On Extremes of Multidimensional Stationary Diffusion Processes in Euclidean Norm

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Abstract

Let $(X_t)_{t\geq 0}$ be a $\mathbb{R}^n$-valued stationary reversible diffusion process. We investigate the asymptotic behavior of $M_T := \max_{0\leq t\leq T} |X_t|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. The aim of this paper is to characterize the tail asymptotics of M_T for fixed $T > 0$ as well as the long time behavior of M_T as $T \to \infty$. This is related to spectral asymptotics of the generator of $(X_t)_{t\geq 0}$ subject to Dirichlet boundary conditions on the ball around the origin with radius R in the limit as $R \to \infty$. We give conditions when sharp spectral asymptotics can be obtained testing with rotationally symmetric functions. Examples include not only rotationally symmetric but also highly non-symmetric processes.

Keywords: absorption probability, diffusion process, Dirichlet form, Dirichlet problem, eigenvalue asymptotics, extreme value theory, generator, semigroup, tail behavior, Temple's inequality, variational principle.

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1 Introduction

We consider a diffusion process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^n$, $n \in \mathbb{N}$, solving the system of SDEs

$$dX_t^i = b^i(X_t)dt + \sum_{j=1}^{n} \sigma^{ij}(X_t)dB_t^j \qquad i = 1, \ldots, n\,. \tag{1.1}$$

where $b^i, \sigma^{ij} : \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \ldots, n$, and $(B_t^j)_{t\geq 0}$, $j = 1, \ldots, n$, are independent one-dimensional standard Brownian motions. We suppose further that the process $(X_t)_{t\geq 0}$ is reversible with respect to a probability measure μ on $\mathbb{R}^n$; μ is then the stationary (or invariant) measure of the process. We will concentrate on diffusion processes *of gradient field type*. In this case the drift is given by the gradient of a potential function Φ and the diffusion term is a constant $\sigma \in \mathbb{R}^n \setminus \{0\}$, i.e. $(X_t)_{t\geq 0}$ solves a SDE the form

$$dX_t^i = -\partial_{x_i}\Phi(X_t)dt + \sigma dB_t^i \qquad i = 1, \ldots, n\,. \tag{1.2}$$

The main interest of this paper is to characterize the running maxima of the process $(X_t)_{t\geq 0}$ with respect to the Euclidean norm $|\cdot|$ in $\mathbb{R}^n$, i.e. we study the random variable

$$M_T := \max_{0\leq t\leq T} |X_t| \qquad T \geq 0\,.$$

M_T is related to absorption at the boundary of the open ball $B_R := \{x \in \mathbb{R}^n : |x| < R\}$ in the following way. We denote by

$$\tau_R := \inf\{s > 0 : X_s \in \mathbb{R}^n \setminus B_R\} \tag{1.3}$$

the *first exit time* off B_R of the process $(X_t)_{t\geq 0}$ and by P_μ the law of the process $(X_t)_{t\geq 0}$ starting with its stationary measure μ. Then for every $R > 0$ and $T > 0$

$$P_\mu(M_T \leq R) = P_\mu(\tau_R > T)\,. \tag{1.4}$$

For one-dimensional stationary diffusion processes, the problem of the characterization of the asymptotic behavior of the maximum has been solved by various authors. An analytic approach was chosen by Newell [New62] using the Focker-Planck equation and a formal eigenvalue expansion of the associated Sturm-Liouville problem. Mandl [Man68] derived the asymptotic distribution of the maximum of a one-dimensional stationary diffusion process within the framework of Laplace transforms. Berman [Ber64] gave an probabilistic proof with the help of discrete approximation techniques. For Gaussian processes we refer to the articles of Qualls and Watanabe [QW72], Pickands [Pic69], and Berman [Ber71]. Davis [Dav82] reduced the asymptotic distribution of the maximum of a general one-dimensional stationary diffusion process to that of an Ornstein-Uhlenbeck process by the method of random time change. A good survey for the characterization of the long time behavior of the running maxima of one-dimensional stationary diffusions, which is essentially the classical extreme value theory approach, is given in Leadbetter et al. [LLR83].

In the multidimensional case however the characterization of the asymptotic behavior of the maximum of general stationary diffusion processes is an open problem. For a multidimensional and even l^2-valued Ornstein-Uhlenbeck process, Iscoe and McDonald [IM92, IM89] evaluated the tail behavior of M_T for fixed $T > 0$. In Lindgren [Lin80a, Lin80b] the long term behavior of extremes of some functions of multidimensional Gaussian processes is considered. For the characterization of extremes of more general Gaussian processes we refer to the books of Berman [Ber92] and Piterbarg [Pit96] and the references therein.

Note the connection with the exit problem of Freidlin-Wentzell and the associated large deviation principle, see Freidlin and Wentzell [FW84] and Dembo and Zeitouni [DZ98]. Instead of considering the probability of the exit off B_R of the process $(X_t)_{t\geq 0}$, one can rescale the process by $X_{R,t} := R^{-1}X_t$, $t \geq 0$, $R > 0$, and analyze the exit probability of $(X_{R,t})_{t\geq 0}$ off the unit ball. If $(X_t)_{t\geq 0}$ is of gradient field type, then $(X_{R,t})_{t\geq 0}$ satisfies for every $R > 0$ the SDE

$$dX^i_{R,t} = -\partial_{x_i}\Phi_R(X_{R,t})dt + R^{-1}\sigma dB^i_t \qquad i = 1, \dots, n\,,$$

where $\Phi_R(x) := R^{-2}\Phi(Rx)$, $x \in \mathbb{R}^n$. For the Ornstein-Uhlenbeck process, Φ is invariant under this scaling. In this case the Freidlin-Wentzell theory provides the asymptotics of the probability that $(X_{R,t})_{t\geq 0}$ does not leave the unit ball before time T as $R \to \infty$ on a logarithmic scale. These asymptotics is determined by the minimum of Φ on the unit sphere.

Our results do not assume that the diffusion process is Gaussian. Further we are able to give not only the asymptotics on a logarithmic scale for the tail of M_T for fixed $T > 0$, but also the *fine* asymptotics. Our approach is inspired by the articles of Newell [New62] and Iscoe and McDonald [IM89, IM92]. The key idea of this approach is to express the probability $P_\mu(M_T \leq R)$ in terms of the backward semigroup associated to the part of the underlying diffusion process on the ball B_R, i.e. to the process killed when it leaves B_R. The generator of this semigroup, denoted by L_R, is given by the generator of the full process subject to Dirichlet boundary conditions on the ball B_R. The asymptotic behavior of the probability $P_\mu(M_T \leq R)$ as $R \to \infty$ is essentially determined by the asymptotics of the bottom eigenvalue λ_R of $-L_R$ in the limit when the radius R of the ball B_R tends to infinity. Restricting to the case of diffusion processes of gradient field type solving the SDE (1.2) with potential Φ, an asymptotic expression of λ_R as $R \to \infty$ can be obtained in the following way: we approximate Φ by a rotationally symmetric potential $x \mapsto \phi(|x|)$. If the potential Φ is already rotationally symmetric, the process can be identified with a one-dimensional process for which the eigenvalue asymptotics is known, see e.g. Newell [New62]. We give conditions such that the terms caused by the asymmetric part of the potential do not destroy the eigenvalue asymptotics for the process associated to the rotationally symmetric potential.

As an example we present a class of diffusion processes of gradient field type, where the asymmetric part of the potential Φ factorizes in radial and spherical components. This class includes the (non-symmetric) Ornstein-Uhlenbeck process and we evaluate explicitly in one and two dimensions the eigenvalue asymptotics and the long term behavior the maximum of this process in Euclidean norm emphasizing the effects of symmetry breaking. Further a diffusion process of gradient field type can be treated with this methods having a bivariate gamma distribution as stationary measure.

The structure of this paper is as follows. The results are stated in section 2. In section 3 we recall some facts from the theory of Markov processes an operator theory. The proofs of the results are given in section 4 and some examples are presented in section 5.

2 Results

The proofs of the theorems and corollaries can be found in sections 4.

Let us first restrict to the case of diffusion processes of gradient field type solving the SDE (1.2). We assume that the reversible (stationary) measure μ has a density w.r.t. the Lebesgue measure on $\mathbb{R}^n$. This density is also denoted by μ. Note that the density μ and the potential Φ are related formally by the relation $\mu(x) = e^{-2\Phi(x)/\sigma^2}$, $x \in \mathbb{R}^n$. We will allow that the zero set of the density μ is not empty. This implies that the potential Φ can take the value $+\infty$. Set

$$\mathcal{Z} := \{x \in \mathbb{R}^n : \Phi(x) = +\infty\}\,, \qquad \mathcal{Z}^c := \mathbb{R}^n \setminus \mathcal{Z}\,. \tag{2.1}$$

We formulate conditions on Φ that guaranty the existence of a weak solution of the SDE (1.2). Assume

$$\Phi \in C(\mathbb{R}^n, \mathbb{R} \cup \{\infty\})\,, \qquad \Phi_{|\mathcal{Z}^c} \in C^1(\mathcal{Z}^c)\,, \tag{2.2}$$

where $\Phi_{|\mathcal{Z}^c}$ denotes the restriction of Φ to the set $\mathcal{Z}^c$. Further assume

$$\int_{\mathcal{Z}^c} e^{-2\Phi(x)/\sigma^2} |\nabla \Phi(x)|^2 dx < \infty\,, \tag{2.3}$$

where ∇ denotes the gradient. Proposition 3.1 states that under these conditions there exists a process $(X_t)_{t\geq 0}$ that is a weak solution of the SDE (1.2) and is reversible w.r.t. the measure μ with density

$$\mu(x) = \begin{cases} e^{-2\Phi(x)/\sigma^2} & x \in \mathcal{Z}^c\,, \\ 0 & x \in \mathcal{Z}\,. \end{cases} \tag{2.4}$$

Further assume that the stationary measure μ is finite, i.e.

$$Z_\sigma := \int_{\mathbb{R}^n} e^{-2\Phi(x)/\sigma^2} dx < \infty\,. \tag{2.5}$$

It will turn out that the asymptotic distribution of $M_T := \max_{0\leq t\leq T} |X_t|$ is determined by spectral properties of the generator of the process $(X_t)_{t\geq 0}$. This generator reads formally

$$Lu = \frac{\sigma^2}{2}\Delta u - \sum_{i=1}^n \partial_{x_i}\Phi\, \partial_{x_i} u = \frac{\sigma^2}{2} e^{2\Phi/\sigma^2} \sum_{i=1}^n \partial_{x_i}\left(e^{-2\Phi/\sigma^2}\partial_{x_i} u\right)\,. \tag{2.6}$$

For $R \in (0, \infty]$, the operator L acting on $L^2(B_R, \mu)$ with Dirichlet boundary conditions on B_R is denoted by L_R (if $R = \infty$ set $B_\infty := \mathbb{R}^n$ and no boundary conditions are present, see section 3 for a proper definition). L_R generates a strongly continuous contraction semigroup $(e^{L_R t})_{t\geq 0}$ on $L^2(B_R,\mu)$. Assume that $-L_\infty$ enjoys the spectral gap property in the sense that

$$\Lambda := \inf \Sigma(-L_\infty) \cap (0,\infty) > 0\,, \tag{2.7}$$

where Σ denotes the spectrum of the operator. In Proposition 3.7 we state a sufficient condition for (2.7) to hold. For $R \in (0,\infty]$, the bottom eigenvalue of the operator $-L_R$ is denoted by $\lambda_R := \inf \Sigma(-L_R)$.

The reason for the above definitions is that the probability $P_\mu(M_T \leq R)$ can be expressed in terms of the semigroup $(e^{L_R t})_{t\geq 0}$. The next proposition provides upper and lower bounds on $P_\mu(M_T \leq R)$ in terms of λ_R and Λ. A sketch of the proof can be found in section 3.

Proposition 2.1 *Let $(X_t)_{t\geq 0}$ be a diffusion process of gradient field type as in Proposition 3.1. Assume that (2.5) and (2.7) hold. Then for every $T > 0$ and sufficiently large $R > 0$*

$$(1-\lambda_R/\Lambda)e^{-\lambda_R T} \leq P_\mu(M_T \leq R) \leq e^{-\lambda_R T}\,.$$

PROOF. Combine Lemma 3.3.(ii) with Lemma 3.4, having (1.4) in mind. □

Remark 2.2 (1) Without loss of generality is suffices to prove Proposition 2.1 under the following additional assumptions: $\sigma^2 = 2$ and the potential Φ is normalized in the sense that the stationary measure $\mu(dx) = e^{-\Phi(x)}dx$ is a probability measure on $\mathbb{R}^n$. To recover the general case, the result for normalized potentials has to be applied to the potential $\Phi_\sigma := (2/\sigma^2)\Phi + \ln Z_\sigma$. Further the bottom eigenvalue λ_R for the normalized problem has to be multiplied by $2/\sigma^2$, since the same holds for the generator, see (2.6). The same simplification is applicable for the proof of Theorem 2.8.

