Asymptotic behavior of the sample autocovariance and autocorrelation function of the AR(1) process with ARCH(1) errors

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Abstract

We study the sample autocovariance and autocorrelation function of the stationary AR(1) process with ARCH(1) errors. In contrast to ARCH and GARCH processes, AR(1) processes with ARCH(1) errors can not be transformed into solutions of linear stochastic recurrence equations. However, we show that they still belong to the class of stationary sequences with regular varying finite-dimensional distributions and therefore the theory of Davis and Mikosch (1998) can be applied.

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1 Introduction

Over the last two decades, there has been a great deal of interest in modelling real data using time series models which exhibit features such as long range dependence, nonlinearity and heavy tails. Many data sets in econometrics, finance or telecommunications have these common characteristics. In particular, they appear to be reconcilable with the assumption of heavy-tailed marginal distributions. Examples are file lengths, CPU time to complete a job or length of on/off cycles in telecommunications and logreturns of stock indices, share prices or exchange rates in finance.

The feature of nonlinearity can be often detected by considering the sample autocorrelation functions (ACFs) of a time series, their absolute values and squares. The reason is the following. A heavy tailed time series that can be represented as an infinite moving average process has the property that the sample ACF at lag h converges in probability to a constant $\rho(h)$ although the mathematical correlation typically does not exist (Davis and Resnick (1985),(1986)). However, for many nonlinear heavy tailed sequences the sample ACF at lag h converges in distribution to a nondegenerate random variable. In Resnick and Van den Berg (1998) a test for (non)linearity of a given infinite variance time series is proposed, based on subsample stability of the sample ACF.

The phenomenon of random limits of sample ACFs was observed first in the context of infinite variance bilinear processes by Davis and Resnick (1996) and Resnick (1997). Davis and Mikosch (1998) studied the weak limit behavior for a large variety of nonlinear processes with regularly varying marginal distributions which satisfy a weak mixing condition and some additional technical assumptions. It is shown in their article that the sample autocovariance function (ACVF) and ACF of such processes with infinite 4th but finite second moments have a rate of convergence to the true ACVF and ACF that become slower the closer the marginal distributions are to an infinite second moment. In cases of an infinite second moment, the limits of the sample ACVF and ACF are nondeterministic. Processes which belong to the framework of Davis and Mikosch (1998) are the ARCH(1) processes, the simple bilinear processes with light-tail noise (Basrak, Davis and Mikosch (1999)) and the GARCH(1,1) processes (Mikosch and Starica (1999)). Finally, Davis, Mikosch and Basrak (1999) embedded the three aforementioned processes to a larger class of processes which still satisfy the conditions for the theory of Davis and Mikosch (1998). These processes have basically the property that they can be transformed to solutions of multivariate linear stochastic recurrence equations. Linear stochastic recurrence equations of this form were considered by Kesten (1973) and Vervaat (1979) and include the important family of the squares of GARCH processes.

The aim of this paper is to apply the general theory of Davis and Mikosch (1998) to a different type of processes with different structure than considered in Davis, Mikosch and Basrak (1999), namely the autoregressive (AR) processes of order 1 with autoregressive conditional heteroscedastic one (ARCH(1)) errors. The class of AR (or more general ARMA) models with ARCH errors were first proposed by Weiss (1984). In the paper of Weiss, they were found to be successful in modelling thirteen different U.S. macroeconomic time series. AR models with ARCH errors are one of the simplest examples of models that can be written by a random recurrence equation of the form

$$X_t = \mu_t + \sigma_t \varepsilon_t\,, \quad t \in \mathbb{N}\,, \tag{1.1}$$

where ε_t are iid innovations with mean zero, μ_t is the conditional expectation of X_t (which may or may not depend on t) and the volatility σ_t describes the change of (conditional) variance. Because of the nonconstant conditional variance models of the form (1.1) are often referred to as conditional heteroscedastic models. Empirical work has confirmed that such models fit many types of financial data (log-returns, exchange rate, etc.). In this paper, we concentrate on the AR(1) process with ARCH(1) in order to have a Markov structure and hence make the model analytically tractable. It is defined by specifying μ_t and σ_t as follows:

$$\mu_t = \alpha X_{t-1} \quad \text{and} \quad \sigma_t^2 = \beta + \lambda X_{t-1}^2\,, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ and $\beta, \lambda > 0$. Note that for $\alpha = 0$ we get just the ARCH(1) model introduced by Engle (1982).

The research of the sample ACVF and ACF of the AR(1) process with ARCH(1) errors is motivated by the following. The AR(1) process with ARCH(1) errors is a natural mixture between an AR(1) and an ARCH(1) process. Therefore results of this paper can be seen as a generalization of results for the aforementioned two processes. The weak limit behavior of the ARCH(1) process was studied by Davis and Mikosch (1998). For $\lambda = 0$, the process defined by (1.1) and (1.2) is an AR(1) process. A summary of results about the asymptotical theory of the sample ACFs of AR processes can be found for instance in Brockwell and Davis (1990), Chapter 7.2 and 13.3, or Embrechts, Klüppelberg and Mikosch (1997), Chapter 7.3.

AR(1) processes with ARCH(1) errors are not solutions of linear stochastic recurrence equations and there is also no obvious way how to transform them to such equations. However, we show that the processes still belong to stationary weak dependent sequences which are jointly regularly varying. One conclusion of this paper is that AR(1) processes with ARCH(1) errors serve as one of the simplest examples of sequences which do not fulfill the framework in Davis, Mikosch and Basrak (1999) but to which the theory of Davis and Mikosch (1998) can still be applied.

The paper is organized as follows. In Section2 we introduce the AR(1) model with ARCH(1) errors and consider some basic theoretical properties of it. The weak convergence of the point processes associated with the sequences (X_t), $(|X_t|)$ and (X_t^2) is investigated in Section 3. Finally, in Section 4 we present the results concerning the weak convergence of the sample ACVF and ACF of the AR(1) process with ARCH(1) errors, the absolute and squared values.

2 Preliminaries

We consider an autoregressive model of order 1 with autoregressive conditional heteroscedastic errors of order 1 (AR(1) model with ARCH(1) errors) which is defined by the stochastic difference equation

$$X_t = \alpha X_{t-1} + \sqrt{\beta + \lambda X_{t-1}^2}\varepsilon_t\,, \quad t \in \mathbb{N}, \tag{2.1}$$

where (ε_t) are i.i.d. random variables, $\alpha \in \mathbb{R}$, $\beta, \lambda > 0$ and the parameters α and λ satisfy in addition the inequality

$$E(\ln|\alpha + \sqrt{\lambda}\,\varepsilon|) < 0\,. \tag{2.2}$$

This condition is necessary and sufficient for the existence and uniqueness of a stationary distribution. Here ε is a generic random variable with the same distribution as ε_t. In what follows we assume the same conditions for ε as in Borkovec and Klüppelberg (1998). These are the so-called *general conditions*:

$$\begin{aligned} &\varepsilon \text{ is symmetric with continuous Lebesgue density } p(x)\,, \\ &\varepsilon \text{ has full support } \mathbb{R}\,, \\ &\text{the second moment of } \varepsilon \text{ exists}\,, \end{aligned} \tag{2.3}$$

and the *technical conditions* $(D.1) - (D.3)$:

(D.1) $p(x) \geq p(x')$ for any $0 \leq x < x'$.

(D.2) For any $c \geq 0$ there exists a constant $q = q(c) \in (0,1)$ and functions $f_+(c,\cdot)$, $f_-(c,\cdot)$ with $f_+(c,x), f_-(c,x) \to 1$ as $x \to \infty$ such that for any $x > 0$ and $t > x^q$

$$p(\frac{x+c+\alpha t}{\sqrt{\beta+\lambda t^2}}) \geq p(\frac{x+\alpha t}{\sqrt{\beta+\lambda t^2}})\, f_+(c,x)\,,$$
$$p(\frac{x+c-\alpha t}{\sqrt{\beta+\lambda t^2}}) \geq p(\frac{x-\alpha t}{\sqrt{\beta+\lambda t^2}})\, f_-(c,x)\,.$$

(D.3) There exists a constant $\eta > 0$ such that

$$p(x) = o(x^{-(N+1+\eta+3q)/(1-q)})\,, \quad \text{as } x \to \infty\,,$$

where $N := \inf\{u \geq 0\,;\, E(|\sqrt{\lambda}\varepsilon|^u) > 2\}$ and q is the constant in $(D.2)$.

There exists a wide class of distributions which satisfy these assumptions. Examples are the normal distribution, the Laplace distribution or the Students distribution. Conditions $(D.1) - (D.3)$ are necessary for determing the tail of the stationary distribution. For further details concerning the conditions and examples we refer to Borkovec and Klüppelberg (1998). Note that the process $(X_n)_{n\in\mathbb{N}}$ is evidently a homogeneous Markov chain with state space $\mathbb{R}$ equipped with the Borel σ-algebra. The next theorem collects some results on (X_t).

Theorem 2.1 *Consider the process (X_t) in (2.1) with (ε_t) satisfying the general conditions (2.3) and with parameters α and λ satisfying (2.2). Then the following assertions hold:*

(a) (X_t) is geometric ergodic. In particular, (X_t) has a unique stationary distribution and satisfies the strong mixing condition with geometric rate of convergence $\gamma_X(h)$, $h \geq 0$. The stationary df is continuous and symmetric.

(b) Let $\overline{F}(x) = P(X > x)$, $x \geq 0$, be the right tail of the stationary df and the conditions $(D.1) - (D.3)$ are in addition fulfilled. Then

$$\overline{F}(x) \sim c\, x^{-\kappa}\,, \quad x \to \infty\,, \tag{2.4}$$

where

$$c = \frac{1}{2\kappa}\frac{E\left(\left|\alpha|X| + \sqrt{\beta+\lambda X^2}\varepsilon\right|^\kappa - \left|(\alpha+\sqrt{\lambda}\varepsilon)|X|\right|^\kappa\right)}{E\left(|\alpha+\sqrt{\lambda}\varepsilon|^\kappa \ln|\alpha+\sqrt{\lambda}\varepsilon|\right)} \tag{2.5}$$

and κ is given as the unique positive solution to

$$E(|\alpha+\sqrt{\lambda}\varepsilon|^\kappa) = 1\,. \tag{2.6}$$

Furthermore, the unique positive solution κ is less than 2 if and only if $\alpha^2 + \lambda\, E(\varepsilon^2) > 1$.

Remark 2.2 (a) Note that $E(|\alpha+\sqrt{\lambda}\varepsilon|^\kappa)$ is a function of κ,α and λ. It can be shown that for fixed λ, the exponent κ is decreasing in $|\alpha|$. This means that the distribution of X gets heavier tails when $|\alpha|$ increases. In particular, the AR(1) process with ARCH(1) errors has for $\alpha\neq 0$ heavier tails than the ARCH(1) process.

