Combined Market and Credit Risk Stress Testing based on the Merton Model*

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Abstract

On the basis of the Merton model for the value of a firm's debt we discuss the problem of stress testing the effects of credit, market and foreign exchange risk on the portfolio of a financial institution. There is no straightforward solution to this problem since the Merton model links only equity and firm value for a single firm, but includes neither interest rate volatility nor foreign exchange rates.

1 Introduction

Since 1996, when the Bank for International Settlements (BIS) allowed banks to use internal market risk models to control their risk, such models have found widespread use. They are generally considered to be quite reliable. This is not the case, however, for credit risk models which the BIS is still reluctant to endorse (Basel Committee on Banking Supervision, 2000). Nevertheless, there is a huge amount of literature on credit risk models, and also a number of quite successful vendors of software for credit portfolio management – in contrast to the situation for literature combining the quantitative aspects of modeling market and credit risks. This is surprising in view of the fact that there is a wide consensus in the economic literature that market and credit risk should not be regarded separately. For our purpose, papers by Iscoe et al. (1999) and Jarrow and Turnbull (2000) seem mainly relevant.

Iscoe et al. suggest a joint market and credit risk model on the basis of the *Merton model* for the evaluation of risky debt (Merton, 1974). This model and its modifications are sometimes called the *structural approach*. The Merton approach has been criticized for several reasons (see section 2.2). Mainly, its assumption of constant risk-free interest rates might seem unrealistic and was

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one of the reasons for the development of the the so-called *reduced form approach* which identifies the credit spread in a certain way with a *default intensity*. The potential of this approach for integrating market and credit risk in one model has been discussed by Jarrow and Turnbull (2000). In this paper, we will rely on the Merton model, both because of its mathematical manageability and its suasive economic interpretation.

The paper is organized as follows: In section 2 we recall the Merton model and some of the criticisms it has attracted. In section 3 we discuss some special estimation problems which appear when the model is applied in the form of KMV-software (`http://www.kmv.com`). Moreover, in this section we briefly discuss how the Merton model can be used to stress-test equity, interest rate and volatility shocks. Section 4 is devoted to the integration of foreign exchange risk into the Merton model. In section 5 we recall, in the spirit of Iscoe et al. (1999), how to extend the Merton model to several firms. And, finally, in section 6, we discuss the impact of different dependence structures in the portfolio on its riskiness.

2 Equity as a European call option on the value of the firm (Merton, 1974)

2.1 The mathematics behind the Merton Model

We consider an economy over the time interval $[0, T]$ with uncertainty represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The flow of information is modeled by a filtration $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F} | t \in [0, T]\}$ which satisfies the usual conditions of being complete, increasing and right continuous (cf. Musiela and Rutkowski, 1997, section 10.1). Corporate debt is here assumed to be a single zero coupon bond with face value $B > 0$ and maturity T. Then corporate equity may be interpreted as a European call option written on the value of the firm with strike B. Formally we adopt the following notations:

- S_t = value of the firm's equity at time $t \in [0, T]$,
- F_t = value of the bond at time $t \in [0, T]$,
- V_t = value of the firm at time $t \in [0, T]$.

We observe that at every epoch $t \in [0, T]$ we have

$$V_t = S_t + F_t . \tag{2.1}$$

At time T the value of the firm's equity is

$$S_T = \max(V_T - B, 0). \tag{2.2}$$

At maturity the bond-holders receive T

$$F_T \;=\; \min(V_T, B) \;=\; B - \max(B - V_T, 0). \tag{2.3a}$$

By definition, the firm *defaults* if it does not completely meet its obligations at time T, i.e., if $V_T < B$. In case of default the bond-holders take over the firm. Its value at this event is called *recovery.* Sometimes, it is convenient to take a rather pessimistic point of view and to assume that there is *no recovery.* This assumption corresponds to

$$F_T \;=\; B\,\mathbf{1}_{\{V_T \geq B\}}. \tag{2.3b}$$

In order to derive evaluation formulas for equity and debt we assume the framework of the well-known Black-Scholes model for option pricing. This means:

- The statistical firm value process is modeled as a geometric Brownian motion with respect to the probability measure $\mathbb{P}$ by

  $$dV_t \;=\; \mu_V V_t dt + \sigma_V V_t dW_t, \tag{2.4}$$

  where $W_t, t \in [0, T]$, is a standard Brownian motion, μ_V is the *instantaneous expected rate of return* on the firm per time unit, and σ_V^2 is the *instantaneous variance* of the firm value per time unit. σ_V is called *volatility* of V_t. The value V_0 is constant and fixed.

- The price of a risk-less zero coupon bond which pays one currency unit for sure at time t in the future is $P(t) = e^{-rt}$ where $r > 0$ is the *instantaneous risk-less rate of interest*, the same for all the time.

In particular, this market model is arbitrage-free and complete, or mathematically speaking, there is a unique probability measure $\mathbb{Q}$ which is equivalent (i.e., has the same sets of measure 0) to $\mathbb{P}$, such that the discounted firm value process $V_t\, e^{-rt}$ is a martingale under $\mathbb{Q}$. In this paper $\mathbb{Q}$ stands for the *risk neutral* probability measure, and $\mathbb{P}$ for the *physical* probability measure. Under $\mathbb{Q}$, the firm value process V_t can be described as a geometric Brownian motion with initial value V_0, a drift equal to the constant interest rate of r and a volatility of σ_V:

$$dV_t \;=\; rV_t dt + \sigma_V V_t dW_t^*. \tag{2.5}$$

Here, $(W_t^*)_{t\geq 0}$ with $W_t^* = W_t - \frac{r-\mu_V}{\sigma_V} t$ by the Girsanov theorem is a standard Brownian motion with respect to the probability measure $\mathbb{Q}$. By Itô's formula we obtain from (2.5) that

$$\begin{aligned} V_t &= V_0 e^{(r-\frac{1}{2}\sigma_V^2)t + \sigma_V W_t^*} \\ \mathbb{E}_{\mathbb{Q}}[V_T] &= V_0 e^{rT} \\ \mathrm{var}_{\mathbb{Q}}[V_T] &= V_0^2 e^{2rT}(e^{\sigma_V^2 T} - 1). \end{aligned} \tag{2.6}$$

The equity value S_t at time t is given by the Black-Scholes value of a European call option (Musiela and Rutkowski, 1997, Theorem 5.1.1)

$$\begin{aligned} S_t &= V_t N(d_1(V_t, T-t)) - Be^{-rT} N(d_2(V_t, T-t)) \qquad (2.7) \\ d_1(v,t) &= \frac{\log \frac{v}{B} + (r + \frac{\sigma_V^2}{2})t}{\sigma_V \sqrt{t}}, \quad d_2(v,t) = d_1(v,t) - \sigma_V \sqrt{t}, \end{aligned}$$

where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz$ denotes the distribution function of the standard normal distribution.

From (2.4) and (2.5) formulas follow for the unconditional probability of default at time 0 under risk neutral and physical probability measures, respectively. Under the risk neutral probability measure $\mathbb{Q}$ we get

$$\begin{aligned} \mathbb{Q}(V_T < B) &= \mathbb{Q}\Big[\frac{W_T^*}{\sqrt{T}} < \frac{\log \frac{B}{V_0} + (\frac{\sigma_V^2}{2} - r)T}{\sigma_V \sqrt{T}}\Big] \\ &= 1 - N(d_2(V_0, T))\,. \qquad (2.8a) \end{aligned}$$

Under the physical probability measure $\mathbb{P}$ we have

$$\begin{aligned} \mathbb{P}(V_T < B) &= \mathbb{P}\Big[\frac{W_T}{\sqrt{T}} < \frac{\log \frac{B}{V_0} + (\frac{\sigma_V^2}{2} - \mu_V)T}{\sigma_V \sqrt{T}}\Big] \\ &= 1 - N[d_2(V_0, T) + \frac{\mu_V - r}{\sigma_V}\sqrt{T}]\,. \qquad (2.8b) \end{aligned}$$

The quantity $\frac{\mu_V - r}{\sigma_V}$ is called *Market Price of Risk* (MPR) as it can be considered the instantaneous premium per unit of risk to be paid to risk-averse investors for holding the firm's assets.

In the case of no recovery the value of the bond can be considered as having the value of a digital option. Hence, by the no-arbitrage pricing principle from (2.3b) we obtain

$$\begin{aligned} F_t &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}\, B\, \mathbf{1}_{\{V_t \geq B\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)}\, B\, \mathbb{Q}[V_T \geq B | \mathcal{F}_t] \\ &= e^{-r(T-t)}\, B\, N(d_2(V_t, T-t)). \qquad (2.9) \end{aligned}$$

In this case the *yield spread* (between risky and risk-less bonds) value Y_t, defined by the relation $B\, e^{-(r+Y_t)\,(T-t)} = F_t$ is

$$\begin{aligned} Y_t &= \frac{-\log \frac{F_t}{B}}{T-t} - r \qquad (2.10a) \\ &= \frac{-\log \mathbb{Q}[V_T \geq B | \mathcal{F}_t]}{T-t} \qquad (2.10b) \\ &= \frac{-\log N(d_2(V_t, T-t))}{T-t}\,. \qquad (2.10c) \end{aligned}$$

In the case where positive recovery is possible we obtain from (2.3a) for the value of the debt

$$\begin{aligned} F_t &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} \min(B, V_T) \,|\, \mathcal{F}_t] \\ &= e^{-r(T-t)} \, B \, \mathbb{Q}[V_T \geq B \,|\, \mathcal{F}_t] + e^{-r(T-t)} \, \mathbb{E}_{\mathbb{Q}}[V_T \, \mathbf{1}_{\{V_T < B\}} \,|\, \mathcal{F}_t] \\ &= e^{-r(T-t)} \, B \, N(d_2(V_t, T-t)) + V_t N(-d_1(V_t, T-t)). \end{aligned} \tag{2.11}$$

In this case the yield spread value Y_t is again given by (2.10a). Observe that in case of possible recovery the value of Y_t is greater than in the case of no recovery.

