Technische Universität München

Zentrum Mathematik

Maximum likelihood estimation of C-vine pair-copula constructions based on bivariate copulas from different families

Diplomarbeit

von

Ulf Schepsmeier

Themensteller/in: Prof. Dr. Claudia Czado
Betreuer/in: Dr. Aleksey Min
Abgabetermin: 15. Juni 2010

Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

Garching, den 15. Juni 2010

Acknowledgments

First, I would like to express my sincere gratitude to Prof. Dr. Claudia Czado for making this diploma thesis possible and her excellent mentoring and valuable support throughout the whole developing process. This thesis gained a lot of her advice, hints and our intensive discussions during our meetings.

Furthermore, I owe special thanks to Dr. Aleksey Min who always had time to answer my questions with his expert guidance and helped me to solve my encountered problems.

Dr. Carlos Almeida supported me by the sometimes tricky implementation of the algorithms and programs.

Last but not least, I would like to thank my parents who unremittingly supported me during my years of study.

Contents

List of Figures

List of Tables

List of Algorithms

Chapter 1

Introduction

In economics, especially on stock exchanges, trading markets for foreign currencies and other financial products, we observe ups and downs of the prices, indices and currencies. If one has a closer look at more than one prize or currency chart one can detect some dependencies between the prices, the ups and downs of different stocks or currencies. Knowing the dependence structure between different assets the asset manager or investor can diversify his risks and hedge his portfolio or if we look at exchange rates his international cash flows. Some of the main portfolio theories like Capital-Asset-Pricing-Model (CAPM) of Sharpe [1964], which is based on the portfolio theory of Markowitz [1952], or other theories depend on the mean, the variance and the correlation between the assets, which is nothing else than the dependence structure.
Based on the pair-copula constructions (PCC) from Aas et al. [2009] with different copula families we consider a canonical vine (C-vine), an intuitively appealing graphical model developed by Bedford and Cooke [2002]. We perform a sequential estimation as well as a joint estimation of the copula parameters by maximum likelihood estimation (MLE) and compare the results against each other. The main work is devoted to develop an explorative data analysis tool for finding C-vine models, which reflect most dependence structure of data. The structure of the C-vine we identify through Kendall's τ correlation matrix and the copula families by graphical methods like contourplots and λ-function, and goodness-of-fit tests.
We analyze the influence of model misspecifications and perform a model comparison based on the Vuong-, Clarke- and other tests. As a data application we execute the model finding process and estimate the model for a data set of 9 exchange rate time series.
The diploma thesis is structured as follows:
In Chapter 2 we introduce the necessary definitions and theorems. We define the different copula families and their properties, the pair-copula construction (PCC) and vines, especially the C-vine. All copulas, vines and definitions are illustrated by some examples. Furthermore some dependence measures, as the correlation coefficient and Kendall's τ, the tail dependence and the λ-function, especially with respect to the observed copula families are defined. For the copula as well as for the model choice we show some goodness-of-fit tests like AIC, BIC and the Vuong- and Clarke-test. At the end of Chapter 2 we concern ourself with the basics of time series analysis, the ARMA- and GARCH-model as well as the Ljung-Box-test.

Chapter 3 deals with the copula parameter estimation based on maximum likelihood for the C-vine. Additionally we describe the simulation algorithm of a C-vine, which we will use for our tests. In the second part of this chapter we cater to the implementation and application in R in detail.
Chapter 4 describes the model selection - the tree selection of the C-vine as well as the copula selection for each pair-copula of the C-vine structure based on the copula properties we introduced in Chapter 2. This is done in two extended and detailed examples. Furthermore, we take a look on the quality of the selected model and finally we compare our selected model with simple models, so called misspecifications.
In Chapter 5 we accomplish all methods and tests from Chapter 2, 3 and 4 in a 9-dimensional data set of exchange rates. Therefore we fit the data with ARMA-GARCH-models, analyze the results and use a 9-dimensional C-vine to model the dependence structure.
The last chapter, Chapter 6, is a short summary and conclusion of the thesis and gives an outlook of possible further research work.

Chapter 2

Background Material

2.1 Copulas

2.1.1 Introduction and definition

A copula is a multivariate distribution function C defined on the unit cube $[0,1]^n$, with uniformly distributed marginals. This definition couples the marginals of different random variables with their joint distribution.
The most used copula is the bivariate copula, i.e. the two-dimensional copula. In this diploma thesis we will deal mainly with the bivariate copula. The mathematical definition and the theorem of Sklar [1959] are taken from the book of Nelson [2006] and the paper of Genest and Favre [2007].

Definition 2.1 (2-dimensional copula)
A 2-dimensional copula is a function $C : [0,1]^2 \to [0,1]$ *with the following properties:*

(i) $C(0,u) = C(u,0) = 0$ *for all* $u \in [0,1]$.

(ii) $C(u,1) = u$ *and* $C(1,u) = u$ *for all* $u \in [0,1]$.

(iii) $C(v_1,v_2) - C(v_1,u_1) - C(u_1,v_2) + C(u_1,u_2) \geq 0$ *for all* $(u_1,u_2),(v_1,v_2) \in [0,1] \times [0,1]$ *with* $u_1 \leq v_1$ *and* $u_2 \leq v_2$.

The connection between the marginals and the joint distribution is expressed in the theorem of Sklar [1959].

Theorem 2.2 (Sklars Theorem)
Let $F : \overline{\mathbb{R}}^2 \to [0,1]$ *with* $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ *be a bivariate distribution with one-dimensional marginals* $F_1, F_2 : \overline{\mathbb{R}} \to [0,1]$. *Then there exists a two-dimensional copula C, such that for all* $(x_1,x_2) \in \overline{\mathbb{R}}^2$

$$F(x_1,x_2) = C(F_1(x_1), F_2(x_2)) \tag{2.1}$$

holds. For continuous F_1 *and* F_2, *C is unique and is defined through*

$$C(x_1,x_2) = F(F_1^{-1}(x_1), F_2^{-1}(x_2)).$$

On the other side, if the have the copula C and the marginals F_1 and F_2 then the function F is a bivariate distribution function with margins F_1 and F_2.

Definition 2.3 (Copula density)
Let C be a two-dimensional two-times partial differentiable copula, then the function $c : [0,1]^2 \to [0,1]$ with

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \tag{2.2}$$

is called the copula density of the copula C.

Having bivariate data the easiest way to check for copula models is the scatter plot. To do so we simulate a huge number of samples of the copula C_θ and draw a picture of this simulated points. In the bivariate case one can use a simple algorithm to simulate the pairs (U,V) of a copula. The Algorithm 2.1 can be found in Genest and Favre [2007].

Algorithm 2.1 Simulation of a bivariate copula

1: Generate U from an uniform distribution on the interval $(0, 1)$.
2: Given $U = u$, generate V from the conditional distribution

$$Q_u(v) = P(V \leq v | U = u) = \frac{\partial}{\partial u} C(u, v),$$

by setting $V = Q_u^{-1}(U^*)$, where U^* is another observation from the uniform distribution on the interval $(0, 1)$. When an explicit formula does not exist for $Q_u^{-1}(U^*)$, the value $v = Q_u^{-1}(u^*)$ can be determined by trial and error or more effectively using the bisection method.

The most important and most common copulas in finance are the Gaussian and the t-copula. Both belong to the so called elliptical copula family. The definition of elliptical copulas can be found in the paper of Embrechts et al. [2003]. In addition to these two copulas there are many more copula families. A few of them will be defined in the following and used in the data set in a later chapter.

Example 2.4 (Gaussian copula)
The Gaussian copula is defined by

$$C(u_1, u_2) = \Phi_2(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \rho),$$

where $\Phi_2(\cdot, \cdot, \rho)$ is the bivariate distribution of two standard normal distributed random variables with correlation ρ, Φ is the $N(0, 1)$ cdf and Φ^{-1} is the functional inverse of Φ of the standard normal distribution. The density of the bivariate Gaussian copula is

$$c(u_1, u_2) = \frac{1}{\sqrt{1-\rho^2}} \exp\left\{ -\frac{\rho^2(x_1^2 + x_2^2) - 2\rho x_1 x_2}{2(1-\rho^2)} \right\},$$

where $x_1 = \Phi^{-1}(u_1)$ *and* $x_2 = \Phi^{-1}(u_2)$.
Using Algorithm 2.1 to simulate data points we have to set

$$v = Q_u^{-1}(U^*) = \Phi(\Phi^{-1}(u^*)\sqrt{1-\rho^2} + \rho\Phi^{-1}(u))$$

with the notation from above as the standard normal distribution function. By setting $\rho = -0.5$ *and* $\rho = 0.5$ *the resulting scatter plots can be seen in Figure 2.1.*

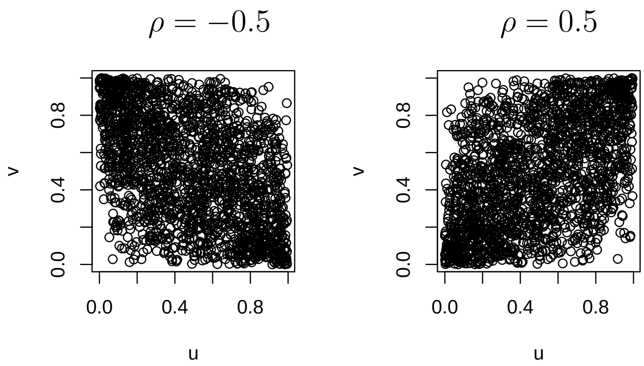


Figure 2.1: Gaussian copula, left: $\rho = -0.5$ and right $\rho = 0.5$

Example 2.5 (t-Copula)
The density of the bivariate t-copula with the parameter ν *and* ρ *is given by*

$$c(u_1, u_2) = \frac{1}{2\pi dt_\nu(x_1) dt_\nu(x_2)\sqrt{1-\rho^2}} \left\{1 + \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\nu(1-\rho^2)}\right\}^{-\frac{\nu+2}{2}},$$

where $x_1 = t_\nu^{-1}(u_1), x_2 = t_\nu^{-1}(u_2)$, $dt(\cdot)$ *is the probability density and* $t_\nu^{-1}(\cdot)$ *is the quantile function of the univariate standard t-distribution with* ν *degrees of freedom. (The expected value of the t-distribution is* 0 *and the variance of the t-distribution is* $\frac{\nu}{\nu-2}$ *for* $\nu > 2$.)
Figure 2.2 shows us the scatter plot of the t-copula.

2.1.2 Archimedean Copulas

Definition 2.6 (Archimedean Copula)
Let Ω *be the set of all continuous, strict monotone decreasing and convex functions* $\varphi : I \rightarrow [0, \infty]$ *with* $\varphi = 0$.
Let $\varphi \in \Omega$, *then*

$$C(u_1, u_2) = \varphi^{[-1]}(\varphi(u_1) + \varphi(u_2)) \tag{2.3}$$

is a copula. C is called Archimedean copula with generator φ.
Is $\varphi(0) = \infty$, *the generator is called strict.*

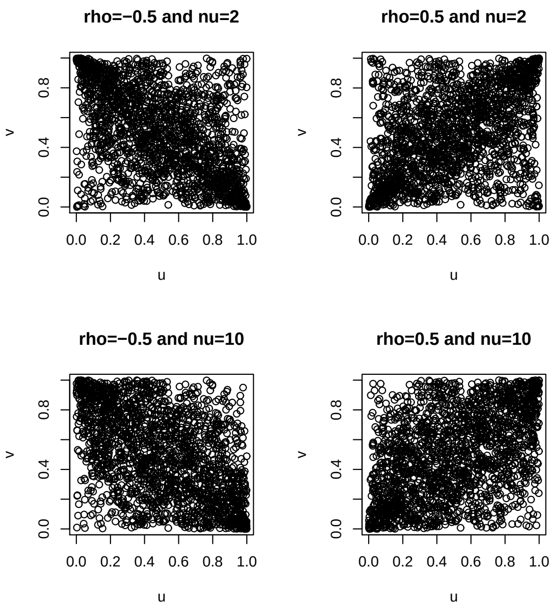


Figure 2.2: t-Copula, left $\rho = -0.5$ and right $\rho = 0.5$, top $\nu = 2$ and bottom $\nu = 10$

Remarks:

(i) $\varphi^{[-1]}$ is the pseudo-inverse of φ and is defined as follows:

$$\varphi^{[-1]} : [0, \infty] \to [0, 1]$$

$$\varphi^{[-1]}(t) := \begin{cases} \varphi^{-1}(t) & , 0 \le t \le \varphi(0) \\ 0 & , \varphi(0) \le t \le \infty. \end{cases}$$

If $\varphi(0) = \infty$, then $\varphi^{[-1]}(t) = \varphi^{-1}(t) \quad \forall t \in [0, \infty]$.

(ii) $\varphi^{[-1]}$ is continuous, non-increasing on $[0, \infty]$ and strict decreasing on $[0, \varphi(0)]$.

(iii) $\varphi^{[-1]}(\varphi(u)) = u$ on $[0, 1]$.

(iv)
$$\varphi(\varphi^{[-1]}(t)) = \begin{cases} t & , 0 \le t \le \varphi(0) \\ \varphi(0) & , \varphi(0) \le t \le \infty. \end{cases}$$

A graphical illustration of a strict and a non-strict generator can be seen in Figure 2.3.

Theorem 2.7
Let $\varphi : [0, 1] \to [0, \infty]$ be a continuous, strict decreasing function, such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Then

$$C \text{ given by (2.3) is a copula} \Leftrightarrow \varphi \text{ is convex.}$$

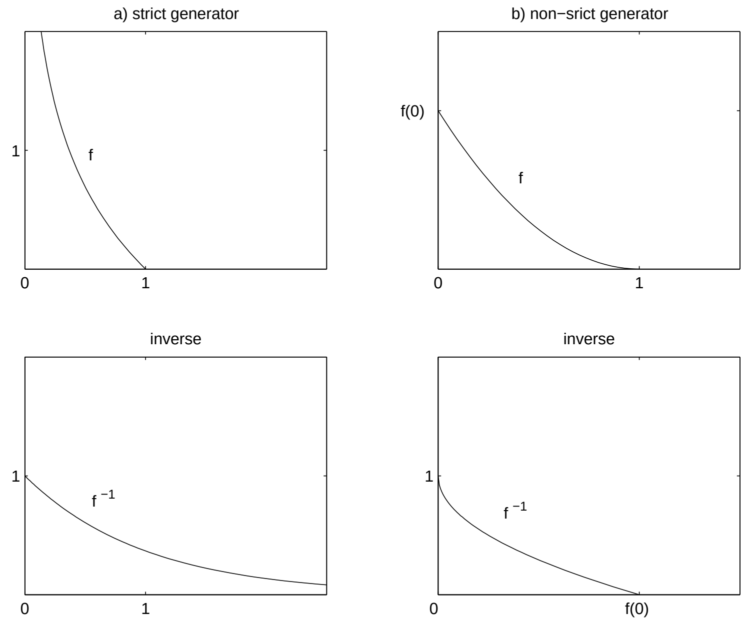


Figure 2.3: Archimedean copula: on the left side with a strict generator (top) and the according inverse (bottom) and on the right side with a non-strict generator (top) and the according inverse (bottom)

Theorem 2.8 (Properties of the Archimedean copula)

(i) C is symmetric.

(ii) C is associative, i.e. $C(C(u_1, u_2), u_3) = C(u_1, C(u_2, u_3))$.

(iii) For $c \geq 0$ is $c\varphi$ a generator of C, too.

Proof:

(i) Obvious.

(ii) With $\varphi(C(u_1, u_2)) = \varphi(u_1) + \varphi(u_2)$ it follows that

$$\begin{aligned} C(C(u_1, u_2), w) &= \varphi^{[-1]}[\varphi(C(u_1, u_2)) + \varphi(u_3)] \\ &= \varphi^{[-1]}[\varphi(u_1) + \varphi(u_2) + \varphi(u_3)] \\ &= \varphi^{[-1]}[\varphi(u_1) + \varphi(C(u_2, u_3))] \\ &= C(u_1, C(u_2, u_3)). \end{aligned}$$

(iii) $c\varphi(C(u_1, u_2)) = c\varphi(u_1) + c\varphi(u_2)) \; \forall c > 0$.

□

See Embrechts et al. [2003].

Corollary 2.9
If C is a continuous Archimedean copula then its density is given as follows

$$c(u_1,u_2) = -\frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} = -\frac{\varphi''(C(u_1,u_2))\varphi'(u_1)\varphi'(u_2)}{[\varphi'(C(u_1,u_2))]^3}. \tag{2.4}$$

Proof:

$$\begin{aligned}
\varphi'(u_1) &= \varphi'(C(u_1,u_2))\frac{\partial}{\partial u_1}C(u_1,u_2),\\
\varphi'(u_2) &= \varphi'(C(u_1,u_2))\frac{\partial}{\partial u_2}C(u_1,u_2),\\
0 &= \varphi''(C(u_1,u_2))\frac{\partial}{\partial u_1}C(u_1,u_2)\frac{\partial}{\partial u_2}C(u_1,u_2) + \varphi'(C(u_1,u_2))\frac{\partial^2}{\partial u_1\partial u_2}C(u_1,u_2)\\
\Rightarrow \quad \frac{\partial^2}{\partial u_1\partial u_2}C(u_1,u_2) &= -\frac{\varphi''(C(u_1,u_2))\frac{\partial}{\partial u_1}C(u_1,u_2)\frac{\partial}{\partial u_2}C(u_1,u_2)}{\varphi'(C(u_1,u_2))} = -\frac{\partial^2 C(u_1,u_2)}{\partial u_1\partial u_2}\\
&= -\frac{\varphi''(C(u_1,u_2))\varphi'(u_1)\varphi'(u_2)}{[\varphi'(C(u_1,u_2))]^3}.
\end{aligned}$$

□

See Embrechts et al. [2003].

Example 2.10 (Clayton copula)
The first example for an Archimedean copula we want to look at is the Clayton copula. The generator of this copula is $\frac{(t^{-\delta}-1)}{\delta}$ *and the copula is*

$$C(u_1,u_2) = (u_1^{-\delta} + u_2^{-\delta} - 1)^{-\frac{1}{\delta}}$$

and the bivariate density is

$$c(u_1,u_2) = (1+\delta)(u_1 \cdot u_2)^{-1-\delta}(u_1^{-\delta} + u_2^{-\delta} - 1)^{-\frac{1}{\delta}-2},$$

where $0 < \delta < \infty$ *control the degree of dependence. One gets full dependence if* $\delta \to \infty$ *and independence if* $\delta = 0$.
The scatter plot of the Clayton copula can be seen in Figure 2.4.

Example 2.11 (Gumbel copula)
The second example is the Gumbel copula. It is an example of an Archimedean copula too, and its generator is $|\ln(t)|^\delta$. *The density of the bivariate Gumbel copula, sometimes called Gumbel-Hougaard, has the form:*

$$\begin{aligned}
c(u_1,u_2) = {} & C(u_1,u_2)(u_1u_2)^{-1}\{-\ln u_1)^\delta + (-\ln u_2)^\delta\}^{-2+\frac{2}{\delta}}(\ln u_1 \ln u_2)^{\delta-1}\\
& \times \{1 + (\delta-1)((-\log u_1)^\delta + (-\ln u_2)^\delta)^{-\frac{1}{\delta}}\},
\end{aligned}$$

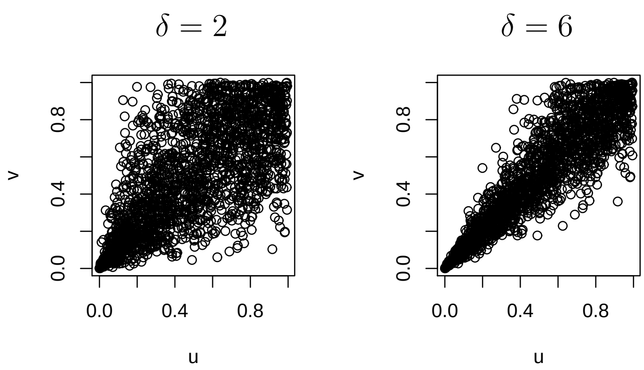


Figure 2.4: Clayton copula: left $\delta = 2$ and right $\delta = 6$

where $C(u_1, u_2)$ *is given by*

$$C(u_1, u_2) = \exp[-\{(-\ln u_1)^{\delta} + (-\ln u_2)^{\delta}\}^{\frac{1}{\delta}}]$$

and $\delta \geq 1$ *is the parameter of dependence (s. Genest and Favre [2007]). For* $\delta \to \infty$ *one gets full dependence, in contrast* $\delta = 1$ *means independence.*
As for the other copulas we plotted a scatter plot for the Gumbel copula too in Figure 2.5.

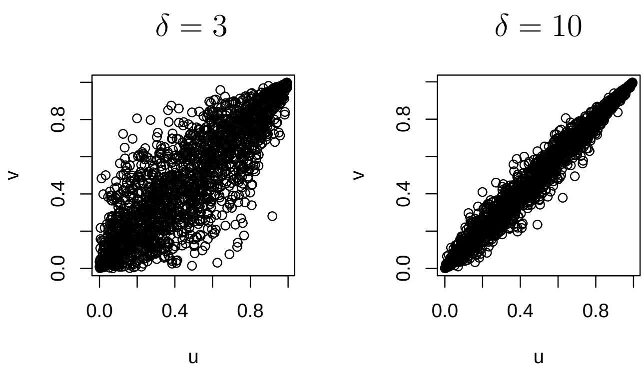


Figure 2.5: Gumbel copula: left $\delta = 3$ and right $\delta = 10$

In the application chapter (Chapter 5) we use the Frank and Joe copula as well. Therefore we will define them shortly in this section. For further information and details see Joe [1997, pp. 139-149].

Example 2.12 (Frank copula)
The Frank copula is also a member of the Archimedean copula family and has the generator $\varphi = -\ln\left(\frac{e^{-\delta t}-1}{e^{-\delta}-1}\right)$. *The copula is defined as*

$$C(u_1, u_2) = -\frac{1}{\delta} \ln\left(\frac{1}{1-e^{-\delta}}[(1-e^{-\delta}) - (1-e^{-\delta u_1})(1-e^{-\delta u_2})]\right)$$

and the density

$$c(u_1, u_2) = \delta(1 - e^{-\delta})e^{-\delta(u_1+u_2)}[(1 - e^{-\delta}) - (1 - e^{-\delta u_1})(1 - e^{-\delta u_2})]^{-2},$$

where $\delta \in [-\infty, \infty]\backslash\{0\}$.
Again we illustrated the copula in a scatter plot (Figure 2.6).

Remark:
Joe [1997, p. 141] defines the Frank copula just for positive parameters $(0, \infty]$, while other authors like Trivedi and Zimmer [2007] define it for $[-\infty, \infty]\backslash\{0\}$. We will see in Chapter 4 that if one uses the negative parameters then negative dependencies can be modeled too.

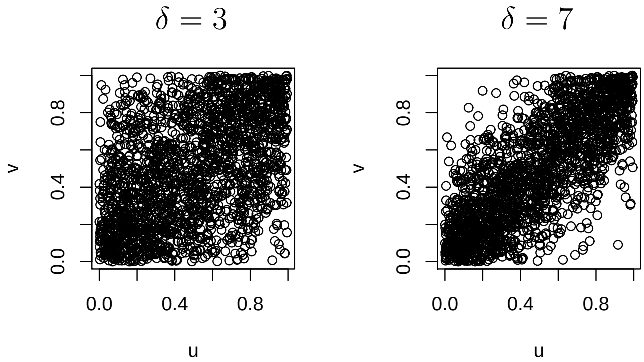


Figure 2.6: Frank copula with $\delta = 3$ and $\delta = 7$

Example 2.13 (Joe copula)
Similar to the Gumbel or Clayton copula, the Joe copula is a member of the Archimedean copulas and has the generator $-\log(1 - (1 - t)^\delta)$ *and is defined for* $\delta \geq 1$ *as*

$$C(u_1, u_2) = 1 - \left((1 - u_1)^\delta + (1 - u_2)^\delta - (1 - u_1)^\delta(1 - u_2)^\delta\right)^{\frac{1}{\delta}}$$

and the density as

$$\begin{aligned} c(u_1, u_2) = &\left((1 - u_1)^\delta + (1 - u_2)^\delta - (1 - u_1)^\delta(1 - u_2)^\delta\right)^{\frac{1}{\delta}-2} \cdot (1 - u_1)^{\delta-1}(1 - u_2)^{\delta-1} \\ &\cdot [\delta - 1 + 1 - u_1)^\delta + (1 - u_2)^\delta - (1 - u_1)^\delta(1 - u_2)^\delta]. \end{aligned}$$

Figure 2.7 is the scatter plot of the Joe copula.

Beside the Archimedean and the elliptical copulas there are some more copula families. One of this is the bivariate two-parameter family, which we will introduce in the next section.

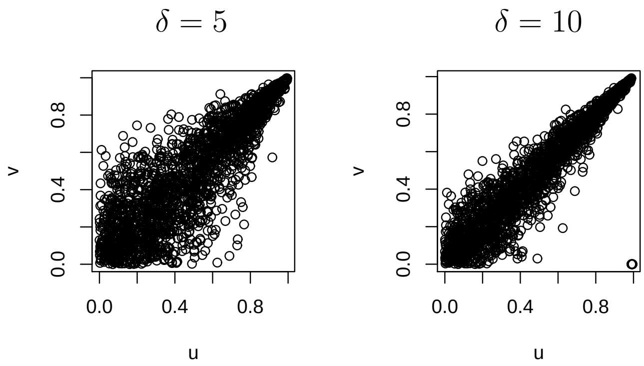


Figure 2.7: Joe copula with $\delta = 5$ and $\delta = 10$

2.1.3 Bivariate two-parameter copula families

Two-parameter copula families might be used to capture more than one type of dependence, e.g. the upper and lower tail dependence or one of the tail dependences and the concordance. The detailed definition and further information may be found in Joe [1997, pp.149-154].

Definition 2.14 (Bivariate two-parameter copula families)
The bivariate two-parameter families are of the form

$$C(u,v) = \Psi(-\log K(e^{-\Psi^{-1}(u)}, e^{-\Psi^{-1}(v)})), \tag{2.5}$$

where K maximal infinite divisible(max-id) [K is max-id if K^α is a cdf for all $\alpha > 0$] and Ψ is a Laplace-transform (LT).
Two-parameter families result if K is parametrized by a parameter δ and Ψ is parametrized by a parameter θ (denoted by ψ_θ). If K increases in concordance as δ increases, then clearly C increases in concordance as δ increases with θ fixed. The concordance ordering for δ fixed and θ varying is harder to check. If K has the form of an Archimedean copula, then C also has the form of an Archimedean copula. That is, if $K(x,y;\delta) = \phi_\delta(\phi_\delta^{-1}(x) + \phi_\delta^{-1}(y))$ for a family ϕ_δ, then

$$\begin{aligned} C(u,v;\theta,\delta) &= \Psi_\theta(-\log \phi_\delta[\phi_\delta^{-1}(e^{-\Psi^{-1}(u)}) + \phi_\delta^{-1}(e^{-\Psi^{-1}(v)})]) \\ &= \eta_{\theta,\delta}(\eta_{\theta,\delta}^{-1}(u) + \eta_{\theta,\delta}^{-1}(v)), \end{aligned} \tag{2.6}$$

where $\eta_{\theta,\delta}^{-1}(s) = \Psi_\theta(-\log \phi_\delta(s))$. For δ fixed and $\theta_2 > \theta_1$ with $\eta_i = \eta_{\theta_i,\delta}, i = 1,2$, the concordance ordering of $C(\cdot;\theta_1,\delta)$ and $C(\cdot;\theta_2,\delta)$ could be established by showing that $\omega = \eta_2^{-1} \circ \eta_1$ is superadditive. (See Joe [1997, pp.149-154])

In the following we want to look at two families more precisely: the BB1- and BB7-copula.

Definition 2.15 (BB1)
The BB1-copula is defined by

$$\begin{aligned} C(u,v;\theta,\delta) &= \left\{1 + [(u^{-\theta} - 1)^\delta + (v^{-\theta} - 1)^\delta]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}} \\ &= \eta(\eta^{-1}(u) + \eta^{-1}(v)), \qquad \theta > 0, \delta \geq 1, \end{aligned}$$

where $\eta(s) = \eta_{\theta,\delta}(s) = (1 + s^{\frac{1}{\delta}})^{-\frac{1}{\theta}}$.
The copula density is given by

$$\begin{aligned} c(u, v; \theta, \delta) = & \left\{1 + [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}-2} \\ & \times [(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{\frac{2}{\delta}-2} \\ & \times \{\theta\delta + 1 + \theta(\delta - 1)[(u^{-\theta} - 1)^{\delta} + (v^{-\theta} - 1)^{\delta}]^{-\frac{1}{\delta}}\} \\ & \times (u^{-\theta} - 1)^{\delta-1} u^{-\theta-1} (v^{-\theta} - 1)^{\delta-1} v^{-\theta-1}. \end{aligned}$$

The BB1-copula can be seen as a bivariate Archimedean copula too, with a generator $\phi(s; \theta, \delta) = (s^{-\theta} - 1)^{\delta}$.
We illustrated the copula in a scatter plot (Figure 2.8) and the density function of the BB1-copula in Figure 2.9 for some special combinations of Kendall's τ *and upper tail dependence (see Table 2.1). Kendall's* τ *and the tail dependence will be defined and explained in Section 2.3.2 and Section 2.4.*

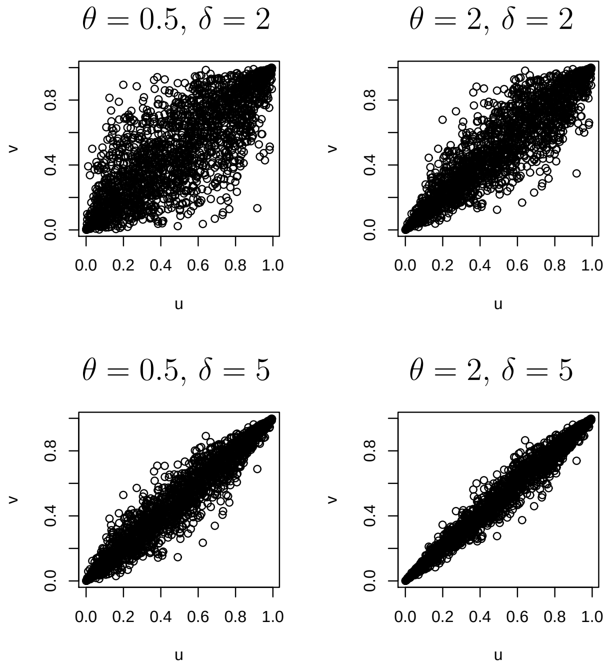


Figure 2.8: BB1-copula: left $\theta = 0.5, \delta = 2$ (top) and $\theta = 0.5, \delta = 5$ (bottom), right $\theta = 2, \delta = 2$ (top) and $\theta = 2, \delta = 5$ (bottom)

Definition 2.16 (BB7)
The BB7-copula, sometimes known as Joe-Clayton copula, has a similar structure as the BB1-copula, beside, that the Archimedean generator is $\phi(s; \theta, \delta) = [1 - (1 - s)^{\theta}]^{-\delta} - 1$ *and it is defined by*

$$\begin{aligned} C(u, v; \theta, \delta) &= 1 - \left(1 - [(1 - (1 - u)^{\theta})^{-\delta} + (1 - (1 - v)^{\theta})^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}} \\ &= \eta(\eta^{-1}(u) + \eta^{-1}(v)), \qquad \theta \geq 1, \delta > 0, \end{aligned}$$

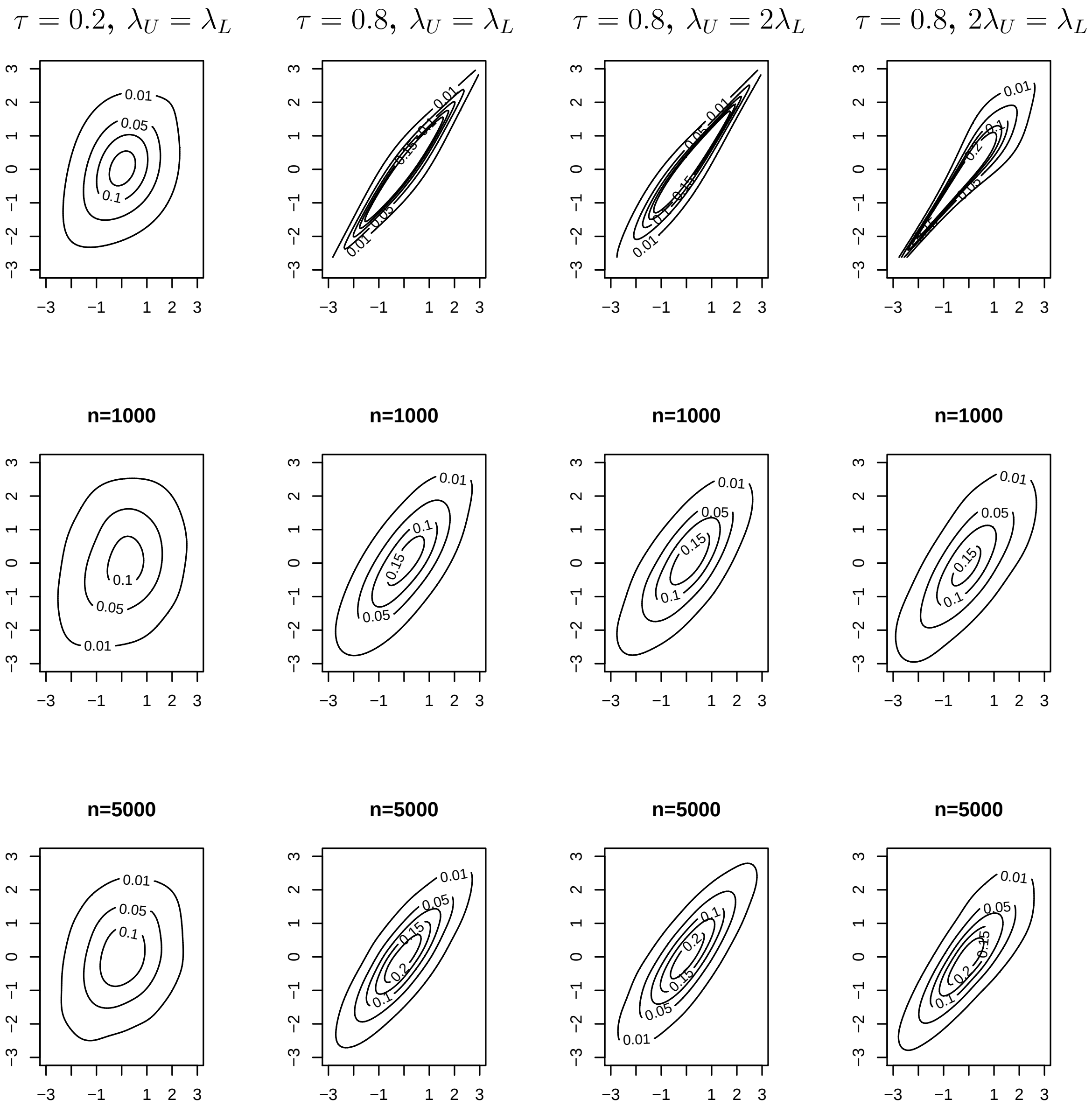


Figure 2.9: Contourplot of the BB1-copula in different combinations of Kendall's τ and tail dependence: theoretical (top), empirical with n=1000 (center) and with n=5000 (bottom) and different combinations = columns (see Table 2.1)

BB1-copula				
Column	**Kendall's τ**	**Tail dependence**	θ	δ
1	$\tau = 0.2$	$\lambda_U = \lambda_L$	0.2910	1.0912
2	$\tau = 0.8$	$\lambda_U = \lambda_L$	0.8048	3.5652
2	$\tau = 0.8$	$\lambda_U = 2\lambda_L$	0.1730	4.6018
3	$\tau = 0.8$	$2\lambda_U = \lambda_L$	4.3067	1.5856

Table 2.1: Combinations of Kendall's τ and tail dependence for the contourplots of the BB1-copula and their appropriate parameters

where $\eta(s) = \eta_{\theta,\delta}(s) = 1 - [1-(1+s)^{-\frac{1}{\delta}}]^{\frac{1}{\theta}}$. *The copula density is*

$$c(u,v;\theta,\delta) = (-\frac{1}{\theta})(\frac{1}{\delta}-1)\cdot h^{\frac{1}{\theta}-2} dvh \cdot duh - \frac{1}{\theta}\cdot h^{\frac{1}{\theta}-1} duvh$$

with

$$\begin{aligned}
h &= 1-((1-(1-u)^{\theta})^{-\delta}-(1-(1-v)^{\theta})^{-\delta}-1)^{\frac{1}{\delta}} \\
duh &= -\theta((1-(1-u)^{\theta})^{-\delta}-(1-(1-v)^{\theta})^{-\delta}-1)^{\frac{1}{\delta}-1}(1-(1-u)^{\theta})^{-\delta-1}(1-u)^{\theta-1} \\
dvh &= -\theta((1-(1-u)^{\theta})^{-\delta}-(1-(1-v)^{\theta})^{-\delta}-1)^{\frac{1}{\delta}-1}(1-(1-v)^{\theta})^{-\delta-1}(1-v)^{\theta-1} \\
duvh &= \frac{1}{\delta}(-\frac{1}{\delta}-1)((1-(1-u)^{\theta})^{-\delta}-(1-(1-v)^{\theta})^{-\delta}-1)^{\frac{1}{\delta}-2} duS\cdot dvS \\
duS &= -\theta\delta(1-(1-u)^{\theta})^{-\delta-1}(1-u)^{\theta-1} \\
dvS &= -\theta\delta(1-(1-v)^{\theta})^{-\delta-1}(1-v)^{\theta-1}.
\end{aligned}$$

As for the BB1-copula we illustrate the density in some contourplots (Figure 2.11) for some special combinations of Kendall's τ and tail dependence (see Table 2.2). Additional we plotted the scatter plot of the BB7-copula in Figure 2.10.

