Subexponential Distributions

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1 Definition and first properties

Subexponential distributions are a special class of heavy–tailed distributions. The name arises from one of their properties, that their tails decrease more slowly than any exponential tail; see (1.3). This implies that large values can occur in a sample with non–negligible probability, and makes the subexponential distributions candidates for modelling situations where some extremely large values occur in a sample compared to the mean size of the data. Such a pattern is often seen in insurance data, for instance in fire, wind–storm or flood insurance (collectively known as catastrophe insurance). Subexponential claims can account for large fluctuations in the surplus process of a company, increasing the risk involved in such portfolios.

Heavy tails are just one of the consequences of the defining property of subexponential distributions, which is designed specially to work well with the probabilistic models commonly employed in insurance applications. The subexponential concept has just the right level of generality to be usable in these models while including as wide a range of distributions as possible. It includes all distributions with regularly varying tails but is considerably wider (see Table 2.6).

Subexponential distributions were first studied in 1964 by Chistyakov [16]. Research during the seventies and eighties was centred around applications in insurance, queueing and branching processes, based on the Pollaczek–Khinchin formula (3.1), linking a subexponential claim size distribution and the ruin probability. Methods were rather more analytic than probabilistic at that time. Properties of subexponential moment generating functions, necessary and sufficient conditions for subexponentiality, and closure properties were investigated. A first textbook treatment can be found in Athreya and Ney [9]. Prominent papers are [17, 18, 20, 21, 22, 24, 27, 35, 36, 37, 46, 49]

Extensions to more general models followed: renewal arrival streams replaced Poisson arrivals. Modelling in that generality required the tracing of subexponential input

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distributions through a Wiener–Hopf factorisation; see [22, 24]. Use of random Markov environments required tracing different input distributions (light– and heavy–tailed), by means of matrix algebra; see [6, 7, 31].

Large deviations were investigated. For subexponential distributions exponential moments do not exist; hence the classical logarithmic approximation is not applicable. Consequently, fine large deviations theory has been called for and is technically very demanding; see [44, 45, 11]. It yields to expressions for the finite time ruin probability [12, 14].

Recently, more probabilistic methods have entered the field. Questions like "how does ruin happen?" or "when is ruin most likely to happen?" given it happens at all were asked and answered. They necessitated novel methods to investigate path properties using the regenerative structure of models, as well as excursion theory for Markov processes and extreme value theory. Such results are in [1, 8]; see also the review in Section 8.4 of [23]

Various review papers appeared on subexponential distributions; see e.g. [28, 48]. Textbook accounts are in [1, 23, 47].

We present two defining properties of subexponential distributions, an analytical one and a probabilistic one.

Definition 1.1. (Subexponential distribution function)
Let $(X_i)_{i\in\mathbb{N}}$ *be iid positive rvs with df* F *such that* $F(x)<1$ *for all* $x>0$. *Denote*

$$\overline{F}(x)=1-F(x)\,,\quad x\geq 0\,,$$

the tail of F *and*

$$\overline{F^{n*}}=1-F^{n*}(x)=P(X_1+\cdots+X_n>x)$$

the tail of the n*–fold convolution of* F. F *is a* subexponential df ($F\in\mathcal{S}$) *if one of the following equivalent conditions holds:*

$$(a)\ \lim_{x\to\infty}\frac{\overline{F^{n*}}(x)}{\overline{F}(x)}=n \text{ for some (all) } n\geq 2\,,$$

$$(b)\ \lim_{x\to\infty}\frac{P(X_1+\cdots+X_n>x)}{P(\max(X_1,\ldots,X_n)>x)}=1 \text{ for some (all) } n\geq 2. \qquad \square$$

Remark 1.2. (i) Definition (a) goes back to Chistyakov [16]. He proved that the limit (a) holds for all $n\geq 2$ if and only if it holds for $n=2$. It was shown in Embrechts and Goldie [21] that (a) holds for $n=2$ if it holds for some $n\geq 2$.

(ii) The equivalence of (a) and (b) was shown in Embrechts and Goldie [20]. A proof goes as follows:

$$P(\max(X_1,\ldots,X_n)>x)=1-F^n(x)=\overline{F}(x)\sum_{k=0}^{n-1}F^k(x)\sim n\overline{F}(x)\,,\quad x\to\infty\,,$$

($\sim$ means that the quotient of lhs and rhs tends to 1). Hence

$$\frac{P(X_1+\cdots+X_n>x)}{P(\max(X_1,\ldots,X_n)>x)}\sim\frac{\overline{F^{n*}}(x)}{n\overline{F}(x)}\to 1 \quad\Longleftrightarrow\quad F\in\mathcal{S}\,.$$

(iii) Definition (b) provides a physical interpretation of subexponentiality: the sum of n iid subexponential rvs is likely to be large if and only if their maximum is likely to be large. This accounts for extremely large values in a subexponential sample.