(b) The strong mixing property includes automatically that the sequence (X_t) satisfies the conditions $\mathcal{A}(a_n)$. The condition $\mathcal{A}(a_n)$ is a frequently used mixing condition in connection with point process theory and was introduced by Davis and Hsing (1995). See (3.7) for the definition. $\square$

In order to investigate the limit behavior of the sample ACVF and ACF of (X_t) we define three auxiliary processes (Y_t), $(\widetilde{X}_t)$ and $(\widetilde{Z}_t)$ as follows: let (Y_t) and $(\widetilde{X}_t)$ be the processes given by the random recurrence equations

$$Y_t=\alpha Y_{t-1}+\sqrt{\lambda Y_{t-1}^2}\,\varepsilon_t\,,\quad t\in\mathbb{N}, \tag{2.7}$$

and

$$\widetilde{X}_t=|\alpha\widetilde{X}_{t-1}+\sqrt{\beta+\lambda\widetilde{X}_{t-1}^2}\varepsilon_t|\,,\quad t\in\mathbb{N}, \tag{2.8}$$

where the notation is the same as in (2.1), $Y_0=X_0$, $\widetilde{X}_0=|X_0|$ a.s. and set

$$\widetilde{Z}_t=\ln(\widetilde{X}_t^2)\,,\quad t\in\mathbb{N}.$$

It is easy to see that the process (Y_t) is not stationary (or at least not non-trivial stationary). However, for $Y_t\approx X_t$ large and $M\in\mathbb{N}$ fixed, we will see that the sequence $Y_t,Y_{t+1},...,Y_{t+M}$ behaves approximately as $X_t,X_{t+1},...,X_{t+M}$ (see also Figure 1). This fact will be very important in order to establish joint regular variation of $X_0,X_1,...,X_M$.

Because of the symmetry of (ε_t), the independence of ε_t and X_{t-1} in (2.1) and the homogeneous Markov structure of (X_t) and $(\widetilde{X}_t)$ it is readily seen that $(\widetilde{X}_t)\stackrel{d}{=}(|X_t|)$. Studying the process $(\widetilde{X}_t)$ instead of $(|X_t|)$ can be often much more convenient. In particular, since $(\widetilde{X}_t)$ follows (2.8) the process $(\widetilde{Z}_t)$ satisfies the stochastic difference equation

$$\widetilde{Z}_t=\widetilde{Z}_{t-1}+\ln\Big((\alpha+\sqrt{\beta\,e^{-\widetilde{Z}_{t-1}}+\lambda}\,\varepsilon_t)^2\Big)\,,\quad t\in\mathbb{N}, \tag{2.9}$$

where $\widetilde{Z}_0$ equals $\ln(X_0^2)$ a.s.. Note that $(\widetilde{Z}_t)\stackrel{d}{=}(\ln(X_t^2))$. Moreover, $(\widetilde{Z}_t)$ does not depend on the sign of the parameter α since ε_t is symmetric. The following lemma shows that $(\widetilde{Z}_t)$ can be bounded above a high threshold by a random walk with negative drift. The proof of this result

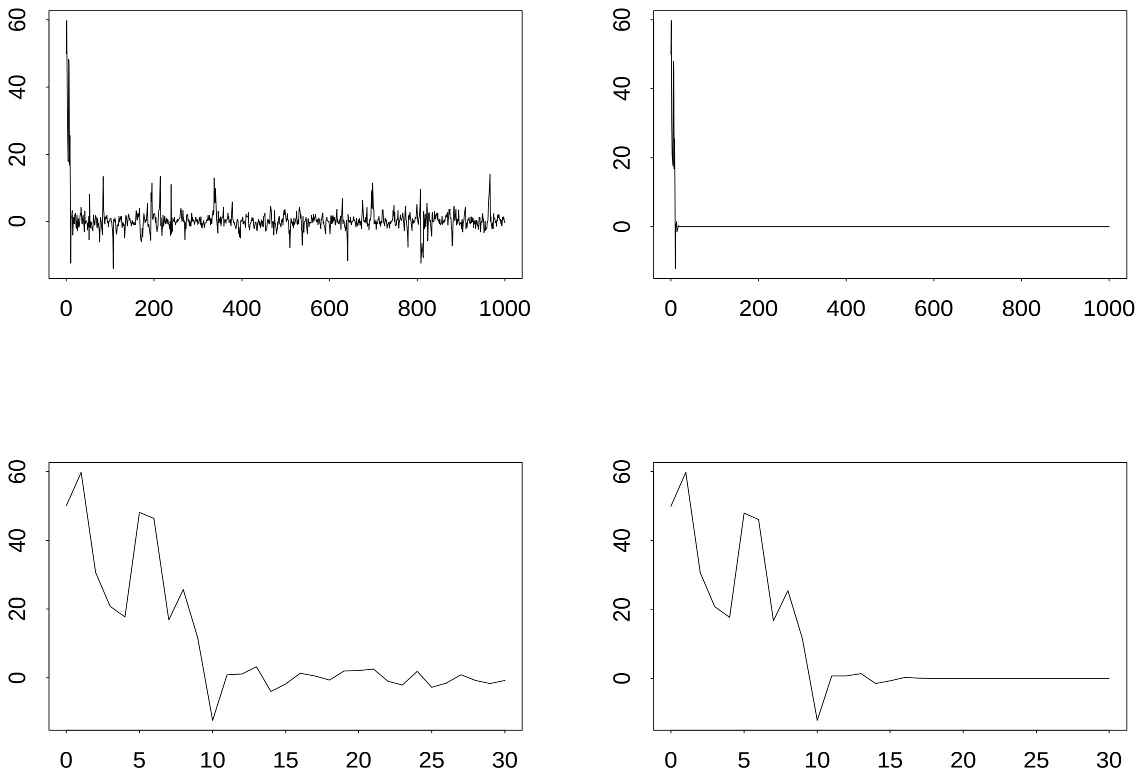

Figure 1: Simulated sample path of (X_t) with initial value $X_0 = 50$ and parameters $\alpha = 0.6, \beta = 1, \lambda = 0.6$ (left) and of (Y_t) with the same initial value and parameters (right) in the case $\varepsilon \sim N(0,1)$. Both simulations are based on the same simulated noise sequence (ε_t). The pictures demonstrate that the processes behave similar for large values.

can be found in Borkovec (2000) and is based basically on the recurrence structure of $(\widetilde{Z}_t)$ in (2.9). The result is crucial for proving Proposition 3.1 .

Lemma 2.3 *Let a be large enough, $N_a := \inf\{\nu \geq 1 \,|\, \widetilde{Z}_\nu \leq a\}$ and $\widetilde{Z}_0 > a$. Then*

$$\widetilde{Z}_t \leq \widetilde{Z}_0 + S_t^a \quad \textit{for any } t \leq N_a \textit{ a.s.}, \tag{2.10}$$

where S_t^a is random walk with negative drift given by

$$S_0^a = 0 \quad \textit{and} \quad S_t^a = S_{t-1}^a + \ln\Big((\alpha + \sqrt{\beta\, e^{-a} + \lambda}\, \varepsilon_t)^2\Big) + \ln\Big(1 - \frac{2\alpha\sqrt{\beta} e^{-a/2} \varepsilon_t}{(\alpha + \sqrt{\beta\, e^{-a} + \lambda}\, \varepsilon_t)^2} 1_{\{\varepsilon_t < 0\}}\Big)\,.$$

Moreover, for $a \uparrow \infty$, we have

$$S_t^a \stackrel{a.s.}{\to} S_t\,,$$

where $S_0 = 0$ *and* $S_t = S_{t-1} + \ln(\alpha + \sqrt{\lambda}\,\varepsilon_t)^2$.

$\square$

3 Weak convergence of some point processes associated with the AR(1) process with ARCH(1) errors

In this section we formulate results on the weak convergence of some point processes of the form

$$N_n = \sum_{t=1}^{n} \delta_{\mathbf{X}_t^{(m)}/a_n}, \quad n = 1, 2, ..., \tag{3.1}$$

where $\mathbf{X}_t^{(m)}$ are random row vectors with dimension $m+1 \in \mathbb{N}$ arbitrary whose components are highly related to the AR(1) process with ARCH(1) errors (X_t) defined in the previous section and (a_n) is a normalizing sequence of real-valued numbers. The main result in this section is summarized in Theorem 3.3. The proof of this result is basically an application of the theory in Davis and Mikosch (1998). Proposition 3.1 collects some properties of $(\mathbf{X}_t^{(m)})$ which we need for the proof of Theorem 3.3.

We follow the notation and the point process theory in Davis and Mikosch (1998) and Kallenberg (1983), respectively. The state space of the point processes considered is $\overline{\mathbb{R}}^{m+1} \setminus \{\mathbf{0}\}$. Write $\mathcal{M}$ for the collection of Radon counting measures on $\overline{\mathbb{R}}^{m+1} \setminus \{\mathbf{0}\}$ with null measure o. This means that $\mu \in \mathcal{M}$ if and only if μ is of the form $\mu = \sum_{i=1}^{\infty} n_i \delta_{\mathbf{x}_i}$, where $n_i \in \{1, 2, 3, ...\}$ and $\mathbf{x}_i \in \overline{\mathbb{R}}^{m+1} \setminus \{\mathbf{0}\}$ distinct and $\#\{i \,|\, |\mathbf{x}_i| > y\} < \infty$ for all $y > 0$.