(2) Proposition 2.1 is taken from Iscoe and McDonald [IM94], theorem 2.13. In their approach the spectral gap property plays an important role to get lower bounds on $P_\mu(M_T \leq R) = P_\mu(\tau_R > T)$. Lower bounds without assuming

spectral gap can be obtained using capacity inequalities as in Iscoe and McDonald [IM90]. More precisely, for every $R, T > 0$ and arbitrary $\theta > 0$ setting $\theta_1 := \theta/T$

$$P_\mu(\tau_R > T) \geq 1 - \theta^{-1} e^\theta T \operatorname{Cap}_{\theta_1}(\mathbb{R}^n \setminus B_R)$$

where the latter is the θ_1-capacity of the set $\mathbb{R}^n \setminus B_R$ w.r.t. the process $(X_t)_{t\geq 0}$ (see e.g. Fukushima et al. [FOT94] for a definition). But this lower bound turns out to be not good enough to yield sharp asymptotics for the long term behavior of M_T as $T \to \infty$ (see Theorem 2.3 and Theorem 2.4).

(3) Proposition 2.1 remains valid if the balls B_R around the origin with radius R are replaced by a more general exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$. Such a family is defined as follows: O_R is an open bounded subset of $\mathbb{R}^n$ with smooth boundary for every $R > R_0$, $O_{R_1} \subset O_{R_2}$ if $R_1 \leq R_2$ and $\bigcup_{R>R_0} O_R = \mathbb{R}^n$. Invoking (1.4), the maximum M_T of $(X_t)_{t\geq 0}$ w.r.t. Euclidean norm has to be replaced by the maximum w.r.t. the distance function $q(x) := \inf\{R : x \in O_R\}$. This means that the maximum of $(X_t)_{t\geq 0}$ has the form $M_T = \max_{0\leq t\leq T} q(X_t)$.

Proposition 2.1 tells us that the asymptotics of $P_\mu(M_T \leq R)$ as $R \to \infty$ is given by the behavior of the bottom eigenvalue λ_R in the limit $R \to \infty$. Evidently $\lambda_R \to 0$ as $R \to \infty$. Unfortunately λ_R is not directly available. It suffices however to have an explicit expression for the convergence $\lambda_R \to 0$ as $R \to \infty$.

We make use of the following *asymptotic notations*: given two real functions a and b, we write $a(t) \sim b(t)$ resp. $a(t) \lesssim b(t)$ as $t \to t_0 \in \mathbb{R} \cup \{\pm\infty\}$ if $\lim_{t\to t_0} a(t)/b(t) = 1$ resp. $\lim_{t\to t_0} a(t)/b(t) \leq 1$ and $a(t) = o(b(t))$ as $t \to t_0$ if $\lim_{t\to t_0} |a(t)/b(t)| = 0$.

We have to find a function $l : \mathbb{R}^+ \to \mathbb{R}^+$, given in terms of the potential Φ and the diffusion coefficient σ, such that

$$\lambda_R \sim l(R) \qquad (R \to \infty)\,. \tag{2.8}$$

The main part of this paper consists of giving conditions which allow to find a function l satisfying (2.8). This will be done in Theorem 2.8.

Assume for the moment that such a function l is already given. Replacing in Proposition 2.1 λ_R by the asymptotic expression $l(R)$, sharp asymptotic upper and lower bounds can be obtained for the tail of the maximum M_T for fixed T.

Theorem 2.3 *Assume the situation of Proposition 2.1. Let l be a function satisfying (2.8). Then for every $T > 0$*

$$T\,l(R) \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda)\,l(R) \quad (R \to \infty)\,.$$

Next the long term behavior of M_T as $T \to \infty$ is analyzed. Passing in Theorem 2.3 to the limit $T \to \infty$, the difference between asymptotic upper and lower bound tends to zero, since the term $1/\Lambda$ vanishes in this limit. Doing this more formally we obtain the following theorem.

Theorem 2.4 *Assume the situation of Proposition 2.1. Let l be a function satisfying (2.8). Then for every sequence $R_T \nearrow \infty$ as $T \to \infty$*

$$|P_\mu(M_T \le R_T) - e^{-l(R_T)T}| \to 0 \qquad (T \to \infty)\,.$$

Remark 2.5 Theorem 2.4 is a multi-dimensional extension of Davis [Dav82].

Theorem 2.4 allows to analyze the the long term behavior of the maximum M_T in the spirit of classical extreme value theory. More precisely, the possibly non-degenerated limit distribution of the properly normalized maximum M_T can be obtained in the limit $T \to \infty$. We recall some facts of this theory (see e.g. chapter 3 of Embrechts et al. [EKM97]). Let F be the cumulative distribution function of some real random variable. F is said to be in the *domain of attraction* of a distribution function H ($F \in \mathrm{DA}(H)$), if there exist norming sequences $(c_T)_{T>0}$ and $(d_T)_{T>0}$ such that

$$\lim_{T\to\infty} F(c_T x + d_T)^T = H(x) \qquad x \in \mathbb{R}\,. \tag{2.9}$$

By the theorem of Fisher and Tippett, H is an extreme value distribution, i.e. $H \in \{\Lambda, \Phi_\alpha, \Psi_\alpha\}$ where Λ is the Gumbel distribution, and Φ_α resp. Ψ_α is the Frechet resp. Weibull distribution with index α. Property (2.9) is preserved under tail equivalence, i.e. F can be replaced by a distribution function F_1 with the same norming constants provided $1 - F(R) \sim 1 - F_1(R)$ as $R \to \infty$. The connection with the long term behavior of running maxima is as follows: let $(\tilde{X}_t)_{t=1,2,\dots}$ be a sequence of i.i.d. random variables distributed according a distribution function F and set $\tilde{M}_T := \max\{\tilde{X}_t : t = 1, \dots, T\}$. Suppose $F \in \mathrm{DA}(H)$ for some extreme value distribution H. Note that by the i.i.d. assumption for every $x \in \mathbb{R}$

$$P(c_T^{-1}(\tilde{M}_T - d_T) \le x) = P(\tilde{X}_t \le c_T x + d_T; t = 1, \dots, T) = F(c_T x + d_T)^T\,.$$

Using (2.9), we obtain the following long term limit, where $\xrightarrow{d}$ denotes convergence in distribution

$$c_T^{-1}(\tilde{M}_T - d_T) \xrightarrow{d} H \qquad (T \to \infty)\,.$$

By Theorem 2.4, the long term behavior of the maximum M_T of the process $(X_t)_{t\geq 0}$ in Euclidean norm can be reduced to that of the maximum of an i.i.d. sequence.

Corollary 2.6 *Assume the situation of Theorem 2.4. Set* $F(R) := e^{-l(R)}$, $R > 0$. *If* $F \in \mathrm{DA}(H)$ *for an extreme value distribution* H *with norming constants* $(c_T)_{T>0}$, $(d_T)_{T>0}$ *according to (2.9), then*

$$c_T^{-1}(M_T - d_T) \xrightarrow{d} H \qquad (T \to \infty)\,.$$

Remark 2.7 Classical extreme value theory characterizes the maximal domain of attraction of each extreme value distribution (see e.g. section 3.3 of Embrechts et al. [EKM97]) and provides methods to calculate the norming constants. In Example 5.4 it is shown that for the two-dimensional Ornstein-Uhlenbeck process $F \in \mathrm{DA}(\Lambda)$ and the norming sequences $(c_T)_{T>0}$, $(d_T)_{T>0}$ are computed explicitly. See also Borkovec and Klüppelberg [BK01] where the long time behavior of one-dimensional stationary diffusions appearing in financial mathematics is studied.

We come back to the evaluation of the eigenvalue asymptotics, which is the main result of this paper. We will give conditions, when an asymptotic expression l for λ_R can be found satisfying (2.8). The idea is to approximate the potential Φ by a rotationally symmetric potential $x \mapsto \phi(|x|)$, where $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$. If the potential Φ is already rotationally symmetric, the process $(X_t)_{t\geq 0}$ can be identified with a one-dimensional process for which the eigenvalue asymptotics is known, see e.g. Newell [New62]. The terms caused by the asymmetric part of the potential

$$\Phi_{as}(x) := \Phi(x) - \phi(|x|) \quad (\in \mathbb{R} \cup \{\infty\}) \qquad x \in \mathbb{R}^n$$

need to be small in a certain sense. The evaluation of the eigenvalue asymptotics is reduced to find a rotationally symmetric potential ϕ satisfying conditions described in the sequel.

Let us introduce the following spherical integral: for a function $f : \mathbb{R}^n \to \mathbb{R}$ set

$$m_R[f] := \int_{S^{n-1}} f(R\xi) d\sigma(\xi)\,,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ the surface measure of S^{n-1}. Note that this corresponds to the integral over the sphere with radius R normalized to the volume γ_n of the unit sphere S^{n-1}. We define two terms measuring the asymmetry of Φ w.r.t. ϕ. The term

$$\delta_{as}(R) := m_R[e^{-2\Phi_{as}/\sigma^2}] = e^{2\phi(R)/\sigma^2} m_R[e^{-2\Phi/\sigma^2}] \qquad R > 0$$

can be interpreted as the spherical mean of the stationary measure of $(X_t)_{t\geq 0}$ relative to the stationary measure corresponding to the rotationally symmetric potential. Note that $\delta_{as}(R) \equiv \gamma_n$, $R > 0$, if Φ is already rotationally symmetric. Also the derivative of Φ_{as} must be controlled. Set

$$\begin{aligned} \Delta_{as}(x) &:= \begin{cases} \frac{1}{|x|}\sum_{i=1}^n x_i\, \partial_{x_i}\Phi(x) - \phi'(|x|) & x \in \mathcal{Z}^c \setminus \{0\}\,, \\ 0 & x \in \mathcal{Z} \setminus \{0\}\,, \end{cases} \\ D_{as}(R) &:= m_R[e^{-2\Phi_{as}/\sigma^2}\Delta_{as}^2] \qquad R > 0\,. \end{aligned}$$

Note that Δ_{as} is essentially the derivative of Φ_{as} in radial direction. Further if Φ is already rotationally symmetric, Δ_{as} and hence D_{as} vanish identically. The term D_{as} can be interpreted as the spherical mean of the square of the radial derivative of the asymmetric part Φ_{as} of the potential weighted with the stationary measure corresponding to Φ_{as}.

The crucial condition on ϕ is the asymptotic relation

$$D_{as}(R) = o\left(\delta_{as}(R)\right) \quad (R \to \infty)\,. \tag{2.10}$$

If Φ is already rotationally symmetric, this condition is trivially satisfied, since $D_{as}(R) \equiv 0$, $R > 0$. For some non-symmetric potentials, condition (2.10) can be shown to hold with the help of Laplace's method, see Lemma 5.1 and also some examples in section 5.

Further some weak growth conditions need to be imposed on ϕ. To this aim we define for a measurable real function g

$$\nu[g](R) := \int_1^R r^{1-n} g(r) e^{2\phi(r)/\sigma^2}\, dr\,, \tag{2.11}$$

whenever the integral exists. Moreover set $\nu(R) := \nu[\mathbf{1}](R)$, where $\mathbf{1}$ is the constant function 1. Note that the integrand is essentially the reciprocal of the stationary measure associated to the rotationally symmetric potential already integrated w.r.t. to the spherical components. Hence ν is expected to grow exponentially. The growth conditions on ϕ are as follows:

$$\nu(R), \nu[\delta_{as}](R) \nearrow \infty\,, \quad \nu[\delta_{as}](R) = o\left(\nu(R)^2\right) \qquad (R \to \infty)\,. \tag{2.12}$$

The second condition implies that δ_{as} does not decay too fast such that the exponential growth of $e^{2\phi/\sigma^2}$, the reciprocal of the stationary measure, is destroyed, whereas the third condition forbids δ_{as} to grow faster than $e^{2\phi/\sigma^2}$.

Theorem 2.8 *Assume the situation of Proposition 2.1. Further suppose that there exists a function $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$ satisfying (2.10) and (2.12). Set*

$$l(R) := \frac{\sigma^2}{2Z_\sigma}\,\delta_{as}(R)\,\nu(R)^{-1} \qquad R > 0\,,$$

where Z_σ the total mass of the stationary measure defined in (2.5). Then the function l satisfies (2.8).

