

Futures pricing in electricity markets based on stable CARMA spot models

Fred Espen Benth¹, Claudia Klüppelberg², Gernot Müller³, Linda Vos⁴

Abstract

We present a new model for the electricity spot price dynamics, which is able to capture seasonality, low-frequency dynamics and extreme spikes in the market. Instead of the usual purely deterministic trend we introduce a non-stationary independent increments process for the low-frequency dynamics, and model the large fluctuations by a non-Gaussian stable CARMA process. The model allows for analytic futures prices, and we apply these to model and estimate the whole market consistently. Besides standard parameter estimation, an estimation procedure is suggested, where we fit the non-stationary trend using futures data with long time until delivery, and a robust L^1 -filter to find the states of the CARMA process. The procedure also involves the empirical and theoretical risk premia which – as a by-product – are also estimated. We apply this procedure to data from the German electricity exchange EEX, where we split the empirical analysis into base load and peak load prices. We find an overall negative risk premium for the base load futures contracts, except for contracts close to delivery, where a small positive risk premium is detected. Peak load contracts, on the other hand, show a clear positive risk premium, when they are close to delivery, while contracts in the longer end also have a negative premium.

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¹Center of Mathematics for Applications (CMA), University of Oslo, P.O. Box 1053, Blindern, N-0316 Oslo Norway, email: fredb@math.uio.no, <http://folk.uio.no/fredb>

²Center for Mathematical Sciences, Technische Universität München, Boltzmannstrasse 3, D-85748 Garching, Germany, email: cklu@ma.tum.de, <http://www-m4.ma.tum.de>

³Institute of Mathematics, Universität Augsburg, Universitätsstrasse 14, D-86159 Augsburg, Germany, email: gernot.mueller@math.uni-augsburg.de, <http://www.math.uni-augsburg.de/prof/csda/>

⁴Center of Mathematics for Applications (CMA), University of Oslo, P.O. Box 1053, Blindern, N-0316 Oslo, Norway and Department of Economics, University of Agder, Serviceboks 422, N-4604 Kristiansand, Norway, email: linda.vos@cma.uio.no, <http://folk.uio.no/lindavos>

1 Introduction

In the last decades power markets have been liberalised world-wide, and there is a large interest for modelling power spot prices and derivatives. Electricity spot prices are known to be seasonally varying and mean-reverting. Moreover, a distinctive characteristic of spot prices is the large spikes that occur due to sudden imbalances in supply and demand, for example, when a large production utility experiences a black-out or temperatures are suddenly dropping. Typically in these markets, different production technologies have big variations in costs, leading to a very steep supply curve. Another characteristic of electricity is the lack of efficient storage possibilities. Many spot price models have been suggested for electricity, and we refer to Eydeland and Wolyniec [22] and Benth, Šaltytė Benth and Koekebakker [3] for a discussion on various models and other aspects of modelling of energy markets.

In this paper we propose a two-factor arithmetic spot price model with seasonality, which is analytically feasible for pricing electricity forward and futures contracts. The spot price model consists of a continuous-time autoregressive moving average factor driven by a stable Lévy process for modelling the stationary short-term variations, and a non-stationary long-term factor given by a Lévy process. We derive futures prices under a given pricing measure, and propose to fit the spot model by a novel optimisation algorithm using spot and futures price data simultaneously. We apply our model and estimation procedure to price data observed at the German electricity exchange EEX.

In a seminal paper by Schwartz and Smith [36] a two-factor model for commodity spot prices is proposed. Their idea is to model the short-term logarithmic spot price variations as a mean-reverting Ornstein-Uhlenbeck process driven by a Brownian motion, reflecting the drive in prices towards their equilibrium level due to changes in supply and demand. But, as argued by Schwartz and Smith [36], there may be significant uncertainty in the equilibrium level caused by inflation, technological innovations, scarceness of fuel resources like gas and coal etc.. To account for such long-term randomness in prices, Schwartz and Smith [36] include a second non-stationary factor being a drifted Brownian motion, possibly correlated with the short-term variations. They apply their model to crude oil futures traded at NYMEX, where the non-stationary part is estimated from futures prices, which are far from delivery. Mean-reversion will kill off the short-term effects from the spot on such futures, and they can thus be applied to filter out the non-stationary factor of the spot prices.

This two-factor model is applied to electricity prices by Lucia and Schwartz [29]. Among other models, they fit an arithmetic two-factor model with deterministic seasonality to electricity spot prices collected from the Nordic electricity exchange NordPool. Using forward and futures prices they fit the model, where the distance between theoretical and observed prices are minimised in a least-squares sense.

A major criticism against the two-factor model considered in Lucia and Schwartz [29] is the failure to capture spikes in the power spot dynamics. By using Brownian motion driven factors, one cannot explain the sudden large price spikes frequently observed in spot data. Multi-factor models, where one or more factors are driven by jump processes, may mend this. For example, Benth, Kallsen and Meyer-Brandis [2] suggest an arithmetic spot model where normal and spike variations in the prices are separated into different factors driven by pure-jump processes. In this way one may model large price increases followed by fast speed of mean-reversion together with a “base component”-behaviour, where price fluctuations are more slowly varying around a mean level. Such multi-factor models allow for analytic pricing of forward and futures contracts.

A very attractive alternative to these multi-factor models is given by the class of continuous-time autoregressive moving-average processes, also called CARMA processes. These processes incorporate in an efficient way memory effects and mean-reversion, and generalise Ornstein-Uhlenbeck processes in a natural way (see Brockwell [11]). As it turns out, (C)ARMA processes fit power spot prices extremely well, as demonstrated in Bernhardt, Klüppelberg and Meyer-Brandis [7] and Garcia, Klüppelberg and Müller [24]. In [7] an ARMA process with stable innovations, and in [24] a CARMA(2,1)-model driven by a stable Lévy process are suggested and empirically fitted to power spot price data collected from the Singapore and German EEX markets, respectively. A CARMA(2,1) process may be viewed on a discrete-time grid as an ARMA(2,1) model (cf. [13]). By using a stable Lévy process to drive the CARMA model, one is blending spikes and small innovations in prices into one process. We remark in passing that a CARMA dynamics has been applied to model crude oil prices at NYMEX by Paschke and Prokopczuk [31] and interest-rates by Zakamouline, Benth and Koekebakker [41].

We propose a generalisation of the stable CARMA model of Garcia et al. [24] by including a long-term non-stationary factor being a general Lévy process. The model allows for analytical pricing of electricity futures, based on pricing measures, which preserve the Lévy property of the driving processes. More precisely, we apply an Esscher measure transform to the non-stationary part, and a transform which maps the stationary stable process into a tempered stable. Due to the semi-affine structure of the model, the futures and forward prices become explicitly dependent on the states of the CARMA model and the non-stationary factor.

The CARMA-based factor in the spot model accounts for the short-term variations in prices and will be chosen stationary. By a CARMA model with a higher order autoregressive part we may include different mean-reversion speeds, such that we can mimic the behaviour of a multi-factor model accounting for spikes and base variations separately. The moving average part is necessary to model the observed dependence structure. The stable Lévy process may have very big jumps, which then can explain spike behaviour in the prices. The smaller variations of the stable Lévy process model the base signal in power prices. As it turns out

from our empirical investigations using market price data from the EEX, the non-stationary long-term behaviour may accurately be modelled using a normal inverse Gaussian (NIG) Lévy process. We filter out the non-stationary part from observing futures prices, which are far from delivery. The influence of the stationary CARMA factor is then not present, and the data shows a significant non-normal behaviour. This is in contrast to the choice suggested in Schwartz and Smith [36] and applied in Lucia and Schwartz [29]. Moreover, we find that a CARMA(2,1) model is accurately explaining mean-reversion and memory effects in spot data.

A novelty of our paper apart from generalising existing one and two factor models, is our estimation procedure. Lucia and Schwartz [29] propose an iterative algorithm for estimating their two-factor model to NordPool electricity data, where they minimise the least-squares distance between the theoretical and observed forward and futures prices to find the risk-neutral parameters. In order to find the theoretical prices, they must have the states of the two factors in the spot model accessible. Since these are not directly observable, they choose an iterative scheme, where they start with a guess on the parameter values, find the states minimising distance, update parameters by estimation, find the states minimising the distance etc. until convergence is reached. We propose a different approach, utilising the idea in Schwartz and Smith [36] that the non-stationary factor is directly observable, at least approximately, from forward prices, which are far from delivery. We apply this to filter out the non-stationary factor. The CARMA-part is then observable from spot prices, where the seasonality and non-stationary term have been subtracted. Since we work with stable processes, which do not have finite second moments, L^2 -filters cannot be used to find the states of the CARMA-process. We propose a simple L^1 -filter being more robust with respect to spikes in spot data to do this. The problem we are facing is to determine, what contracts to use for filtering out the non-stationary part. To find an optimal “time-to-maturity” which is sufficiently far from delivery, so that the futures prices behave as the non-stationary factor and at the same time provide a sufficiently rich set of data, we use an optimisation algorithm, which minimises the least-square distance between the empirical and theoretical risk premium. In order to find the risk premia, we must have all the parameters of the model available, which in turn can only be found, if we know which futures contracts can be used for filtering the long-term factor. We implement an algorithm, which estimates all model parameters for futures contracts with different times to maturities and minimises the distance to the empirical risk premium.

We apply our model and estimation scheme to data from the German EEX (European Energy Exchange) market where we use spot prices as well as futures prices of contracts with a delivery period of one month. Our empirical studies cover both base load and peak load contracts, where base load contracts are settled against the average of all hourly spot prices in the delivery period. Peak load futures contracts are settled against the average of

hourly spot prices in peak periods of the delivery period. The peak load period is the period between 8 a.m. and 8 p.m., during every working day. As a first summary, we can say that the results for both base and peak load data are in general rather similar. However, peak load data show a more extreme behaviour. The average risk premium decays when time to maturity increases, and is negative for contracts in the longer end of the futures curve. This points towards a futures market, where producers use the contracts for hedging and in return accept to pay a premium to insure their production, in accordance with the theory of normal backwardation. The risk premium is completely determined by the effect of the long-term factor, which induces a close to linear decay as a function of “time-to-maturity”.

We see that for base load contracts the risk premium in the short end of the curve is only slightly positive. The risk premium is negative for contracts starting to deliver in about two months or later. On the other hand, peak load contracts have a clear positive risk premium, which turns to a negative one for contracts with delivery in about four months or later. The positive risk premium for contracts close to delivery tells us that the demand side (retailers and consumers) of the market is willing to pay a premium for locking in electricity prices as a hedge against spike risk (see Geman and Vasicek [23]).

Due to the German “Energiewende”, where carbon-intensive energy production is gradually substituted with renewable power sources like wind and photovoltaic, one may expect different results on the risk premium in the short end of the German forward and futures market. Power generated from wind and sun has priority into the German power system and has recently caused prices to spike *downwards* and even become negative at some instances. Hence, retailers may not have strong incentives to hedge short-term price risk any longer, and we may expect the risk premium to be negative also in the short end of the forward market. The data analysis performed in this paper covers a period, where renewable power generation in the EEX area was small compared to the current situation. However, the model and methods developed in the present paper go beyond the specific data sets analysed in this paper. Moreover, given sufficiently long data history, it will be of interest in the future to fit our model to data from the new market regime.

Our results are presented as follows. In Section 2 we present the two-factor spot model, and we compute analytical futures prices along with a discussion of pricing measures in Section 3. Section 4 explains in detail the estimation steps and the procedure applied to fit the model to data. The results of this estimation procedure applied to EEX data is presented and discussed in Section 5. We conclude in Section 6.

Throughout we use the following notation. For a matrix D we denote by D^* its transposed, and I is the identity matrix. For $p \in \mathbb{N}$ we denote by \mathbf{e}_p the p -th unit vector. The matrix exponential e^{At} is defined by its Taylor expansion $e^{At} = I + \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$ with identity matrix I . We also denote by $\log^+ x = \max(\log x, 0)$ for $x \in \mathbb{R}$.

2 The spot price dynamics

In most electricity markets, like the EEX, hourly spot prices for the delivery of 1 MW of electricity are quoted. As is usual in the literature on electricity spot price modelling, one assumes a continuous-time model and estimates it on the discretely observed *daily* average spot prices. We refer, for instance, to Lucia and Schwartz [29] and Benth, Šaltytė Benth and Koekebakker [3] for more details.

We generalise the α -stable (C)ARMA model of Bernhardt, Klüppelberg and Meyer-Brandis [7] and Garcia, Klüppelberg and Müller [24] by adding a non-stationary stochastic component in the trend of the spot dynamics. By modelling the trend as a combination of a stochastic process and a deterministic seasonality function rather than only a deterministic seasonality function, which seems common in most models, we are able to describe the low frequency variations of the spot dynamics quite precisely. As it turns out, this trend will explain a significant part of the futures price variations and lead to an accurate estimation of the risk premium in the EEX market.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space satisfying the usual conditions of completeness and right continuity. We assume the spot price dynamics

$$S(t) = \Lambda(t) + Z(t) + Y(t), \quad t \geq 0, \quad (2.1)$$

where Λ is a deterministic trend/seasonality function and Z is a Lévy process with zero mean. The process Z models the low-frequency non-stationary dynamics of the spot, and can together with Λ be interpreted as the long-term factor for the spot price evolution. The process Y accounts for the stationary short-term variations. We will assume that Y and Z are independent processes. We follow Garcia et al. [24] and Bernhardt et al. [7] and suppose that Y is a stationary CARMA-process driven by an α -stable Lévy process.