In what follows we suppose that (X_t) is the stationary AR(1) process with ARCH(1) given by (2.1). (ε_t) satisfies the general conditions (2.3) and $(D.1) - (D.3)$ and the parameters α and λ are chosen such that (2.2) holds. We start by specifying the random row vectors $(\mathbf{X}_t^{(m)})$ and the normalising constants (a_n) in (3.1) and by introducing some auxilary quantities in order to be in the framework of Davis and Mikosch (1998). For $m \in \mathbb{N}_0$, define

$$\begin{aligned}
\mathbf{X}_t^{(m)} &= (X_t, X_{t+1}, ..., X_{t+m}), \quad t \in \mathbb{Z}, \\
\mathbf{Z}_0^{(m)} &= \left(r_0, (\alpha r_0 + \sqrt{\lambda}\varepsilon_1), ..., (\alpha r_0 + \sqrt{\lambda}\varepsilon_1) \textstyle\prod_{s=1}^{m-1}(\alpha + \sqrt{\lambda} r_s \varepsilon_{s+1})\right)
\end{aligned}$$

and

$$\mathbf{Z}_t^{(m)} = (\alpha r_0 + \sqrt{\lambda}\varepsilon_1) \left(\prod_{s=1}^{t-1} (\alpha + \sqrt{\lambda} r_s \varepsilon_{s+1}), ..., \prod_{s=1}^{m+t-1} (\alpha + \sqrt{\lambda} r_s \varepsilon_{s+1}) \right), \quad t \in \mathbb{N},$$

where $r_s = \operatorname{sign}(Y_s)$ is independent of $|Y_s|$, (Y_s) is the process in (2.7) and $\prod_{i=1}^{0} = 1$. Besides, for $k \in \mathbb{N}_0$ arbitrary but fixed, define the stochastic vectors

$$\mathbf{X}_{-k}^{(m)}(2k+1) = (\mathbf{X}_{-k}^{(m)}, \mathbf{X}_{-k+1}^{(m)}, ..., \mathbf{X}_{k}^{(m)})$$

and

$$\mathbf{Z}_{0}^{(m)}(2k+1) = (\mathbf{Z}_{0}^{(m)}, \mathbf{Z}_{1}^{(m)}, ..., \mathbf{Z}_{2k}^{(m)})\,.$$

Analogously to Davis and Mikosch (1998) we take $|\cdot|$ to be the max-norm in $\mathbb{R}^{m+1}$, i.e.

$$|\mathbf{x}| = |(x_0, ..., x_m)| = \max_{i=0,\dots,m} |x_i|\,.$$

Now we are able to define the sequence (a_n) in (3.1). Let $(a_n^{(k,m)})$ be a sequence of positive numbers such that

$$P(|X| > a_n^{(k,m)}) \sim (n\, E(|\mathbf{Z}_0^{(m)}(2k+1)|^{\kappa}))^{-1}\,, \quad \text{as } n \to \infty\,. \tag{3.2}$$

For $k = 0$, we write $a_n = a_n^{(0,m)}$ in the following. Note that because of (2.4) one can choose $a_n^{(k,m)}$ as

$$a_n^{(k,m)} = \Big(2\, c\, E(|\mathbf{Z}_0^{(m)}(2k+1)|^{\kappa})\Big)^{1/\kappa}\, n^{1/\kappa}\,, \quad n \geq 1\,. \tag{3.3}$$

With this notation we can state the following proposition.

Proposition 3.1 *Let (X_t) be the stationary AR(1) process with ARCH(1) given by (2.1) and and assume that the conditions of Theorem 2.1 hold. Then*

(a) $(\mathbf{X}_t^{(m)})$ is strongly mixing with a geometric rate of convergence. To be more specific, there exist constants $\rho \in (0,1)$ and $C > 0$ such that for any $h \geq m$

$$\begin{aligned}\sup_{A\in\sigma(\mathbf{X}_s^{(m)},\, s\leq 0),\, B\in\sigma(\mathbf{X}_s^{(m)},\, s\geq h)} |P(A\cap B) - P(A)\,P(B)| \quad &=: \quad \gamma_{\mathbf{X}^{(m)}}(h) \\ &= \quad \gamma_X(h-m) \quad \leq \quad C\,\rho^{h-m}\,.\end{aligned}$$

(b) $\mathbf{X}_{-k}^{(m)}(2k+1)$ is jointly regularly varying with index $\kappa > 0$, more precisely

$$\begin{aligned}& n\,P(|\mathbf{X}_{-k}^{(m)}(2k+1)| > t\,a_n^{(k,m)}, \mathbf{X}_{-k}^{(m)}(2k+1)/|\mathbf{X}_{-k}^{(m)}(2k+1)| \in \cdot\,) \\ & \quad \xrightarrow{v} \quad t^{-\kappa}\, E(|\mathbf{Z}_0^{(m)}(2k+1)|^{\kappa}\, 1_{\{\mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)|\in\,\cdot\,\}})/E(|\mathbf{Z}_0^{(m)}(2k+1)|^{\kappa})\,, \quad t > 0\,,\end{aligned} \tag{3.4}$$

as $n \to \infty$, where the symbol $\xrightarrow{v}$ stands for vague convergence.

(c) Let (p_n) be an increasing sequence such that

$$\frac{p_n}{n} \to 0 \quad \textit{and} \quad \frac{n\gamma_{\mathbf{X}^{(m)}}(\sqrt{p_n})}{p_n} \to 0 \quad \textit{as } n \to \infty\,. \tag{3.5}$$

Then for any $y > 0$

$$\lim_{p\to\infty} \limsup_{n\to\infty} P\left(\bigvee_{p\le|t|\le p_n} |\mathbf{X}_t^{(m)}| > a_n y \,\middle|\, |\mathbf{X}_0^{(m)}| > a_n y \right) = 0\,. \tag{3.6}$$

Remark 3.2 *(a) In the spirit of Davis and Mikosch (1998) the jointly regular varying property of $\mathbf{X}_{-k}^{(m)}(2k+1)$ can also be expressed in the more familiar way*

$$n\,P(|\mathbf{X}_{-k}^{(m)}(2k+1)| > t\,a_n^{(k,m)}, \mathbf{X}_{-k}^{(m)}(2k+1)/|\mathbf{X}_{-k}^{(m)}(2k+1)| \in \cdot\,) \xrightarrow{v} t^{-\kappa}\,P_{\mathbf{\Theta}}(\,\cdot\,)\,, \quad \textit{as } n \to \infty\,,$$

where $P_{\mathbf{\Theta}}(\,\cdot\,) = \widetilde{P} \circ (\theta_{-k}^{(k)}, ..., \theta_k^{(k)})^{-1}$, $\theta_j^{(k)} = \mathbf{Z}_{k+j}^{(m)}/|\mathbf{Z}_0^{(m)}(2k+1)|$, $j = -k, ..., k$, and $d\widetilde{P} = |\mathbf{Z}_0^{(m)}(2k+1)|^\kappa / E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa)\,dP$. In the following we will basically use this notation.
(b) Due to statement (b) in Proposition 3.1 the positive sequence (a_n) in (3.2) with $k = 0$ can be also characterized by

$$\lim_{n\to\infty} n\,P(|\mathbf{X}_0^{(m)}| > a_n) = 1\,.$$

Hence a_n can be interpreted as the (approximated) $(1-n^{-1})$-quantile of $\mathbf{X}_0^{(m)}$.
(c) In the case of a strong mixing process, the conditions in (3.5) are sufficient to guarantee that (p_n) is a $\mathcal{A}(a_n)$-separating sequence, i.e.

$$E\exp\left(-\sum_{t=1}^{n} f(\mathbf{X}_t^{(m)}/a_n)\right) - \left(E\exp\left(-\sum_{t=1}^{p_n} f(\mathbf{X}_t^{(m)}/a_n)\right)\right)^{k_n} \to 0\,, \quad \textit{as } n \to \infty\,, \tag{3.7}$$

where $k_n = [n/p_n]$ and f is an arbitrary bounded non-negative step function on $\overline{\mathbb{R}}^m \setminus \{\mathbf{0}\}$. Note that (p_n) is in the case of a strong mixing process independent of (a_n).

Proof. (a) This is an immediate consequence of the strong mixing property of (X_n) and the fact that strong mixing is a property of the underlying σ-field.
(b) Fix $t > 0$ and let $\delta > 0$ be small enough such that $t - 2\delta > 0$. Moreover, choose $B \in \mathcal{B}(\mathcal{S}^{(2k+1)(m+1)-1})$ arbitrary, where $\mathcal{S} := \mathcal{S}^{(2k+1)(m+1)-1}$ denotes the unit sphere in $\mathbb{R}^{(2k+1)(m+1)}$ with respect to the max-norm $|\cdot|$. Define $B_\delta = \{\mathbf{x} \in \mathcal{S} \,|\, \exists\, \mathbf{y} \in B : |\mathbf{x}-\mathbf{y}| \le 2\delta/(t-2\delta)\}$ and $B_{-\delta} = \{\mathbf{x} \in B \,|\, |\mathbf{x}-\mathbf{y}| \ge 2\delta/(t-2\delta)\ \ \forall \mathbf{y} \in \mathcal{S} \setminus B\}$. Note that $B_{-\delta} \subseteq B \subseteq B_\delta$.
Next set $\mathbf{Y}_t^{(m)} := (Y_t, Y_{t+1}, ..., Y_{t+m})$, $t \in \mathbb{N}_0$, and $\mathbf{Y}_0^{(m)}(2k+1) = (\mathbf{Y}_0^{(m)}, \mathbf{Y}_1^{(m)}, ..., \mathbf{Y}_{2k}^{(m)})$, where

(Y_t) is the process given in (2.7). Using the definition of the process (Y_t) and of the stochastic vectors $\mathbf{Z}_t^{(m)}$, it can be readily seen that

$$\mathbf{Y}_0^{(m)}(2k+1) = |X_0|\,\mathbf{Z}_0^{(m)}(2k+1)\,. \tag{3.8}$$

The basic idea of proving (3.4) is now to approximate $\mathbf{X}_{-k}^{(m)}(2k+1)$ by $\mathbf{Y}_0^{(m)}(2k+1)$. Because of the stationarity of (X_t) it is sufficient to compare $\mathbf{X}_0^{(m)}(2k+1)$ with $\mathbf{Y}_0^{(m)}(2k+1)$. First we bound the probability in the left hand side of (3.4) from above as follows.

$$\begin{aligned}
&n\,P\Big(|\mathbf{X}_{-k}^{(m)}(2k+1)| > t\,a_n^{(k,m)},\ \mathbf{X}_{-k}^{(m)}(2k+1)/|\mathbf{X}_{-k}^{(m)}(2k+1)| \in B\Big) \\
&= \quad n\,P\Big(|\mathbf{X}_{0}^{(m)}(2k+1)| > t\,a_n^{(k,m)},\ \mathbf{X}_{0}^{(m)}(2k+1)/|\mathbf{X}_{0}^{(m)}(2k+1)| \in B\Big) \\
&\leq \quad n\,P\Big(|\mathbf{X}_{0}^{(m)}(2k+1) - \mathbf{Y}_0^{(m)}(2k+1)| > \delta\,a_n^{(k,m)}\Big) \\
&\qquad + n\,P\Big(|\mathbf{X}_{0}^{(m)}(2k+1)| > t\,a_n^{(k,m)},\ \mathbf{X}_{0}^{(m)}(2k+1)/|\mathbf{X}_{0}^{(m)}(2k+1)| \in B\,, \\
&\qquad\qquad |\mathbf{X}_{0}^{(m)}(2k+1) - \mathbf{Y}_0^{(m)}(2k+1)| \leq \delta\,a_n^{(k,m)}\Big) \\
&\leq \quad n\,P\Big(|\mathbf{X}_{0}^{(m)}(2k+1) - \mathbf{Y}_0^{(m)}(2k+1)| > \delta\,a_n^{(k,m)}\Big) \\
&\qquad + n\,P\Big(|\mathbf{Y}_{0}^{(m)}(2k+1)| > (t-\delta)\,a_n^{(k,m)},\ \mathbf{Y}_{0}^{(m)}(2k+1)/|\mathbf{Y}_{0}^{(m)}(2k+1) \in B_\delta\,, \\
&\qquad\qquad |\mathbf{Y}_{0}^{(m)}(2k+1) - \mathbf{Y}_0^{(m)}(2k+1)| \leq \delta\,a_n^{(k,m)}\Big) \\
&=: \quad (I_1) + (I_2)\,,
\end{aligned}$$

where the last inequality follows from the fact that for any $\mathbf{x},\mathbf{y} \in \mathbb{R}^{(2k+1)(m+1)}$ the inequalities $|\mathbf{x}-\mathbf{y}| \leq \delta a_n^{(k,m)}$ and $|\mathbf{x}| > t a_n^{(k,m)}$ imply $|\mathbf{y}| > (t-\delta)a_n^{(k,m)}$, $|\mathbf{x}/|\mathbf{x}| - \mathbf{y}/|\mathbf{x}|| \leq \delta/t$ and $||\mathbf{y}|/|\mathbf{x}| - 1| \leq \delta/(t-2\delta)$. The rest is a triangular argument.