Remark 2.9 (1) Heuristically large fluctuations of the process $(X_t)_{t\geq 0}$ occur in regions where the potential Φ is flat. Hence the asymptotic behavior of $P_\mu(M_T \leq R)$ and also of λ_R as $R \to \infty$ should be determined by the minimal slope of the potential Φ. Thus ϕ in Theorem 2.8 should be given by the spherical minimum of Φ, i.e. $\phi(R) := \min\{\Phi(y) : |y| = R\}$. However this is not always the right choice for ϕ. For a counterexample, where ϕ is not the spherical minimum, see Example 5.5.

(2) If ϕ is the spherical minimum as in (1), the condition $\nu[\delta_{as}](R) \nearrow \infty$ as $R \to \infty$ also implies the two other conditions in (2.12). This is due to the fact that $0 \leq \delta_{as}(R) \leq 1$ for every $R > 0$ by the choice of ϕ. Hence also $0 \leq \nu[\delta_{as}](R) \leq \nu(R)$ for every $R > 0$.

(3) The condition $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$ of Theorem 2.8 is not necessarily satisfied if ϕ is the spherical minimum as in (1). A counterexample can be constructed by means of the potential $\Phi(x_1, x_2) := 1 + x_1(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2 + 1)^{-1}$. Obviously $\Phi \in C^1(\mathbb{R}^2, \mathbb{R})$ but ϕ is not differentiable at $R = 1$.

(4) Since $\nu(R) \nearrow \infty$ as $R \to \infty$ by (2.12), the definition of ν in (2.11) is independent of the lower limit of the integral (here chosen to be 1). This is an easy consequence of Hopital's rule.

(5) We use the spectral gap assumption (2.7) to simplify the control of the asymptotics of the lower bound on λ_R in Temple's inequality (see Proposition 3.5 and Remark 3.6). The spectral gap assumption can be abandoned if one can show that $\lambda_R = o(\lambda_{R,2})$ as $R \to \infty$, where $\lambda_{R,2}$ is the second eigenvalue of the operator $-L_R$ (see also (3.3)).

(6) Theorem 2.8 gives conditions when sharp eigenvalue asymptotics can be obtained testing with rotationally symmetric test-functions (see step 1 in the proof of Theorem 2.8 in section 4). If these conditions fail or if we replace the balls $(B_R)_{R>0}$ by an arbitrary exhausting family of $\mathbb{R}^n$ as in Remark 2.2.(3), we can not use rotationally symmetric test-functions any more. In Kunz [Kun02b] the case is treated where the exhausting family $(O_R)_{R>R_0}$ of $\mathbb{R}^n$ is given by the level sets of the potential itself, i.e. $O_R := \{x \in \mathbb{R}^n : \Phi(x) < R\}$. This choice of the exhausting family of $\mathbb{R}^n$ is more adapted to the geometry of the problem. In this case we use test-functions for the eigenvalue asymptotics, which are constant on the iso-level sets of the potential Φ.

Up to now we have restricted ourselves to diffusion processes of gradient field type. There is a straight forward generalization towards uniformly elliptic reversible diffusions. Set $a^{ij}(x) := (1/2)(\sigma(x)\sigma(x)^T)^{ij}$, where $(\sigma_{ij})^{ij}$ is the diffusion matrix in the SDE (1.1). Assume the following uniform ellipticity condition: there exist constants $0 < \alpha_* < \alpha^*$ such that

$$\alpha_*|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \alpha^*|\xi|^2 \qquad x, \xi \in \mathbb{R}^n\,. \tag{2.13}$$

The reversibility of the process implies that there exists a positive function μ such that the drift in the SDE (1.1) reads formally for $i = 1, \dots, n$

$$b^i(x) = \frac{1}{\mu(x)} \sum_{j=1}^{n} \partial_{x_j}\left(a^{ij}(x)\mu(x)\right) \qquad x \in \{x \in \mathbb{R}^n : \mu(x) > 0\}\,. \tag{2.14}$$

To an uniformly elliptic reversible diffusion we associate a diffusion process of gradient field type solving the SDE (1.2) with $\Phi = -\ln\mu$ and $\sigma = \sqrt{2}$. Assume that Φ satisfies the conditions (2.2)-(2.5). An extension of Proposition 3.1 guarantees the existence of a weak solution of the SDE (1.1) which is reversible w.r.t. the measure with Lebesgue density μ on $\mathbb{R}^n$ (see also the last paragraph of section 3). There is a generalization of Theorem 2.3

Corollary 2.10 *Let $(X_t)_{t\geq 0}$ be a uniformly elliptic reversible diffusion process. Suppose there exists a function l satisfying $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R corresponds to the associated process of gradient field type. Then for every $T > 0$*

$$\alpha_* T\, l(R) \lesssim P_\mu(M_T > R) \lesssim \alpha^*(T + 1/\Lambda)\, l(R) \qquad (R \to \infty)\,.$$

Remark 2.11 For Theorem 2.4, which allows to analyze the long term behavior of the maximum M_T, there is no straight forward generalization to uniformly elliptic reversible diffusions. This is due to the fact that in the limit $T \to \infty$, the difference between asymptotic upper and lower bound in Corollary 2.10 does not tend to zero (see also the motivation to Theorem 2.4).

3 Preliminaries: Markov Processes and Operator Theory

The behavior of the maximum M_T in Euclidean norm up to time T of a reversible diffusion process is related to spectral properties of its generator. These properties are better to handle working with Hilbert spaces. We concentrate on diffusion processes of gradient field type solving a SDE of the form (1.2). According to Remark 2.2.(1) we assume that $\sigma^2 = 2$ and that the measure μ with Lebesgue density (also denoted by μ) defined in (2.4) is a probability measure on $\mathbb{R}^n$.

We use the following notations: for $R \in (0, \infty]$ we denote by μ_R the restriction of μ to the ball B_R with radius R (where we set $B_\infty := \mathbb{R}^n$). We write for short $L^2_{\mu_R}$ for $L^2(B_R, \mu_R)$ and $\|\cdot\|_{2,R}$ resp. $(\cdot,\cdot)_R$ for norm resp. scalar product in $L^2_{\mu_R}$. Further the characteristic function of a set A is denoted by I_A.

We construct a weak solution of the SDE (1.2) which is symmetric (and hence reversible) w.r.t. the measure μ. Recall that for a μ-symmetric process $(X_t)_{t\geq 0}$ the associated backward semigroup $(P_t)_{t\geq 0}$ with $P_t f(x) := E_x[f(X_t)]$, $x \in \mathbb{R}^n$, $t \geq 0$, extends (under some regularity conditions) to a strongly contraction semigroup on $L^2_{\mu_\infty}$. The generator of this semigroup stands in one-to-one correspondence with a Dirichlet form, see e.g. chapter 1.4 of Fukushima et al. [FOT94].

We define the following operators and quadratic forms: for $R \in (0,\infty]$ set

$$\mathcal{E}'_R(u,v) := \sum_{i=1}^n \int_{B_R} \partial_{x_i} u \, \partial_{x_i} v \, \mu \, dx \qquad u,v \in C_0^2(B_R)\,, \tag{3.1}$$

where $C_0^2(B_R)$ is the set of two times continuously differentiable functions having value 0 at the boundary of B_R. Note that $\mu^{-1} = e^{\Phi}$ is continuous in $\mathcal{Z}^c$ by condition (2.2) and hence is an element of $L^1_{loc}(\mathcal{Z}^c)$. Thus for every $R \in (0,\infty]$, the quadratic form $(\mathcal{E}'_R, C_0^2(B_R))$ is closable in $L^2_{\mu_R}$ and its closure $(\mathcal{E}_R, \mathcal{D}(\mathcal{E}_R))$ is a symmetric Dirichlet form, see e.g. section II.2.(a) in Ma and Röckner [MR92]. Let $(-L_R, \mathcal{D}(L_R))$ be the positive, selfadjoint operator on $L^2_{\mu_R}$ associated to $(\mathcal{E}_R, \mathcal{D}(\mathcal{E}_R))$. Using the continuity of $\mu = e^{-\Phi}$ in $\mathbb{R}^n$ and the differentiability in $\mathcal{Z}^c$ (stated in condition (2.2)) we have for a function $u \in C_0^2(B_R)$

$$u \in \mathcal{D}(L_R) \quad \text{if} \quad Lu \cdot I_{\mathcal{Z}^c} \in L^2_{\mu_R}\,, \tag{3.2}$$

where L is the differential operator defined in (2.6). For every $R \in (0,\infty]$ the operator L_R induces a strongly continuous contraction semigroup $(e^{L_R t})_{t\geq 0}$ on $L^2_{\mu_R}$.

The connection between the semigroup $(e^{L_\infty t})_{t\geq 0}$ and weak solutions of the SDE (1.2) is provided in the next proposition. A proof can be found in Meyer and Zheng [MZ85], see also section 6.3 of Fukushima et al. [FOT94]. Set $\Omega := C([0,\infty), \mathbb{R}^n)$, equipped with the canonical projections $(X_t)_{t\geq 0}$ and the natural filtration $(\mathcal{F}_t)_{t\geq 0}$.

Proposition 3.1 *Assume that the potential Φ satisfies (2.2) and (2.3). There exists a μ-symmetric diffusion process $\mathbf{X} := (\Omega, (\mathcal{F}_t), (X_t), (P_x)_{x\in\mathbb{R}^n})$ with life time ζ satisfying $P_\mu(\zeta < \infty) = 0$, where μ is defined in (2.4) and $P_\mu := \int_{\mathbb{R}^n} P_x \mu(x) dx$. Moreover $\mathbf{X}$ never hits the set $\mathcal{Z}$ defined in (2.1) in the sense that $P_\mu(\dot\sigma_{\mathcal{Z}} < \infty) = 0$, where $\dot\sigma_{\mathcal{Z}} := \inf\{t \geq 0 : X_t \in \mathcal{Z}\}$. $\mathbf{X}$ is associated to the Dirichlet form $\mathcal{E}_\infty$, i.e. the $L^2_{\mu_\infty}$-extension of its backward semigroup $(P_t)_{t\geq 0}$ coincides with the semigroup $(e^{L_\infty t})_{t\geq 0}$. $\mathbf{X}$ solves the SDE (1.2) in the following sense: there exists an increasing sequence $(T_n)_{n\in\mathbb{N}}$ of stopping times with $T_\infty := \lim_n T_n$ such that $P_\mu(T_\infty < \infty) = 0$ and $\mathbf{X}$ is a weak solution of the SDE (1.2) on $[0, T_n)$ for every $n \in \mathbb{N}$.*

Remark 3.2 A slightly stronger formulation of $\mathbf{X}$ being a weak solution of the SDE (1.2) is the following, see section II of Meyer and Zheng [MZ85]: there exists a polar set N such that for every $x \notin N$ the process $(X_t - X_0 + \int_0^t \nabla\Phi(X_s) ds)_{t\geq 0}$

is a standard Brownian motion for the law P_x. Recall that a set $N \subset \mathbb{R}^n$ is polar if there exists a Borel set $N' \subset \mathbb{R}^n$ with $N \subset N'$ such that $P_x(\sigma_{N'} < \infty) = 0$ for every $x \in \mathbb{R}^n$, where $\sigma_{N'} := \inf\{t > 0 : X_t \in N'\}$.

For $R \in (0,\infty)$, we denote by $(X_t^R)_{t\geq 0}$ the part of $(X_t)_{t\geq 0}$ on B_R, i.e. the process $(X_t)_{t\geq 0}$ killed when it hits the set $\mathbb{R}^n \backslash B_R$. $(X_t^R)_{t\geq 0}$ is μ_R-symmetric and the backward semigroup $(P_t^R)_{t\geq 0}$ of $(X_t^R)_{t\geq 0}$ is given by $P_t^R f(x) := E_x[f(X_t) I_{\{\tau_R > t\}}]$, $x \in \mathbb{R}^n$, $t \geq 0$, where τ_R is defined in (1.3). We denote by $\mathbf{1}$ the constant function 1.

Lemma 3.3 *Let $R \in (0,\infty)$.*
(i) $(X_t^R)_{t\geq 0}$ is associated to the Dirichlet form $\mathcal{E}_R$ in the sense that the $L^2_{\mu_R}$-extension of the backward semigroup $(P_t^R)_{t\geq 0}$ of $(X_t^R)_{t\geq 0}$ coincides with the semigroup $(e^{L_R t})_{t\geq 0}$.
(ii) $P_\mu(\tau_R > T) = (e^{L_R T}\mathbf{1}, \mathbf{1})_R$ for every $T > 0$.