2.1 The stable CARMA-process

We introduce stationary CARMA(p, q)-Lévy processes (see Brockwell [11]) and discuss its relevant properties.

Definition 2.1 (CARMA(p, q)-Lévy process).

A CARMA(p, q)-Lévy process $\{Y(t)\}_{t \geq 0}$ (with $0 \leq q < p$) driven by a Lévy-process L is defined as the solution of the state space equations

$$Y(t) = \mathbf{b}^* \mathbf{X}(t) \quad (2.2)$$

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + \mathbf{e}_p dL(t), \quad (2.3)$$

with

$$\mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{pmatrix}, \quad \mathbf{e}_p = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \vdots & -a_1 \end{pmatrix}.$$

where $a_1, \dots, a_p, b_0, \dots, b_{p-1}$ are possibly complex-valued coefficients such that $b_q = 1$ and $b_j = 0$ for $q < j \leq p$. For $p = 1$ the matrix A is to be understood as $A = -a_1$.

The driving process L of Y will be a non-Gaussian α -stable Lévy process $\{L(t)\}_{t \geq 0}$ with characteristic function given by $\ln \mathbb{E}e^{izL(t)} = t\phi_L(z)$ for $z \in \mathbb{R}$, where,

$$\phi_L(z) = \begin{cases} -\gamma^\alpha |z|^\alpha (1 - i\beta(\text{sign } z) \tan(\frac{\pi\alpha}{2})) + i\mu z & \text{for } \alpha \neq 1, \\ -\gamma |z| (1 + i\beta \frac{2}{\pi} (\text{sign } z) \log |z|) + i\mu z & \text{for } \alpha = 1. \end{cases} \quad (2.4)$$

The sign function is defined by $\text{sign } z = -1$ for $z < 0$, $\text{sign } z = 1$ for $z > 0$ and $\text{sign } 0 = 0$, respectively. Further, $\alpha \in (0, 2)$ is the shape parameter, $\gamma > 0$ the scale, $\beta \in [-1, 1]$ the skewness, and μ the location parameter. If $\gamma = 1$ and $\mu = 0$, then L is called standardised.

The parameter $\alpha \in (0, 2)$ determines the tail of the distribution function of $L(t)$ for all $t \geq 0$. Moreover, only moments strictly less than α are finite, so that no second moment exists. This implies also that the autocorrelation function does not exist. For further properties on stable processes and Lévy processes, we refer to the monographs Samorodnitsky and Taqqu [34] and Sato [35].

The solution of the SDE (2.3) is a p -dimensional Ornstein-Uhlenbeck (OU) process given by

$$\mathbf{X}(t) = e^{A(t-s)} \mathbf{X}(s) + \int_s^t e^{A(t-u)} \mathbf{e}_p dL(u), \quad 0 \leq s < t, \quad (2.5)$$

where the stable integral is defined as in Ch. 3 of Samorodnitsky and Taqqu [34]. From (2.2) we find that Y is given by

$$Y(t) = \mathbf{b}^* e^{A(t-s)} \mathbf{X}(s) + \int_s^t \mathbf{b}^* e^{A(t-u)} \mathbf{e}_p dL(u), \quad 0 \leq s < t. \quad (2.6)$$

Equations (2.2) and (2.3) constitute the state-space representation of the formal p -th order SDE

$$a(D)Y(t) = b(D)DL(t), \quad t \geq 0, \quad (2.7)$$

where D denotes differentiation with respect to t , and

$$a(z) := z^p + a_1 z^{p-1} + \dots + a_p \quad (2.8)$$

$$b(z) := b_0 + b_1 z + \dots + b_q z^q \quad (2.9)$$

are the characteristic polynomials. Equation (2.7) is a natural continuous-time analogue of the linear difference equations, which define an ARMA process (cf. Brockwell and Davis [13]).

Throughout we assume that Y and \mathbf{X} are stationary in the sense that all finite dimensional distributions are shift-invariant. Based on Proposition 2.2 of Garcia et al. [24] (which summarises results by Brockwell and Lindner [15]) we make the following assumptions to ensure this:

Assumptions 2.2. *Stationarity of CARMA-process.*

- (i) *The polynomials $a(\cdot)$ and $b(\cdot)$ defined in (2.8) and (2.9), resp., have no common zeros.*
- (ii) $\mathbb{E} [\log^+ |L(1)|] < \infty$.
- (iii) *All eigenvalues of A are distinct and have strictly negative real parts.*

Assumptions (ii) and (iii) imply that \mathbf{X} is a causal p -dimensional OU process, hence also Y is causal.

Remark 2.3. Our model is a significant generalisation of the two-factor dynamics of Schwartz and Smith [36] and Lucia and Schwartz [29]. Among various models, Lucia and Schwartz [29] suggested a two factor dynamics of the spot price evolution based on a short term Gaussian OU process and a long-term drifted Brownian motion. In our framework a Gaussian OU process would correspond to a Gaussian CARMA(1,0) process. It is clear that such a Gaussian model cannot capture the large fluctuations in the spot price, like for example spikes, and jump processes seem to be the natural extension. Based on the studies of Bernhardt et al. [7] and Garcia et al. [24], α -stable processes are particularly suitable for the short-term dynamics in the spot price evolution. Furthermore, empirical analysis of electricity spot price data from Singapore and Germany in [7] and [24] shows strong statistical evidence for full CARMA processes to capture the dependency structure of the data. As for the long-term baseline trend, we shall see in Section 4 that a normal inverse Gaussian Lévy process is preferable to a Gaussian process in a data study from the German electricity exchange EEX. \square

2.2 Dimensionality of CARMA-processes

A more standard model in electricity is to describe the spot by a sum of several OU processes, where some summands describe the spike behaviour and others the baseline dynamics (see

for example the model in Benth et al. [2]). A CARMA process is in a sense comparable to such models, as we now discuss.

By Assumption 2.2(iii) A has full rank, i.e. it is diagonalisable with $e^{At} = Ue^{Dt}U^{-1}$. Here, D is a diagonal matrix with the eigenvalues $\lambda_1, \dots, \lambda_p$ of A on the diagonal and U is the full rank matrix having the eigenvectors of A as columns. Since all eigenvalues have negative real parts, all components of e^{At} are mean-reverting. Each component of the vector $e^{A(t-s)}\mathbf{X}(s)$ from (2.6) will therefore mean-revert at its own speed, where the speed of mean-reversion is a linear combination of the diagonal elements of e^{Dt} .

As we shall see in a simulation example of a CARMA(2,1)-process in Section 4.5, the model captures the situation, where a first component has a slower rate of mean-reversion than the second (see Fig. 5). This is similar to a two-factor spot model, where base and spike components of the spot price evolution are separated into two OU processes with different speeds of mean-reversion. The advantage of working with a stable CARMA process, as we propose, is that it is possible to capture the distribution of the small and large jumps in one distribution. Since extreme spikes are rather infrequently observed, it is difficult to calibrate the spike component in a conventional two-factor model; this has been observed in Klüppelberg et al. [26]. With our CARMA-model, we avoid the difficult question of spike identification and filtering.

3 The futures price dynamics

In commodity markets, futures contracts are commonly traded on exchanges, including electricity, gas, oil, and coal. In this section we derive the futures price dynamics based on the α -stable CARMA spot model (2.1). Appealing to general arbitrage theory (see e.g. Duffie [20], Ch.1), we define the futures price $f(t, \tau)$ at time t for a contract maturing at time τ by

$$f(t, \tau) = \mathbb{E}_Q [S(\tau) | \mathcal{F}_t] , \quad 0 \leq t \leq \tau < \infty , \quad (3.1)$$

where Q is a risk-neutral probability measure. This definition is valid as long as $S(\tau) \in L^1(Q)$. In the electricity market, the spot cannot be traded, and every $Q \sim P$ will be a risk-neutral probability (see Benth et al. [3]). For example, $Q = P$ is a valid choice of a pricing measure. In that case, the condition $S(\tau) \in L^1(P)$ is equivalent to a tail parameter α of the stable process L being strictly larger than one, and a process Z with finite expectation. In real markets one expects a risk premium and hence it is natural to use a pricing measure $Q \neq P$. We will discuss possible choices of risk-neutral probability measures Q in Section 3.1.

Based on our spot price model, we find the following explicit dynamics of the futures price for a given class of risk-neutral probability measures:

Theorem 3.1. *Let S be the spot dynamics as in (2.1), and suppose that $Q \sim P$ is such that L and Z are independent Lévy processes under Q . Moreover, assume that the processes Z and L have finite first moments under Q . Then the futures price dynamics for $0 \leq t \leq \tau$ is given by*

$$f(t, \tau) = \Lambda(\tau) + Z(t) + \mathbf{b}^* e^{A(\tau-t)} \mathbf{X}(t) + (\tau - t) \mathbb{E}_Q[Z(1)] + \mathbf{b}^* A^{-1} (I - e^{A(\tau-t)}) \mathbf{e}_p \mathbb{E}_Q[L(1)].$$

Proof. Using (3.1), $f(t, \tau) = \mathbb{E}_Q[S(\tau) | \mathcal{F}_t] = \Lambda(\tau) + \mathbb{E}_Q[Z(\tau) | \mathcal{F}_t] + \mathbb{E}_Q[Y(\tau) | \mathcal{F}_t]$. Since Z is a Lévy process under Q , we find

$$\mathbb{E}_Q[Z(\tau) | \mathcal{F}_t] = Z(t) + \mathbb{E}_Q[Z(\tau) - Z(t) | \mathcal{F}_t] = Z(t) + (\tau - t) \mathbb{E}_Q[Z(1)].$$

Now denote by $M(u) = L(u) - \mathbb{E}_Q[L(1)]u$ for $t \leq u \leq \tau$, which has zero mean. Then, by partial integration,

$$\begin{aligned} & \mathbb{E}_Q \left[\int_t^\tau \mathbf{b}^* e^{A(\tau-u)} \mathbf{e}_p dM(u) \right] \\ &= \mathbb{E}_Q \left[\mathbf{b}^* \mathbf{e}_p M(\tau) - \mathbf{b}^* e^{A(\tau-t)} \mathbf{e}_p M(t) \right] - \int_t^\tau \mathbf{b}^* A e^{A(\tau-u)} \mathbf{e}_p \mathbb{E}_Q[M(u)] du = 0, \end{aligned}$$

which implies
$$\mathbb{E}_Q \left[\int_t^\tau \mathbf{b}^* e^{A(\tau-u)} \mathbf{e}_p dL(u) \right] = \mathbb{E}_Q[L(1)] \int_t^\tau \mathbf{b}^* e^{A(\tau-u)} \mathbf{e}_p du.$$

Hence, the CARMA part of the spot dynamics converts to

$$\begin{aligned} \mathbb{E}_Q[Y(\tau) | \mathcal{F}_t] &= \mathbb{E}_Q \left[\mathbf{b}^* e^{A(\tau-t)} \mathbf{X}(t) + \int_t^\tau \mathbf{b}^* e^{A(\tau-u)} \mathbf{e}_p dL(u) | \mathcal{F}_t \right] \\ &= \mathbf{b}^* e^{A(\tau-t)} \mathbf{X}(t) + \mathbb{E}_Q \left[\int_t^\tau \mathbf{b}^* e^{A(\tau-u)} \mathbf{e}_p dL(u) \right] = \mathbf{b}^* e^{A(\tau-t)} \mathbf{X}(t) + \int_t^\tau \mathbf{b}^* e^{A(\tau-u)} \mathbf{e}_p du \mathbb{E}_Q[L(1)] \\ &= \mathbf{b}^* e^{A(\tau-t)} \mathbf{X}(t) + \mathbf{b}^* A^{-1} (I - e^{A(\tau-t)}) \mathbf{e}_p \mathbb{E}_Q[L(1)]. \end{aligned}$$

Combining the terms yields the result. □

In electricity markets the futures contracts deliver the underlying commodity over a period rather than at a fixed maturity time τ . For instance, in the German electricity market contracts for delivery over a month, a quarter or a year, are traded. These futures are sometimes referred to as swaps, since during the delivery period a fixed (futures) price of energy is swapped against a floating (uncertain) spot price. The futures price is quoted as the price of 1 MWh of power and, therefore, it is settled against the average spot price over the delivery period. Hence, the futures price $F(t, T_1, T_2)$ at time $0 \leq t \leq T_1 < T_2$ for a contract with delivery period $[T_1, T_2]$ is defined as

$$F(t, T_1, T_2) = \mathbb{E}_Q \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(\tau) d\tau \middle| \mathcal{F}_t \right], \quad (3.2)$$

where we have assumed that settlement of the contract takes place at the end of the delivery period, T_2 .