First, we consider (I_1). By the definition of the max-norm and due to the Boolean and Markov inequalities, we derive

$$\begin{aligned}
(I_1) &= n\,P\big(\max_{0\leq s\leq 2k+m} |X_s - Y_s| > \delta a_n^{(k,m)}\big) \\
&= n\,P\big(\max_{0\leq s\leq 2k+m} |\sqrt{\beta + \lambda\,X_{s-1}^2} - \sqrt{\lambda\,X_{s-1}^2}|\,|\varepsilon_s| > \delta a_n^{(k,m)}\big) \\
&\leq n\,P\big(\max_{0\leq s\leq 2k+m} |\varepsilon_s| > \delta a_n^{(k,m)}/\sqrt{\beta}\big) \\
&\leq (2k+m+1)\,\frac{n\,E(|\varepsilon|^{\kappa+\nu})}{(\delta a_n^{(k,m)}/\sqrt{\beta})^{\kappa+\nu}},
\end{aligned}$$

where $\nu > 0$ is chosen such that $E(|\varepsilon|^{\kappa+\nu}) < \infty$. This is possible because of the assumption $(D.3)$. Using (3.3) the right hand side converges to zero.

Now we estimate (I_2). By (3.8) , we have

$$\begin{aligned}(I_2) &\leq n\,P\Big(|X_0|\,|\mathbf{Z}_0^{(m)}(2k+1)| > (t-\delta)a_n^{(k,m)},\ \mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)| \in B_\delta\Big)\\ &= n\,P\Big(|X_0|\,|\mathbf{Z}_0^{(m)}(2k+1)|\,1_{\{\mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)|\in B_\delta\}} > (t-\delta)a_n^{(k,m)}\Big)\end{aligned}$$

Note that $|X_0|$ and $\mathbf{Z}_0^{(m)}(2k+1)$ are independent, nonnegative random variables. Moreover, $|X_0|$ is regularly varying with index $\kappa > 0$ and $E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa) < \infty$. Thus, a result of Breiman (1965) yields that (I_2) behaves asymptotically as

$$\begin{aligned}&n\,P\Big(|X_0| > (t-\delta)a_n^{(k,m)}\Big)\,E\Big(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa\,1_{\{\mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)|\in B_\delta\}}\Big)\\ &\sim\ (t-\delta)^{-\kappa}\frac{E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa\,1_{\{\mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)|\in B_\delta})}{E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa)}\,,\quad \text{as } n\to\infty\,,\end{aligned}$$

where we used in the second line (3.2). Because $\delta > 0$ is arbitrary and $B_\delta \downarrow B$ as $\delta \downarrow 0$ we have found that

$$\begin{aligned}&\limsup_{n\to\infty} n\,P\Big(|\mathbf{X}_{-k}^{(m)}(2k+1)| > t\,a_n^{(k,m)},\mathbf{X}_{-k}^{(m)}(2k+1)/|\mathbf{X}_{-k}^{(m)}(2k+1)| \in\ \cdot\Big)\\ &\quad\leq\ t^{-\kappa}\,E\Big(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa\,1_{\{\mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)|\in\,\cdot\,\}})/E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa)\Big)\,. \qquad (3.9)\end{aligned}$$

Next we proceed to establish that the inequality (3.9) also holds in the converse direction for $\liminf$. With similar arguments as above, we have

$$\begin{aligned}&n\,P\Big(|\mathbf{X}_{-k}^{(m)}(2k+1)| > t\,a_n^{(k,m)},\ \mathbf{X}_{-k}^{(m)}(2k+1)/|\mathbf{X}_{-k}^{(m)}(2k+1)| \in\ B\Big)\\ &\geq\ n\,P\Big(|\mathbf{Y}_0^{(m)}(2k+1)| > (t+\delta)\,a_n^{(k,m)},\ \mathbf{Y}_0^{(m)}(2k+1)/|\mathbf{Y}_0^{(m)}(2k+1) \in\ B_{-\delta}\,,\\ &\qquad\qquad |\mathbf{X}_0^{(m)}(2k+1) - \mathbf{Y}_0^{(m)}(2k+1)| \leq \delta\,a_n^{(k,m)}\Big)\\ &\geq\ n\,P\Big(|\mathbf{Y}_0^{(m)}(2k+1)| > (t+\delta)\,a_n^{(k,m)},\ \mathbf{Y}_0^{(m)}(2k+1)/|\mathbf{Y}_0^{(m)}(2k+1) \in\ B_{-\delta}\Big)\\ &\qquad\qquad -\ n\,P\Big(|\mathbf{X}_0^{(m)}(2k+1) - \mathbf{Y}_0^{(m)}(2k+1)| > \delta\,a_n^{(k,m)}\Big)\\ &\sim\ (t+\delta)^{-\kappa}\frac{E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa\,1_{\{\mathbf{Z}_0^{(m)}(2k+1)/|\mathbf{Z}_0^{(m)}(2k+1)|\in B_\delta\}})}{E(|\mathbf{Z}_0^{(m)}(2k+1)|^\kappa)}\,,\quad \text{as } n\to\infty\,.\end{aligned}$$

Since again $\delta > 0$ is arbitrary the statement follows.

(c) We start by rewriting the probability in statement (c).

$$\begin{aligned}&P\Big(\bigvee_{p\leq|t|\leq p_n} |\mathbf{X}_t^{(m)}(2k+1)| > a_n y\ \Big|\ |\mathbf{X}_0^{(m)}(2k+1)| > a_n y\Big)\\ &=\ P\Big(\max_{-p_n\leq t\leq -p+2k+1}|X_t| > a_n y\ \Big|\ \max_{0\leq j\leq 2k+1}|X_j| > a_n y\Big)\\ &\qquad +\ P\Big(\max_{p\leq t\leq p_n+2k+1}|X_t| > a_n y\ \Big|\ \max_{0\leq j\leq 2k+1}|X_j| > a_n y\Big)\\ &=:\ (J_1)\ +\ (J_2)\,.\end{aligned}$$

In what follows we consider only (J_1). (J_2) can be treated in a similar way. First note that

$$
\begin{aligned}
(J_1) &\le \sum_{j=0}^{2k+1} \frac{P(\max_{-p_n\le t\le -p+2k+1}|X_t|>a_ny, |X_j|>a_ny)}{P(|X_j|>a_ny)} \frac{P(|X_j|>a_ny)}{P(\max_{0\le j\le 2k+1}|X_j|>a_ny)} \\
&\le \sum_{j=0}^{2k+1} P\Big(\max_{-p_n-j\le t\le -p+2k+1-j}|X_t|>a_ny \,\Big|\, |X_0|>a_ny\Big) \\
&\le 2(k+1)\, P\Big(\max_{-p_n-(2k+1)\le t\le -p+2k+1}|X_t|>a_ny \,\Big|\, |X_0|>a_ny\Big) \\
&\le 2(k+1) \sum_{t=-p_n-(2k+1)}^{-p+2k+1} P\Big(|X_t|>a_ny \,\Big|\, |X_0|>a_ny\Big)\,.
\end{aligned}
$$

Moreover, using again the property of conditional probability together with the stationarity of (X_t) and substituting t by $-t$, we get that (J_1) is bounded by

$$
\begin{aligned}
& 2(k+1) \sum_{t=-p_n-(2k+1)}^{-p+2k+1} P\Big(|X_{-t}|>a_ny \,\Big|\, |X_0|>a_ny\Big) \\
&= 2(k+1) \sum_{t=p-(2k+1)}^{p_n+(2k+1)} P\Big(|X_t|>a_ny \,\Big|\, |X_0|>a_ny\Big)\,.
\end{aligned}
$$

Recalling that $(\widetilde{Z}_t) = (\ln(\widetilde{X}_t)^2) \stackrel{d}{=} (\ln(X_t)^2)$ it follows that the last expression can be also expressed by

$$
2(k+1) \sum_{t=p-(2k+1)}^{p_n+(2k+1)} P\Big(\widetilde{Z}_t > \ln(a_ny)^2 \,\Big|\, \widetilde{Z}_0 > \ln(a_ny)^2\Big)\,.
$$