In the sequel we consider only the case of no recovery because the recovery distribution deduced from the Merton model is not a realistic distribution for practitioners. With a view towards application in risk management it makes better sense to proceed on the pessimistic assumption of no recovery at all.

2.2 Problems with the Merton approach in practice

(i) Direct observation of the firm value process is impossible. Even if the firm's stocks are traded on an exchange, one can only observe the equity value process.

(ii) The same applies to its volatility.

(iii) The Merton model for bond prices has the same theoretical deficiencies as the Black-Scholes model for the price of a European call-option. In particular, it is unrealistic to assume that volatility and risk-less interest rates remain constant over time.

(iv) Other deficiencies are specific to the Merton model. One concerns the interest spreads on a risk-less interest rate. If spreads followed the dynamics given by the model, they would tend to zero as the maturity of the bond approaches. In practice, experience shows that this is not the case. Spreads are usually bounded away from zero over all the time to maturity of the bond. This observation has stimulated the development of many other default models for corporate bonds (cf. Lando, 1997).

(v) Another problem with the Merton model is the phenomenon that, in its view, the value of the firm's debt as given by (2.9) or (2.11) should decrease whenever the risk-less interest rate increases. This is contrary to daily experience.

Despite the deficiencies enumerated above, the Merton model is popular because of its relative simplicity. We will see that points (i), (ii) and (v) can at least be repaired to some extent.

3 Parameter estimation and simple shock scenarios

3.1 Implicit firm value and firm value volatility

With the KMV (`http://www.kmv.com`) software *Credit Monitor*®, values are delivered for the face value B of debt of a firm (weighted average of all the firm's obligations), the physical default probability EDF ("Expected Default Frequency"), where $\mathrm{EDF} = \mathbb{P}[V_T < B]$, and the actual equity value of the firm under consideration is S_0. The Merton model says that these quantities are connected through equations (2.7) for $t = 0$ and (2.8b).

Values for the value V_0 of the firm at time 0 and the volatility σ_V of the firm value are also available in this software. Nevertheless, in order to get values which are better consistent with the Merton model one can try to determine implicit values for V_0 and σ_V. The problem here is the simultaneous solution of equations (2.7), (2.8b) for V_0 and σ_V. Observe that – assuming that the risk-less interest rate r is known – by (2.7), (2.8b) we have two equations for the unknown values V_0, σ_V and μ_V.

One way to find a unique solution in this situation is to fix the Market Price of Risk $\mathrm{MPR} = \frac{\mu_V - r}{\sigma_V}$ – which seems reasonable by the CAPM – and to solve the resulting equation system for two variables. This can be done effectively using the fact that (2.8b) can be analytically solved for σ_V, i.e.,

$$\sigma_V = \mathrm{MPR} - \frac{N^{-1}(1-\mathrm{EDF})}{\sqrt{T}} + \sqrt{\Big(\mathrm{MPR} - \frac{N^{-1}(1-\mathrm{EDF})}{\sqrt{T}}\Big)^2 + 2\Big(r + \frac{\log\frac{V_0}{B}}{T}\Big)}. \tag{3.1}$$

If we insert this expression for σ_V into (2.7), equation (2.7) can be numerically solved for V_0. Now, inserting this value into equation (3.1) yields the exact solution of equation system (2.7), (2.8b).

On the whole, this methodology for determining the volatility σ_V of the firm value is not completely satisfactory, as the same value for MPR is used for all firms. We suggest improving the methodology by relying on supplementary market data, namely yield spreads of bonds with ratings equal to those of the firms under consideration. If we denote the actual yield spread by Y_0, we have to simultaneously solve the two equations (2.7) and (2.10a). Again, we note that (2.10a) can be solved analytically for σ_V in dependence of V_0, i.e.,

$$\sigma_V = -\frac{N^{-1}(e^{-YT})}{\sqrt{T}} + \sqrt{\frac{N^{-1}(e^{-YT})^2}{T} + 2\Big(r + \frac{\log\frac{V_0}{B}}{T}\Big)}. \tag{3.2}$$

Proceeding as with equation (3.1) we arrive at an exact solution pair (V_0, σ_V) of equation system (2.7), (2.10a).

3.2 A simple approach to shock scenarios

As we have just seen in Section 3.1, a crucial point for fixing a model for the value of a firm's debt is to determine an implicit firm value V_0. This suggests the following approach for combining equity, interest rate, and volatility shocks. We use equation (2.7) as a basis and calculate numerically a changed value V_0 when the values for S_0, r, or σ_V have been changed. The way to shock the firm value volatility is quite obvious. Simply define

$$\sigma_V^{\text{new}} \;=\; \sigma_V^{\text{old}}\,(1+\Delta\sigma_V). \tag{3.3}$$

Concerning equity and interest rate shock, we suggest leaving the value of r unchanged and to apply changes only to equity by setting

$$S_0^{\text{new}} \quad=\quad S_0^{\text{old}}\,(1+\Delta S)\,e^{-\Delta r\,T}\,. \tag{3.4}$$

We understand the shock value in (3.4) in the way that, ceteris paribus, the interest rates rise if the equities rise, and vice versa. At the balance sheet level of a single firm we can say that its equity grows if the interest rate rises because the debt of the firm will increase. That is why we multiply by the factor $e^{-\Delta r\,T}$.

4 Integrating foreign exchange risk into the credit default model for a single firm

4.1 Modeling by decomposing the firm value process

In order to model the influence of foreign exchange rates on the value of a firm, in this section we assume that the assets of the firm are spread over several countries with different currencies. Nevertheless, the firm has a home country – say, the country where its headquarters are or the country whose *(domestic currency)* it uses for the balance sheet. Let V_t denote the value of the firm at time t as in section 2.1. Then our assumption means

$$V_t \;=\; V_t^{(d)} + \sum_{i=1}^{n} X_t^{(f,i)}\,V_t^{(f,i)}\,. \tag{4.1}$$

Here $V_t^{(d)}$ denotes the value of the *domestic* part of the firm's assets, i.e., the value of that part of the assets which is evaluated in the currency of the home country.
$V_t^{(f,i)}$, $i=1,\dots,n$, is the value of the firm's assets which are quoted in currency i.
$X_t^{(f,i)}$, $i=1,\dots,n$, is the exchange rate of currency i, i.e., at time t one unit of currency i is traded for $X_t^{(f,i)}$ units of domestic currency. Thus

$$\widetilde{V}_t^{(f,i)} \;=\; X_t^{(f,i)}\,V_t^{(f,i)}\,, \qquad i=1,\dots,n, \tag{4.2}$$

is the value of the firm's assets converted from currency i to domestic currency.

We assume here, as in section 2.1, that all the firm's obligations are summarized in a single zero bond in domestic currency, with face value B and maturity T. Of course (2.1) is still valid in the present extended context. Hence, in order to derive an evaluation formula for the value F_t at time t of the firm's debt in domestic currency, we again face the problem of evaluating the European call option (2.2) where V_t is now given by (4.1).

Again, we are going to work within the classic Black–Scholes framework. Hence, we rely on the following assumptions (cf. Musiela and Rutkowski, 1997, sec. 7.1):

- The domestic risk-free interest rate r_d and the foreign risk-free interest rates $r_{f,i}$, $i = 1, \ldots, n$, are positive constants.
- The domestic and foreign parts of the firm value, and the exchange rates are modeled by correlated geometric Brownian motions.
- In order to avoid perfect correlation between the firm value processes and the exchange rate processes, the underlying noise process will be modeled by means of a multidimensional Brownian motion.

As a consequence of these assumptions we specify the dynamics of the model by fixing initial values

$$V_0^{(d)},\ V_0^{(f,i)},\ X_0^{(f,i)},\ i = 1, \ldots, n,$$

and by the following stochastic differential equation system

$$\begin{aligned} d\,V_t^{(d)} &= V_t^{(d)}\,(\mu_d\,dt + a_0 \cdot d\,W_t) \\ d\,V_t^{(f,i)} &= V_t^{(f,i)}\,(\mu_{f,i}\,dt + a_i \cdot d\,W_t)\,, \quad i = 1, \ldots, n \\ d\,X_t^{(f,i)} &= X_t^{(f,i)}\,(\nu_{f,i}\,dt + a_{n+i} \cdot d\,W_t)\,, \quad i = 1, \ldots, n. \end{aligned} \tag{4.3}$$

As in (2.4) the μ_d, $\mu_{f,i}, \nu_{f,i}$, $n = 1, \ldots, n$, are the *instantaneous expected rates of return.*

$$W_t \;=\; (W_t^{(0)}, W_t^{(1)}, \ldots, W_t^{(2n)})'$$

is a (column-)vector of independent standard Brownian Motions under the *physical* probability $\mathbb{P}$.

$$a_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,2n}), \qquad i = 0, \ldots, 2n,$$

are the rows of a suitable *dispersion matrix* A used to describe the dependencies between the asset value processes and the exchange rate processes.

