A universal constant for exponential Riesz sequences

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Abstract

The aim of this paper is to study certain correlations between lower and upper bounds of exponential Riesz sequences, in particular between the sharp lower and upper bounds, where we show that the product of the sharp bounds of an exponential Riesz sequence is bounded from above by a universal constant. The result is applied to the norms of coefficient and frame operators and their inverses.

Keywords: exponential Riesz sequence, exponential Riesz basis, exponential frame, sharp bounds
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1 Introduction

Let H be a separable Hilbert space over $\mathbb{C}$. A sequence $\Phi = (\varphi_n)_{n\in\mathbb{Z}}$ of elements in H is called a *Riesz-Fischer-sequence*, resp. a *Bessel sequence* for H, if there is $A > 0$, resp. $B > 0$, such that for all natural numbers n and all complex numbers $c_{-n}, \ldots, c_n$:

$$A \cdot \sum_{j=-n}^{n} |c_j|^2 \leq \left\| \sum_{j=-n}^{n} c_j \varphi_j \right\|_H^2, \tag{1}$$

$$\text{resp.} \quad \left\| \sum_{j=-n}^{n} c_j \varphi_j \right\|_H^2 \leq B \cdot \sum_{j=-n}^{n} |c_j|^2, \tag{2}$$

holds. If Φ is both a Bessel and a Riesz-Fischer-sequence, it is called a *Riesz sequence* for H. The constants A and B are called *lower* and *upper bounds*, respectively. It is an easy matter to check that the supremum of all lower bounds, resp. the infimum of all upper bounds, of a Riesz sequence is again a lower bound, resp. an upper bound, which we denote by $A_{opt}(\Phi)$, resp. $B_{opt}(\Phi)$. The

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constants $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ are called the *sharp* lower and the *sharp* upper bounds for Φ.

In this paper, we shall be concerned with *exponential* Riesz sequences for $L^2(-\sigma,\sigma)$, i.e. with Riesz sequences for $L^2(-\sigma,\sigma)$ of the form $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$, where $\sigma > 0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers. From (1) and (2) it follows readily that if $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma,\sigma)$, then the sequence $(\Im\lambda_n)_{n\in\mathbb{Z}}$ of imaginary parts must be uniformly bounded.

Young [9, Prop. 1, Cor. 1], [10, Ch. 4, Props. 3 + 2, Th. 3; Ch. 2, Rem. following Th. 17] has shown that for a sequence $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ of exponentials to be a Riesz sequence for $L^2(-\sigma,\sigma)$ it is sufficient that it be a Riesz-Fischer-sequence for $L^2(-\sigma,\sigma)$, provided $(\Im\lambda_n)_{n\in\mathbb{Z}}$ is uniformly bounded. We shall have a closer look at this result and the occuring bounds. In particular, if $\sigma,\tau > 0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers such that

$$\sup_{n\in\mathbb{Z}} |\Im\lambda_n| \le \tau, \tag{3}$$

and $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz-Fischer-sequence for $L^2(-\sigma,\sigma)$ with lower bound A, we shall construct an upper bound B for $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$, depending only on A, σ and τ (Proposition 1). From this we shall obtain a universal constant for the product of the sharp bounds of exponential Riesz sequences (Theorem 1).

The results of this paper are part of the author's doctoral thesis [5, Ch. 4].

2 Results

We need the following

Definition 1. A sequence $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numers is called *separated by* $\delta > 0$, if $\inf_{n\ne m} |\lambda_n - \lambda_m| \ge \delta$. The sequence is called *separated*, if there is some $\delta > 0$ such that $(\lambda_n)_{n\in\mathbb{Z}}$ is separated by δ.