BB7-copula				
Column	**Kendall's τ**	**tail dependence**	θ	δ
1	$\tau = 0.2$	$\lambda_U = \lambda_L$	1.1237	0.3614
2	$\tau = 0.8$	$\lambda_U = \lambda_L$	5.9265	5.2324
3	$\tau = 0.2$	$\lambda_U = 2\lambda_L$	1.1779	0.3
4	$\tau = 0.8$	$\lambda_U = 2\lambda_L$	8.0799	0.88

Table 2.2: Combinations of Kendall's τ and tail dependence for the contourplots of the BB7-copula and their appropriate parameters

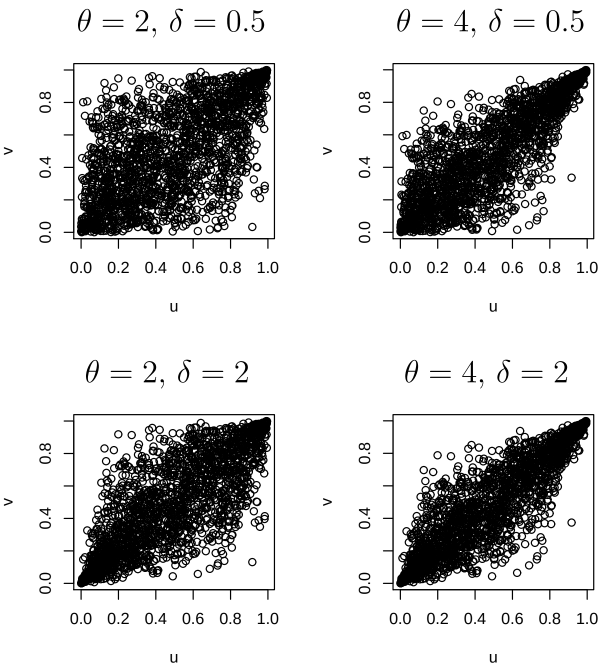


Figure 2.10: BB7-copula: left $\theta = 2, \delta = 0.5$ (top) and $\theta = 2, \delta = 2$ (bottom), right $\theta = 4, \delta = 0.5$ (top) and $\theta = 4, \delta = 2$ (bottom)

2.2 Pair-copula constructions and vines

2.2.1 Modeling dependence with bivariate copulas

Using two-dimensional copulas one might construct general multivariate distributions by specifying the dependence and conditional dependence of selected pairs of random variables and all marginal distribution functions. We will define such decomposition in this section. Our presentation of the concept follows Aas et al. [2009], but the idea was first developed by Bedford and Cooke [2002].

By Sklar we can write every multivariate distribution F with the marginals $F_1(x_1), \ldots, F_n(x_n)$ as

$$F(x_1, \ldots, x_n) = C\big(F_1(x_1), \ldots, F_n(x_n)\big) \tag{2.7}$$

for some appropriate n-dimensional copula C.
Now we use the chain rule to get the density function f. Thus we get

$$f(x_1, \ldots, x_n) = c_{1,\ldots,n}\{F_1(x_1), \ldots, F(x_n)\} \cdot f_1(x_1) \cdot \ldots \cdot f_n(x_n) \tag{2.8}$$

for some (uniquely identified) n-variate copula density $c_{1\ldots n}(\cdot)$. In the bivariate case we get the pair-copula decomposition:

$$f(x_1, x_2) = c_{12}\{F_1(x_1), F_2(x_2)\} \cdot f_1(x_1) \cdot f_2(x_2), \tag{2.9}$$

where $c_{12}(X_1, X_2)$ is the appropriate pair-copula density of X_1 and X_2.

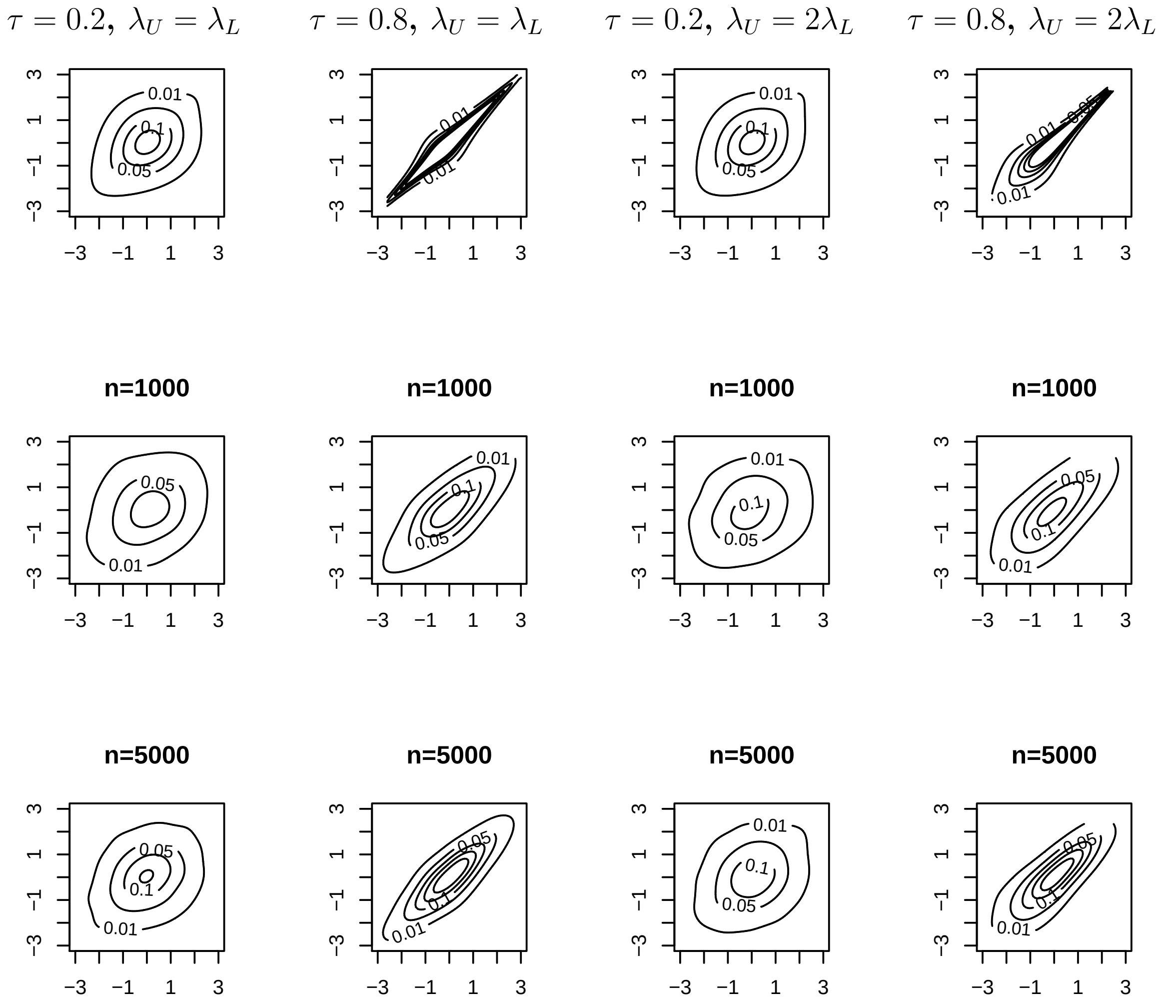


Figure 2.11: Contourplot of the BB7-copula in different combinations of Kendall's τ and tail dependence: theoretical (top), empirical with n=1000 (center) and with n=5000 (bottom) and different combinations = columns (see Table 2.2)

Our goal is to find a factorization of the joint density function $f(x_1, \ldots, x_n)$ for the random variables $X = (X_1, \ldots, X_n)^T$. We know that we can factorize f as follows:

$$f(x_1, \ldots, x_n) = f_n(x_n) \cdot f(x_{n-1}|x_n) \cdot f(x_{n-2}|x_{n-1}, x_n) \cdot \ldots \cdot f(x_1|x_2 \ldots, x_n). \tag{2.10}$$

To get the conditional densities of this factorization we use the definition of conditional density and (2.9). For the 2-dimensional case we get

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \tag{2.11}$$

for the same pair-copula c_{12}.
This argumentation can be stretched to the 3-dim case as

$$f(x_1|x_2, x_3) = c_{12|3}(F(_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3)) \cdot f(x_1|x_3) \tag{2.12}$$

or as an alternative decomposition

$$\begin{aligned} f(x_1|x_2, x_3) &= c_{13|2}(F(_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot f(x_1|x_2) \\ &= c_{13|2}(F(_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1) \end{aligned} \tag{2.13}$$

for the appropriate pair-copula $c_{12|3}$, applied to the transformed variables $F_{1|3}(x_1|x_3)$ and $F_{2|3}(x_2|x_3)$.
So with the decomposition (2.12), we can factorize the joint density of X_1, X_2 and X_3 in the form

$$\begin{aligned} f(x_1, x_2, x_3) &= f_3(x_3)f(x_2|x_3)f(x_1|x_2, x_3) \\ &= c_{12|3}(F(x_1|x_3), F(x_2|x_3))c_{13}(F_1(x_1), F_3(x_3))c_{23}(F_2(x_2), F_3(x_3)) \\ &\cdot f_1(x_1)f_2(x_2)f_3(x_3), \end{aligned}$$

where we used $f(x_2|x_3) = c_{23}(F_2(x_2), F_3(x_3))f_2(x_2)$.
For the d-dim case we can use the general formula for the conditional density

$$f(x|v) = c_{x,v_j|v_{-j}}\{F(x|v_{-j}), F(v_j|v_{-j})\} \cdot f(x|v_{-j})$$

with $v = (v_1, \ldots, v_d), v_{-j} = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_d)$.

In conclusion, under appropriate regularity conditions, a multivariate density can be expressed as a product of pair-copulae, acting on several different conditional probability distributions. It is also clear that the construction is iterative by nature, and that given a specific factorization, there are still many different re-parametrisations.
For the pair-copula construction we need marginal conditional distributions of the form $F(x|v)$. For every v_j in the vector v we can write $F(x|v)$ as

$$F(x|v) = \frac{\partial C_{x,v_j|v_{-j}}\{F(x|v_{-j}), F(v_j|v_{-j})\}}{\partial F(v_j|v_{-j})} \tag{2.14}$$

with $C_{i,j|k}$ as a bivariate copula density function.

In the following sections we will use the function $h(x, v, \theta)$ to represent this conditional distribution function with x and v uniform distributed, i.e. $f(x) = f(v) = 1, F(x) = x$ and $F(v) = v$. That is,

$$h(x, v, \theta) := F(x|v) = \frac{\partial C_{x,v}(x, v, \theta)}{\partial v}. \tag{2.15}$$

Example 2.17 (h-function of the bivariate Gaussian copula)
The density of the bivariate Gaussian copula is given by

$$c(u_1, u_2) = \frac{1}{\sqrt{1-\rho_{12}^2}} \exp\left\{-\frac{\rho_{12}^2(x_1^2 + x_2^2) - 2\rho_{12}x_1x_2}{2(1-\rho_{12}^2)}\right\},$$

where ρ_{12} is the parameter of the copula and $x_1 = \Phi^{-1}(u_1), x_2 = \Phi^{-1}(u_2)$ and $\Phi^{-1}(\cdot)$ is the inverse of the standard normal distribution function.
For this copula the h-function is as follows:

$$h(u_1, u_2, \rho_{12}) = \Phi\left(\frac{\Phi^{-1}(u_1) - \rho_{12}\Phi^{-1}(u_2)}{\sqrt{1-\rho_{12}^2}}\right)$$

and the inverse of the h-function is given by

$$h_{12}^{-1}(u_1, u_2, \rho_{12} = \Phi\left\{\Phi^{-1}(u_1)\sqrt{1-\rho_{12}^2} + \rho_{12}\Phi^{-1}(u_2)\right\}.$$

Example 2.18 (h-function of the bivariate t-copula)
For the bivariate t-copula with parameters ρ and ν the h-function is given by

$$h(x, v, \rho, \nu) = t_{\nu+1}\left\{\frac{t_\nu^{-1}(x) - \rho t_\nu^{-1}(v)}{\sqrt{\frac{(\nu+(t_\nu^{-1}(v))^2)(1-\rho^2)}{\nu+1}}}\right\}$$

and the inverse of the h-function

$$h^{-1}(u, v, \rho, \nu) = t_\nu\left\{t_{\nu+1}^{-1}(u)\sqrt{\frac{(\nu + (t_\nu^{-1}(v))^2)(1-\rho^2)}{\nu+1}} + \rho t_\nu^{-1}(v)\right\},$$

where $t_\nu^{-1}(\cdot)$ is the quantile-function of the univariate standard t distribution with ν degrees of freedom, expected value 0 and variance $\frac{\nu}{\nu-2}$ for $\nu > 2$.

For some copula families like the Gumbel, Frank, Joe, BB1- and BB7-copula we can't derive the inverse of the h-function directly. In this cases we have to use a numerically optimaziation algorithm on the basis of Newton-Raphson to solve the inverse. Therefore we need the derivative of the h-function. Because of limited research in the field of the bivariate two-parameter families we will focus on the BB1- and BB7-copula.

Corollary 2.19 (BB1-copula)
Let $C(u,v;\theta,\delta)$ the BB1-copula from Definition 2.15. The h-function is then

$$h = \left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}-1}(v^{-\theta}-1)^{\delta-1}v^{-\theta-1} \tag{2.16}$$

and the derivative with respect to v of the h-function is

$$\begin{aligned} h' = h \cdot \Bigg(& (\theta-1)\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}-1} \\ & \cdot (v^{-\theta}-1)^{\delta-1}v^{-\theta-1} \\ & +(\delta-1)\theta((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{-1}(v^{-\theta}-1)^{\delta-1}v^{-\theta-1} \\ & +(1-\delta)\theta(v^{-\theta}-1)^{-1}v^{-\theta-1}+(-\theta-1)v^{-1}\Bigg). \end{aligned} \tag{2.17}$$

Proof:

$$\begin{aligned} h &= \frac{\partial C(u,v;\theta,\delta)}{\partial v} = \frac{\partial\left\{1+[(u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta}]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}}}{\partial v} \\ &= \frac{\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}(v^{-\theta}-1)^{\delta}v^{-\theta}}{v(v^{-\theta}-1)((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)} \\ &= \left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}-1}(v^{-\theta}-1)^{\delta-1}v^{-\theta-1} \end{aligned}$$

$$\begin{aligned} h' = \frac{\partial h}{\partial v} &= (-\frac{1}{\theta}-1)\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}-2}\frac{1}{\delta}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta-1}} \\ & \cdot \delta(v^{-\theta}-1)^{\delta-1}(-\theta)v^{-\theta-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}-1}(v^{-\theta}-1)^{\delta-1}v^{-\theta-1} \\ & +\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta-2}} \\ & \cdot \delta(v^{-\theta}-1)\delta-1(-\theta)v^{-\theta-1}(v^{-\theta}-1)\delta-1v^{-\theta-1} \\ & +\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta-2}} \\ & \cdot (\delta-1)(v^{-\theta}-1)^{\delta-2}v^{-\theta-1}(-\theta)v^{-\theta-1} \\ & +\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta-2}} \\ & \cdot (v^{-\theta}-1)^{\delta-1}(-\theta-1)v^{-\theta-2} \end{aligned}$$

$$\overset{factor\ out}{=} h \cdot \Bigg((\theta-1)\left(1+((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}}\right)^{-1}((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{\frac{1}{\delta}-1}$$
$$\cdot (v^{-\theta}-1)^{\delta-1}v^{-\theta-1}+(\delta-1)\theta((u^{-\theta}-1)^{\delta}+(v^{-\theta}-1)^{\delta})^{-1}(v^{-\theta}-1)^{\delta-1}v^{-\theta-1}$$
$$+(1-\delta)\theta(v^{-\theta}-1)^{-1}v^{-\theta-1}+(-\theta-1)v^{-1}\Bigg).$$

□

Corollary 2.20 (BB7-copula)
Let $C(u, v; \theta, \delta)$ the BB7-copula from Definition 2.16. The h-function is then

$$\begin{aligned} h &= \left(1-[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-1} \\ &\cdot [(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}-1}(1-(1-v)^{\theta})^{-\delta-1}(1-v)^{\theta-1} \end{aligned} \tag{2.18}$$

and the derivative with respect to v of the h-function is

$$\begin{aligned} h' = h \cdot \Bigg(&(1-\theta)\left(1-[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}}\right)^{-1} \\ &\cdot [(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}-1}(1-(1-v)^{\theta})^{-\delta-1}(1-v)^{\theta-1} \\ &+(\delta+1)\theta[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-1}(1-(1-v)^{\theta})^{-\delta-1}(1-v)^{\theta-1} \\ &+(-\delta-1)\theta(1-(1-v)^{\theta})^{-1}(1-v)^{\theta-1}+(1-\theta)(1-v)^{-1}\Bigg). \end{aligned} \tag{2.19}$$

Proof:

$$\begin{aligned} h &= \frac{\partial C(u,v;\theta,\delta)}{\partial v} = \frac{\partial\left(1-\left(1-[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}}\right)}{\partial v} \\ &= -\frac{1}{\theta}\left(1-[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-1} \\ &\cdot \frac{1}{\delta}[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}-1} \\ &\cdot(-\delta)(1-(1-v)^{\theta})^{-\delta-1}(-(1-v)^{\theta-1})\theta(-1) \\ &= \left(1-[(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-1} \\ &\cdot [(1-(1-u)^{\theta})^{-\delta}+(1-(1-v)^{\theta})^{-\delta}-1]^{-\frac{1}{\delta}-1}(1-(1-v)^{\theta})^{-\delta-1}(1-v)^{\theta-1} \end{aligned}$$

$$
\begin{aligned}
h' = \frac{\partial h}{\partial v} = (\frac{1}{\theta} & \left(1 - [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-2} \\
& \cdot (-\frac{1}{\delta})[(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}-1}(-\delta)(1-(1-v)^\theta)^{-\delta-1} \\
& \cdot \theta(-(1-v)^{\theta-1})(-1)(1-v)^{\theta-1} \\
& + \left(1 - [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-1} \\
& \cdot (-\frac{1}{\delta})[(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}-2}(-\delta)(1-(1-v)^\theta)^{-\delta-1} \\
& \cdot \theta(-(1-v)^{\theta-1})(-1)(1-v)^{\theta-1} \\
& + \left(1 - [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-1} \\
& \cdot [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}-1}(-\delta-1)(1-(1-v)^\theta)^{-\delta-1} \\
& \cdot \theta(-(1-v)^{\theta-1})(-1)(1-v)^{\theta-1} \\
& + \left(1 - [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}-1} \\
& \cdot [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}-1}(-\delta-1)(1-(1-v)^\theta)^{-\delta-1} \\
& \cdot (\theta-1)(1-v)^{\theta-2}(-1)
\end{aligned}
$$

$$
\begin{aligned}
& \overset{factor\ out}{=} h \cdot \Bigg((1-\theta)\left(1 - [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}}\right)^{-1} \\
& \cdot [(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-\frac{1}{\delta}-1}(1-(1-v)^\theta)^{-\delta-1}(1-v)^{\theta-1} \\
& + (\delta+1)\theta[(1-(1-u)^\theta)^{-\delta} + (1-(1-v)^\theta)^{-\delta} - 1]^{-1}(1-(1-v)^\theta)^{-\delta-1}(1-v)^{\theta-1} \\
& + (-\delta-1)\theta(1-(1-v)^\theta)^{-1}(1-v)^{\theta-1} + (1-\theta)(1-v)^{-1}\Bigg).
\end{aligned}
$$

□

Remark:
Haff et al. [2009] showed in their paper that pair-copula construction in the way we defined it above, are in most cases simplified versions of the real pair-copula constructions. If we want to define the pair-copula decomposition correct and not in the simplified version, the pair-copula depend on the conditioning variables merely through the two conditional distribution functions that constitute their arguments and directly. The direct dependence are skipped in the simplified version. But Haff et al. [2009] showed that the simplified version is a good approximation for the real decomposition. In our case we need the simplified version for the practical application. For this it is sufficient.

2.2.2 Vines

Kurowicka and Cooke [2006] and Bedford and Cooke [2002] introduced the so called regular vines as a graphical method to represent the multitude of the possible pair-copula

decompositions. For example, there are 240 possible decompositions for a 5 dimensional distribution. The vines help to organize them.
There are two special classes of vines, the canonical (C-vine) and the D-vine. In this diploma thesis we will concentrate on the C-vine. Every model decomposes the density in a different way. The different decompositions are represented by a set of trees.
A tree is an acyclic undirected graph; when trees are used to describe dependence structures in high-dimensional distributions, these are called dependence trees.
The following definitions can be found in Kurowicka and Cooke [2006].

Definition 2.21 (Tree)
$T = (N, E)$ is a tree with nodes $N = 1, 2, \ldots, n$ and edges E, where E is a subset of unordered pairs of N with no cycle; that is, there does not exist a sequence $a_1, \ldots, a_k$ $(k > 2)$ of elements of N such that

$$a_1, a_2 \in E, \ldots, a_{k-1}, a_k \in E, a_k, a_1 \in E.$$

The degree of node $a_i \in N$ is $\#(a_j \in N | a_i, a_j \in E)$; that is, the number of edges attached to a_i.

Definition 2.22 (Vine, regular vine)
$\mathcal{V}$ is a vine on n elements if

(i) $\mathcal{V} = (T_1, \ldots, T_{n-1})$.

(ii) *T_1 is a connected tree with nodes $N_1 = 1, \ldots, n$ and edges E_1; for $i = 2, \ldots, n-1$, T_i is a connected tree with nodes $N_i = E_{i-1}$.*
$\mathcal{V}$ is a regular vine on n elements if additionally

(iii) *(proximity) For $i = 2, \ldots, n-1$, if $a, b \in E_i$, then $\#a\Delta b = 2$, where Δ denotes the symmetric difference. In other words, if a and b are nodes of T_i connected by an edge in T_i, where $a = \{a_1, a_2\}, b = \{b_1, b_2\}$, then exactly one of the a_i equals one of the b_i.*

An example of a regular vine can be seen in Figure 2.12.
One special form of the regular vine is the D-vine. In the D-vine every node has maximum 2 edges, i.e. neighbors. The other special case is the canonical vine (C-vine), which has one unique node which is connected to all other nodes. This node is the so called root node. The mathematical definition of this two vines are in the book of Kurowicka and Cooke [2006] too and is written below in Definition 2.23.

Definition 2.23 (D-vine, C-vine)
A regular vine is called a

- *D-vine if each node in T_1 has a degree of at most 2.*
- *Canonical or C-vine if each tree T_i has a unique node of degree $n-i$. The node with maximal degree in T_1 is the root.*

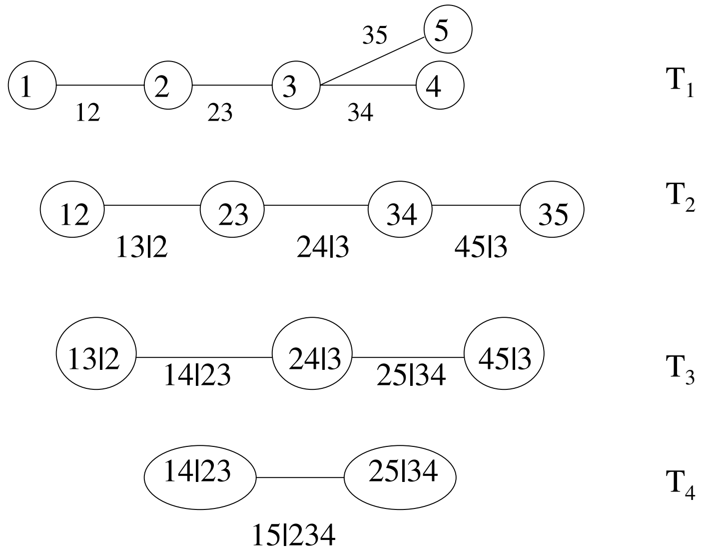


Figure 2.12: A regular vine with 5 variables, 4 trees and 10 edges

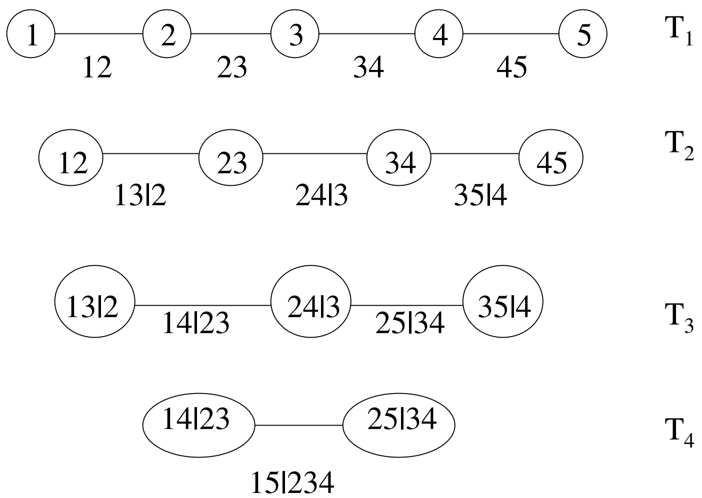


Figure 2.13: Tree representation of a five dimensional D-vine

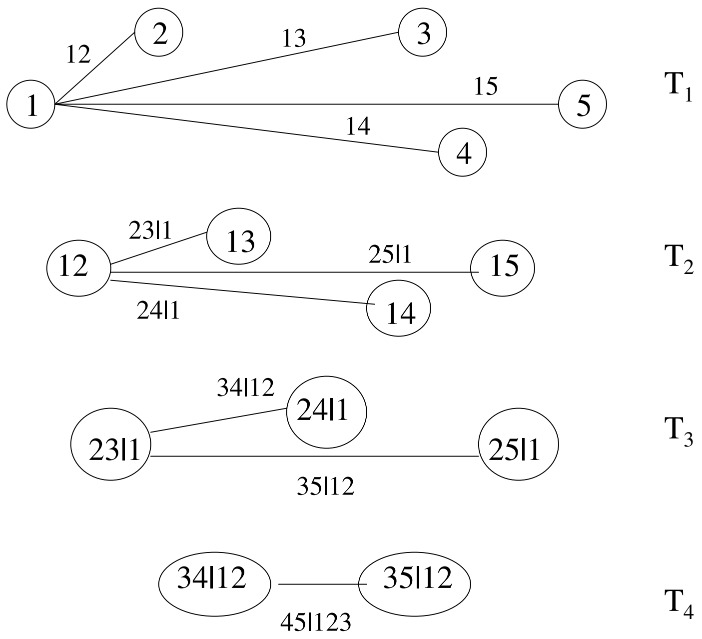


Figure 2.14: Tree representation of a five dimensional C-vine

Examples of a D-vine and a C-vine can be found in Figure 2.13 and 2.14.

If it happens for a data set that one variable has a key role and dominates the other, one might model it with a canonical vine.
Bedford and Cooke [2002] gave the density of an n-dimensional distribution in terms of a regular vine. Aas et al. [2009] specialized this to a D-vine and a canonical vine.
The density $f(x_1, \ldots, x_n)$ corresponding to a D-vine may be written as

$$f(x_1, \ldots, x_n) = \prod_{k=1}^{n} f(x_k) \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,j+i|i+1,\ldots,i+j-1}\{F(x_i|x_{i+1}, \ldots, x_{i+j-1}), F(x_{j+i}|x_{i+1}, \ldots, x_{i+j-1})\}, \tag{2.20}$$

where index j identifies the tree, while i runs over the edges in each tree.

The n-dimensional density of a canonical vine is given by Aas et al. [2009], too.

$$f(x_1, \ldots, x_n) = \prod_{k=1}^{n} f(x_k) \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|1,\ldots,j-1}\{F(x_j|x_1, \ldots, x_{j-1}), F(x_{j+i}|x_1, \ldots, x_{j-1})\}. \tag{2.21}$$

In the following examples we want to provide an insight into the pair-copula constructions and the differences between a C-vine and a D-vine in their decompositions.
The first example is for 3 variables and the PCC of a C-vine is the same as for a D-vine.
In the second and third example with 4 and 5 variables the PCCs differ.

Example 2.24 (3 variables)

$$\begin{aligned} f(x_1, x_2, x_3) = & f_1(x_1)f_2(x_2)f_3(x_3) \cdot c_{12}\{F_1(x_1), F_2(x_2)\} \\ & \cdot c_{23}\{F_2(x_2), F_3(x_3)\} \cdot c_{13|2}\{F(x_1|x_2), F(x_3|x_2)\}. \end{aligned}$$

There exist six different permutations of x_1, x_2 and x_3 in this representation of the density, but only one tree leads to different decompositions. Moreover, each of the tree decompositions is both a canonical vine and a D-vine.

Example 2.25 (4 variables)
D-vine:

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = & f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4) \cdot c_{12}\{F_1(x_1), F_2(x_2)\} \cdot c_{23}\{F_2(x_2), F_3(x_3)\} \\ & \cdot c_{34}\{F_3(x_3), F_4(x_4)\} \cdot c_{13|2}\{F(x_1|x_2), F(x_3|x_2)\} \\ & \cdot c_{24|3}\{F(x_2|x_3), F(x_4|x_3)\} \cdot c_{14|23}\{F(x_1|x_2, x_3), F(x_4|x_2, x_3)\}. \end{aligned}$$

C-vine:

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = & f_1(x_1)f_2(x_2)f_3(x_3)f_4(x_4) \cdot c_{12}\{F_1(x_1), F_2(x_2)\} \cdot c_{13}\{F_1(x_1), F_3(x_3)\} \\ & \cdot c_{14}\{F_1(x_1), F_4(x_4)\} \cdot c_{23|1}\{F(x_2|x_1), F(x_3|x_1)\} \\ & \cdot c_{24|1}\{F(x_2|x_1), F(x_4|x_1)\} \cdot c_{34|12}\{F(x_3|x_1, x_2), F(x_4|x_1, x_2)\}. \end{aligned}$$

Altogether there are 12 different D-vine and canonical vine decompositions, and none of the D-vine decompositions is a canonical vine decomposition. There are no further decompositions in a regular vine. Thus in the four-dimension case, we have 24 possible compositions of bivariate copula densities. (see Aas et al. [2009])

In the five-dimensional case there are regular vines too, which are not canonical or D-vines. Altogether there are 60 different D-vines and 60 different C-vines, which are not identical with anyone of the D-vines and additional 120 regular vines. Thus we get 240 possible decompositions for the copula density.

For the general case of n variables we look at Figure 2.14. We see that the conditioning sets of the edges in each of the trees T_2, T_3 and T_4 are the same. For example in T_3 the conditioning set is always $\{12\}$. Extending this idea to n nodes, we see that there are n choices for the conditioning set $\{i_2\}$ in T_2, $n-1$ choices for the conditioning set $\{i_2, i_3\}$ in T_3 once i_2 is chosen in T_2. Finally, we have three choices for the conditioning set $\{i_{n-1}, i_{n-2}, \ldots, i_2\}$ when $i_2, \ldots, i_{n-2}$ are chosen before. So altogether we have $n(n-1) \cdot \ldots \cdot 3 = \frac{n!}{2}$ different canonical vines on n nodes.
For an n-dimensional D-vine, there are $n!$ possible ways of ordering the variables in tree T_1. Since we have undirected edges for all pairs i, j and arbitrary conditioning sets for D-vines, we can reverse the order in tree T_1 for a D-vine without changing the corresponding vine. Therefore we have only $n!/2$ different trees on the first level. Given such a tree T_1, the trees $T_2, T_3, \ldots, T_{n-1}$ are completely determined. This implies that the number of distinct D-vines on nodes n is given by $n!/2$.

2.3 Dependence measures

In this section we want to introduce briefly two different dependence measures. These measures are needed to determine the dependence between the variables on the one side and to estimate the copula parameter from the empirical data on the other side.
The most important measure for us will be Kendall's τ. Kendall's τ is an easy to calculate, implement and estimate dependence measure.
Beside Kendall's τ there are many more dependence measures, e.g. Spearman's ρ. But here we will concentrate on the correlation coefficient and Kendall's τ. More information and details can be found in Genest and Favre [2007].

2.3.1 Correlation coefficient

The correlation coefficient is the simplest and most common measure for the degree of dependence between two random variables. Its values are between -1 and 1, where -1 identifies fully linear negative dependence, +1 fully linear positive dependence. If there is no (linear) dependence between the random variables X and Y the correlation coefficient is zero.

Definition 2.26 (Correlation coefficient)
The linear correlation coefficient between X and Y is defined as

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}, \tag{2.22}$$

where $Cov(X,Y)$ is the covariance between X and Y, $Var(X)$ and $Var(Y)$ is the variance of X and Y, respectively.

Remarks: (Properties of the correlation coefficient ρ)

(i) $|\rho(X,Y)| \leq 1$.

(ii) If X and Y are independent, than $\rho(X,Y) = 0$.

(iii) If and only if $|\rho(X,Y)| = 1$ than $\exists a$ and $\exists b \neq 0$ such that $P(X = a + bY) = 1$.

(iv) $\rho(\alpha X + \beta Y, \gamma Y + \delta) = sgn(\alpha\gamma)\rho(X,Y)$.

(v) If X and Y have a joint bivariate normal distribution with standard normal margins, the correlation coefficient ρ is then uniquely defined by the joint distribution.

Definition 2.27 (Estimator for ρ)
The correlation coefficient ρ can be estimated by

$$\hat{\rho} = \frac{\widehat{Cov}(X,Y)}{\hat{\sigma}_x\hat{\sigma}_y}, \tag{2.23}$$

where

$$\widehat{Cov}(X,Y) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}),$$
$$\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad \hat{\sigma}_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2,$$
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n}Y_i.$$

2.3.2 Kendall's τ

In contrast to the correlation coefficient Kendall's τ, developed by Kendall [1938], is a so called rank correlation coefficient, i.e. τ measures how good an arbitrary monotone function can describe the dependence between two variables. Thereby one doesn't make any assumptions of the distribution. Hence we can measure non-linear dependencies with Kendall's τ too. See also Genest and Favre [2007] or Nelson [2006].

Definition 2.28 (Kendall's τ)
Let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables with a joint distribution F and marginal distributions F_X and F_Y. Then Kendall's τ is defined by

$$\begin{aligned}\tau &= P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)\\ &= E[sgn(X_1 - X_2)sgn(Y_1 - Y_2)]\\ &= P(X_1 < X_2, Y_1 > Y_2) - P(X_1 > X_2, Y_1 < Y_2),\end{aligned} \tag{2.24}$$

where sgn is the sign-function.

Definition 2.29 (Estimator of Kendall's τ)
For the estimation of Kendall's τ of given data we don't have to order them by rank. Given N data points (x_i, y_i) we examine all $\frac{N(N-1)}{2}$ unordered pairs of data points.
We call a pair concordant if the order of the relative values of the two x are concordant with the order of the relative values of the y.
With $N_{concordant} := |(x_i, y_i)(x_j, y_j), i, j = 1, \ldots, N, i \neq j|(x_i - x_j)(y_i - y_j) > 0|$ we name the number of pairs which are concordant.
A pair is called discordant if both x are ordered the other way around than the both y. The number of these pairs are
$N_{discordant} := |(x_i, y_i)(x_j, y_j), i, j = 1, \ldots, N, i \neq j|(x_i - x_j)(y_i - y_j) < 0|$.
If both x are the same, than we call the pair $extra_y$, are both y the same we call the pair $extra_x$. The according numbers of pairs are
$N_{extra_y} := |(x_i, y_i)(x_j, y_j), i, j = 1, \ldots, N, i \neq j|x_i = x_j|$ and
$N_{extra_x} := |(x_i, y_i)(x_j, y_j), i, j = 1, \ldots, N, i \neq j|y_i = y_j|$.
If both x and both y are the same, we don't call the pair at all.
After ordering and counting all pairs we estimate Kendall's τ by

$$\hat{\tau} = \frac{N_{concordant} - N_{discordant}}{\sqrt{N_{concordant} + N_{discordant} + N_{extra_y}}\sqrt{N_{concordant} + N_{discordant} + N_{extra_x}}} \tag{2.25}$$

The numerator in (2.25) estimates the number of all pairs as defined above. Thus, $\hat{\tau}$ is a good approximation for (2.24).

Remarks: (Properties of Kendall's τ)

(i) $\tau(X_1, Y_1)$ is symmetric.

(ii) If X_1 and Y_1 are independent, then $\tau(X, Y) = 0$.

(iii) $\tau(T_x(X_1), T_y(Y_1)) = \tau(X_1, Y_1)$ for all increasing maps T_x and T_y.

(iv) $\tau = 4 \int\int F(x, y) dF(x, y) - 1$.

(v) If the vector (X_1, Y_1) is bivariate normal distributed with correlation coefficient ρ and standard normal distributed margins, then $\rho = \sin(\frac{\pi}{2}\tau)$.

The last two remarks are particularly interesting for the copulas. It is valid for copulas too, see Nelson [2006], Lindskog et al. [2003] and Embrechts et al. [2003]. Thus we can calculate Kendall's τ by

$$\tau = 4 \int\int C(u_1, u_2) dC(u_1, u_2) - 1,$$

where $C(\cdot, \cdot)$ is the copula distribution function.
Using this equation for the Gaussian and t-copula we can calculate Kendall's τ or the inverse of Kendall's τ as follows:

$$\rho = \sin(\frac{\pi}{2}\tau).$$

The proof of the following theorem for the calculation of Kendall's τ for Archimedean copulas can be found in Embrechts et al. [2003].

Theorem 2.30 (Genest und MacKay)
If φ is a generator of an Archimedean copula then

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt. \tag{2.26}$$

For the Clayton copula one can determine:

$$\tau = \frac{\delta}{\delta + 2}$$

and for the Gumbel copula

$$\tau = 1 - \frac{1}{\delta}.$$

Obviously, we have different Kendall's τ for different copulas. Thus we try to identify the copula by the characteristic τ
Now we will collect the theoretical Kendall's τ of the most important copulas in Table 2.3.