Since the face value of the firm's debt is denoted in domestic currency, we need a market model in domestic currency. Note that one unit of currency i invested at time 0 in the assets of the firm in country i will be worth $\widetilde{V}_t^{(f,i)}$ at time t. One unit of currency i invested at time 0 in a risk-free asset in country i will be worth

$$\widetilde{X}_t^{(f,i)} \;=\; e^{r_{f,i}\,t}\,X_t^{(f,i)} \tag{4.4}$$

at time t. Applying Itô's formula to the products $X_t^{(f,i)}\, V_t^{(f,i)}$ and $e^{r_{f,i}\, t}\, X_t^{(f,i)}$ we now derive from (4.3) the following modified stochastic differential system for the market model in domestic currency:

$$\begin{aligned} d\, V_t^{(d)} &= V_t^{(d)}\, (\mu_d\, dt + a_0 \cdot d\, W_t) \\ d\, \widetilde{V}_t^{(f,i)} &= \widetilde{V}_t^{(f,i)} \Big((\mu_{f,i} + \nu_{f,i} + a_i \cdot a_{n+i})\, dt + (a_i + a_{n+i}) \cdot d\, W_t \Big)\,, \quad i = 1, \ldots, n \\ d\, \widetilde{X}_t^{(f,i)} &= \widetilde{X}_t^{(f,i)} \Big((\nu_{f,i} + r_{f,i})\, dt + a_{n+i} \cdot d\, W_t \Big)\,, \quad i = 1, \ldots, n. \end{aligned} \tag{4.5}$$

Recall that the market described by (4.5) is arbitrage-free and is complete if, and only if, the underlying model admits a unique probability $\mathbb{Q}$ which is equivalent to $\mathbb{P}$, and such that all asset value and exchange rate processes discounted with the domestic risk-free rate are martingales under $\mathbb{Q}$. By Girsanov's Theorem we see from (4.5) that this is the case if, and only if, the equation system

$$\begin{aligned} a_0 \cdot \eta &= r_d - \mu_d \\ (a_i + a_{n+i}) \cdot \eta &= r_d - (\mu_{f,i} + \nu_{f,i} + a_i \cdot a_{n+i}), \quad i = 1, \ldots, n \\ a_{n+i} \cdot \eta &= r_d - (\nu_{f,i} + r_{f,i}), \quad i = 1, \ldots, n. \end{aligned} \tag{4.6}$$

has a unique solution $\eta \in \mathbb{R}^{2n+1}$. Obviously, there is a unique solution to (4.6) if, and only if, the dispersion matrix A has full rank. We will assume this for the rest of the section.

Denote by η^* the unique solution to (4.6). Then by Girsanov's Theorem the process

$$W_t^* \;=\; W_t - t\, \eta^*$$

is a standard Brownian Motion under $\mathbb{Q}$ and the dynamics of the market is described by

$$\begin{aligned} d\, V_t^{(d)} &= V_t^{(d)}\, (r_d\, dt + a_0 \cdot d\, W_t^*) \\ d\, \widetilde{V}_t^{(f,i)} &= \widetilde{V}_t^{(f,i)}\, (r_d\, dt + (a_i + a_{n+i}) \cdot d\, W_t^*)\,, \quad i = 1, \ldots, n \\ d\, \widetilde{X}_t^{(f,i)} &= \widetilde{X}_t^{(f,i)}\, (r_d\, dt + a_{n+i} \cdot d\, W_t^*)\,, \quad i = 1, \ldots, n. \end{aligned} \tag{4.7}$$

Since the market in this section is complete, by (4.1) and (4.2) we can determine the value S_t of the firm's equity at time $t \leq T$ with the formula

$$\begin{aligned} S_t &= e^{-r_d\, (T-t)}\, \mathbb{E}_{\mathbb{Q}}[\max(V_T - B, 0)\,|\, \mathcal{F}_t] \\ &= e^{-r_d\, (T-t)}\, \mathbb{E}_{\mathbb{Q}}\Big[\max \Big(V_t^{(d)} + \sum_{i=1}^{n} \widetilde{V}_t^{(f,i)} - B, 0 \Big) \,\Big|\, \mathcal{F}_t \Big] \end{aligned} \tag{4.8}$$

(see section 2.1 for the definition of $\mathcal{F}_t$). In principle, by (4.7), we are in a position to evaluate (4.8). Unfortunately, this leads to the problem of determining the distribution of a sum of jointly log-normally distributed random variables – a problem for which no analytical solution is known.

Hence, we have to rely on numerical solutions or on an approximation. We will briefly discuss here the approximation suggested by Gentle (1993) (see also Musiela and Rutkowski, 1997, sec. 9.9). The key to the application of Gentle's result is the observation that by

$$\max\Big(V_t^{(d)} + \sum_{i=1}^{n} \widetilde{V}_t^{(f,i)} - B, 0\Big) \;=\; (n+1)\,\max\Big((n+1)^{-1}\Big(V_t^{(d)} + \sum_{i=1}^{n} \widetilde{V}_t^{(f,i)}\Big) - (n+1)^{-1}B, 0\Big) \tag{4.9}$$

equation (4.8) may be interpreted as the evaluation formula for a basket option, i.e., the underlying is a weighted sum of asset value processes where the positive weights sum up to 1.

Denote by $(U_t^{(0)}, U_t^{(1)}, \ldots, U_t^{(2n)})$ a multidimensional Brownian Motion with standard marginals such that with

$$\sigma_i \;=\; \begin{cases} \|a_i\|, & i = 0, n+1, \ldots, 2n \\ \|a_i + a_{n+i}\|, & i = 1, \ldots, n \end{cases} \tag{4.10a}$$

we have for all $t > 0$

$$\rho_{i,j} \;=\; \mathrm{corr}[U_t^{(i)}, U_t^{(j)}] \;=\; \begin{cases} \frac{a_i \cdot a_j}{\sigma_i\,\sigma_j}, & i,j \in \{0, n+1, \ldots, 2n\} \\ \frac{a_i \cdot (a_j + a_{n+j})}{\sigma_i\,\sigma_j}, & i \in \{0, n+1, \ldots, 2n\} \\ & \text{and } j \in \{1, \ldots, n\} \\ \frac{(a_i + a_{n+i}) \cdot (a_j + a_{n+j})}{\sigma_i\,\sigma_j}, & i,j \in \{1, \ldots, n\}\,. \end{cases} \tag{4.10b}$$

Then the solutions to the stochastic differential equations (4.7) and

$$\begin{aligned} d\,V_t^{(d)} &= V_t^{(d)}\,(r_d\,dt + \sigma_0\,d\,U_t^{(0)}) \\ d\,\widetilde{V}_t^{(f,i)} &= \widetilde{V}_t^{(f,i)}\,(r_d\,dt + \sigma_i\,d\,U_t^{(i)})\,, \quad i = 1, \ldots, n \\ d\,\widetilde{X}_t^{(f,i)} &= \widetilde{X}_t^{(f,i)}\,(r_d\,dt + \sigma_{n+i}\,d\,U_t^{(n+i)})\,, \quad i = 1, \ldots, n. \end{aligned} \tag{4.11}$$

are identical in distribution as long as they are considered with the same initial conditions. In comparison with (4.7) representation (4.11) has the advantage that the quantities σ_i and $\rho_{i,j}$ are observable volatilities and correlations.

In order to get an approximation formula for (4.8) – Gentle (1993, adapted to our context for the case $t = 0$) suggested to replace the weights $(n+1)^{-1}$ from (4.9) by

$$\widehat{\omega}^{(d)} = \frac{V_0^{(d)}}{V_0^{(d)} + \sum_{i=1}^{n} \widetilde{V}_0^{(f,i)}} \quad \text{for } V^{(d)} \tag{4.12a}$$

and

$$\widehat{\omega}^{(f,i)} = \frac{V_0^{(f,i)}}{V_0^{(d)} + \sum_{i=1}^{n} \widetilde{V}_0^{(f,i)}} \quad \text{for } V^{(f,i)},\; i = 1, \ldots, n\,. \tag{4.12b}$$

He would then rewrite the left-hand side of (4.9) as

$$\max\Big(V_t^{(d)} + \sum_{i=1}^n \widetilde{V}_t^{(f,i)} - B, 0\Big) = e^{r\,T}\Big(V_0^{(d)} + \sum_{i=1}^n \widetilde{V}_0^{(f,i)}\Big)\max\Big(\widetilde{A}_T - \widetilde{B}, 0\Big)\,, \tag{4.13a}$$

where

$$\widetilde{A}_T = e^{-r\,T}\Big(\widehat{\omega}^{(d)}\frac{V_T^{(d)}}{V_0^{(d)}} + \sum_{i=1}^n \widehat{\omega}^{(f,i)}\frac{\widetilde{V}_T^{(f,i)}}{\widetilde{V}_0^{(f,i)}}\Big), \tag{4.13b}$$

$$\widetilde{B} = \frac{(n+1)^{-1}\,e^{-r\,T}\,B}{V_0^{(d)} + \sum_{i=1}^n \widetilde{V}_0^{(f,i)}}\,. \tag{4.13c}$$

Note that $e^{r\,T}\,V_0^{(d)}$ and $e^{r\,T}\,\widetilde{V}_0^{(f,i)}$ are the forward prices at time 0 for the settlement date T of $V_t^{(d)}$ and $\widetilde{V}_t^{(f,i)}$, respectively. Hence, by the transition from (4.9) to $\max(\widetilde{A}_T - \widetilde{B}, 0)$ one does not only change the weights in the basket portfolio but also replaces the asset value processes by the ratios of the values and the corresponding forward prices.