Proof. *(i)* Since B_R is open, this follows e.g. from Theorem 4.4.2 and Theorem 4.4.3(i) of Fukushima et al. [FOT94]. *(ii)* Obviously $\mathbf{1} \in L^2_{\mu_R}$. Hence

$$\begin{aligned} P_\mu(\tau_R > T) &= \int_{B_R} P_x(\tau_R > T)\,\mu(x)dx = \int_{B_R} E_x[I_{\{\tau_R > T\}}]\,\mu(x)dx \\ &= \int_{B_R} P_T^R \mathbf{1}(x)\,\mu(x)dx = (e^{L_R T}\mathbf{1}, \mathbf{1})_R\,. \quad \Box \end{aligned}$$

Next we analyze some spectral properties of L_R. Recall the definition of the bottom eigenvalue $\lambda_R := \inf \Sigma(-L_R)$, $R \in (0,\infty]$, where $\Sigma(-L_R)$ is the spectrum of the operator $-L_R$ (with respect to $L^2_{\mu_R}$). Obviously $\Sigma(-L_R) \subset [0,\infty)$ for every $R \in (0,\infty]$ and $\lambda_\infty = 0$. Further we set

$$\lambda_{R,2} := \inf \Sigma(-L_R) \cap (\lambda_R, \infty) \qquad R \in (0,\infty]\,. \tag{3.3}$$

For $R \in (0,\infty)$, the operator $-L_R$ has discrete spectrum, since the domain B_R is bounded, and hence λ_R is an eigenvalue and is known to be simple. Further $\lambda_{R,2} > \lambda_R \geq 0$ for every $R \in (0,\infty]$. Since $\mathcal{D}(\mathcal{E}_R) \subset \mathcal{D}(\mathcal{E}_\infty)$ for $R \in (0,\infty]$ we get by the min-max principle that $\lambda_{R,2} \geq \lambda_{\infty,2}$ and also that $\lambda_R \searrow 0$ as $R \to \infty$. Note that the spectral gap property (2.7) just states that $\Lambda = \lambda_{\infty,2} > 0$.

We cite a result of Iscoe and McDonalds [IM94], theorem 2.13, giving upper and lower bounds for the quantity $(e^{L_R T}\mathbf{1},\mathbf{1})_R$ (appearing in Lemma 3.3.(ii)). Since it is of some importance here, a sketch of the proof is given.

Lemma 3.4 *Assume (2.7) and that μ is a probability measure on $\mathbb{R}^n$. Then for every $T>0$ and sufficiently large $R>0$*

$$(1-\lambda_R/\Lambda)e^{-\lambda_R T} \le (e^{L_R T}\mathbf{1},\mathbf{1})_R \le e^{-\lambda_R T}\,.$$

Sketch of the Proof. For the upper bound we estimate using Cauchy-Schwartz inequality and the fact that μ is a probability measure on $\mathbb{R}^n$

$$(e^{L_R T}\mathbf{1},\mathbf{1})_R \le \|e^{L_R T}\mathbf{1}\|_{2,R}\,\|\mathbf{1}\|_{2,R} \le \|e^{L_R T}\|_{L^2_{\mu_R}\to L^2_{\mu_R}}\,\|\mathbf{1}\|^2_{2,\infty} \le e^{-\lambda_R T}\,.$$

For the last inequality note that the norm of the operator $e^{L_R T}$ can be estimated by $\inf\{e^{-\lambda T} : \lambda\in\Sigma(-L_R)\} = e^{-\lambda_R T}$ using the spectral theorem.

To obtain the lower bound, let $\phi\in L^2_{\mu_R}$ be an eigenfunction of $-L_R$ corresponding to the simple eigenvalue λ_R (extended to be 0 outside B_R). $\mathbf{1}\in L^2_{\mu_\infty}$ and hence there exists $\psi\in L^2_{\mu_\infty}$ such that $\mathbf{1}=\phi+\psi$ and $(\phi,\psi)_\infty=0$. Since $e^{L_R T}$ is a positive operator we obtain

$$\begin{aligned}(e^{L_R T}\mathbf{1},\mathbf{1})_R &= \|\phi\|^2_{2,R}e^{-\lambda_R T} + (e^{L_R T}\psi,\psi)_R \\ &\ge \|\phi\|^2_{2,\infty}e^{-\lambda_R T} = (1-\|\psi\|^2_{2,\infty})e^{-\lambda_R T}\,.\end{aligned}$$

We need an upper bound for $\|\psi\|^2_{2,\infty}$. Let $\{E_\lambda : \lambda\in\Sigma(-L_\infty)\}$ be the family of spectral projections associated to $-L_\infty$ and set $\mu_\psi(d\lambda) = d(E_\lambda\psi,\psi)_\infty$. With the simplifying assumption that $-L_\infty$ is bounded (which can be abandoned) we obtain

$$\begin{aligned}\|\psi\|^2_{2,\infty} &= \int_{\Sigma(-L_\infty)} \mu_\psi(d\lambda) = (E_{\{0\}}\psi,\psi)_\infty + \int_\Lambda^\infty \mu_\psi(d\lambda) \\ &\le (\psi,\mathbf{1})^2_\infty + \frac{1}{\Lambda}\int_\Lambda^\infty \lambda\,\mu_\psi(d\lambda) = \|\psi\|^4_{2,\infty} + \frac{1}{\Lambda}\mathcal{E}_\infty(\psi,\psi)\,.\end{aligned} \tag{3.4}$$

Further $\mathcal{E}_\infty(\psi,\psi)=\mathcal{E}_\infty(\phi,\phi)=\lambda_R\|\phi\|^2_{2,R}=\lambda_R(1-\|\psi\|^2_{2,R})$. Plugging this in (3.4), we obtain a quadratic inequality in $\|\psi\|^2_{2,\infty}$, which yields $\|\psi\|^2_{2,\infty}\le\min(\lambda_R/\Lambda,1)$.

□

To estimate the bottom of the spectrum λ_R of $-L_R$ we use the variation principle for upper bounds and Temple's inequality for lower bounds. For $R \in (0,\infty]$ and a function $v \in \mathcal{D}(L_R)$ we define the Rayleigh quotient ρ_R and the expression l_R by

$$\rho_R(v) := \|v\|_{2,R}^{-2}\mathcal{E}_R(v,v)\,, \quad l_R(v) := \|v\|_{2,R}^{-2}\|L_R v\|_{2,R}^2\,. \tag{3.5}$$

We summarize the bounds on λ_R in the following proposition (for a proof see e.g. theorems XIII.2 and XIII.5 of Reed and Simon [RS78]).

Proposition 3.5 *Let $R \in (0,\infty]$. Then for every $v \in \mathcal{D}(L_R)$ with $\rho_R(v) < \lambda_{R,2}$*

$$\rho_R(v) - \frac{l_R(v) - \rho_R(v)^2}{\lambda_{R,2} - \rho_R(v)} \le \lambda_R \le \rho_R(v)\,.$$

Remark 3.6 For the lower bound we need to show that $\rho_R(v) < \lambda_{R,2}$. The situation simplifies if we assume the spectral gap property (2.7). Since $\lambda_{R,2} \ge \lambda_{\infty,2} = \Lambda$ for every $R \in (0,\infty)$, we can replace $\lambda_{R,2}$ by Λ in Proposition 3.5.

Next we state a condition on the potential Φ, such that the spectral gap assumption (2.7) holds. We will make use of the fact, that the operator $-L_\infty$ is unitarily equivalent to the Schrödinger operator $-\Delta + V_\Phi$ on $\mathbb{R}^n$ with potential

$$V_\Phi(x) := \tfrac{1}{4}|\nabla\Phi(x)|^2 - \tfrac{1}{2}\Delta\Phi(x) \qquad x \in \mathbb{R}^n\,, \tag{3.6}$$

provided $\Phi \in C^2(\mathbb{R}^n,\mathbb{R})$. We use the notation $\liminf_{|x|\to\infty} V(x) := \lim_{R\to\infty}\inf_{|x|>R} V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$.

Proposition 3.7 *Suppose $\Phi \in C^2(\mathbb{R}^n,\mathbb{R})$ and $\liminf_{|x|\to\infty} V_\Phi(x) > 0$. Then the spectral gap property (2.7) holds.*

Proof. Consider the unitary transform

$$U : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, d\mu),\ f \mapsto e^{\Phi/2}f\,,$$

where dx is the Lebesgue measure on $\mathbb{R}^n$. For functions $u, v \in C_0^2(\mathbb{R}^n)$ we get after some simple calculations using the integration by parts theorem

$$\mathcal{E}_\infty(Uu, Uv) = \sum_{i=1}^n \int_{\mathbb{R}^n} (\partial_{x_i}u\,\partial_{x_i}v + V_\Phi uv)\,dx =: Q_\Phi(u,v)\,. \tag{3.7}$$

Q_Φ is the quadratic form of the Schrödinger operator $H_\Phi := -\Delta + V_\Phi$ on $L^2(\mathbb{R}^n, dx)$. A standard result in the theory of Schrödinger operators (see e.g. theorem 3.1 of

Berezin and Shubin [BS91]) tells us that $c := \liminf_{|x|\to\infty} V_\Phi(x) > 0$ implies that H_Φ has discrete spectrum in $(-\infty, c)$. Since the transform U is unitary we deduce from (3.7) that the spectrum of H_Φ on $L^2(\mathbb{R}^n, dx)$ and the spectrum of $-L_\infty$ on $L^2_{\mu_R}$ coincide. Hence $-L_\infty$ has also discrete spectrum in $(-\infty, c)$. Thus $\lambda_\infty = 0$ is an eigenvalue and since $c > 0$, the result follows. $\square$

This section is closed with a few remarks on the extension towards uniformly elliptic reversible diffusion process. The same techniques are working for this class of processes. We analyze a process $(X_t)_{t\geq 0}$ solving the SDE (1.1), where we assume the uniform ellipticity condition (2.13) and suppose that the drift in the SDE (1.1) is of the form (2.14). Let us define the associated operators and quadratic forms. Set

$$\mathcal{E}'_{a,R}(u,v) := \sum_{i,j=1}^n \int_{B_R} a^{ij} \partial_{x_i} u \, \partial_{x_j} v \, \mu \, dx\,, \quad u,v \in C_0^2(B_R),\ R \in (0,\infty]\,,$$

where the functions μ and a^{ij}, $i,j = 1,\dots,n$ are defined in (2.13) and (2.14). This quadratic form is also closable in $L^2_{\mu_R}$ (see e.g. section II.2.(b) in Ma and Röckner [MR92]) and extends to a symmetric Dirichlet form $(\mathcal{E}_{a,R}, \mathcal{D}(\mathcal{E}_{a,R}))$ with associated positive, selfadjoint operator $(-L_{a,R}, \mathcal{D}(L_{a,R}))$ and strongly continuous contraction semigroup $(e^{L_{a,R}t})_{t\geq 0}$ on $L^2_{\mu_R}$.

Assume in addition that $\Phi := -\ln\mu$ satisfies the condition (2.2)-(2.5). By a modification of Proposition 3.1 it can be shown that there exists a weak solution of the SDE (1.1) which is μ-symmetric. Further the $L^2_{\mu_\infty}$-extension of the associated backward semigroup coincides with the semigroup $(e^{L_{a,\infty}t})_{t\geq 0}$ (just replace the Brownian motion $\mathbf{W}$ in the proof of Proposition 3.1 by the strong solution $\mathbf{V}$ of the SDE $dV_t^i = \sum_j \sigma^{ij}(V_t) dW_t^i$, $i = 1,\dots,n$). Lemma 3.3, Lemma 3.4, and hence Proposition 2.1 remain valid if we replace the operator L_R by $L_{a,R}$ and λ_R by $\lambda_R^a := \inf \Sigma(-L_{a,R})$. Comparing $\mathcal{E}_{a,R}$ with the Dirichlet form $\mathcal{E}_R$ (defined in (3.1)) of the associated diffusion process of gradient field type solving the SDE (1.2) with $\Phi = -\ln\mu$ and $\sigma = \sqrt{2}$, we obtain

$$\alpha_* \mathcal{E}_R(u,v) \leq \mathcal{E}_{a,R}(u,v) \leq \alpha^* \mathcal{E}_R(u,v) \quad u,v \in \mathcal{D}(\mathcal{E}_R),\ R \in (0,\infty]\,.$$

Hence we deduce that

$$\alpha_* \lambda_R \le \lambda_R^a \le \alpha^* \lambda_R \qquad R \in (0, \infty]\,, \tag{3.8}$$

where $\lambda_R := \inf \Sigma(-L_R)$ and $-L_R$ is the self adjoint operator associated to $\mathcal{E}_R$.

4 Proofs

In this section we prove the results stated in section 2. Assume the situation of Proposition 3.1 that guarantees the existence of a diffusion process $(X_t)_{t\ge 0}$ of gradient field type solving the SDE (1.2). Suppose that there exists a function l satisfying (2.8).