Copula-Family	**Kendall's τ**	$\tau \in \ldots$
Gaussian Copula	$\tau = \frac{2}{\pi}\arcsin(\rho)$	[-1,1]
t-Copula	$\tau = \frac{2}{\pi}\arcsin(\rho)$	[-1,1]
Gumble Copula	$\tau = 1 - \frac{1}{\delta}$	[0,1]
Clayton Copula	$\tau = \frac{\delta}{\delta+2}$	[0,1]
Frank Copula	$\tau = 1 - \frac{4}{\delta} + 4\frac{D_1(\delta)}{\delta}$ with $D_1(\delta) = \int_o^\delta \frac{x/\delta}{e^x-1}dx$ (Debye-Function)	[-1,1]
Joe Copula	$\tau = 1 + \left(\frac{-2+2\gamma+2\ln(2)+\Psi(\frac{1}{\delta})+\Psi(\frac{1}{2}\frac{2+\delta}{\delta})+\delta}{-2+\delta}\right)$	[0,1]
	with Euler's constant $\gamma = \lim_{n\to\infty}(\sum_{i=1}^n \frac{1}{i} - \ln(n)) \approx 0,57721$	
	and Digamma-function $\Psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{d}{dx}\Gamma(x)/\Gamma(x)$	
BB1	$\tau = 1 - \frac{2}{\delta(\theta+2)}$	[0,1]
BB7	$\tau = 1 - \frac{2}{\delta(2-\theta)} + \frac{4}{\theta^2\delta}B(\frac{2-2\theta}{\theta}+1, \delta+2)$	[0,1]
	with Beta-function $B(x,y) = \int_0^1 t^{x+1}(1-t)^{y-1}dt$	

Table 2.3: The different copula families and their Kendall's τ

Proof:(Kendall's τ)

- For Gaussian and t-copula see remark above and Lindskog et al. [2003] and Embrechts et al. [2003]
- Clayton:

$$\tau \overset{(2.26)}{=} 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt = 1 + 4\int_0^1 \frac{(t^{-\delta}-1)}{\delta}(-t^{\delta+1})dt = 1 - \frac{4}{\delta}\int_0^1 (t - t^{\delta+1})dt$$
$$= 1 - \frac{4}{\delta}\left[\frac{1}{2} - \frac{1}{\delta+2}\right] = 1 - \frac{2}{\delta} + \frac{4}{\delta(\delta+2)} = \frac{\delta}{\delta+2}.$$

- Gumbel:

$$\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt = 1 + 4\int_0^1 \frac{t}{\delta}\log(t)dt = 1 + \frac{4}{\delta}\left[-\frac{1}{2}\int_0^1 tdt - \frac{1}{2}\right]$$
$$= 1 + \frac{4}{\delta}\left[-\frac{1}{2} + \frac{1}{2}[0,5x^2]_0^1\right] = 1 + \frac{4}{\delta}\left[-\frac{1}{2} + \frac{1}{4}\right] = 1 - \frac{1}{\delta}$$

because of

$$\int_0^1 t\ln(t)dt = t\cdot[-t + t\ln(t)] - \int_0^1 (-t + t\ln(t))dt = -1 + \int_0^1 tdt - \int_0^1 t\ln(t)dt$$
$$\Leftrightarrow \int_0^1 t\ln(t) = -\frac{1}{2} + \frac{1}{2}\int_0^1 tdt.$$

- Frank:

$$\begin{aligned}
\varphi &= -\ln(\frac{e^{-\delta t}-1}{e^{-\delta}-1}) \\
\varphi' &= \frac{\delta}{1-e^{\delta t}} \\
\tau &= 1+4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt = 1+\frac{4}{\delta}\int_0^1 \ln(e^{-\delta}-1)(1-e^{\delta t})dt - \frac{4}{\delta}\int_0^1 \ln(e^{-\delta t}-1)(1-e^{\delta t})dt \\
&= 1+\frac{4\ln(e^{-\delta-1}-1)}{\delta^2}(1+\delta-e^{\delta}) - \frac{4}{\delta}\int_0^1 \ln(e^{-\delta t}-1)dt + \frac{4}{\delta}\int_0^1 \ln(e^{-\delta t}-1)e^{\delta t}dt \\
&\overset{substitution}{=} 1+\frac{4\ln(e^{-\delta-1}-1)}{\delta^2}(1+\delta-e^{\delta}) - \frac{4}{\delta^2}\int_0^{e^{\delta}} \frac{\ln(t)}{1-t}dt - \frac{4}{\delta} - \frac{4\ln(1-e^{\delta})}{\delta} + 4 \\
&- \frac{4}{\delta^2}(\ln(1-e^{\delta})(1-e^{\delta})+e^{\delta}\delta) \\
&\overset{substitution}{=} 1-\frac{4}{\delta}+\frac{4}{\delta}\underbrace{\int_0^{\delta}\frac{x/\delta}{e^x-1}dx}_{=D_1(\delta)}
\end{aligned}$$

with

$$D_1(\delta) = \int_o^{\delta} \frac{x/\delta}{e^x-1}dx \quad (Debye-function).$$

- Joe

$$\begin{aligned}
\varphi &= -\ln(1-(1-t)^{\delta}) \\
\varphi' &= -\frac{\delta(1-t)^{\delta-1}}{1-(1-t)^{\delta}} \\
\tau &= 1+4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt = 1+4\int_0^1 \frac{\ln(1-(1-t)^{\delta})(1-(1-t)^{\delta})}{\delta(1-t)^{\delta-1}}
\end{aligned}$$

with substitution $s = 1-(1-t)^{\delta}$ and $ds = \delta(1-t)^{\delta-1}$

$$\tau = 1+\frac{4}{\delta^2}\int_0^1 \ln(s)(1-s)^{\frac{2(1-\delta)}{\delta}}sds$$

and using the math-program MAPLE we get

$$\tau = 1+\frac{4}{\delta^2}\left(\frac{-2\delta^2+2\delta^2\gamma+2\delta^2\ln(2)+\delta^2\Psi(\frac{1}{\delta})+\delta^2\Psi(\frac{1}{2}\frac{2+\delta}{\delta})+\delta^3}{-8+4\delta}\right)$$

with Euler's constant

$$\gamma = \lim_{n\to\infty}(\sum_{i=1}^{n}\frac{1}{i}-\ln(n)) \approx 0,57721$$

and Digamma-function

$$\Psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{d}{dx}\Gamma(x)/\Gamma(x) = \int_0^{\infty}\left(\frac{e^{-t}}{t}-\frac{e^{-xt}}{1-e^{-t}}\right)dt.$$

Remark: The MAPLE-function can be found in the appendix.

- BB1:

$$\begin{aligned}
\varphi &= (u^{-\theta}-1)^{\delta} \\
\varphi' &= -\theta\delta(u^{-\theta}-1)^{\delta-1}u^{-1-\theta} \\
\tau &= 1+4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt = 1-\frac{4}{\theta\delta}\int_0^1 \frac{(u^{-\theta}-1)}{u^{-1-\theta}}du = 1-\frac{4}{\theta\delta}\int_0^1 udu+\frac{4}{\theta\delta}\int_0^1 u^{\theta+1}du \\
&= 1-\frac{2}{\theta\delta}+\frac{4}{\theta\delta}\left[\frac{1}{\theta+2}u^{\theta+2}\right]_0^1 = 1-\frac{2}{\theta\delta}+\frac{4}{\theta\delta(\theta+2)} = 1-\frac{2}{\delta(\theta+2)}.
\end{aligned}$$

- BB7:

$$\begin{aligned}
\varphi &= \left(1-(1-u)^{\theta}\right)^{-\delta}-1 \\
\varphi' &= -\theta\delta(1-u)^{\theta-1}\left(1-(1-u)^{\theta}\right)^{-\delta-1} \\
\tau &= 1+4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt = 1+4\int_0^1 \frac{\left(1-(1-u)^{\theta}\right)^{-\delta}-1}{-\theta\delta(1-u)^{\theta-1}\left(1-(1-u)^{\theta}\right)^{-\delta-1}}du \\
&= 1-\frac{4}{\theta\delta}\int_0^1\left(\frac{1-(1-u)^{\theta}}{(1-u)^{\theta-1}}-\frac{1}{(1-u)^{\theta-1}\left(1-(1-u)^{\theta}\right)^{-\delta-1}}\right)du \\
&= 1-\frac{4}{\theta\delta}\int_0^1\frac{1}{(1-u)^{\theta-1}}du+\frac{4}{\theta\delta}\int_0^1(1-u)du+\frac{4}{\theta\delta}\int_0^1\frac{1}{(1-u)^{\theta-1}\left(1-(1-u)^{\theta}\right)^{-\delta-1}}du \\
&= 1+\frac{4}{\theta\delta(2-\theta)}+\frac{2}{\theta\delta}+\frac{4}{\theta\delta}\int_0^1\frac{1}{(1-u)^{\theta-1}\left(1-(1-u)^{\theta}\right)^{-\delta-1}}du \\
&= 1-\frac{2}{\delta(2-\theta)}+\frac{4}{\theta\delta}\underbrace{\int_0^1\frac{1}{(1-u)^{\theta-1}\left(1-(1-u)^{\theta}\right)^{-\delta-1}}du.}_{=(**)}
\end{aligned}$$

Compute $(**)$ by using substitution $s=(1-u)^{\theta}, ds=-\theta s^{\frac{\theta-1}{\theta}}du$

$$\begin{aligned}
(**) &= \int_0^1 \frac{s^{\frac{1-\theta}{\theta}}(1-s)^{1+\delta}}{\theta s^{\frac{\theta-1}{\theta}}}ds = \frac{1}{\theta}\int_0^1 s^{\frac{1-\theta}{\theta}-\frac{(\theta-1)}{\theta}}(1-s)^{1+\delta}ds \\
&= \frac{1}{\theta}\int_0^1 s^{\frac{2-2\theta}{\theta}}(1-s)^{1+\delta}ds \overset{Beta-function}{=} \frac{1}{\theta}B(\frac{2-2\theta}{\theta}+1,\delta+2)
\end{aligned}$$

with Beta-function

$$B(x,y)=\int_0^1 t^{x+1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and Gamma-function

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt.$$

□

Figure 2.15 - 2.17 should illustrate Kendall's τ of the different copula families depending on the appropriate copula parameter(s). It also shows the relation between Kendall's τ and the copula parameter(s), which we calculated and tabulated in Table 2.3. Note: Only for the Gaussian, the t- and the Frank copula one can compute negative Kendall's τ.

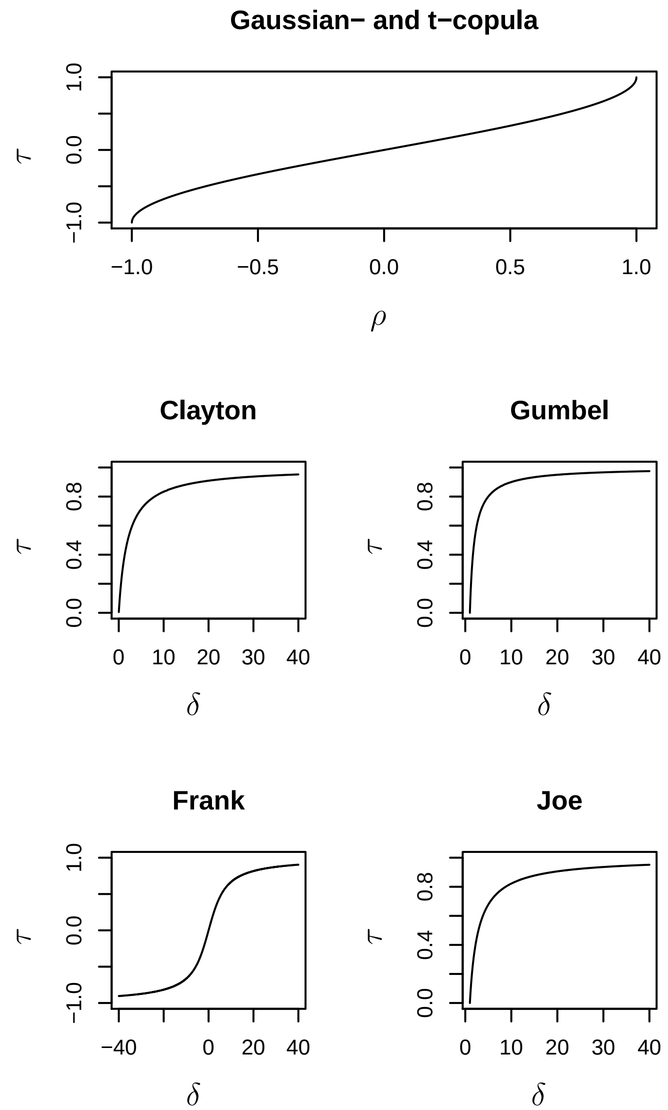


Figure 2.15: Kendall's τ of the Gaussian, t-, Clayton, Gumbel, Frank and Joe copula depending on the copula parameter.

2.4 Tail dependence

In the previous section we introduced the dependence measures ρ (correlation coefficient) and Kendall's τ, which are global measures of dependence related to the joint distribution

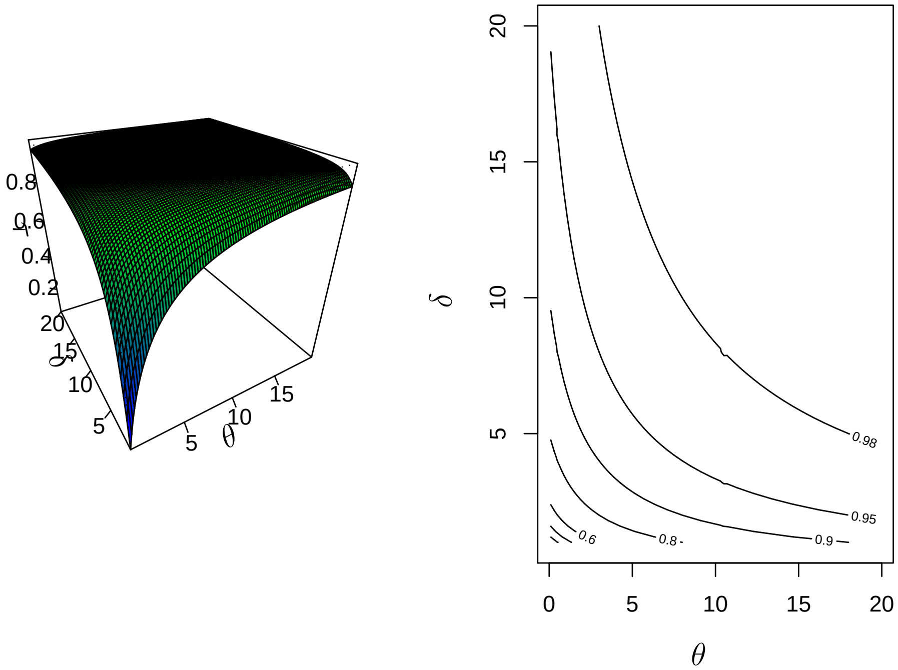


Figure 2.16: Kendall's τ of the BB1-copula as a 3D-plot (left) and as a contourplot (right)

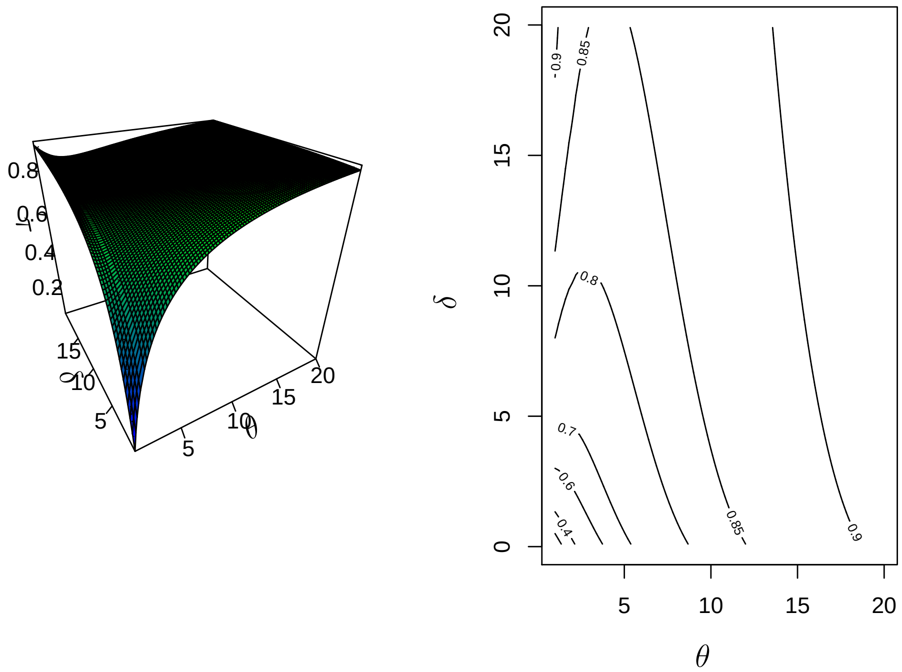


Figure 2.17: Kendall's τ of the BB7-copula as a 3D-plot (left) and as a contourplot (right)

of two variables. But sometimes it is interesting to look at local dependencies and especially to study dependence between extreme values. The concept of tail dependence relates to the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. It turns out that tail dependence between two continuous random variables X and Y is a copula property and hence the amount of tail dependence is invariant under strict increasing transformations of X and Y. But first we will address the definition of tail dependence.

Definition 2.31 (Upper tail dependence)
Let $(X, Y)^T$ be a vector of continuous random variables with marginal distribution functions F and G. The coefficient of upper tail dependence of $(X, Y)^T$ is

$$\lambda_U = \lim_{u \nearrow 1} P(Y > G^{-1}(u) | X > F^{-1}(u)) \tag{2.27}$$

provided that the limit $\lambda_U \in [0, 1]$ exists. If $\lambda_U \in (0, 1]$, X and Y are said to be asymptotically dependent in the upper tail; if $\lambda_U = 0$, X and Y are said to be asymptotically independent in the upper tail.

Definition 2.32 (Upper tail dependence for copulas)
If a bivariate copula C is such that

$$\lambda_U = \lim_{u \nearrow 1} \frac{1 - 2u + C(u, u)}{1 - u} \tag{2.28}$$

exists. Then C has upper tail dependence if $\lambda_U \in (0, 1]$, and upper tail independence if $\lambda_L = 0$.

Example 2.33 (Gumbel copula)
Consider the bivariate Gumbel family of copulas given by Example 2.11. Then

$$\frac{1 - 2u + C(u, u)}{1 - u} = \frac{1 - 2u + exp(2^{\frac{1}{\delta}} \ln(u))}{1 - u} = \frac{1 - 2u + u^{2^{1/\delta}}}{1 - u}$$

and hence

$$\lambda_U = \lim_{u \nearrow 1} \frac{1 - 2u + C(u, u)}{1 - u} = 2 - \lim_{u \nearrow 1} 2^{\frac{1}{\delta}} u^{2^{1/\delta} - 1} = 2 - 2^{\frac{1}{\delta}}.$$

Thus for $\delta > 1, C_\delta$ has upper tail dependence.

The concept of lower tail dependence can be defined in a similar way.

Definition 2.34 (Lower tail dependence for copulas)
If a bivariate copula C is such that

$$\lambda_L = \lim_{u \searrow 0} \frac{C(u, u)}{u} \tag{2.29}$$

exists. Then C has lower tail dependence if $\lambda_L \in (0, 1]$, and lower tail independence if $\lambda_L = 0$.

If λ_U and λ_L are the same we simply name it λ. For example the t-copula.

Example 2.35 (t-copula)
For continuous distributed random variables with the t-copula C with parameters ρ and ν the tail dependence is given by

$$\lambda = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right),$$

where $t_{\nu+1}$ is the univariate t distribution function with $\nu+1$ degrees of freedom. The value of λ depends on the parameters and the t-copula has both upper and lower tail dependence.

In contrast, some copulas have no tail dependence at all, neither upper nor lower tail dependence. The Gaussian copula is an example for this kind of copula.

Example 2.36
Let C be a Gaussian copula with parameter $\rho \in (-1, 1)$, then the tail dependence is

$$\lambda = \lim_{x\to\infty} 2\left(1 - \Phi\left(\frac{x\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right) = 0$$

Thus the Gaussian copula has no tail independence.

For Archimedean copulas, tail dependence can be expressed in terms of the generator. The following theorems can be found in Embrechts et al. [2003].

Theorem 2.37
Let φ be a strict generator such that φ^{-1} belongs to the class of Laplace transforms of strictly positive random variables. If $\varphi^{-1\prime}(0)$ is finite then

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

does not have upper tail dependence. If C has upper tail dependence, $\varphi^{-1\prime}(0) = -\infty$ and the coefficient of upper tail dependence is given by

$$\lambda_U = 2 - 2\lim_{s\searrow 0}\left(\frac{\varphi^{-1\prime}(2s)}{\varphi^{-1\prime}(s)}\right). \tag{2.30}$$

Proof:
see Embrechts et al. [2003]. □

Theorem 2.38
Let φ be as in Theorem 2.37. The coefficient of lower tail dependence for the copula $C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ is equal to

$$\lambda_L = 2\lim_{s\to\infty}\left(\frac{\varphi^{-1\prime}(2s)}{\varphi^{-1\prime}(s)}\right). \tag{2.31}$$

Proof:
see Embrechts et al. [2003]. □

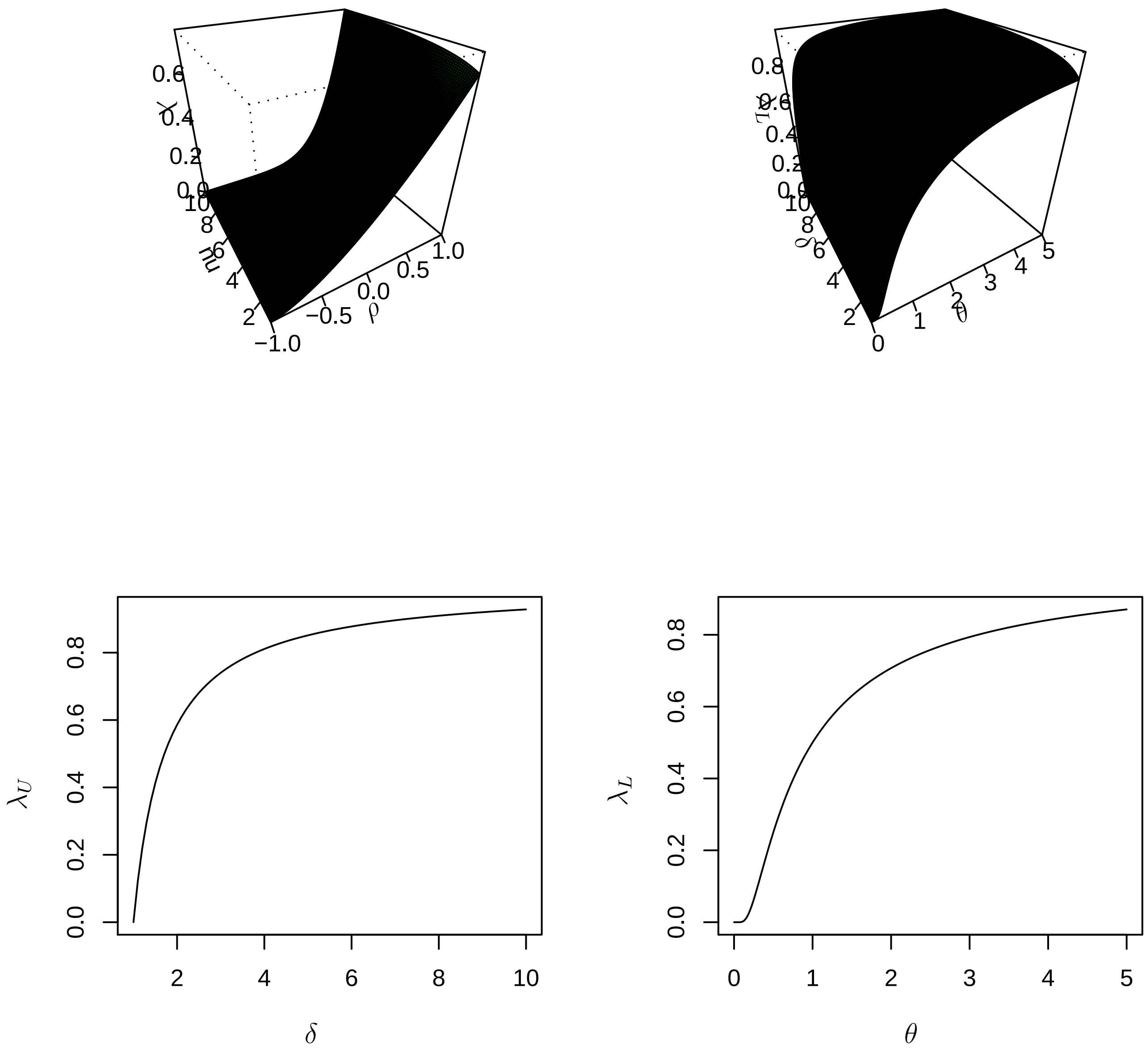


Figure 2.18: Upper and lower tail dependence behavior of the t-copula (top, left), lower tail dependence behavior for the BB1-copula (top, right), upper tail dependence behavior of the Gumbel, Joe, BB1- and BB7-copula (bottom, left) and lower tail dependence behavior for the Clayton and BB7-copula (bottom, right)

Copula family	Upper tail dependence	Lower tail dependence
Gaussian copula	-	-
t-Copula	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$
Gumbel copula	$2-2^{1/\delta}$	-
Clayton copula	-	$2^{-1/\delta}$
Frank copula	-	-
Joe copula	$2-2^{1/\delta}$	-
BB1-copula	$2-2^{1/\delta}$	$2^{-1/(\delta\theta)}$
BB7-copula	$2-2^{1/\theta}$	$2^{-1/\delta}$

Table 2.4: Tail dependence of the different copula families

2.5 The λ-Function

We have seen, that we can distinguish the different copulas by their Kendall's τ as well as by their tail dependence behavior. Now we introduce a further method - the λ-function. The λ-function is characteristic for each copula family and is defined by Kendall's cdf (see definition below).

Definition 2.39 (λ-function)
The λ-function is defined as a shifted copula distribution function given by

$$\begin{aligned} \lambda(v,\theta) &:= v - K(v,\theta) \\ K(v,\theta) &:= P(C_\theta(u_1,u_2) \le v) \quad v \in [0,1] \quad (Kendall's\ cdf) \end{aligned} \tag{2.32}$$

For Archimedean copulas we have the following proposition (see Genest and Rivest [1993]).

Proposition 2.40
Let X and Y be uniform random variables whose dependence function $C(x,y)$ has the properties of an Archimedean copula. Set $U = \frac{\varphi(X)}{\varphi(X)+\varphi(Y)}$, $V = C(X,Y)$ and $\lambda(v,\theta) = \frac{\varphi(v)}{\varphi'(v)}$ for $0 < v < 1$ and $\varphi(\cdot)$ generator of C_θ.
Then

- *U is uniformly distributed on (0,1).*
- *V is distributed as $K(v,\theta) = v - \lambda(v,\theta)$ on (0,1).*
- *U and V are independent random variables.*

Hence we can calculate the λ-function for the Archimedean copulas with the formula

$$\lambda(v,\theta) = \frac{\varphi(v)}{\varphi'(v)}.$$

For large data sets the λ-function can be easily estimated empirically by using the empirical copula function.

In Table 2.5 we can see a few λ-functions for different copula families and their generator. Additionally we computed the limits of the λ-functions, i.e. the λ-function $\lambda(t,\tau)$ with the parameters $t \in [0,1]$ and Kendall's τ, which is a function of the parameter(s), with Kendall's τ converges to zero or one. If τ converges to one the λ-function converges for all copula families against zero. ($\lim_{\tau\to 1} \lambda(t,\tau) = 0$)

Copula family	**Generator** $\varphi(t)$	λ**-function**	$\lim_{\tau\to 0}\lambda(t,\tau)$
Gumbel copula	$\lvert\ln(t)\rvert^\delta$	$\frac{t\ln(t)}{\delta}$	$t\ln(t)$
Clayton copula	$\frac{t^{-\delta}-1}{\delta}$	$-\frac{t(1-t^\delta)}{\delta}$	$t\ln(t)$
Frank copula	$\varphi = -\ln\left(\frac{e^{-\delta t}-1}{e^{-\delta}-1}\right)$	$-\frac{1-e^{-\delta t}}{\delta e^{-\delta t}}\ln\left(\frac{1-e^{-\delta}}{1-e^{-\delta t}}\right)$	$-t\ln(\frac{1}{t})$
Joe copula	$-\ln(1-(1-t)^\delta)$	$\frac{\ln(1-(1-t)^\delta)(1-(1-t)^\delta)}{\delta(1-t)^{\delta-1}}$	$t\ln(t)$
BB1-copula	$(t^{-\theta}-1)^\delta$	$-\frac{1}{\theta\delta}\frac{(t^{-\theta}-1)}{t^{-1-\theta}}$	$t\ln(t)$
BB7-copula	$[1-(1-t)^\theta]^{-\delta}-1$	-$\frac{\left(1-(1-t)^\theta\right)^{-\delta}-1}{\theta\delta(1-t)^{\theta-1}\left(1-(1-t)^\theta\right)^{-\delta-1}}$	$t\ln(t)$

Table 2.5: λ-function of the different copula families and their limits, i.e. the λ-functions if Kendall's τ converges to zero

Proof: (Of the limits)
Let $t \in [0,1]$ for all copula families.

- Gumbel:
  If $\tau \to 0$ then $\delta \to 1$
  $$\Rightarrow \lim_{\tau\to 0}\lambda(t,\tau) = \lim_{\delta\to 1}\lambda(t,\delta) = t\ln(t).$$
- Clayton:
  If $\tau \to 0$ then $\delta \to 0$
  $$\Rightarrow \lim_{\tau\to 0}\lambda(t,\tau) = \lim_{\delta\to 0}\lambda(t,\delta) \overset{L'Hopital}{=} \lim_{\delta\to 0} t^{\delta+1}\ln(t) = t\ln(t).$$
- Frank:
  If $\tau \to 0$ then $\delta \to 0$
  $$\Rightarrow \lim_{\tau\to 0}\lambda(t,\tau) = \lim_{\delta\to 1}\lambda(t,\delta) = \lim_{\delta\to 0} -\frac{1-e^{-\delta t}}{\delta e^{-\delta t}} \cdot \lim_{\delta\to 0}\ln\left(\frac{1-e^{-\delta}}{1-e^{-\delta t}}\right)$$
  $$\overset{L'Hopital}{=} \lim_{\delta\to 0} -\frac{te^{-\delta t}}{e^{-\delta t}-\delta^2 e^{-\delta t}} \cdot \ln\left(\lim_{\delta\to 0}\frac{e^{-\delta}}{te^{-\delta t}}\right) = -t\ln\left(\frac{1}{t}\right).$$
- Joe:
  If $\tau \to 0$ then $\delta \to 1$
  $$\Rightarrow \lim_{\tau\to 0}\lambda(t,\tau) = \lim_{\delta\to 1}\lambda(t,\delta) = t\ln(t).$$

- BB1:
$$\tau = 1 - \frac{2}{\delta(\theta+2)} = 0 \Leftrightarrow 2 = \delta(\theta+2).$$
Set $\delta = 1$ and $\theta \to 0$
$$\Rightarrow \lim_{\tau\to 0} \lambda(t,\tau) = \lim_{\theta\to 1} \lambda(t,\theta) = \lim_{\theta\to 0} -\frac{1}{\theta}\frac{(t^{-\theta}-1)}{t^{-1-\theta}} \overset{L'Hopital}{=} \lim_{\theta\to 0} \frac{t^{-\theta}\ln(t)}{t^{-1-\theta} - \theta t^{-1-\theta}\ln(t)} = t\ln(t).$$

- BB7:
Set $\theta = 1$ and $\delta \to 0$
$$\Rightarrow \lim_{\tau\to 0} \lambda(t,\tau) = \lim_{\delta\to 1} \lambda(t,\delta) = \lim_{\delta\to 0} -\frac{1}{\delta}\frac{(t^{-\delta}-1)}{t^{-1-\delta}} \overset{L'Hopital}{=} \lim_{\delta\to 0} \frac{t^{-\delta}\ln(t)}{t^{-1-\delta} - \delta t^{-1-\delta}\ln(t)} = t\ln(t).$$

□

A nonparametric estimator of $K(v)$ is given by:

$$K_n(v) = \frac{1}{n}\sum_{i=1}^{n} 1_{(V_i \le v)},$$

where $V_i = \#((X_j,Y_j) : X_j < X_i, Y_j < Y_i)\frac{1}{n-1}$, for $1 \le i \le n$ and random vectors $X = (X_1,\dots,X_n), Y = (Y_1,\dots,Y_n)$.

Proposition 2.41
Let $C(x,y)$ be a dependence function that is absolutely continuous with respect to Lebesgue measure, and suppose that $(X_1,Y_1),\dots,(X_n,Y_n)$ form a random sample from C. An $o(\frac{1}{n})$ approximation of the asymptotic variance of $K_n(v)$ is given by:

$$\sigma^2_{\hat\theta}(v) = (K(v)\overline{K(v)} + k(v)(k(v)R(v) - 2v\overline{K(v)}))\frac{1}{n},$$

where $\overline{K(v)} = 1 - K(v), k(v) = K'(v)$,
$R(v) = E[C(min(X_1,X_2), min(Y_1,Y_2)) - v^2|C(X_1,Y_1) = C(X_2,Y_2) = v]$
and $\hat\theta$ is the estimated copula parameter (vector) of C.

Remarks:

- For general copulas the calculation of $K(v), k(v)$ and $R(v)$ are involved.
- There exists a closed form for the function $R(v)$ (see Genest and Rivest [1993]).
- The Clayton family is the only one for which an explicit algebraic expression could be found for $R(v)$.

- Confidence bands for the unknown λ are constructed using

$$\lambda_n(v) \pm c\sigma_{\hat{\theta}}(v),$$

where c = 4.72 as suggested by Kotel'nikova and Chmaladze (1982) for an experimentwise coverage of 95% and $\sigma_{\hat{\theta}}$ is the sample variance of $K_n(v)$ with estimated parameter $\hat{\theta}$ of the Clayton family.
Caution: The construction of confidence bands for the unknown λ is incorrect because one uses for all copula families the asymptotic variance $\sigma^2_{\hat{\theta}}(v)$ of the Clayton copula, because the Clayton copula is the only one for which we can calculate $R(v)$.

In Figure 2.19 we plotted the λ-function of the Clayton copula for different parameters. Additionally the included the limits, i.e. Kendall's $\tau \to 0$ and $\tau \to 1$.

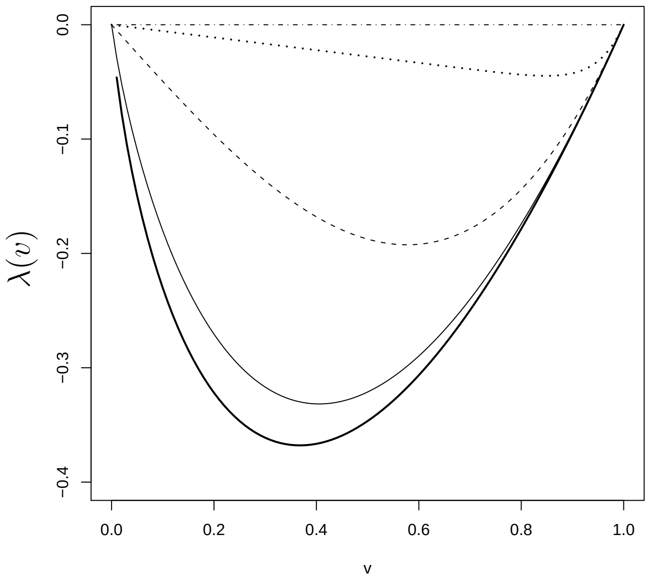


Figure 2.19: λ-function of the Clayton copula. We draw one λ-function with $\tau = 0.1$ ($\delta = 0.22$) as solid line, $\tau = 0.5$ ($\delta = 2$) as dashed line, $\tau = 0.9$ ($\delta = 18$) as dotted line, fully independent ($\tau = 0$) as thick solid line and fully dependent ($\tau = 1$) as dot dashed line.

2.6 Goodness-of-fit

2.6.1 Goodness-of-fit test for copula selection

A goodness-of-fit of a statistical model describes how well it fits a set of observations. Measures of goodness-of-fit typically summarize the discrepancy between the expected and the observed values under the model in question. These tests are often used in statistical hypothesis testing, in our case we test the hypothesis if a chosen copula fits the underlying copula of the data.

$$H_0 : C \in \mathcal{C} = \{C_\theta : \theta \in \Theta\} \quad \text{vs.} \quad H_1 : C \notin \mathcal{C} = \{C_\theta : \theta \in \Theta\},$$

where $\mathcal{C}$ is the set of copulas and Θ is the parameter space.
And the test statistic we use, the Cramér-von Mises statistic, which is implemented in the R-package "copula", is defined as follows:

$$S_n^{\mathbb{C}_n} = \int_{[0,1]^d} \mathbb{C}_n^2(u) dC_n(u) \tag{2.33}$$

with $U_1, \ldots, U_n$ be observations from an unknown distribution

$$C_n(u) = \frac{1}{n} \sum_{i=1}^{n} 1(U_{i1} \leq u_1, \ldots, U_{id} \leq u_d) \tag{2.34}$$

as the empirical distribution, $u = (u_1, \ldots, u_d) \in [0,1]^d$, $1(\cdot)$ is the indicator function and

$$\mathbb{C}_n = \sqrt{n}(C_n - C_{\theta_n}).$$

Large values of this statistic lead to a rejection of H_0. Genest and Rémillard proved the convergence of the process $\mathbb{C}_n$ and showed that the test based on $S_n^{\mathbb{C}_n}$ are consistent, i.e. if $C \notin \mathcal{C}$, then H_0 is rejected with probability 1 as $n \to \infty$. (see Genest et al. [2007] and Genest et al. [2005]).

Remarks:
In statistical hypothesis testing, the p-value is the probability of obtaining a test statistic at least as extreme as the one that was actually observed, assuming that the null hypothesis is true. The fact that p-values are based on this assumption is crucial to their correct interpretation.
The lower the p-value, the less likely the result, assuming the null hypothesis, the more "significant" the result, in the sense of statistical significance. One often rejects a null hypothesis if the p-value is less than 0.05 or 0.01, corresponding to a 5% or 1% chance respectively of an outcome at least that extreme, given the null hypothesis.
The p-value is not the probability that the null hypothesis is true.
The significance level of the test is not determined by the p-value. It is set by the statistician, often 0.05 or 0.01.

To compute p-values for our test statistic based on the empirical process one requires generating a large number, N, of independent samples of size n from C_{θ_n} and computing the corresponding values of the statistic.
The test statistic $S_n^{\mathbb{C}_n}$ depends on the copula under H_0 and the unknown parameter θ. Therefore a distribution of the test statistic can not be tabulated and the p-value can only be obtained via Monte Carlo methods.
A possible procedure to compute p-values is Algorithm 2.2 (quoted from Genest et al.

Algorithm 2.2 Computation of the goodness-of-fit p-value

Compute C_n as per formula (2.34) and estimate θ with θ_n.
if there is an analytical expression for C_n **then**
  Compute the value of $S_n^{\mathbb{C}_n}$, as defined in (2.33).
else
  Proceed by Monte Carlo approximation. Specially, choose $m \geq n$ and carry out the following steps:
  Generate a random sample $U_1^*, \ldots, U_m^*$ from distribution C_{θ_n}.
  Approximate C_{θ_n} by $B_m^*(u) = \frac{1}{m}\sum_{i=1}^{m} 1(U_i^* \leq u),\ u \in [0,1]^d$.
  Approximate $S_n^{\mathbb{C}_n}$ by $S_n^{\mathbb{C}_n} = \sum_{i=1}^{n}(C_n(U_i) - B_m^*(U_i))^2$.
end if
for $k = 1$ **to** N **do**
  Generate a random sample $Y_{1,k}^*, \ldots, Y_{n,k}^*$ from distribution C_{θ_n} and compute their associated rank vectors $R_{1,k}^*, \ldots, R_{n,k}^*$.
  for $i = 1$ **to** n **do**
    Compute $U_{i,k}^* = R_{i,k}^*/(n+1)$.
  end for
  Let $C_{n,k}^*(u) = \frac{1}{n}\sum_{i=1}^{n} 1(U_{i,k}^* \leq u), \quad u \in [0,1]^d$
  and estimate θ by $\theta_{n,k}^*$.
  if there is an analytical expression for C_θ **then**
    let $S_{n,k}^* = \sum_{i=1}^{n}(C_{n,k}^*(U_{i,k}^*) - C_{\theta_{n,k}^*}(U_{i,k}^*))^2$
  else
    Proceed as follows:
    Generate a random sample $Y_{1,k}^{**}, \ldots, Y_{n,k}^{**}$ from distribution $C_{\theta_{n,k}^*}$.
    Approximate $C_{\theta_{n,k}^*}$ by $B_{m,k}^{**}(u) = \frac{1}{m}\sum_{i=1}^{m} 1(Y_{i,k}^{**} \leq u),\ u \in [0,1]^d$
    and let $S_{n,k}^* = \sum_{i=1}^{n}(C_{n,k}^*(U_{i,k}^*) - B_{m,k}^{**}(U_{i,k}^*))^2$.
  end if
end for
An approximate p-value for the test is then given by $\sum_{i=1}^{N} 1(S_{n,k}^* > S_n)/N$.