The key step in the approximation now is to replace the arithmetic mean $\widetilde{A}_T$ from (4.13b) with the geometric mean

$$\widetilde{G}_T = e^{-r\,T}\left(\frac{V_T^{(d)}}{V_0^{(d)}}\right)^{\widehat{\omega}^{(d)}}\prod_{i=1}^n\left(\frac{\widetilde{V}_T^{(f,i)}}{\widetilde{V}_0^{(f,i)}}\right)^{\widehat{\omega}^{(f,i)}} \tag{4.14a}$$

and to correct the strike $\widetilde{B}$ by the difference of the expected values, i.e.,

$$\widehat{B} = \widetilde{B} + \mathbb{E}_{\mathbb{Q}}[\widetilde{G}_T - \widetilde{A}_T]\,. \tag{4.14b}$$

Since the asset value processes under consideration are a multidimensional geometric Brownian motion, the random variable $\widetilde{G}_T$ has a logarithmic normal distribution. Thus the evaluation of

$$\mathbb{E}_{\mathbb{Q}}[\max(\widetilde{G}_T - \widehat{B}, 0)]$$

is a feasible task leading to the result

$$S_0 \;\approx\; \Big(V_0^{(d)} + \sum_{i=1}^n \widetilde{V}_0^{(f,i)}\Big)\Big(c\,N(l_1(T)) - (\widetilde{B} + c - 1)\,N(l_1(T))\Big) \tag{4.15a}$$

with

$$c \;=\; \exp\Big\{T\Big(\sigma_0\,\widehat{\omega}^{(d)}\sum_{i=1}^n \rho_{0,i}\,\widehat{\omega}^{(f,i)}\,\sigma_i + \sum_{i=1}^n\sum_{j=i+1}^n \rho_{i,j}\,\widehat{\omega}^{(f,i)}\,\widehat{\omega}^{(f,j)}\,\sigma_i\,\sigma_j - 1/2\,v^2\Big)\Big\}\,, \tag{4.15b}$$

$$v^2 \;=\; \sigma_0^2\,(\widehat{\omega}^{(d)})^2 + \sum_{i=1}^n \sigma_i^2\,(\widehat{\omega}^{(f,i)})^2\,, \tag{4.15c}$$

$$l_{1,2}(T) \;=\; \frac{\log c - \log(\widetilde{B} + c - 1) \pm T/2\,v^2}{\sqrt{T}\,v}\,. \tag{4.15d}$$

Here $\widetilde{B}$, σ_i, $\rho_{i,j}$, $\widehat{\omega}^{(d)}$, $\widehat{\omega}^{(f,i)}$ are given by (4.13c), (4.10a), (4.10b), (4.12a), and (4.12b), respectively. The correlations between the foreign exchange rate returns could be taken as approximations for the correlations of the firm's asset value processes in different countries. The value F_0 of the firm's debt can now be calculated by (2.1) and (4.15a).

4.2 Practical aspects of the decomposition

As in the classical model of the firm value, the domestic and the foreign asset values of the firm from (4.1) will not usually be directly observable. As a pragmatic solution to this problem, we suggest first calculating an induced total value of the firm's assets V_0 via the classical methodology, as described in section 3.1. The problem then is to find reasonable weights $w_0, \ldots, w_n$ with $\sum_{i=0}^{n} w_i = 1$ such that the values

$$V_0^{(d)} = w_0\, V_0, \quad \widetilde{V}_0^{(f,i)} = w_i\, V_0,\ i = 1, \ldots, n, \tag{4.16}$$

can be plugged in to the equations (4.15a), (4.15b), and (4.15c). One way to determine the weights is to read them off from the balance sheet of the firm – as is done when the *Credit Monitor*® software by KMV (http://www.kmv.com) is used.

Another approach would be to take the determination coefficients of the regression of the logarithmic return of the firm value on the logarithmic returns of the foreign exchange rates as proxies for the weights w_i. Here, we consider in more detail the case of only one foreign currency influencing the firm value, i.e., the case $n = 1$ in (4.1). In this case, the foreign component $\widetilde{V}_0^{(f,1)} \approx w_1\, V_0$ would be obtained by multiplying the total firm value V_0 with the determination coefficient w_1 of the regression given in (4.17) of the firm's equity log-returns s_t (as a proxy for the firm value returns) on the log-returns x_t of the foreign currency rate X_t:

$$\begin{aligned} s_t &= a + b\, x_t + \epsilon_t, \\ w_1 &= \frac{b^2 \mathrm{var}(x_t)}{\mathrm{var}(s_t)}. \end{aligned} \tag{4.17}$$

A further problem arises if the firm's debt is partially denominated in foreign currency. This question calls for further research.

Once a model for the value of the firm's equity and debt has been established shocks on the firm value caused by shocks of the foreign exchange rates can be introduced by shocking the foreign currency part of the assets in the spirit of section 3.2. Let us denote by V_t^{new} the firm value after a foreign exchange shock. This means that we compute the value of the firm after a shock by

$$V_t^{\text{new}} = (1 - w_1)V_t + w_1 V_t (1 + \Delta K)\,, \tag{4.18}$$

where ΔK denotes the relative change in the foreign exchange rate.

We illustrate this approach with a data example. Some data studies on joint distributions for Nestlé stock returns in USD (US-Dollar) and foreign exchange rate returns CHF (Swiss-francs)/

USD were performed by Filip Lindskog from RiskLab Switzerland (`http://www.risklab.ch`). He used a data set covering the time period 1990–2000.

The joint daily-log-return distribution of this data best fits a t_{11} distribution with linear correlation coefficient between -0.15 and -0.17, a variance of the Nestlé returns between $1e-04$ and $1.5e-04$, and a variance of the exchange rate returns between $3.6e-0.5$ and $5.8e-05$. The Nestlé returns have a skewness of about -0.25, the skewness of the exchange rate returns is about 0.25.

The joint weekly-log-return distribution best fits a t_{17} distribution with linear correlation coefficient between -0.23 and -0.30, a variance of Nestlé returns between $5.9e-04$ and $9.8e-04$, and an exchange rate variance between $2.2e-04$ and $2.9e-04$. The Nestlé returns have a skewness of about -0.6, the skewness of the exchange rate returns is abaout 0.14.

By fitting a linear regression of the daily log-returns of Nestlé stock prices in CHF on to the daily log-returns of the exchange rates CHF / USD we get a determination coefficient of 2%. This means that about 2% of the variation of the daily equity log returns can be explained by the variation of the daily log returns of the exchange rates.

4.3 Modeling by decomposing the firm value volatility

The approach described in sections 4.1 and 4.2 entails some problems, the most important of which is probably the fixing of the weights in (4.16). This difficulty motivates the alternative approach introduced in this subsection. The basic idea is to replace the decomposition of the firm value process in a domestic part and some foreign parts by a decomposition of the firm value volatility in a domestic part and some foreign parts.

Observe that by (4.7) and (4.11) the volatility of the domestic part of the firm value process can be written alternatively as σ_0 or $\sqrt{\sum_{i=0}^{2n} a_{0,i}^2}$. This suggests a natural decomposition of the instantaneous variance of $V^{(d)}$ into a component $a_{0,0}^2$ stemming from a white noise specific for $V^{(d)}$ and further components $a_{0,i}^2$ from the white noises corresponding to the foreign parts of the firm value and the foreign exchange rates.

To be more concrete: Denote as before by V_t the value of all the assets of the firm, quoted in domestic currency. Define the value $\widetilde{X}_t^{(f,i)}$ at time t of one unit of currency i, $i = 1, \ldots, n$, invested at time 0 in a risk-free asset in country i by (4.4). We can then, analogously to (4.7), describe the risk-neutral dynamics of the common evolution of V_t and the $\widetilde{X}_t^{(f,i)}$ by

$$\begin{aligned} d\,V_t &= V_t\,(r_d\,dt + a_0 \cdot d\,W_t^*) \\ d\,\widetilde{X}_t^{(f,i)} &= \widetilde{X}_t^{(f,i)}\,(r_d\,dt + a_i \cdot d\,W_t^*)\,, \quad i = 1, \ldots, n, \end{aligned} \tag{4.19a}$$

where $W_t^* = (W^{*,0}, \ldots, W^{*,n})'$ is a vector of independent standard Brownian motions, and $a_i = (a_{i0}, \ldots, a_{in})$, $i = 0, \ldots, n$ are the rows of a dispersion matrix A. Another description, suggested by (4.11), is

$$
\begin{aligned}
d\,V_t &= V_t\,(r_d\,dt + \sigma_0\,d\,U_t^{(0)}) \\
d\,\widetilde{X}_t^{(f,i)} &= \widetilde{X}_t^{(f,i)}\,(r_d\,dt + \sigma_i\,d\,U_t^{(i)})\,, \quad i = 1,\ldots,n,
\end{aligned} \tag{4.19b}
$$

with volatilities $\sigma_0,\ldots,\sigma_n > 0$ and an $(n+1)$-dimensional Brownian motion $(U^{(0)},\ldots,U^{(n)})'$ whose components are standard Brownian motions with correlations

$$\mathrm{corr}[U_t^{(i)}, U_t^{(j)}] \;=\; \rho_{ij},\;\; i,j = 0,\ldots,n,\;\; t > 0. \tag{4.20}$$

Denote by Σ the covariance matrix of $(\sigma_1\,U_1^{(0)},\ldots,\sigma_n\,U_1^{(n)})'$, namely

$$\Sigma \;=\; (\Sigma_{ij}) \;=\; (\sigma_i\,\sigma_j\,\rho_{ij}). \tag{4.21}$$

Equations (4.19a) and (4.19b) – under the same initial conditions – give rise to the same distribution for $(V, \widetilde{X}^{(f,0)},\ldots,\widetilde{X}^{(f,n)})$ if and only if

$$A' \cdot A \;=\; \Sigma. \tag{4.22}$$

In particular, (4.22) implies

$$\sigma_0^2 \;=\; \sum_{i=0}^{n} a_{0i}^2\,. \tag{4.23}$$

Of course, (4.22) does not uniquely determine A for given Σ. Nevertheless, we get a one-to-one correspondence between A and Σ if we require A to be symmetric and positively definite. In fact, this is a reasonable assumption that we fix for the sequel. Furthermore, we assume that the covariance matrix Σ is non-degenerate, i.e., that it is positive definite.