Using Theorem 3.1 we derive by straightforward integration the swap price dynamics $F(t, T_1, T_2)$ from (3.2).

Corollary 3.2. *Suppose all assumptions of Theorem 3.1 are satisfied. Then,*

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(\tau) d\tau + Z(t) + \frac{\mathbf{b}^* A^{-1}}{T_2 - T_1} (e^{AT_2} - e^{AT_1}) e^{-At} \mathbf{X}(t) + \Gamma_Q(t, T_1, T_2)$$

where

$$\begin{aligned} \Gamma_Q(t, T_1, T_2) = & \left(\frac{1}{2}(T_2 + T_1) - t \right) \mathbb{E}_Q[Z(1)] - \frac{\mathbf{b}^* A^{-2}}{T_2 - T_1} (e^{AT_2} - e^{AT_1}) e^{-At} \mathbf{e}_p \mathbb{E}_Q[L(1)] \\ & + \mathbf{b}^* A^{-1} \mathbf{e}_p \mathbb{E}_Q[L(1)]. \end{aligned}$$

Proof. By Fubini's theorem, we find

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t, \tau) d\tau.$$

Applying Theorem 3.1 and integrating yield the desired result. \square

The risk premium is defined as the difference between the futures price and the predicted spot, that is, in terms of electricity futures contracts,

$$R_{pr}(t, T_1, T_2) = F(t, T_1, T_2) - \mathbb{E} \left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(\tau) d\tau \mid \mathcal{F}_t \right]. \quad (3.3)$$

From Cor. 3.2 we find that the theoretical risk premium for a given pricing measure Q is

$$\begin{aligned} R(t, T_1, T_2) &= \Gamma_Q(t, T_1, T_2) - \Gamma_P(t, T_1, T_2) \\ &= \left(\frac{1}{2}(T_2 + T_1) - t \right) \mathbb{E}_Q[Z(1)] - \frac{\mathbf{b}^* A^{-2}}{T_2 - T_1} (e^{AT_2} - e^{AT_1}) e^{-At} \mathbf{e}_p (\mathbb{E}_Q[L(1)] - \mathbb{E}[L(1)]) \\ &\quad + \mathbf{b}^* A^{-1} \mathbf{e}_p (\mathbb{E}_Q[L(1)] - \mathbb{E}[L(1)]). \end{aligned} \quad (3.4)$$

Here, we used the assumption that Z has zero mean under P . The first term gives a trend in "time to maturity" implied by the non-stationarity part Z in the spot price dynamics. "Time to maturity" is here interpreted as time left to the middle of the delivery period. The two last terms are risk premia contributions from the CARMA short-term spot dynamics. They involve an explicit dependence on the speeds of mean-reversion of the autoregressive parts and the memory in the moving-average part. We will estimate the risk premium in the empirical analysis of spot and futures data from the EEX.

3.1 Equivalent measure transforms for Lévy and α -stable processes

In this subsection we discuss a class of pricing measures that will be used for the specification of the futures price dynamics.

We require from the pricing measure that Z and L preserve the Lévy property and independence. For this purpose, we consider probability measures $Q = Q_L \times Q_Z$, where Q_L and Q_Z are measure changes for L and Z , respectively (leaving the other process unchanged). Provided Z has exponential moments, a standard choice of measure change is given by the Esscher transform (see Benth et al. [3], Section 4.1.1). Note that L , the α -stable process in the CARMA-dynamics, does not have exponential moments.

We define the density process of the Radon-Nikodym derivative of Q_Z as

$$\left. \frac{dQ_Z}{dP} \right|_{\mathcal{F}_t} = \exp(\theta_Z Z(t) - \phi_Z(\theta_Z)t), \quad t \geq 0, \quad (3.5)$$

for a constant $\theta_Z \in \mathbb{R}$ and ϕ_Z being the log-moment generating function of $Z(1)$ (sometimes called the cumulant function of Z). In order to make this density process well-defined, exponential integrability of the process Z up to the order of θ_Z must be assumed. Under this change of measure, the Lévy measure of Z will be exponentially tilted by θ_Z , that is, if we denote the Lévy measure of Z (under P) by $\nu(dx)$, then its Lévy measure under Q_Z becomes $\nu_{Q_Z}(dx) = \exp(\theta_Z)\nu(dx)$ (see Benth et al. [3], Section 4.1.1-4.1.2 for details).

To choose a risk-neutral measure Q_L is a more delicate task. We know from Sato [35], Theorems 33.1 and 33.2, that equivalent measures Q exist for stable processes, however, it seems difficult to construct one which preserves the stable property. As an alternative, one may introduce the class of *tempered stable processes* (see e.g. Cont and Tankov [17], Chapter 9), and apply standard Esscher transformation on these.

A tempered stable process is a pure jump Lévy process, where the stable-like behaviour is preserved for the small jumps. However, the tails are tempered and, therefore, extreme spikes are less likely to be modelled with the tempered stable process. The Lévy measure is given by

$$\nu_{TS}(dx) = \frac{c_+ e^{\theta_L x}}{x^{1+\alpha}} 1_{(0,\infty)}(x) dx + \frac{c_- e^{\theta_L |x|}}{|x|^{1+\alpha}} 1_{(-\infty,0)}(x) dx. \quad (3.6)$$

Here, $\theta_L \leq 0$ and $c_-, c_+ \in \mathbb{R}_+$. A consequence of the tempering is that certain exponential moments exist. Tempering of a stable distribution results in a *tempered* stable distribution, and is analogous to taking an Esscher transform of the stable process using a negative parameter θ_L on the positive jumps, and a positive parameter $-\theta_L$ on the negative jumps.

In particular, define $q : \mathbb{R} \mapsto \mathbb{R}$ as $q(x) := e^{\theta_L x} 1_{(0,\infty)}(x) + e^{\theta_L |x|} 1_{(-\infty,0)}(x)$ for some constant $\theta_L < 0$. Suppose the stable process L has (under P) the characteristic triplet $(\gamma_L, 0, \nu_L)$, where

$$\nu_L(dx) = \frac{c_+}{x^{1+\alpha}} 1_{(0,\infty)}(x) dx + \frac{c_-}{|x|^{1+\alpha}} 1_{(-\infty,0)}(x) dx$$

is the Lévy measure of L . The parameters c_+, c_- can be matched to the parameters in (2.4) using Example 2.3.3 of Samorodnitsky and Taqqu [34]. Then the tempered stable measure Q_L with characteristic triplet $(\gamma_{TS}, 0, \nu_{TS})$ is *equivalent* to the physical probability measure P (see Cont and Tankov [17], Proposition 9.8), with drift parameter γ_{TS} . For our application it is of particular value to have finite expectations and to know the expectations of $L(1)$ under P and Q_L , corresponding to $\alpha \in (1, 2)$. In this case

$$\gamma_{TS} = \gamma_L + \int_{\{|x|<1\}} x(q(x) - 1)\nu_L(dx) \quad (3.7)$$

and the Lévy measure ν_{TS} is given by $\nu_{TS}(dx) = q(x)\nu_L(dx)$.

Lemma 3.3. *Let L be an α -stable Lévy process under P with $\alpha \in (1, 2)$. Find Q_L by stable tempering for $\theta_L < 0$ as in (3.6). Then the difference in mean of $L(1)$ under Q_L and P is given by*

$$\mathbb{E}_{Q_L}[L(1)] - \mathbb{E}[L(1)] = \Gamma(1 - \alpha)(-\theta_L)^{\alpha-1} (c_+ - c_-) , \quad (3.8)$$

where Γ is the gamma function.

Proof. Using (3.7) and the Lévy-Khintchine formula (e.g. Cont and Tankov [17], Prop. 3.13) for $1 < \alpha < 2$ we obtain

$$\begin{aligned} \mathbb{E}[L(1)] - \mathbb{E}_{Q_L}[L(1)] &= \gamma_L - \gamma_{TS} + \int_{\{|x|>1\}} x(\nu_L - \nu_{TS})(dx) \\ &= \gamma_L - \gamma_{TS} + \int_{\{|x|>1\}} x(1 - q(x))\nu_L(dx) \\ &= c_- \int_{-\infty}^0 (1 - e^{\theta_L |x|}) \frac{dx}{x^{1+\alpha}} + c_+ \int_0^{\infty} (1 - e^{\theta_L x}) \frac{dx}{x^{1+\alpha}} \\ &= -\Gamma(1 - \alpha)(-\theta_L)^{\alpha-1} (c_+ - c_-) , \end{aligned} \quad (3.9)$$

where we have used partial integration on the two integrals and l'Hospital's rule to obtain the last identity. This proves the result. \square

Remark 3.4. By altering θ_L one can match any relevant change of mean in the risk premium $\mathbb{E}_{Q_L}[L(1)] - \mathbb{E}[L(1)]$, as long as this can be obtained by a negative choice of θ_L . This turns out to be appropriate for our applications. \square

4 Fitting the model to German electricity data

Our data are daily spot and futures prices from July 1, 2002 to June 30, 2006 (available from <http://eex.com>). We fitted our model both to base load and peak load data, respectively. The futures contracts considered in this analysis are the Phelix-Base-Month-Futures and the Phelix-Peak-Month-Futures. *Base load futures contracts* are settled against the average of all hourly spot prices in the delivery period. *Peak load futures contracts*, on the other hand, are settled against the average of the hourly spot prices in peak periods of the delivery period. The peak period is counted as the hours between 8 a.m. and 8 p.m. every working day during the delivery period. The time series of daily spot prices used for our combined statistical analysis is taken to match the futures contracts: for the base load contracts we use the full time series consisting of daily observations including weekends (i.e., we have 7 observations per week), while in the case of peak load contracts the weekends are excluded (i.e., we have 5 observations per week).

Figures 1 and 2 show the spot and futures prices for both base load and peak load. From these plots we can see similar patterns of the base and peak load data, however, peak load data are more extreme. Note that all plots cover the same time period; however, for the base spot data we have 1461 observations, whereas for the peak spot data we have only 1045 observations in the same period, due to the missing weekends.

The estimation procedure for our model consists of several steps, which are explained in the following.

4.1 Seasonality function Λ

The estimation of the deterministic trend component Λ is a delicate question. A misspecification of the trend has a significant effect on the subsequent analysis, in particular, on the risk premium. Motivated by the seasonality functions used in Bernhardt et al. [7] and Garcia et al. [24], we take the seasonality function of the peak load contracts as a combination of a linear trend and some periodic function

$$\Lambda_p(t) = c_1 + c_2 t + c_3 \cos\left(\frac{2\pi t}{261}\right) + c_4 \sin\left(\frac{2\pi t}{261}\right). \quad (4.1)$$

Note that we choose a slightly simpler seasonality function than Bernhardt et al. [7] and Garcia et al. [24], only taking the mean level, a linear trend and a yearly periodicity (modelling the weather difference between summer and winter) into account. Weekly periodicity in peak load contracts is not that pronounced, since weekends are not considered in peak

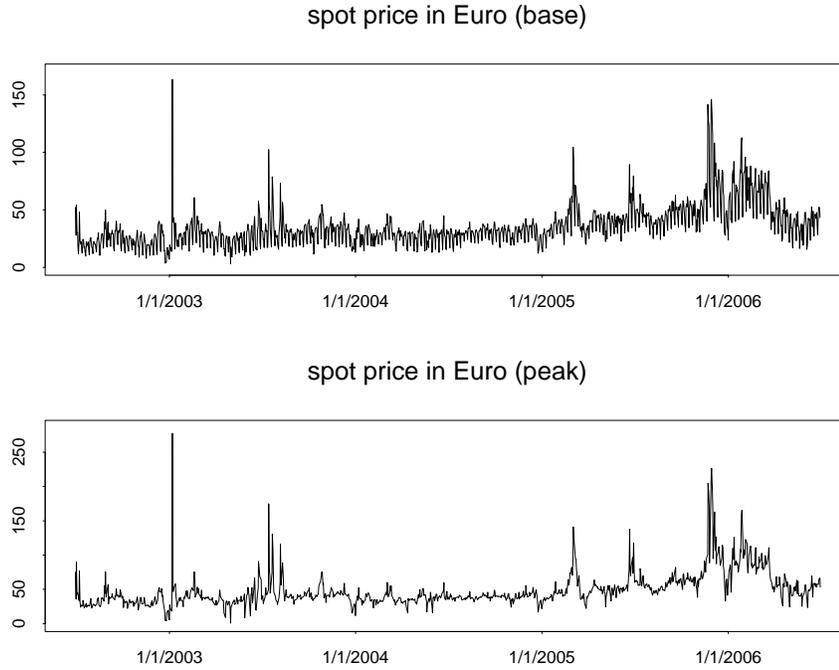


Figure 1: *Daily spot prices from July 1, 2002 to June 30, 2006, base load (top) and peak load (bottom).*

load data. Since no new trading information is entering during the weekend (trading takes place during weekdays), we will adjust the periodicity to 261 and consider the peak load contracts as a continuous process on all non-weekend days.