[2007](Appendix A)).

Remarks:

- The goodness-of-fit test shows only tendencies in the data and is not a test which identifies the copula. Notice that if one runs the goodness-of-fit test of the "copula"-package for the same data several times, one gets different results. (see example below)
- The goodness-of-fit test is implemented in the R-package "copula" for the Gaussian, t-, Clayton, Gumbel and Frank copula.

Genest et al. [2007] mentioned other statistics to compute the goodness-of-fit test, which are based on Kendall's transform:

$$X \mapsto V = H(X) = C(U_1; \ldots; U_d),$$

where $U_i = F_i(X_i)$ for $i \in \{1, \ldots, d\}$ and the joint distribution of $U = (U_1, \ldots, U_d)$ is C. Now under H_0, the vector $U = (U_1, \ldots, U_d)$ is distributed as C_θ for some θ, and hence the Kendall transform $C_\theta(U)$ has distribution K_θ. Through a measure of distance between K_n and a parametric estimation K_{θ_n} of K, one can test

$$H_0 : K \in K_0 = \{K_\theta : \theta \in \Theta\}.$$

Consequently, tests based on the empirical process

$$\mathbb{K}_n = \sqrt{n}(K_n - K_{\theta_n})$$

are the test statistics

$$S_n^{(K)} = \int_0^1 \mathbb{K}_n(v)^2 dK_{\theta_n}(v)$$

and

$$T_n^{(K)} = sup_{v \in [0,1]} |\mathbb{K}_n(v)|,$$

where θ_n is the estimate of θ.

2.6.2 Akaike information criterion

The Akaike information criterion (AIC), developed by Akaike [1974], is one of the mostly used model selection criterion. The AIC is not a test of the model in the sense of hypothesis testing; rather it is a test between models - a tool for model selection. Given a data set, several competing models may be ranked according to their AIC, with the one having the lowest AIC being the best. The AIC is defined by

$$AIC = -2l(\hat{\theta}|x) + 2k,$$

where $l(\cdot)$ is the log-likelihood function, $\hat{\theta}$ the estimated parameter vector of the model, x the empirical data and k the length of the parameter vector. (see Burnham and Anderson [2004])
The first part $(-2l(\hat{\theta}|x))$ is a measure of the goodness-of-fit of the selected model and the second part is a penalty term, penalizing the complexity of the model. Originally the AIC was developed to compare nested regression models with different number of covariables (length of $\hat{\theta} = k$).
The intuition behind this approach is that the goodness-of-fit increases if one increases the number of covariables. But concurrently the complexity of the model increases. At the beginning, out of the null-model, the AIC decreases with increasing goodness-of-fit/log-likelihood and increases if the number of parameters becomes greater than the gain of the log-likelihood.
One disadvantage of this criterion is that it doesn't account the number of observations in the the penalty term. If the number of observation is large than the improvement of the log-likelihood is "easier" as for a small data set.

2.6.3 Bayesian information criterion

In contrast to the AIC the Bayesian information criterion (BIC) or Schwarz Criterion (SBC), developed by Schwarz [1978], comprise the number of observations in the penalty term. Thus in BIC, the penalty for additional parameters is stronger than that of the AIC. Apart from that the BIC is similar to the AIC and is defined as

$$BIC = -2l(\hat{\theta}|x) + k\log(n),$$

where $l(\cdot)$ is the log-likelihood function, $\hat{\theta}$ the estimated parameter vector of the model with length k and x the data vector of length n (see Burnham and Anderson [2004]).
As the AIC, the first term is a measure of the goodness-of-fit and the second part is a penalty term, comprising the number of parameters as well as the number of observations.

2.6.4 Vuong- and Clarke-test

The Vuong- and the Clarke-test (Vuong [1989], Clarke [2007]) are tests to compare two models, which are not necessarily nested. Both are based on the likelihood or rather on their likelihood ratio and the Kullback-Leibner information criterion (KLIC). The KLIC is a measure for the distance between two statistical models and is defined as

$$KLIC := E_0[\log h_0(Y_i|x_i)] - E_0[\log f(Y_i|x_i,\hat{\beta})], \tag{2.35}$$

where $h_0(\cdot|\cdot)$ is the unknown true conditional probability function of Y_i given x_i. Here E_0 is the expected value under the true model and $\hat{\beta}$ is the estimator (vector) for the parameter (vector) β in the model $f(Y_i|x_i,\hat{\beta})$, which has not to be the true model.
The model with the minimum KLIC, i.e. the smallest distance, is the best one.
The goal is to compare two models with probability functions $f_1((Y_i|x_{i,1},\hat{\beta}_1)$ and $f_2((Y_i|x_{i,2},\hat{\beta}_2)$.
If the model 1 is better than the model 2 it has to be true that

$$E_0\left[\log\frac{f_1(Y_i|x_{i,1},\hat{\beta}_1)}{f_2(Y_i|x_{i,2},\hat{\beta}_1)}\right] > 0. \tag{2.36}$$

Vuong-test

To compare two models and on the basis of the likelihood ratio from above, Vuong [1989] defined and calculated the following statistics: (see also Erhardt [2006])

$$m := (m_1, \dots, m_n)^t$$

with

$$m_i := \log \frac{f_1(Y_i|x_{i,1}, \hat{\beta}_1)}{f_2(Y_i|x_{i,2}, \hat{\beta}_1)}$$

for $i = 1 \dots n$ and the expected value

$$E_0[m] = \mu_0^m = (\mu_1^m, \dots, \mu_n^m)^t.$$

The null-hypothesis of Vuong is

$$H_0 : \mu_0^m = 0 \text{ versus } H_1 : \mu_0^m \neq 0,$$

where μ_0^m is known. Additionally he defined the test statistic

$$\nu := \frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n m_i\right)}{\sqrt{\frac{1}{n}\sum_{1=1}^n (m_i - \bar{m})^2}} \tag{2.37}$$

with $\bar{m} = \frac{1}{n}\sum_{i=1}^n m_i$ and showed that under H_0 ν converges in distribution to a standard normal distribution, i.e.

$$\nu \xrightarrow{D} N(0,1).$$

One big disadvantage of the Vuong-test is that he does not account for the number of parameters in the models and that the number of parameters can differ between the models. Therefore Vuong suggests to correct the log-likelihood ration with the correction term of either Schwarz's Bayesian information criteria or Akaike's information criteria (see Vuong [1989])

$$\frac{p}{2n}\log n - \frac{q}{2n}\log n \quad \text{(Schwarz's Bayesian information criteria)}$$

or

$$p - q \quad \text{(Akaike's information criteria)},$$

where p is the number of parameters of model 1 and q the number of parameters of model 2. n is the number of observations.

We chose the Vuong-test with the Schwarz's Bayesian information criteria. Thus the log-likelihood ratio is as follows:

$$\log f_1(Y_i|x_{i,1}, \hat{\beta}_1) - \log f_2(Y_i|x_{i,2}, \hat{\beta}_2) - \left(\frac{p}{2n}\log n - \frac{q}{2n}\log n\right).$$

As for every other test one is interested in the quality of the test, which is usually be measured by the p-value.

Remember, the null hypothesis was $H_0 : \mu_0^m = 0$ vs. $H_1 : \mu_0^m \neq 0$ and we can reject the hypothesis at the significance level of $\alpha\%$ if $|\nu| \geq z_{1-\frac{\alpha}{2}}$, i.e. the smallest α for which we can reject the null hypothesis is

$$\alpha = 2\Phi(-|\nu|),$$

where Φ is the distribution function of the standard normal distribution.
Thus the p-value for the Vuong test is

$$p - value = 2\Phi(-|\nu|).$$

Clarke test

The Clarke-test is as the Vuong-test a test for model selection based on KLIC. The difference between the Vuong- and the Clarke-test is the null hypothesis. The null hypothesis of the Clarke-test is:

$$H_0 : P\left(\log\left(\frac{f_1(Y_i|x_{i,1}, \hat{\beta}_1)}{f_2(Y_i|x_{i,2}, \hat{\beta}_2)}\right) > 0\right) = p.$$

If the models are equivalent then p has to be 0.5.
The intuition behind this null hypothesis is, that the log-likelihood ration of the single observation are uniformly distributed around zero and in expectation one half of the log-likelihood ratios has to be greater than zero.

The resulting test statistic is

$$B = \sum_{i=1}^{n} 1_{(0,+\infty)}(d_i), \tag{2.38}$$

where $1_{()}$ is the indicator function and

$$d_i = \log(f_1(Y_i|x_{i,1}, \hat{\beta}_1)) - \log(f_1(Y_i|x_{i,1}, \hat{\beta}_1)).$$

Clarke [2007] proved that B can be interpreted as a binomial distributed random variable with parameter n and $p = 0.5$, i.e.

$$B \sim Binom(n, p).$$

Therefore the model 1 is equivalent to model 2 if B is equal to the expected value $np = \frac{n}{2}$.

One can assume the equivalence of both models at significance level α, if

$$B \in \left(\frac{n}{2} - \epsilon_\alpha, \frac{n}{2} + \epsilon_\alpha\right)$$

with c_α small. Using notation $c_{\alpha+} = \frac{n}{2} + \epsilon_\alpha$ and $c_{\alpha-} = \frac{n}{2} - \epsilon_\alpha$ we can formulate the expression

$$B \in (c_{\alpha-}, c_{\alpha+}).$$

Generally, it is difficult to construct a two sided test for the Clarke testing problem. Thus Djunushalieva [2010] split the testing problem into two cases: the upper tail test

$$H_0 : B = \frac{n}{2} \text{ vs. } H_1 : B > \frac{n}{2}$$

and the lower tail test

$$H_0 : B = \frac{n}{2} \text{ vs. } H_1 : B < \frac{n}{2}.$$

For the upper tail testing problem one can determine the error probability of type 1 as

$$P(B \geq c_{\alpha+}) \leq \alpha$$

and thus

$$c_{\alpha+} = 1 + z_{bin}(1 - \alpha),$$

where z_{bin} is the quantile function of the binomial distribution function with parameters n and $p = 0.5$.
The p-value is the smallest significance level at which the test rejects H_0. Hence the p-value for the upper tail test is:

$$p - value = 1 - \mathcal{B}(B - 1),$$

where $\mathcal{B}$ is the binomial distribution function. Analogue one can compute the p-value for the lower tail test as:

$$p - value = \mathcal{B}(B).$$

Remarks:

- As in the Vuong-test, the Clarke-test consider only the likelihoods and not the model specifications, i.e. the different number of parameters.
- One can correct this problem by Schwarz's Bayesian information criteria or by Akaike's information criteria.
- It turns out, that the Vuong-test is more conservative than the Clarke-test.
- In the paper of Clarke [2007] are some efficients tests between the Vuong- and the distribution free test (Clarke-test), i.e. which test deliver more often the correct model. It turns out that the quality of the test with respect to the other test depends on the the form of the distribution of the individual log-likelihoods, which is be set up by the kurtosis.
  More to this topic can be found in Erhardt [2006].

2.6.5 Goodness-of-fit test based on Vuong and Clarke

The Vuong- as well as the Clarke-test compare two models against each other and as a result of their hull-hypothesis, test-statistic and p-value one can say if one model is better than the other or if non of them can be favored. In the goodness-of-fit test Djunushalieva [2010] developed, we assume that we can use this results for bivariate copula selection. If we favor model 1 we get a score value of "+1" and if we favor model 2 we get a score value of "-1" and a "0" if non of them can be favored.
The models we compare against each other are just bivariate models, i.e. one copula, with one parameter, or two for the t-, BB1- and BB7-copula. We want to make a proposition whether one copula fits the data better than the other copulas.
Therefore we compare all possible copulas, in our case the Gaussian, t-, Clayton, Gumbel, Frank, BB1- and BB7-copula, against each other in a Vuong- and Clarke-test. E.g. if one wants to test the hypothesis that the Gaussian copula fits the data best, then we compare the Gaussian copula against all other possible copulas. For every test in which the Gaussian copula is preferred we get a score of "+1". If we sum up all the single scores of the tests ($n-1$ tests if we have n possible copulas) we get a score for the Gaussian copula.
Now we compute this score for all copulas, i.e. for each copula we set the hypothesis "Copula C_i is better than all other copulas", which is nothing else than $n-1$ times the hypothesis

$$H_0 : \text{Copula } C_i \text{ is better than copula } C_j \quad \forall j \in [1, \dots, i-1, i+1, \dots, n],$$

where n is the number of possible copulas, i.e. the number of copulas one want to compare. And thus we get a score for the different copula families based on the Vuong- and the Clarke-test, which we can compare.

Example 2.42
If we get the following table (Table 2.6) as an output of our test, we make the assumption that the t-copula fits the data best, because the t-copula has the highest score.

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	3	**6**	-6	2	0	-1	-4
Clarke	3	**6**	-6	1	2	-2	-4

Table 2.6: Example for a goodness-of-fit table based on the Vuong- and Clarke-test

As a further result one can see that the Gaussian copula is the second best copula family, which is not very surprising if we remember that the t-copula converges against the Gaussian copula.
The score value of 6 means that in the 6 Vuong-tests the t-copula was six times better than the other copula.
The negative score of the Clayton, BB1- and BB7-copula is the result of single tests, i.e. the Clayton, BB1- and BB7-copula are more often beaten by the comparing copula than vise versa.

2.7 Time series analysis

In this section we introduce some basics of the time series analysis, which we will need in the following to model our data. Therefore we will launch some basic definitions and two important models, the ARMA- and the GARCH-model. The leading references are the books of Brockwell and Davis [2003] and Brockwell and Davis [1991].

2.7.1 Introduction and basics

First we will introduce some basics, which are important for the main idea of the time series analysis. In the following $(X_t)_{t\in\mathbb{Z}}$ will denote a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and T a time domain (here $T = \mathbb{Z}$).
A time series is a set of observations x_t, each one being recorded at a specified time $t \in T$. Its most characteristic feature is the chronical order, which must be taken into account when modeling/analyzing time series.

Definition 2.43 (Autocovariance function)
Let $(X_t)_{t\in T}$ be a stochastic process and let $E[X_t^2] < \infty, t \in T$. The autocovariance function of $(X_t)_{t\in T}$ is defined as

$$\gamma_X : T \times T \to \mathbb{R}, (s,t) \longmapsto Cov(X_s, X_t)$$

Properties of γ_X:

- $\gamma_X(s,t) = \gamma_X(t,s)$ (symmetry)
- $\gamma_X(t,t) = Var(X_t) > 0$
- $\gamma_X(s,t)^2 \leq \gamma_X(s,s)\gamma_X(t,t) \quad s,t \in T$

Empirical analogue of the autocovariance function are used as an estimator for γ_X.

Definition 2.44
Let $(X_t)_{t\in\mathbb{Z}}$ of which we observe $x_1, \ldots, x_n$ for some $n \in \mathbb{N}$.
The sample/empirical autocovariance function based on $x_1, \ldots, x_n$ is the random function $\hat{\gamma} := \hat{\gamma}_{x_1,\ldots,x_n} : -n+1, \ldots, n-1 \to \mathbb{R}, h \mapsto \frac{1}{n}\sum_{j=1}^{n-|h|}(x_{j+|h|} - \bar{x})(x_j - \bar{x})$,
where $\bar{x} = \sum_{i=1}^{n} x_i$ is the sample mean of $x_1, \ldots, x_n$.

Definition 2.45 (Stationarity)
Let $(X_t)_{t\in\mathbb{Z}}$ be a $\mathbb{R}$-valued process (time series) on $(\Omega, \mathcal{F}, \mathcal{P})$ with time domain $\mathbb{Z}$.
If $(X_{t_1}, \ldots, X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, \ldots, X_{t_n+h})$ for all $n \in \mathbb{N}$, $t_1, \ldots, t_n, h \in \mathbb{Z}$ then the time series $(X_t)_{t\in\mathbb{Z}}$ is called strictly stationary.

$(X_t)_{t\in\mathbb{Z}}$ is called (weak) stationary if it satisfies

(i) $E[X_t^2] < \infty$ for all $t \in \mathbb{Z}$,

(ii) $E[X_t] \equiv m$ *for some* $m \in \mathbb{R}, \forall t \in \mathbb{Z}$,

(iii) $\gamma_X(s,t) = \gamma_X(s+h, t+h) \quad \forall s,t \in \mathbb{Z}, h \in \mathbb{Z}$.

Lemma 2.46
Let $(X_t)_{t\in\mathbb{Z}}$ *be strictly stationary* $\Rightarrow (X_t)_{t\in\mathbb{Z}}$ *stationary if* $E[X_t^2] < \infty \quad \forall t \in T$.

We will only use the weak form of stationarity. From the definition it follows that $\gamma_X(r,s) = \gamma_X(r-s, 0) \quad \forall r,s \in \mathbb{Z}$. Hence we define

$$\gamma_X(h) = \gamma_X(h,0) = Cov(X_{t+h}, X_t) \quad \forall t,h \in \mathbb{Z},$$

i.e. $\gamma_X(h)$ is the covariance between observations with distance h. We call $\gamma_X(h)$ the autocovariance function of the weak stationary stochastic process $(X_t)_{t\in\mathbb{Z}}$.

Definition 2.47 (White noise)
Let $(Z_t)_{t\in\mathbb{Z}}$ *be a weak stationary stochastic process with* $E[Z_t] = 0 \quad \forall t \in \mathbb{Z}$ *and the autocovariance function*

$$\gamma_Z(h) = \begin{cases} \sigma_Z^2 & for\ h = 0, \\ 0 & for\ h \neq 0. \end{cases}$$

Then $(Z_t)_{t\in\mathbb{Z}}$ *is called white noise with expected value 0 and variance* σ_Z^2.
Notation: $(X_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma_Z^2)$

Definition 2.48 (Backward shift operator)
Let $(X_t)_{t\in\mathbb{Z}}$ *be a stochastic process. Then the backward shift operator B is defined by*

$$BX_t = X_{t-1} \quad \forall t \in \mathbb{Z}.$$

The powers of B are defined iteratively by $B^j X_t = X_{t-j}$.

2.7.2 The ARMA-model

Autoregressive Moving Average-Models (ARMA-models) became very popular in the last years to model time series. Now we want to introduce this model shortly. More information and details can be found in the books of Brockwell and Davis [2003] and Brockwell and Davis [1991].

Definition 2.49 (The ARMA(p,q) process)
Let $(Z_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma^2)$ *for some* $\sigma^2 \in \mathbb{R}_+$ *be a white noise sequence on* $(\Omega, \mathcal{F}, \mathcal{P})$. *Let* $p, q \in \mathbb{N}_0$ *and* $\Phi_1, \ldots, \Phi_p, \Theta_1, \ldots, \Theta_q \in \mathbb{R}$. *Then any stationary time series* $(X_t)_{t\in\mathbb{Z}}$ *on* $(\Omega, \mathcal{F}, \mathcal{P})$ *satisfying* $E[X_0] = 0$ *and*

$$X_t - \Phi_1 X_{t-1} - \ldots - \Phi_p X_{t-p} = Z_t + \Theta_1 Z_{t-1} + \ldots + \Theta_q Z_{t-q} \quad \forall t \in \mathbb{Z} \tag{2.39}$$

is called an ARMA(p,q) process (w.r.t. $(Z_t)_{t\in\mathbb{Z}}$*).*
A solution to (2.39) is called causal, if $X_t = f(Z_t, Z_{t-1}, \ldots)$ *with f an appropriate measurable function.*

Remarks:

(i) More compact notation based on the backward shift operator B.
Defining the polynomials

$$\Phi(z) = 1 - \Phi_1 z - \ldots - \Phi_p z^p \quad \text{“autoregressive polynom of degree p”}$$
$$\Theta(z) = 1 + \Theta_1 z + \ldots + \Theta_q z^q \quad \text{“moving average polynom of degree q”}$$

we have, for all $t \in \mathbb{Z}$

$$\Phi(B)X_t = \Theta(B)Z_t \quad \forall t \in \mathbb{Z}. \tag{2.40}$$

(ii) If $E[X_0] = \mu$ and $(X_t - \mu)_{t\in\mathbb{Z}}$ is an ARMA(p,q) process, then $(X_t)_{t\in\mathbb{Z}}$ is called an ARMA(p,q) process with mean μ.

Sometimes one uses only the moving average or autoregressive part of the ARMA equation.

Example 2.50 (AR(p)-process)
In the setting of 2.39 let $q = 0$ and hence $\Theta(z) = 1$. Then the equation simplifies to

$$X_t - \Theta_1 X_{t-1} - \ldots - \Theta_p X_{t-p} = Z_t \quad \forall t \in \mathbb{Z}$$
$$\Theta(B)X_t = Z_t \quad \forall t \in \mathbb{Z}.$$

And for $p = 1$ we get the AR(1)-process

$$X_t - \Theta_1 X_{t-1} = Z_t.$$

One can show that for $|\theta_1| < 1$ there is exactly one stationary zero-mean solution $(X_t)_{t\in\mathbb{Z}}$: $X_t = \sum_{j=0}^{\infty} \Theta_1^j Z_{t-j}$.
If $|\theta_1| > 1$ the AR(1)-process is $(X_t)_{t\in\mathbb{Z}} : X_t = -\sum_{j=0}^{\infty} \Theta_1^{-j} Z_{t+j}$.
And for $|\Theta_1| = 1$ there exists no solution.

Theorem 2.51
If $\Phi(z) \neq 0\ \forall z \in \mathbb{C}$ with $|z| \leq 1$, then the ARMA-equation (2.40) has the unique stationary solution

$$X_t = \sum_{j=-\infty}^{\infty} \Psi_j Z_{t-j}, \quad \forall t \in \mathbb{Z},$$

where the coefficients $(\Psi_j)_{j\in\mathbb{Z}}$ are defined by $\Psi(z) = \sum_{j=-\infty}^{\infty} \Psi_j z^j = \frac{\Theta(z)}{\Phi(z)}$ and the sum converges absolutely for $r^{-1} < |z| < r$ with $r > 1$.

Proof:
See Brockwell and Davis [1991] □

Theorem 2.52
Let $(Z_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma_Z^2)$ for some $\sigma_Z^2 \in \mathbb{R}_+$ and $X_t = \sum_{j=-\infty}^{\infty} \Psi_j Z_{t-j},\ \forall t \in \mathbb{Z}$ be a weak stationary stochastic process. Then the autocovariance function of $(X_t)_{t\in\mathbb{Z}}$ is given by

$$\gamma_X(h) = \sum_{k=-\infty}^{\infty} \Psi_k \Psi_{k+h} \sigma_Z^2, \quad h \in \mathbb{Z}.$$

Proof:
Because of the assumption $E[Z_t] = 0\ \forall t \in \mathbb{Z}$ it follows that $E[X_t] = 0\ \forall t \in \mathbb{Z}$. Hence

$$\begin{aligned}
\gamma_X(h) = E[X_{t+h}X_t] &= E\left[\sum_{j=-\infty}^{\infty} \Psi_j Z_{t+h-j} \sum_{k=-\infty}^{\infty} \Psi_k Z_{t-k}\right] \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \Psi_j \Psi_k E[Z_{t+h}Z_{t-k}] = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \Psi_j \Psi_k \gamma_Z(h-j+k) \\
&\overset{j=h+k}{=} \sum_{k=-\infty}^{\infty} \Psi_k \Psi_k + h\sigma_z^2.
\end{aligned}$$

□

Example 2.53 (ARMA(1,1))
Let $(X_t)_{t\in\mathbb{Z}}$ be an ARMA(1,1) process with

$$X_t - 0.5X_{t-1} = Z_t + 0.4Z_{t-1}, \quad t \in \mathbb{Z},\ (Z_t)_{t\in\mathbb{Z}} \sim WN(0, \sigma_z^2),$$

i.e. $\Phi_1 = 0.5$ and $\Theta_1 = 0.4$. The characteristic polynoms are

$$\Phi(z) = 1 - 0.5z \text{ and } \Theta(z) = 1 + 0.4z,$$

where $\phi(z)$ has the only zero for $z = 2$ and so there exists a unique stationary solution by Theorem 2.51. The needed coefficients we get by

$$(1 - 0.5z) \sum_{j=-\infty}^{\infty} \Psi_j z^j = 1 + 0.4z$$

and coefficient composition

$$\begin{aligned}
1 \cdot \Psi_0 = 1 &\Rightarrow \Psi_0 = 1, \\
-0.5z + \Psi_1 z = 0.4z &\Rightarrow \Psi_1 = 0.9, \\
\Psi_2 z^2 - 0.5\Psi_1 z^2 = 0 &\Rightarrow \Psi_2 = 0.5 \cdot 0.9 = 0.45, \\
\Psi_3 z^3 - 0.5\Psi_3 z^3 = 0 &\Rightarrow \Psi_3 = 0.5 \cdot 0.5 \cdot 0.9, \\
&\vdots
\end{aligned}$$

Hence

$$X_t = Z_t + 0.9 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j-1} Z_{t-j},\ t \in \mathbb{Z}.$$

2.7.3 The GARCH-model

The ARMA-models are on the basis of white noise with variance σ_Z^2. But sometimes, especially in finance, observed time series exhibit features not in line with the behavior of ARMA processes with i.i.d. noise. At first sight it might even look being non-stationary.

They have clusters of high volatility and clusters of low volatility, the magnitude of the changes (=variance/volatility) changes over time, i.e. the variance is conditional. This is in contrast to the constant variance of noise in ARMA-processes. To model this behavior Bollerslev [1986] introduced the Generalized Autoregressive Conditional Heteroskedasticity model (GARCH-model), which is a generalized form of the ARCH-models introduced by Engle [1982].
The following definition of GARCH (2.41) can be found in Franke et al. [2001, pp.219-220].

Definition 2.54 (GARCH(p,q))
The real-valued stochastic process $(\epsilon_t)_{t\in\mathbb{Z}}$ *is a GARCH(p,q), if* $E[\epsilon_t|F_{t-1}] = 0$,

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{2.41}$$

and

- $Var(\epsilon_t|F_{t-1}) = \sigma_t^2$, $\epsilon_t = \sigma_t Z_t$ *and* Z_t *is i.i.d. (strong GARCH),*
- $Var(\epsilon_t|F_{t-1}) = \sigma_t^2$ *(semi-strong GARCH) or*
- $P(\epsilon_t^2|1, \epsilon_{t-1}, \epsilon_{t-2}, \ldots, \epsilon_{t-1}^2, \epsilon_{t-2}^2, \ldots) = \sigma_t^2$ *(weak GARCH)*

with

$$p \geq 0, \quad q > 0, \quad \omega > 0, \quad \alpha_i \geq 0,\ i = 1, \ldots, q, \quad \beta_j \geq 0,\ j = 1, \ldots, p,$$

where F_t *is the information set at time t.*

In the following we assume that the assumptions of a strong GARCH are fulfilled.
Using the backward shift operator L we can transform the equation (2.41) to

$$\sigma_t^2 = \omega + A(L)\epsilon_t^2 + B(L)\sigma_t^2 \tag{2.42}$$

with

$$\begin{aligned} A(L) &= \alpha_1 L + \alpha_2 L^2 + \ldots + \alpha_q L^q \\ B(L) &= \beta_1 L + \beta_2 L^2 + \ldots + \beta_p L^p. \end{aligned} \tag{2.43}$$

Remarks:

(i) Usually one assumes $E[Z_t] = 0, Var(Z_t) = 1$ and often even $Z_t \sim \mathcal{N}(0,1)$ (or $t_\nu(0,1)$). But this is not needed for many results.

(ii) Assume σ_t^2 is a measurable function of $Z_{t-1}, Z_{t-2}, \ldots$ and stationary with $E[\sigma_t^2] < \infty$ and $E[Z_0] = 0$. Then X_t is stationary and $E[X_t|\sigma_t^2] = 0, Var(X_t|\sigma_t^2) = \sigma_t^2$. It is easy to see that X is white noise (even so called martingale difference sequence).

(iii) Hence, X can be used as the noise term in an ARMA process, i.e. ARMA with GARCH noise (ARMA-GARCH).
A very popular time series model in various applications. We will use this model later in our data set.

Theorem 2.55
The GARCH(p,q)-process (2.41) is stationary with $E[\epsilon_t] = 0, Var(\epsilon_t) = \omega(1 - A(1) - B(1))^{-1}$ *and* $Cov(\epsilon_t, \epsilon_s) = 0$ *for* $t \neq s$, *if and only if* $A(1) + B(1) < 1$.

An equivalent description of the GARCH(p,q) is given by

$$\epsilon_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \epsilon_{t-j}^2 - \sum_{j=1}^{p} \beta_j \nu_{t-j} + \nu_t \tag{2.44}$$

and

$$\nu_t = \epsilon_t^2 - \sigma_t^2 = (Z_t^2 - 1)\sigma_t^2. \tag{2.45}$$

Example 2.56 (GARCH(1,1))
The GARCH(1,1) is given by

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad \omega > 0, \alpha_1 \geq 0, \beta_1 \geq 0$$

and is stationary for $\alpha_1 + \beta_1 < 1$ *by Theorem 2.55.*
A necessary and sufficient condition for the existence of the 2m-th moments is by Bollerslev [1986]

$$\mu(\alpha_1, \beta_1, m) = \sum_{j=0}^{m} \binom{m}{j} a_j \alpha_1^j \beta_1^{m-j} < 1,$$

with

$$a_0 = 1, \quad a_j = \prod_{i=1}^{j} (2j - 1), \quad j = 1, 2, \ldots$$

The 2m-th moment can be determined by the following recursive formula:

$$E[\epsilon_t^{2m}] = a_m \left[\sum_{n=0}^{m-1} a_n^{-1} E[\epsilon_t^{2m}] \omega^{m-n} \binom{m}{m-n} \mu(\alpha_1, \beta_1, n) \right] \cdot [1 - \mu(\alpha_1, \beta_1, m)]^{-1}.$$

If the 2m-th moment exists it follows by symmetry that $E[\epsilon_t^{2m-1}] = 0$. *Thus we can calculate the four first moments of the GARCH(1,1) process.*

$$\begin{aligned} &E[\epsilon_t] = 0, \\ &E[\epsilon_t^2] = Var(\epsilon_t) = \omega(1 - \alpha_1 - \beta_1)^{-1}, \\ &E[\epsilon_t^3] = 0, \\ &E[\epsilon_t^4] = 3\frac{\omega^2 + 2\omega^2(\beta_1 + \alpha_1)(1 - \alpha_1 - \beta_1)^{-1}}{1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2}. \end{aligned}$$

And the covariance function simplifies to

$$\gamma_{\epsilon^2}(n) = Cov(\epsilon_t^2, \epsilon_{t-n}^2) = (\alpha_1 + \beta_1)\gamma_{\epsilon^2}(n - 1) = (\alpha_1 + \beta_1)^n \gamma_{\epsilon^2}(0).$$

2.7.4 GARCH-skew-t model

Financial data often show three main properties: volatility clustering, negative skewness and leptokurtosis (thick tails). These facts are the 2nd, 3rd and 4th moment. As we have seen in the GARCH part above we can capture the volatility clustering very well by the GARCH or ARMA-GARCH models with Gaussian error. But these models are unable to account asymmetric mass in the tail parts of distribution.
Thus some alternative error distributions underlying the GARCH-model are explored to overcome the limitation of traditional GARCH-models. To mention a few of them:

- Symmetric Stable distribution by McCulloch (1985)
- Student-t by Bollerslev (1987)
- Normal Inverse Gaussian by Barndorff-Nielsen (1997), Andersson (2001), Jensen & Lunde (2001)
- The Skew-t by Aas & Haff (2006), Kim & McCulloch (2007)
- and many other.

Here we give only a brief definition of the skew-t model.
The following definition can be found in Kim [2008, Chapter 2].

Definition 2.57 (GARCH(1,1) with skew-t errors)
Consider the assumptions of the GARCH(1,1) process from above.
We have a GARCH(1,1) model with skew-t errors if

$$\begin{aligned} \epsilon_t &= \mu + \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ Z_t &\sim Skewt(\mu^*, \sigma^*, \nu, \bar{\beta}) \end{aligned}$$

with $E[Z_t] = 0$ *and* $Var(Z_t) = 1$
where μ^* *and* σ^* *are set so that* ϵ_t *has zero mean and unit variance. The specification allows* σ_t *to be interpreted as the conditional standard deviation of return* ϵ_t*. Note that* $\bar{\beta} = \beta\sigma^*$ *and* ν *are invariant parameters which make* $Skewt(\mu^*, \sigma^*, \nu, \bar{\beta})$ *the location-scale family - i.e.* $(Z_t - \mu^*)/\sigma^* \sim Skewt(0, 1, \nu, \bar{\beta})$.

Hence we have a more appropriate instrument to fit financial data with volatility clustering, negative skewness and thick tails.
We will use this model for some time series of our data set.

2.7.5 Ljung-Box-Test

To test the goodness-of-fit of the estimated ARMA models for given time series Ljung and Box [1978] developed a test, the so called Ljung-Box-Test, which tests the independence of the residuals. If there is no dependence among between these residuals, then we can

regard them as observations of independent random variables, and there is no further modeling to be done. Otherwise we have to look for more complex stationary time series model for the noise that accounts for the dependence.
Let (X_t) be a discrete time series, which is generated by a stationary ARMA(p,q) process

$$\Phi(B)X_t = \Theta(B)Z_t,$$

where (Z_t) is a sequence of independent and identical distributed $\mathcal{N}(0, \sigma^2)$ random variables. After fitting a model of such form for a time series $X_1, \ldots, X_n$ we look at the residuals $\hat{Z}_t, \ldots, \hat{Z}_t$ and their autocorrelation

$$\hat{r}_k = \sum_{t=k+1}^{n} \hat{Z}_t \hat{Z}_{t-k} \Big/ \sum_{t=1}^{n} \hat{Z}_t^2 \quad (k = 1, 2, \ldots)$$

to test the goodness-of-fit. If the model is chosen adequate and the parameters are known, then the statistic from Ljung and Box [1978]

$$\tilde{Q}(r) = n(n+2) \sum_{k=1}^{m} (n-k)^{-1} r_k^2$$

with

$$r_k = \sum_{t=k+1}^{n} Z_t Z_{t-k} \Big/ \sum_{t=1}^{n} Z_t^2$$

is $\mathcal{X}_m^2$ distributed for large n (m = degree of freedom of the Chi-distribution function).
Sketch of proof: The marginal distribution of $r = (r_1, \ldots, r_m)^T$ is multivariate normal with expected value vector 0, $Var(r_k) = (n-k)/[n(n+2)]$ and $Cov(r_k, r_l) = 0$ for $k \neq l$.
If the $p+q$ parameter of the ARMA-model are unknown and have to be estimated one has to substitute r_k by $\hat{r}_k$. If we have a good estimator the statistic $\tilde{Q}(\hat{r}_k)$ should be $\mathcal{X}_{m-p-q}^2$ distributed, see Ljung and Box [1978].
The aim of a good ARMA-model is to have a p-value of this statistic > 0.05,
i.e. if p-value < 0.05 the null hypothesis can be rejected.

Chapter 3

Copula parameter estimation via pseudo likelihood for C-Vine models

In this chapter we want to estimate the copula parameters of a C-Vine on the basis of maximum likelihood. Therefore we will define first the likelihood in general and then for the C-Vine in particular. Developing an appropriate representation of the likelihood an algorithm can be build for fast and efficient calculation.

3.1 Definitions

Definition 3.1
Let $(\Omega, \mathcal{F}, \mathcal{P} = (P_\theta)_{\theta\in\Theta})$ be a discrete or continuous statistic model with density φ_θ of $P_\theta, \theta \in \Theta$. If

$$T(x) = \arg\max_{\theta\in\Theta} \varphi_\theta(x) \qquad \forall x \in \Omega,$$

T is called maximum-likelihood estimator for θ. The function $\theta \longmapsto \varphi_\theta(x)$ is called likelihood-function for the sample $x \in \Omega$.

Remark:
In many cases it is easier to maximize $\log \varphi_\theta(x)$ instead of $\varphi_\theta(x)$.

Properties: (see Fahrmeir et al. [2008, p.475])

(i) ML-estimators are consistent, i.e. $\lim \hat{\theta} \overset{P}{=} \theta$.

(ii) ML-estimators are asymptotically normal distributed:
$\sqrt{N}(\hat{\theta} - \theta) \xrightarrow{d} N(\theta, I^{-1}(\theta))$ with I=information matrix.

(iii) ML-estimators are asymptotically unbiased.

(iv) ML-estimators are asymptotically efficient, i.e. the variance of $\hat{\theta}$ reaches asymptotically the minimum variance, which a unbiased estimator can reach.

(v) Invariance: Let $g(\theta)$ be a continuous function. Then the ML-estimator of $g(\theta)$ will be $g(\hat{\theta})$.

But let us once more focus on the canonical vine.
Let $x_i = (x_{i,1}, \ldots, x_{i,T})^T$, $i = 1, \ldots, n$ be a set of data with T points of time.
To make things easier we assume that the T observations of every variable are independent over time. Additionally we assume that every random variable $x_{i,t}$ is uniformly distributed on [0,1]. Having this assumption in mind, we will call the likelihood pseudo-likelihood in the following.
Aas et al. [2009] calculated the log-likelihood for the C-vine as follows:

$$\sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\sum_{t=1}^{T} \log[c_{j,j+i|1,\ldots,j-1}\{F(x_{j,t}|x_{1,t},\ldots,x_{j-1,t}), F(x_{j+i,t}|x_{1,t},\ldots,x_{j-1,t}); \theta_{j,i}\}]. \quad (3.1)$$

One gets this representation of the log-likelihood easily from the Definition 3.1 and the definition of the density of the C-Vine (2.21).
For each copula in the sum there is at least one parameter to be determined. The number depends on which copula type is used, e.g. Gaussian-copula has one parameter and the t-copula has two.
The conditional distributions are determined using the relationship (2.14) and (2.15).
The log-likelihood must be numerically maximized over all parameters.
To develop an algorithm, we transform the log-likelihood in such a way that we get a recursion.