Next, set $N_a = \inf\{s\in\mathbb{N};\ \widetilde{Z}_s \le a\}$ as in Lemma 2.3. Choose the threshold a large enough in order to guarantee that $E\Big(\Big((\alpha+\sqrt{\beta\, e^{-a}+\lambda}\,\varepsilon)^2 - 2\alpha\sqrt{\beta}e^{-a/2}\varepsilon 1_{\{\varepsilon<0\}}\Big)^{\kappa/4}\Big) \le \eta$ for a fixed $\eta\in(0,1)$. This is possible because of (2.2) which implies that $E(|\alpha+\sqrt{\lambda}\,\varepsilon|^u) < 1$ for all $u\in(0,\kappa)$ and the fact that

$$
E\Big(\Big((\alpha+\sqrt{\beta e^{-a}+\lambda}\,\varepsilon)^2 - 2\alpha\sqrt{\beta}e^{-a/2}\varepsilon 1_{\{\varepsilon<0\}}\Big)^{\kappa/4}\Big) \to E\Big(|\alpha+\sqrt{\lambda}\,\varepsilon|^{\kappa/2}\Big)\,, \quad a\to\infty\,,
$$

by the dominated convergence theorem. We derive

$$
\begin{aligned}
(J_1) \quad \le \quad & 2(k+1)\Big(\sum_{\nu=1}^{p-2(k+1)-1} \sum_{t=p-(2k+1)}^{p_n+(2k+1)} P(\widetilde{Z}_t > \ln(a_ny)^2, N_a=\nu \mid \widetilde{Z}_0 > \ln(a_ny)^2) \\
&+ \sum_{\nu=p-(2k+1)}^{p_n+2k} \sum_{t=p-(2k+1)}^{p_n+(2k+1)} P(\widetilde{Z}_t > \ln(a_ny)^2, N_a=\nu \mid \ln\widetilde{Z}_0 > \ln(a_ny)^2)
\end{aligned}
$$

$$
+ \sum_{\nu=p_n+(2k+1)}^{\infty} \sum_{t=p-(2k+1)}^{p_n+(2k+1)} P(\widetilde{Z}_t > \ln(a_n y)^2, N_a = \nu \mid \widetilde{Z}_0 > \ln(a_n y)^2)\Big)
$$

$$
=: \quad 2(k+1)\Big((K_1) + (K_2) + (K_3) \Big)
$$

It can be shown now (see Borkovec (2000)) that the summands (K_1), (K_2) and (K_3) tend to zero as $n \to \infty$ and then $p \to \infty$. The basic idea underlying this result is to use Lemma 2.3 and the fact that the expression $n\,P(\widetilde{Z}_t > \ln(a_n y)^2 \mid \widetilde{Z}_0 = x)$ is uniformly bounded for any $n \in \mathbb{N}$, $x \in [e^{-n}, e^a]$ and $t \in \mathbb{N}$. This finishes the proof. $\square$

Proposition 3.1 provides some properties for $(\mathbf{X}_t^{(m)})$ and $(\mathbf{X}_t^{(m)}(2k+1))$ which are just the required assumptions in Davis and Mikosch (1998) for weak convergence of point processes of the form (3.1). If we define

$$
\widetilde{\mathcal{M}} = \{\mu \in \mathcal{M} \mid \mu(\{\mathbf{x} \mid |\mathbf{x}| > 1\}) = 0 \quad \text{and} \quad \mu(\{\mathbf{x} \mid \mathbf{x} \in \mathcal{S}^m\}) > 0\})
$$

and if we let $\mathcal{B}(\widetilde{\mathcal{M}})$ be the Borel σ-field of $\widetilde{\mathcal{M}}$ then the following theorem is an immediate consequence of Proposition 3.1.

Theorem 3.3 *Assume (X_t) is the stationary AR(1) process with ARCH(1) errors satisfying the conditions of Theorem 2.1. Then*

$$
N_n^{\mathbf{X}} = \sum_{t=1}^{n} \delta_{\mathbf{X}_t/a_n} \xrightarrow{\nu} N^{\mathbf{X}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{P_i \mathbf{Q}_{ij}}\,, \tag{3.10}
$$

where $\mathbf{X}_t = \mathbf{X}_t^{(m)}$, $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process on $(0,\infty]$ with intensity

$$
\nu(dy) = \kappa\, E(\sup_{k\ge 1} \prod_{s=1}^{k} (\alpha + \sqrt{\lambda}\varepsilon_s) \le P^{-1})\, y^{-\kappa-1} dy
$$

and P is a Pareto(κ)-distributed random variable, independent of (ε_s). The process $\sum_{i=1}^{\infty} \delta_{P_i}$ is independent of the sequence of iid point processes $\sum_{j=1}^{\infty} \delta_{\mathbf{Q}_{ij}}$, $i \ge 1$, with joint distribution Q on $(\widetilde{\mathcal{M}}, \mathcal{B}(\widetilde{\mathcal{M}}))$, where Q is the weak limit of

$$
\widetilde{E}\Big(\Big(|\theta_0^{(k)}|^\kappa - \bigvee_{j=1}^{k} |\theta_j^{(k)}|^\kappa\Big)_+ 1_{\{\,\cdot\,\}}\Big(\sum_{|t|\le k} \delta_{\theta_t^{(k)}}\Big)\Big) / \widetilde{E}\Big(|\theta_0^{(k)}|^\kappa - \bigvee_{j=1}^{k} |\theta_j^{(k)}|^\kappa\Big)_+ \tag{3.11}
$$

as $k \to \infty$, and the limit exists. $\widetilde{E}$ is the expectation with respect to the probability measure $d\widetilde{P}$ defined in Remark 3.2(a).

Remark 3.4 *Analogous results can be found for the vectors*

$$|\mathbf{X}_t| = |\mathbf{X}_t^{(m)}| = (|X_t|, ..., |X_{t+m}|) \quad \textit{and} \quad \mathbf{X}_t^2 = \mathbf{X}_t^{(m)^2} = (X_t^2, ..., X_{t+m}^2)\,, \quad t \in \mathbb{Z}\,, m \in \mathbb{N},$$

by using (3.10) and the continuous mapping theorem. Thus, under the same assumptions as in Theorem 3.3, we have

$$N_n^{|\mathbf{X}|} = \sum_{t=1}^{n} \delta_{|\mathbf{X}_t|/a_n} \xrightarrow{\nu} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{P_i|\mathbf{Q}_{ij}|}$$

and

$$N_n^{\mathbf{X}^2} = \sum_{t=1}^{n} \delta_{\mathbf{X}_t^2/a_n} \xrightarrow{\nu} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{P_i\mathbf{Q}_{ij}^2}$$

where the sequences (P_i), $(\mathbf{Q}_{ij})$ are the same as above and

$$|\mathbf{Q}_{ij}|^l = (|Q_{ij}^{(0)}|^l, |Q_{ij}^{(1)}|^l, ..., |Q_{ij}^{(m)}|^l)\,, \quad l = 1, 2\,.$$

Proof. The proof is simply an application of Theorem 2.8 in Davis and Mikosch (1998). The assumptions of this theorem are satisfied because of Proposition 3.1. Finally, the extremal index $\gamma = \lim_{k\to\infty} E\left(|\theta_0^{(k)}|^\kappa - \bigvee_{j=1}^k |\theta_j^{(k)}|^\kappa\right)_+ / E|\theta_0^{(k)}|^\kappa$ of the AR(1) process with ARCH(1) errors is not zero and is specified by the formula (see Borkovec (2000))

$$\gamma = E(\sup_{k\geq 1} \prod_{i=1}^{k} (\alpha + \sqrt{\lambda}\varepsilon_i) \leq P^{-1})\,,$$

where P is a Pareto(κ)-distributed random variable, independent of (ε_s). Hence the statement follows. □

4 Asymptotic behavior of the sample ACVF and ACF

In what follows we derive the limit behaviour of the sample ACVF and ACF of the stationary AR(1) process with ARCH(1) considered in the previous sections. The point process results of the section 3 will be crucial.

Define the sample ACVF of (X_t) by

$$\gamma_{n,X}(h) = \frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h}\,, \quad h = 0, 1, ...,$$

and the corresponding sample ACF by

$$\rho_{n,X}(h) = \gamma_{n,X}(h)/\gamma_{n,X}(0)\,, \quad h = 0, 1, \ldots .$$

The sample ACVF and ACF for $(|X_t|)$ and (X_t^2) are given in the same way. Moreover, we write

$$\gamma_X(h) = E(X_0 X_h)\,, \quad \gamma_{|X|^l}(h) = E(|X_0|^l |X_h|^l)\,,$$

and

$$\rho_X(h) = \gamma_X(h)/\gamma_X(0)\,, \quad \rho_{|X|^l}(h) = \gamma_{|X|^l}(h)/\gamma_{|X|^l}(0)\,, \quad l = 1,2 \quad h = 0, 1, \ldots$$

if these quantities exist. If this is the case straightforward calculations yield

$$\gamma_X(h) = \alpha^h \gamma_X(0) = \alpha^h \beta/(1 - \alpha^2 - \lambda E(\varepsilon^2))$$

and

$$\gamma_{X^2}(h) = (\alpha^2 + \lambda E(\varepsilon^2))^h \gamma_{X^2}(0) + \beta E(\varepsilon^2)\gamma_X(0) \sum_{j=0}^{h-1} (\alpha^2 + \lambda E(\varepsilon^2))^j\,, \quad h \geq 0\,,$$

where

$$\gamma_{X^2}(0) = 2\beta\gamma_X(0)(2\alpha^2 E(\varepsilon^2) + \lambda E(\varepsilon^4))/(1 - \alpha^4 - 6\alpha^2\lambda E(\varepsilon^2) - \lambda^2 E(\varepsilon^4))\,.$$

Mean-corrected versions for the sample ACVF and ACF can also be investigated. However one can show (with the same approach as in the proof of Theorem 4.1) that the limits stay the same (see also Remark 3.6 of Davis and Mikosch (1998)).

In order to state our results we have to introduce several mappings. Let $\delta > 0$, $\mathbf{x}_t = (x_t^{(0)}, \ldots, x_t^{(m)}) \in \overline{\mathbb{R}}^{m+1} \setminus \{0\}$ and define the mappings

$$T_{h,k,\delta} : \quad \mathcal{M} \to \overline{\mathbb{R}}$$

by

$$\begin{aligned} T_{-1,-1,\delta}\Big(\textstyle\sum_{t=1}^\infty n_t \delta_{\mathbf{x}_t}\Big) &= \textstyle\sum_{t=1}^\infty n_t 1_{\{|x_t^{(0)}| > \delta\}}\,, \\ T_{h,k,\delta}\Big(\textstyle\sum_{t=1}^\infty n_t \delta_{\mathbf{x}_t}\Big) &= \textstyle\sum_{t=1}^\infty n_t x_t^{(h)} x_t^{(k)} 1_{\{|x_t^{(0)}| > \delta\}}\,, \quad h, k \geq 0\,, \end{aligned}$$

where $n_t \in \mathbb{N}_0$ for any $t \geq 1$. Since the set $\{\mathbf{x} \in \overline{\mathbb{R}}^{m+1} \setminus \{0\} \,|\, |x^{(h)}| > \delta\}$ is bounded for any $h = 0, \ldots, m$ the mappings are a.s. continuous with respect to the limit point processes $N^{\mathbf{X}}$, $N^{|\mathbf{X}|}$ and $N^{\mathbf{X}^2}$. Consequently, by the continuous mapping theorem, we have in particular

$$T_{-1,-1,\delta}(N_n^{\mathbf{X}}) = \sum_{t=1}^n 1_{\{|X_t^{(0)}| > \delta\}} \xrightarrow{d} T_{-1,-1,\delta}(N^{\mathbf{X}}) = \sum_{i=1}^\infty \sum_{j=1}^\infty 1_{\{|P_i Q_{ij}^{(0)}| > \delta\}} \tag{4.1}$$

and for any $h,k \geq 0$

$$T_{h,k,\delta}(N_n^{\mathbf{X}}) = \sum_{t=1}^{n} X_t^{(h)} X_t^{(k)} 1_{\{|X_t^{(0)}|>\delta\}} \xrightarrow{d} T_{h,k,\delta}(N^{\mathbf{X}}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(h)} Q_{ij}^{(k)} 1_{\{|P_i Q_{ij}^{(0)}|>\delta\}} . \quad (4.2)$$

Note that, with obvious modifications, (4.1) and (4.2) hold also for $N_n^{|\mathbf{X}|}$ and $N^{|\mathbf{X}|}$ respectively $N_n^{\mathbf{X}^2}$ and $N^{\mathbf{X}^2}$. The following theorem collects the weak limit results of the sample ACVF and ACF of (X_t), $(|X_t|)$ and (X_t) depending on the tail index $\kappa > 0$. The weak limits turn to be infinite variance stable random vectors. However, they are only functionals of point processes and have no explicit representation. Therefore, the results are only of qualitative nature and explicit asymptotic confidence bounds for the sample ACVFs and ACFs can't be constructed.