In the base load prices a clear weekly seasonality is visible. Weekend prices are in general lower than during the rest of the week, and over the week one observes the pattern that Monday and Friday have prices lower than in the middle of the week. Therefore we include a weekly term in the base load seasonality function:

$$\Lambda_b(t) = c_1 + c_2 t + c_3 \cos\left(\frac{2\pi t}{365}\right) + c_4 \sin\left(\frac{2\pi t}{365}\right) + c_5 \cos\left(\frac{2\pi t}{7}\right) + c_6 \sin\left(\frac{2\pi t}{7}\right). \quad (4.2)$$

Since time t is running through the weekends, a yearly periodicity of 365 is chosen.

In the following, we will analyse both data sets, base load data and peak load data (spot and futures, respectively). For simplicity we will suppress the indices p for peak load and b for base load.

The seasonality functions can be estimated using a robust least-squares estimate on the data; see Table 4.1 for the resulting estimates. The first plot of Figure 3 shows the 1461

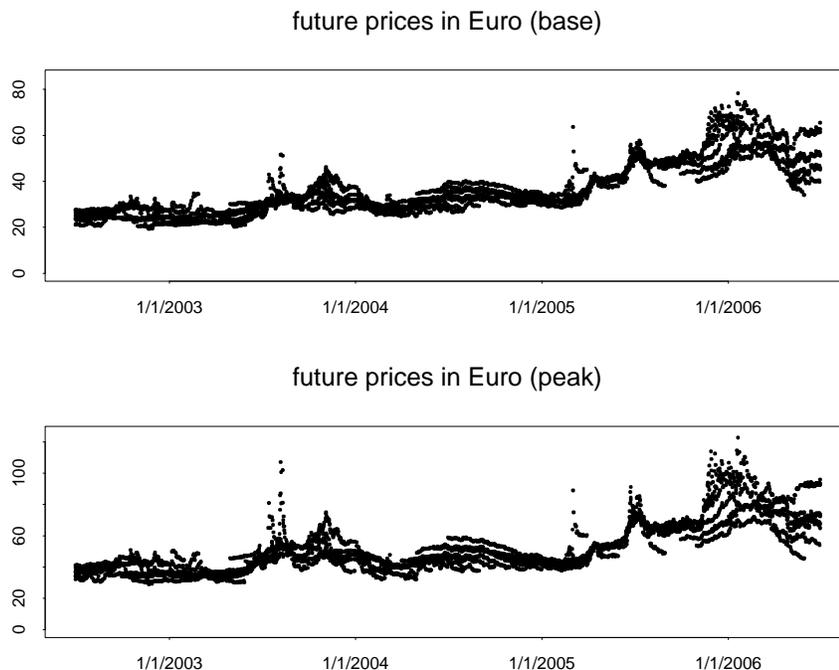


Figure 2: *Daily futures prices from July 1, 2002 to June 30, 2006, base load (top) and peak load (bottom).*

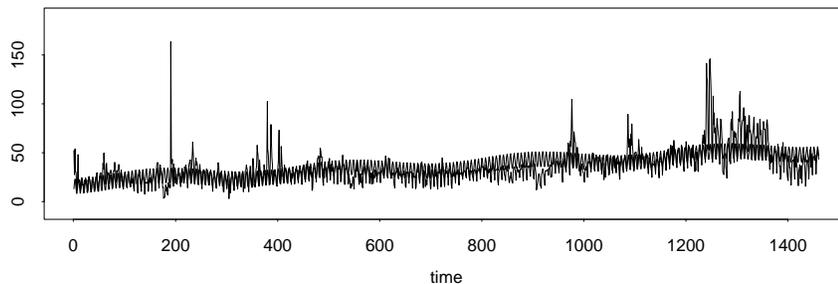
observations of the base data set together with the estimated trend and seasonality curve, the second plot zooms into this plot and shows these curves for the first 200 observations. From this second plot we can clearly see that the daily seasonality is captured by Λ quite accurately. For the peak data over the same period (consisting of 1045 observations) we get similar plots (hence, we omit them here), except of the fact that Λ does not show a weekly seasonality in this case. The overall growth rate was very small during the data period, justifying a linear term as a first order approximation. Note that we have introduced the non-stationary stochastic process Z to absorb all stochastic small term effects in the seasonality. This term will play a prominent role for the futures prices later.

	c_1	c_2	c_3	c_4	c_5	c_6
Base	19.4859	0.0217	-2.8588	0.6386	-6.7867	2.8051
Peak	30.7642	0.0349	-2.5748	1.5762		

Table 4.1: *Estimated parameters of the seasonality function $\Lambda(\cdot)$.*

Subtracting the estimated seasonality function from the spot data leaves us with the reduced model $Z(\cdot) + Y(\cdot)$, where we have neglected in our notation the fact that we have subtracted only an estimator of $\Lambda(\cdot)$.

spot price and estimated seasonality / July 1, 2002 to June 30, 2006 (base)



spot price and estimated seasonality / July 1, 2002 to Jan 16, 2003 (base)

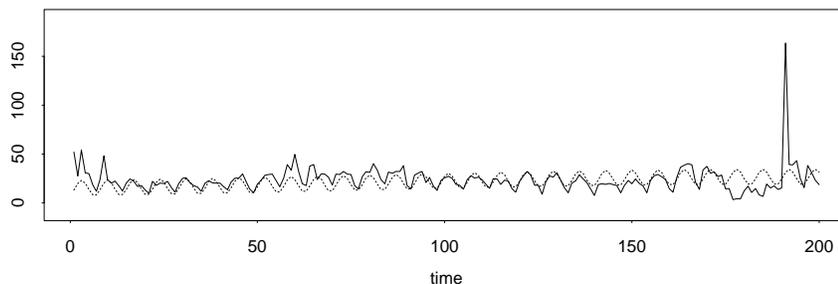


Figure 3: *Base spot prices and estimated seasonality function. Top: whole period (1461 observations). Bottom: first 200 observations.*

Next we want to estimate both components Z and Y applying the deseasonalised spot price data and the futures prices. We will exploit the fact that the futures prices far from delivery will have a dynamics approximately given by the non-stationary trend component Z . Only relatively close to delivery, large fluctuations in the spot price dynamics are reflected in the futures prices. Since it is not clear how far away from delivery we need to be before the approximation of futures prices by Z works well, we will use an optimisation routine to find the optimal distance. For this purpose, we introduce the notation $u := \frac{1}{2}(T_1 + T_2) - t$, which will be referred to as “*time to maturity*”.

Denote by \hat{u}^* the optimal time to maturity (we will define what we understand by “optimal” below), where futures contracts with time to maturity $u \geq \hat{u}^*$ have a dynamics approximately behaving like the non-stationary term. How big to choose \hat{u}^* is not possible to determine *a priori*, since we must analyse the error in an asymptotic consideration of the futures prices (see (4.3) and (4.4) below). This error is highly dependent on the parameters of the spot

price model, which we do not yet know. In the end, \hat{u}^* will be chosen so that the error in the risk premium estimation is minimal; cf. Section 4.6.

The estimation will, therefore, be repeated using different values of u^* (this parameter will sometimes be called *threshold* in the following); i.e., we choose a subset $U^* := [u_{\min}^*, u_{\max}^*] \subseteq [v/2, M_f]$, where v is the average delivery period and M_f is the maximal time to maturity observed in the futures data set, and perform the steps of the Sections 4.2–4.6 below repeatedly for $u^* \in U^*$. For each value $u^* \in U^*$ the error in the risk premium is calculated, and \hat{u}^* is the value which minimises this error among all $u^* \in U^*$. This *optimal* threshold \hat{u}^* is then considered as *final* choice of u^* for the calculation of all estimates including the processes Z and Y and for the CARMA parameters.

One should keep in mind that for too large u^* there are only a few data points available with $u \geq u^*$, which yields unreliable estimates. Since we count the time to maturity as number of trading days until the mid of the delivery period (which has length v), time to maturity is always at least $v/2$; hence we do not consider any u^* smaller than $v/2$. Overall, we decided to choose $u_{\min}^* = \lceil v/2 \rceil$ (which is 16 for the base load and 11 for the peak load data) and $u_{\max}^* = M_f/2$ (note that M_f is 200 days for the base load and 144 days for the peak load contracts). As we will see later, the optimal \hat{u}^* in our data examples is quite small, so that this choice of u_{\max}^* is completely satisfying.

Next, we want to explain in detail, how we separate Z and Y for a given fixed time to maturity. Consequently, we perform the model estimation for all $\frac{1}{2}(T_1 + T_2) \leq u^* \leq 200(144)$ and take all futures prices for the estimation procedure, whose time to maturity $u \geq u^*$.

We explain each step in the estimation procedure in detail:

4.2 Filtering the realisation of the non-stationary stochastic process Z

Recall the futures price $F(t, T_1, T_2)$ in Corollary 3.2. Since we assume that the high-frequency CARMA term Y is stationary, it holds for fixed length of delivery $T_2 - T_1$ that

$$\lim_{T_1, T_2 \rightarrow \infty} \frac{\mathbf{b}^* A^{-1}}{T_2 - T_1} (e^{AT_2} - e^{AT_1}) e^{-At} \mathbf{X}(t) = 0 \quad (4.3)$$

$$\lim_{T_1, T_2 \rightarrow \infty} \frac{\mathbf{b}^* A^{-2}}{T_2 - T_1} (e^{AT_2} - e^{AT_1}) e^{-At} \mathbf{e}_p \mathbb{E}_Q[L(1)] = 0. \quad (4.4)$$

Hence, in the long end of the futures market, the contribution from Y to the futures prices may be considered as negligible. In particular, from the futures price dynamics (Corol-

lary 3.2) we find for $[T_1, T_2]$ far into the future (that is, t much smaller than T_1) that

$$\begin{aligned}\tilde{F}(t, T_1, T_2) &:= F(t, T_1, T_2) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(\tau) d\tau \\ &\approx Z(t) + \mathbf{b}^* A^{-1} \mathbf{e}_p \mathbb{E}_Q[L(1)] + \left(\frac{1}{2}(T_1 + T_2) - t \right) \mathbb{E}_Q[Z(1)].\end{aligned}\quad (4.5)$$

Recalling the notation $u := \frac{1}{2}(T_1 + T_2) - t$ coined ‘‘time-to-maturity’’, we slightly abuse the notation and introduce $\tilde{F}(t, u) := \tilde{F}(t, T_1, T_2)$.

For $u \geq u^*$, we approximate

$$\begin{aligned}\mu_{\tilde{F}}(u) &:= \mathbb{E}[\tilde{F}(t, u)] \\ &\approx \mathbb{E}[Z(t)] + \mathbf{b}^* A^{-1} \mathbf{e}_p \mathbb{E}_Q[L(1)] + u \mathbb{E}_Q[Z(1)] \\ &= \mathbf{b}^* A^{-1} \mathbf{e}_p \mathbb{E}_Q[L(1)] + u \mathbb{E}_Q[Z(1)] \\ &=: C + u \mathbb{E}_Q[Z(1)],\end{aligned}\quad (4.6)$$

where we have used the zero-mean assumption of Z under P . This approximative identity can now be used for a robust linear regression on the time to maturity u , in order to estimate the real numbers C and $\mathbb{E}_Q[Z(1)]$. Knowing these two parameters enables us to filter out the realisation of the process Z . According to Equation (4.5) we obtain

$$\hat{Z}(t) = \hat{Z} \left(\frac{1}{2}(T_1 + T_2) - u \right) = \frac{1}{\text{card } U(t, u^*)} \sum_{(u, T_1, T_2) \in U(t, u^*)} \left[\tilde{F}(t, T_1, T_2) - \hat{C} - u \hat{\mathbb{E}}_Q[Z(1)] \right],\quad (4.7)$$

where $U(t, u^*) := \{(u, T_1, T_2) \in \mathbb{R}^3 \mid u \geq u^* \text{ and } \exists F(t, T_1, T_2) : \frac{1}{2}(T_1 + T_2) - t = u\}$.

Remark 4.1. Note that after estimating the CARMA parameters, we can also find an estimate for $\mathbb{E}_Q[L(1)]$ simply by taking $\hat{\mathbb{E}}_Q[L(1)] = \hat{C}(\hat{\mathbf{b}}^* \hat{A}^{-1} \mathbf{e}_p)^{-1}$. \square

Remark 4.2. We recall that the futures market at EEX is not open for trade during the weekend. Therefore, using our estimation procedure, we do not get any observations of Z during weekends. We will assume that Z is constant and equal to the Friday value over the weekend, when filtering the non-stationary part of the spot in the base load model. One may argue that this strategy could lead to observed large jumps of Z on Monday morning, when all information accumulated over the weekend is subsumed at once. We will return to this question in Section 5.1. \square

4.3 Estimation of the CARMA parameters

Recall our spot price model (2.1)

$$S(t) = \Lambda(t) + Y(t) + Z(t), \quad t \geq 0.$$

After $\Lambda(\cdot)$ and $Z(\cdot)$ have been estimated in Subsections 4.1 and 4.2, respectively, a realisation of the CARMA-process Y can be found by subtracting both from the spot price. Figure 4 shows the estimated processes Z (bold line) and $Y + Z$ (fine line) for both the base and the peak load data, for the full period July 1, 2002 to June 30, 2006 and for $u^* = 16$ exemplarily. Obviously, the process Z captures the medium-range fluctuations and Y the short-range fluctuations of the detrended and deseasonalised process $Y + Z$.