3.2 Log-likelihood recursion

The log-likelihood (3.1) can be written as:

$$l(x,\theta) = \sum_{t=1}^{T}\left[\sum_{i=1}^{n-1} log\{c(x_{1,t}, x_{i+1,t}; \theta_{1,i})\} + \sum_{j=2}^{n-1}\sum_{i=1}^{n-j} log\{c(v_{j-1,i+1,t}, v_{j-1,1,t}; \theta_{j,i})\}\right] \quad (3.2)$$

for all $t = 1, \ldots, T$ and

$$\begin{aligned} v_{1,i,t} &= h(x_{i+1}, x_1; \theta_{1,i}) \quad i = 1, \ldots, n-1 \\ v_{j,i,t} &= h(v_{j-1,i+1,t}, v_{j-1,1,t}; \theta_{j,i}) \quad j = 2, \ldots, n-1 \quad i = 1, \ldots, n-j, \end{aligned}$$

where θ is the vector of the parameters we are looking for and $\theta_{j,i}$ is the set of parameters of the corresponding copula density $c_{j,j+i|1,\ldots,j-1}(\cdot,\cdot)$. Further, $h(\cdot)$ as in (2.15) and

$$v_{j,i,t} = F(x_{i+j,t}|x_{1,t}, \ldots, x_{j,t}; \theta_{j,i}). \quad (3.3)$$

Proof:

$$
\begin{aligned}
l(x,\theta) &\overset{(3.1)}{=} \sum_{j=1}^{n-1}\sum_{i=1}^{n-j}\sum_{t=1}^{T}\log[c_{j,j+i|1,\ldots,j-1}\{F(x_{j,t}|x_{1,t},\ldots,x_{j-1,t}),F(x_{j+i,t}|x_{1,t},\ldots,x_{j-1,t});\theta_{j,i}\}] \\
&= \sum_{j=1}^{n-1}\sum_{i=1}^{n-j} log[c\{F(x_{1,t}),F(x_{i+1,t});\theta_{1,i}\}+ \\
&+ \sum_{j=2}^{n-1}\sum_{i=1}^{n-j}\sum_{t=1}^{T}\log[c\{F(x_{j,t}|x_{1,t},\ldots,x_{j-1,t}),F(x_{j+i,t}|x_{1,t},\ldots,x_{j-1,t});\theta_{j,i}\}] \\
&\overset{(3.3)}{=} \sum_{j=1}^{n-1}\sum_{i=1}^{n-j} log[c\{F(x_{1,t}),F(x_{i+1,t});\theta_{1,i}\}+ \\
&+ \sum_{j=2}^{n-1}\sum_{i=1}^{n-j}\sum_{t=1}^{T}\log[c\{v_{j-1,i+1,t},v_{j-1,1,t};\theta_{j,i}\}].
\end{aligned}
$$

□

The first index j of $v_{j,i,t}$ corresponds to the $j+1$ tree of the C-vine, the second index i corresponds to the node, which is observed related to the root node (here $i=1$).

Example 3.2 (n=4)
The log-likelihood, in the representation of (3.2), for one observation ($T=1$) and $n=4$ is

$$
\begin{aligned}
l(x,\theta) &= l(x_1,x_2,x_3,x_4,\theta) \\
&= \log\{c(x_1,x_2;\theta_{1,1})\}+\log\{c(x_1,x_3;\theta_{1,2})\}+\log\{c(x_1,x_4;\theta_{1,3})\} \\
&+ \log\{c(v_{1,2},v_{1,1},\theta_{2,1})\}+\log\{c(v_{1,3},v_{1,1},\theta_{2,2})\}+\log\{c(v_{2,2},v_{2,1},\theta_{3,1})\} \\
&= \log\{c_{1,2}(x_1,x_2)\}+\log\{c_{1,3}(x_1,x_3)\}+\log\{c_{1,4}(x_1,x_4)\} \\
&+ \log\{c_{2,3|1}(h(x_3,x_1;\theta_{1,2}),h(x_2,x_1;\theta_{1,1}))\}+\log\{c_{2,4|1}(h(x_4,x_1;\theta_{1,3}),h(x_2,x_1;\theta_{1,1}))\} \\
&+ \log\{c_{3,4|1,2}(h[h(x_4,x_1;\theta_{1,3}),h(x_2,x_1;\theta_{1,1});\theta_{2,2}],h[h(x_3,x_1;\theta_{1,2}),h(x_2,x_1;\theta_{1,1});\theta_{2,2}])\}.
\end{aligned}
$$

Example 3.3 (n=5)

$$
\begin{aligned}
l(x,\theta) &= \log\{c_{1,2}(x_1,x_2)\}+\log\{c_{1,3}(x_1,x_3)\}+\log\{c_{1,4}(x_1,x_4)\}+\log\{c_{1,5}(x_1,x_5)\} \\
&+ \log\{c_{2,3|1}(h(x_3,x_1;\theta_{1,2}),h(x_2,x_1;\theta_{1,1}))\}+\log\{c_{2,4|1}(h(x_4,x_1;\theta_{1,3}),h(x_2,x_1;\theta_{1,1}))\} \\
&+ \log\{c_{2,5|1}(h(x_5,x_1;\theta_{1,4}),h(x_2,x_1;\theta_{1,1}))\}+ \\
&+ \log\{c_{3,4|1,2}(h[h(x_4,x_1;\theta_{3,1}),h(x_2,x_1;\theta_{1,1});\theta_{2,2}],h[h(x_3,x_1;\theta_{1,2}),h(x_2,x_1;\theta_{1,1});\theta_{2,1}])\} \\
&+ \log\{c_{3,5|1,2}(h[h(x_5,x_1;\theta_{1,4}),h(x_2,x_1;\theta_{1,1});\theta_{2,3}],h[h(x_3,x_1;\theta_{2,1}),h(x_2,x_1;\theta_{1,1});\theta_{2,1}])\} \\
&+ \log\{c_{4,5|1,2}(h[h(h(x_5,x_1;\theta_{1,4}),h(x_2,x_1;\theta_{1,1});\theta_{2,2}),h(h(x_3,x_1;\theta_{2,1}),h(x_2,x_1;\theta_{1,1});\theta_{2,1}), \\
&h[h(h(x_4,x_1;\theta_{1,3}),h(x_2,x_1;\theta_{1,1});\theta_{2,3}),h(h(x_3,x_1;\theta_{1,2}),h(x_2,x_1;\theta_{1,1});\theta_{2,1});\theta_{3,1}])\}
\end{aligned}
$$

On the basis of the representation (3.2) an algorithm can be build easily. (See Aas et al. [2009])

Assuming the notation of (3.2) and

$$L(x, v, \theta) = \sum_{t=1}^{T} \log\{c(x_t, v_t, \theta)\} \tag{3.4}$$

we get the Algorithm 3.1.

Algorithm 3.1 Computation of the log-likelihood

log-likelihood $= 0$
for $i = 1$ **to** n **do**
  $v_{0,i} = x_i$
end for
for $j = 1$ **to** $n - 1$ **do**
  for $i = 1$ **to** $n - j$ **do**
    log-likelihood = log-likelihood + $L(v_{j-1,1}, v_{j-1,i+1}, \theta_{j,i})$
  end for
  if $j == n - 1$ **then**
    STOP
  end if
  for $i = 1$ **to** $n - j$ **do**
    $v_{j,i} = h(v_{j-1,i+1}, v_{j-1,1}, \theta_{j,i})$
  end for
end for

Starting values of the parameters needed in the numerical maximization of the log-likelihood may be determined as follows:

(i) Estimate the parameters of the copulae in tree 1 from the original data.

(ii) Compute observations (i.e. conditional distribution functions) for tree 2 using the copula parameter from tree 1 and the h-function.

(iii) Estimate the parameters of the copulae in tree 2 using the observations from (ii).

(iv) Compute observations for tree 3 using the copula parameters at level 2 and the h-function.

(v) Estimate the parameters of the copulae in tree 2 using the observations from (iv).

(vi) etc.

Remarks:

(i) $L(x, v, \theta)$ and $h(\cdot)$ depend on the chosen copula.

Algorithm 3.2 Computation of the log-likelihood: mathematical representation

```
1: The mathematical representation of the calculation of the conditional distribution
   function follows, as above, from the density function (3.2).
2: for i = 1 to n do
3:     Set v_{0,i} = F(x_i).
4: end for
5: for j = 1 to n − 1 do
6:     for i = 1 to n − j do
7:         Determine v_{j,i} = h(v_{j−1,i+1}, v_{j−1,1}, θ_{j,i}).
8:     end for
9: end for
```

(ii) The calculation of $L(x, v, \theta)$ is easy to implement.

(iii) The algorithm determine $\frac{n(n-1)}{2} - 1$ $v_{j,i}$'s.

The dependence structure between the values $v_{j,i}$, generated from the Algorithm 3.2, is graphically demonstrated in Figure 3.1. The arrows symbolize the dependence structure of the values $v_{j,i}$, e.g. $v_{2,1}$ depending on $v_{1,1}$ and $v_{1,2}$.
It can also be discovered, that a disturbance or a new arrangement of one of the nodes has an effect on other nodes in the tree above and beyond that.
To be more precise, if the value of a node $v_{j,i}$ change, we have to update all $v_{k,l}$ lying on the diagonal $v_{j,i} - v_{n-i-1,1}$ and all rows beneath of $v_{n-i-1,1}$.
Altogether $i + \frac{(n-j-i+1)(n-j-i)}{2} - 2$ updates have to be done.

The dependence diagram reflects as well, that every row of the diagram, as every tree of the C-vine, has a root node. In the first tree the node “1” is the root and in the diagram $v_{0,1}$. Analog for the second tree. Here is the node “1,2” the root node and in the diagram $v_{1,1}$. In the k-th tree it is “k-1,k—1,. . . ,k-2” as root and $v_{k+1,1}$ as root in the diagram.

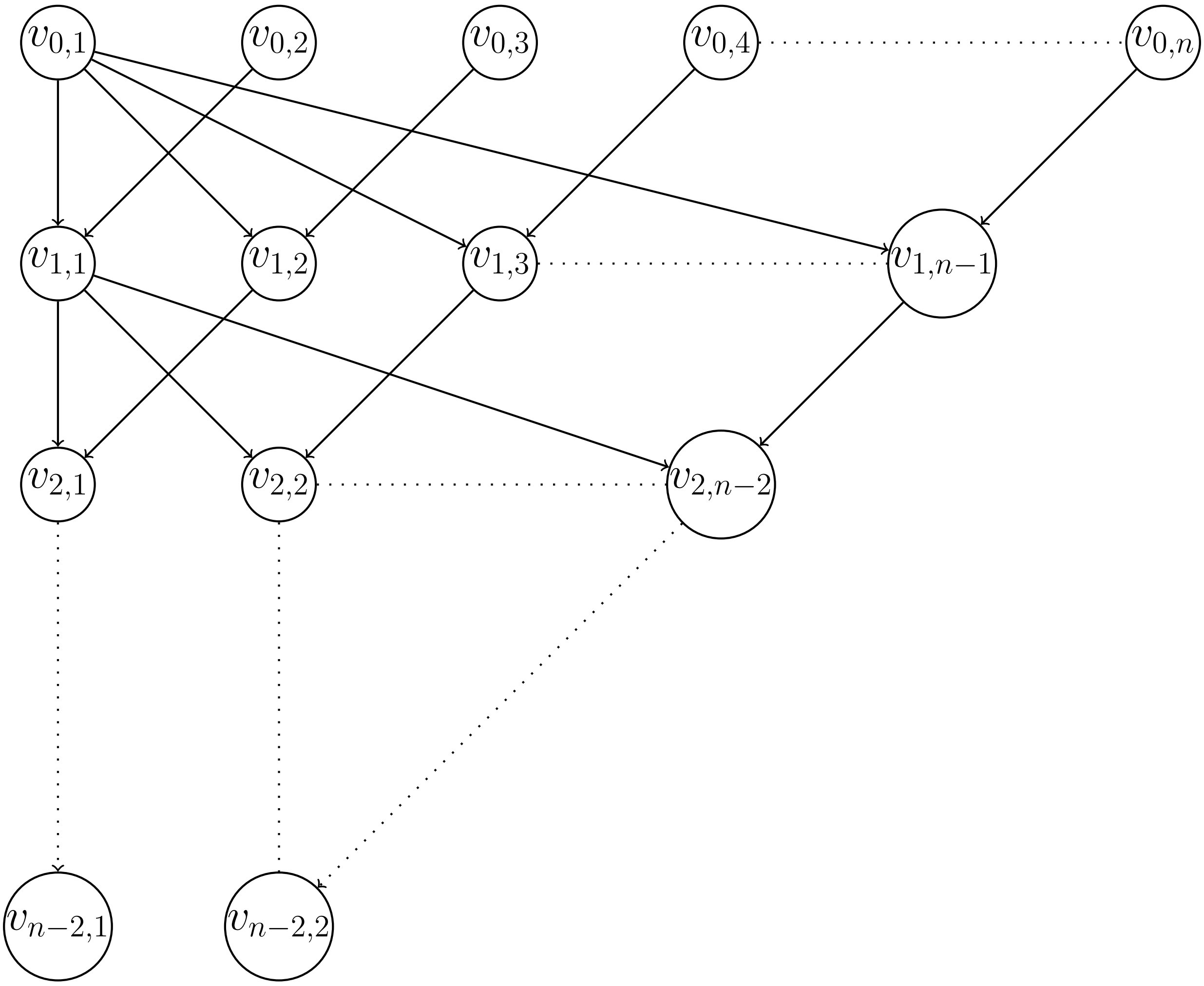


Figure 3.1: Dependence structure of the $v_{j,i}$ for n dimensions

Example 3.4 (n=6)
A disturbance or a new arrangement of the node $v_{1,3}$ *results in*

$$3 + \frac{(6-1-3+1)(6-3-1)}{2} - 2 = 4$$

additional updates, namely in the nodes $v_{2,2}, v_{3,1}, v_{4,1}$ *and* $v_{4,2}$.

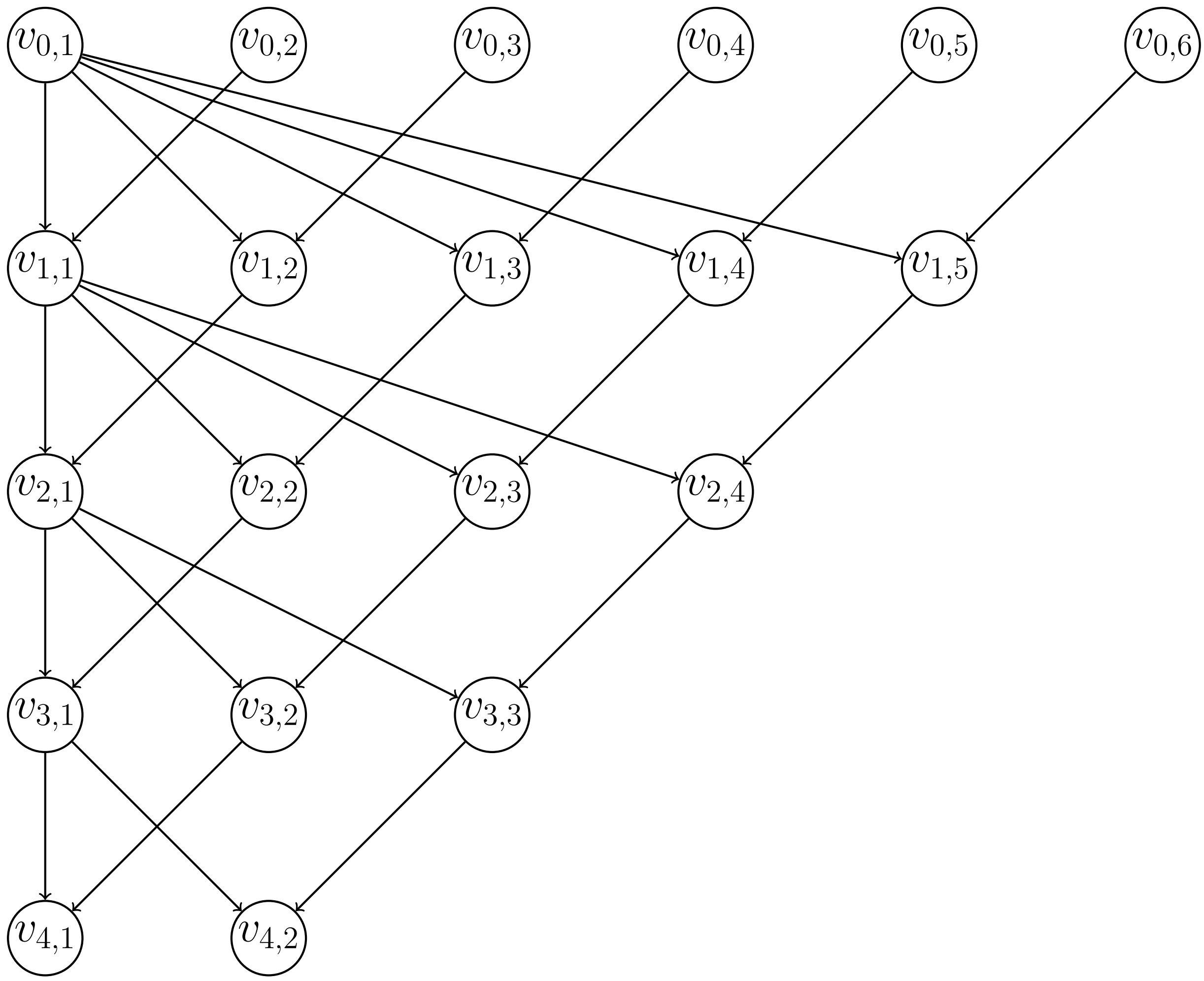


Figure 3.2: Dependence structure of the $v_{j,i}$ for $n = 6$

3.3 Simulation of a C-vine

Now we look at the simulation of a C-vine. Sometimes one needs simulated data to check if the computed results are correct or other tests or examples.

The algorithm of the canonical and D-vine are straightforward and can be easy implemented. The general algorithm for sampling n dependent uniform [0,1] variables is common for the canonical and the D-vine:

$$
\begin{aligned}
&Sample\ w_1, \ldots, w_n\ independent\ uniform\ on\ [0,1].\\
&Set\ x_1 = w_1\\
&x_2 = F^{-1}(w_2|x_1)\\
&x_3 = F^{-1}(w_3|x_1, x_2)\\
&\ldots\\
&x_n = F^{-1}(w_n|x_1, \ldots, x_{n-1}).
\end{aligned}
$$

Using the definition of the h-function of (2.15) we can determine $F(x_j|x_1, \ldots, x_{j-1})$ for each j recursively for the canonical vine as for the D-vine, too. For the canonical vine the $F(x_j|x_1, \ldots, x_{j-1})$ looks like this:

$$F(x_j|x_1, \ldots, x_{j-1}) = \frac{\partial C_{j,j-1|1,\ldots,j-2}(F(x_j|x_1, \ldots, x_{j-2}), F(x_{j-1}|x_1, \ldots, x_{j-2}))}{\partial F(x_1|x_2, \ldots, x_{j-1})}.$$

The following algorithm (Algorithm 3.3) gives the procedure for sampling from a canonical vine. The variables which should be sampled are represented by the outer for-loop. This loop consists of two other for-loops. In the first, the i'th variable is sampled, while in the other, the conditional distribution functions needed for sampling the $(i+1)$th variable are computed. To compute these conditional distribution functions, we repeatedly use the h-function defined in (2.15) with previously computed conditional distribution functions, $v_{i,j} = F(x_i|x_1, \ldots, x_{j-1})$, as the first two arguments. The last argument is the copula parameter.

Algorithm 3.3 Simulation algorithm for a canonical vine

Sample $w_1, \ldots, w_n$ independent uniform on [0,1].
$x_1 = v_{1,1} = w_1$
for $i = 2$ **to** n **do**
    $v_{i,1} = w_i$
    for $k = i-1$ **to** 1 **do**
        $v_{i,1} = h^{-1}(v_{i,1}, v_{k,k}, \theta_{k,i-k})$
    end for
    $x_i = v_{i,1}$
    if $i == n$ **then**
        STOP
    end if
    for $j = 1$ **to** $i-1$ **do**
        $v_{i,j+1} = h(v_{i,j}, v_{j,j}, \theta_{j,i-j})$
    end for
end for

3.4 Implementation and application

Given the representation of Section 3.2 we implemented the Algorithm 3.2 in the R-package "VineMLE" using the program language C and the statistical program R.
This package is a collection of functions for the D-vine as well as for the C-vine. Using this functions in the statistical program R we can calculate the log-likelihood and the MLE and simulate the vine with different copula families. In Chapter 5 we will use this functions to estimate our parameters of a exchange rate data set application. In this section we want to give a brief introduction into this function collection and the implementation of the Algorithm 3.2 of the canonical vine.
The algorithm can be implemented as it is shown in Algorithm 3.2. The needed h-functions and summations are written in separate functions and implemented in the main function to compute the log-likelihood.
For the calculation of the (pseudo-) maximum log likelihood we need starting values. These we determine by estimating Kendall's τ and using Table 2.3 of the Kendall's τ's we calculated in Chapter 2. For that we invite the formulas analytically or numerically.
The log likelihood can not be maximized analytically by solving the derivative of the log likelihood, but by using optimizing steps.
Now we want to give a brief example how to use the R-Package and the functions.

Example 3.5 (4-dim C-Vine with mixed copula-families)
As a first step we have to load the package, provided that it is installed. (How to install a R-package please read the manual and the help of the R software.)

```
load("VineMLE")
```

Now we simulate a data set, of which we want to compute the log-likelihood and the MLE. The algorithm for the simulation of a C-vine we implemented as well as the last algorithm in the R-package "VineMLE".
For both algorithms one has to set a few parameters, which are explained in Table 3.1.
Using the function $SimulateCopulaVine$, we simulate our data with the parameters and families we set in Table 3.2.

```
n=1000
d=4
fam=c(1,2,3,4,5,1)
par=c(0.4,0.6,2,3,3,0.4)
nu=c(0,5,0,0,0,0)
U<-SimulateCopulaVine(n,d,fam,par,nu, type=1)
```

As result we get a 4 dimensional matrix with simulated data of the C-vine we defined by setting the copula families and the copula parameters.

Variable	**Description**
d	Dimension of the D- or C-vine
fam	Copula family 1 = Gaussian copula 2 = t-copula 3 = Clayton copula 4 = Gumbel copula 5 = Frank copula 6 = Joe copula 7 = BB1-copula 8 = BB7-copula
par	vector of copula parameters
nu	vector of degrees of freedom of the t-copulas or second parameter of the BB1- and BB7-copula
type	copula type 1 = C-vine 2 = D-vine

Table 3.1: Table of the variables and their description, which are implemented in the R-package VineMLE

Copula c_{12}	Gaussian copula	$\rho = 0.4$
Copula c_{13}	t-copula	$\rho = 0.6, \nu = 5$
Copula c_{14}	Clayton copula	$\theta = 2$
Copula $c_{23\|1}$	Gumbel copula	$\theta = 3$
Copula $c_{24\|1}$	Frank copula	$\theta = 3$
Copula $c_{34\|12}$	Gaussian copula	$\rho = 0.4$

Table 3.2: Parameter setting for Example 3.5

```
head(U)
          [,1]       [,2]       [,3]       [,4]
[1,] 0.7066986 0.82601029 0.82942057 0.97389049
[2,] 0.2983142 0.02707463 0.03720185 0.13762051
[3,] 0.9451951 0.98093823 0.97794778 0.65571578
[4,] 0.1308285 0.14430086 0.11847408 0.08089933
[5,] 0.8388829 0.92371351 0.87507507 0.80806076
[6,] 0.5050555 0.87250171 0.88477126 0.88672860
```

Now we calculate the log-likelihood for this simulated data. Therefore we combine the parameter vector and the vector of the degrees of freedom to one vector and call the function $VineLogLik$.

```
par2=c(0.4,0.6,2,3,3,0.4,5)
LogLik <- VineLogLik(par=par2,U,family=fam, type=1)
LogLik$loglik
[1] 1765.517
```

To determine the MLE we use the function $VineMLE$. The resulting estimators should be near the real parameters of the set above.

```
est<-VineMLE(U, fam, type=1)
est$par
[1] 0.42 0.61 2.08 3.00 3.09 0.40 5.33
est$value
[1] 1767
```

In this case we only needed a few seconds for the calculation, but it can increase rapidly if we increase the dimensions and the number of observations, e.g. for a C-vine with 7 nodes and 4900 observations we needed around 19 hours.

To test the robustness and stability of our estimators we made a few tests of the package "VineMLE" with different scenarios. In each scenario we tested the VineMLE-function in four different combinations of Kendall's τ. As a common setup we chose a **4 dimensional C-vine** with **n=500** observations and **r=100** replications.

The combinations are in Table 3.3, where H means $\tau = 0.8$ and L stands for $\tau = 0.2$.

Tree 1			**Tree 2**		**Tree 3**
τ_{12}	τ_{13}	τ_{14}	$\tau_{23\|1}$	$\tau_{24\|1}$	$\tau_{34\|12}$
H	H	H	H	H	H
L	L	L	L	L	L
H	H	H	L	L	L
L	L	L	H	H	H

Table 3.3: Four different scenarios for the MLE test with different Kendall's τ's and different copula families

The general test procedure was as follows: we fixed the Kendall's tau accordingly to the scenarios we set above and calculated the according parameter(s) of the copulas we chose. Then we simulated 500 observations with the R-simulator "SimulateCopulaVine", we mentioned in the example. This simulation we did 100 times and estimated for each 100 data set the according parameter(s) with the function "VineMLE" (see example above). Afterwards we calculated the following moments and statistics (Table 3.4) to check the stability.
The detailed results of these tests can be found in the appendix. But here we will give a short summary of the results and our conclusions.

Conclusions of the tests:

- In the first test all copulas of the four-dimensional C-vine were of one copula family. We tested all implemented copula families and for all copula families we results were satisfying, i.e. the results of the MLE-algorithm are nearly the same as the setting parameters.

mean	Mean of the parameter	$\bar{\theta} := \frac{1}{r}\sum_{i=1}^{r}\hat{\theta}_i$
trim mean	Trimmed mean, i.e mean without the 5%-highest and lowest values	$\bar{\theta}_{0.05}$
bias	BIAS of the parameter	$\hat{b}(\bar{\theta}) := \bar{\theta} - \theta$
Variance	Variance of the parameter	$s^2(\bar{\theta}) := \frac{1}{r-1}\sum_{i=1}^{r}(\hat{\theta}_i - \bar{\theta})^2$
MSE	Mean squared error	$\widehat{mse}(\bar{\theta}) := s^2(\bar{\theta}) + \hat{b}(\bar{\theta})^2$
mean2	Mean of Kendall's τ	$\bar{\tau} := \frac{1}{r}\sum_{i=1}^{r}\hat{\tau}_i$
trim mean2	Trimmed mean of τ, i.e mean without the 5%-highest and lowest values	$\bar{\tau}_{0.05}$
bias2	BIAS of Kendall's τ	$\hat{b}(\bar{\tau}) := \bar{\tau} - \tau$
Variance2	Variance of Kendall's τ	$s^2(\bar{\tau}) := \frac{1}{r-1}\sum_{i=1}^{r}(\hat{\tau}_i - \bar{\tau})^2$
MSE2	Mean squared error of Kendall's τ	$\widehat{mse}(\bar{\tau}) := s^2(\bar{\tau}) + \hat{b}(\bar{\tau})^2$

Table 3.4: Notation and definitions of the test-statistics for the MLE test

- The only abnormalities were the degrees of freedom of the t-copulas. It may be a trend that the degrees of freedom are over-estimated systematically. The difference between the setting and the estimated degrees of freedom grew if the degree of freedom grows.

- For the BB1- and BB7-copula one can detect that in some cases the estimation differs from the setting especially for scenarios with high Kendall's τ. But the differences are not essential and become smaller if one enlarges the number of observations.

- In the second block of tests we tested some combinations of different copula families. Here we detected that in some cases the estimation of the last copula $c_{34|12}$ is worse than the other copulas. One explanation for this phenomena may be that one needs for the computation of the last copula more computation steps and numerical errors can sum up to a relative big error. Some tests with 5-dimensional C-vines confirmed this thesis (not in this diploma thesis).

- Especially the combinations with BB1- and BB7-copulas seem to be vulnerable to numerical errors. Some additional tests (not in this diploma thesis) showed that the computation of the h-function and of the density function of the BB1- and BB7-copula is not stable for all parameter and copula family combinations.

Chapter 4

Model selection

If one has decided to model his data and their dependence structure by a C-vine, one has to find the tree and the copula families in each level of the C-vine. To do so, we will present a method to select a tree and different methods to select an appropriate but maybe not perfect copula family for a pair-copula. We will describe three procedures, the contourplots, the λ-function and the goodness-of-fit test. The definition for the first two methods are already in the Chapter 2.3.2 and 2.5. The goodness-of-fit test is introduced and defined in Section 2.6.1.
In this chapter we will just point out how these three methods can be a help to identify the underlying copula. At this point we want to suggest that these methods are just a help to find a copula not a mathematical procedure or algorithm to identify the underlying copula of a pair-copula composition.

4.1 Tree selection

In every C-vine the root node is the decisive node. If this node is chosen then all the other nodes of the tree are chosen too, because the root node is the only one which is connected to all the other nodes and has more than one edge to another node. All the other nodes have just one edge, the edge to the root node. They are not connected to each other in any way. (see Definition 2.23)
To select the root node we look at the correlation matrix with respect to Kendall's τ. In our context and with respect to the application in Chapter 5 we decided that the root node should be the one which has the highest dependence structure to the other nodes, i.e. the highest sum of absolute Kendall's τ.
Therefore we summarize all the absolute Kendall's τ in each row of the correlation matrix. The rows as well as the columns stand for the nodes of the tree. As the root node we choose the row (node), which has the highest sum of Kendall's τ. In our following example with a four dimensional C-vine based on simulated data we get the node A as our root node (see Figure 4.1)
The thickness of the edges illustrate the degree of dependence between two nodes, i.e. the absolute value of Kendall's τ with respect to the two data columns represented by the two nodes.

Example 4.1 (4-dim C-vine)
As an example we use a 4 dimensional C-vine with n=1000 observations. To get this data we simulate a C-vine as in Chapter 3 by using the R-package "VineMLE". As underlying copulas we used the following copulas with the according parameters in brackets:
c_{12}*: t-copula,* $\tau = 0.6$ *(0.80, 5),* c_{13}*: Gumbel copula,* $\tau = 0.55$ *(2.22),* c_{14}*: Frank copula,* $\tau = -0.35$ *(-3.51),* $c_{23|1}$*: Gaussian copula,* $\tau = 0.2$ *(0.31),* $c_{24|1}$*: BB1-copula,* $\tau = 0.2$ *(0.12, 1.18) and* $c_{34|12}$*: Clayton copula,* $\tau = 0.05$ *(0.10)*

```
n=1000
d=4
tau=c(0.6,0.55,-0.35,0.2,0.2,0.05)
fam=c(2,4,5,1,7,3)
copula=c("t","G","F","N","BB1","C")
par=rep(0,6)
nu=rep(0,6)
for(i in 1:6)
{
par[i]=parameter(copula[i],tau[i],0.2)[1]
}
nu=rep(0,6)
nu[1]=5
nu[5]=parameter("BB1",tau[5],0.2)[2]

data_example=SimulateCopulaVine(n,d,fam,par,nu, type=1)
colnames(data_example)=c("A","B","C","D")
```

	A	B	C	D	Sum
A	1.00000	0.59955	0.55507	-0.32477	**2.4794**
B	0.59955	1.00000	0.50901	-0.14144	2.25
C	0.55507	0.50901	1.00000	-0.19728	2.2614
D	-0.32477	-0.14144	-0.19728	1.00000	1.6635

Table 4.1: Empirical Kendall's τ matrix and the sum over each row for the example C-vine

In this example we constructed the simulated C-vine such that we have the node A as root node. In a second example at the end of this section we will illustrated an example in which this will not be the case because one can choose the parameters of the simulated C-vine in such a way, that one gets an other root node when one uses the method described above.

Remark:
There are other methods to choose the tree of a C-vine. Nikoloulopoulos et al. [2010] developed empirically three different rules: (Quoted from Nikoloulopoulos et al. [2010])
First select a pilot variable 1 that has strong dependence with all other variables. Then

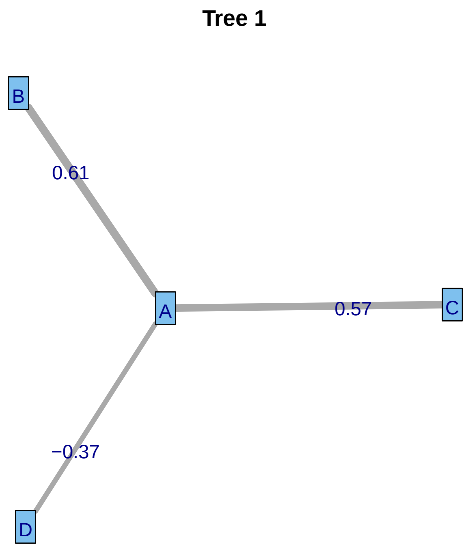


Figure 4.1: Tree 1 of our example C-vine (4-dim)

(i) list the most dependent variables with the pilot in decreasing in dependence order;

(ii) list the least dependent variables with the pilot in increasing in dependence order;

(iii) sequentially list the least dependent variable with the previous selected.

In the next example we want to show that our procedure of finding a tree is not always the best one. Our method finds the tree with the highest degree of dependence, but sometimes one wants an other C-vine.

Example 4.2 (counterexample)
As in Example 4.1 we simulate a 4-dim. C-vine with different copula families and Kendall's τ's. The setting of the C-vine is in Table 4.3.

```
n=1000
d=4
fam=c(1,2,3,4,5,1)
par=c(0.4,0.6,2,3,3,0.4)
nu=c(0,5,0,0,0,0)
data_example=SimulateCopulaVine(n,d,fam,par,nu, type=1)
colnames(data_example)=c("A","B","C","D")
```

One can see in Table 4.2 that the node with the highest sum of Kendall's τ is the node C (see Figure 4.2), but we chose the node A in our simulation setting as the root node. This setting for a C-vine is an extreme example and not very realistic, because we chose

	A	B	C	D	Sum
A	1.000	0.261	0.404	0.468	2.13
B	0.261	1.000	0.679	0.379	2.32
C	0.404	0.679	1.000	0.495	**2.58**
D	0.468	0.379	0.495	1.000	2.34

Table 4.2: Empirical Kendall's τ matrix and the sum over each row for the example C-vine (example 2)

the parameters of the second and third tree that high, such that the resulting Kendall's τ is high too. A high Kendall's τ in the second and third tree means that the conditional dependencies are also high and not only the direct ones. This scenario is not very realistic as we will see in our data set in Chapter 6. But this extreme example should illustrate that the selected tree may not the best one and the underlying structure of the dependencies or independencies are different.

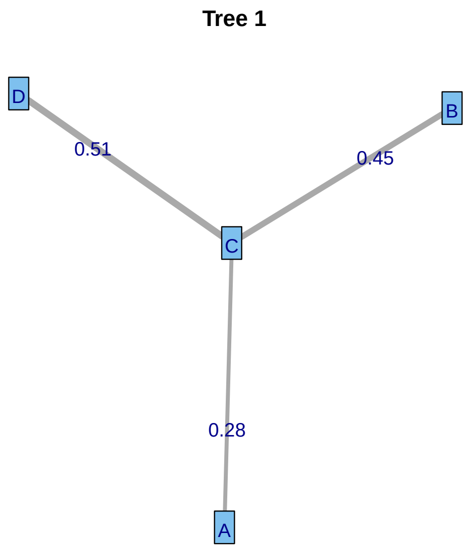


Figure 4.2: Tree 3 of our example C-vine (4-dim, example 2)

The next question is, how far away is this model from the true model? This will be discussed in the sections "Quality of the selected model" and "Model comparison" (Section 4.3 and 4.4)
Therefore we looked at the copula families and the trees of the resulting model. Notice, the copulas we look at are different from the one we set in the example settings (Table 4.3) because of different Kendall's τ's and a different first tree. As a result we get the following copula families (Table 4.4) by using the methods we will introduce in the next section.

Copula	Family	Parameter(s)
$c_{A,B}$	Gaussian	0.4
$c_{A,C}$	t	0.6, 4
$c_{A,D}$	Clayton	2
$c_{B,C\|A}$	Gumbel	3
$c_{B,D\|A}$	Frank	3
$c_{C,D\|A,B}$	Gaussian	0.4

Table 4.3: Setting of the second example (4-dim C-vine)

Copula	Family	Est. parameter(s)
$c_{A,C}$	Gaussian	0.58
$c_{B,C}$	Gumbel	3.08
$c_{D,C}$	t	0.68, 11.79
$c_{A,B\|C}$	Frank	-2.37
$c_{A,D\|C}$	Clayton	0.91
$c_{B,D\|A,C}$	t	-0.07, 20.51

Table 4.4: Copula selection for the second example (4-dim C-vine)

4.2 Copula selection

After finding the tree for the first level we have to assign the copula families of the first level to get the other levels of the canonical vine. Therefore we tried three methods to select an appropriate copula for the underlying data. Two of them are graphical methods, the contourplots and the λ-function. The third one will be a statistic test. But notice, all these methods give only a hint of the true copula family and are not methods to identify the copula family uniquely. With these three methods together we try to reduce the set of possible copula families and end up with a good guess.
Again we will illustrate the coming methods in an example.

4.2.1 Copula selection based on contourplots

The first method is to look at the behavior of the empirical data with respect to their Kendall's τ and density function. The density function can be illustrated in the so called contourplot. We derive the empirical density function of the data and plot it in a contourplot. Then we compare this density with the theoretical density of possible copula families by comparing their contourplots. The theoretical densities are given in the definitions in Chapter 2. Their parameters are set by the empirical Kendall's τ. To get these parameters one can invert the formulas of Kendall's τ we summarized in Table 2.3. For the bivariate copula families as BB1 and BB7 we set the upper tail dependence equal to 0.2 to get a second equation and improve this results by a bivariate maximum likelihood estimation. A better way to get a second equation would be to calculate the empirical upper tail dependence and set this equal to the according equation. But the calculation of the tail dependence in general is very instable and thus we decided not to use it. This one has to be in mind when one compares the contourplots of the empirical density with the density of the BB1- and BB7-copula. For the t-copula one has to estimate the degree of freedoms as a second parameter. Again we compute them by a ML estimation.
Every copula family has a characteristic form of its density and can theoretically be identified by this. In praxis the contours blur and get diffuse and an unique identification is nearly impossible. A second problem is that some copula families have for special parameter combinations a very similar contourplot. Furthermore all copula families have for low Kendall's τ, i.e. $\tau \approx 0$, nearly the same contourplot.