Proof of Theorem 2.3. Fix $T > 0$. Using Proposition 2.1 and the inequality $1 - x \le e^{-x}$ for every $x \in \mathbb{R}$, we get for sufficiently large $R > 0$

$$\begin{aligned} 1 - e^{-\lambda_R T} \le P_\mu(M_T > R) &\le 1 - (1 - \lambda_R/\Lambda) e^{-\lambda_R T} && (4.1) \\ &\le 1 - (1 - \lambda_R/\Lambda)(1 - T\lambda_R) \\ &= (T + 1/\Lambda)\lambda_R + (T/\Lambda)\lambda_R^2\,. && (4.2) \end{aligned}$$

We deduce that

$$T\,\lambda_R \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda)\lambda_R \quad (R \to \infty)\,. \tag{4.3}$$

The left asymptotic inequality follows since $T\lambda_R P_\mu(M_T > R)^{-1} \le T\lambda_R(1 - e^{-\lambda_R T})^{-1}$ by (4.1). The latter converges to 1 as $R \to \infty$. Dividing inequality (4.2) by the term $(T + 1/\Lambda)\lambda_R$ and passing to the limit yields the right asymptotic inequality. Since by assumption $\lim_{R\to\infty} \lambda_R / l(R) = 1$, the result follows. □

Proof of Theorem 2.4. By assumption $\lambda_R \sim l(R)$ as $R \to \infty$ and hence $\lambda_R = l(R) + \epsilon(R)$ where $\epsilon(R) = o(l(R))$ as $R \to \infty$. Using Proposition 2.1 and the inequality $|e^x - 1| \le |x|(1 + e^{|x|})$ for every $x \in \mathbb{R}$, we estimate for fixed $R, T > 0$ with $\lambda_R \ge 0$ for every $R > 0$ in mind

$$\begin{aligned} \left| P_\mu(M_T \le R) - e^{-l(R)T} \right| &\le e^{-l(R)T} \max\left\{ \left| e^{-\epsilon(R)T} - 1 \right|, \left| \left(1 - \frac{\lambda_R}{\Lambda}\right) e^{-\epsilon(R)T} - 1 \right| \right\} \\ &\le e^{-l(R)T} \left(|e^{-\epsilon(R)T} - 1| + \frac{\lambda_R}{\Lambda} e^{-\epsilon(R)T} \right) \end{aligned}$$

$$
\begin{aligned}
&= e^{-\lambda_R T}\left(|1-e^{\epsilon(R)T}|+\frac{\lambda_R}{\Lambda}\right)\\
&\le e^{-\lambda_R T}\,|\epsilon(R)|T\left|1+e^{|\epsilon(R)|T}\right|+\frac{\lambda_R}{\Lambda}=:I(R)+\frac{\lambda_R}{\Lambda}\,.
\end{aligned}
$$

Now let $(R_T)_{T\ge 0}$ be an arbitrary sequence with $R_T \nearrow \infty$ as $R \to \infty$ and we replace in the above estimations R by R_T. Since $\lim_{R\to\infty}\lambda_R = 0$ we also have $\lim_{T\to\infty}\lambda_{R_T} = 0$.

It remains to show that also $\lim_{T\to\infty} I(R_T) = 0$. We choose an arbitrary sequence $(T_i)_{i\in\mathbb{N}}$ with $T_i \nearrow \infty$ as $i \to \infty$ and we write for short $\lambda_i := \lambda_{R_{T_i}}$, $\epsilon_i := \epsilon_{R_{T_i}}$, and $I_i := I(R_{T_i})$. Assume for the moment that $\{\lambda_i T_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ is bounded. Then $\lim_{i\to\infty}\epsilon_i T_i = 0$, since $\lim_{i\to\infty}\lambda_i^{-1}\epsilon_i = 0$ by definition of $\epsilon(R)$. Since $\lambda_i T_i \ge 0$, it follows

$$
I_i \le |\epsilon_i T_i|(1+e^{|\epsilon_i T_i|}) \to 0 \quad (i\to\infty)\,.
$$

If $\{\lambda_i T_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ is unbounded, then $\lambda_i T_i \nearrow \infty$ as $i \to \infty$ after extraction of a subsequence. Since $\lim_{i\to\infty}\lambda_i^{-1}\epsilon_i = 0$, we have $|\epsilon_i| \le \lambda_i/2$ for large i. Hence for large i

$$
I_i \le (\lambda_i/2)T_i(e^{-\lambda_i T_i}+e^{-(\lambda_i/2)T_i}) \le \lambda_i T_i e^{-(\lambda_i/2)T_i} \to 0 \quad (i\to\infty)\,.
$$

Since the choice of the sequence $(T_i)_{i\in\mathbb{N}}$ was arbitrary, the result follows. □

Proof of Corollary 2.6. Assume that $F := e^{-l} \in \mathrm{DA}(H)$ for an extreme value distribution H with norming constants $(c_T)_{T>0}$, $(d_T)_{T>0}$ according to (2.9). We have for every $x\in\mathbb{R}$ setting $R_T := c_T x + d_T$

$$
\begin{aligned}
&\left|P_\mu\left(c_T^{-1}(M_T-d_T)\le x\right)-H(x)\right|\\
&\quad\le \left|P_\mu(M_T\le R_T)-F(R_T)^T\right|+\left|F(R_T)^T-H(x)\right|\,.
\end{aligned}
$$

The first term vanishes by Theorem 2.4 and the second by (2.9) as $T\to\infty$. □

The main result of this paper, Theorem 2.8, determines the fine asymptotics of λ_R as $R \to \infty$ in the sense that a function l is given satisfying (2.8). To this aim suitable test-functions $(v_R)_{R>0}$ must be found such that the bounds on λ_R in Proposition 3.5 get sharp in the limit $R\to\infty$. We work with rotationally symmetric test-functions. According to Remark 2.2.(1) we may assume w.l.o.g. that

$$\sigma^2 = 2 \quad \text{and} \quad \mu(dx) = e^{-\Phi(x)}dx \text{ is a probability measure on } \mathbb{R}^n. \tag{4.4}$$

For a function $v \in \mathcal{D}(\mathcal{E}_R)$ we write for short $\mathcal{E}_R(v)$ for $\mathcal{E}_R(v,v)$ and the norm in $L^2_{\mu_R} := L^2(B_R, \mu_R)$ is again denoted by $\|\cdot\|_{2,R}$.

Proof of Theorem 2.8.

Step 1: Construction of the test-functions. By assumption $\phi \in C^1(\mathbb{R}^+, \mathbb{R})$ and we deduce from definition (2.11) that $\nu \in C^2((1,\infty), \mathbb{R}^+)$. ν can be extended to a function $\tilde{\nu} \in C^2_c((0,\infty), \mathbb{R}^+)$ with compact support in $(0,\infty)$. For $R > 1$ we define the test-functions by

$$v_R(x) := 1 - \tilde{\nu}(|x|)/\nu(R) \qquad |x| \le R\,.$$

Note that $\nu(R) > 0$ for $R > 1$ and hence v_R is well defined. By the above construction we have obviously $v_R \in C^2_0(B_R)$ for every $R > 1$. In step 4 it is shown that $Lv_R \cdot I_{\mathcal{Z}^c} \in L^2_{\mu_R}$. Hence $v_R \in \mathcal{D}(L_R)$ for $R > 1$ by (3.2).

Step 2: $\mathcal{E}_R(v_R) \sim \nu[\delta_{as}](R)\,\nu(R)^{-2} \to 0$ as $R \to \infty$. Using step 1 and definition (2.11) of ν we calculate for $R > 1$

$$\begin{aligned} \nu(R)^2 \mathcal{E}_R(v_R) &= \nu(R)^2 \sum_{i=1}^n \int_{B_R} |\partial_{x_i} v_R|^2 e^{-\Phi} dx \\ &= \int_0^R r^{n-1} \tilde{\nu}'(r)^2 m_r[e^{-\Phi}]\, dr \\ &= \left(\kappa_1 + \int_1^R r^{1-n} e^{2\phi(r)} m_r[e^{-\Phi}]\, dr\right) \\ &= (\kappa_1 + \nu[\delta_{as}](R))\ , \end{aligned}$$

where $\kappa_1 = \int_0^1 r^{n-1}\tilde{\nu}'(r)^2 m_r[e^{-\Phi}] dr < \infty$. Further $\kappa_1 + \nu[\delta_{as}](R) \sim \nu[\delta_{as}](R)$ as $R \to \infty$ since $\nu[\delta_{as}](R) \nearrow \infty$ as $R \to \infty$ by assumption (2.12). This proves the asymptotic equivalence and the convergence to 0 also follows from assumption (2.12).

Step 3: It suffices to show

$$\|Lv_R\|^2_{2,R} = o\,(\mathcal{E}_R(v_R)) \quad (R \to \infty)\,. \tag{4.5}$$

We need to prove that $\rho_R(v_R) \to 0$ as $R \to \infty$, where ρ_R is defined in (3.5). Then, using Remark 3.6 (having in mind that the spectral gap condition (2.7) holds), Proposition 3.5 is applicable for large R. We can deduce that $\lambda_R \sim \rho_R(v_R)$ as $R \to \infty$ if we can show (using again Remark 3.6)

$$\frac{l_R(v_R) - \rho_R(v_R)^2}{\Lambda - \rho_R(v_R)} = o\left(\rho_R(v_R)\right) \quad (R \to \infty)\,. \tag{4.6}$$

Since $\nu(R) \nearrow \infty$ as $R \to \infty$ by assumption (2.12), we deduce that $v_R \nearrow 1$ μ-a.s. as $R \to \infty$ (where v_R is extended to a function on $\mathbb{R}^n$ by setting 0 on $\mathbb{R}^n \setminus B_R$). Since μ is a probability measure on $\mathbb{R}^n$, we have

$$\|v_R\|_{2,R}^2 \to 1 \qquad (R \to \infty)\,. \tag{4.7}$$

Hence we get together with step 2 and the definition of ρ_R that $\rho_R(v_R) \to 0$ as $R \to \infty$. Further (4.7) allows to replace in (4.6) in the limit $R \to \infty$ the terms $l_R(v_R)$ resp. $\rho_R(v_R)$ (defined in (3.5)) by $\|Lv_R\|_{2,R}^2$ resp. $\mathcal{E}_R(v_R)$. But then (4.6) follows from (4.5) using again $\rho_R(v_R) \to 0$ as $R \to \infty$ and we get $\lambda_R \sim \mathcal{E}_R(v_R)$ as $R \to \infty$. In order to obtain the asymptotic expression $l(R)$ for λ_R we use step 2 and the fact that $\nu[\delta_{as}](R)\,\nu(R)^{-1} \sim \delta_{as}(R)$ as $R \to \infty$ (which is a consequence of Hopital's rule, applicable since $\nu(R) \nearrow \infty$ as $R \to \infty$ by assumption (2.12)). To obtain the general asymptotic expression $l(R)$ without assuming the simplifying condition (4.4) see Remark 2.2.(1).

Step 4: Condition (4.5) holds. We evaluate the term $\|Lv_R\|_{2,R}^2$. Recalling the alternative form of the operator L defined in (2.6), we calculate for $x \in \mathcal{Z}^c$ with $1 < |x| < R$

$$\begin{aligned}
\nu(R)^2 L v_R(x) &= \nu(R)^2 e^{\Phi(x)} \sum_{i=1}^n \partial_{x_i}\left(e^{-\Phi}\partial_{x_i} v_R\right)(x) \\
&= e^{\Phi(x)} \sum_{i=1}^n \partial_{x_i}\left(e^{-\Phi(x)}|x|^{-n} e^{\phi(|x|)} x_i\right) \\
&= e^{\phi(|x|)}\Big\{\frac{n}{|x|^n} + \sum_{i=1}^n x_i\Big[-\frac{n}{|x|^{n+1}}\frac{x_i}{|x|} + \frac{1}{|x|^n}\Big(\phi'(|x|)\frac{x_i}{|x|} - \partial_{x_i}\Phi(x)\Big)\Big]\Big\} \\
&= -|x|^{1-n} e^{\phi(|x|)} \Delta_{as}(x)\,.
\end{aligned}$$

Setting $\kappa_2 := \|L\tilde{\nu}\|_{2,1}^2 < \infty$ (the function $x \mapsto \tilde{\nu}(|x|)$ is also denoted by $\tilde{\nu}$), we have for $R > 1$

$$\nu(R)^2 \|Lv_R I_{\mathcal{Z}^c}\|_{2,R}^2 \;=\; \kappa_2 + \int_{B_R \setminus B_1} |x|^{2(1-n)} e^{2\phi(|x|) - \Phi(x)} \Delta_{as}(x)^2 dx$$

$$
\begin{aligned}
&= \kappa_2 + \int_1^R r^{1-n} e^{2\phi(r)} m_r[e^{-\Phi}\Delta_{as}^2]\, dr \\
&= \kappa_2 + \nu[D_{as}](R)\,.
\end{aligned}
$$

Using step 2 we see that condition (4.5) is satisfied if $\nu[D_{as}](R) = o\,(\nu[\delta_{as}](R))$ as $R \to \infty$. Since $\nu[\delta_{as}](R) \nearrow \infty$ as $R \to \infty$ by assumption (2.12), this follows immediately from the crucial condition (2.10) by an application of Hopital's rule. $\square$

Corollary 2.10 extends Theorem 2.3 to the case, where $(X_t)_{t\geq 0}$ is a uniformly elliptic reversible diffusion process. We use the notation of the last paragraph of section 3. Suppose there exists a function l satisfying $\lambda_R \sim l(R)$ as $R \to \infty$, where λ_R corresponds to the associated process of gradient field type.