Theorem 4.1 *Assume (X_t) is the stationary AR(1) process with ARCH(1) errors satsifying the conditions of Theorem 2.1 with $E(\varepsilon^2) = 1$. Let $\kappa > 0$ be the tail index in (2.6) and (a_n) be the sequence satisfying* (3.3) *for $k = 0$. Then the following statements hold:*

(1) (a) If $\kappa \in (0,2)$, then

$$\left(n a_n^{-2} \gamma_{n,X}(h)\right)_{h=0,\dots,m} \xrightarrow{d} (V_h^X)_{h=0,\dots,m} ,$$
$$\left(\rho_{n,X}(h)\right)_{h=1,\dots,m} \xrightarrow{d} (V_h^X / V_0^X)_{h=1,\dots,m} ,$$

and

$$\left(n a_n^{-2} \gamma_{n,|X|}(h)\right)_{h=0,\dots,m} \xrightarrow{d} (V_h^{|X|})_{h=0,\dots,m} ,$$
$$\left(\rho_{n,|X|}(h)\right)_{h=1,\dots,m} \xrightarrow{d} (V_h^{|X|} / V_0^{|X|})_{h=1,\dots,m} ,$$

where the vectors $(V_0^X, \dots, V_m^X)$ and $(V_0^{|X|}, \dots, V_m^{|X|})$ are jointly $\kappa/2$-stable in $\mathbb{R}^{m+1}$ with point process representation

$$V_h^X = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i^2 Q_{ij}^{(0)} Q_{ij}^{(h)}$$

and

$$V_h^{|X|} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i^2 |Q_{ij}^{(0)}||Q_{ij}^{(h)}| , \quad h = 0, \dots, m , \quad \textit{respectively}.$$

(b) If $\kappa \in (0,4)$, then

$$\left(n a_n^{-4} \gamma_{n,X^2}(h)\right)_{h=0,\dots,m} \xrightarrow{d} (V_h^{X^2})_{h=0,\dots,m} ,$$
$$\left(\rho_{n,X^2}(h)\right)_{h=1,\dots,m} \xrightarrow{d} (V_h^{X^2} / V_0^{X^2})_{h=1,\dots,m} ,$$

where $(V_0^{X^2},...,V_m^{X^2})$ *is jointly* $\kappa/4$*-stable in* $\mathbb{R}^{m+1}$ *with point process representation*

$$V_h^{X^2} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i^4 (Q_{ij}^{(0)} Q_{ij}^{(h)})^2 , \quad h=0,...,m\,.$$

respectively,

(2) (a) If $\kappa \in (2,4)$ *and* $E(\varepsilon^4)<\infty$*, then*

$$\big(na_n^{-2}(\gamma_{n,X}(h)-\gamma_X(h))\big)_{h=0,\ldots,m} \xrightarrow{d} (V_h^X)_{h=0,\ldots,m}\,,$$
$$\big(na_n^{-2}(\rho_{n,X}(h)-\rho_X(h))\big)_{h=1,\ldots,m} \xrightarrow{d} \gamma_X^{-1}(0)(V_h^X-\rho_X(h)V_0^X)_{h=1,\ldots,m}$$

and

$$\big(na_n^{-2}(\gamma_{n,|X|}(h)-\gamma_{|X|}(h))\big)_{h=0,\ldots,m} \xrightarrow{d} (V_h^{|X|})_{h=0,\ldots,m}\,,$$
$$\big(na_n^{-2}(\rho_{n,|X|}(h)-\rho_{|X|}(h))\big)_{h=1,\ldots,m} \xrightarrow{d} \gamma_{|X|}^{-1}(0)(V_h^{|X|}-\rho_{|X|}(h)V_0^{|X|})_{h=1,\ldots,m}\,,$$

where the vectors $(V_0^X,...,V_m^X)$ *and* $(V_0^{|X|},...,V_m^{|X|})$ *are jointly* $\kappa/2$*-stable in* $\mathbb{R}^{m+1}$ *with*

$$V_0^X = \widetilde{V}_0^X\left(1-(\alpha^2+\lambda)\right)^{-1}\,,\quad V_m^X=\widetilde{V}_m^X+\alpha V_{m-1}^X\,,\quad m\geq 1\,,$$

and

$$V_0^{|X|}=V_0^X\,,\quad V_m^{|X|}=\widetilde{V}_m^{|X|}+E(|\alpha+\sqrt{\lambda}\varepsilon|)V_{m-1}^{|X|}\,,\quad m\geq 1\,.$$

Furthermore, $(\widetilde{V}_0^X,...,\widetilde{V}_m^X)$ *and* $(\widetilde{V}_0^{|X|},...,\widetilde{V}_m^{|X|})$ *are the distributional limits of*

$$\Big(T_{1,1,\delta}(N^{\mathbf{X}})-(\alpha^2+\lambda)T_{0,0,\delta}(N^{\mathbf{X}}),\Big(T_{0,h,\delta}(N^{\mathbf{X}})-\alpha T_{0,h-1,\delta}(N^{\mathbf{X}})\Big)_{h=1,\ldots,m}\Big)$$

and

$$\Big(T_{1,1,\delta}(N^{|\mathbf{X}|})-(\alpha^2+\lambda)T_{0,0,\delta}(N^{|\mathbf{X}|}),\Big(T_{0,h,\delta}(N^{|\mathbf{X}|})-E(|\alpha+\sqrt{\lambda}\varepsilon|)T_{0,h-1,\delta}(N^{|\mathbf{X}|})\Big)_{h=1,\ldots,m}\Big)\,,$$

respectively, as $\delta\to 0$.

(b) If $\kappa\in(4,8)$ *and* $E(\varepsilon^8)<\infty$*, then*

$$\big(na_n^{-4}(\gamma_{n,X^2}(h)-\gamma_{X^2}(h))\big)_{h=0,\ldots,m} \xrightarrow{d} (V_h^{X^2})_{h=0,\ldots,m}\,,$$
$$\big(na_n^{-4}(\rho_{n,X^2}(h)-\rho_{X^2}(h))\big)_{h=1,\ldots,m} \xrightarrow{d} \gamma_{X^2}^{-1}(0)(V_h^{X^2}-\rho_X(h)V_0^{X^2})_{h=1,\ldots,m}\,,$$

where $(V_0^{X^2},...,V_m^{X^2})$ *is jointly* $\kappa/4$*-stable in* $\mathbb{R}^{m+1}$ *with*

$$V_0^{X^2}=\widetilde{V}_0^{X^2}\left(1-(\alpha^4+6\alpha^2\lambda+\lambda^2E(\varepsilon^4))\right)^{-1}\,,\quad V_m^{X^2}=\widetilde{V}_m^{X^2}+(\alpha^2+\lambda)V_{m-1}^{X^2}\,,\quad m\geq 1\,,$$

and $(\widetilde{V}_0^{X^2}, ..., \widetilde{V}_m^{X^2})$ *is the distributional limit of*

$$\left(T_{1,1,\delta}(N^{\mathbf{X}^2}) - (\alpha^4 + 6\alpha^2\lambda + \lambda^2 E(\varepsilon^4))T_{0,0,\delta}(N^{\mathbf{X}^2}), \left(T_{0,h,\delta}(N^{\mathbf{X}^2}) - (\alpha^2 + \lambda)T_{0,h-1,\delta}(N^{\mathbf{X}^2})\right)_{h=1,\dots,m}\right),$$

as $\delta \to 0$.

(3) (a) If $\kappa \in (4, \infty)$, *then*

$$\left(n^{1/2}(\gamma_{n,X}(h) - \gamma_X(h))\right)_{h=0,\dots,m} \xrightarrow{d} (G_h^X)_{h=0,\dots,m},$$
$$\left(n^{1/2}(\rho_{n,X}(h) - \rho_X(h))\right)_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)(G_h^X - \rho_X(h)G_0^X)_{h=1,\dots,m}$$

and

$$\left(n^{1/2}(\gamma_{n,|X|}(h) - \gamma_{|X|}(h))\right)_{h=0,\dots,m} \xrightarrow{d} (G_h^{|X|})_{h=0,\dots,m},$$
$$\left(n^{1/2}(\rho_{n,X}(h) - \rho_X(h))\right)_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)(G_h^{|X|} - \rho_X(h)G_0^{|X|})_{h=1,\dots,m},$$

where the limits are multivariate Gaussian with mean zero.

(b) If $\kappa \in (8, \infty)$, *then*

$$\left(n^{1/2}(\gamma_{n,X^2}(h) - \gamma_{X^2}(h))\right)_{h=0,\dots,m} \xrightarrow{d} (G_h^{X^2})_{h=0,\dots,m},$$
$$\left(n^{1/2}(\rho_{n,X^2}(h) - \rho_{X^2}(h))\right)_{h=1,\dots,m} \xrightarrow{d} \gamma_{X^2}^{-1}(0)(G_h^{X^2} - \rho_{X^2}(h)G_0^{X^2})_{h=1,\dots,m},$$

where the limits are multivariate Gaussian with mean zero.