Proposition 1. *Let $\tau \ge 0, \sigma > 0, A > 0$ and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers satisfying (3), such that $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz-Fischer-sequence for $L^2(-\sigma,\sigma)$ with lower bound A. Then holds:*

1. *$(\lambda_n)_{n\in\mathbb{Z}}$ is separated by*

$$\delta := \delta(A,\sigma,\tau) := \frac{1}{\sigma}\log\left(1 + e^{-\sigma\tau}\sqrt{A/\sigma}\right).$$

2. *$(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma,\sigma)$ with upper bound B, where*

$$B := B(A,\sigma,\tau) := \frac{2}{\sigma}\cdot(e^{2\sigma(\tau+1)} - 1)\cdot\left(1 + \frac{2}{\delta}\right)^2 =$$

$$\frac{2}{\sigma} \cdot (e^{2\sigma(\tau+1)} - 1) \cdot \left(1 + \frac{2\sigma}{\log\left(1 + e^{-\sigma\tau}\sqrt{A/\sigma}\right)}\right)^2. \tag{4}$$

Remark 1. Proposition 1 states more explicitly a result of Young [9, Prop. 1, Cor. 1], [10, Ch. 4, Props. 3 + 2, Th. 3; Ch. 2, Rem. following Th. 17], who proved that, under the assumptions of Proposition 1, the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ must be separated and that $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence.

From Proposition 1 we shall obtain:

Theorem 1. *For every $\sigma > 0$ and every $\tau \geq 0$, there exists a positive constant $C(\sigma, \tau)$, such that the following holds:*
If $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers, satisfying (3), such that $\Phi := (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(-\sigma, \sigma)$, then the product of the sharp bounds $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ is bounded from above by $C(\sigma, \tau)$:

$$A_{opt}(\Phi) \cdot B_{opt}(\Phi) \leq C(\sigma, \tau). \tag{5}$$

The constant $C(\sigma, \tau)$ can be choosen as

$$C(\sigma, \tau) := 256 \cdot e^{4\sigma\tau + 2\sigma} \cdot (\sigma + 1/8)^2. \tag{6}$$

Remark 2. There is no universal constant $D(\sigma, \tau) > 0$, such that $A_{opt}(\Phi) \cdot B_{opt}(\Phi) \geq D(\sigma, \tau)$ for all exponential Riesz sequences $\Phi = (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ for $L^2(-\sigma, \sigma)$ satisfying (3).
Counterexample: For $0 < \varepsilon < 1$, we define

$$\lambda_n^\varepsilon := \begin{cases} n & \text{for } n \in \mathbb{Z}\backslash\{0\} \\ 1 - \varepsilon & \text{for } n = 0 \end{cases}$$

and $\Phi^\varepsilon := (e^{i\lambda_n^\varepsilon \bullet})_{n\in\mathbb{Z}}$. From the orthonormality of $(\frac{1}{\sqrt{2\pi}} e^{in\bullet})_{n\in\mathbb{Z}}$ in $L^2(-\pi, \pi)$ it follows that, for $0 < \varepsilon < 1$, Φ^ε is a Riesz sequence for $L^2(-\pi, \pi)$, and that $B_{opt}(\Phi^\varepsilon) \leq 4\pi$. However, from $\|e^{i\lambda_0^\varepsilon \bullet} - e^{i\lambda_1^\varepsilon \bullet}\|^2_{L^2(-\pi,\pi)} \to 0$, for $\varepsilon \to 0$, we conclude $A_{opt}(\Phi^\varepsilon) \to 0$, for $\varepsilon \to 0$. This shows $A_{opt}(\Phi^\varepsilon) \cdot B_{opt}(\Phi^\varepsilon) \to 0$, for $\varepsilon \to 0$.

Definition 2. A sequence $\Phi = (\varphi_n)_{n\in\mathbb{Z}}$ in a separable Hilbert space H over $\mathbb{C}$ is called a *frame* for H (cf. Duffin–Schaeffer [3, Sect. 3]), if there exist positive constants A, B, such that for all $f \in H$:

$$A \cdot \|f\|_H^2 \leq \sum_{n\in\mathbb{Z}} |(f, \varphi_n)_H|^2 \leq B \cdot \|f\|_H^2.$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. The operators

$$\begin{aligned} T_\Phi \quad &: \quad H \to l^2(\mathbb{Z}), \;\; f \mapsto ((f, \varphi_n)_H)_{n\in\mathbb{Z}} \;\; \text{and} \\ S_\Phi \quad &: \quad H \to H, \;\; f \mapsto \sum_{n\in\mathbb{Z}} (f, \varphi_n)_H \varphi_n \end{aligned}$$

are called the *coefficient operator*, resp. the *frame operator*, corresponding to the frame Φ.