Proof of Corollary 2.10. In this situation the asymptotic estimation (4.3) reads

$$
T\,\lambda_R^a \lesssim P_\mu(M_T > R) \lesssim (T + 1/\Lambda)\lambda_R^a \quad (R \to \infty)\,.
$$

The result follows from (3.8), since by assumption $\lambda_R \sim l(R)$ as $R \to \infty$. $\square$

5 Examples

We give some examples of diffusion processes of gradient field type for which the sharp eigenvalue asymptotics can be evaluated by Theorem 2.8 and hence the asymptotics of the maximum M_T of the process in Euclidean norm is given by Theorem 2.3 and Theorem 2.4. The long term behavior of the normalized maximum is explicitly given for the Ornstein-Uhlenbeck process, as mentioned in Corollary 2.6. We present also a situation where the eigenvalue asymptotics is not determined by the spherical minimum of the potential.

In some concrete cases, the crucial condition (2.10) of Theorem 2.8 can be shown by means of Laplace's method stated in the next lemma (for a proof see e.g. theorem 7.1 of Olver [Olv74])

Lemma 5.1 (Laplace's method) *Let $I \subset \mathbb{R}$ be an open interval containing 0 and $p \in C^1(I)$, $q \in C(I)$, where p attains its minimum only at 0. Assume further that there exist constants $P, \varpi, \eta > 0$ and $Q \in \mathbb{R}$ such that $p(\theta) - p(0) \sim P\theta^{\varpi}$, $p'(\theta) \sim$*

$\varpi P\theta^{\varpi-1}$, and $q(\theta) \sim Q\theta^{\eta-1}$ as $\theta \to \pm 0$. If $J(x) := \int_I e^{-xp(\theta)}q(\theta)\,d\theta$ converges absolutely for large x, then $J(x) \sim 2Q\varpi^{-1}\Gamma(\eta/\varpi)(Px)^{-\eta/\varpi}e^{-xp(0)}$ as $x \to \infty$.

Remark 5.2 Laplace's method is also applicable if the only minimum of p occurs at an endpoint of the interval I. Then the term J has to be multiplied by $1/2$.

The following lemma is used for the asymptotic evaluation of integrals over exponential terms appearing in the function ν defined in (2.11).

Lemma 5.3 *Let $A, \gamma > 0$ and $\delta \in \mathbb{R}$. Then*

$$\int_1^R r^\delta e^{Ar^\gamma}\,dr \sim (\gamma A)^{-1}R^{\delta-\gamma+1}e^{AR^\gamma} \quad (R \to \infty)\,.$$

PROOF. Apply Hopital's rule to the quotient. □

5.1 Rotationally Symmetric Case

Assume that the potential Φ in the SDE (1.2) has the property that there exist $R_0 > 0$ and $\phi \in C^2([R_0,\infty),\mathbb{R})$ such that

$$\Phi(x) = \phi(|x|) \qquad |x| > R_0\,. \tag{5.1}$$

Suppose further that Φ satisfies (2.2) and (2.3). Then Proposition 3.1 guarantees the existence of a weak solution $(X_t)_{t\geq 0}$ of the SDE (1.2).

Set $V(R) := (1/4)\phi'(R)^2 - (1/2)(\phi''(R) + (n-1)R^{-1}\phi'(R))$, $R > R_0$. To satisfy the spectral gap condition, we assume

$$\liminf_{R\to\infty} V(R) > 0\,. \tag{5.2}$$

Note that this condition is satisfied if ϕ has polynomial form $\phi(R) = R^\alpha$ where $\alpha \geq 1$. The volume of the unit sphere S^{n-1} in $\mathbb{R}^n$ is denoted by γ_n.

Theorem 5.4 *Let $(X_t)_{t\geq 0}$ be a diffusion process of gradient field type solving the SDE (1.2) with potential Φ of the form (5.1). Assume (2.5) and (5.2). Then the result of Theorem 2.8 holds with ϕ defined in (5.1) and $\delta_{as}(R) \equiv \gamma_n$ for every $R > R_0$.*

Remark 5.5 (a) The lower integration limit in the definition of ν in (2.11) is replaced by R_0 (see Remark 2.9.(4)).

(b) This result has already been established in articles analyzing the extreme behavior of one-dimensional diffusions (see e.g. [New62, Man68, Ber64, Dav82]). Note that condition (5.2), insuring the spectral gap property, was not needed in this context (see also Remark 2.2.(2) and Remark 2.9.(5)).

PROOF. We have to show that conditions (2.7), (2.12), and (2.10) are satisfied. The spectral gap condition (2.7) holds by Proposition 3.7 and assumption (5.2). As stated already in section 2 in the motivation for Theorem 2.8, $\delta_{as}(R) \equiv \gamma_n$ and $D_{as}(R) \equiv 0$ for $R > R_0$. Hence the crucial condition (2.10) holds immediately. Since $r^{n-1}e^{-2\phi(r)/\sigma^2} \to 0$ as $r \to \infty$ by (2.5), the growth condition (2.12) is also satisfied (see Remark 2.9.(2)). $\square$

5.2 Special Non-Symmetric Process

We consider a situation where the potential Φ in the SDE (1.2) is not finally rotationally symmetric. For notational convenience we restrict ourselves to the two-dimensional case and use polar coordinates writing $\mathbb{R}^2 \setminus \{0\} \ni x = Re_\theta$ where $R = |x| > 0$ and $e_\theta = (\cos\theta, \sin\theta)$, $\theta \in [\theta_0, \theta_0 + 2\pi)$. Assume $\Phi \in C^2(\mathbb{R}^2, \mathbb{R})$. If Φ satisfies also (2.2) and (2.3), Proposition 3.1 guarantees the existence of a weak solution $(X_t)_{t\geq 0}$ of the SDE (1.2).

We set $\phi(R) := \min\{\Phi(y) : |y| = R\}$ as in Remark 2.9.(1). Assume further that the asymmetric part of the potential factorizes in radial and spherical component, i.e. there exist $R > R_0$ and functions $\psi \in C^2((R_0, \infty), \mathbb{R}^+)$ and $p \in C^2(S^1, \mathbb{R}^+)$ such that

$$\Phi_{as}(R,\theta) = \Phi(R,\theta) - \phi(R) = p(\theta)\psi(R) \qquad R > R_0,\, \theta \in [\theta_0, \theta_0 + 2\pi)\,. \tag{5.3}$$

Note that $\Phi_{as} \geq 0$ by definition and the minimum in the definition of ϕ is attained. Hence $\psi \geq 0$, $p \geq 0$ and the zero-set $\mathcal{N}(p)$ of p is not empty. We assume that $\mathcal{N}(p)$ is finite, i.e.

$$\mathcal{N}(p) = \{\theta_1, \ldots, \theta_N\}\,, \tag{5.4}$$

where w.l.o.g. $\theta_0 < \theta_1 < \ldots < \theta_N < \theta_0 + 2\pi$. By assumption (5.4) we can find open disjoint intervals I_i containing θ_i such that $[\theta_0, \theta_0 + 2\pi] = \bigcup_{i=1}^N \bar{I}_i$, where $\bar{I}_i$ denotes the closure of I_i. Further we assume that for every $i = 1, \ldots, N$

$$\begin{aligned} &p(\cdot + \theta_i) \text{ satisfies the conditions of Lemma 5.1 on } \{\theta : \theta + \theta_i \in I_i\} \\ &\text{with corresponding constants } P_i, \varpi_i > 0. \end{aligned} \tag{5.5}$$

We set $\varpi_* := \max\{\varpi_i : i = 1, \ldots, N\}$ and $J_* := \{i : \varpi_* = \varpi_i\}$. Further we need that

$$\psi(R) \to \infty \qquad (R \to \infty)\,. \tag{5.6}$$

To assure the spectral gap condition (2.7) we assume that

$$\begin{aligned} &\liminf_{R\to\infty} \phi'(R) > 0\,, \\ &\left\{\phi''(R), \psi''(R), R^{-1}\psi'(R), R^{-2}\psi(R)\right\} \text{ are } o\left(\phi'(R)^2\right) \quad (R \to \infty)\,. \end{aligned} \tag{5.7}$$

In this setting, the crucial condition (2.10) of Theorem 2.8 takes the form of a regularity condition on ψ

$$\psi'(R) = o\left(\psi(R)\right) \quad (R \to \infty)\,. \tag{5.8}$$

The growth conditions (2.12) read in our situation

$$\int_{R_0}^{R} r^{-1}\psi(r)^{-1/\varpi_*} e^{2\phi(r)/\sigma^2} dr \nearrow \infty \quad (R \to \infty)\,. \tag{5.9}$$

Theorem 5.6 *Let $(X_t)_{t\geq 0}$ be a diffusion process of gradient field type solving the SDE (1.2), where Φ is of the form (5.3). Assume (2.5) and (5.4)-(5.8). Set*

$$l(R) \;:=\; C\psi(R)^{-1/\varpi_*} \left(\int_{R_0}^{R} r^{-1} e^{2\phi(r)/\sigma^2} dr\right)^{-1} \qquad R > R_0\,,$$

where $C := 2(Z_\sigma \varpi_*)^{-1}(\sigma^2/2)^{1+1/\varpi_*}\Gamma(1/\varpi_*)\sum_{i\in J_*} P_i^{-1/\varpi_*}$. Then l satisfies (2.8).

Remark 5.7 (a) Assume that ϕ, ψ are of polynomial form, i.e. $\phi(R) = R^\alpha$, $\psi(R) = R^\beta$ for large R. Then conditions (5.7) and (5.9) are satisfied if $\alpha \geq 1$ and $\beta \in (0, 2\alpha)$.

(b) If ψ is regularly varying with index $\gamma > 0$ as $R \to \infty$, then ψ satisfies condition (5.8), since by Karamatas theorem $R\psi'(R)\psi(R)^{-1} \to 1 + \gamma$ as $R \to \infty$. Especially (5.8) holds, if ψ is of polynomial form.

PROOF. We have to show that conditions (2.7), (2.12), and (2.10) are satisfied. First we evaluate the term δ_{as}. Invoking (5.5) and Laplace's method (Lemma 5.1), where by (5.6) the limes $x \to \infty$ can be replaced by $\psi(R) \to \infty$ as $R \to \infty$, we calculate

$$\begin{aligned} \delta_{as}(R) &= m_R[e^{-2\Phi_{as}/\sigma^2}] \\ &= \sum_{i=1}^{N} \int_{I_i} e^{-2p(\theta+\theta_i)\psi(R)/\sigma^2} d\theta \\ &\sim \sum_{i=1}^{N} \frac{2}{\varpi_i} \Gamma\left(\frac{1}{\varpi_i}\right) \left(\frac{2P_i}{\sigma^2}\psi(R)\right)^{-1/\varpi_i} \\ &\sim \frac{1}{\varpi_*} \left(\frac{\sigma^2}{2}\right)^{1/\varpi_*} \Gamma\left(\frac{1}{\varpi_i}\right) \left(\sum_{i\in J_*} P_i^{-1/\varpi_*}\right) \psi(R)^{-1/\varpi_*} \quad (R \to \infty) \,. \end{aligned} \tag{5.10}$$

The growth condition (2.12) follows immediately from assumption (5.9) and Remark 2.9.(2). To show the crucial condition (2.10), we need to evaluate asymptotically the term $D_{as}(R)$. Δ_{as} reads in polar coordinates

$$\Delta_{as}(R,\theta) = \partial_R \Phi(R,\theta) - \phi'(R) = p(\theta)\psi'(R) \,.$$