(iv) From Definition (a) and the fact that $\mathcal{S}$ is closed with respect to tail–equivalence (see after Proposition 2.4) we conclude that $F\in\mathcal{S}$ implies $F^{n*}\in\mathcal{S}$ for all $n\in\mathbb{N}$. Furthermore, from Definition (b) and the fact that F^n is the df of the maximum of n iid rvs with df F, we obtain that $F\in\mathcal{S}$ implies $F^n\in\mathcal{S}$ for all $n\in\mathbb{N}$. Hence $\mathcal{S}$ is closed with respect to taking sums and maxima of iid rvs.

(v) Definition (b) demonstrates the heavy–tailedness of subexponential dfs. It is further substantiated by the implications (first proved by Chistyakov [16])

$$F\in\mathcal{S} \implies \lim_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)}=1 \quad \forall\, y\in\mathbb{R} \tag{1.1}$$

$$\implies \int_0^\infty e^{\varepsilon x}\,dF(x)=\infty \quad \forall\,\varepsilon>0 \tag{1.2}$$

$$\implies \overline{F}(x)/e^{-\varepsilon x}\to\infty \quad \forall\,\varepsilon>0\,. \tag{1.3}$$

Property (1.3) accounts for the name subexponential df: the tail of F decreases more slowly than any exponential tail. Property (1.2) shows that subexponential dfs have no exponential moments. This prevents any method being applicable that requires the existence of exponential moments. □

2 Conditions for subexponentiality

It should be clear from the definition that a characterisation of subexponential dfs or even of dfs whose integrated tail df is subexponential (as needed in the risk and queueing models) will not be possible in terms of simple expressions involving the tail.

Recall that all subexponential dfs have property (1.1), hence the class of such dfs provides potential candidates for subexponentiality. The class is named as follows.

Definition 2.1. (The class $\mathcal{L}$)
Let F be a df on $(0,\infty)$ such that $F(x)<1$ for all $x>0$. We say $F\in\mathcal{L}$ if

$$\lim_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)}=1 \quad \forall\, y\in\mathbb{R}\,.$$ □

Unfortunately, $\mathcal{S}$ is a proper subset of $\mathcal{L}$. Examples for dfs in $\mathcal{L}$ but not in $\mathcal{S}$ can be found in Embrechts and Goldie [20] and Pitman [46].

A famous subclass of $\mathcal{S}$ is the class of dfs with regularly varying tail. For a positive measurable function f we write $f \in \mathcal{R}(\alpha)$ for $\alpha \in \mathbb{R}$ (f is *regularly varying with index* α) if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^{\alpha} \quad \forall\, t > 0\,.$$

A function $f \in \mathcal{R}(0)$ is called *slowly varying*. For further properties of regularly varying functions we refer to the monograph by Bingham, Goldie and Teugels [15].

Example 2.2. (Distribution functions with regularly varying tails)
Let $\overline{F} \in \mathcal{R}(-\alpha)$ for $\alpha \geq 0$, then it has the representation

$$\overline{F}(x) = x^{-\alpha}\ell(x)\,, \quad x > 0\,,$$

for some $\ell \in \mathcal{R}(0)$. Notice first that $F \in \mathcal{L}$, hence it is a candidate for $\mathcal{S}$. To check Definition 1.1(a), let X_1, X_2 be iid rvs with df F. Then

$$\frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2\int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)}\, dF(y) + \frac{\overline{F}^2(x/2)}{\overline{F}(x)}\,.$$

Immediately, by the definition of $\mathcal{R}(-\alpha)$, the last term tends to 0. The integrand satisfies $\overline{F}(x-y)/\overline{F}(x) \leq \overline{F}(x/2)/\overline{F}(x)$ for $0 \leq y \leq x/2$, hence Lebesgue dominated convergence applies and, since $F \in \mathcal{L}$, the integral on the rhs tends to 1 as $x \to \infty$.

Examples of dfs with regularly varying tail are Pareto, Burr, transformed beta (also called generalized F), log–gamma and stable dfs (see Table 2.6); see Hogg and Klugman [30] or Johnson, Kotz and Balakrishnan [32, 33] for details.

