.6 and the fact that $m'(x)s'(x) = 2x^{-2\gamma}/\sigma^2$,

$$\frac{m'_{ou}(s_{ou}^{-1}(s(x)))}{s'_{ou}(s_{ou}^{-1}(s(x)))}\frac{s'(x)}{m'(x)} \quad \sim \quad \frac{\sigma^2}{4}\frac{s'(x)}{s(x)}\frac{x^{2\gamma-1}}{\ln(|s(x)|)}\left(\frac{s''(x)}{s'(x)}x+\gamma\right), \quad x \uparrow \infty \text{ or } x \downarrow 0.$$

If we further distinguish between left and right endpoint we derive

$$\frac{m'_{ou}(s_{ou}^{-1}(s(x)))}{s'_{ou}(s_{ou}^{-1}(s(x)))}\frac{s'(x)}{m'(x)} \to \begin{cases} 0 & \gamma < 0.5 \\ \frac{\sigma^2}{4}\frac{\chi}{2} & \gamma = 0.5 \\ \infty & \gamma > 0.5 \end{cases}, \quad x \uparrow \infty,$$

and

$$\frac{m'_{ou}(s_{ou}^{-1}(s(x)))}{s'_{ou}(s_{ou}^{-1}(s(x)))}\frac{s'(x)}{m'(x)} \to \begin{cases} \infty & \gamma < 1.5 \\ \frac{\sigma^2}{4}\frac{\psi}{2} & \gamma = 1.5 \\ 0 & \gamma > 1.5 \end{cases}, \quad x \downarrow 0.$$

Hence by Remark 3.5(c), we may conclude that in the case $0.5 \le \gamma \le 1.5$ the assumptions of the Theorem 3.4 are fulfilled while in the other cases condition (3.21) of Lemma 3.6 does not hold.

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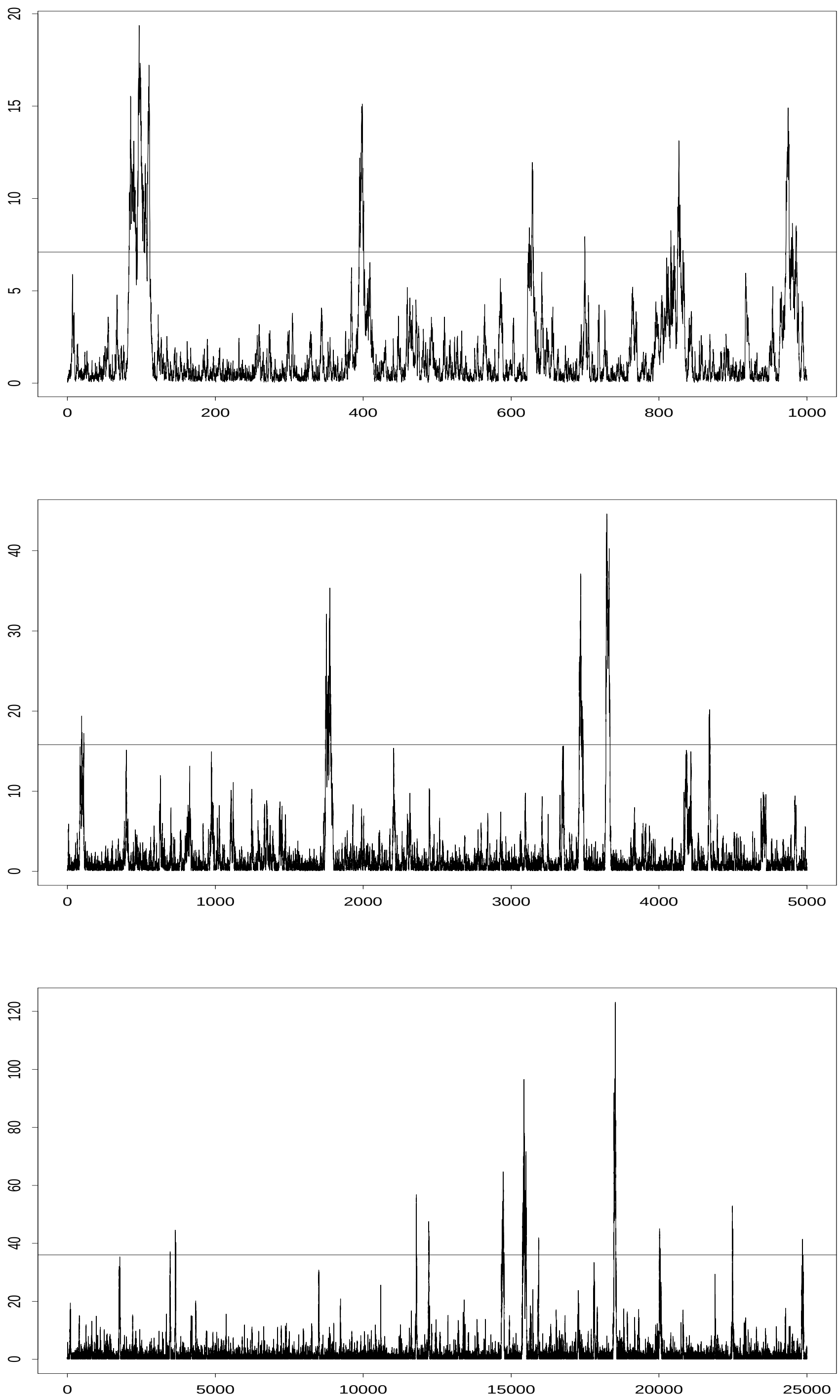