Using Laplace's method (Lemma 5.1) and (5.6) we calculate

$$\begin{aligned} D_{as}(R) &= m_R[e^{-2\Phi_{as}/\sigma^2} \Delta_{as}^2] \\ &= \psi'(R)^2 \sum_{i=1}^{N} \int_{I_i} e^{-2p(\theta+\theta_i)\psi(R)/\sigma^2} p(\theta+\theta_i)^2 d\theta \\ &\sim \psi'(R)^2 \sum_{i=1}^{N} K_i \psi(R)^{-(2\varpi_i+1)/\varpi_i} \\ &\sim K\left((\psi'/\psi)^2 \psi^{-1/\varpi_*}\right)(R) \quad (R \to \infty) \,, \end{aligned}$$

where K and K_i are positive constants. Having (5.10) in mind, condition (5.8) implies the crucial condition (2.10). It remains to show that the spectral gap condition (2.7) holds. We will do this with the help of Proposition 3.7. To obtain lower bonds on the function V_Φ defined in (3.6), we can estimate $|\nabla\Phi(R,\theta)| \geq \phi'(R)^2$ uniformly in θ. Using the fact that p and p'' are bounded on S^1, we can find a constant $K > 0$ such that the following estimation holds uniformly in θ for every $R > 0$

$$|\Delta\Phi(R,\theta)| \leq K\left\{|\phi''(R)| + |\psi''(R)| + \frac{|\phi'(R)| + |\psi'(R)|}{R} + \frac{|\psi(R)|}{R^2}\right\} \,.$$

By condition (5.7) the terms in braces are all $o\left(\phi'(R)^2\right)$ as $R \to \infty$. Since also $\liminf_{R\to\infty} \phi'(R) > 0$ by condition (5.7), we have $\liminf_{|x|\to\infty} V_\Phi(x) > 0$ and the spectral gap condition (2.7) holds by Proposition 3.7. Thus the result of Theorem 2.8 holds and by (5.10) we also obtain the desired form of the asymptotic expression $l(R)$ for the eigenvalue asymptotics. $\square$

5.3 Diffusion Process with Gamma Distribution

We present a two-dimensional stationary diffusion process living only in the positive quadrant. The stationary measure μ of this process is given by the product measure of two gamma distributions with density

$$\mu(x) := \begin{cases} \prod_{i=1}^{2} \left(\beta_i^{\alpha_i}\Gamma(\alpha_i)\right)^{-1} x_i^{\alpha_i - 1} e^{-x_i/\beta_i} & x_1, x_2 > 0\,, \\ 0 & \text{otherwise,} \end{cases} \tag{5.11}$$

where $\alpha_1, \alpha_2 \geq 1$ and $0 < \beta_2 \leq \beta_1$. In order to construct a stationary diffusion process of gradient field type with $\sigma^2 = 2$ having stationary measure μ as above, the potential Φ needs to be set

$$\Phi(x_1, x_2) := \begin{cases} \sum_{i=1}^{2} x_i/\beta_i - (\alpha_i - 1)\ln x_i & x_1, x_2 > 0, \\ \infty & \text{otherwise.} \end{cases} \tag{5.12}$$

If $\alpha_1, \alpha_2 > 3/2$, Φ satisfies (2.2) and (2.3). By Proposition 3.1 there exists a weak solution $(X_t)_{t\geq 0}$ of the SDE (1.2) with Φ as above and $\sigma = \sqrt{2}$. In polar coordinates Φ reads for $R > 0$

$$\Phi(R,\theta) = R\left(\frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2}\right) - (\alpha_1 + \alpha_2 - 2)\ln R - \ln\left((\cos\theta)^{\alpha_1 - 1}(\sin\theta)^{\alpha_2 - 1}\right)$$

for $\theta \in (0, \pi/2)$ and ∞ otherwise. We choose ϕ as the spherical minimum of the non-logarithmic term of Φ, i.e.

$$\phi(R) := R/\beta_1 \qquad R > 0\,.$$

Note that ϕ coincides at least asymptotically with the spherical minimum of Φ, see also Remark 2.9.(1).

Theorem 5.8 *Let $(X_t)_{t\geq 0}$ be the two-dimensional stationary diffusion process of gradient field type with $\sigma^2 = 2$ and Φ defined in (5.12) where $0 < \beta_2 \leq \beta_1$ and $\alpha_1, \alpha_2 > 3$. Set*

$$l(R) := \beta_1^{-(\alpha_1+1)} \left(\frac{R^{\alpha_1-1}}{\Gamma(\alpha_1)} + \delta_{\beta_1\beta_2} \beta_1^{\alpha_1-\alpha_2} \frac{R^{\alpha_2-1}}{\Gamma(\alpha_2)} \right) e^{-R/\beta_1} \,,$$

where $\delta_{\beta_1\beta_2} = 1$ if $\beta_1 = \beta_2$ and $= 0$ otherwise. Then l satisfies (2.8).

Remark 5.9 (a) The condition $\alpha_1, \alpha_2 > 3$ insures the spectral gap property (2.7). This condition may be relaxed, see also Remark 2.2.(2) and Remark 2.9.(5).

(b) The stationary measure of this process is the product measure of two independent gamma distributions. This measure can be replaced by a bivariate distribution with gamma distributed marginals implementing spatial depencence. Such a distrubution can be created by means of copula techniques, we refer to Joe [Joe97]. In Kunz [Kun02a] the asymptotic behavior of the maximum in Euclidean norm of the diffusion process of gradient field type is analyzed, which has the above distribution as stationary measure. In this case, the crucial condition (2.10) is shown by numerical methods.

Proof. We show that conditions (2.5), (2.7), (2.12), and (2.10) are satisfied. Condition (2.5) clearly holds with $Z_\sigma = \prod_{i=1}^2 \beta_i^{\alpha_i} \Gamma(\alpha_i)$, see (5.11). First we evaluate the term δ_{as}.

$$\begin{aligned} \delta_{as}(R) &= \int_0^{\pi/2} R^{\alpha_1+\alpha_2-2} (\cos\theta)^{\alpha_1-1} (\sin\theta)^{\alpha_2-1} e^{-R\left(\frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2} - \frac{1}{\beta_1}\right)} d\theta \\ &= R^{\alpha_1+\alpha_2-2} \int_0^{\pi/2} (\cos\theta)^{\alpha_1-1} (\sin\theta)^{\alpha_2-1} e^{-Rp(\theta)} d\theta \,, \end{aligned}$$

where $p(\theta) := \beta_1^{-1}(\cos\theta - 1) + \beta_2^{-1}\sin\theta$, $\theta \in [0, \pi/2]$. To evaluate this integral asymptotically using Laplace's method (Lemma 5.1), we need to know the zero points $\mathcal{N}(p)$ of p. It can be seen that $\mathcal{N}(p) = \{0, \pi/2\}$ if $\beta_1 = \beta_2$ and $\mathcal{N}(p) = \{0\}$ if $\beta_1 > \beta_2$. Note that as $\theta \searrow 0$

$$\begin{aligned} &p(\theta) \sim \theta/\beta_2 \,, \quad (\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1} \sim \theta^{\alpha_2-1} \,, \\ &p(\tfrac{\pi}{2} - \theta) \sim \theta/\beta_1 \,, \quad \left(\cos(\tfrac{\pi}{2} - \theta)\right)^{\alpha_1-1} \left(\sin(\tfrac{\pi}{2} - \theta)\right)^{\alpha_2-1} \sim \theta^{\alpha_1-1} \,, \end{aligned}$$

if $\beta_1 = \beta_2$ is the latter case. We obtain invoking Laplace's method (Lemma 5.1 and Remark 5.2)

$$\delta_{as}(R) \quad \sim \quad R^{\alpha_1+\alpha_2-2} \left(\Gamma(\alpha_2) \left(\frac{R}{\beta_2}\right)^{-\alpha_2} + \delta_{\beta_1\beta_2} \Gamma(\alpha_1) \left(\frac{R}{\beta_1}\right)^{-\alpha_1} \right)$$

$$(5.13) \qquad = \beta_2^{\alpha_2}\Gamma(\alpha_2)R^{\alpha_1-2} + \delta_{\beta_1\beta_2}\beta_1^{\alpha_1}\Gamma(\alpha_1)R^{\alpha_2-2} \quad (R\to\infty)\,.$$

It is easily seen by the definition of ϕ that the growth condition (2.12) holds, see also Remark 2.9.(2). To show the crucial condition (2.10), the term $D_{as}(R)$ must be evaluated asymptotically. Δ_{as} reads in polar coordinates

$$\Delta_{as}(R,\theta) = \frac{\cos\theta}{\beta_1} + \frac{\sin\theta}{\beta_2} - \frac{\alpha_1+\alpha_2-2}{R} - \frac{1}{\beta_1} = p(\theta) - \frac{\alpha_1+\alpha_2-2}{R}\,.$$

Hence we get analogously to the calculation in (5.13)

$$\begin{aligned} D_{as}(R) = & \int_0^{\pi/2} R^{\alpha_1+\alpha_2-2}(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}\times \\ & \times e^{-Rp(\theta)}\left(p(\theta) - \frac{\alpha_1+\alpha_2-2}{R}\right)^2 d\theta \\ \leq & \;\kappa_1\left\{\frac{\delta_{as}(R)}{R^2} + R^{\alpha_1+\alpha_2-2}\int_0^{\pi/2}(\cos\theta)^{\alpha_1-1}(\sin\theta)^{\alpha_2-1}p(\theta)^2e^{-Rp(\theta)}d\theta\right\} \\ \lesssim & \;\kappa_2\left\{\frac{\delta_{as}(R)}{R^2} + R^{\alpha_1+\alpha_2-2}\left(R^{-(\alpha_2+2)} + \delta_{\beta_1\beta_2}R^{-(\alpha_1+2)}\right)\right\} \\ = & \;\kappa_2\left\{\frac{\delta_{as}(R)}{R^2} + R^{\alpha_1-4} + \delta_{\beta_1\beta_2}R^{\alpha_2-4}\right\} \qquad (R\to\infty)\,, \end{aligned}$$

where κ_1, κ_2 are positive constants. Comparing this with (5.13) we see that the crucial condition (2.10) is satisfied. It remains to show that the spectral gap condition (2.7) holds. Since the two components of $(X_t)_{t\geq 0}$ are independent, it suffices to prove the spectral gap condition for the generator $L^{(1)}$ of the first component of the process. As in the proof of Proposition 3.7, $-L^{(1)}$ is unitarily equivalent to the Schrödinger operator $Hu := -u'' + Vu$ on $\mathbb{R}$ with

$$V(x) = \begin{cases} \frac{1}{4}(\frac{1}{\beta_1} - \frac{\alpha_1-1}{x})^2 - \frac{\alpha_1-1}{2x^2} = \frac{1}{4\beta_1^2} - \frac{\alpha_1-1}{2x} + (\frac{\alpha_1-1}{2} - 1)\frac{\alpha_1-1}{2x^2} & x > 0, \\ \infty & x \leq 0. \end{cases}$$

Note that $\lim_{x\to\infty} V(x) = (2\beta_1)^{-2} > 0$ and $\lim_{x\searrow 0} V(x) = \infty$ since $\alpha_1 > 3$. Hence H and also $-L^{(1)}$ have spectral gap by Proposition 3.7. Thus Theorem 2.8 is applicable and we obtain the eigenvalue asymptotics using (5.13) and Lemma 5.3

$$\begin{aligned} \lambda_R \sim & \; Z_\sigma^{-1}\delta_{as}(R)\left(\int_1^R r^{-1}e^{r/\beta_1}dr\right)^{-1} \\ \sim & \left(\frac{R^{\alpha_1-2}}{\beta_1^{\alpha_1}\Gamma(\alpha_1)} + \delta_{\beta_1\beta_2}\frac{R^{\alpha_2-2}}{\beta_2^{\alpha_2}\Gamma(\alpha_2)}\right)\left(\beta_1R^{-1}e^{R/\beta_1}\right)^{-1} \\ = & \; \beta_1^{-(\alpha_1+1)}\left(\frac{R^{\alpha_1-1}}{\Gamma(\alpha_1)} + \delta_{\beta_1\beta_2}\beta_1^{\alpha_1-\alpha_2}\frac{R^{\alpha_2-1}}{\Gamma(\alpha_2)}\right)e^{-R/\beta_1} \qquad (R\to\infty)\,. \end{aligned}$$

□

5.4 Ornstein-Uhlenbeck Process

The eigenvalue asymptotics is evaluated explicitly for an Ornstein-Uhlenbeck process (OU process) in one and two dimensions emphasizing the different behavior in the rotationally symmetric and non-symmetric case. In addition the long term behavior of the normalized maxima in Euclidean norm is presented (see Corollary 2.6).

The OU process is of gradient field type, where the potential Φ is given in two dimensions by

$$\Phi(x_1, x_2) = \tfrac{1}{2}(\alpha x_1^2 + \beta x_2^2) \qquad x_1, x_2 \in \mathbb{R},\ 0 < \alpha \le \beta\,,$$

and in one dimension by $\Phi(x) = (\alpha/2)x^2$, $x \in \mathbb{R}$.