Remark 3. From the definition it follows easily that the coefficient operator, corresponding to a frame, is an injective, bounded linear operator, with bounded inverse on ist range. Furthermore, it can be shown that S_Φ is a well-defined, bijective, bounded linear map (cf. Duffin–Schaeffer [3, Sect. 3]; the sum converges in the norm of H).

Definition 3. A frame in a separable Hilbert space over $\mathbb{C}$ is called an *exact* frame (or a *Riesz basis*), if it is no longer a frame after any of its elements are removed.

Remark 4. Every exact frame is a Riesz sequence with the same bounds (cf. Duffin–Schaeffer [3, Lemma X] and Kölzow [4, Sect. II.1, Cor. 1 to Th. 8]).

As a consequence of Theorem 1, we have

Corollary 1. *For the constant $C(\sigma,\tau)$ of Theorem 1 holds:*
If $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers, satisfying (3), such that $\Phi := (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is an exact frame for $L^2(-\sigma,\sigma)$, then we have the following inequalities for the norms of the coefficient and frame operator and their inverses:

$$\begin{aligned}
\|T_\Phi\| &\leq \sqrt{C(\sigma,\tau)} \cdot \|T_\Phi^{-1}\|, \\
\|S_\Phi\| &\leq C(\sigma,\tau) \cdot \|S_\Phi^{-1}\|, \\
\|T_\Phi\|^2 &\leq C(\sigma,\tau) \cdot \|S_\Phi^{-1}\|.
\end{aligned}$$

Remark 5. An analogue statement to Theorem 2 does not hold for arbitrary (non exact) exponential frames.
Counterexample: We define the sequence $\Phi_m := (e^{i\frac{n}{m}\bullet})_{n\in\mathbb{Z}}$, where $m \in \mathbb{N}$. It can be shown that Φ_m is a frame for $L^2(-\pi,\pi)$, but

$$\|T_{\Phi_m}\|^2 = \|T_{\Phi_m}^{-1}\|^{-2} = \|S_{\Phi_m}\| = \|S_{\Phi_m}^{-1}\|^{-1} = 2\pi m.$$

Definition 4. Denote by PW_σ^2 the *Paley–Wiener–space*, consisting of all entire functions of exponential type at most σ, whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. The norm on PW_σ^2 is the usual $L^2(\mathbb{R})$-norm.
For a sequence $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, the operator I_Λ is defined by

$$I_\Lambda : PW_\sigma^2 \to \mathbb{C}^{\mathbb{Z}}, \;\; F \mapsto (F(\lambda_n))_{n\in\mathbb{Z}}.$$

Remark 6. If, for $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$, $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is an exact frame for $L^2(-\sigma,\sigma)$, then I_Λ defines a bijective, bounded linear operator from PW_σ^2 onto $l^2(\mathbb{Z})$ (i.e. Λ is a *complete interpolating sequence*, cf. Young [10, Ch. 4, Th. 9]).

From Corollary 1 we obtain

Corollary 2. *For the constant $C(\sigma,\tau)$ of Theorem 1 holds:*
If $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers, satisfying (3), such that $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is an exact frame for $L^2(-\sigma,\sigma)$, then

$$\|I_\Lambda\| \le \frac{\sqrt{C(\sigma,\tau)}}{2\pi}\,\|I_\Lambda^{-1}\|.$$

Consequently, $\|I_\Lambda^{-1}\|$ is large if $\|I_\Lambda\|$ is large.