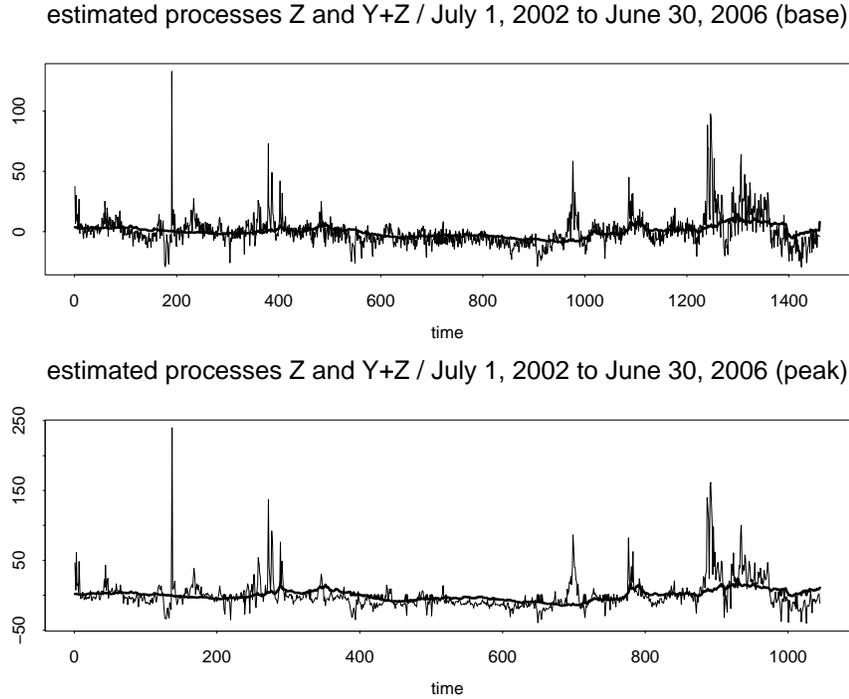


Figure 4: *Estimated processes Z (bold line) and $Y + Z$ (fine line) in the period July 1, 2002, to June 30, 2006, for both the base load data (top) and the peak load data (bottom).*

Again we keep in mind that the process Y is the result of some estimation procedure. There exists a number of papers devoted to the estimation of the CARMA parameters in L^2 (see for instance Brockwell et al. [14], Tsai and Chan [37]). Methods can be based either directly on the continuous-time process or on a discretised version. The latter relates the continuous-time dynamics to a discrete time ARMA process. The advantage of this method is obvious, since standard packages for the estimation of ARMA processes may be used in order to estimate the parameters of the corresponding CARMA process. Some care, however, is needed since this approach does not work in all cases. Brockwell and collaborators devote several papers to the embedding of ARMA processes in a CARMA process; cf. [10, 12]. Not every $\text{ARMA}(p, q)$ process is embeddable in a $\text{CARMA}(p, q)$ process.

From Assumptions 2.2, it follows by Proposition 2 of Brockwell et al. [14] (cf. also Garcia et al. [24], Prop. 2.5) that every CARMA(p, q) process Y observed at discrete times can be represented as an autoregressive process of order p with a more complex structure than a moving average process for the noise. For such a discretely observed CARMA process Y on a grid with grid size h , denoting the sequence of observations by $\{y_n\}_{n \in \mathbb{N}}$; i.e. $y_n \triangleq Y(nh)$, Prop. 3 of Brockwell et al. [14] gives

$$\prod_{i=1}^p (1 - e^{\lambda_i h} B) y_n = \varepsilon_n. \quad (4.8)$$

Here, B is the usual backshift operator and $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is the noise process, which has representation

$$\varepsilon_n = \sum_{i=1}^p \kappa_i \prod_{j \neq i} (1 - e^{\lambda_j h} B) \int_{(n-1)h}^{nh} e^{\lambda_i(nh-u)} dL(u). \quad (4.9)$$

The constants κ_i are given by $\kappa_i := b(\lambda_i)/a'(\lambda_i)$ and $\lambda_1, \dots, \lambda_p$ are the eigenvalues of A . The process $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is $(p-1)$ -dependent. When L has finite variance, ε_n has a moving average representation; cf. Brockwell et al. [13], Proposition 3.2.1. However, for the case of infinite variance this is no longer true and causes problems.

Davis [18], Davis and Resnick [19] and Mikosch et al. [30] have proved that ordinary L^2 -based estimation methods for ARMA parameters may be used for α -stable ARMA processes, although they have no finite second moments. Moreover, Davis and Resnick [19] showed that the empirical autocorrelation function of an α -stable ARMA process yields a consistent estimator of the linear filter of the model, although the autocorrelation function of the process does not exist. In [19] it has also been shown that the rate of convergence is faster than in the L^2 -case. Since [18], [19], and [30] all assume a strong ARMA process, and Proposition 3.2.1 in [13] is not applicable here, these results cannot be transferred to our situation, where $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is $(p-1)$ -dependent. However, the simulation study in Garcia et al. [24] gives strong evidence that the estimates of the CARMA parameters are quite accurate. A formal proof of the consistency is subject of current research.

Already in the analysis performed in Garcia et al. [24] the CARMA(2,1) process has been found to be optimal. Although our model is slightly different, it turns out that this CARMA dynamics is still preferable for Y based on the AIC model selection criterion. Hence, in the following example we spell out the above equations for the case of a stable CARMA(2,1) model.

Example 4.3. [The CARMA(2,1) process]

By applying (4.8) and (4.9) for the case of a CARMA(2,1) process, we find the discrete-time representation for a gridsize $h > 0$,

$$y_n = (e^{\lambda_1 h} + e^{\lambda_2 h}) y_{n-1} - e^{(\lambda_1 + \lambda_2)h} y_{n-2} + \varepsilon_n,$$

where ε_n is given by

$$\begin{aligned}\varepsilon_n &= \int_{(n-1)h}^{nh} (\kappa_1 e^{\lambda_1(nh-u)} + \kappa_2 e^{\lambda_2(nh-u)}) dL(u) \\ &\quad + \int_{(n-2)h}^{(n-1)h} (\kappa_1 e^{\lambda_2 h} e^{\lambda_1(nh-u)} + \kappa_2 e^{\lambda_1 h} e^{\lambda_2(nh-u)}) dL(u).\end{aligned}$$

The two integrals in the noise are independent. It is, however, not possible to recover the noise by simple multiplication and subtraction as in the ARMA case. The actual relation of two successive noise terms ε_n and ε_{n+1} is based on the continuous realisation of $\{L(t)\}_{t \geq 0}$ in the relevant intervals, which is unobservable. \square

For the mapping of the estimated ARMA parameters to the corresponding CARMA parameters we observe that equation (4.8) is a complex way to express that $\{e^{-\lambda_i h}\}_{i=1, \dots, p}$ are the roots of the autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ of the ARMA process. We proceed, therefore, as follows for identifying the CARMA parameters from the estimated ARMA process:

- Estimate the coefficients ϕ_1, \dots, ϕ_p of the ARMA process
- Determine the distinct roots ξ_i for $i = 1, \dots, p$ of the characteristic polynomial.
- Set $\lambda_i = -\log(\xi_i)/h$, where we recall that h denotes the grid size.

Because of the simple structure of the autoregressive matrix A of the CARMA process we can calculate the characteristic polynomial P of the matrix A as

$$P(\lambda) = (-1)^p (\lambda^p + a_1 \lambda^{p-1} + \dots + a_p).$$

Since the λ_i are the eigenvalues of A , we know that $P(\lambda_i) = 0$ for $i = 1, \dots, p$. Hence, given the eigenvalues λ_i of the matrix A we obtain the coefficients a_1, \dots, a_p by the relation $a(z) = (z - \lambda_1) \dots (z - \lambda_p)$ with $a(\cdot)$ given in (2.8).

We estimate the moving average parameters based on the empirical autocorrelation function, which exists always. We apply a least absolute deviation algorithm to fit the linear filter of the CARMA process. This linear filter is in the L^2 -case the autocorrelation function of y and takes the form

$$\gamma_y(s) = \mathbf{b}^* e^{A|s|} \Sigma \mathbf{b}, \quad s > 0,$$

where the matrix Σ is given by

$$\Sigma = \int_0^\infty e^{Au} \mathbf{e}_p \mathbf{e}_p^* e^{A^* u} du = -\mathbf{A}^{-1} \mathbf{e}_p \mathbf{e}_p^*.$$

In this representation, \mathbf{A}^{-1} is the inverse of the operator $\mathbf{A} : X \mapsto AX + XA^*$ and can be represented as $\text{vec}^{-1} \circ ((A \otimes I_p) + (I_p \otimes A))^{-1} \circ \text{vec}$ (see Pigorsch and Stelzer [32]). Using the above procedure for the estimation of the moving average parameter \mathbf{b} is based on second order structure and, therefore, not straightforward for stable processes. In practice, however, this procedure works and we can use it to estimate the moving average parameter \mathbf{b} .

4.4 Estimation of the stable parameters

After estimating the autoregressive parameters, the noise $\{\varepsilon_n\}_{n \in \mathbb{N}}$ can be recovered. Recall that, by $(p-1)$ -dependence, the noise terms of lags $m > p-1$ are independent. Motivated by results for discrete-time stable ARMA processes, Garcia et al. [24] have applied estimation methods for independent noise variables $\{\varepsilon_n\}_{n \in \mathbb{N}}$. They have also shown in a simulation study that this results in reliable estimates. This procedure is also theoretically justified by composite likelihood theory (e.g. Varin, Reid and Firth [40], based on arguments going back to Wald [38]), which shows that under quite general regularity conditions the dependence in the likelihood function can be reduced or even ignored. Maximum composite likelihood estimators are usually consistent and often satisfy a central limit theorem. A formal proof of these results will appear elsewhere.

By a simple computation we can relate the estimated stable parameters of $\{\varepsilon_n\}_{n \in \mathbb{N}}$ to the α -stable process L . We show this for the CARMA(2,1) model in the next example:

Example 4.4. [Continuation of Example 4.3, cf. [24]]

Using Samorodnitsky and Taqqu [34], Property 3.2.2, a relation in distribution between the α -stable process L and the noise process $\{\varepsilon_n\}_{n \in \mathbb{N}}$ of the ARMA(2,1) model sampled on a grid with grid size h can be established. To avoid ambiguities we will write $(\alpha_L, \gamma_L, \beta_L, \mu_L)$ for the parameters of the α -stable process L , if necessary. In particular, ε_n has an α -stable distribution with parameters $(\alpha_\varepsilon, \gamma_\varepsilon, \beta_\varepsilon, \mu_\varepsilon)$ given by

$$\begin{aligned} \alpha_\varepsilon &= \alpha_L = \alpha \\ \gamma_\varepsilon &= \left(\int_0^h |\kappa_1 e^{\lambda_1(h-u)} + \kappa_2 e^{\lambda_2(h-u)}|^\alpha + |\kappa_1 e^{\lambda_2 h} e^{\lambda_1(h-u)} + \kappa_2 e^{\lambda_1 h} e^{\lambda_2(h-u)}|^\alpha du \right)^{1/\alpha} \gamma_L \\ \beta_\varepsilon &= \beta_L \frac{\gamma_L^\alpha}{\gamma_\varepsilon^\alpha} \left(\int_0^h (\kappa_1 e^{\lambda_1(h-u)} + \kappa_2 e^{\lambda_2(h-u)})^{\langle \alpha \rangle} + (\kappa_1 e^{\lambda_2 h} e^{\lambda_1(h-u)} + \kappa_2 e^{\lambda_1 h} e^{\lambda_2(h-u)})^{\langle \alpha \rangle} du \right) \\ \mu_\varepsilon &= \mu_L = \mu \quad \text{for } \alpha_\varepsilon \neq 1 \end{aligned}$$

Note that, for a and p being real numbers, $a^{(p)} := |a|^p \text{sign}(a)$ denotes the signed power (Samorodnitsky and Taqqu [34], eq. (2.7.1)). Moreover, we can easily see that $\beta_\varepsilon = \beta_L$, if both κ_1 and κ_2 are positive.

4.5 Recovering the states

In order to calculate the theoretical futures prices derived in Corollary 3.2 it is necessary to recover the states \mathbf{X} of the CARMA-process. Brockwell et al. [14] describe a rather ad-hoc method to do this by using an Euler approximation.

In the linear state space model (2.2), the Kalman filter is the best linear predictor provided the driving noise is in L^2 . Since α -stable Lévy processes for $\alpha \in (0, 2)$ do not have finite second moments, the Kalman filter will perform unsatisfactorily. One possibility to resolve this is to apply a particle filter, which does not require a finite second moment of the noise process. However, the particle filter requires a density function instead, which poses a new problem for α -stable processes. Integral approximations of α -stable densities exist, but they are time consuming to calculate and simple expressions do not exist. One can use a particle filter by simulating from the α -stable distribution, but this is also very time consuming. A large number of paths need to be simulated in order to get a reasonable estimation (even when using appropriate variance reducing methods like importance sampling). As an attractive alternative, we introduce a simple L^1 -filter applicable to CARMA processes with finite mean.