As we shall see in Section 3 below the *integrated tail* df plays an important role in ruin theory. It is defined for a df F with finite mean μ as

$$F_I(x) = \frac{1}{\mu}\int_{-\infty}^{x} \overline{F}(y)dy\,, \quad x \in \mathbb{R}\,. \tag{2.1}$$

If F has regularly varying tail with index $\alpha > 1$ then F has finite mean and, by Karamata's theorem, $F_I \in \mathcal{R}(-(\alpha-1))$, giving $F_I \in \mathcal{S}$ as well. □

In order to unify the problem of finding conditions for $F \in \mathcal{S}$ and $F_I \in \mathcal{S}$, the following class was introduced in Klüppelberg [35].

Definition 2.3. (The class $\mathcal{S}^*$)
Let F be a df on $(0,\infty)$ such that $F(x) < 1$ for all $x > 0$. We say $F \in \mathcal{S}^*$ if F has finite mean μ and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{F}(y)\, dy = 2\mu\,. \qquad \square$$

The next result makes the class useful for applications.

Proposition 2.4. *If $F \in \mathcal{S}^*$, then $F \in \mathcal{S}$ and $F_I \in \mathcal{S}$.* □

Remark 2.5. (i) The class $\mathcal{S}^*$ is "almost" $\mathcal{S} \cap \{F : \mu(F) < \infty\}$, where $\mu(F)$ is the mean of F. A precise formulation can be found in Klüppelberg [35]. An example where $F \in \mathcal{S}$ with finite mean, but $F \notin \mathcal{S}^*$ can be found in Klüppelberg and Villaseñor [41]

(ii) The tails of dfs in $\mathcal{S}^*$ are subexponential densities.

Name	*Tail $\overline{F}$ or density f*	*Parameters*
Pareto	$\overline{F}(x) = \left(\dfrac{\kappa}{\kappa + x}\right)^{\alpha}$	$\alpha, \kappa > 0$
Burr	$\overline{F}(x) = \left(\dfrac{\kappa}{\kappa + x^{\tau}}\right)^{\alpha}$	$\alpha, \kappa, \tau > 0$
Log–gamma	$f(x) = \dfrac{\alpha^{\beta}}{\Gamma(\beta)} (\ln x)^{\beta-1} x^{-\alpha-1}$	$\alpha > 1, \beta > 0$
Transformed beta	$f(x) = \dfrac{\lvert a \rvert x^{ap-1}}{B(p,q)(1+x^{a})^{p+q}}$	$a \in \mathbb{R},\ p, q > 0$
Truncated α–stable	$\overline{F}(x) = P(\lvert X \rvert > x)$ where X is an α–stable rv	$0 < \alpha < 2$
Lognormal	$f(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma x} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$\mu \in \mathbb{R},\ \sigma > 0$
Benktander–type–I	$\overline{F}(x) = c(\alpha + 2\beta \ln x)$ $e^{-(\beta (\ln x)^2 + (\alpha+1)\ln x)}$	$c, \alpha, \beta > 0$
Benktander–type–II	$\overline{F}(x) = c\alpha x^{-(1-\beta)} e^{-\alpha\, x^{\beta}/\beta}$	$c, \alpha > 0$ $0 < \beta < 1$
Weibull	$\overline{F}(x) = e^{-x^{\tau}}$	$0 < \tau < 1$
"Almost" exponential	$\overline{F}(x) = e^{-x(\ln x)^{-\alpha}}$	$\alpha > 0$

Table 2.6. *Subexponential dfs. All of them are in $\mathcal{S}^*$ provided they have finite mean.*

In much of the present discussion we are dealing only with the right tail of a df. This notion can be formalised by denoting two dfs F and G with support unbounded to the right *tail–equivalent* if $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = c \in (0, \infty)$.

The task of finding easily verifiable conditions for $F \in \mathcal{S}$ or/and $F_I \in \mathcal{S}$ has now been reduced to the finding of simple conditions for $F \in \mathcal{S}^*$. We formulate some of them in terms of the *hazard function* $Q = -\ln \overline{F}$ and its density q, the *hazard rate* of F. Recall that (provided $\mu < 0$) $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{L}$. Then it can be shown that each $F \in \mathcal{L}$ is tail-equivalent to an absolutely continuous df with hazard rate q which tends to 0 Since $\mathcal{S}$ is closed with respect to tail–equivalence (Teugels [49]) as well as $\mathcal{S}^*$ it is of interest to find conditions on the hazard rate such that the corresponding df or/and integrated tail df is subexponential.