Hence, we have to determine the positive square root of Σ, i.e., a symmetric, positive definite $(n+1)\times(n+1)$-matrix A such that

$$A \cdot A \;=\; \Sigma. \tag{4.24}$$

Let $v_0,\ldots,v_n$ be a basis of $\mathbb{R}^{n+1}$ consisting of eigenvectors of Σ and denote by D a diagonal matrix whose diagonal elements are the eigenvalues of Σ, i.e. $D = (d_{ij})_{i,j=0,\ldots,n}$ with

$$d_{ij} \;=\; \begin{cases} \lambda_i, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases} \tag{4.25}$$

where $\lambda_0,\ldots,\lambda_n > 0$ are the eigenvalues corresponding to $v_0,\ldots,v_n$. We define $\sqrt{D}$ as

$$\sqrt{D} \;=\; (\sqrt{d_{ij}})_{i,j=0,\ldots,n}. \tag{4.26}$$

Denote by M the matrix whose columns are just the vectors $v_0,\ldots,v_n$. Then the solution A to (4.24) is given by

$$A \;=\; M \cdot \sqrt{D} \cdot M^{-1}. \tag{4.27}$$

Once we have determined a representation (4.24) for Σ and therefore also a representation (4.23) for σ_0 we can model the effect of exchange rate volatility shocks on the firm value volatility σ_0 as follows: If the volatility σ_i of the exchange rate $X^{f,i}$, $i = 1, \ldots, n$, becomes

$$\sigma_i^{\text{new}} = \sigma_i^{\text{old}}(1 + \Delta\sigma_i), \tag{4.28}$$

we calculate the changed value of σ_0 as

$$\sigma_0^{\text{new}} = \sqrt{(1 + \Delta\sigma_i)^2\, a_{0i}^2 + \sum_{j=0,\, j\neq i}^{n} a_{0j}^2}. \tag{4.29}$$

Note that the shock scenarios for exchange rate volatilities have to be defined with care (see Wystup, 2001).

4.4 An example for the decomposition of the firm value volatility

We are now going to study the approach of section 4.3 in the case of $n = 1$ in more detail. Let us assume that the volatility σ_0 of the firm value process and volatility σ_1 of a foreign exchange rate, as well as their correlation ρ_{01} as in (4.19b) and (4.20), are given.

If we adminster a shock to the exchange rate volatility, according to (4.28) we obtain

$$\sigma_1^{\text{new}} = \sigma_1^{\text{old}}(1 + \Delta\sigma_1). \tag{4.30}$$

The individual components a_{11}, a_{01} of σ_1 will then also be shocked in the sense of (4.23) by the factor $1 + \Delta\sigma_1$, and the influence of this shock to σ_0 can be modeled following (4.29) by

$$(\sigma_0^{\text{new}})^2 = a_{00}^2 + (1 + \Delta\sigma_1)^2 a_{01}^2. \tag{4.31}$$

Equation (4.31) shows the dampened effect of a shock like that in (4.30) to the firm value volatility. That means that a shock to the exchange rate and its volatility only influence a part of the firm value volatility, because only a part of the firm value is in foreign currency.

In order to make this methodology work in practice, one first has to estimate the volatilities σ_1, σ_0 and the correlation ρ_{01} between firm value returns and foreign exchange rate returns. Given these estimates, the decomposition (4.24) can be computed as the positive square root of the covariance matrix $\begin{pmatrix} \sigma_1^2 & \sigma_1\,\sigma_0\,\rho_{01} \\ \sigma_1\,\sigma_0\,\rho_{01} & \sigma_0^2 \end{pmatrix}$. This means that we look for a symmetric matrix $\begin{pmatrix} a_{11} & a_{01} \\ a_{01} & a_{00} \end{pmatrix}$ with positive eigenvalues such that

$$\begin{pmatrix} a_{11} & a_{01} \\ a_{01} & a_{00} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{01} \\ a_{01} & a_{00} \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_1\,\sigma_0\,\rho_{01} \\ \sigma_1\,\sigma_0\,\rho_{01} & \sigma_0^2 \end{pmatrix}. \tag{4.32}$$

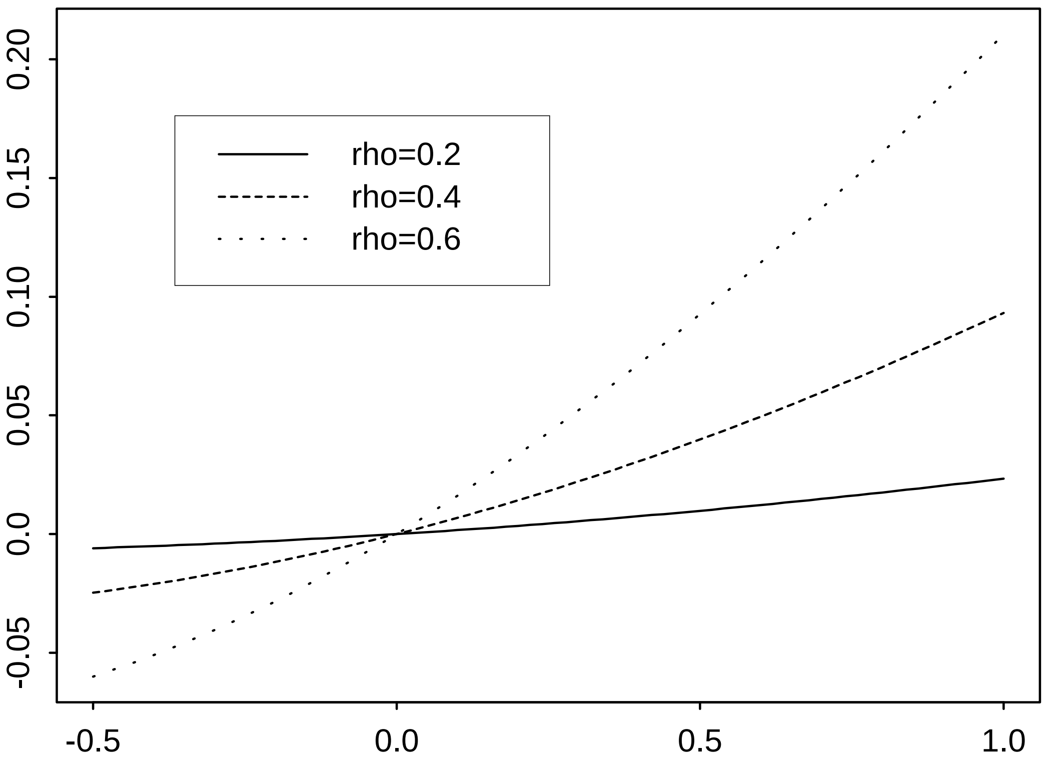

Figure 1: $\Delta\sigma_0$ as function of $\Delta\sigma_1$ for different values of ρ_{01}.

The solution to (4.32) is given by

$$\begin{aligned}
H &= \sqrt{(\sigma_1^2 - \sigma_0^2)^2 + 4\,\sigma_1^2\,\sigma_0^2\,\rho_{01}^2} \\
a_{11} &= 2^{-3/2}\,H^{-1}\left(\sqrt{\sigma_1^2+\sigma_0^2-H}\,(\sigma_0^2-\sigma_1^2+H)+\sqrt{\sigma_1^2+\sigma_0^2+H}\,(\sigma_1^2-\sigma_0^2+H)\right) \\
a_{01} &= \frac{\sqrt{2}\,\sigma_1\,\sigma_0\,\rho_{01}}{\sqrt{\sigma_1^2+\sigma_0^2-H}+\sqrt{\sigma_1^2+\sigma_0^2+H}} \qquad (4.33) \\
a_{00} &= 2^{-3/2}\,H^{-1}\left(\sqrt{\sigma_1^2+\sigma_0^2-H}\,(\sigma_1^2-\sigma_0^2+H)+\sqrt{\sigma_1^2+\sigma_0^2+H}\,(\sigma_0^2-\sigma_1^2+H)\right).
\end{aligned}$$

Obviously, (4.33) is only valid where $H > 0$. Since we assume that the covariance matrix $\begin{pmatrix} \sigma_1^2 & \sigma_1\,\sigma_0\,\rho_{01} \\ \sigma_1\,\sigma_0\,\rho_{01} & \sigma_0^2 \end{pmatrix}$ is non-degenerate, $H = 0$ can only occur if $\rho_{01} = 0$. In this case the solution to (4.32) can be written as

$$a_{11} = \sigma_1 \qquad a_{01} = 0 \qquad a_{00} = \sigma_0\,. \qquad (4.34)$$

Define the relative difference $\Delta\sigma_0$ in the firm value volatility by

$$\Delta\sigma_0 \quad = \quad \frac{\sigma_0^{\text{new}}}{\sigma_0^{\text{old}}} - 1 \quad = \quad (\sigma_0^{\text{old}})^{-1} \sqrt{a_{00}^2 + (1+\Delta\sigma_1)^2\, a_{01}^2} - 1\,. \tag{4.35}$$

Figure 1 demonstrates the effect of shocks to the exchange rate volatility on the firm value volatility. The strength of damping depends perspicuously on the correlation between exchange rate and firm value. The weaker the correlation, the stronger the damping of the exchange rate volatility shock. Even for higher correlations the induced shocks in the firm value volatility are far weaker than the currency volatility shocks we provoke.