Remark 4.2 *(a) Theorem 4.1 is a generalization of results for the ARCH(1) process (see Davis and Mikosch (1998)). They use a different approach which does not extend to the general case because of the autoregressive part of* (X_t).
(b) The assumption $\sigma^2 := E(\varepsilon^2) = 1$ *in the theorem is not a restriction. In cases where the second moment is different from one consider the process* $(\widehat{X}_t)$ *defined by the stochastic recurrence equation*

$$\widehat{X}_t = \alpha \widehat{X}_{t-1} + \sqrt{\beta/\sigma^2 + \lambda \widehat{X}_{t-1}^2}\,\varepsilon_t/\sigma, \quad t \in \mathbb{N},$$

where the notation is the same as for the process (X_t) *in (2.1). Note that* $(\widehat{X}_t) = (X_t/\sigma^2)$. *Since the assumptions in the theorem do not dependent on the parameter* β *the results hold for* $(\widehat{X}_t)$ *and hence they also hold for* (X_t) *replacing the limits* $(V_h^X, V_h^{|X|}, V_h^{X^2})_{h=0,\dots,m}$ *by* $\sigma^4\,(V_h^X, V_h^{|X|}, V_h^{X^2})_{h=0,\dots,m}$ *and* $(G_h^X, G_h^{|X|}, G_h^{X^2})_{h=0,\dots,m}$ *by* $\sigma^4\,(G_h^X, G_h^{|X|}, G_h^{X^2})_{h=0,\dots,m}$, *respectively.*

(c) Note that the description of the distributional limits in part (2) *of Theorem 4.1 is different than in Theorem 3.5 of Davis and Mikosch (1998). In the latter theorem the condition*

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \operatorname{var}\left(a_n^{-2}\sum_{t=1}^{n-h} X_t X_{t+h} 1_{\{|X_t X_{t+h}|\le a_n^2\delta\}}\right) = 0$$

is required. However, this condition is very strong and does not seem to be in general fulfilled when (X_t) *is correlated (see e.g. Theorem 1.1 of Rio (1993) for a possible justification). Therefore, we choose another way and establish the convergence in distribution of the sample ACVF directly from the point process convergence in Theorem 3.3.*

Proof. Statements $(1a)$ and $(1b)$ are immediate consequences of Theorem 3.5(1) of Davis and Mikosch (1998). Note that all conditions in this theorem are fulfilled because of Proposition 3.1 and Theorem 3.3. Statements $(3a)$ and $(3b)$ for the sample ACVFs follows from standard limit theorems for strongly mixing sequences (see e.g. Theorem 3.2.1 of Zhengyan and Chuanrong (1996)). The limit behavior for the ACFs can be shown in the same way as e.g. in Davis and Mikosch (1998), p.2062 .

It remains to show $(2a)$ and $(2b)$. We restrict ourselves to the case $(|X_t|)$ and only establish joint convergence of $(\gamma_{n,|X|}(0), \gamma_{n,|X|}(1))$. All other cases can be treated similar or even easier. Recall that $(\widetilde{X}_t) \stackrel{d}{=} (|X_t|)$, where the process $(\widetilde{X}_t)$ is defined in (2.8). Thus it is sufficient to study the sample ACVF of the process $(\widetilde{X}_t)$.

We start by rewriting $\gamma_{n,\widetilde{X}}(0)$ using the recurrence structure of $(\widetilde{X}_t)$

$$\begin{aligned} n a_n^{-2}\left(\gamma_{n,\widetilde{X}}(0) - \gamma_{\widetilde{X}}(0)\right) &= a_n^{-2}\sum_{t=1}^{n}\left(\widetilde{X}_{t+1}^2 - E(\widetilde{X}^2)\right) \\ &= (\alpha^2+\lambda) a_n^{-2}\sum_{t=1}^{n}\left(\widetilde{X}_t^2 - E(\widetilde{X}^2)\right) \\ &\quad + a_n^{-2}\sum_{t=1}^{n}\left(2\alpha\widetilde{X}_t\sqrt{\beta+\lambda\widetilde{X}_t^2}\,\varepsilon_{t+1} + (\beta+\lambda\widetilde{X}_t^2)(\varepsilon_{t+1}^2 - 1)\right) . \end{aligned}$$

We conclude that for any $\delta > 0$,

$$\begin{aligned} &\left(1-(\alpha^2+\lambda)\right) n a_n^{-2}\left(\gamma_{n,\widetilde{X}}(0) - \gamma_{\widetilde{X}}(0)\right) \\ &= a_n^{-2}\sum_{t=1}^{n}(\beta+\lambda\widetilde{X}_t^2)(\varepsilon_{t+1}^2-1) 1_{\{\widetilde{X}_t\le a_n\delta\}} \\ &\qquad + 2\alpha a_n^{-2}\sum_{t=1}^{n}\widetilde{X}_t\sqrt{\beta+\lambda\widetilde{X}_t^2}\,\varepsilon_{t+1} 1_{\{\widetilde{X}_t\le a_n\delta\}} \end{aligned}$$

$$
\begin{aligned}
&+ a_n^{-2}\sum_{t=1}^{n}\left(\widetilde{X}_t\sqrt{\beta+\lambda\widetilde{X}_t^2}\varepsilon_{t+1}+(\beta+\lambda\widetilde{X}_t^2)(\varepsilon_{t+1}^2-1)\right)1_{\{\widetilde{X}_t>a_n\delta\}}+o_P(1)\\
=: &\ (I_1)\,+\,(I_2)+\,(I_3)\,+\,o_P(1)\,.
\end{aligned}
$$

We show first that (I_1) and (I_2) converge in probability to zero. Note that the summands in (I_1) are uncorrolated. Therefore,

$$
\begin{aligned}
\operatorname{var}(I_1) &= a_n^{-4}\sum_{t=1}^{n}\operatorname{var}\left((\beta+\lambda\widetilde{X}_t^2)\,1_{\{|\widetilde{X}_t|\le a_n\delta\}}\,(\varepsilon_{t+1}^2-1)\right)\\
&\le \quad a_n^{-4}\sum_{t=1}^{n}E\left((\beta+\lambda\widetilde{X}_t^2)^2\,1_{\{|\widetilde{X}_t|\le a_n\delta\}}\right)\,E\left((\varepsilon_{t+1}^2-1)^2\right)\\
&\sim \quad const\,\delta^{4-\kappa}\,,\quad \text{as } n\to\infty\,,\\
&\to \quad 0\,,\quad \text{as } \delta\downarrow 0\,,
\end{aligned}
$$

where the asymptotic equivalence comes from Karamatas theorem on regular variation and the tail behavior of the stationary distribution of $(\widetilde{X}_t)$. Note that the condition $E(\varepsilon^4)<\infty$ is crucial. Analogously, one can show that

$$
\lim_{\delta\downarrow 0}\lim_{n\to\infty}\operatorname{var}(I_2)=0\,.
$$

Now we consider (I_3). (2.8) yields

$$
\begin{aligned}
(I_3) \quad &= \quad a_n^{-2}\sum_{t=1}^{n}\widetilde{X}_{t+1}^2 1_{\{\widetilde{X}_t>a_n\delta\}}-(\alpha^2+\lambda)a_n^{-2}\sum_{t=1}^{n}\widetilde{X}_t^2 1_{\{\widetilde{X}_t>a_n\delta\}}-\beta a_n^{-2}\sum_{t=1}^{n}1_{\{\widetilde{X}_t>a_n\delta\}}\\
&\stackrel{d}{=} \quad T_{1,1,\delta}(N_n^{|\mathbf{X}|})-(\alpha^2+\lambda)T_{0,0,\delta}(N_n^{|\mathbf{X}|})-\beta a_n^{-2}T_{-1,-1,\delta}(N_n^{|\mathbf{X}|})\\
&\stackrel{d}{\to} \quad T_{1,1,\delta}(N^{|\mathbf{X}|})-(\alpha^2+\lambda)T_{0,0,\delta}(N^{|\mathbf{X}|})\,,
\end{aligned}
\tag{4.3}
$$

where the limit has expectation zero. Finally, following the same arguments as in Davis and Hsing (1995), pp. 897-898, the right hand side in (4.3) converges in distribution to a $\kappa/2$-stable random variable, as $\delta\to 0$.

Now consider $\gamma_{n,\widetilde{X}}(1)$. We proceed as above and write

$$
\begin{aligned}
&na_n^{-2}\left(\gamma_{n,\widetilde{X}}(1)-\gamma_{\widetilde{X}}(1)\right)=a_n^{-2}\sum_{t=1}^{n-1}\widetilde{X}_t\widetilde{X}_{t+1}-E(\widetilde{X}_0\widetilde{X}_1)\\
&\quad = \quad a_n^{-2}\sum_{t=1}^{n-1}\left(f_{\varepsilon_{t+1}}(\widetilde{X}_t)-E(f_{\varepsilon_{t+1}}(\widetilde{X}_t))\right)+a_n^{-2}\sum_{t=1}^{n-1}\left(\widetilde{X}_t^2|\alpha+\sqrt{\lambda}\varepsilon_{t+1}|-E(\widetilde{X}^2)E(|\alpha+\sqrt{\lambda}\varepsilon|)\right)\\
&\quad =: \quad (J_1)\,+\,(J_2)\,,
\end{aligned}
$$

where $f_z(y) = |y| \left(|\alpha y + \sqrt{\beta + \lambda y^2}\, z| - |\alpha y + \sqrt{\lambda} y z| \right)$ for any $y, z \in \mathbb{R}$. First, we show that (J_1) converges in probability to zero. Observe for that purpose that

$$\begin{aligned} & \operatorname{var}\left(|a_n^{-2} \sum_{t=1}^{n-1} f_{\varepsilon_{t+1}}(\widetilde{X}_t) - E(f_{\varepsilon_{t+1}}(\widetilde{X}_t))| \right) \\ \leq \ & \operatorname{var}\left(a_n^{-2} \sum_{t=1}^{n} |f_{\varepsilon_{t+1}}(\widetilde{X}_t)| \right) \qquad (4.4) \\ = \ & a_n^{-4} \sum_{t=1}^{n} \sum_{s=1}^{n} \operatorname{cov}(|f_{\varepsilon_{t+1}}(\widetilde{X}_t)|, |f_{\varepsilon_{s+1}}(\widetilde{X}_s)|)\,. \end{aligned}$$

Now note that $|f_z(y)| \leq |y|\sqrt{\beta}|z|$ for any $y, z \in \mathbb{R}$. Therefore and since $\kappa > 2$ there exists a $\mu > 0$ such that

$$E(|f_\varepsilon(\widetilde{X})|^{2+\mu}) \leq \sqrt{\beta} E(|\varepsilon|^{2+\mu}) E(|\widetilde{X}|^{2+\mu}) < \infty\,. \qquad (4.5)$$