Define

$$r = \limsup_{x\to\infty} \frac{xq(x)}{Q(x)}\,.$$

Proposition 2.7. *Let $\rho \in (0,1)$ and $v \geq 1$. Then the following are equivalent.*

(*a*) *$Q(x)$ is absolutely continuous on $x \geq v$ with Lebesgue density $q(x) \to 0$ as $x \to \infty$, and $xq(x)/Q(x) \leq \rho$ for all $x \geq v$.*

(*b*) *$Q(xy) \leq y^\rho Q(x)$ for all $x \geq v$ and $y \geq 1$.*

(*c*) *$Q(x)/x^\rho$ decreases on $x \geq v$.*

Theorem 2.8. (Conditions for $F \in \mathcal{S}$ or $\mathcal{S}^*$)

(*a*) *If $\limsup_{x\to\infty} xq(x) < \infty$, then $F \in \mathcal{S}$. If additionally $\mu < \infty$ then $F \in \mathcal{S}^*$.*

(*b*) *If $r < 1$, then $F \in \mathcal{S}$.*

(*c*) *If $r < 1$ and $\overline{F}^{1-r-\varepsilon}$ is integrable for some $\varepsilon > 0$, then $F \in \mathcal{S}^*$.*

(*e*) *If q is eventually decreasing to 0, then $F \in \mathcal{S}$ if and only if $\lim_{x\to\infty} \int_0^x e^{yq(x)}\,dF(y) = 1$. Moreover, $F \in \mathcal{S}^*$ if and only if $\lim_{x\to\infty} \int_0^x e^{yq(x)}\overline{F}(y)\,dy = \mu < \infty$.* □

There are many more conditions for $F \in \mathcal{S}$ or $F_I \in \mathcal{S}$ to be found in the literature. We mention Chistyakov [16], Cline [17], Goldie [27], Klüppelberg [35, 37], Pitman [46], Teugels [49]; the selection above is taken from Baltrūnas, Daley and Klüppelberg [13] and Baltrūnas and Klüppelberg [14].

3 Ruin probabilities

We consider the *Sparre-Andersen model*, which is defined by the following quantities:

(i) The claim times constitute a renewal process, i.e. the interclaim times $(T_n)_{n\in\mathbb{N}}$ are iid rvs and we assume that they have finite second moment.

(ii) The claim sizes $(X_k)_{k\in\mathbb{N}}$ are iid rvs with common df F with $EX_1 = \mu < \infty$. The claim sizes and interclaim times are independent.

(iii) We denote by $R_0 = u$ the initial risk reserve and by $c > 0$ the intensity of the risk premium. We also assume that $m = EX_1 - cET_1 = \mu - c/\lambda < 0$.

Define the *risk process* for $u \geq 0$ as

$$R(t) = u + ct - \sum_{k=1}^{N(t)} X_k\,, \quad t \geq 0\,, \tag{3.1}$$

where (with the convention $\sum_{i=1}^{0} a_i = 0$)

$$N(0) = 0\,, \quad N(t) = \sup\{k \geq 0 : \sum_{i=1}^{k} T_i \leq t\}\,, \quad t > 0\,.$$

Defining by R_n the level of the *risk process* immediately after the n-th claim, the embedded random walk structure becomes visible. Note that

$$R_0 = u\,, \quad R_{n+1} = R_n + cT_{n+1} - X_{n+1}\,, \quad n \in \mathbb{N}\,. \tag{3.2}$$

Then setting $Z_k = X_k - cT_k$, $k \in \mathbb{N}$,

$$S_0 = 0\,, \quad S_n = u - R_n = \sum_{k=1}^{n} Z_k\,, \quad n \in \mathbb{N}\,, \tag{3.3}$$

defines a random walk starting in 0 with mean $m < 0$.

The *ruin probability in finite time* is defined as

$$\psi(u,T) = P(R(t) < 0 \text{ for some } 0 \leq t \leq T \,|\, R(0) = u)\,, \quad 0 < T < \infty\,, \quad u \geq 0\,. \tag{3.4}$$

The *ruin probability in infinite time* is then

$$\psi(u) = \psi(u,\infty)\,, \quad u \geq 0\,.$$

By definition of the risk process, ruin can occur only at the claim times, hence for $u \geq 0$,

$$\begin{aligned}
\psi(u) &= P(R(t) < 0 \text{ for some } t \geq 0 \,|\, R(0) = u) \\
&= P(u + \sum_{k=1}^{n}(cT_k - X_k) < 0 \text{ for some } n \in \mathbb{N}) \\
&= P(\max_{n \geq 1} S_n > u)\,.
\end{aligned}$$

If the interclaim times are exponential, i.e. the claim arrival process is a Poisson process, then the last probability has an explicit representation, called Pollacek-Khinchine formula; see e.g. Feller [25] or Embrechts, Klüppelberg and Mikosch [23]. This model is called the *Cramér-Lundberg model.*

Theorem 3.1. (Pollacek-Khinchine formula)
Consider the Cramér-Lundberg model, where the claims arrival process constitutes a Poisson process. Let $S_0 = 0$ and $S_n = \sum_{k=1}^n Z_k$, $n \in \mathbb{N}$, be the random walk as in (3.3) with negative drift $EZ_1 = m = \mu - c/\lambda < 0$, then

$$1 - \psi(u) = (1-\rho) \sum_{n=0}^{\infty} \rho^n F_I^{n*}(u) \,, \quad u \geq 0 \,,$$

where $\rho = \lambda\mu/c$ and $F_I(u) = \mu^{-1} \int_0^u P(X_1 > y) dy$, $u \geq 0$, is the integrated tail distribution of X_1.