Let us take a closer look on two special cases. First, assume that $\rho_{01} = 0$. It then follows from (4.34) and (4.35) that $\Delta\sigma_0 = 0$ regardless of the value of $\Delta\sigma_1$, as indeed it should be since, in this case, the firm value and foreign exchange rate are independent.

The other point of interest is $|\rho_{01}| = 1$. From (4.33) we then obtain

$$\begin{aligned} H &= \sigma_1^2 + \sigma_0^2 & \qquad a_{11} &= \frac{\sigma_1^2}{\sqrt{H}} \\ a_{01} &= \pm\frac{\sigma_1\,\sigma_0}{\sqrt{H}} & \qquad a_{00} &= \frac{\sigma_0^2}{\sqrt{H}}\,. \end{aligned} \tag{4.36}$$

Inserting (4.36) into (4.35) yields

$$\Delta\sigma_0 \;=\; \frac{\sqrt{\sigma_0^2 + (1+\Delta\sigma_1)^2\,\sigma_1^2}}{\sqrt{H}} - 1\,. \tag{4.37}$$

In the case of an FX rate volatility σ_1 reduced to 0 (i.e., $\Delta\sigma_1 = -1$) from (4.37) we obtain

$$\Delta\sigma_0 \;=\; \frac{\sigma_0}{\sqrt{H}} - 1 \;\in (-1, 0)\,. \tag{4.38}$$

Hence, we still have a damping effect even if the log returns of the firm value and the FX rate are linearly dependent.

Finally, we can conclude from (4.37) that

$$\Delta\sigma_0 \approx \Delta\sigma_1\,\frac{\sigma_1^2}{H}, \qquad \text{as } \Delta\sigma_1 \to 0\,, \tag{4.39}$$

$$\Delta\sigma_0 \sim \Delta\sigma_1\,\frac{\sigma_1}{\sqrt{H}}, \quad \text{as } \Delta\sigma_1 \to \pm\infty\,. \tag{4.40}$$

5 The multi-firm Merton model

So far, we have considered the modeling of the value of a single firm's debt, focusing on its dependence on equity, risk-free interest, and foreign exchange rates. In this section we are going to extend this methodology to the case of more than one firm. Note that this approach is quite common. For instance, it is used in the *Portfolio Manager*™ by KMV (`http://www.kmv.com`) or in the *CreditManager* by RiskMetrics (`http://www.creditmetrics.com`) (cf. Iscoe et al., 1999; Nyfeler, 2000).

5.1 Extending the Merton model to several firms

We consider a portfolio of n firms. For a single firm, the extended model will coincide with the model described in section 2.1. At first glance the new model looks exactly like that in section 2.1 with the only exception that almost all occurring quantities wear an index $i \in \{1, \dots, n\}$. Nevertheless, we will face two new features: the firm values processes are driven by a multi-dimensional Brownian motion with correlated components, and the debt maturities of the firms may differ from one another. As, in actual fact, most quantities are defined in exactly the same way as in section 2.1, we give only a short enumeration of them.

- We are interested in determining the value of the debt portfolio of firms $i = 1, \dots, n$ at a fixed time horizon $T > 0$.
- $S_t^{(i)}$ is the value of the equity of firm $i, i = 1, .., n$ at time $t \in [0, T]$.
- The liabilities of firm i are considered to consist of a single zero-bond with some maturity T_i which may be less or greater than T, and some face value $B^{(i)}$. Obviously, this is not realistic. But imagine the bond to be a summing-up of the firm's liabilities by some mapping procedure. For instance, T_i could be the duration (i.e. the average time to maturity) of the liabilities and $B^{(i)}$ could be some weighted average of the due payments. Hence, $F_t^{(i)}$ denotes the value of this bond at time $t \in [0, T]$,
- $F_t^{\text{port}} = \sum_{i=1}^n F_t^{(i)}$ is the value of the debt portfolio at time $t \in [0, T]$. The distribution of the random variable F_T^{port} is of primary interest.
- $V_t^{(i)} = S_t^{(i)} + F_t^{(i)}$ is the value of firm i at time $t \in [0, T]$.
- As in section 2.1, the price of a risk-less zero coupon bond, which pays one currency unit for sure at time t in the future, is $P(t) = e^{-rt}$ where $r > 0$ is the *instantaneous risk-less rate of interest* and constant over time.
- $S_{T_i}^{(i)} := \max(V_{T_i}^{(i)} - B^{(i)}, 0)$ is the amount that the equity holders would receive at maturity T_i in case of liquidation of firm i.
- $F_{T_i}^{(i)} := \min(V_{T_i}^{(i)}, B^{(i)}) = B^{(i)} - \max(B^{(i)} - V_{T_i}^{(i)}, 0)$ is the amount that the bond holders of firm i receive at maturity T_i. Default of firm i takes place if $V_{T_i}^{(i)} < B^{(i)}$.
- The stochastic firm value processes are modeled as geometric Brownian motions with respect to the *physical* probability measure $\mathbb{P}$ as

$$dV_t^{(i)} = \mu_{V_i} V_t^{(i)} dt + \sigma_{V_i} V_t^{(i)} dW_t^{(i)}, \quad i = 1, \dots, n, \tag{5.1}$$

  where μ_{V_i} is the instantaneous expected rate of return to firm i per time unit, $\sigma_{V_i}^2$ is the instantaneous variance of the value of firm i per time unit, and $W_t = (W_t^{(1)}, \dots, W_t^{(n)})$ is an n-dimensional Brownian motion. Each component of $W_t = (W_t^{(1)}, \dots, W_t^{(n)})$ is a standard Brownian motion, but the correlations between them may be non-zero. σ_{V_i} is called *volatility* of $V_t^{(i)}$. The initial values $V_0^{(i)}$ are fixed.

We again assume that the market under consideration is complete, i.e., that there is a unique probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that all discounted firm value processes $e^{-r\,t}\,V_t^{(i)}$ are martingales under $\mathbb{Q}$. In particular, under $\mathbb{Q}$ the dynamics in the model can be described by the stochastic differential system

$$dV_t^{(i)} = r\,V_t^{(i)}dt + \sigma_{V_i}V_t^{(i)}dW_t^{*,(i)}, \quad i = 1, \ldots, n, \tag{5.2}$$

where $\sigma_{V_i}^2$ is as above and $W_t^* = (W_t^{*,(1)}, \ldots, W_t^{*,(n)})$ is again an n-dimensional Brownian motion (not identical with $W(t)$) whose marginal distributions are standard.

The assumption of completeness implies that there are unique prices $S_t^{(i)}$ and $F_t^{(i)}$ for equities and bonds respectively at every epoch t of time in $[0, T]$. These prices are given by

$$S_t^{(i)} = e^{r\,(t-T_i)}\,\mathrm{E}_{\mathbb{Q}}\left[S_{T_i}^{(i)}\,|\,\mathcal{F}_t\right] \tag{5.3a}$$

and

$$F_t^{(i)} = e^{r\,(t-T_i)}\,\mathrm{E}_{\mathbb{Q}}\left[F_{T_i}^{(i)}\,|\,\mathcal{F}_t\right], \tag{5.3b}$$

where $\mathcal{F}_t$ denotes the history up to time t of $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$. Note that for $t \geq T_i$ equations (5.3a) and (5.3b) simplify to

$$S_t^{(i)} = e^{r\,(t-T_i)}\,S_{T_i}^{(i)} \tag{5.4a}$$

and

$$F_t^{(i)} = e^{r\,(t-T_i)}\,F_{T_i}^{(i)}. \tag{5.4b}$$

In fact, when using (5.3b) one implicitly assumes that there will be some recovery in case of default of the firm. We take here the more pessimistic point of view of no recovery. This means that we replace (5.3b) by

$$F_t^{(i)} \;=\; e^{r\,(t-T_i)}\,B^{(i)}\,\mathbb{Q}\left[V_{T_i}^{(i)} \geq B^{(i)}\,|\,\mathcal{F}_t\right]. \tag{5.5}$$

In particular, by (2.9), in case $t = T$ equation (5.5) yields

$$F_T^{(i)} \;=\; \begin{cases} e^{r\,(T-T_i)}\,B^{(i)}\,\mathbf{1}_{\{V_{T_i}^{(i)} \geq B^{(i)}\}}, & T \geq T_i \\[2ex] e^{r\,(T-T_i)}\,B^{(i)}\,N\Big(\frac{\log\frac{V_T^{(i)}}{B^{(i)}} + (r - \frac{\sigma_{V_i}^2}{2})(T_i - T)}{\sigma_{V_i}\sqrt{T_i - T}}\Big), & T < T_i. \end{cases} \tag{5.6}$$

Whether $F_t^{(i)}$ be defined by (5.3b) or by (5.5), we can define for $t < T_i$ the *yield spread* $Y_t^{(i)}$ of bond i by

$$F_t^{(i)} = e^{(Y_t^{(i)} + r)\,(t-T_i)}\,B^{(i)} \tag{5.7a}$$

or equivalently

$$Y_t^{(i)} = \frac{\log\left(F_t^{(i)}/B^{(i)}\right)}{t - T_i} - r\,. \tag{5.7b}$$

5.2 The distribution of the portfolio value

In section 5.1 we specified a model that uniquely determines the distribution of the debt portfolio value

$$F_T^{\text{port}} \;=\; \sum_{i=1}^{n} F_T^{(i)} \tag{5.8}$$

under the physical probability measure $\mathbb{P}$ at time T, as soon the necessary initial values and parameters are given, such as actual firm values, volatilities, and correlations between the driving Brownian motions etc. Assuming that this is the case we can perform stress tests along the lines of sections 3 and 4 in order to gain insight into the effects of changing marginal conditions on the portfolio value distribution.