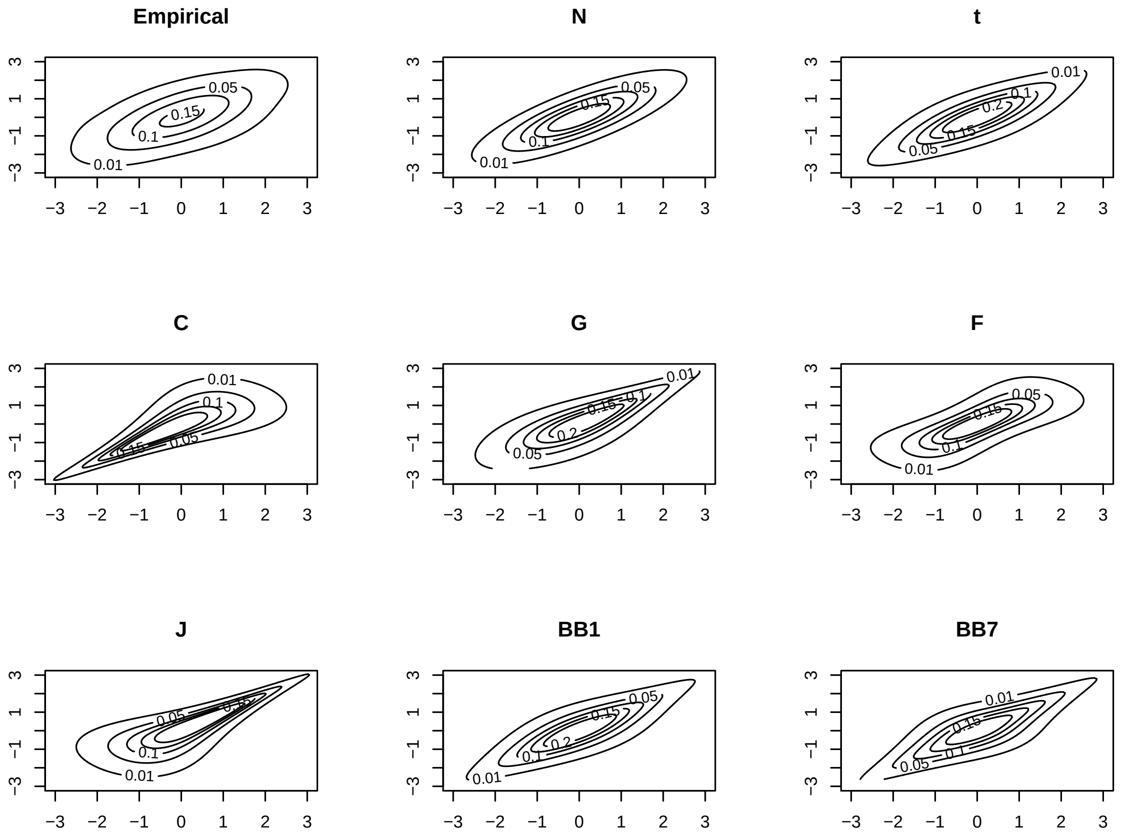


Figure 4.3: Possible Contourplots of the first pair-copula in the example (4-dim C-vine). The empirical contourplot of the pair-copula A-B (first plot) and the theoretical Gaussian (N), t- , Clayton (C), Gumbel (G), Frank (F), Joe (J), BB1- (BB1) and BB7-copula (BB7)

In Figure 4.3 one can see immediately that some of the possible copulas are definitely discard. In this case we can eliminate the Clayton, the Gumbel, the Frank and the Joe copula. The BB1- and BB7-copula does not look good at all, but as we mentioned it above there are some difficulties for these two copulas and thus we can't reject them in this state of case.
As the most likely seem to be the Gaussian and the t-copula with higher degree of freedom. Remember: The t-copula converges for high degrees of freedom to the Gaussian copula. Thus this result is not very surprising. The pair "A,B" is in our original simulation set the t-copula with 5 degrees of freedom. Thus our first choice was correct.

Remark:
If the Kendall's τ of a pair is negative then one can only use the Gaussian, the t- and the Frank copula for comparison.

4.2.2 Copula selection based on the λ-function

As we saw for the BB1- and BB7-copula and the flouting intersections of the contourplots we need other methods too. One another graphical method is the λ-function. We defined the λ-function already in Chapter 2.5 and explained that we can derive the λ-function only for the Archimedean copulas. A summary of all the λ-function are given in Table 2.5.
For the Gaussian and t-copula we simulate data with Gaussian or t-density, respectively and compute the empirical λ-function of it. Thus we get a competitive λ-function of the Gaussian and t-copula.
The plots of the λ-function should be a help to identify the underlying copula as well as the contourplots graphically. Like for the contourplots one derives the empirical Kendall's τ and the empirical λ-function for the data. Then one calculates the λ-function for the possible copulas theoretically and compare the resulting plots.
In the plots we draw in each theoretical plot (thick grey line) the empirical one too and in dashed lines the extreme values of the λ-function, i.e Kendall's $\tau = 0$ and Kendall's $\tau = 1$ (horizontal dashed line)

When we look again on our example from above we can again reject the Clayton, the Gumbel, the Frank and the Joe copula (Figure 4.4). The BB1- and BB7-copula can not be rejected. Especially the λ-function of the BB1-copula fits the empirical λ-function well, i.e. the grey (theoretical) and black (empirical) line in the figure overlap. The Gaussian- and the t-copula are not Archimedean copulas but we derive their λ-function with a simulation method as we described it above. We can't reject them. Or in other words, the λ-functions of the Gaussian and t-copula fit the empirical λ-function well. As mentioned for the contourplots, it is difficult to choose a copula just by these plots, because there are, as in in the contourplots, some comparison problems. Because of too few data or because of calculation errors the empirical λ-function may be not correct. Thus these plots have to treated cautiously.

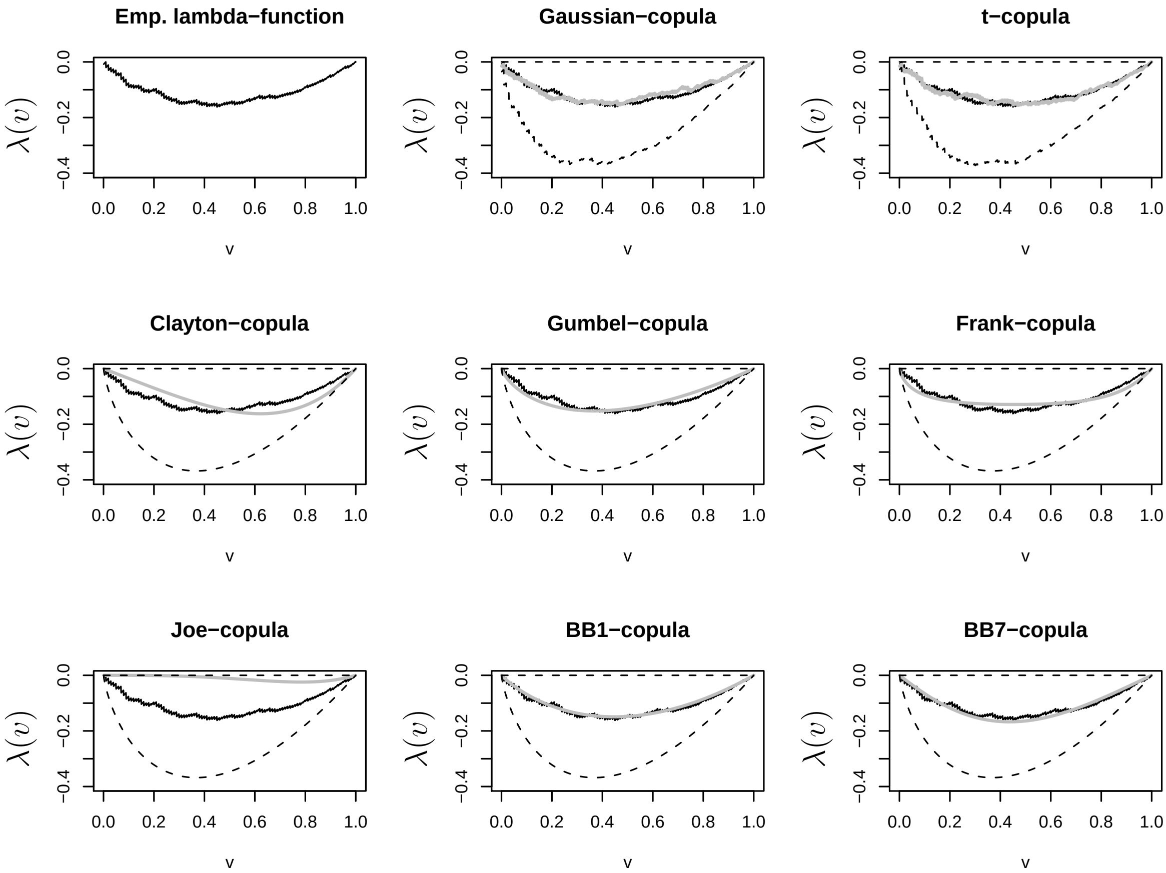


Figure 4.4: λ-function of the 4-dim. example C-vine. The empirical λ-function of the pair-copula A-B (first plot) and the theoretical Gaussian, t-, Clayton, Gumbel, Frank, Joe, BB1- and BB7-copula. In each plot we draw the empirical version (black line), the theoretical version (grey line) and the extreme values of the λ-function (dashed lines)

4.2.3 Copula selection based on the goodness-of-fit test

Beside the two graphical methods from above there is the goodness-of-fit test we introduced in Section 2.6.1. As we said already in the introduction chapter the goodness-of-fit test we look at is based on the Cramér-von Mises statistic and the according p-value.
The goodness-of-fit test is implemented for the Gaussian, the t-, the Clayton, the Gumbel and for the Frank copula in the R-package "copula".
The statistical test is the third method which should give us a hint of the true underlying copula. As we said already for the contourplots and the λ-functions we can only reject or favor some copula families on the basis of this test. Because the calculation of the p-value is based on a Monte Carlo simulations we have to be cautious by the choice of our copula family.

Example 4.3 (continued)
When we look again on our example of a 4-dim C-vine, we can calculate the goodness-of-fit test for the implemented copulas.
The results are:

	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0238	0.0211	0.6977	0.0462	0.0785
p-value	**0.1283**	**0.1423**	0.0004	0.004	0.0004

Table 4.5: Goodness-of-fit test for the example C-vine (first copula)

If we run the test again we get these results:

	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0205	0.0168	0.5651	0.0619	0.0908
p-value	**0.2632**	**0.3341**	0.0004	0.0004	0.0004

Table 4.6: Second goodness-of-fit test for the example C-vine (first copula)

We highlighted the p-values which are greater than 0.05, because we chose the 5%-level as the critical level for our hypothesis test.
The results of the second run are a little bit different of the results of the first run, but the tendencies in the results and their conclusions are the same.

Another method to get a goodness-of-fit test is the Vuong- or Clarke-test. Both are explained in Section 2.6.4 and in the diploma thesis of Djunushalieva [2010]. If one changes these two tests for a bivariate copula density one gets a kind of goodness-of-fit test. (see Section 2.6.5).
We use this test just as a further goodness-of-fit test to check if our copula selection was correct. Further information to this test can be found in the diploma thesis of Djunushalieva [2010].

A higher score value of the tests means that it is more probably that the underlying copula is the selected copula. We illustrated this in the definition/introduction in Section 2.6.5. The following copulas are implemented by Djunushalieva [2010] : Gaussian, t- Clayton, Gumbel, Frank, BB1-, BB7 and Plackett copula. We don't use the Plackett copula in this diploma thesis, thus we skipped it in the tests.

Example 4.4 (Vuong and Clarke test)
Again we use the simulated data set form the first example.

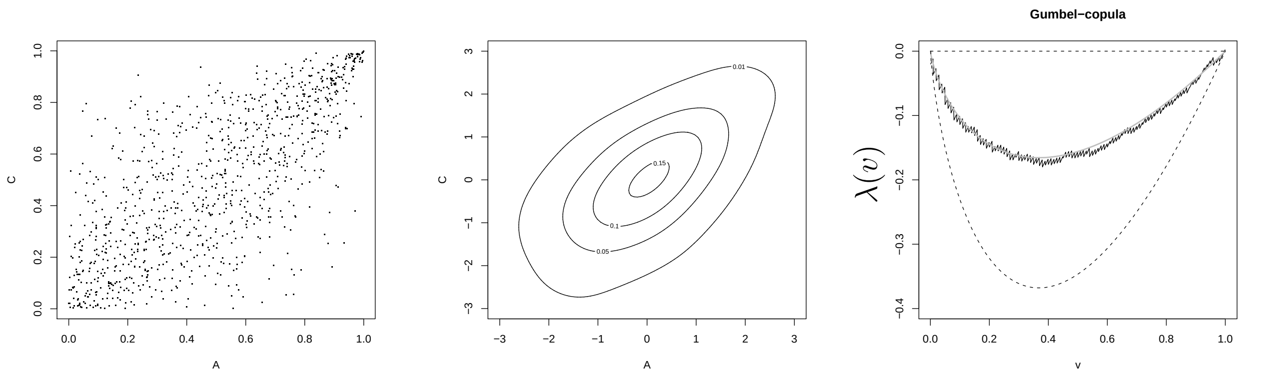
	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	3	**6**	-6	2	0	-1	-4
Clarke	3	**6**	-6	1	2	-2	-4

Table 4.7: Goodness-of-fit test for the example C-vine on the basis of the Vuong- and Clarke-tests. (first copula)

As the final copula we choose the t-copula as underlying copula, because of the good contourplot and the goodness-of-fit test, which has a p-value grater than 0.05. The λ-function is good too, but for the t-copula it is based on simulations and not on a theoretical function.

For the copulas c_{AC} and c_{AD} we can do the same and get the following results (Table 4.8 - 4.9). Remember, for the negative Kendall's τ in copula c_{AD} we can only compare the data with the Gaussian, the t- and the Frank copula as possible copula families.
In Table 4.8 - 4.9 we summarize all the necessary information about a copula pair. In the first line we write the according copula pair and her empirical Kendall's τ.
In the second line we draw the pairs-plot, the empirical contourplot and the plot of the empirical λ-function (black line) with the theoretical λ-function of our choice as a grey line. The pairs-plot should give us a first impression of the data and their dependence behavior. If the points cluster around a diagonal line one can say that there is a positive dependence. If they cluster additionally in one or two corners of the quad we know there may be tail dependence. The second plot, the contourplot, should give us a hint of the density. To find the underlying copula we compared this empirical contourplot with several possible theoretical contourplots as we described it above. But we do not draw all the possible plots. For the third plot, the λ-function, we do the same. Here we plot in one figure the empirical λ-function as well as the selected one.
The table under the plots summarizes the goodness-of-fit test with the Cramér-von Mises statistic and the according p-value. We highlight the p-values, which are greater than 0.05 because we can not reject the null-hypothesis on the 5%-level. (see explaination above to the topic goodness-of-fit).
The next sub-table is the goodness-of-fit test based on the Vuong- and Clarke-tests. Here we highlight the copula with the highest score.
At the end of each table we write the possible copula families, i.e. the copula families which fit for one or more of the three criteria, and the selected copula family, i.e. the copula family which we think fits the empirical data best.

Copula c_{AC} $(\tau = 0.55)$



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0597	0.0820	1.1306	0.0165	0.0972
p-value	0.0004	0.0004	0.0004	**0.3691**	0.0004

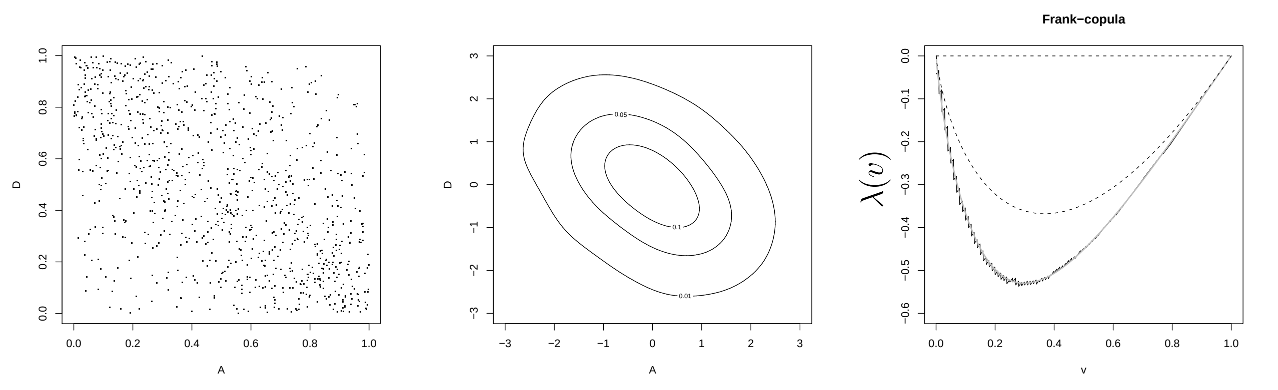
	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	3	3	-6	**6**	0	-2	-4
Clarke	1	**5**	-6	3	3	-2	-4

True copula:	Possible copula(s):	Selected copula:
G	G	G

Table 4.8: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula c_{AC} of the example C-vine

With the selected copulas for tree 1 one can construct the second tree. Therefore we estimate the parameters of the copulas by inverting Kendall's τ and for the t- ,BB1-, and BB7-copula by a simple bivariate maximum likelihood estimator. The method of inverting Kendall's τ are for these copulas not possible because they have two parameters and with Kendall's τ one has only one equation.
If we use our method for tree selection (Section 4.1) we get the following trees (Figure 4.5 and 4.6) and using the methods from Section 4.2 we get for the next copulas the Tables 4.10 - 4.12.

Copula c_{AD} $(\tau = -0.35)$



	Gaussian	t	Frank
Cramer-von Mises statistic	0.0532	0.0430	0.0132
p-value	0.0045	0.0074	**0.7447**

	Gaussian	t	Frank
Vuong	1	-2	**1**
Clarke	-2	**2**	0

True copula:	Possible copula(s):	Selected copula:
F	F	F

Table 4.9: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula c_{AD} of the example C-vine

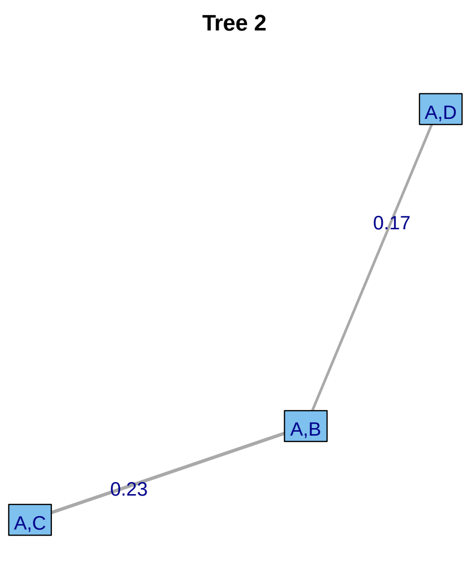


Figure 4.5: Tree 2 of our example C-vine (4-dim)

Copula $c_{BC|A}$ $(\tau = 0.2)$

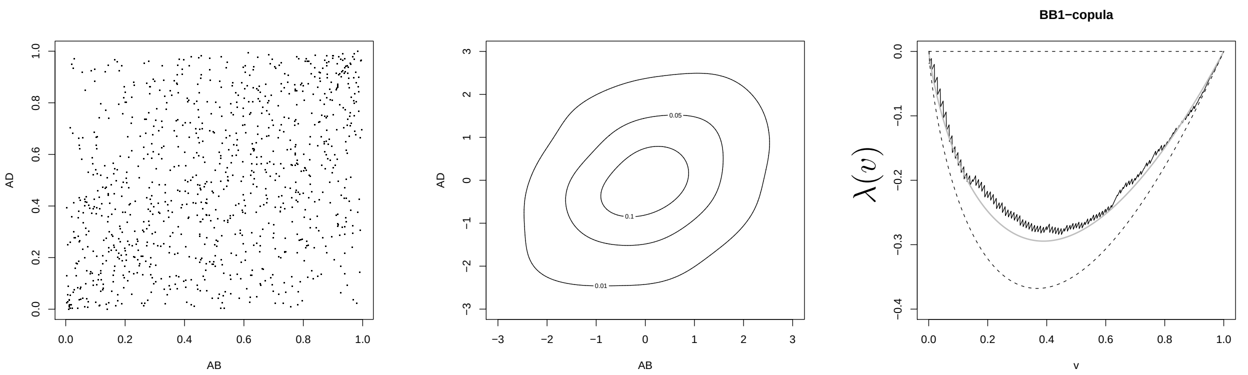
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0153	0.0159	0.1476	0.0675	0.0163
p-value	**0.6978**	**0.6698**	0.0004	0.0004	**0.5349**

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	**6**	3	-5	-3	2	0	-3
Clarke	-4	**6**	-4	2	0	1	-1

True copula:	Possible copula(s):	Selected copula:
N	N, t, F, BB1	N

Table 4.10: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{BC|A}$ of the example C-vine

Copula $c_{BD|A}$ $(\tau = 0.2)$



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0347	0.0304	0.1107	0.0222	0.0464
p-value	**0.0644**	**0.0814**	0.0004	**0.3351**	0.0034

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	1	0	-2	-2	0	**2**	1
Clarke	-6	**6**	-2	4	-1	0	-1

True copula:	Possible copula(s):	Selected copula:
BB1	N, t, G, BB1, BB7	BB1

Table 4.11: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{BD|A}$ of the example C-vine

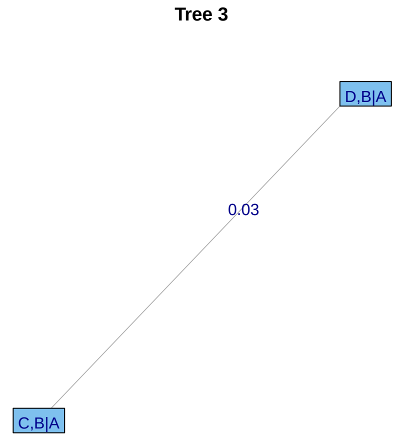
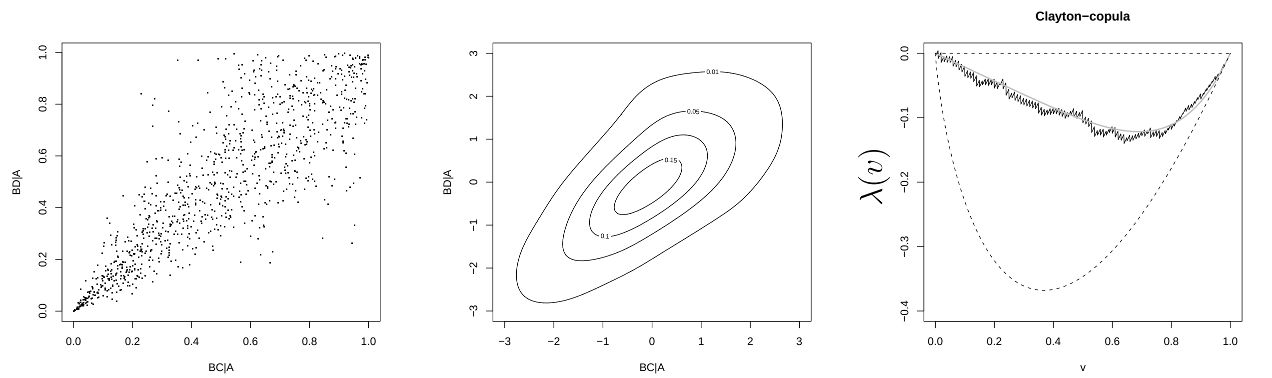


Figure 4.6: Tree 3 of our example C-vine (4-dim)

Copula $c_{CD|AB}$ $(\tau = 0.05)$



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0353	0.0350	0.0312	0.0429	0.0359
p-value	0.0494	0.0374	**0.1433**	**0.0594**	0.0244

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	-3	-1	**4**	-6	-2	**4**	**4**
Clarke	2	0	**5**	-6	-4	4	3

True copula:	Possible copula(s):	Selected copula:
C	BB1, BB7, C	C

Table 4.12: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CD|AB}$ of the example C-vine

4.3 Quality of the selected model

As we saw in the last examples the selection of the trees and the copulas are difficult and can sometimes be misleading or indifferent between two or more copulas. But what about the dependence structure? Describes the selected model the Kendall's τ's correct? Therefore we compute the maximum likelihood of the C-vine and get the parameters for the copulas. Then we calculate the theoretical conditional and unconditional Kendall's τ's by the formulae from Table 2.3 and compare them with the empirical ones.

Example 4.5 (continue of example 4.1)
We calculate the MLE of our selected model and compare it with the likelihood of the model we set before.

```
VineMLE(data_example,c(2,4,5,1,7,1),1000,type=1)
par
[1]  0.8194  2.2349 -3.2442  0.2920  0.0969  0.0344  4.6558  1.1964

value
[1] 1252.1

VineLogLik(c(par,5,nu[5]),data_example,c(2,4,5,1,7,3),type=1)
loglik
[1] 1249.9
```

Kendall's τ is a measure in the middle of the distribution and by this fact a less good measure for the quality of a model. But it gives a first hint of the quality of the model. Our model fits the conditional and unconditional Kendall's τ very well (see Table 4.13 for the first example and Figure 4.7 for the second example).

	Tree 1			**Tree 2**		**Tree 3**
	τ_{12}	τ_{13}	τ_{14}	$\tau_{23\|1}$	$\tau_{24\|1}$	$\tau_{34\|12}$
True value:	0.6000	0.5500	-0.3500	0.2000	0.2000	0.0500
Estimated value:	0.6004	0.5502	-0.3281	0.1886	0.1923	0.0169
Difference:	0.0004	0.0002	**0.0218**	0.0113	0.0076	**0.0330**

Table 4.13: True and estimated conditional Kendall's τ and their difference of the C-vine example (Example 4.1)

The estimated conditional Kendall's τ are quite good only the third and the last one are less exact.
For the second example we get, as we mentioned it already above, different copulas. Therefore we can not compare the conditional Kendall's τ's as is Table 4.13. But we can compare the unconditional Kendall's τ's we have in Table 4.2 and the Monte Carlo estimated unconditional Kendall's τ we get by our model selection. (r = 100 replications)
In Figure 4.7 we illustrated the difference between the the empirical and the Monte Carlo

estimated unconditional Kendall' τ's with respect to the selected model and his MLE parameters. We can see that the errors we made by selecting a different first tree and hence different copulas is very small in this example. There may be other examples and data sets where this misspecified model has a greater impact on the modeled Kendall's τ's.

The log-likelihood is a better measure because it depends on all data and over the whole distribution. To make a point with respect to the quality of a model one has to compare the log-likelihood of two models (see next section "Model comparison", in which we compare the model with simple models, so called misspecifications). A better comparison would be under consideration of the number of parameters of the two models.
In this first case we compare the true, simulated model with the selected model. The number of parameters are equal, thus we can ignore them. The log-likelihood of the selected model is 1252.1 and thus close to the log-likelihood of the true model (1249.9).

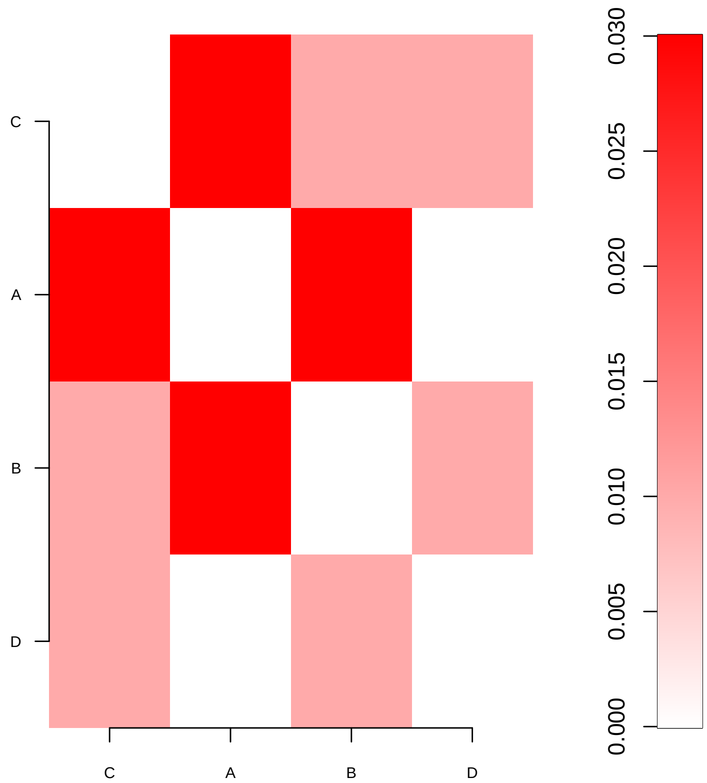


Figure 4.7: Absolute difference between the empirical and the Monte Carlo estimated unconditional Kendall' τ's with respect to the selected model and his MLE parameters (Example 4.2)

4.4 Model comparison

An important question is, are these methods necessary or would it be enough to set all pair-copulas equal to the Gaussian or t-copula to get an acceptable result and model? To test this assumption, we calculated the maximum likelihood for the two examples from above with Gaussian and t-copulas for all pair-copulas in a C-vine.
For the simulated data set from Example 4.1 we get the following unconditional Kendall's τ's (Table 4.15 for the C-vine Gaussian copula model (M2) and Table 4.16 for the C-vine t-copula model (M3)) and the difference between the empirical and the Monte Carlo estimated ones can be seen in Figure 4.8 and Figure 4.9, respectively.
The unconditional Kendall's τ matrix of the original data set can be found in Table 4.1 for comparison.

Short	**Model**	**Trees**	**Pair-copulas**
M1	mixed C-vine (true=selected model)	Figure 4.1 - 4.6	mixed copula
M2	C-vine Gaussian copula	Figure 4.1 - 4.6	bivariate Gaussian copula
M3	C-vine t-copula	Figure 4.1 - 4.6	bivariate t-copula

Table 4.14: Compendium of the compared models (Example 4.1)

	A	B	C	D
A	1.00	0.59	0.53	-0.33
B	0.59	1.00	0.66	-0.11
C	0.53	0.66	1.00	0.07
D	-0.33	-0.11	0.07	1.00

Table 4.15: Monte Carlo estimated unconditional Kendall's τ's of the C-vine Gaussian copula model (M2). (Example 4.1)

	A	B	C	D
A	1.00	0.58	0.53	-0.36
B	0.58	1.00	0.65	-0.15
C	0.53	0.65	1.00	0.04
D	-0.36	-0.15	0.04	1.00

Table 4.16: Monte Carlo estimated unconditional Kendall's τ's of the C-vine t-copula model (M3) (Example 4.1)

	M1	**M2**	**M3**
log-likelihood	2350.9	1906	2042
# of parameters	8	6	12
AIC	-4685.8	-3800	-4060
BIC	-4677.8	-3793	-4048
Test-statistic (Vuong)		12.4	11.8
p-value (Vuong)		0	0
Test-statistic (Clarke)		749	686
p-value (Clarke)		0	0

Table 4.17: Log-Likelihoods, number of parameters, AIC, BIC, Vuong- and Clarke-tests of the three models (Example 4.1)

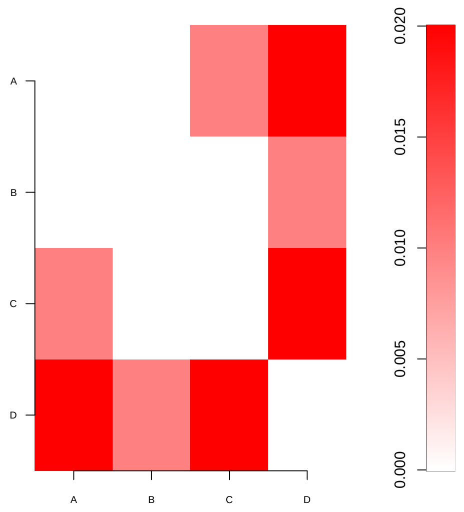


Figure 4.8: Absolute difference between the empirical and the Monte Carlo estimated unconditional Kendall's τ's for the C-vine Gaussian copula model (M2) (Example 4.1)

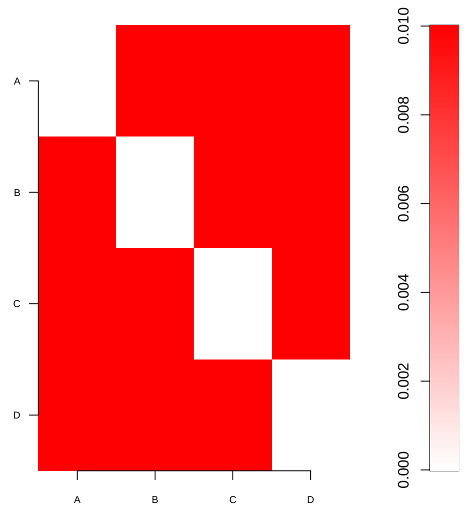


Figure 4.9: Absolute difference between the empirical and the Monte Carlo estimated unconditional Kendall's τ's for the C-vine t-copula model (M3) (Example 4.1)

For the second example we get these results (Table 4.19 and 4.20 and Figure 4.10 and 4.11) of the C-vines (M3') and (M4') and their error with respect to Kendall's τ.

Short	**Model**	**Trees**	**Pair-copulas**
M1'	mixed C-vine (true model)	-	mixed copula
M2'	mixed C-vine (selected model)	Figure 4.2	mixed copula
M3'	C-vine Gaussian copula	Figure 4.2	bivariate Gaussian copula
M4'	C-vine t-copula	Figure 4.2	bivariate t-copula

Table 4.18: Compendium of the compared models (Example 4.2)

Conclusions:
For the first example both tests, the Vuong- and the Clarke-test, favor the true model (M1)(see Table 4.17), although the model fits the unconditional Kendall's τ's very well in both cases, the Gaussian (M2) and the t-scenario (M3)(see Figure 4.8 and 4.9)
The log-likelihood of the Gaussian- and the t-scenario are also worse than the true one, but the log-likelihood does not implicate the number of parameters. But the AIC and the BIC does, and both favor the true model.
But we have to be cautiously by using AIC and BIC, because the AIC as well as the BIC are developed for regression models to compare nested models. In the vine circumstances the AIC and the BIC should only give us some hints if one model may be better than an other. The problem of the AIC and the BIC and other statistics based on the log-likelihood is that they assume increasing likelihood if one increases the number of parameters, adding

	C	A	B	D
C	1.00	0.41	0.67	0.49
A	0.43	1.00	0.27	0.45
B	0.67	0.27	1.00	0.36
D	0.49	0.45	0.36	1.00

Table 4.19: Monte Carlo estimated unconditional Kendall's τ's of the C-vine Gaussian copula (M3')(Example 4.2)

	C	A	B	D
C	1.00	0.41	0.68	0.51
A	0.43	1.00	0.26	0.47
B	0.68	0.26	1.00	0.37
D	0.51	0.47	0.37	1.00

Table 4.20: Monte Carlo estimated unconditional Kendall's τ's of the C-vine t-copula (M4')(Example 4.2)

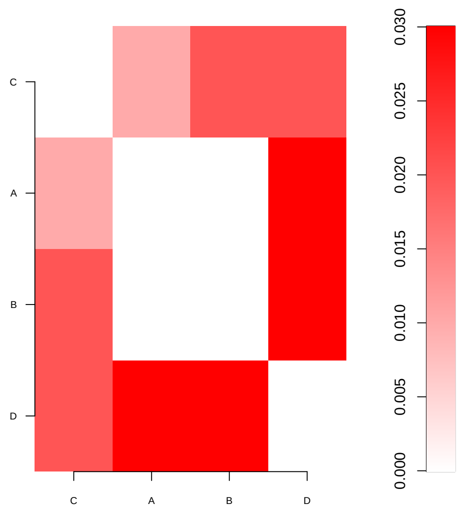


Figure 4.10: Absolute difference between the empirical and the Monte Carlo estimated unconditional Kendall's τ's for the C-vine Gaussian copulas (M3')(Example 4.2)

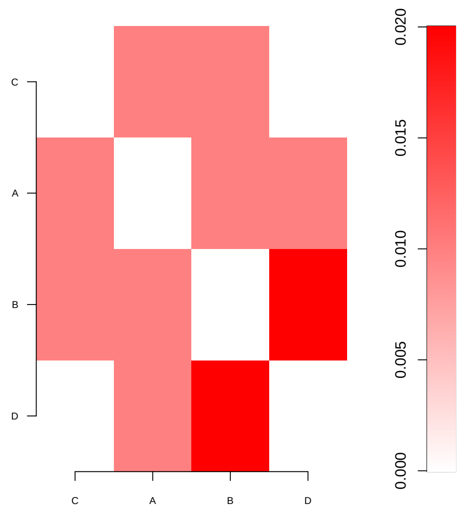


Figure 4.11: Absolute difference between the empirical and the Monte Carlo estimated unconditional Kendall's τ's for the C-vine t-copulas (M4')(Example 4.2)

	M1'	M2'	M3'	M4'
log-likelihood	1734.4	1583.1	1439	1520
# of parameters	7	8	6	12
AIC	-3454.8	-3150.2	-2866	-3016
BIC	-3447.8	-3142.2	-2860	-3004
Test-statistic (Vuong)		25.9	18.9	25.9
p-value (Vuong)		0	0	0
Test-statistic (Clarke)		844	837	817
p-value (Clarke)		0	0	0

Table 4.21: Log-Likelihoods, number of parameters, AIC, BIC, Vuong- and Clarke-tests of the compared models (Example 4.2)

a new covariable to the model in the regression sense. In the vine environments this assumption don't have to be true, because an one-parametric copula family can fit copula data better than a two-parametric copula family. Thus the log-likelihood don't have to increase if we increase the number of parameters. E.g. if an underlying copula is a Clayton copula a t-copula with two parameters (ρ and ν) don't fit the data better than a Clayton copula with only one parameter.
In Table 4.17 we can also see that the t-scenario (M3) is better than the Gaussian scenario (M2) because of the better log-likelihood, the test-statistics and the better fit of the Kendall's τ's. A possible explanation may be that the second parameter, the degrees of freedom, in each t-copula lead to a better fit.
In the second example we had three models, the model with the different first root (M2'), the model with Gaussian copula each (M3') and the one with the t-copulas (M4'). All three models lead to a good fit of the unconditional Kendall's τ's (see Figure 4.7, 4.10 and 4.11).
The tests with the Vuong- and Clarke-test lead us to the conclusion, that neither the model we called (M2'), with the different root, nor the models with just Gaussian or t-copula are the right one, although (M2') has a good fit with respect to Kendall's τ and a similar high log-likelihood. But we have to mention that the AIC and the BIC of (M2') is much better than the AIC or BIC of the models (M3') and (M4') (see Table 4.21). But again, the AIC and the BIC don't have a strong explanatory power because of the problem we mentioned above.