The following result is due to Embrechts, Goldie and Veraverbeke [22], Embrechts and Goldie [21], and Cline [18].

Theorem 3.2. (Random sums of iid subexponential rvs)
Suppose $(p_n)_{n\geq 0}$ defines a probability measure on $\mathbb{N}_0$ such that $\sum_{n=0}^{\infty} p_n(1+\varepsilon)^n < \infty$ for some $\varepsilon > 0$ and $p_k > 0$ for some $k \geq 2$. Let

$$G(x) = \sum_{n=0}^{\infty} p_n F^{n*}(x) \,, \quad x \geq 0 \,. \tag{3.5}$$

Then

$$F \in \mathcal{S} \iff \lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = \sum_{n=1}^{\infty} n p_n \iff G \in \mathcal{S} \text{ and } \overline{F}(x) \neq o(\overline{G}(x)) \,. \qquad \square$$

Remark 3.3. Let $(X_i)_{i\in\mathbb{N}}$ be iid with df F and let N be a rv taking values in $\mathbb{N}_0$ with distribution $(p_n)_{n\geq 0}$. Then G is the df of the random sum $\sum_{i=1}^N X_i$ and the result of Theorem 3.2 translates into

$$P\left(\sum_{i=1}^{N} X_i > x\right) \sim ENP(X_1 > x) \,, \quad x \to \infty \,.$$

If $(p_n)_{n\geq 0}$ is a Poisson or geometric distribution the condition $\overline{F}(x) \neq o(\overline{G}(x))$ in (c) is unnecessary (Cline [18]). $\square$

For the ruin probability this translates to the following.

Theorem 3.4. (Ruin probability in the Cramér-Lundberg model; subexponential claims)
Consider the Cramér-Lundberg model, where the claims arrival process constitutes a Poisson process and the claim sizes are subexponential with df F. Then the following assertions are equivalent:

(a) $F_I \in \mathcal{S}$,
(b) $1 - \psi \in \mathcal{S}$,
(c) $\lim_{u\to\infty} \psi(u)/\overline{F}_I(u) = \rho/(1-\rho)$.

The following definition describes a more general class of dfs.

Definition 3.5. (The classes $\mathcal{S}(\gamma)$, $\gamma \geq 0$)
A df F on $(0,\infty)$ such that $F(x) < 1$ for all $x > 0$ belongs to the class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, if

(a) $\displaystyle\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2d < \infty,$

(b) $\displaystyle\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\gamma y}, \quad y \in \mathbb{R}.$

One can show that $d = \widehat{f}(\gamma) = \int_0^\infty e^{\gamma y} dF(y)$, so that $\mathcal{S} = \mathcal{S}(0)$. The classes $\mathcal{S}(\gamma)$ for $\gamma > 0$ are "nearly exponential" in the sense that their tails are only slightly modified exponential. The slight modification, however, results in a moment generating function, which is finite at the point γ. An important class of dfs belonging to $\mathcal{S}(\gamma)$ is the generalized inverse Gaussian distribution (see Embrechts [19] and Klüppelberg [36]).

Theorem 3.4 generalises as follows.

Theorem 3.6. (Ruin probability in the Cramér-Lundberg model; claims in $\mathcal{S}(\gamma)$, $\gamma > 0$)
Consider the Cramér-Lundberg model, where the claims arrival process constitutes a Poisson process and the claim size df $F \in \mathcal{S}(\gamma)$. Assume furthermore that the Lundberg coefficient does not exist, i.e. $\rho\widehat{f}_I(\gamma) < 1$, where $\widehat{f}_I(\gamma)$ denotes the moment generating function of F_I at γ. Then the following assertions are equivalent:

(a) $F \in \mathcal{S}(\gamma)$,

(b) $1 - \psi \in \mathcal{S}(\gamma)$,

(c) $\displaystyle\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}(u)} = \frac{\rho(1-\rho)}{\mu\gamma(1-\rho\widehat{f}_I(\gamma))^2}.$