Figure 8: The Poisson approximation for ε-upcrossings of the GIG diffusion with parameters $\gamma = 0.5, \chi = 0, \psi = 1, \sigma = 1, \lambda = -1$ as in Figure 6. The threshold increases with the sample size. For the calculation of the thresholds we used $\tau = 10$, i.e. on average there are 10 ε–upcrossings for large t and fixed small $\varepsilon > 0$. The first figure shows a realisation of the process X_t for $0 \le t \le 1\,000$, the last two figures represent continuations of this realisation to $t = 5\,000$ and $t = 25\,000$, respectively.

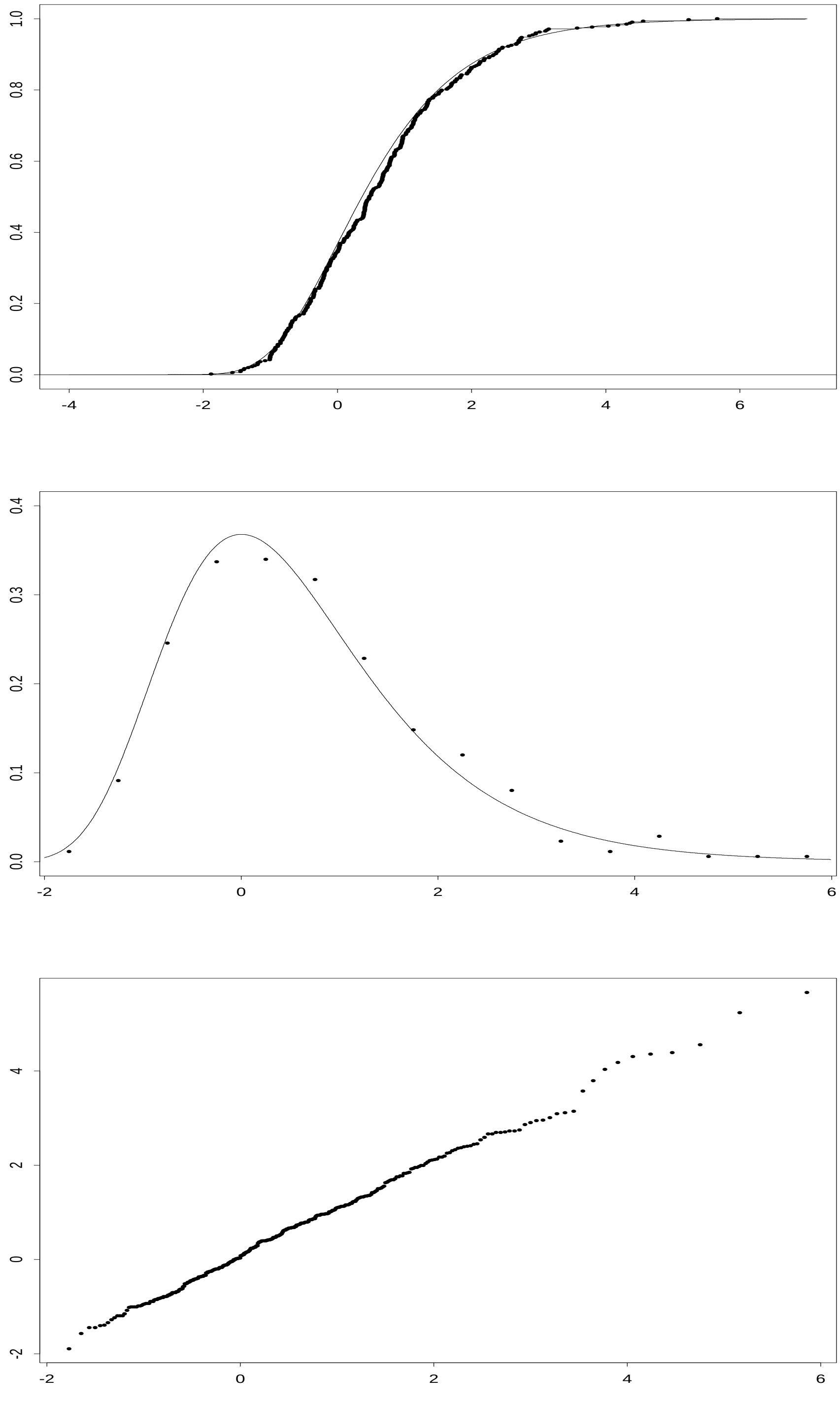

Figure 9: The empirical df (top), the empirical density (middle) and the QQ-plot (bottom) of the normalised maxima of the GIG model and the Gumbel df and density (solid line), based on 350 simulations with $t = 25\,000$ and parameters $\gamma = 0.5, \chi = \psi = \sigma = \lambda = 1$ as in Figure 7.