Chapter 5

Application: Exchange rates

In this chapter we will test our functions and methods we introduced in the previous chapters on a data set with exchange rates and look for dependence structures between the rates and how we can model them.

5.1 Data description

Our application is a data set, which contains 9 time series of exchange rates of different countries from 22. July 2005 until 17. July 2009, all in all 1007 data points of each time series. The exchange rates are notated in the home currency, e.g. 1 US-Dollar = 0.8466 Euro.
For the further notation we named the exchange rates as follows: EUR (Euro-area), UK (United Kingdom), CAN (Canada), AUS (Australia), BRA (Brazil), CH (China), JPN (Japan), SZ (Switzerland) and IN (India).
A graphical illustration of the exchange rates and their moments can be seen in Figure 5.1 and Table 5.1.

	Euro	UK	Canada	Australia	Brazil	China	Japan	Switzerland	India
Mean	0.75	0.55	1.12	1.27	2.04	7.48	110.38	1.18	44.20
Std. dev.	0.06	0.06	0.08	0.13	0.23	0.49	9.03	0.08	3.23
Skewness	-0.29	1.20	-0.14	0.17	-0.41	-0.19	-0.72	-0.42	0.18
Kurtosis	-0.81	0.45	-0.83	-0.37	-0.92	-1.60	-0.61	-0.82	-0.78

Table 5.1: Descriptive statistics for the exchange rates

For the following calculations we need independent data. Therefore we will use the ARMA(1,1)-GARCH(1,1) model we introduced in Section 2.7. Then we get for each time series i the following equations for the values X_i:

$$X_{i,t} = \mu_i + \Phi_i X_{i,t} + \sigma_{i,t} Z_{i,t} + \Theta_i \sigma_{i,t-1} Z_{i,t-1},$$
$$\sigma_{i,t}^2 = \omega + \alpha_i \epsilon_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2,$$

where $E[Z_{i,t}] = 0,\ Var(Z_{i,t}) = 1$ and $\epsilon_{i,t-1} = \sigma_{i,t} Z_{i,t}$.
We tested several models with Gaussian-, t- and skewed-t-innovations and GARCH(1,1),

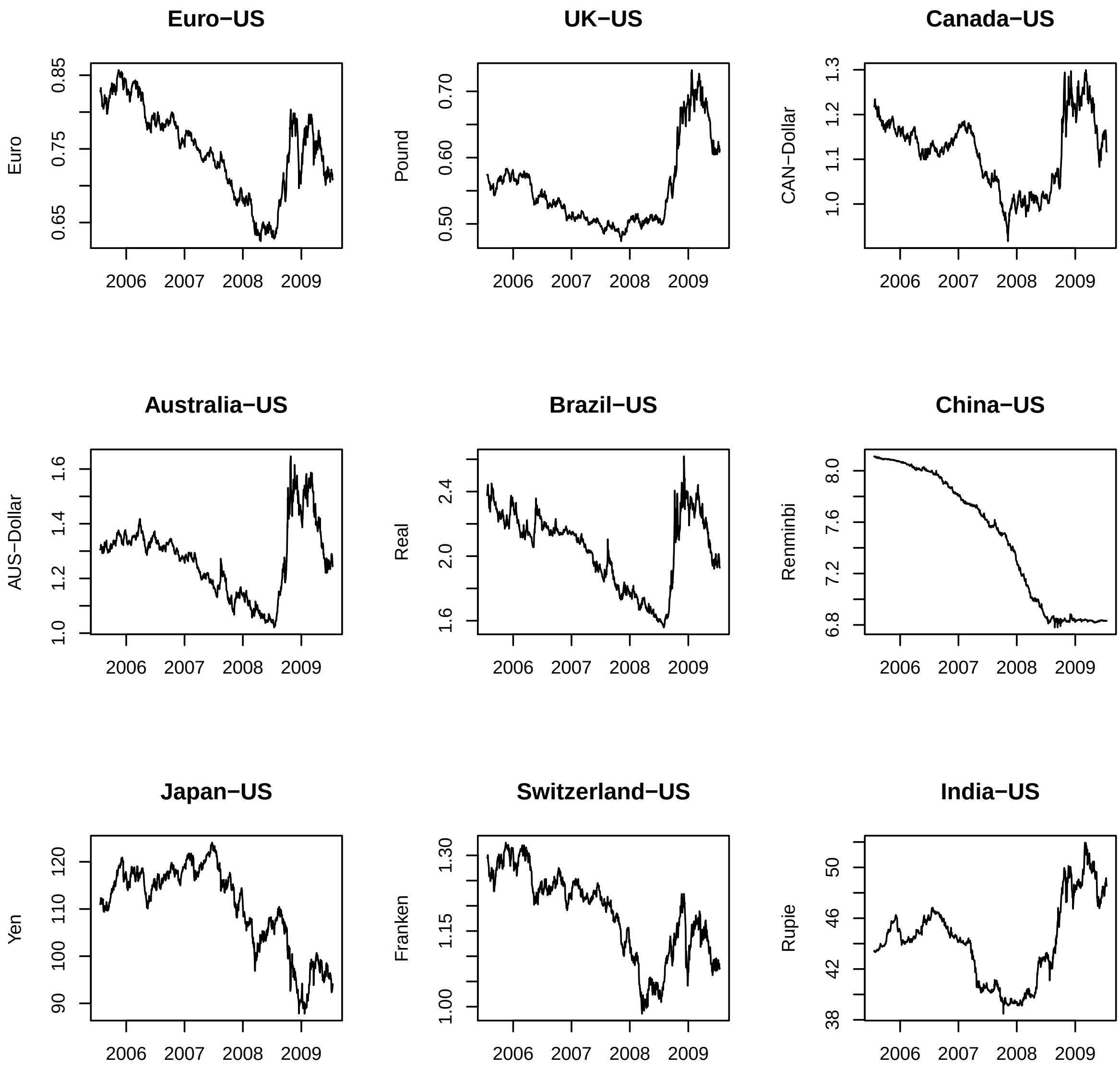


Figure 5.1: Exchange rates from 22. July 2005 to 17. July 2009

AR(1)+GARCH(1,1) and ARMA(1,1)+GARCH(1,1). For the analysis we will use the standardized residuals Z_i of these models. They can be estimated by

$$\hat{Z}_{i,t} = \frac{1}{\hat{\sigma}_{i,t}}(X_{i,t} - \hat{\mu}_i - \hat{\Phi}_i X_{i,t-1} - \hat{\Theta}_i \hat{\sigma}_{i,t-1}\hat{Z}_{i,t-1}),$$

where $\hat{\mu}_i, \hat{\sigma}_i, \hat{\Phi}_i, \hat{\Theta}_i, \hat{Z}$ are the estimated parameters of $\mu_i, \sigma_i, \Phi_i, \Theta_i$ and Z. To test the independence of these residuals we use the Ljung-Box-Test we introduced in Section 2.6.5. The results of these test can be seen in in the Table 5.2. To choose the best model for our data we also looked at the QQ-Plots of the standardized residuals of the different innovations. (see Appendix). In Table 5.3 we benchmarked the different models with respect to their QQ-Plots and their degree of fit to the underlying data. The benchmark has three levels. + means that the fit is OK but not very good, ++ means, that the fit is sufficient and +++ symbols that the fit is very good. Because of the results of Table 5.2, i.e. we detected that for all time series the ARMA(1,1)+GARCH(1,1) model is sufficient on the 5% level, we looked only at the QQ-Plots of the ARMA(1,1)+GARCH(1,1) model.

Country	**GARCH(1,1)**			**AR(1)+GARCH(1,1)**			**ARMA(1,1)+GARCH(1,1)**		
	N	t	skew-t	N	t	skew-t	N	t	skew-t
EUR	0.00	0.00	0.00	0.63	0.65	0.65	**0.84**	0.47	0.48
UK	0.00	0.00	0.00	0.86	0.92	0.92	0.91	**0.97**	0.94
CAN	0.00	0.00	0.00	0.25	0.24	0.24	0.78	**0.97**	0.96
AUS	0.00	0.00	0.00	0.47	0.54	0.53	**0.69**	0.56	0.19
BRA	0.00	0.00	0.00	**0.26**	0.26	0.25	0.17	**0.25**	0.08
CH	0.00	0.00	0.00	0.00	0.00	0.00	0.93	**1.00**	0.98
JPN	0.00	0.00	0.00	0.72	0.69	0.70	0.56	**0.72**	0.58
SZ	0.00	0.00	0.00	0.43	0.40	0.41	**0.90**	0.76	0.63
IN	0.00	0.00	0.00	0.05	0.09	0.09	**0.99**	0.36	0.35

Table 5.2: p-values of the Ljung-Box-test of the different models and exchange rates

Country	**Innovations**		
	N	t	skew-t
EUR	+++	++	++
UK	+++	+++	+++
CAN	+++	+++	+++
AUS	++	++	++
BRA	++	+++	+++
CH	+	++	++
JPN	++	+	++
SZ	+++	++	+++
IN	+	++	++

Table 5.3: Benchmark of the QQ-Plots for different innovations (Gaussian, t and skewed-t)

On this basis we choose our models for the different exchange rates. For a good model we have to have in mind both criterias to make an appropriate choice. The final model choice is tabled in Table 5.4.

Country	Models	
EUR	ARMA(1,1)+GARCH(1,1)	Gaussian innovations
UK	ARMA(1,1)+GARCH(1,1)	t innovations
CAN	ARMA(1,1)+GARCH(1,1)	t innovations
AUS	ARMA(1,1)+GARCH(1,1)	Gaussian innovations
BRA	ARMA(1,1)+GARCH(1,1)	t innovations
CH	ARMA(1,1)+GARCH(1,1)	t innovations
JPN	ARMA(1,1)+GARCH(1,1)	Skewed-t innovations
SZ	ARMA(1,1)+GARCH(1,1)	Gaussian innovations
IN	ARMA(1,1)+GARCH(1,1)	Skewed-t innovations

Table 5.4: Final model choice for the exchange rate data set.

The resulting standardizised residuals of these models are transformed with the "Empirical Cumulative Distribution Function" and a scaling factor $\frac{1}{n-1}$ to copula data on $[0, 1]$. In Figure 5.2 we have drawn the pairs-plots on the right side of the diagonal and the according Kendall's τ's on the left side of the diagonal. We can detect some dependencies between the exchange rates, especially between the EURO and UK, EURO and AUS, EURO and SZ and some other dependencies. Further, we can see that some exchange rates are almost independent as for example CH and JPN and CH and SZ. A next interesting fact is, that almost all dependencies are positive beside the pair BRA - JPN and CAN - JPN.

With these copula data we want to build our C-vine-model.

5.2 Model selection

5.2.1 Tree selection on the first level

Now we will use the sequential method developed in Chapter 4 to select a C-vine structure for our data. Using this as a starting point we will then perform an optimization to find the parameters which maximize the likelihood of the distribution.
As in Chapter 4 we will not draw all steps here for the selection of the trees and copula families. But we will show the method on the first tree (first level) exemplarily and summarize the results of our model in some plots and detailed tabulars.

The first step in every C-vine is to find the first tree. As we explained in Chapter 4 we found our first tree by summarizing the Kendall's τ's over each row and choosing the maximum as root node. (see Table 5.5)
As we expected from the pairs-plots the exchange rate between Euro and the US-Dollar is the root-node.

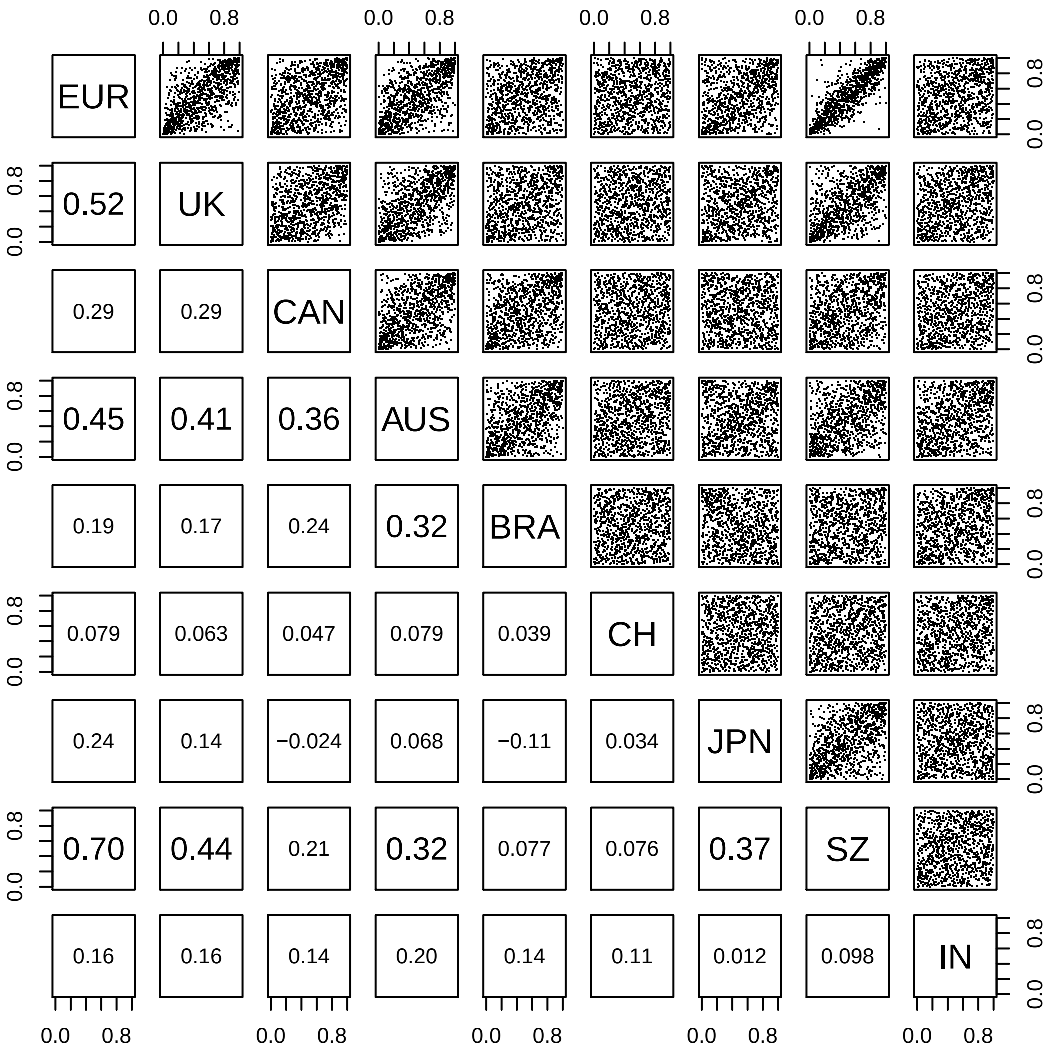


Figure 5.2: Pairsplot of the copula data (top, right) and the according Kendall's τ (bottom, left)

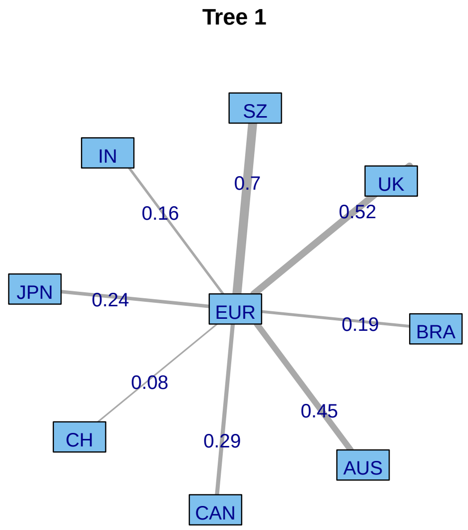


Figure 5.3: First tree of the C-vine of the exchange rate data

	EUR	UK	CAN	AUS	BRA	CH	JPN	SZ	IN	Sum
EUR	1.00	0.51	0.29	0.44	0.19	0.07	0.24	0.69	0.16	**3.63**
UK	0.51	1.00	0.28	0.41	0.17	0.06	0.13	0.43	0.15	3.18
CAN	0.29	0.28	1.00	0.35	0.24	0.04	-0.02	0.20	0.14	2.60
AUS	0.44	0.41	0.35	1.00	0.31	0.07	0.06	0.32	0.19	3.19
BRA	0.19	0.17	0.24	0.31	1.00	0.03	-0.11	0.07	0.14	2.30
CH	0.07	0.06	0.04	0.07	0.03	1.00	0.03	0.07	0.10	1.52
JPN	0.24	0.13	-0.02	0.06	-0.11	0.03	1.00	0.37	0.01	2.00
SZ	0.69	0.43	0.20	0.32	0.07	0.07	0.37	1.00	0.09	3.28
IN	0.16	0.15	0.14	0.19	0.14	0.10	0.01	0.09	1.00	2.02

Table 5.5: Empirical Kendall's τ matrix and the sum over each row for the exchange rate data set

5.2.2 Copula selection on the first level

The next step is to select an adequate copula family for every pair-copula in the first tree (first level). In Chapter 4 we described how we try to select an adequate copula family by using contourplots, λ-functions and the goodness-of-fit test.
These methods we will use to find the copula families for the first tree and in the next sections the copula families of the other trees. But now we will concentrate on the first tree and explain in detail how we choose our copula families. In the next section we don't will do this in this detailed way.
For the first copula pair (EUR-UK) Figure 5.4 illustrates the empirical density in a contourplot on the left side and the empirical λ-function on the right side. Because of some experience we could limit the number of possible copula families for comparison, but in this first step we want to show all the possibilities and why we can eliminate them.

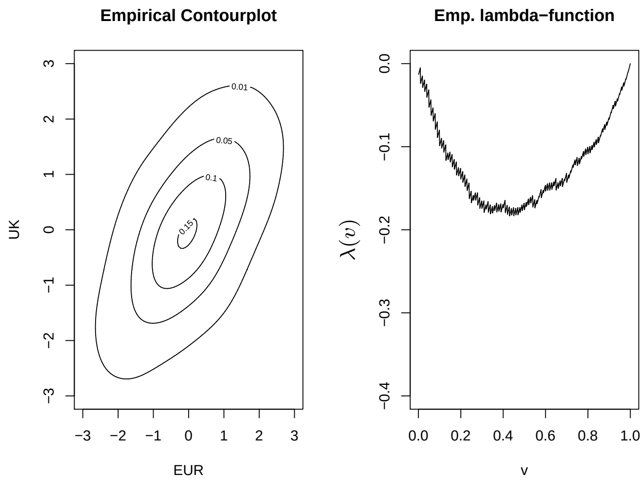


Figure 5.4: Empirical contourplot (left) and λ-function (right) of the first copula pair (EUR-UK)

We choose the Gaussian, t- Clayton, Gumbel, Frank, Joe, BB1- and BB7-copula to compare with the empirical contourplot.

When we look at the contourplots in Figure 5.5 we can reject the Clayton, Gumbel, Frank and Joe copula. Although the empirical contourplot hat a small tendency to V-notch in the lower left corner, we decide that only a Gaussian or t-copula may fit the data well. The BB1- and BB7-copula have a lower tail dependence which we can not detect in the empirical contourplot. But the experience we get by several tests lead us to the assumption that the BB1- and BB7-copula may also be a good fit.

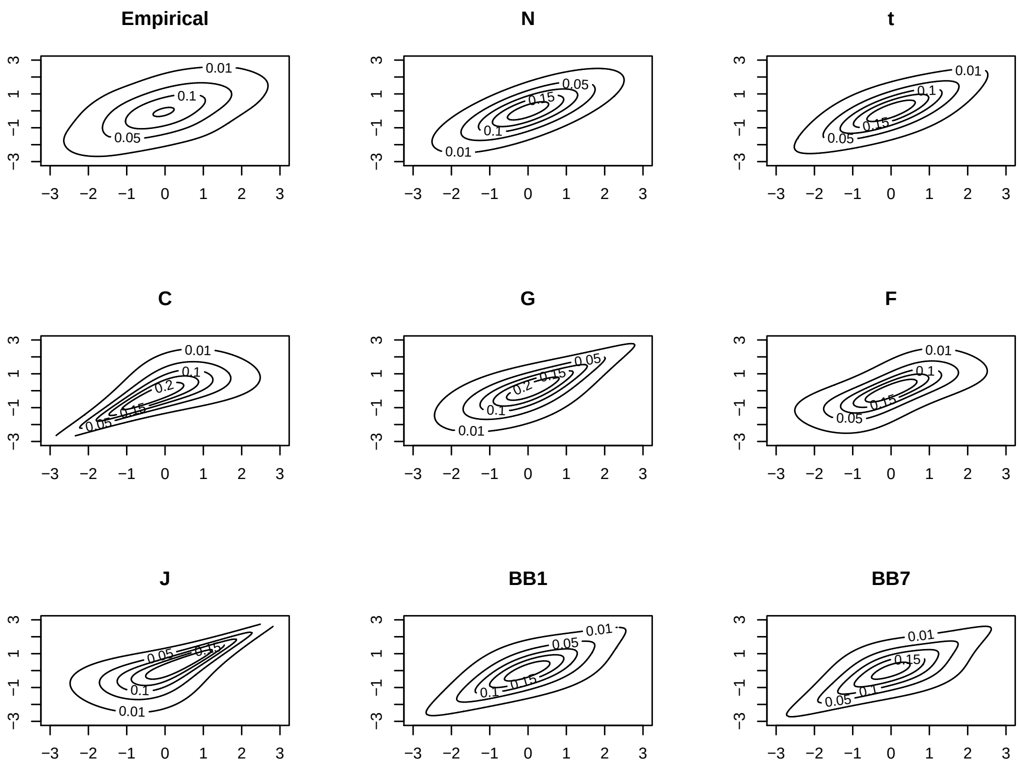


Figure 5.5: The empirical contourplot of the pair copula EUR-UK and the theoretical Gaussian (N), t-, Clayton (C), Gumbel (G), Frank (F), Joe (J), BB1- and BB7-copula

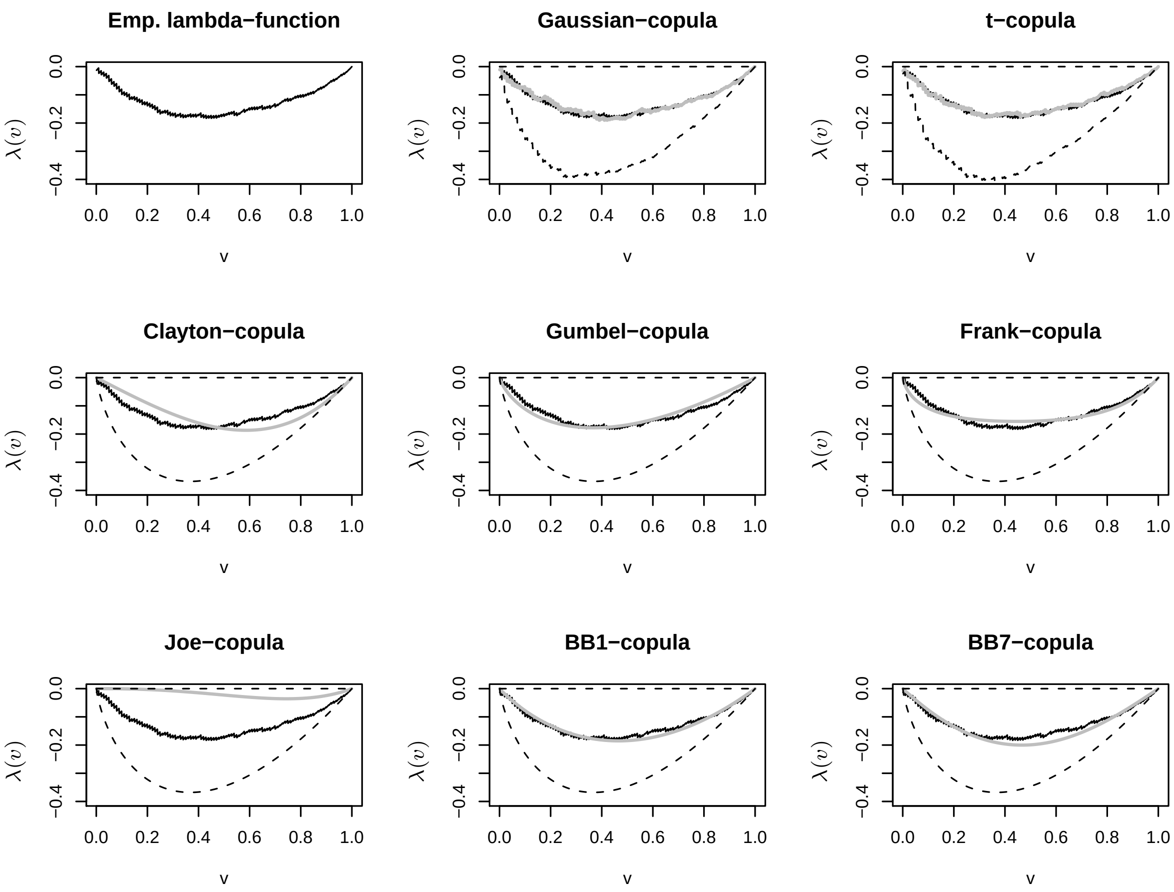


Figure 5.6: The empirical λ-function of the pair copula EUR-UK and the theoretical Gaussian, t-, Clayton, Gumbel, Frank, Joe, BB1- and BB7-copula

The second method was the λ-function. The hypothesis we made out of the contourplots seems to be confirm. The Gaussian, t-, BB1- and BB7-copula are near the empirical λ-function (see Figure 5.6). These results are also be proven by the goodness-of-fit test in Table 5.6 and 5.7, which favor the Gaussian and the t-copula.

	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0193	0.0149	0.4070	0.1194	0.0711
p-value	**0.3021**	**0.5239**	0.0004	0.0004	0.0004

Table 5.6: Goodness-of-fit test for the exchange rate data set (first copula)

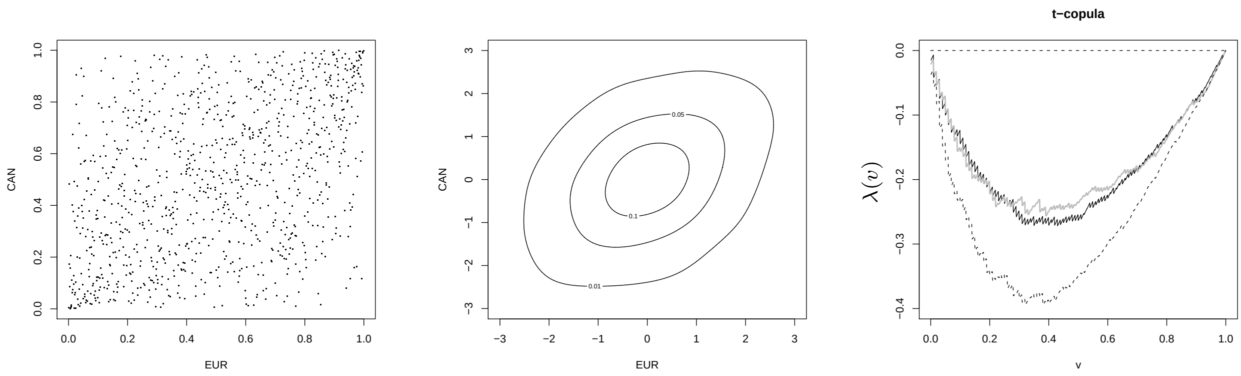
As a second goodness-of-fit test we have introduced the one with respect to the Vuong- and Clarke-test. As we mentioned it already in Chapter 4, the test is implemented for the Gaussian, the t-, the Clayton, the Gumbel, the Frank, the BB1- and the BB7-copula.
For the Joe copula there is no goodness-of-fit implemented neither in the package "copula" nor by Djunushalieva [2010]. But the Joe copula is in some kind similar to the Gumbel copula with some more extreme behavior in the tails. In this data set we never detected structures which would lead us to the assumption of a Joe copula.

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	**5**	**5**	-5	-2	-1	0	-2
Clarke	0	**6**	-5	0	1	1	-3

Table 5.7: Goodness-of-fit test for the C-vine on the basis of Vuong and Clarke (first copula)

For the first copula we choose the t-copula, because of the good contourplot and the goodness-of-fit test. The Gaussian copula is also a good choice based on our plots and tests. The other copulas can be rejected because of minimum two criteria except of the BB1- and BB7-copula which have to be handled cautiously.
For the other 7 copulas we do we same as for the first copula, but we will skip all the plots and summarize the results in the Tables 5.8 - 5.14.

Copula $c_{EUR,CAN}$ $(\hat{\tau} = 0.29)$



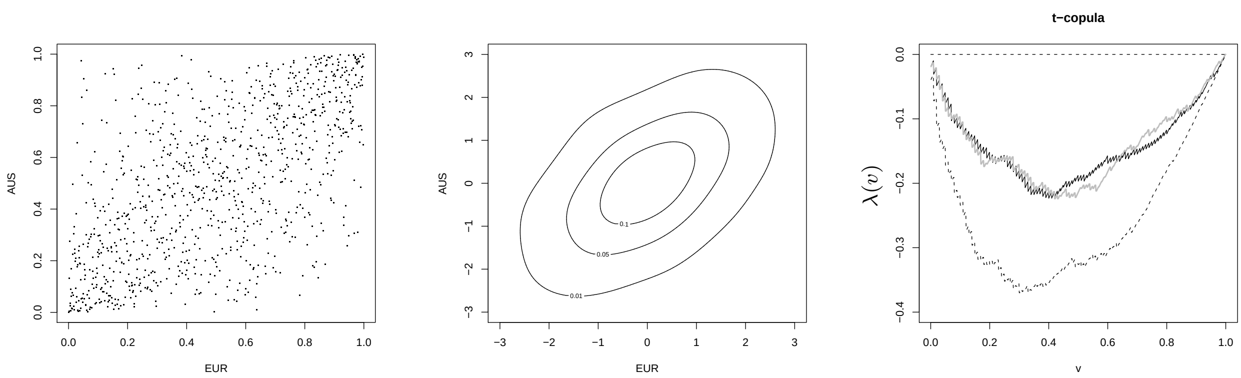
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0244	0.0201	0.2054	0.0348	0.0570
p-value	**0.2112**	**0.3261**	0.0004	0.0454	0.0004

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	2	2	-5	1	-3	**3**	0
Clarke	-6	**6**	-4	4	1	1	-2

Possible copula(s):	Selected copula:
N, t, BB1	t

Table 5.8: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,CAN}$ of the exchange rate data C-vine

Copula $c_{EUR,AUS}$ $(\hat{\tau} = 0.45)$



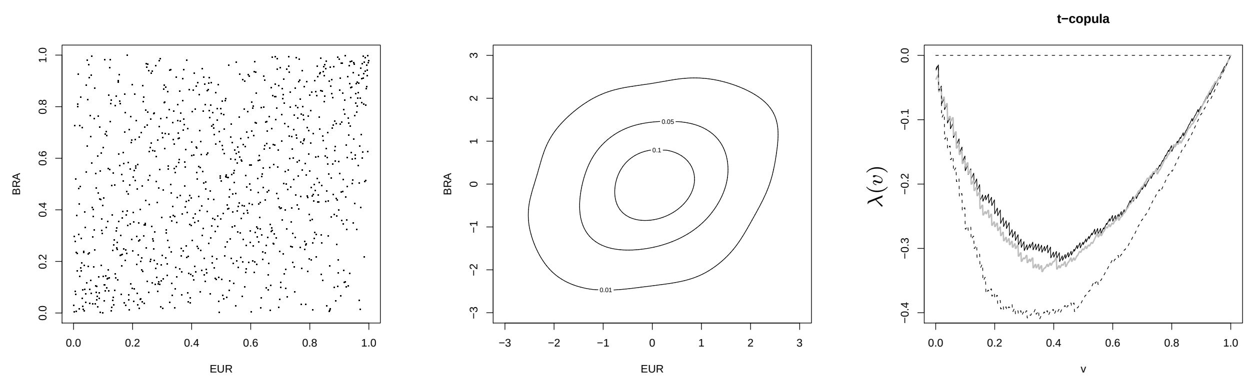
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0215	0.0189	0.4498	0.0903	0.0536
p-value	**0.2662**	**0.3421**	0.0004	0.0004	0.0014

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	**4**	**4**	-6	-1	2	0	-3
Clarke	-5	**6**	-5	3	3	0	-2

Possible copula(s):	Selected copula:
N, t	t

Table 5.9: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,AUS}$ of the exchange rate data C-vine

Copula $c_{EUR,BRA}$ $(\hat{\tau} = 0.19)$



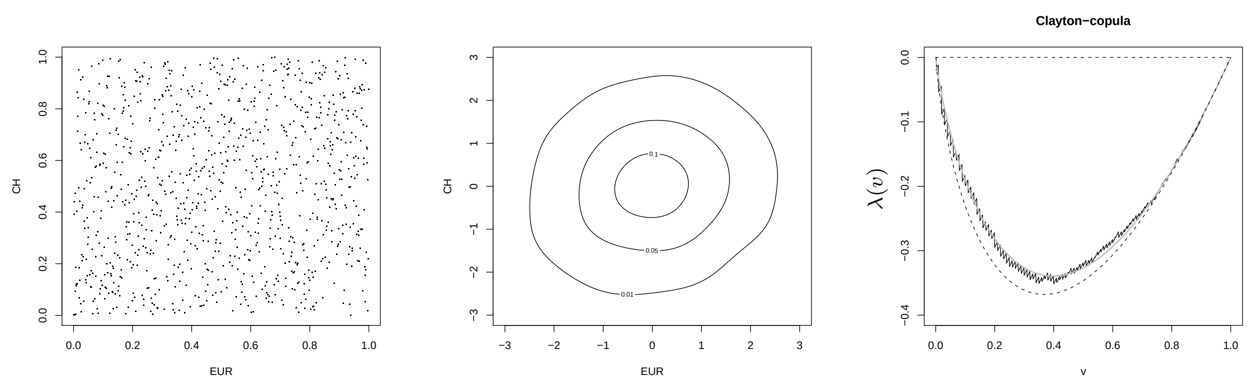
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0307	0.0232	0.0988	0.0433	0.0511
p-value	**0.0944**	**0.2052**	0.0004	0.0124	0.0004

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	1	**1**	-4	0	0	1	1
Clarke	-6	**6**	-3	4	-3	1	1

Possible copula(s):	Selected copula:
N, t, BB1, BB7	t

Table 5.10: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,BRA}$ of the exchange rate data C-vine

Copula $c_{EUR,CH}$ $(\hat{\tau} = 0.08)$



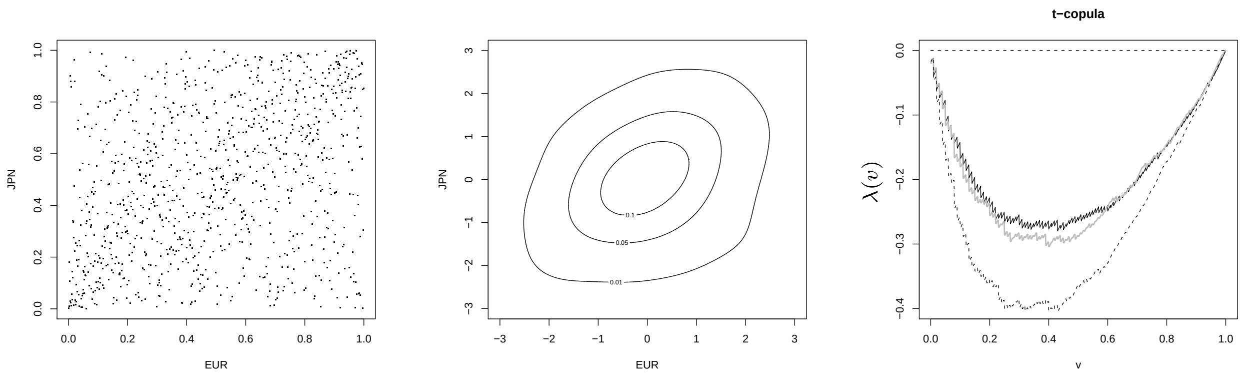
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0152	0.0143	0.0118	0.0381	0.0176
p-value	0.7297	0.7697	**0.9545**	0.0464	0.4760

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	3	-1	**3**	-1	1	-5
Clarke	-3	**5**	2	2	-1	-5

Possible copula(s):	Selected copula:
N, t, C, F	C

Table 5.11: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,CH}$ of the exchange rate data C-vine

Copula $c_{EUR,JPN}$ $(\hat{\tau} = 0.24)$



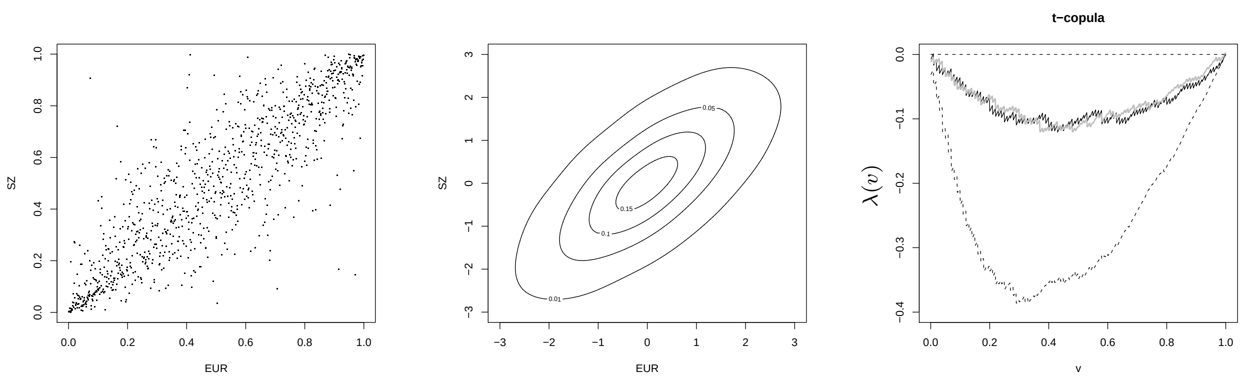
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0315	0.0133	0.4876	0.1082	0.0722
p-value	0.0074	**0.4130**	0.0004	0.0004	0.0004

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	-2	**6**	-3	-1	-1	1	0
Clarke	-6	**6**	-4	4	0	1	-1

Possible copula(s):	Selected copula:
t, F, BB1, BB7	t

Table 5.12: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,JPN}$ of the exchange rate data C-vine

Copula $c_{EUR,SZ}$ $(\hat{\tau} = 0.7)$



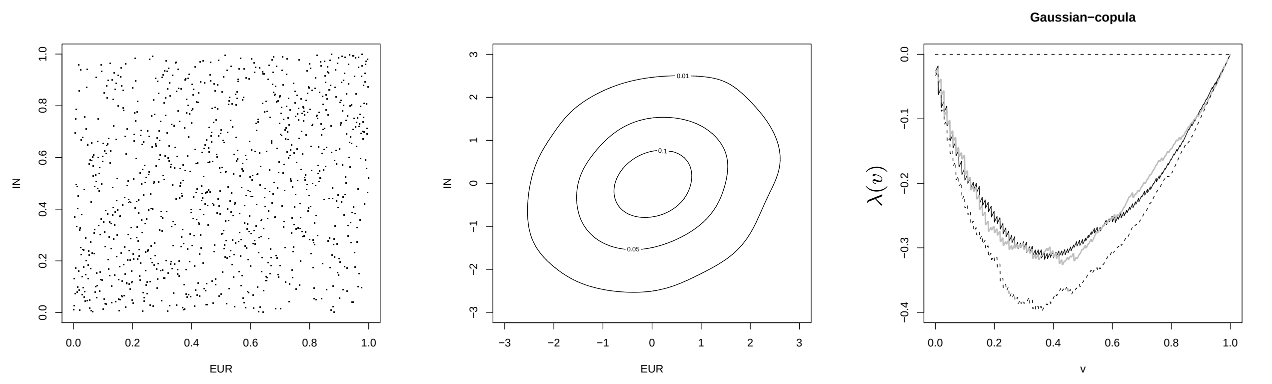
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0315	0.0133	0.4876	0.1082	0.0722
p-value	0.0074	**0.4130**	0.0004	0.0004	0.0004

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	2	**6**	-6	1	2	-1	-4
Clarke	4	**6**	-4	1	1	-2	-6

Possible copula(s):	Selected copula:
N, t	t

Table 5.13: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,SZ}$ of the exchange rate data C-vine

Copula $c_{EUR,IN}$ ($\hat{\tau} = 0.16$)



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises stat.	0.0123	0.0119	0.0836	0.0414	0.0160
p-value	**0.8946**	**0.9105**	0.0004	0.0194	**0.5579**

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	**4**	-2	-1	1	0	-2	0
Clarke	-6	**6**	-1	4	-1	-1	-1

Possible copula(s):	Selected copula:
N, t, F	N

Table 5.14: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{EUR,IN}$ of the exchange rate data C-vine

For some copulas we are indifferent between two or more copula families. As we mentioned it in Chapter 4, it is difficult to select the right one. For the copula $c_{EUR,CH}$ we selected the Clayton copula but because of the very low Kendall's τ the independent copula would be a good choice too.