Remark 3.7. The results of Theorems 3.4 and 3.6 can be extended to the Sparre-Anderson model, but also to various more sophisticated models. As an important example we mention Markov modulation of the risk process; see Asmussen, Fløe Henriksen and Klüppelberg [6], Jelenkovič and Lazar [31] and Asmussen and Højgaard [7]. Moreover, modelling interest to be received on the capital is another natural extension. The ruin probability decreases slightly, which has been shown by Klüppelberg and Stadtmüller [39] for regularly varying claims, and for general subexponential claims by Asmussen [3]; see also related work by Gaier and Grandits [26]

4 Large deviations and finite time ruin probabilities

A further question immediately arises from Definition 1.1, namely what happens if n varies together with x. Hence *large deviations* theory is called for. Notice that the usual "rough" large deviations machinery based on logarithms cannot be applied to dfs in $\mathcal{S}$, since exponential moments do not exist.

We introduce the following conditions on the quantities of the Sparre-Anderson model from Section 3.

Condition A *The inter-arrival time df A is such that for every increasing function $g(n)$ satisfying $g(n)/n \to 0$ and $g(n)/\sqrt{n} \to \infty$ as $n \to \infty$, there is a positive constant c_A and an integer n_g such that for $n \geq n_g$,*

$$P\big(|S_n^Y - nEY| > g(n)\big) \leq \exp\big(-c_A g^2(n)/n\big). \tag{4.1}$$

Condition A is for instance satisfied when the inter arrival times have any finite exponential moment.

Conditions B *The claim size df F is absolutely continuous with density f so that its hazard function $Q = -\log \overline{F}$ has a hazard rate $q = Q' = f/\overline{F}$ satisfying*

(i) $r := \limsup_{t\to\infty} tq(t)/Q(t) < 1/2$;

(ii) $\liminf_{t\to\infty} tq(t) > a(r) = \begin{cases} 2 & \text{if } r = 0\,, \\ 4/(1-r) & \text{if } r \neq 0\,. \end{cases}$

Conditions B imply that X_1 has at least a finite second moment and that $F \in \mathcal{S}^*$. Lemma 3.6 of Baltrunas et al. [13] and (ii) imply that

$$\alpha = \sup\{k : E[X_1^k; X_1 > 0] < \infty\} = \liminf_{t\to\infty} Q(t)/\log t > a(r)\,.$$

Hence (ii) is a moment condition on the positive part of the increment and limits the pathological cases. In all non-pathological cases the case $r \neq 0$ corresponds to dfs with moments of all order, hence (ii) is satisfied, whereas such dfs with infinite moments correspond to $r = 0$ and then (ii) requires a finite second moment.

A version of the following result can be found in Baltrunas et al. [13], a slightly weaker version is stated in Baltrunas and Klüppelberg [14].

Theorem 4.1. (Large deviations property for rvs with subexponential tail)
*Assume that for the random walk $(S_n)_{n\geq 0}$ in (3.3) conditions **A** and **B** hold. Then for any sequence $(t_n)_{n\in\mathbb{N}}$ satisfying*

$$\limsup_{n\to\infty} \sqrt{n} Q(t_n)/t_n < \infty\,, \tag{4.2}$$

the centered random walk $(S_n)_{n\in\mathbb{N}}$, satisfies

$$\lim_{n\to\infty} \sup_{t\geq t_n} \left| \frac{P(S_n - ES_n > t)}{n\overline{F}(t)} - 1 \right| = 0. \tag{4.3}$$

A classical version of this result states that for $\overline{F} \in \mathcal{R}(-\alpha)$ relation (4.3) is true for any sequence $t_n = \alpha n$ for all $\alpha > 0$; see Klüppelberg and Mikosch [38] for further references. In that paper extensions of (4.3) for $\overline{F} \in \mathcal{R}(-\alpha)$ towards random sums with applications to insurance and finance are treated, an important example concerning the modelling and pricing of catastrophe futures. A very general treatment of large deviation results for subexponentials is given in Pinelis [45]; see also Mikosch and Nagaev [42, 43].

Theorem 4.1 has been applied in Baltrūnas [12] to prove the following results for the *ruin time*

$$\tau(u) = \inf\{t > 0 : R(t) < 0 \,|\, R(0) = u\}\,, \quad u \geq 0\,.$$

Theorem 4.2. (Asymptotic behaviour of ruin time)
(a) Under conditions **A** *and* **B**, *for fixed* $u > 0$,

$$P(\infty > \tau(u) > t) \sim P(\infty > \tau(0) > t)V(u) \sim \sum_{n=0}^{\infty} \overline{F}_I(|\mu| n) P(N(t) = n) V(u)\,, \quad t \to \infty$$