Recall from (2.2)-(2.6) that we can work with the following state-space representation of the CARMA process

$$y_n = \mathbf{b}^* \mathbf{x}_n, \quad (4.10)$$

$$\mathbf{x}_n = e^{Ah} \mathbf{x}_{n-1} + \mathbf{z}_n \quad \text{with} \quad \mathbf{z}_n = \int_{(n-1)h}^{nh} e^{A(nh-u)} \mathbf{e}_p dL(u) \quad (4.11)$$

Here, y_n and \mathbf{x}_n are discrete observations of Y and \mathbf{X} , respectively, on a grid with grid size h .

Notice that given y_n and \mathbf{x}_{n-1} the value of $\mathbf{b}^* \mathbf{z}_n$ is determined and given by

$$\mathbf{b}^* \mathbf{z}_n | y_n, \mathbf{x}_{n-1} = y_n - \mathbf{b}^* e^{Ah} \mathbf{x}_{n-1}. \quad (4.12)$$

This will come to use in a moment when deriving the filter.

First, we make an "Euler" approximation of the stochastic integral defining \mathbf{z}_n by

$$\mathbf{z}_n \approx \frac{1}{h} \int_{(n-1)h}^{nh} e^{A(nh-u)} \mathbf{e}_p du \Delta L(n, h) = -A^{-1} (I - e^{Ah}) \frac{\Delta L(n, h)}{h},$$

where $\Delta L(n, h) = L(nh) - L((n-1)h)$. Note that a traditional Euler approximation (see Kloeden and Platen [25]) would use the left end-point value of the integrand in the approximation, whereas here we use the average value of the integrand over the integration interval. We find

$$\mathbb{E}[\mathbf{z}_n | y_n, \mathbf{x}_{n-1}] \approx -\mathbb{E}[\Delta L(n, h)/h | y_n, \mathbf{x}_{n-1}] A^{-1} (I - e^{Ah}) \mathbf{e}_p. \quad (4.13)$$

Multiplying (4.13) with \mathbf{b}^* and combining it with (4.12) gives

$$\mathbb{E}[\Delta L(n, h)/h \mid y_n, \mathbf{x}_{n-1}] \approx \frac{y_n - \mathbf{b}^* e^{Ah} \mathbf{x}_{n-1}}{-\mathbf{b}^* A^{-1} (I - e^{Ah}) \mathbf{e}_p}. \quad (4.14)$$

By plugging (4.14) into (4.13) we find

$$\mathbb{E}[\mathbf{z}_n \mid y_n, \mathbf{x}_{n-1}] \approx -A^{-1} (I - e^{Ah}) \mathbf{e}_p \frac{y_n - \mathbf{b}^* e^{Ah} \mathbf{x}_{n-1}}{-\mathbf{b}^* A^{-1} (I - e^{Ah}) \mathbf{e}_p}. \quad (4.15)$$

We can use this as an L^1 -filter for \mathbf{z}_n . Applying (4.15), we can filter the states \mathbf{X} of the CARMA-process. Using the state equation (4.11) we find

$$\mathbb{E}[\mathbf{x}_n \mid y_n, \mathbf{x}_{n-1}] \approx e^{Ah} \mathbf{x}_{n-1} + \mathbb{E}[\mathbf{z}_n \mid y_n, \mathbf{x}_{n-1}]. \quad (4.16)$$

We tested the filter on simulated data from a CARMA(2,1) process with the same parameters as we estimate from our model for the base spot prices (see Table 5.3). The path of the CARMA(2,1) process was simulated based on an Euler scheme on a grid size of 0.01 for $0 \leq t \leq 1461$, and the α -stable Lévy process was simulated using the algorithm suggested in Chambers, Mallows and Stuck [16]. The estimation of the states is done on a grid with grid size $h = 1$. Figure 5 shows the estimated states (bold line) for both state components together with the simulated states (fine line, mostly covered by the bold line). It is clearly visible that the L^1 -filter gives a good approximation of the true states \mathbf{X} driving the α -stable CARMA process Y .

4.6 Risk premium comparison

In order to find the optimal threshold \hat{u}^* for filtering out the non-stationary process Z from the futures data, we compare the empirically observed risk premium with its theoretical counterpart.

Recall the risk premium R_{pr} in (3.4) implied by the futures price dynamics in Corollary 3.2. By using $v = T_2 - T_1$ and recalling the notation $u = \frac{1}{2}(T_1 + T_2) - t$, we can rewrite R_{pr} for $u \geq \frac{1}{2}(T_2 - T_1)$ and fixed v (being one month in our studies) to

$$\begin{aligned} R_{pr}(u^*, u, v) &= -\frac{1}{v} \mathbf{b}^* A^{-2} \left(e^{\frac{1}{2}Av} - e^{-\frac{1}{2}Av} \right) e^{Au} \mathbf{e}_p \left(\mathbb{E}_Q[L(1)] - \mathbb{E}[L(1)] \right) \\ &\quad + \mathbf{b}^* A^{-1} \mathbf{e}_p \left(\mathbb{E}_Q[L(1)] - \mathbb{E}[L(1)] \right) + u \mathbb{E}_Q[Z(1)]. \end{aligned} \quad (4.17)$$

Note that we can estimate all parameters in (4.17) only depending on a chosen threshold u^* . Hence, the risk premium in Equation (4.17) also depends on u^* . In order to find an

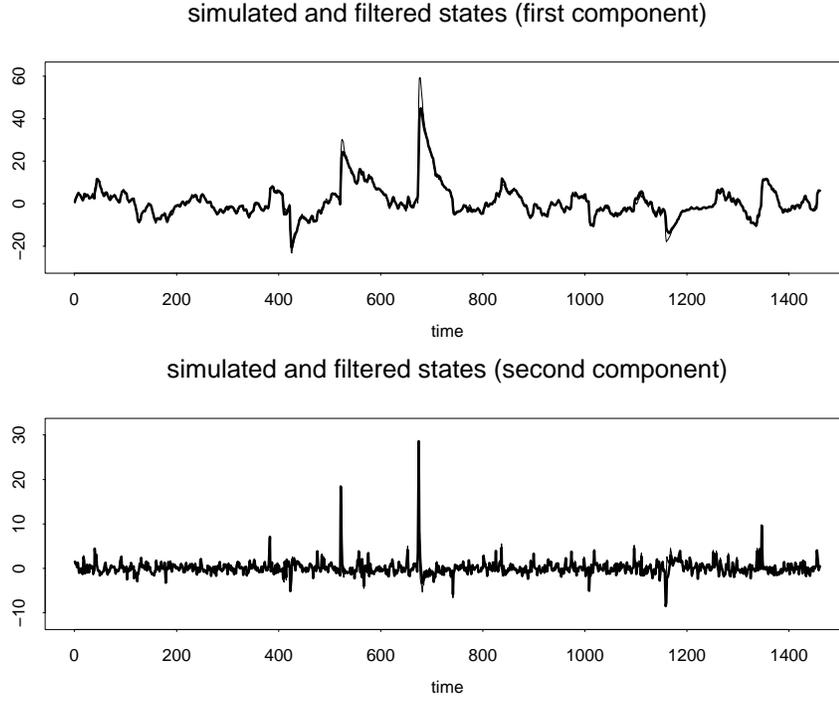


Figure 5: *Estimated states (bold line) and true states (fine line) of a simulated CARMA(2,1) process using the L^1 -filter.*

optimal \hat{u}^* we compare the risk premium (4.17), which has been estimated on our model assumptions, with the mean empirical risk premium based on the futures prices given by

$$\begin{aligned}
\tilde{R}_{pr}(u^*, u, v) &:= \frac{1}{v} \mathbf{b}^* A^{-2} \left(e^{\frac{1}{2}Av} - e^{-\frac{1}{2}Av} \right) e^{Au} \mathbf{e}_p \mathbb{E}[L(1)] - \mathbf{b}^* A^{-1} \mathbf{e}_p \mathbb{E}[L(1)] \\
&+ \frac{1}{\text{card } U(u, v)} \sum_{t, T_1, T_2 \in U(u, v)} \left[F(t, T_1, T_2) - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(\tau) d\tau \right. \\
&\left. - \frac{\mathbf{b}^* A^{-1}}{T_2 - T_1} (e^{AT_2} - e^{AT_1}) e^{-At} \mathbf{X}(t) - Z(t) \right]. \quad (4.18)
\end{aligned}$$

Here,

$$U(u, v) := \left\{ t, T_1, T_2 \in \mathbb{R}; \frac{1}{2}(T_2 + T_1) - t = u, T_2 - T_1 = v \text{ and } F(t, T_1, T_2) \text{ exists} \right\}.$$

The dependence on the threshold u^* is only implicit. The estimated sample paths of Z and Y depend on u^* , therefore, also the CARMA parameters A, \mathbf{b} , the stable parameters $(\alpha, \beta, \gamma, \mu)$ and the estimated sample paths of the states \mathbf{X} also depend on u^* . In order to compute \tilde{R}_{pr} and R_{pr} all these estimated parameters are used. Consequently, by using

different thresholds we will get different estimates and different risk premia. We want to choose an optimal threshold \hat{u}^* , such that the mean empirical risk premium \tilde{R}_{rp} is as close as possible to the model based risk premium R_{pr} . We apply a least-squares method, i.e. we minimise (for fixed v) the mean square error between the two functions $(\tilde{R}_{pr}(u^*, u, v))_{u \geq \frac{1}{2}v}$ and $(R_{pr}(u^*, u, v))_{u \geq \frac{1}{2}v}$ with respect to all chosen thresholds $u^* \in U^*$ for the estimation procedure, cf. Section 4.1.

$$\hat{u}^* = \operatorname{argmin}_{u^* \in U^*} \sum_{u=v/2}^{M_f} |\tilde{R}_{pr}(u^*, u, v) - R_{pr}(u^*, u, v)|^2$$

Here the dependence of the error function

$$f(u^*, v) := \sum_{u=v/2}^{M_f} |\tilde{R}_{pr}(u^*, u, v) - R_{pr}(u^*, u, v)|^2 \quad (4.19)$$

on u^* is only implicit. In our data v corresponds to the average number of days per month (i.e. $v = 1461/48 = 30.44$ for the base data and $v = 1045/48 = 21.77$ for the peak data), and the number M_f is the longest time to maturity, which we recall from Section 4.1 being 200 for the base load contracts and 144 for the peak load. In order to calculate this minimum we calculate the values of $f(u^*, v)$ for all $u^* \in [v/2, M_f/2] \cap \mathbb{N}$. Figure 6 shows the risk premium error function $f(u^*, v)$ for base load (left) and peak load (right). In both cases, the minimum is attained at $\hat{u}^* = 16$. So our estimation procedure considers only base load forward contracts with delivery at least (about) two weeks away, and peak load contracts with delivery at least (about) three weeks away.

We remark that the optimal threshold being so small as $\hat{u}^* = 16$ for base load contracts suggests that the stationary CARMA component in the spot price dynamics has very little impact on forward prices in the market. The delivery period seems to average out the influence of the spike component. For peak load contracts the influence of the stationary spike component is more apparent, although not very big either. However, we must look at contracts at least one week farther out on the forward curve in this case. These results hint to an approximative estimation procedure, where one simply uses *all* forward contracts in the filtering of the non-stationary component. This should be working well for the base load data, as in fact only very few contracts are left out of the filtering.

Below we present a summary of the estimation algorithm.

The algorithm

Estimate $\Lambda(\cdot)$ as in (4.2) for the base load model and as in (4.1) for peak load model, and subtract from $S(\cdot)$.

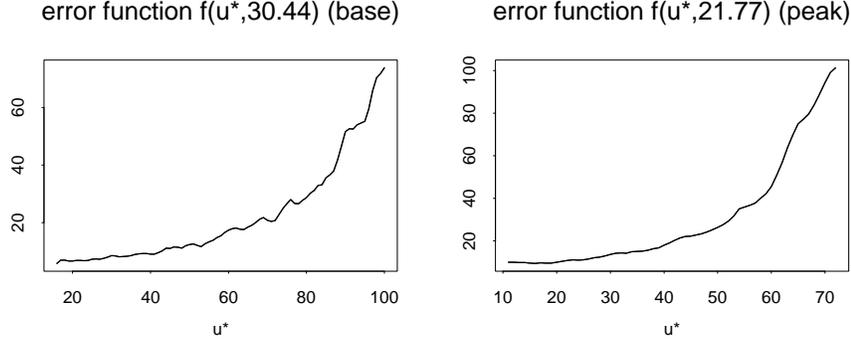


Figure 6: *The risk premium error function for base data (left) and peak data (right), for $v/2 \leq u^* \leq M_f/2$. In both cases, f has its minimum at $\hat{u}^* = 16$.*

For each threshold $u^* \in U^*$:

- Approximate $\mu_{\tilde{F}}(u) = C + u \mathbb{E}_Q[Z(1)]$ for $u \geq u^*$ and estimate $C, \mathbb{E}_Q[Z(1)]$ by linear regression (4.6);
- filter Z by (4.7);
- model $Y = S - \Lambda - Z$ as CARMA(2,1) process, estimate the coefficients a_1, a_2, b_0 (recall that $b_1 = 1$) and estimate the parameters $(\alpha_L, \gamma_L, \beta_L, \mu_L)$ of L ;
- estimate $\mathbb{E}_Q[L(1)]$ using (4.7);
- filter states of $\mathbf{X} = (X_1, X_2)^*$ using (4.16);
- calculate $R_{pr}(u^*, u, v)$ as in (4.17) using the estimated parameters and states from the former steps;
- calculate $\tilde{R}_{pr}(u^*, u, v)$ as in (4.18) using the estimated parameters and states from the former steps and the futures data.