• *The symmetric case*, i.e. the one-dimensional and the two-dimensional case for $\alpha = \beta$. Setting $\phi(R) = (\alpha/2)R^2$, the conditions of Theorem 5.4 are obviously satisfied (for dimension $n = 1, 2$). For $n = 1$ we obtain form Theorem 5.4 using Lemma 5.3 (here $\gamma_1 = 2$, $Z_\sigma = \sqrt{\sigma^2 \pi \alpha^{-1}}$)

$$\begin{aligned} \lambda_R &\sim \sigma^2 \sqrt{\frac{\alpha}{\sigma^2 \pi}} \left(\int_1^R e^{\alpha r^2/\sigma^2} dr \right)^{-1} \\ &\sim \sqrt{\frac{\sigma^2 \alpha}{\pi}} \left(\frac{\sigma^2}{2\alpha} R^{-1} e^{\alpha R^2/\sigma^2} \right)^{-1} = 2\sqrt{\frac{\alpha^3}{\sigma^2 \pi}} R e^{-\alpha R^2/\sigma^2} \quad (R \to \infty)\,. \end{aligned}$$

Hence the function l_α satisfying (2.8) can be chosen

$$l_\alpha(R) = C_\alpha R e^{-\alpha R^2/\sigma^2}\,, \qquad \text{where } C_\alpha = 2\sqrt{\frac{\alpha^3}{\sigma^2 \pi}}\,. \tag{5.14}$$

Note that this characterizes the maximum of the *absolute value* of a one-dimensional OU process. Similarly for $n = 2$ and $\alpha = \beta$ (here $\gamma_2 = 2\pi$, $Z_\sigma = \sigma^2 \pi \alpha^{-1}$)

$$\begin{aligned} \lambda_R &\sim 2\pi \frac{\sigma^2}{2} \frac{\alpha}{\sigma^2 \pi} \left(\int_1^R r^{-1} e^{\alpha r^2/\sigma^2} dr \right)^{-1} \\ &\sim \alpha \left(\frac{\sigma^2}{2\alpha} R^{-2} e^{\alpha R^2/\sigma^2} \right)^{-1} = \frac{2\alpha^2}{\sigma^2} R^2 e^{-\alpha R^2/\sigma^2} \quad (R \to \infty)\,. \end{aligned}$$

Hence the function $l_{\alpha\alpha}$ satisfying (2.8) can be chosen

$$l_{\alpha\alpha}(R) = C_{\alpha\alpha} R^2 e^{-\alpha R^2/\sigma^2}\,, \qquad \text{where } C_{\alpha\alpha} = \frac{2\alpha^2}{\sigma^2}\,. \tag{5.15}$$

• *The non-symmetric case*, i.e. the two-dimensional case for $\alpha < \beta$. The potential Φ in the present setting written in polar coordinates is of the form (5.3) with

$$\phi(R) = \frac{\alpha}{2} R^2,\ \psi(R) = \frac{\beta - \alpha}{2} R^2,\ p(\theta) = \sin^2 \theta \qquad R > 0,\ \theta \in \left[-\tfrac{\pi}{2}, \tfrac{3\pi}{2} \right)\,.$$

We show that the conditions of Theorem 5.6 are satisfied. Condition (2.5) holds with $Z_\sigma = \sigma^2\pi(\alpha\beta)^{-1/2}$ and (5.4) is also satisfied since $\mathcal{N}(p) = \{0, \pi\}$. (5.5) holds with $\varpi_i = 2$, $P_i = 1$ for $i = 1, 2$ and hence $\varpi_* = 2$. (5.6) is obvious and (5.7)-(5.8) also hold (see Remark 5.7). Thus we get from Theorem 5.6 using Lemma 5.3

$$\begin{aligned}
\lambda_R &\sim \pi^{-1}\sqrt{\frac{\sigma^2\alpha\beta}{2}}\,\Gamma\left(\frac{1}{2}\right)\left(\frac{\beta-\alpha}{2}R^2\right)^{-1/2}\left(\int_1^R r^{-1}e^{\alpha r^2/\sigma^2}dr\right)^{-1} \\
&\sim \sqrt{\frac{\sigma^2\alpha\beta}{\pi(\beta-\alpha)}}R^{-1}\left(\frac{\sigma^2}{2\alpha}R^{-2}e^{\alpha R^2/\sigma^2}\right)^{-1} \\
&= 2\sqrt{\frac{\alpha^3\beta}{\sigma^2\pi(\beta-\alpha)}}Re^{-\alpha R^2/\sigma^2} \qquad (R\to\infty)\,.
\end{aligned}$$

Hence the function $l_{\alpha\beta}$ satisfying (2.8) can be chosen

$$l_{\alpha\beta}(R) = C_{\alpha\beta}Re^{-\alpha R^2/\sigma^2}\,, \qquad \text{where } C_{\alpha\beta} = 2\sqrt{\frac{\alpha^3\beta}{\sigma^2\pi(\beta-\alpha)}}\,. \tag{5.16}$$

- *Effects of symmetry breaking:* Comparing the two-dimensional rotationally symmetric and non-symmetric OU process, we see that $l_{\alpha\alpha}$ and $l_{\alpha\beta}$ differs in the pre-exponential factor and the constant. Especially the pre-exponential factor is reduced from R^2 to R. On the other hand, the extreme fluctuation of a two-dimensional non-symmetric OU process behave like those of an one-dimensional OU process (compare (5.16) with (5.14)), just the constant is different. But in the limit $\beta \to \infty$, i.e. if we make the steeper direction of Φ infinitely steep, we get the expected convergence $C_{\alpha\beta} \to C_\alpha$ as $\beta \to \infty$ of the constants in (5.16) and (5.14). In the contrary, if the asymmetric potential tends to the symmetric one (i.e. $\beta \searrow \alpha$), there is no convergence $l_{\alpha\alpha} \to l_{\alpha\beta}$ as $\beta \searrow \alpha$.

A similar effect of symmetry breaking can be observed looking at the tail of the Euclidean norm of a bivariate normally distributed random variable. More precisely, given a bivariate normal random variable $X \sim N(0, \Sigma)$, where $\Sigma = \mathrm{diag}(\alpha, \beta)$ with $0 < \alpha \leq \beta$, the asymptotic behavior of $P(|X| > R)$ as $R \to \infty$ is different in the symmetric case ($\alpha = \beta$) and in the non-symmetric case ($\alpha < \beta$).

- *Long term behavior of normalized maxima:* In view of Corollary 2.6 we show that $F := e^{-l} \in \mathrm{DA}(\Lambda)$, $l \in \{l_\alpha, l_{\alpha\beta}, l_{\alpha\alpha}\}$, where Λ is the Gumbel distribution. Note that every $l \in \{l_\alpha, l_{\alpha\beta}, l_{\alpha\alpha}\}$ is of the form $CR^\gamma e^{-\alpha R^2/\sigma^2}$, where $C > 0$ is a constant and $\gamma \in \{1, 2\}$. For results of classical extreme value theory used in the sequel we refer to §3.3.3 of Embrechts et al. [EKM97]. It can be shown that F is a so called *Van Mises* function (this follows e.g. from $\lim_{R\to\infty}(1-F(R))F''(R)/(F'(R))^2 = -1$) and hence $F \in \mathrm{DA}(\Lambda)$. The norming constants in (2.9) can be obtained from the relations

$$F(d_T) = 1 - 1/T\,, \quad c_T = (1 - F(d_T))/F'(d_T)\,.$$

Careful asymptotic expansion of these relations as $T \to \infty$ leads to the following choice of the norming constants

$$c_T = \frac{1}{2}\sqrt{\frac{\sigma^2}{\alpha \ln T}}\,, \quad d_T = \sqrt{\frac{\sigma^2 \ln T}{\alpha}} + \frac{\gamma}{4}\sqrt{\frac{\sigma^2}{\alpha \ln T}}\Big(\ln\ln T + \ln(C^{2/\gamma}\sigma^2/\alpha)\Big)\,.$$

Hence we get from Corollary 2.6 that $P_\mu(c_T^{-1}(M_T - d_T) \le x) \to \Lambda(x)$ for every $x \in \mathbb{R}$ as $T \to \infty$ with c_T and d_T as above, where in d_T the right values have to be plugged in for the the constants γ and C depending on $l \in \{l_\alpha, l_{\alpha\beta}, l_{\alpha\alpha}\}$.

5.5 A Counterexample

We present a two-dimensional situationfor which the eigenvalue asymptotics is not governed by the spherical minimum of the potential.

Set $\sigma = \sqrt{2}$ in the SDE (1.2) and choose the potential Φ as follows. Using polar coordinates, Φ is constructed in a similar way as in (5.3), but here the function p is no longer independent of R. Let $g \in C_c^\infty((-1,1),[0,1])$ be a function with $g(\theta) = 1 - \theta^2$ in a neighborhood of 0 and the maximum of g is only attained in 0. We set for $R > 0$

$$p(R,\theta) := \begin{cases} g(e^{R^2}\theta) & |\theta| < e^{-R^2} \\ 0 & e^{-R^2} \le |\theta| \le \pi \end{cases}$$

Note that $p(R,\cdot) \in C^\infty(S^1)$ for every $R > 0$. We define the potential by

$$\Phi(R,\theta) := R^2 - (R^2 - R)p(R,\theta) \qquad R > 1, \theta \in [-\pi,\pi)\,,$$

and Φ can be extended to $C^\infty(\mathbb{R}^2,\mathbb{R})$. Observe that

$$\min_{\theta\in[-\pi,\pi]} \Phi(R,\theta) = R \qquad \max_{\theta\in[-\pi,\pi]} \Phi(R,\theta) = R^2 \qquad R > 1\,.$$

We will show that the corresponding eigenvalue asymptotics is governed by the spherical *maximum* of Φ, i.e. we have to choose $\phi(R) = R^2$ in Theorem 2.8. The interpretation of this fact is that the gap, where the minimum of Φ occurs, becomes too narrow as $R \to \infty$ to influence large fluctuation of the process.

Obviously condition (2.5) is satisfied since $\int_{|x|>1} e^{-\Phi} dx \leq \int_1^\infty re^{-r} dr < \infty$. We evaluate the term $\delta_{as}(R)$ using Laplace's method (Lemma 5.1)

$$\begin{aligned}
\delta_{as}(R) &= \int_{-\pi}^{\pi} e^{(R^2-R)p(R,\theta)} d\theta \\
&= (2\pi - 2e^{-R^2}) + \int_{-e^{-R^2}}^{e^{-R^2}} e^{(R^2-R)g(e^{R^2}\theta)} d\theta \\
&= (2\pi - 2e^{-R^2}) + e^{-R^2} \int_{-1}^{1} e^{(R^2-R)g(\eta)} d\eta \\
&= (2\pi - 2e^{-R^2}) + e^{-R} \int_{-1}^{1} e^{-(R^2-R)(1-g(\eta))} d\eta \\
&\sim 2\pi + e^{-R} \sqrt{\frac{\pi}{R^2-R}} \sim 2\pi \qquad (R \to \infty)\,.
\end{aligned} \tag{5.17}$$

Hence the growth conditions (2.12) holds since we obviously have $\nu(R) \nearrow \infty$ as $R \to \infty$ by the choice of ϕ. Further we have to evaluate the term $D_{as}(R)$. Note that

$$\Delta_{as}(R,\theta) = \begin{cases} -(2R-1)g(e^{R^2}\theta) - 2R(R^2-R)e^{R^2}g'(e^{R^2}\theta)\theta & |\theta| < e^{-R^2} \\ 0 & e^{-R^2} \leq |\theta| \leq \pi \end{cases}$$

Hence we can calculate

$$\begin{aligned}
D_{as}(R) &= \int_{-e^{-R^2}}^{e^{-R^2}} e^{(R^2-R)g(e^{R^2}\theta)} \Delta_{as}(R,\theta)^2 d\theta \\
&\leq K_1 e^{-R^2} \int_{-1}^{1} e^{(R^2-R)g(\theta)} \Big[(2R-1)^2 g(\eta)^2 + R^2(R^2-R)^2 e^{2R^2} g'(\eta)^2 e^{-2R^2}\eta^2\Big] d\eta \\
&\leq K_2 R^6 e^{-R} \int_{-1}^{1} e^{-(R^2-R)(1-g(\theta))} \left[g(\eta)^2 + g'(\eta)^2\eta^2\right] d\eta\,,
\end{aligned}$$

where K_1, K_2 are suitable positive constants. By Laplace's method (Lemma 5.1) the last integral is asymptotically equivalent to $\Gamma(5/2)(R^2-R)^{-5/2}$ as $R \to \infty$. Hence $D_{as}(R) \to 0$ as $R \to \infty$ and invoking (5.17) the crucial condition (2.10) of Theorem 2.8 is satisfied.

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