where $V(u)$ *is the renewal function of the strictly increasing ladder heights of* $(S_n)_{n\geq 0}$.
(b) Under conditions **A** *and* **B**, *provided that the hazard rate* q_I *of* F_I *satisfies* $q_I(t) = O(1)q(t)$,

$$P(\infty > \tau(0) > t) \sim e^{-\widetilde{B}} \frac{m}{|\mu|} \overline{F}_I(|\mu|(\lambda/c)t) = e^{-\widetilde{B}} \frac{m}{|\mu|} \overline{F}_I\left((1 - \frac{EX_1}{cET_1})t \right)\,, \quad t \to \infty\,,$$

where $\widetilde{B} = \sum_{n=1}^{\infty} P(S_n > 0)/n$ *(which is finite according to Feller [25], Theorem XII, 2).*

5 When and how ruin occurs: a sample path analysis

Consider the continuous time version of the random walk (3.3) of the Sparre-Anderson model

$$S(t) = \sum_{i=1}^{N(t)} X_i - ct\,, \quad t \geq 0\,,$$

where the increment S_1 has df B and we require that

$$\overline{B}(x) = P(X_1 - cT_1 > x) \sim \overline{F}(x)\,, \quad x \to \infty\,.$$

Since $(S(t))_{t\geq 0}$ has drift $m = ES_1 < 0$, $M = \max_{t\geq 0} S(t) < \infty$ a.s. Then the *ruin time*

$$\tau(u) = \inf\{t > 0 : S(t) > u\}\,, \quad u \geq 0\,,$$

yields an expression for the ruin probability as

$$\psi(u) = P(\tau(u) < \infty) = P(M > u)\,.$$

From Section 2 we know the asymptotic form of $\psi(u)$ as $u \to \infty$, and in this section we investigate properties of a sample path leading to ruin, i.e. an upcrossing of a high level u. Let

$$P^{(u)} = P(\cdot \mid \tau(u) < \infty)\,,$$

then we are interested in the $P^{(u)}$–distribution of the path

$$S([0, \tau(u))) = (S(t))_{0 \le t \le \tau(u)}$$

leading to ruin.

In particular, we study the following quantities:

$Y(u) = S(\tau(u))$	the level of the process after the upcrossing,
$Z(u) = S(\tau(u)-)$	the level of the process just before the upcrossing,
$Y(u) - u$	the size of the overshoot,
$W(u) = Y(u) + Z(u)$	the size of the increment leading to the upcrossing.

In this subexponential set-up an upcrossing happens as a result of one large increment whereas the process behaves in a typical way until the rare event happens. The following result describes the behaviour of the process before an upcrossing, and the upcrossing event itself; see Asmussen and Klüppelberg [8] and Asmussen [2].

We first have to introduce some notions from extreme value theory; see Embrechts et al. [23] or any other book on extreme value theory for background.

For iid rvs $(X_i)_{i\in\mathbb{N}}$ with df F we say $F \in \mathrm{MDA}(H)$ (F belongs to the maximum domain of attraction of H), if there exist norming constants $c(n) > 0$ and $d(n) \in \mathbb{R}$ such that

$$c(n)^{-1}(\max(X_1, \ldots, X_n) - d(n)) \stackrel{d}{\to} H\,, \quad n \to \infty\,,$$

where $\stackrel{d}{\to}$ denotes convergence in distribution; the df H is called extreme value distribution. In the large claims case we are only interested in dfs F with infinite right endpoint, i.e. $F(x) < 1$ for all $x \in \mathbb{R}$. Then

$$H(x) = \Phi_\alpha(x) = \exp(-x^{-\alpha}) I_{\{x>0\}} \quad \alpha \in (0, \infty) \quad \text{or} \quad G(x) = \Lambda(x) = \exp(-e^{-x})\,, \quad x \in \mathbb{R}\,.$$

The following characterization plays an important role.

Proposition 5.1. (Characterization of $\mathrm{MDA}(G)$)
Assume that $F(x) < 1$ for all $x \in \mathbb{R}$. Then $F \in \mathrm{MDA}(G)$ if and only if there exists some measurable function $a : \mathbb{R} \to \mathbb{R}^+$ sucht that

$$\lim_{u\to\infty} P\left(\frac{X_1 - u}{a(u)} > x \,|X_1 > u\right) = \overline{G}_\alpha(x)\,, \quad x \in \mathbb{R}\,.$$

Here $\alpha \in (0,\infty)$ corresponds to $H = \Phi_\alpha$ and $\alpha = \infty$ to $H = \Lambda$. The right hand limit function is the tail of the generalized Pareto distribution, given by

$$\overline{G}_\alpha(x) = \begin{cases} (1+x/\alpha)^{-\alpha} & \alpha \in (0,\infty)\,, \\ & \qquad\qquad x > 0\,. \\ e^{-x} & \alpha = \infty\,, \end{cases}$$

The function a is unique up to asymptotic equivalence and can be chosen as $a(u) = \int_u^\infty \overline{F}(y)dy/\overline{F}(u)$ as $u \to \infty$.