Unfortunately, due to (5.6) the random variables summed up to F_T^{port} in (5.8) will usually be fairly heterogenous. Moreover, they will be more or less dependent. Consequently, in general, it will be impossible to arrive at an analytical formula for the distribution function of F_T^{port}.

In the sequel, we discuss two ways to deal with this difficulty. First we examine how to estimate the distribution of F_T^{port} by means of Monte-Carlo simulations. In section 5.3 we then consider an approximation approach.

Concerning the Monte-Carlo simulations, we observe that the stochastic differential equation system (5.1) under $\mathbb{P}$ has the (pathwise) unique solution

$$V_t^{(i)} \;=\; V_0^{(i)} \exp\left(\mu_{V_i}\, t - \frac{(\sigma_{V_i})^2}{2}\, t + \sigma_{V_i}\, W_t^{(i)}\right), \quad t \geq 0, i = 1, \ldots, n, \tag{5.9}$$

where the randomness is caused only by the Brownian motion $W = (W^{(1)}, \ldots, W^{(n)})$ with correlated components and marginals which are standard Brownian motions. By (5.9), (5.6), and (5.8) all we have to do for the simulations is to generate realizations of $W_{\min(T_1,T)}, \ldots, W_{\min(T_n,T)}$. Without loss of generality we may assume that there are integers $1 \leq k_1 < k_2 < \ldots < k_l \leq n$ such that

$$0 \leq T_1 = \ldots = T_{k_1} < T_{k_1+1} = \ldots = T_{k_2} < \ldots < T_{k_{l-1}+1} = \cdots = T_{k_l} \leq T < T_{k_l+1} \leq \ldots \leq T_n.$$

Hence, by the independence of the increments of a Brownian motion, it is sufficient to generate random vectors $Y_1, \ldots, Y_l, Y_{l+1}$ with

$$\begin{aligned} &Y_1 \sim \mathcal{N}(0, T_{k_1}\, C), Y_2 \sim \mathcal{N}(0, (T_{k_2} - T_{k_1})\, C), \\ &\qquad \ldots, Y_l \sim \mathcal{N}(0, (T_{k_l} - T_{k_{l-1}})\, C), Y_{l+1} \sim \mathcal{N}(0, (T - T_{k_l})\, C). \end{aligned} \tag{5.10}$$

Here, C is the covariance matrix of $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ at $t = 1$. Note that Y_{l+1} will be constant 0 in case $T_{k_l} = T$.

Since $\sqrt{t}\,Y \sim \mathcal{N}(0, t\,C)$ for $t \geq 0$ and $Y \sim \mathcal{N}(0, C)$, by (5.10) we must generate $(l+1)\,m$ independent random vectors with distribution $\mathcal{N}(0, C)$ if we want to get a Monte-Carlo sample of F_T^{port} with size m.

For conceptual reasons, it can be useful to represent the Brownian motion $W = (W^{(1)}, \ldots, W^{(n)})$ which drives (5.1) as

$$W_t^{(i)} \;=\; \sum_{j=1}^{k} a_{ij}\, U_t^{(j)} + b_i\, \xi_t^{(i)}, \quad t \geq 0, i = 1, \ldots, n. \tag{5.11}$$

Here, a_{ij} and b_i are constants such that

$$\sum_{j=1}^{k} a_{ij}^2 + b_i^2 \;=\; 1, \quad i = 1, \ldots, n,$$

and $U^{(1)}, \ldots, U^{(k)}$, $\xi^{(1)}, \ldots, \xi^{(n)}$ are standard Brownian motions such that $U = (U^{(1)}, \ldots, U^{(k)})$, $\xi^{(1)}, \ldots, \xi^{(n)}$ are independent, whereas the components of U may be dependent. A representation as in (5.11) – called *factor model* – could result from a regression of W on some normalized economic index returns or from a principal component analysis of C.

In (5.11), $U^{(1)}, \ldots, U^{(k)}$ are interpreted as *systematic* risk factors, common to all firms, whereas $\xi^{(1)}, \ldots, \xi^{(n)}$ are *firm-specific* risk factors. Note that, given $U^{(1)}, \ldots, U^{(k)}$, the processes $W^{(1)}, \ldots,$ $W^{(n)}$ are conditionally independent.

On the one hand, this motivates the considerations noted below in section 5.3. On the other hand, it offers the opportunity to define worst-case scenarios for the systematic risk factors. We will consider a particular choice for the factor model (5.11) in more detail in section 6.

5.3 Approximating the portfolio value distribution

Assume that (5.8) can be written as

$$F_n \;=\; \sum_{i=1}^{n} w_{in}\, X_i\,, \quad n \in \mathbb{N}, \tag{5.12}$$

where $X_1, X_2, \ldots$ are random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $w_{1n}, \ldots, w_{nn}$ are constants for each $n \in \mathbb{N}$. The factor model (5.11) suggests the following approximation result which slightly generalizes a result in section 1.6 from Nyfeler (2000).

Proposition 5.1 *Let $X_1, X_2, \ldots$ be a sequence of square integrable real random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that there is a σ-algebra $\mathcal{A} \subset \mathcal{F}$ such that, conditional* *on $\mathcal{A}$, the sequence $X_1, X_2, \ldots$ is independent and identically distributed with a non-degenerate distribution on an event of positive probability. Let for each $n \in \mathbb{N}$ numbers $w_{1n}, \ldots, w_{nn}$ be given with $\sum_{i=1}^n w_{in} = 1$. Then, with F_n defined by (5.12), we have that*

$$F_n \xrightarrow[n\to\infty]{\mathrm{L}_2} \mathbb{E}[X_1 \,|\, \mathcal{A}] \tag{5.13a}$$

if and only if

$$\lim_{n\to\infty} \sum_{i=1}^n w_{in}^2 \;=\; 0\,. \tag{5.13b}$$

Proof. By the conditional independence and the identical distribution property of $X_1, X_2, \ldots$ given $\mathcal{A}$ we obtain

$$\begin{aligned}
\mathbb{E}[(F_n - \mathbb{E}[X_1 \,|\, \mathcal{A}])^2] &= \sum_{i=1}^n w_{in}^2\, \mathbb{E}[(X_i - \mathbb{E}[X_i \,|\, \mathcal{A}])^2] \\
&\quad + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{in}\, w_{jn}\, \mathbb{E}\Big[\mathbb{E}\big[X_i - \mathbb{E}[X_i \,|\, \mathcal{A}]\big]\, \mathbb{E}\big[X_j - \mathbb{E}[X_j \,|\, \mathcal{A}]\big]\Big] \\
&= \mathbb{E}[(X_1 - \mathbb{E}[X_1 \,|\, \mathcal{A}])^2] \sum_{i=1}^n w_{in}^2\,.
\end{aligned}$$

Observe that $\mathbb{E}[(X_1 - \mathbb{E}[X_1 \,|\, \mathcal{A}])^2] > 0$ since the distribution of X_1 given $\mathcal{A}$ is non-degenerate on a set of positive probability. This proves the result. $\square$

Of course, the L_2-convergence in Proposition 5.1 implies convergence in distribution. For the purpose of illustration, consider the following special case of (5.11), (5.6), and (5.8).

As for (5.6), assume that $T_i = T$ and $B^{(i)} = B$ for all i. This yields

$$F_T^{(i)} \;=\; B\, \mathbf{1}_{\{V_T^{(i)} \geq B\}}, \qquad i = 1, \ldots, n. \tag{5.14}$$

Concerning (5.11) and $V_T^{(i)}$ assume that $V_0^{(i)} = V_0$, $\mu_{V_i} = \mu$, and $\sigma_{V_i} = \sigma$ for all i and that the $W_T^{(i)}$ are given by

$$W_T^{(i)} \;=\; \sqrt{\rho}\, W_T + \sqrt{1-\rho}\, \xi_T^{(i)}, \tag{5.15}$$

where W and the ξ are independent standard Brownian motions and $\rho \in (0,1)$. A portfolio with a factor model like (5.15) is called *equi-correlated*. From (5.14) and (5.15) we get

$$X_i \;=\; F_T^{(i)} \;=\; B\, \mathbf{1}_{\{W\sqrt{\rho} + \xi_i \sqrt{1-\rho} \geq c\}}, \qquad i = 1, \ldots, n, \tag{5.16}$$

where $W, \xi_1, \ldots, \xi_n$ are independent standard normal variables and c is given by

$$c \;=\; \frac{\log(B/V_0) - (\mu - \sigma^2/2)\, T}{\sigma\, \sqrt{T}}. \tag{5.17}$$

With $\mathcal{A} = \sigma(W)$ and $w_{in} = \frac{B^{(i)}}{\sum_{j=1}^n B^{(j)}} = (n\,B)^{-1}$ we are in a position to apply Proposition 5.1 in order to get a limit for the distribution of the normalized portfolio values $(n\,B)^{-1}\,F_T^{\mathrm{port}}$. As

$$\mathrm{E}[X_1\,|\,\sigma(W)] \;=\; B\,N\Big(\frac{\sqrt{\rho}\,W - c}{\sqrt{1-\rho}}\Big)$$

we obtain

$$(n\,B)^{-1}\,F_T^{\mathrm{port}} \;\xrightarrow[n\to\infty]{\mathrm{L}_2}\; N\Big(\frac{\sqrt{\rho}\,W - c}{\sqrt{1-\rho}}\Big), \tag{5.18}$$

where N as usual denotes the distribution function of the standard normal distribution. Nyfeler (2000, sec. 1.6), who follows KMV (`http://www.kmv.com`), calls the distribution of the limiting variable $N\Big(\frac{\sqrt{\rho}\,W-c}{\sqrt{1-\rho}}\Big)$ a *Normal Inverse* distribution. As this notion might be confounded with *inverse Gaussian* distributions – which have a completely different meaning – the notion of *Probit-Normal mixture-distribution* as introduced by Frey and McNeil (2001) is to be preferred.