3 Proofs

We shall need the following

Lemma 1 ([6, Lemma 1]). *Let $\delta,\sigma > 0$, $\tau \ge 0$, and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers, separated by δ and satisfying (3). Then, for all functions F of the Paley-Wiener-space PW^2_σ, the following inequality holds:*

$$\sum_{n\in\mathbb{Z}} |F(\lambda_n)|^2 \le \frac{e^{2\sigma(\tau+1)}-1}{\pi\sigma}\cdot\left(1+\frac{2}{\delta}\right)^2 \int_{\mathbb{R}} |F(x)|^2 dx. \tag{7}$$

It should be noted that, for separated real sequences, the first inequality of type (7) (with a different constant) was given by Plancherel and Pólya [8, p. 126].

Proof of Proposition 1.
1. (For the following proof that $(\lambda_n)_{n\in\mathbb{Z}}$ is separated we use use similar arguments as Pavlov did in [7, Th. 1]). Let $k,n\in\mathbb{Z}$, where $k\ne n$. Then we have from (1):

$$2A = A\cdot(|1|^2+|-1|^2) \le \int_{-\sigma}^{\sigma} |e^{i\lambda_k x} - e^{i\lambda_n x}|^2\,dx =$$

$$= \int_{-\sigma}^{\sigma} |e^{i\lambda_k x}|^2 \cdot |1-e^{i(\lambda_n-\lambda_k)x}|^2\,dx \le e^{2\sigma\tau}\cdot \int_{-\sigma}^{\sigma} |1-e^{i(\lambda_n-\lambda_k)x}|^2\,dx. \tag{8}$$

For $x\in(-\sigma,\sigma)$, we have

$$|1-e^{i(\lambda_n-\lambda_k)x}| = \left|\sum_{k=1}^{\infty}\frac{(i(\lambda_n-\lambda_k)x)^k}{k!}\right| \le \sum_{k=1}^{\infty}\frac{(|\lambda_n-\lambda_k|\cdot|\sigma|)^k}{k!} = e^{|\lambda_n-\lambda_k|\sigma}-1.$$

Hence we conclude from (8):

$$2A \le e^{2\sigma\tau}\cdot 2\sigma(e^{|\lambda_n-\lambda_k|\sigma}-1)^2.$$

From this it follows easily that

$$|\lambda_n-\lambda_k| \ge \frac{1}{\sigma}\log\left(1+e^{-\sigma\tau}\sqrt{A/\sigma}\right) = \delta(A,\sigma,\tau),$$

i.e. $(\lambda_n)_{n\in\mathbb{Z}}$ is separated by $\delta = \delta(A, \sigma, \tau)$.

2. From Lemma 1 and assertion 1 we derive (7) with $\delta = \delta(A, \sigma, \tau)$. By the Paley-Wiener-Theorem, this is equivalent to

$$\sum_{n\in\mathbb{Z}} |(f, e^{i\lambda_n \bullet})|^2 \leq \frac{2}{\sigma}(e^{2\sigma(\tau+1)} - 1) \cdot \left(1 + \frac{2}{\delta}\right)^2 \cdot \|f\|^2,$$

for all $f \in L^2(-\sigma, \sigma)$. By a theorem of Boas [2, Th. 1] (cf. Young [10, Ch. 4, Th. 3]), the latter inequality is equivalent to $(e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ being a Bessel sequence with upper bound $B = B(A, \sigma, \tau)$, as defined by (4). □

Proof of Theorem 1. Let $\sigma > 0$, $\tau \geq 0$, and $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers, satisfying (3), such that $\Phi := (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ is a Riesz sequence with sharp bounds $A := A_{opt}(\Phi)$ and $B_{opt}(\Phi)$. From Proposition 1, we conclude

$$B_{opt}(\Phi) \leq \frac{2(e^{2\sigma(\tau+1)} - 1)}{\sigma} \cdot (1 + \frac{2\sigma}{\log(1 + e^{-\sigma\tau}\sqrt{A/\sigma})})^2. \tag{9}$$