Because of (4.5) and the geometric strong mixing property of $(\widetilde{X}_t)$ all assumptions of Lemma 1.2.5 of Zhengyan and Chuanrong (1996) are satisfied and we can bound (4.4) by

$$const\, a_n^{-4}\, n \sum_{s=1}^{n} (\rho^{2/(2+\delta)})^s \qquad (4.6)$$

which converges to zero as $n \to \infty$ since $\kappa < 4$. Next we rewrite (J_2) and get

$$\begin{aligned} (J_2) &= E(|\alpha + \sqrt{\lambda}\varepsilon|) n a_n^{-2} \left(\gamma_{n,\widetilde{X}}(0) - \gamma_{\widetilde{X}}(0) \right) \\ &+ \ a_n^{-2} \sum_{t=1}^{n-1} \widetilde{X}_t^2 1_{\{\widetilde{X}_t \leq a_n \delta\}} \left(|\alpha + \sqrt{\lambda}\varepsilon_{t+1}| - E(|\alpha + \sqrt{\lambda}\varepsilon|) \right) \\ &+ \ a_n^{-2} \sum_{t=1}^{n-1} \widetilde{X}_t^2 1_{\{\widetilde{X}_t > a_n \delta\}} \left(|\alpha + \sqrt{\lambda}\varepsilon_{t+1}| - E(|\alpha + \sqrt{\lambda}\varepsilon|) \right) \\ &= \ (K_1) + (K_2) + (K_3)\,. \end{aligned}$$

By (4.3), $(K_1) \xrightarrow{d} T_{1,1,\delta}(N^{|\mathbf{X}|}) - (\alpha^2 + \lambda) T_{0,0,\delta}(N^{|\mathbf{X}|})$. Moreover, using the same arguments as before one can show that $\lim_{\delta \downarrow 0} \lim_{n \to \infty} \operatorname{var}(K_2) = 0$. Hence $(K_2) = o_P(1)$. It remains to consider (K_3). We begin with the decomposition

$$\begin{aligned} (K_3) \ &= \ a_n^{-2} \sum_{t=1}^{n-1} \widetilde{X}_t 1_{\{\widetilde{X}_t > a_n \delta\}} \left(|\alpha \widetilde{X}_t + \sqrt{\lambda} \widetilde{X}_t \varepsilon_{t+1}| - |\alpha \widetilde{X}_t + \sqrt{\beta + \lambda \widetilde{X}_t^2} \varepsilon_{t+1}| \right) \\ &+ a_n^{-2} \sum_{t=1}^{n-1} \widetilde{X}_{t+1} \widetilde{X}_t 1_{\{\widetilde{X}_t > a_n \delta\}} \ - \ a_n^{-2} \sum_{t=1}^{n-1} \widetilde{X}_t^2 1_{\{\widetilde{X}_t > a_n \delta\}} E(|\alpha + \sqrt{\lambda}\varepsilon|)\,. \end{aligned}$$

Proceeding the same way as in (4.4)-(4.6) the first term converges in probability to zero. Thus,

$$\begin{aligned} (K_3) \ &\stackrel{d}{=} \ o_P(1) + T_{0,1,\delta}(N_n^{|\mathbf{X}|}) - E(|\alpha + \sqrt{\lambda}\varepsilon|) T_{0,0,\delta}(N_n^{|\mathbf{X}|}) \\ &\xrightarrow{d} \ T_{0,1,\delta}(N^{|\mathbf{X}|}) - E(|\alpha + \sqrt{\lambda}\varepsilon|) T_{0,0,\delta}(N^{|\mathbf{X}|})\,, \end{aligned}$$

where the limit has zero mean and converges again to a $\kappa/2$-stable random variable as $\delta \downarrow 0$. Since for the distributional convergence only the point process convergence and the continuous mapping theorem has been used, it is immediate that the same kind of argument yields the joint convergence of the sample autocovariances to a $\kappa/2$-stable limit as described in the statement. Finally, the asymptotic behavior of the sample ACF can be shown in the same way as in Davis and Mikosch (1998), p.2062 . $\square$

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References

[1] Basrak, B., Davis, R.A. and Mikosch, T. (1999) The sample ACF of a simple bilinear process. *Stoch. Proc. Appl.* **83**, 1–14.

[2] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987) *Regular Variation.* Cambridge University Press, Cambridge.

[3] Borkovec, M. and Klüppelberg, C. (1998) The tail of the stationary distribution of an autoregressive process with ARCH(1) errors. *Preprint.*

[4] Borkovec, M. (2000) Extremal behavior of the autoregressive process with ARCH(1) errors. *Stoch. Proc. Appl.* **85/2**, 189-207.

[5] Breiman, L. (1965) On some limit theorems similar to the arc-sin law. *Theory Probab. Appl.* **10**, 323–331.

[6] Brockwell, P.J. and Davis, R.A. (1990) *Time Series: theory and Methods.* Springer, New York.

[7] Davis, R.A. and Hsing, T. (1995) Point process and partial sum convergence for weakly dependent random variables with infinite variance. *Ann. Probab.* **23**, 879–917.

[8] Davis, R.A. and Mikosch, T. (1998) The sample autocorrelations of heavy-tailed processes with application to ARCH. *Ann. Statist.* **26**, 2049–2080.

[9] Davis, R.A. Mikosch, T., and Basrak, B. (1998) The sample ACF of solutions to multivariate stochastic recurrence equations. Technical Report.

[10] Davis, R.A. and Resnick, S.I. (1985) More limit theory for the sample correlation function of moving averages. *Stoch. Proc. Appl.* **20**, 257–279.

[11] Davis, R.A. and Resnick, S.I. (1986) Limit theory for the sample covariance and correlation functions of moving averages. *Ann. Statist.* **14**, 533–558.

[12] Davis, R.A. and Resnick, S.I. (1996) Limit theory for bilinear processes with heavy tailed noise. *Ann. Appl. Probab.* **6**, 1191–1210.

[13] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997) *Modelling Extremal Events for Insurance and Finance.* Springer, Heidelberg.

[14] Engle, R. F. (1982) Autoregressive conditional heteroscedasticity with estimates of the variance of U.K. inflation. *Econometrica* **50**, 987–1007.

[15] Kallenberg, O. (1983) *Random Measures,* 3rd edition. Akademie-Verlag, Berlin.

[16] Kesten, H. (1973) Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.

[17] Mikosch, T. and Starica, C. (1998) Limit theory for the sample autocorrelations and extremes of a GARCH(1,1) process. Technical report.

[18] Resnick, S.I. (1997) Heavy tail modeling and teletraffic data. With discussion and a rejoinder by the author. *Ann. Stat.* **25**, 1805–1869.

[19] Resnick, S.I. and Van den Berg, E. (1998) A test for nonlinearity of time series with infinite variance. Technical report.

[20] Rio, E. (1993) Covariance inequalities for strongly mixing processes. *Ann. Inst. Henri Poincaré* **29**, 587–597.

[21] Vervaat, W. (1979) On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. Appl. Prob.* **11**, 750–783.

[22] Weiss, A.A. (1984) ARMA models with ARCH errors. *J. T. S. A.* **3**, 129–143.

[23] Zhengyan, L. and Chuanrong, L. (1996) *Limit Theory for Mixing Dependent Random Variables,* Kluwer Academic Publishers, Dordrecht.

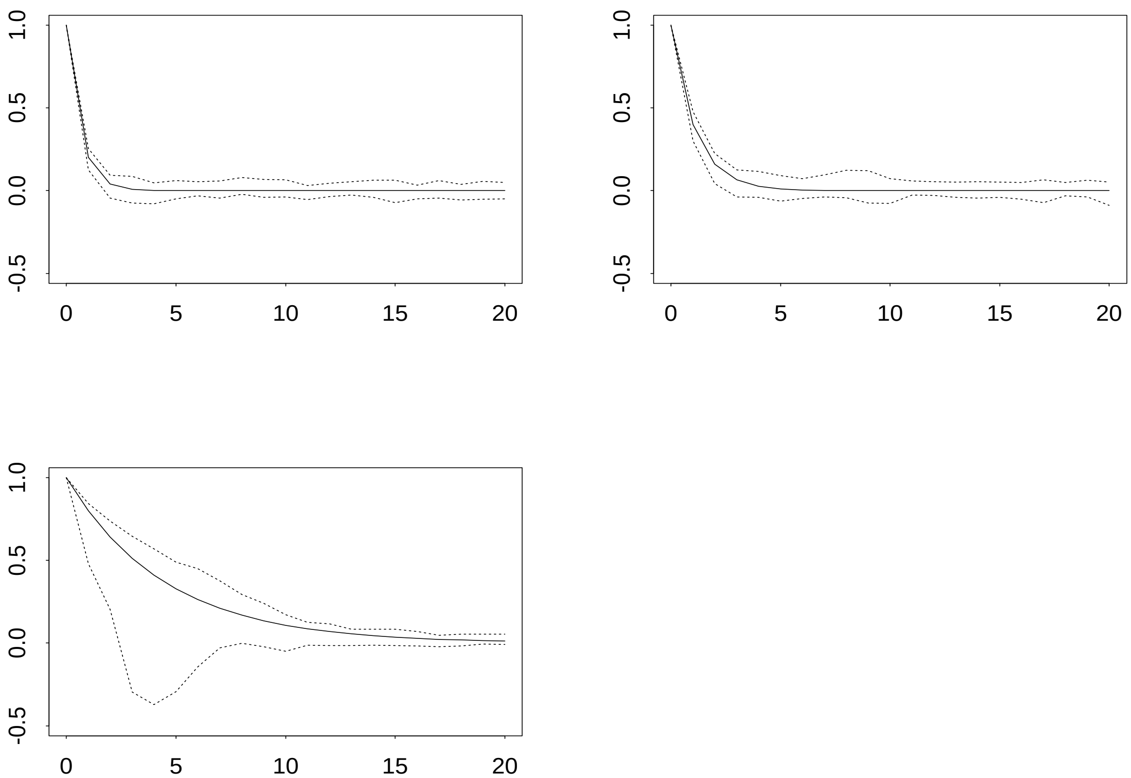

Figure 2: ACFs of the AR(1) process with ARCH(1) errors with standard normal distributed innovations (ε_t) and parameters $\alpha = 0.2, \beta = 1$ and $\lambda = 0.4$ (top, left), $\alpha = 0.4$, $\beta = 1$ and $\lambda = 0.6$ (top, right) and $\alpha = 0.8$, $\beta = 1$, $\lambda = 0.6$ (bottom). In the first case $\kappa = 5.49$, in the second $\kappa = 2.87$ and in the last $\kappa = 1.35$. The dotted lines indicate the $5\%-$ and 95%-quantiles of the empirical distributions of the sample ACFs at different lags. The underlying simulated sample paths have length 1000. The confidence bands were derived from 1000 independent simulations of the sample ACFs at these lags. The plots confirm the different limit behavior of the sample ACFs as described in this article.