With (5.6) in mind, it seems quite unreasonable to hope that a portfolio of bonds might be homogeneous enough for an application of Proposition 5.1. Nevertheless, it might be possible to split the portfolio into a number of sub-portfolios consisting of bonds which are similar in maturity and face value. The proposition could then be applied to each of the sub-portfolios. The resulting limit distribution for the portfolio itself would be that of a linear combination of conditional expectations of the kind appearing in (5.13a).

6 Comparison of different portfolio structures

In this section we describe another construction of a factor model as an alternative to (5.11). It is due to Nicole Bäuerle (University of Ulm, Germany, guest at RiskLab (`http://www.risklab.ch`) in September 2000). This construction provides an additional useful result on the comparison of different portfolio structures with respect to concentration effects. The idea of this model is to allow the default of a counter-party to be caused by three factors:

(i) an *individual factor* or *management factor* which is different for each company.

(ii) a *sector factor* which may be geographical or related to the type of industry or a combination of both.

(iii) an economic *global factor* which influences all companies in the same way.

All factors are assumed to be independent. Let us now suppose that there are n counter-parties in the portfolio classified into k different sectors. Then the vector of the correlated Brownian motions $W = (W^{(1)}, \ldots, W^{(n)})$ in (5.1) can be represented by

$$W^{(i)} \;=\; \sqrt{1-\rho_*}\,S^{(i)} + \sqrt{\rho_*}\,G, \quad i = 1, \ldots, n. \tag{6.1}$$

Here, G is a standard Brownian motion which models the global effect. $S = (S^{(1)}, \ldots, S^{(n)})$ is a vector-valued Brownian motion with standard marginals whose correlation matrix at any point in time is given by

$$\Lambda(n) = \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_k \end{pmatrix}, \tag{6.2}$$

where $\Lambda_u = (\lambda_{ij}(u))$ is a correlation matrix with $\lambda_{ij}(u) = \frac{\rho_u - \rho_*}{1-\rho_*}$ if $i \neq j$ and 1 if $i = j$. We assume that $0 \leq \rho_* \leq \rho_1, \ldots, \rho_k \leq 1$. Hence, the correlation matrix $R(n) = (\rho_{ij}(n))$ of W at any point in time is given by

$$\rho_{ij}(n) = \begin{cases} 1, & \text{if } i = j \\ \rho_h, & \text{if } i \text{ and } j \text{ are both in sector } h \\ \rho_*, & \text{if } i \text{ and } j \text{ are in different sectors.} \end{cases} \tag{6.3}$$

ρ_h characterizes the dependence of the counter-parties in sector h, and ρ_* rules the dependence on the global economic situation, common to all counter-parties. Hence, $\rho_h - \rho_*$ is a measure for the rise in dependence due to common membership in sector h.

For the remainder of the section we assume that

$$\rho_1 \;=\; \ldots \;=\; \rho_k \tag{6.4}$$

even if the value of k may vary. We now want to compare two portfolio structures with different concentrations specified by the number of counter-parties in sectors 1 to k. Let s and s' be two n-dimensional vectors with

$$s = (s_1, \ldots, s_r, 0, \ldots, 0), \quad s' = (s'_1, \ldots, s'_l, 0, \ldots, 0), \tag{6.5}$$

where $1 \leq r, l \leq n$ are fixed. r and l denote the number of sectors in these two different portfolio structures. $s_i, s'_j \in \mathbb{N}$ are the numbers of firms in sector i of the first structure and in sector j of the second structure, respectively. Of course, we require that

$$\sum_{i=1}^{n} s_i \;=\; \sum_{i=1}^{n} s'_i \;=\; n. \tag{6.6}$$

The next thing we need is an appropriate order relation to compare the structures s and s'. The following notion of majorization is best suited for this purpose.

Let $s, s' \in \mathbb{N}_0^n$ and denote by $s_{[1]} \geq \ldots \geq s_{[n]}$ the decreasing rearrangement of s, analogously for s'. We say that s' *majorizes* s ($s \prec s'$) if and only if

$$\sum_{i=1}^{m} s_{[i]} \leq \sum_{i=1}^{m} s'_{[i]}, \text{ for } m = 1, \ldots, n. \tag{6.7}$$

Intuitively speaking $s \prec s'$ means that in s' the sectors are larger. In order to assess how concentrations influence possible losses in a portfolio, we examine the size of the losses in homogeneous portfolios which only differ in their concentration structures. For given portfolio structures s and s' with numbers of sectors r and l, respectively, and a given non-increasing function $f \geq 0$ define random variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ by

$$X_i \;=\; f(W_1^{(i)}), \quad i = 1, \ldots, n, \tag{6.8a}$$

where $W^{(i)}$ is the Brownian motion specified by the structure s via (6.1), (6.2), (6.3), and (6.4), and

$$Y_i \;=\; f(W_1^{(i)}), \quad i = 1, \ldots, n, \tag{6.8b}$$

where $W^{(i)}$ is a different Brownian motion this time specified by s' the same way as for (6.8a). We interpret X_i and Y_i as losses of firm i in the portfolio structures s and s', respectively. With

$$L \;=\; \sum_{i=1}^{n} X_i \quad \text{and} \quad L' \;=\; \sum_{i=1}^{n} X_i \tag{6.9}$$

Theorem 3.2 from Bäuerle and Müller (1998) now states:

If $s \prec s'$, then

$$\mathbb{E}[\max(L - c, 0)] \;\leq\; \mathbb{E}[\max(L' - c, 0)] \tag{6.10}$$

for all thresholds $c \geq 0$.

This means that well-balanced portfolios are less risky in the sense that the expectation of the loss shortfall over a threshold c (c arbitrary) is lower. The least risky portfolio is given by the one which contains only companies of different sectors, whereas the most risky portfolio is that with all the firms in only one sector (Bäuerle and Müller, 1998, Corollary 3.3).

The following example is taken from Bäuerle and Müller (1998) and illustrates the effect of dependencies. In order to keep the computation simple, we suppose that in (6.8a) and (6.8b) we have $f = a\,\mathbf{1}_{(-\infty,z]}$ with $a = 4$ and z such that $\mathbb{P}[X_i = 1] = \mathbb{P}[Y_i = 1] = 0.06$ for all $i = 1, \ldots, n$. The portfolio consists of 20 risks ($n = 20$). For the correlations in (6.3) and (6.4) we assume $\rho_* = 0$ and $\rho_1 = \ldots = \rho_k = 1$, i.e., there is no global influence but linear dependence within the sectors. Bäuerle and Müller have computed the relative expected loss excesses for 8 different scenarios which are given by their sector group structures $s^{(m)}$, $m = 1, \ldots, 8$, listed in Table 1. Scenario 1 in Table 1 corresponds to the safest portfolio with 20 independent risks in different sectors. Scenario 8 is the most risky portfolio, where the same risk occurs 20 times. Note that not all portfolios can be compared with respect to the majorization ordering.

Table 2 now contains the relative expected loss excesses (the expectations from (6.10) divided by the value from the independent case $m = 1$) multiplied by 100 for several thresholds, i.e.,

$$100\,\frac{\mathbb{E}[\max(L^{(m)} - c, 0)]}{\mathbb{E}[\max(L^{(1)} - c, 0)]}. \tag{6.11}$$

Note that the expectation of the total loss (threshold = 0) equals 4.8 and the outcomes range between 0 and 80. We know that, given a threshold, the relative expected loss excess increases

scenario m	$s^{(m)}$
1	$(1,1,1,\ldots,1,1,1)$
2	(4,3,3,2,2,1,1,1,1,1,1)
3	(8,2,2,2,2,2,2)
4	(4,4,4,3,3,2)
5	(15,2,1,1,1)
6	(5,5,5,5)
7	(10,5,5)
8	(20)

Table 1: Portfolio structures for comparison of concentration effects

	scenario							
threshold	$s^{(1)}$	$s^{(2)}$	$s^{(3)}$	$s^{(4)}$	$s^{(5)}$	$s^{(6)}$	$s^{(7)}$	$s^{(8)}$
0	100	100	100	100	100	100	100	100
1	100	105	109	110	111	112	113	116
2	100	113	121	124	126	129	132	139
3	100	124	140	145	150	155	161	173
4	100	144	173	182	191	200	210	233
6	100	174	210	229	272	272	295	347
8	100	270	330	385	537	506	572	717
10	100	327	478	480	830	700	834	1128

Table 2: Relative expected loss excesses for different portfolio structures

in s. Table 2 shows that the increase is moderate if $s^{(i)}$ and $s^{(j)}$ are, in some sense, nearby as, for example, $s^{(6)}$ and $s^{(7)}$. In the cases where we were not able to establish the comparison theoretically, as, for example, for scenarios 5 and 6, we find that the order can change when the threshold increases.

An important conclusion to be drawn from the computation is that the increase in the relative expected loss excess can be dramatic in the presence of positive dependence. Even a minor occurrence of dependence, as in scenario 2, has a severe effect. Moreover, when a portfolio allows for positive dependence between the risks, the larger the number of risks becomes the higher will be the increase in relative expected loss excess.

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