From

$$A \leq \|e^{i\lambda_n \bullet}\|^2_{L^2(-\sigma,\sigma)} \leq 2\sigma \cdot e^{2\sigma\tau} \quad \forall\, n \in \mathbb{Z}$$

we derive

$$\frac{1}{\sqrt{8}} \cdot e^{-\sigma\tau}\sqrt{\frac{A}{\sigma}} \leq \frac{1}{2}. \tag{10}$$

Using $\log(1 + x) \geq x/2$ for $x \in [0, 1/2[$, we thus obtain

$$\log(1 + e^{-\sigma\tau}\sqrt{A/\sigma}) \geq \log(1 + \frac{1}{\sqrt{8}}\, e^{-\sigma\tau}\sqrt{A/\sigma}) \geq \frac{1}{2} \cdot \frac{1}{\sqrt{8}} \cdot e^{-\sigma\tau}\sqrt{A/\sigma}.$$

Using (9) and (10), we conclude

$$B_{opt}(\Phi) \leq \frac{2}{\sigma} \cdot (e^{2\sigma(\tau+1)} - 1) \cdot (1 + 4\sqrt{8}\,\sigma\, e^{\sigma\tau}\sqrt{\sigma/A})^2 \leq$$

$$\leq \frac{2}{\sigma}\cdot(e^{2\sigma(\tau+1)} - 1)\cdot\left(\frac{1}{8} + \sigma\right)^2\cdot\left(4\sqrt{8}\, e^{\sigma\tau}\sqrt{\sigma/A}\right)^2 \leq 256\cdot e^{4\sigma\tau+2\sigma}\cdot\left(\frac{1}{8} + \sigma\right)^2\cdot\frac{1}{A}.$$

This shows $A_{opt}(\Phi) \cdot B_{opt}(\Phi) \leq C(\sigma, \tau)$, for $C(\sigma, \tau)$ defined by (6). □

Proof of Corollary 1. Let $\Phi := (e^{i\lambda_n \bullet})_{n\in\mathbb{Z}}$ be an exact frame for $L^2(-\sigma, \sigma)$. Since the frame bounds of the exact frame Φ coincide with the bounds of Φ as a Riesz sequence (Remark 4), we conclude that we have, for all $f \in L^2(-\sigma, \sigma)$,

$$A_{opt}(\Phi) \cdot \|f\|^2 \leq \sum_{n\in\mathbb{Z}} |(f, e^{i\lambda_n \bullet})|^2 \leq B_{opt}(\Phi) \cdot \|f\|^2, \tag{11}$$

and that $A_{opt}(\Phi)$ and $B_{opt}(\Phi)$ are the best possible constants in this inequality. From this we derive

$$\|T_\Phi\| = \sqrt{B_{opt}(\Phi)}, \quad \|T_\Phi^{-1}\| = \frac{1}{\sqrt{A_{opt}(\Phi)}}. \tag{12}$$

The norms of S_Φ and S_Φ^{-1} can also be expressed by the best constants occuring in (11). It holds

$$\|S_\Phi\| = B_{opt}(\Phi), \quad \|S_\Phi^{-1}\| = \frac{1}{A_{opt}(\Phi)} \tag{13}$$

(cf. Benedetto and Walnut [1, Th. 3.2 a]). Thus, the theorem follows from inequalities (5), (12) and (13). □

Proof of Corollary 2. This follows from Corollary 1 and the Paley–Wiener–theorem (cf. Young [10, Ch. 4, Th. 18, p. 100]). □

4 Open questions

1. For what classes of exact frames (not of exponential type) do analogues to Theorems 1 and 2 exist? It can be thought, e.g., of the class of all exact Gabor-frames with the discretisation parameters t_0, ω_0, such that $t_0 \cdot \omega_0 = 1$.
2. What is the best constant $C_{opt}(\sigma, \tau)$, fulfilling inequality (5)?
   Conjecture: $C_{opt}(\pi, 0) = 4\pi^2$.

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