Theorem 5.2. (Sample path leading to ruin)
Assume that the increment S_1 has df B with finite mean $m < 0$. Assume furthermore that $\overline{B}(x) \sim \overline{F}(x)$ as $x \to \infty$ for some $F \in \mathcal{S}^* \cap \mathrm{MDA}(G)$ for some extreme value distribution G. Let $a(u) = \int_u^\infty \overline{F}(y)\,dy/\overline{F}(u)$. Then, as $u \to \infty$,

$$\left(\frac{Z(u)}{a(u)}, \frac{\tau(u)}{a(u)}, \frac{Y(u)-u}{a(u)}, \left(\frac{S_{t\tau(u)}}{\tau(u)}\right)_{0\le t<1} \right) \longrightarrow \left(V_\alpha, \frac{V_\alpha}{\mu}, T_\alpha, (mt)_{0\le t<1} \right)$$

in $P^{(u)}$–distribution in $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{D}[0,1)$, where $\mathbb{D}[0,1)$ denotes the space of cadlag functions on $[0,1)$, and V_α and T_α are positive rvs with df satisfying for $x, y > 0$

$$P(V_\alpha > x, T_\alpha > y) = \overline{G}_\alpha(x+y) = \begin{cases} \left(1 + \dfrac{x+y}{\alpha}\right)^{-\alpha} & \text{if} \quad \overline{F} \in \mathcal{R}(-\alpha-1)\,, \\ e^{-(x+y)} & \text{if} \quad F \in \mathrm{MDA}(\Lambda)\,. \end{cases}$$

(*Here α is a positive parameter, the latter case when $F \in \mathrm{MDA}(\Lambda)$ being considered as the case $\alpha = \infty$.*) $\square$

Remark 5.3. (i) Extreme value theory is the basis of this result: recall first that $\overline{F} \in \mathcal{R}(-(\alpha+1))$ is equivalent to $F \in \mathrm{MDA}(\Phi_{\alpha+1})$, and hence to $F_I \in \mathrm{MDA}(\Phi_\alpha)$ by Karamata's theorem. Furthermore, $F \in \mathrm{MDA}(\Lambda) \cap \mathcal{S}^*$ implies $F_I \in \mathrm{MDA}(\Lambda) \cap \mathcal{S}$. Extreme value theory then provides the form of G_α as the only possible limit df for the excess distribution (Balkema and de Haan [10]). G_α is called a *generalised Pareto distribution*. The normalising function $a(u)$ tends to infinity as $u \to \infty$. For $\overline{F} \in \mathcal{R}(-(\alpha+1))$ Karamata's theorem gives $a(u) \sim u/\alpha$. For $F \in \mathrm{MDA}(\Lambda)$ this is Lemma 2.1 in Goldie and Resnick [29].
(ii) The limit result for $(S(t\tau(u)))_{t\ge 0}$ given in Theorem 5.2 substantiates the assertion that the process $(S(t))_{t\ge0}$ evolves typically up to time $\tau(u)$.
(iii) Generalizations of Theorem 5.2 to general Lévy processes can be found in Klüppelberg, Kyprianou and Maller [40]. $\square$

As a complementary result to Theorem 4.2 of Section 4, where u is fix and $t \to \infty$, we obtain here a result on the ruin probability within a fixed time interval $(0, T]$ for $u \to \infty$.

Example 5.4. (Finite time ruin probability)
For $u \geq 0$ and $T \in (0, \infty)$ define the ruin probability before time T by

$$\Psi(u, T) = P(\tau(u) \leq T).$$

From the limit result on $\tau(u)$ given in Theorem 5.2 one finds the following:
if $\overline{F} \in \mathcal{R}(-(\alpha+1))$ for some $\alpha > 0$ then

$$\lim_{u \to \infty} \frac{\psi(u, uT)}{\psi(u)} = 1 - \big(1 + (1-\rho)T\big)^{-\alpha},$$

and if $F \in \mathrm{MDA}(\Lambda) \cap \mathcal{S}^*$ then

$$\lim_{u \to \infty} \frac{\psi(u, a(u)T)}{\psi(u)} = 1 - e^{-(1-\rho)T}. \qquad \square$$

Such results can also be used for simulation of ruin probabilities; see Asmussen, Binswanger and Højgaard [5] and Juneja and Shahabuddin [34].

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