Now define the mean square error of the estimated $R_{pr}(u^*, u, v)$ and $\tilde{R}_{pr}(u^*, u, v)$ based on all different thresholds $u^* \in U^*$. The optimal threshold is found to be \hat{u}^* .

5 Estimation results

We now report the other results from the estimation procedure, when using the optimal threshold $\hat{u}^* = 16$ both for base and peak load data. In this section we discuss the estimated values and their implications.

5.1 Distributional properties of the filtered sample path of Z

For the filtered Z which was found using (4.7) we can derive certain properties. Both for the base and the peak data, the realisation of Z shows uncorrelated increments, see Figure 7.

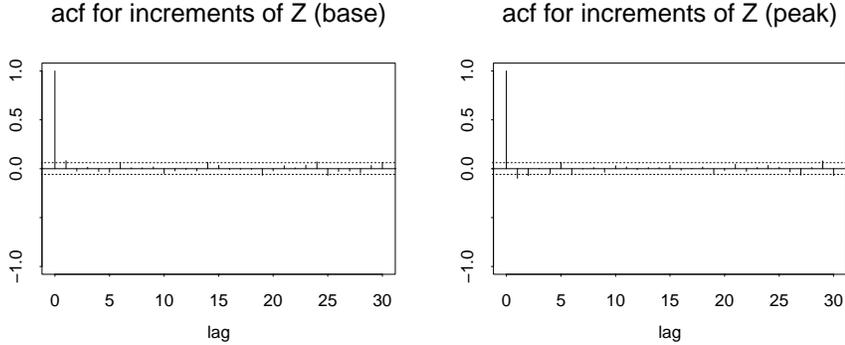


Figure 7: *Empirical autocorrelation functions for the increments of Z , for base data (left) and peak data (right).*

Figure 8 shows QQ-plots for the increments of Z versus a corresponding normal distribution, both for base data (left) and peak data (right). The empirical variances of the increments are 0.35 and 2.78 for the base and peak data, respectively. From these plots we conclude that for both data sets the increments of Z have heavier tails than the Gaussian distribution, and that this feature is even more pronounced for the peak data.

Kernel density estimates suggest that the increments of Z can be described quite well using a normal inverse Gaussian (NIG) distribution (we refer to Barndorff-Nielsen [1] for a discussion on the NIG distribution and its properties). The bold lines in Figure 9 show log-density estimates for the increments of Z for the base data (left) and the peak data (right), respectively. For comparison, we also plot the log-density curves of NIG distributions (fine solid lines) and normal distributions (dashed curves) that have been fitted to the increments of Z via maximum likelihood (the parameters for the NIG distributions can be found in Table 5.2). Clearly, the NIG distribution gives a much better fit than the normal distribution. Hence, we identify the non-stationary process Z with a normal inverse Gaussian Lévy process.

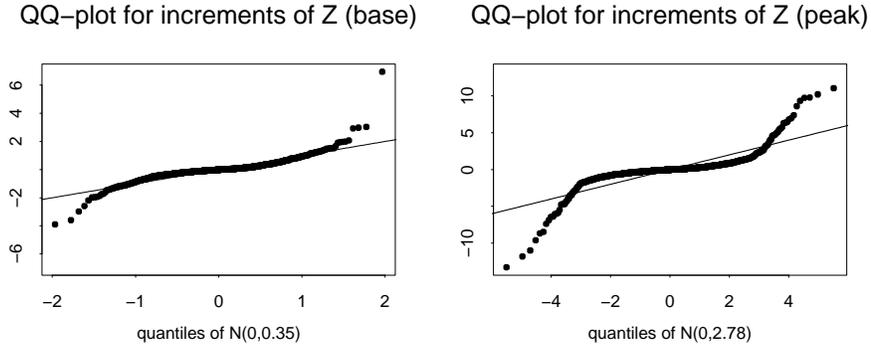


Figure 8: *QQ-plots for the increments of Z against suitable normal distributions, for base data (left) and peak data (right). The reference distributions are $N(0,0.35)$ and $N(0,2.78)$ for base and peak data, respectively.*

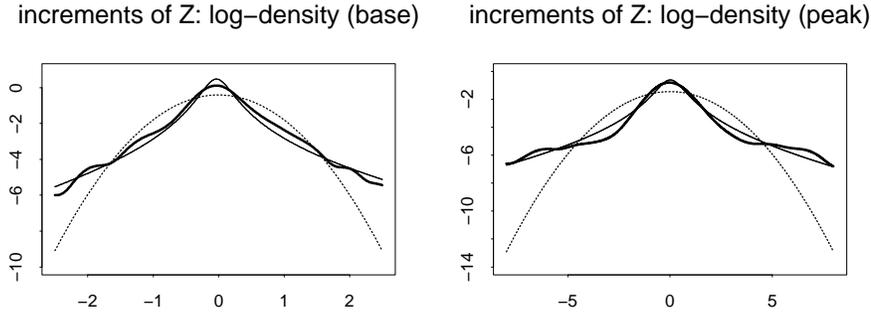


Figure 9: *Log-densities for the increments of Z (bold solid) as well as for NIG distributions (fine solid) and normal distributions (dashed) that have been fitted to these increments, for base data (left) and peak data (right).*

Remark 5.1. Recall that futures contracts are only traded on weekdays and, therefore, no variability in Z during weekends is observed. For base load contracts we have thus assumed that Z is constant during weekends for filtering purposes, but only considered weekdays data for analysing the distributional properties of Z . As we already mentioned in Section 4.2, this strategy could lead to larger up- or downward movements of Z on Mondays, when all information accumulated over the weekend is subsumed at once. However, we do not find such a behaviour in the estimated increments of Z . We conducted a Brown-Forsythe test to check the null hypothesis of variance homogeneity between the group of all Friday/Monday increments and the group of the remaining daily increments. For our situation this test seems to be a better choice than the Levene test since we have data which is heavier tailed than the normal distribution. On the usual 5% level, the test does not reject the hypothesis of variance homogeneity between the Friday/Monday increments and the remaining daily

	α_Z	β_Z	δ_Z	μ_Z
Base load	0.6451	0.0998	0.2206	-0.0346
Peak load	0.2371	-0.0083	0.6582	0.0230

Table 5.2: *Estimated parameters of the NIG distribution Q for the increments of Z . Since we assume that $\mathbb{E}[Z] = 0$, the parameters have been estimated conditionally on this assumption.*

increments in the base data. This supports our reasoning that no relevant information enters the futures market over the weekends. Indeed, this can be expected, since the futures deal with a product to be delivered *quite far* in the future; hence, only *really influential* news with a long-range impact should affect the market. \square

5.2 Estimation of the CARMA parameters

The autocorrelation and partial autocorrelation function of the data suggest that there are two significant autoregressive lags, but also a relevant moving average component. Also the AIC and BIC criterion confirm that a CARMA(2,1) model leads to the best fit. The estimated parameters of a CARMA(2,1) model are given in Table 5.3.

	CARMA parameters			Stable parameters			
	a_1	a_2	b_0	α_L	β_L	γ_L	$\mathbb{E}[L(1)]$
Base load	1.4854	0.0911	0.2861	1.6524	0.3911	6.4072	0.0566
Peak load	2.3335	0.2263	0.6127	1.3206	0.0652	6.5199	-0.0448

Table 5.3: *Estimates of the CARMA parameters and of the parameters of the stable process L .*

For the estimated parameters (\hat{a}_1, \hat{a}_2) in the autoregression matrix A the eigenvalues of A are real and strictly negative, being $\lambda_1 = -0.0641, \lambda_2 = -1.4213$ for the base load and $\lambda_1 = -0.1014, \lambda_2 = -2.2319$ for the peak load. Our parameters satisfy Assumptions 2.2. Hence, the estimated model is stationary.

The estimates of α_L (1.6524 for base and 1.3206 for peak) confirm that extreme spikes are more likely in the peak load data. As we conclude from the positive signs of the skewness parameter β_L , positive spikes are more likely to happen than negative spikes for both data sets. Note, however, that a direct comparison of the values of β_L for the base and the peak load data is misleading, due to the significantly different parameters α_L . Indeed, if we calculate the empirical skewness for the estimated ε_n (cf. Section 4.4) directly, we get a

	$\widehat{\mathbb{E}_Q[Z(1)]}$	\widehat{C}	$\widehat{\mathbb{E}_Q[L(1)]}$	$\widehat{\theta}_Z$	$\widehat{\theta}_L$
Base load	-0.0243	1.6587	-0.5282	-0.1093	-0.0021
Peak load	-0.0382	3.5678	-1.3178	-0.0168	-0.0552

Table 5.4: *Estimates of parameters determining the risk.*

value of 0.28 for the base load data and 1.59 for the peak load data (cf. the comment on $\beta_\varepsilon = \beta_L$ in Example 4.4; for the base load data $(\kappa_1, \kappa_2) = (0.1636, 0.8364)$, and for the peak load data $(\kappa_1, \kappa_2) = (0.2400, 0.7600)$).

5.3 Market price of risk and risk premium

We next present the results on the risk premium and the parameters for the market price of risk, based on our statistical analysis of base and peak load contracts with threshold $\widehat{u}^* = 16$ days in both cases. Estimates of the relevant parameters are presented in Table 5.4.

Recall that $\widehat{\mathbb{E}_Q[Z(1)]}$ and \widehat{C} are estimated by regression based on the approximation (4.6); using the estimates of the CARMA model from Table 5.3, we derive an estimate of $\mathbb{E}_Q[L(1)]$. Having both $\mathbb{E}_Q[L(1)]$ and $\mathbb{E}_Q[Z(1)]$ estimated, we can compute the parameters in the respective measure transforms of the NIG Lévy process Z and the stable process L . For an NIG Lévy process we use the fact that an Esscher transformed $\text{NIG}(\alpha_Z, \beta_Z, \delta_Z, \mu_Z)$ random variable Z is again NIG distributed with parameters $(\alpha_Z, \beta_Z + \theta_Z, \delta_Z, \mu_Z)$ (see e.g. Benth et al. [3], p. 99). Using the mean of an NIG distributed random variable it holds that

$$\mathbb{E}_Q[Z(1)] = \mu_Z + \frac{\delta_Z(\beta_Z + \theta_Z)}{\sqrt{\alpha_Z^2 - (\beta_Z + \theta_Z)^2}},$$

where θ_Z is the market price of risk for Z . Since estimates for the parameters $\alpha_Z, \beta_Z, \delta_Z$ and μ_Z are known from Table 5.2, we can use the estimate for $\widehat{\mathbb{E}_Q[Z(1)]}$ together with the above equality to obtain an estimate of θ_Z , which results here in $\widehat{\theta}_Z = -0.1093$ for the base load and in $\widehat{\theta}_Z = -0.0168$ for the peak load data. Since θ_Z is estimated negative, more emphasis is given to the negative jumps and less emphasis to the positive jumps of Z in the risk-neutral world Q . We see from the estimate on the risk-neutral expectation of Z that the contribution from the non-stationarity factor of the spot on the overall risk premium is negative. This is natural from the point of view of the hedging needs of producers. The non-stationary factor induces a long-term risk, which is the risk producers want to hedge using futures contracts.

Lucia and Schwartz [29] also find a negative market price of risk associated to the non-stationary term in their two-factor models, when analysing data from the NordPool market.

We recall that they propose a two-factor model, where the non-stationary term is a drifted Brownian motion. The negative market price of risk appears as a negative risk-neutral drift, which corresponds to a contribution to the risk premium similar to our model. We refer to Benth and Sgarra [6] for a theoretical analysis of the Esscher transform in factor models applied to power markets.

Using the relations $\gamma^\alpha = c_+ + c_-$ and $\beta = (c_+ - c_-)/(c_+ + c_-)$ in the stable parameters as calculated in Example 2.3.3 of Samorodnitsky and Taqqu [34] and plugging in the estimated parameters from Table 5.3, we find

$$\widehat{c}_+ = \frac{1}{2}(1 + \beta_L)\gamma_L^{\alpha_L} = 14.9715 \quad \text{and} \quad \widehat{c}_- = \frac{1}{2}(1 - \beta_L)\gamma_L^{\alpha_L} = 6.5532$$

for the base load data, and

$$\widehat{c}_+ = \frac{1}{2}(1 + \beta_L)\gamma_L^{\alpha_L} = 6.3342 \quad \text{and} \quad \widehat{c}_- = \frac{1}{2}(1 - \beta_L)\gamma_L^{\alpha_L} = 5.5587$$

for the peak load data. Then by using (3.9) we can derive an estimate for θ_L by

$$\widehat{\theta}_L = - \left(\frac{\mathbb{E}_Q[\widehat{L}(1)] - \mathbb{E}[\widehat{L}(1)]}{\Gamma(1 - \widehat{\alpha}_L)(\widehat{c}_+ - \widehat{c}_-)} \right)^{\frac{1}{\widehat{\alpha}_L - 1}}$$

which leads to $\widehat{\theta}_L = -0.0021$ for the base load data and to $\widehat{\theta}_L = -0.0552$ for the peak load data. The market price of risk for the CARMA-factor noise L is also negative, however, unlike the non-stationary factor a negative sign does not necessarily lead to a negative contribution to the risk premium. As we already see in the estimate of the constant C in the regression (4.6), we get a positive contribution to the risk premium. There will also be a term involving time to maturity, which will converge to zero in the long end of the futures curve. This part of the risk premium may contribute both positively or negatively. The CARMA factor is thus giving a positive risk premium for contracts, which start delivering reasonably soon. Since this factor is accounting for the short term variations, and in particular the spike risk of the spot, we may view this as a result of consumers and retailers hedging their price risk and, therefore, accepting to pay a premium for this. This conclusion is in line with the theoretical considerations of Benth, Cartea and Kiesel [4], who showed – using the certainty equivalence principle – that presence of jumps in the spot price dynamics will lead to a positive risk premium in the short end of the futures curve. Bessembinder and Lemmon [8] explain the existence of a positive premium in the short end of the futures market by an equilibrium model, where skewness in spot prices induced by spikes is a crucial driver.

As we know from the third summand in Equation (4.17), the risk premium is dominated by a linear trend for most times to maturity except very short ones. In the latter case, the first summand of Equation (4.17) is dominating, leading to a small exponential decay when time

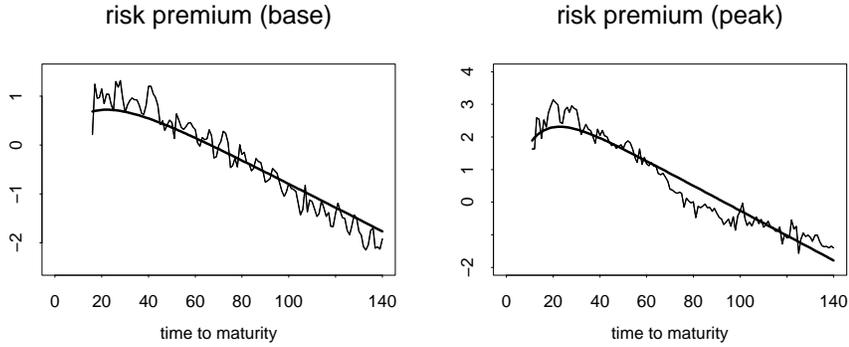


Figure 10: *The estimated risk premium R_{pr} (bold smooth line) and the empirical risk premium \tilde{R}_{pr} (fine line), for base load data (left) and peak load data (right).*

to maturity tends to 0. A plot of the empirical risk premium versus the theoretical one is given in Figure 10.

We see that for base load contracts the positive risk premium in the short end of the curve is not that pronounced, however, it is detectable. The risk premium is negative for contracts starting to deliver in about two months. On the other hand, peak load contracts have a clear positive risk premium, which changes to a negative one for contracts starting to deliver in about four months. This form of the risk premium is in line with the analysis of Geman and Vasicek [23]. Interesting here is the difference between base and peak load contracts. Base load futures have a longer delivery period than peak loads, since they are settled against more hours. This means that extreme prices are more smoothed out for base load contracts. The sensitivity towards spikes is more pronounced for peak load contracts since they settle during those hours when spikes typically occur. Hence, peak load contracts are much more spike sensitive than base load contracts, which we see reflected in the risk premium having a bigger and more visible positive part in the short end of the futures curve. The study of Longstaff and Wang [28] on the PJM (Pennsylvania, New Jersey and Maryland) market shows that the risk premium may vary over time, and indeed change sign. Their analysis is performed on hourly prices in the balancing market as being the spot, and the day ahead hourly prices as being futures contracts. Hence, the analysis in Longstaff and Wang [28] is valid for the very short end of the futures curve.

Our findings for the EEX are in line with the empirical studies in Benth et al. [4], which applies a two factor model to analyse the risk premium in the EEX market. Their model consists of two stationary processes, one for the short term variations, and another for stationary variations mean-reverting at a slower speed. Their studies confirm a change in sign of the risk premium as we observe for our model. Moreover, going back to Lucia and

Schwartz [29], they find a positive contribution to the risk premium from their short-term variation factor, when applying their analysis to NordPool data. This shows that also in this market there is a tendency towards hedging of spike risk in the short end of the futures curve. On the other hand, our results for the EEX market are at stake with the findings in Kolos and Ronn [27]. They perform an empirical study of many power markets, where they estimate market prices of risk for a two-factor Schwartz and Smith model. In the EEX market, they find that both the short and the long term factors contribute negatively⁵ for the EEX market. However, interestingly, the PJM market in the US, which is known to have huge price variations with many spikes observed, they find results similar to ours. We find our results natural in view of the spike risk fully accounted for in the short term factor, and the natural explanation of the hedging pressure from producers in the non-stationary factor. Our statistical analysis also strongly suggest non-Gaussian models for both factors, which is very different to the Gaussian specification of the dynamics in [27].

As mentioned in the Introduction, the German EEX market has in recent years undergone a structural change on the supply side as a result of the so-called “Energiewende”. The building of renewable power generation plants, like wind farms and photovoltaic installations, has been strongly politically supported and is now responsible for a large share of the capacity. Of course, these renewable power sources can only produce, when the weather conditions allow, which makes the generation dependent on wind speed and cloud cover. On the other hand, renewable power has priority into the German grid at a minimum tariff, leading to market prices spiking downwards, or even becoming negative over short periods. This poses a major risk for gas- and coal-fired power utilities as they may have to generate electricity at a loss. Consequently, retailers have a significantly reduced short-term price risk exposure. One may conjecture from this in the current situation that there may be a *negative* risk premium also in the short end of the forward market, as carbon-based power companies hedge short-term price risk. This will certainly be checked when the market conditions have been long enough in power to allow for statistical assessment.

5.4 Simulation using parameters from fitted model

To check whether our model is able to reproduce the important characteristics of the empirical data (as seasonality, spikes, and low-frequency fluctuations) we finally simulate from our model using the parameters estimated in the empirical analysis. Figure 11 shows simulated spot prices for base and peak period and, hence, can be considered a simulation-based analogue to Figure 1. The model obviously produces spikes and short-term fluctuations, low-frequency fluctuations as well as the correct seasonal pattern for both the base and the

⁵In their paper, the signs are positive due to the choice of parametrisation of the market price of risk

peak data. The exact location of the spikes and the low-frequency fluctuations is, of course, random.

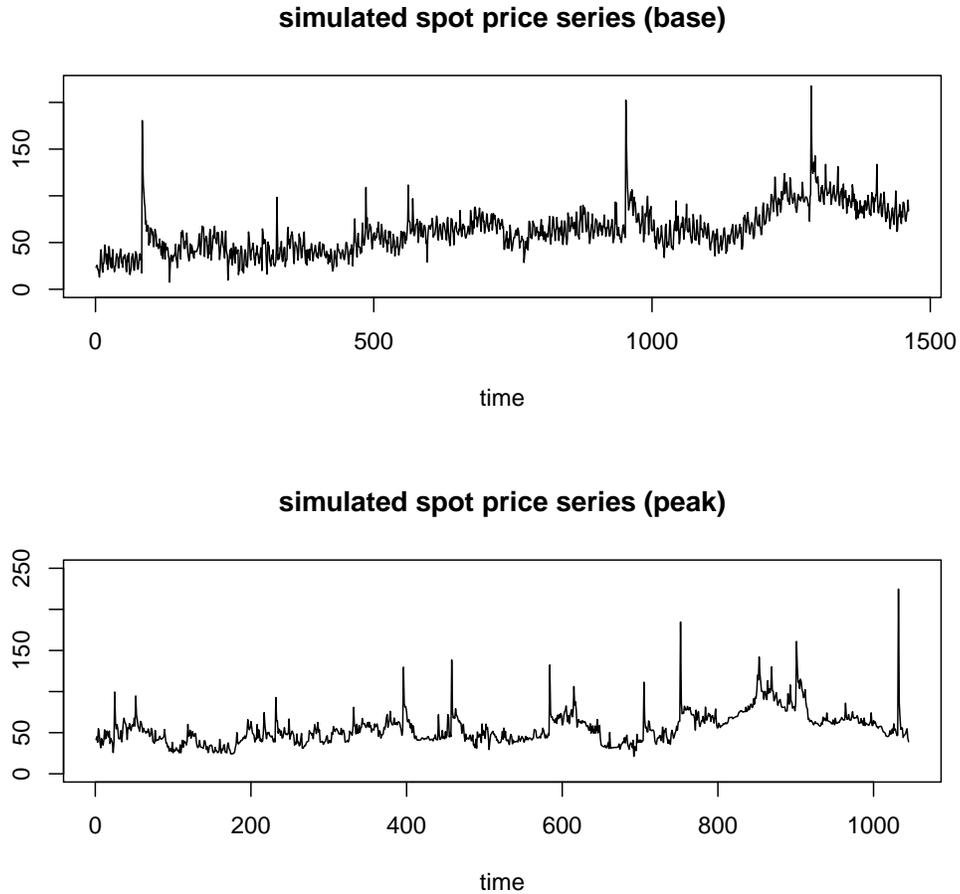


Figure 11: *Simulated spot prices using the estimates from the base data analysis (top) and the peak data analysis (bottom).*

6 Conclusion

In this paper we suggest a two-factor arithmetic spot model to analyse power spot and futures prices. After removal of seasonality, a non-stationary long term factor is modelled as a Lévy process, while the short term variations in the spot price is assumed to follow a stationary stable CARMA process. An empirical analysis of spot price data from the German power exchange EEX shows that a stable CARMA processes is able to capture the

extreme behaviour of electricity spot prices, as well as the more normal variations when the market is in a quiet period.

As in Lucia and Schwartz [29] we use a combination of a deterministic function and a non-stationary term to model the low frequency long term dynamics of the spot. Empirical data suggests that futures curves and spot prices are driven by a common stochastic trend, and it turns out that this is very well described by a normal inverse Gaussian Lévy process. This leads to realistic predictions of futures prices. Moreover, a CARMA(2,1) process is statistically the best model for the short term variations in the spot dynamics. Since α -stable processes are not in L^2 , we introduce a robust L^1 -filter in order to recover the states of the CARMA process required for the estimation of the risk premium.

We apply the Esscher transform to produce a parametric class of market prices of risk for the non-stationary term. The α -stable Lévy process driving the CARMA-factor is transformed into a tempered stable process in the risk-neutral setting. The spot price dynamics and the chosen class of risk-neutral probabilities allow for analytic pricing of the futures. A crucial insight in the futures price dynamics is that the stationary CARMA effect from the spot price is vanishing for contracts far from delivery, where prices essentially behave like the non-stationary long-term factor.

We propose a statistical method to fit our new spot and futures model to real data. The estimation is performed using spot and futures data together, where we applied futures prices in the far end of the market to filter out the non-stationary factor in the spot. We choose a threshold for what is sufficiently “far out” on the futures curve by minimising the error in matching the theoretical risk premium to the empirical version. For this minimisation over thresholds, we need to re-estimate the whole model for all reasonable thresholds.

Our model and calibration technique is used on spot and futures data collected at the EEX. Moreover, in order to gain full insight into the risk premium structure in this market, we study both peak load and base load futures contracts with delivery over one month. The base load futures are settled against the hourly spot price over the whole delivery period, while the peak load contracts only deliver against the spot price in the peak hours from 8 a.m. to 8 p.m. on working days. Our model and estimation technique seem to work well in both situations.

We find that base load futures contracts have a risk premium, which is close to linearly decaying with time to delivery. The risk premium is essentially governed by the long term factor. There is evidence of a positive premium in the short end of the futures curve. For peak load contracts, which are much more sensitive to spikes, the positive premium in the short end is more distinct, but also here the premium decays close to linearly in the long end of the market. These observations are in line with other theoretical and empirical studies

of risk premia in electricity markets, which argue that the risk premia in power markets are driven by hedging needs. Our findings also show that we should have an exponential dampening of the premium towards maturity, resulting from the CARMA factor of the spot.

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The STABLE package is a collection of algorithms for computing densities, distribution functions, quantiles, simulating and estimating for α -stable distributions. A free basic version is available from academic2.american.edu/~jpnolan; an advanced version is available from www.robustanalysis.com. We are grateful to John Nolan for making the packages available.

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