Remark:
The Clayton copula converges for $\theta \to 0$ against the independent copula as well as the Gaussian copula or other copulas. The advantage of setting a copula the independent copula is that one does not have to fit a parameter and the calculation of the maximum likelihood is easier and faster.

If we choose the t-copula for the first, second, third, forth, sixth and seventh copula, the Clayton copula for fifth copula and the Gaussian copula for the last copula, when we get the second tree (Figure 5.7) by calculating the h-functions of these copulas. Therefore we need the estimated parameters we get by inverting Kendall's τ of the appropriate copula and solving the resulting equation with the empirical τ. For the t-copula we need the bivariate MLE as we mentioned it in Chapter 4. But notice, if we choose other copulas in the first tree we may get a different second and different trees in the following levels. The results may be improved if we use the maximum-likelihood estimator for all parameters, but the tests we made before lead us to the assumption, that the estimators we get by inverting τ are good enough.
For the second and sixth copula the BB1-copula and for the forth copula the BB7-copula

may be a good choice too. We tested this scenario with the statistics we explained in Chapter 4.3 and 4.4, but the results are worse than the results we get if we choose the t-copula for this copula families.

Remark:
The advantage of the method of inverting Kendall's τ is the fast computation.

For many copula families we get several possible copulas and we had to decide which one we use. To make a good choice we looked at all criterias and tried to find the differences between the copulas and which one may fit the data best. Again, these criteria are subjective criteria and an other person may choose other copulas.

The result of our copula choice is the second tree (Figure 5.7).

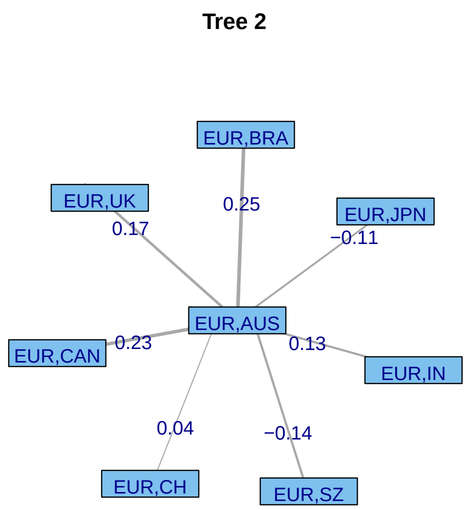
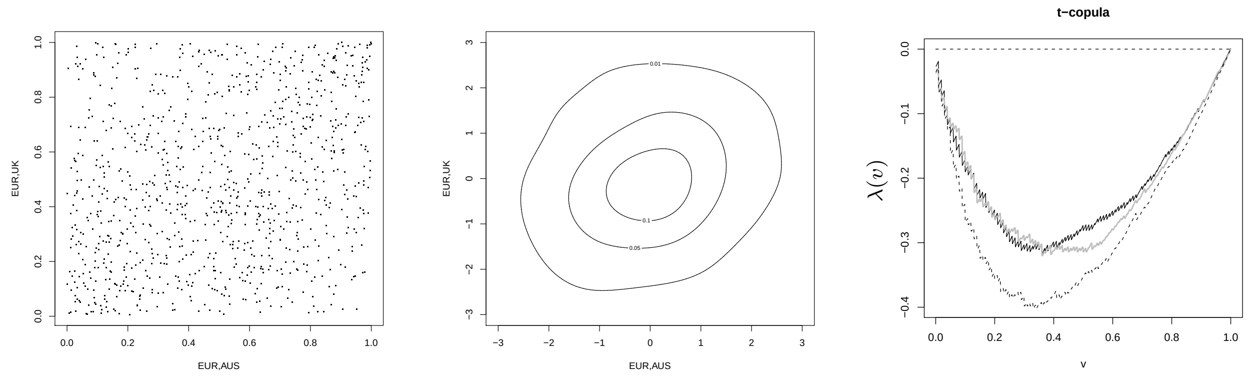


Figure 5.7: Second tree of the C-vine of the exchange rate data

5.2.3 Model selection for the next levels

For the following trees and copulas we do the same routines and select the copulas and trees by the criteria and methods we explained above and in Chapter 4.

Copula $c_{UK,AUS|EUR}$ $(\hat{\tau} = 0.17)$



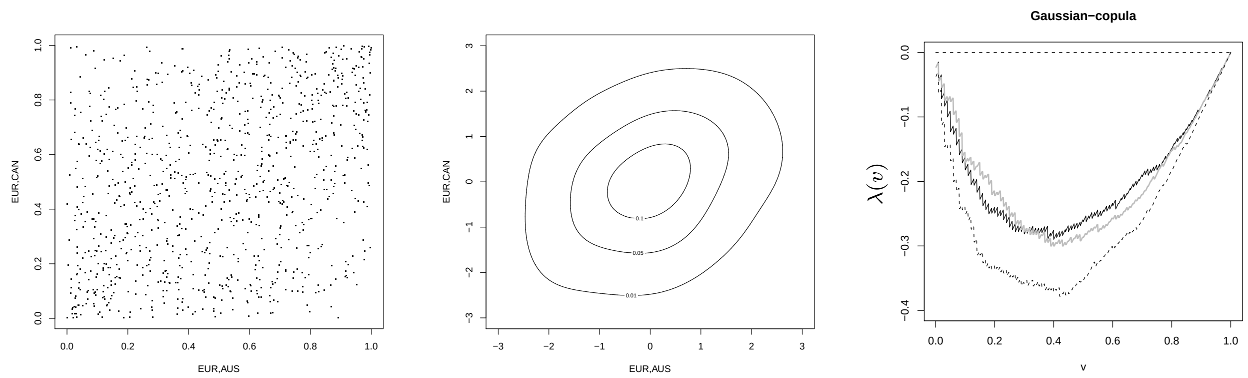
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0176	0.0163	0.1159	0.0252	0.0234
p-value	**0.5789**	**0.6118**	0.0004	**0.2252**	**0.1803**

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	1	0	-2	0	1	0	0
Clarke	-6	**6**	-3	4	-2	1	0

Possible copula(s):	Selected copula:
N, t, G, F	t

Table 5.15: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{UK,AUS|EUR}$ of the exchange rate data C-vine

Copula $c_{CAN,AUS|EUR}$ $(\hat{\tau} = 0.23)$



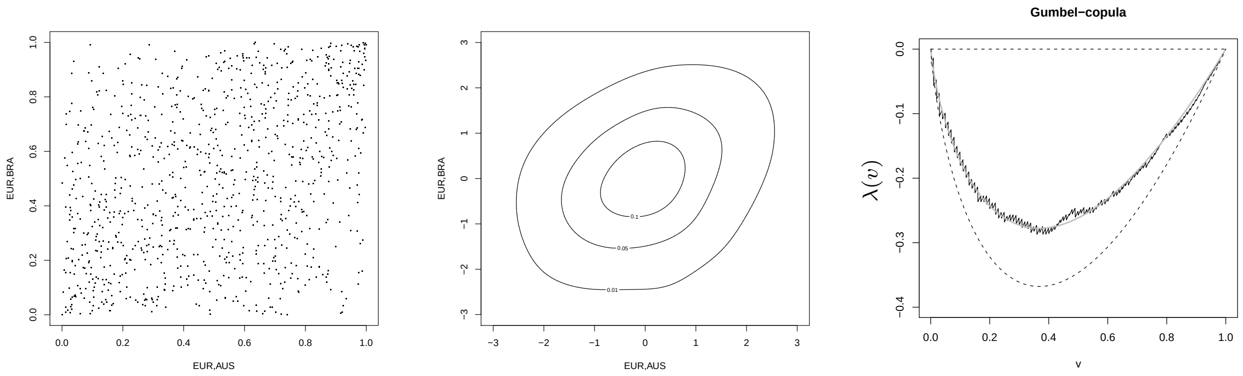
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0200	0.0220	0.1920	0.0275	0.0279
p-value	**0.4140**	**0.2672**	0.0004	**0.1323**	0.0714

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	1	1	-6	1	1	1	0
Clarke	-6	**6**	-3	4	-1	1	-1

Possible copula(s):	Selected copula:
N, t, F, BB1, BB7	N

Table 5.16: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CAN,AUS|EUR}$ of the exchange rate data C-vine

Copula $c_{BRA,AUS|EUR}$ ($\hat{\tau} = 0.25$)



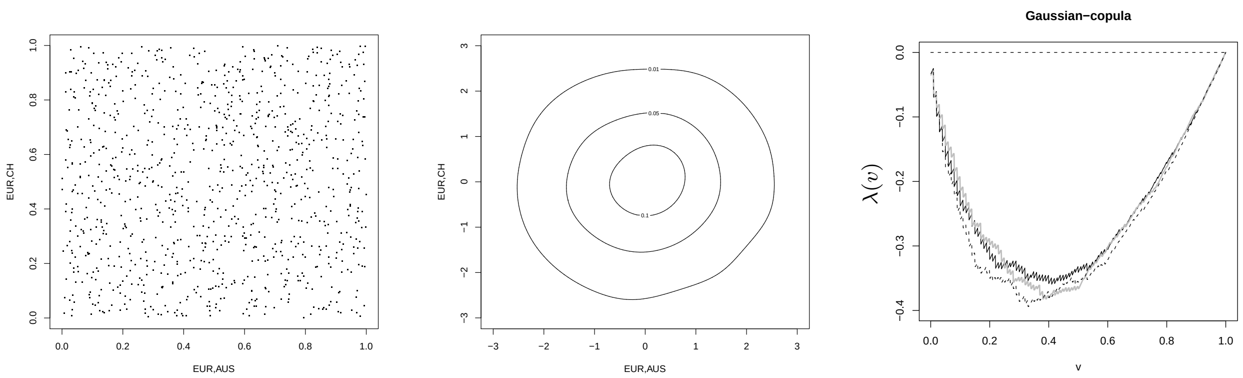
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0337	0.0351	0.2622	0.0193	0.0475
p-value	0.0494	0.0364	0.0004	**0.4480**	0.0034

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	1	1	-6	1	1	**2**	0
Clarke	-6	**6**	-3	4	-1	1	-1

Possible copula(s):	Selected copula:
G, BB1, BB7	G

Table 5.17: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{BRA,AUS|EUR}$ of the exchange rate data C-vine

Copula $c_{CH,AUS|EUR}$ ($\hat{\tau} = 0.04$)



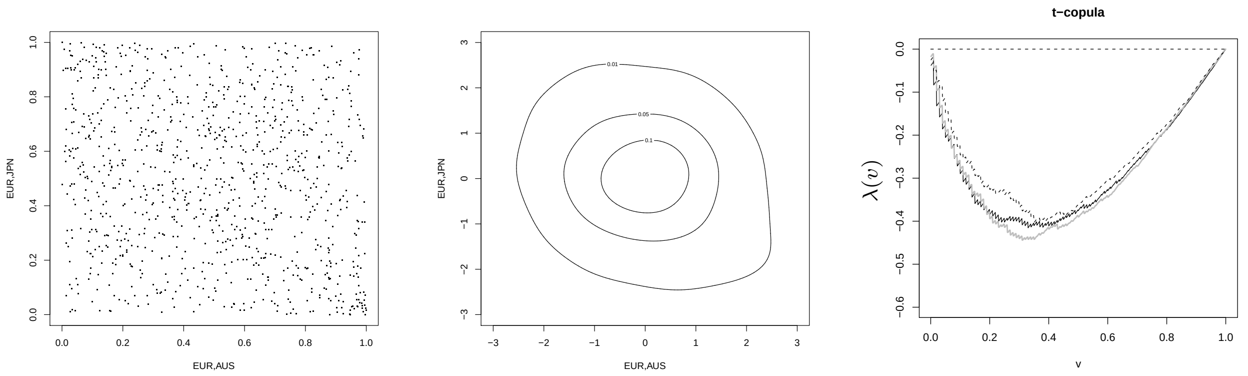
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0112	0.0112	0.0221	0.0117	0.0107
p-value	**0.9475**	**0.9525**	**0.3961**	**0.9545**	**0.9455**

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	2	-3	2	2	2	-5
Clarke	-3	**5**	2	2	-1	-5

Possible copula(s):	Selected copula:
N, t, C, G, F	N (indifferent)

Table 5.18: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CH,AUS|EUR}$ of the exchange rate data C-vine

Copula $c_{JPN,AUS|EUR}$ ($\hat{\tau} = -0.11$)



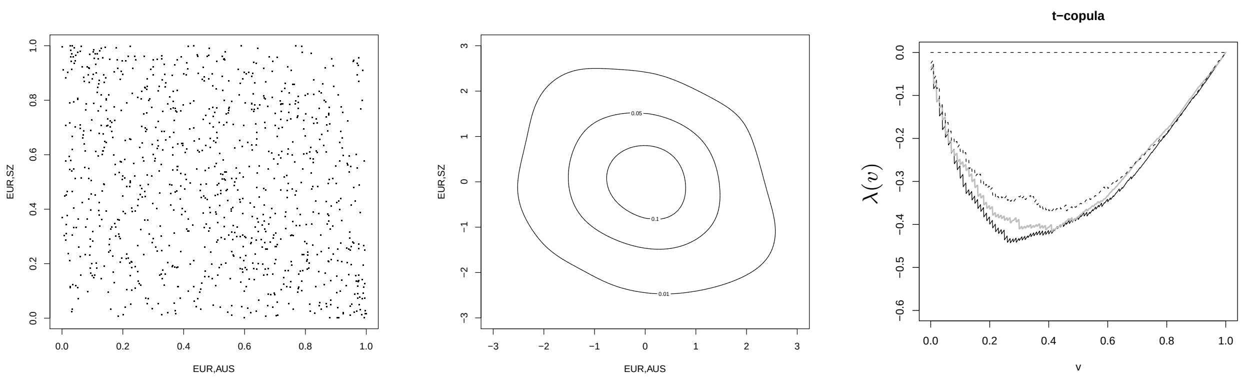
	Gaussian	t	Frank
Cramer-von Mises statistic	0.0790	0.0496	0.0673
p-value	0.0004	0.0034	0.0004

	Gaussian	t	Frank
Vuong	1	0	-1
Clarke	-1	2	-1

Possible copula(s):	Selected copula:
N, t, F	t

Table 5.19: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{JPN,AUS|EUR}$ of the exchange rate data C-vine

Copula $c_{SZ,AUS|EUR}$ ($\hat{\tau} = -0.14$)



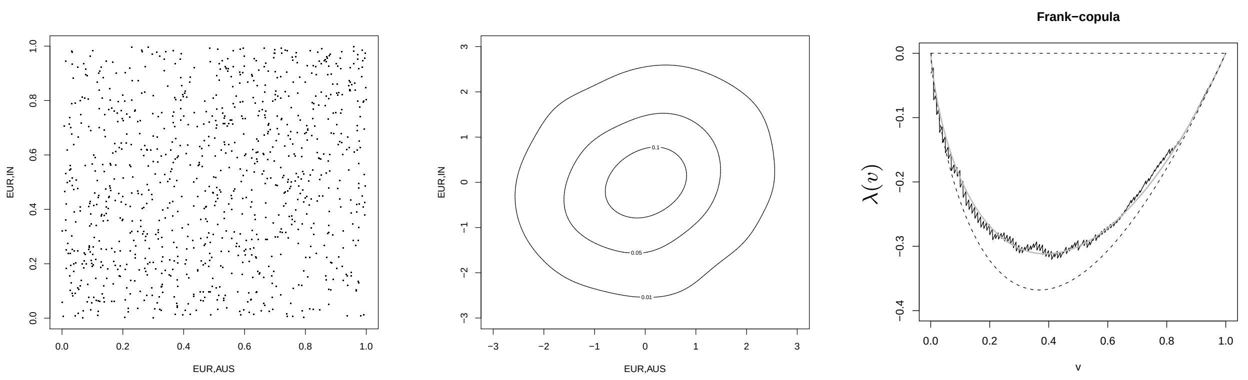
	Gaussian	t	Frank
Cramer-von Mises statistic	0.0388	0.0266	0.0418
p-value	0.0264	**0.1643**	0.0084

	Gaussian	t	Frank
Vuong	1	0	-1
Clarke	-2	**2**	0

Possible copula(s):	Selected copula:
N, t, F	t

Table 5.20: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{SZ,AUS|EUR}$ of the exchange rate data C-vine

Copula $c_{IN,AUS|EUR}$ $(\hat{\tau} = 0.13)$



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0257	0.0244	0.1224	0.0391	0.0135
p-value	**0.1593**	**0.1983**	0.0004	0.0294	**0.7727**

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	4	-2	-1	3	**4**	-6	-2
Clarke	-4	**6**	0	4	0	-6	0

Possible copula(s):	Selected copula:
N, t, F, BB1	F

Table 5.21: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{IN,AUS|EUR}$ of the exchange rate data C-vine

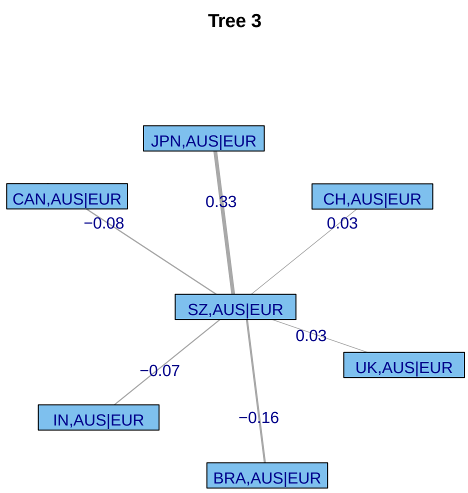
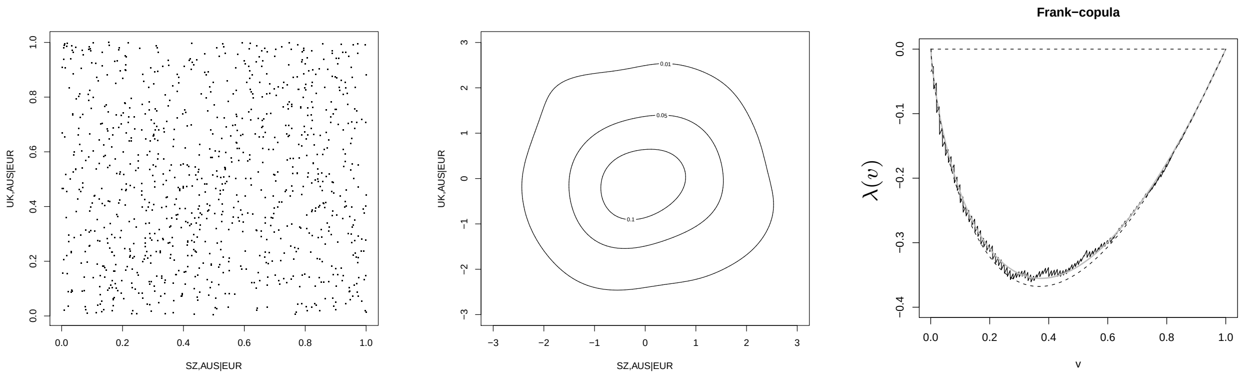


Figure 5.8: Third tree of the C-vine of the exchange rate data

Copula $c_{UK,SZ|EUR,AUS}$ $(\hat{\tau} = 0.03)$



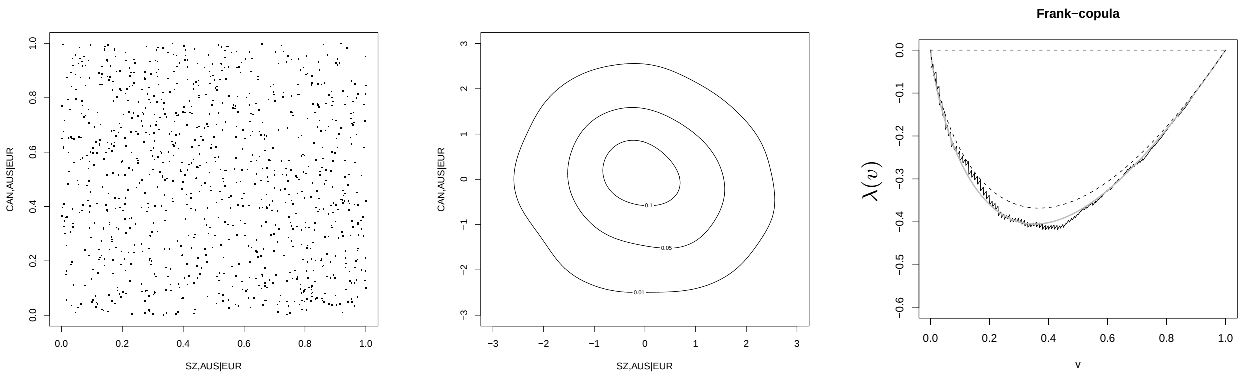
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0819	0.0493	0.0661	0.0625	0.0448
p-value	0.0004	0.0014	0.0044	0.0034	0.0074

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	0	1	1	1	2	-5
Clarke	-3	**5**	1	2	0	-5

Possible copula(s):	Selected copula:
N, t, C, F	F

Table 5.22: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{UK,SZ|EUR,AUS}$ of the exchange rate data C-vine

Copula $c_{CAN,SZ|EUR,AUS}$ $(\hat{\tau} = -0.08)$



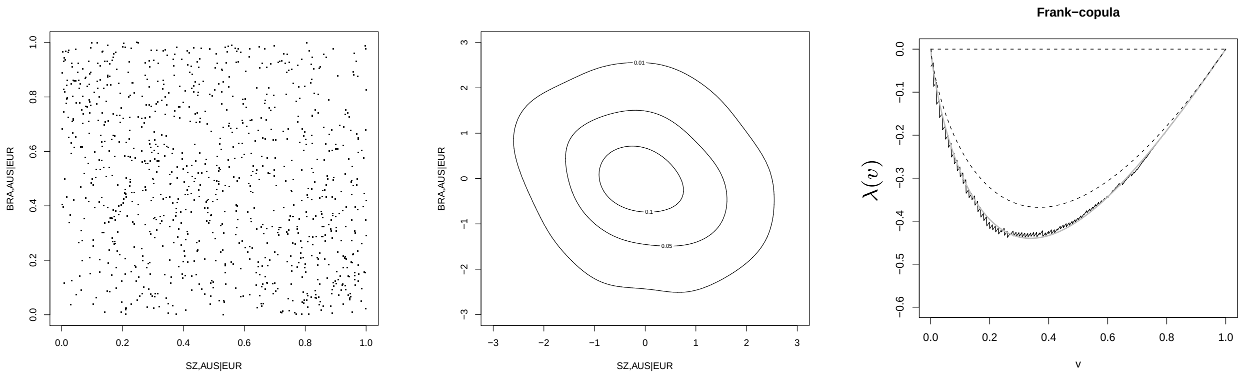
	Gaussian	t	Frank
Cramer-von Mises statistic	0.0384	0.0380	0.0287
p-value	0.0274	0.0264	**0.0754**

	Gaussian	t	Frank
Vuong	0	-2	**2**
Clarke	-2	2	0

Possible copula(s):	Selected copula:
N, t, F	F

Table 5.23: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CAN,SZ|EUR,AUS}$ of the exchange rate data C-vine

Copula $c_{BRA,SZ|EUR,AUS}$ $(\hat{\tau} = -0.16)$



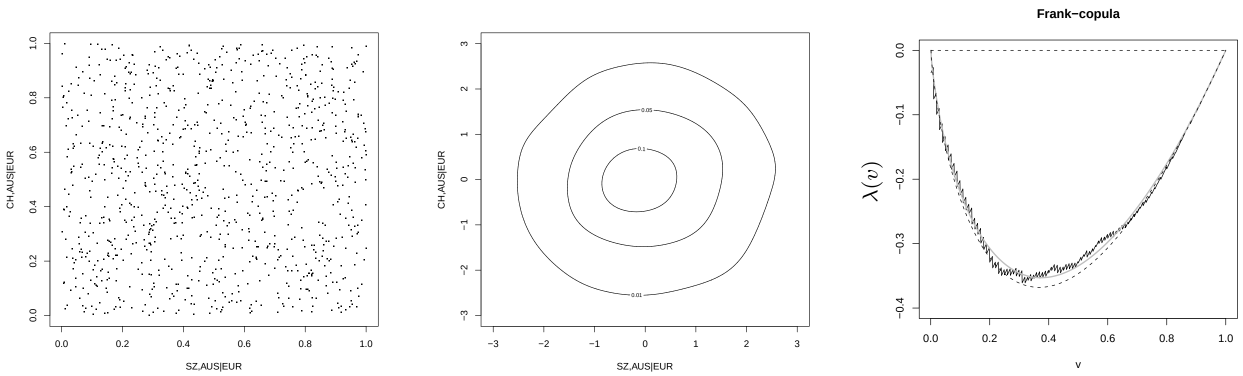
	Gaussian	t	Frank
Cramer-von Mises statistic	0.0281	0.0272	0.0240
p-value	**0.1173**	**0.1153**	**0.1503**

	Gaussian	t	Frank
Vuong	1	-2	1
Clarke	-2	**2**	0

Possible copula(s):	Selected copula:
N, t, F	F

Table 5.24: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{BRA,SZ|EUR,AUS}$ of the exchange rate data C-vine

Copula $c_{CH,SZ|EUR,AUS}$ $(\hat{\tau} = 0.03)$



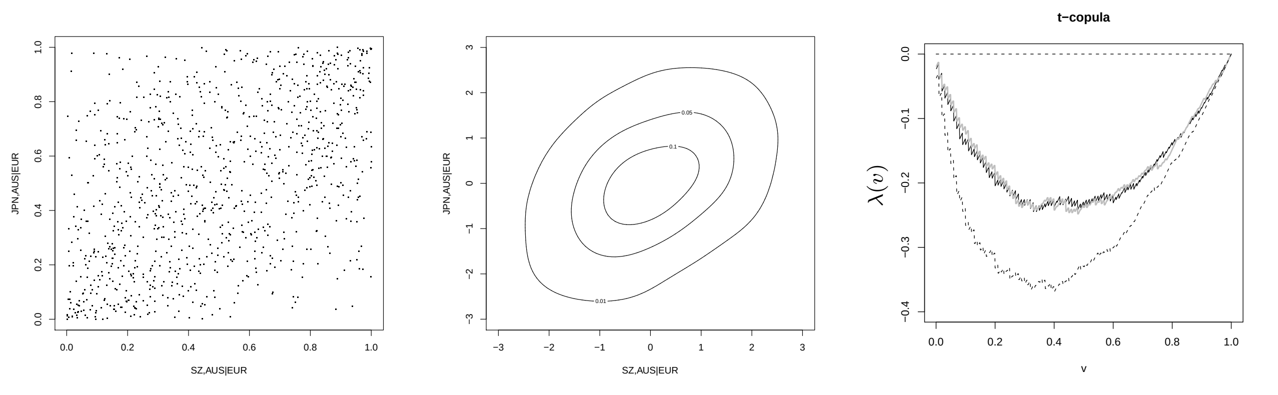
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0171	0.0167	0.0243	0.0272	0.0136
p-value	**0.5889**	**0.6288**	**0.2962**	**0.2302**	**0.7637**

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	2	-3	2	2	2	-5
Clarke	-2	**5**	1	2	-1	-5

Possible copula(s):	Selected copula:
N, t, C, G, F	F (indifferent)

Table 5.25: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CH,SZ|EUR,AUS}$ of the exchange rate data C-vine

Copula $c_{JPN,SZ|EUR,AUS}$ $(\hat{\tau} = 0.33)$



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0251	0.0217	0.2226	0.1189	0.0301
p-value	**0.2022**	**0.2732**	0.0004	0.0004	**0.0534**

	Gaussian	t	Clayton	Gumbel	Frank	BB1	BB7
Vuong	2	2	-5	-2	1	2	0
Clarke	-6	**6**	-4	4	2	0	-2

Possible copula(s):	Selected copula:
N, t, F, BB1	t

Table 5.26: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{JPN,SZ|EUR,AUS}$ of the exchange rate data C-vine

Copula $c_{IN,SZ|EUR,AUS}$ $(\hat{\tau} = -0.07)$

	Gaussian	t	Frank
Cramer-von Mises statistic	0.0178	0.0174	0.0173
p-value	**0.5559**	**0.5879**	**0.5059**

	Gaussian	t	Frank
Vuong	1	-2	1
Clarke	-2	**2**	0

Possible copula(s):	Selected copula:
N, t, F	t

Table 5.27: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{IN,SZ|EUR,AUS}$ of the exchange rate data C-vine

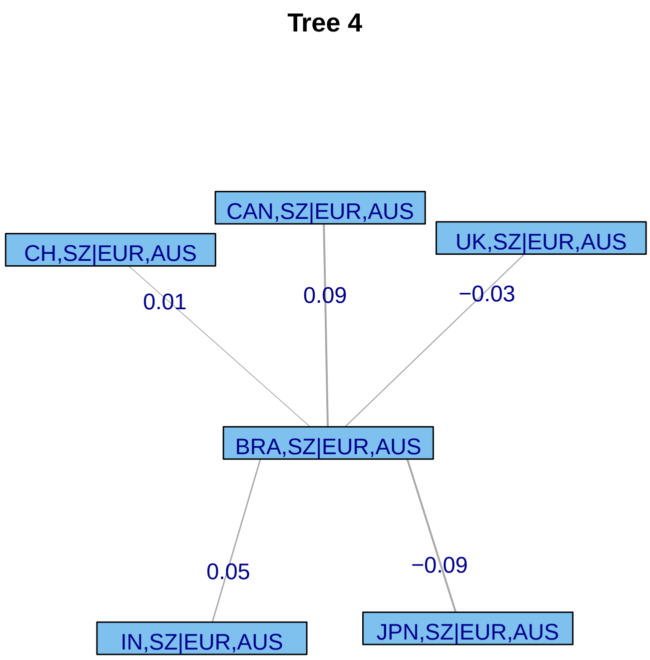


Figure 5.9: Forth tree of the C-vine of the exchange rate data

Copula $c_{UK,BRA|EUR,AUS,SZ}$ $(\hat{\tau} = -0.03)$

	Gaussian	t	Frank
Cramer-von Mises statistic	0.0108	0.0107	0.0113
p-value	**0.9555**	**0.9685**	**0.9155**

	Gaussian	t	Frank
Vuong	1	-2	1
Clarke	-2	2	0

Possible copula(s):	Selected copula:
N, t, F	N (indifferent)

Table 5.28: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{UK,BRA|EUR,AUS,SZ}$ of the exchange rate data C-vine

Copula $c_{CAN,BRA|EUR,AUS,SZ}$ $(\hat{\tau} = 0.09)$

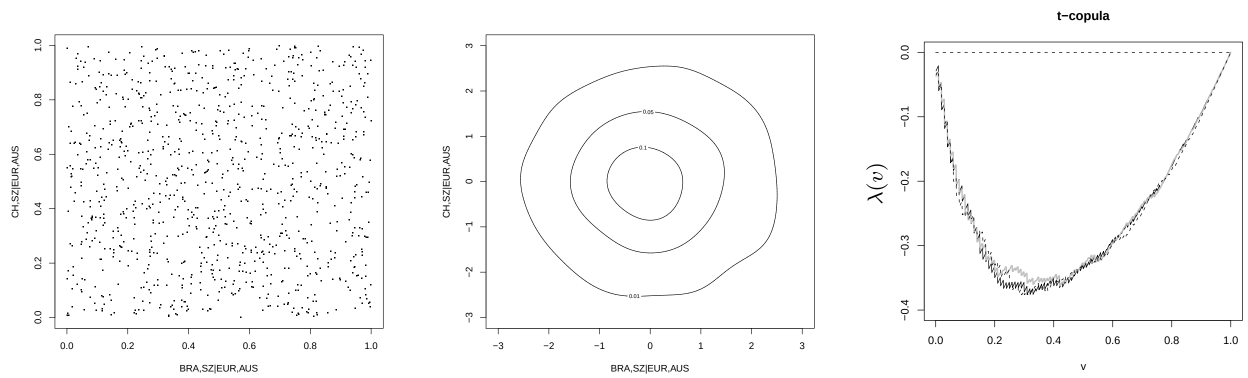
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0242	0.0238	0.0596	0.0204	0.0273
p-value	**0.2592**	**0.2312**	0.0064	**0.4360**	**0.0864**

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	1	1	1	1	1	-5
Clarke	-2	**4**	-1	**4**	0	-5

Possible copula(s):	Selected copula:
N, t, G, F	G (indifferent)

Table 5.29: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CAN,BRA|EUR,AUS,SZ}$ of the exchange rate data C-vine

Copula $c_{CH,BRA|EUR,AUS,SZ}$ $(\hat{\tau} = 0.01)$



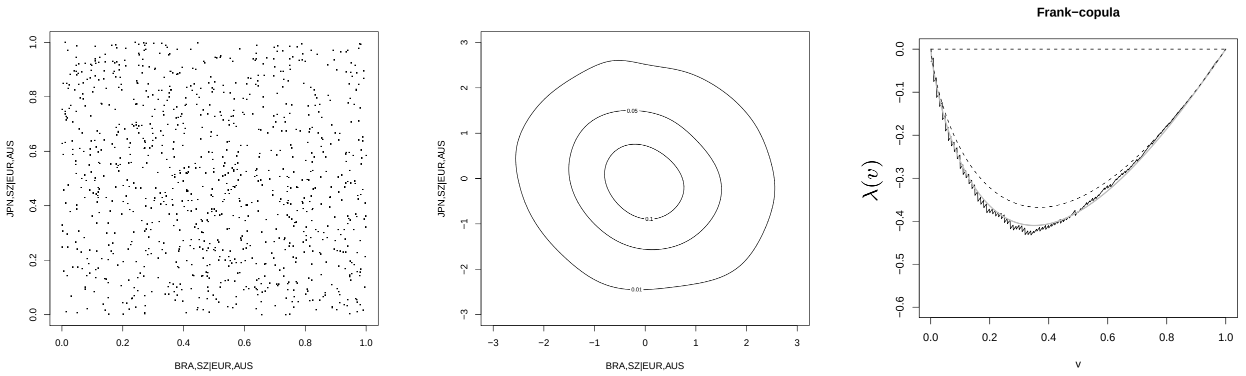
	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0263	0.0262	0.0292	0.0237	0.0259
p-value	**0.1743**	**0.1523**	**0.2432**	**0.3221**	**0.1133**

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	2	-3	2	2	2	-5
Clarke	-1	**5**	-1	3	-1	-5

Possible copula(s):	Selected copula:
N, t, C, G, F	t (indifferent)

Table 5.30: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{CH,BRA|EUR,AUS,SZ}$ of the exchange rate data C-vine

Copula $c_{JPN,BRA|EUR,AUS,SZ}$ $(\hat{\tau} = -0.09)$



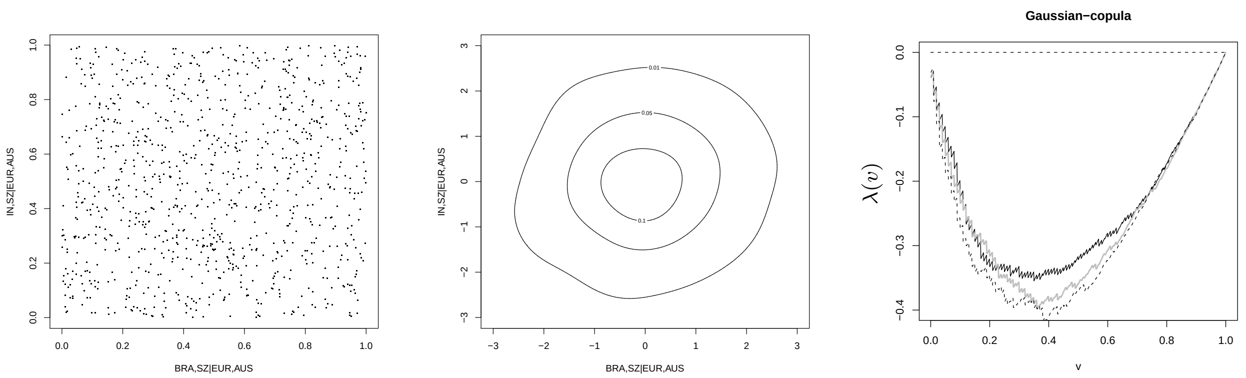
	Gaussian	t	Frank
Cramer-von Mises statistic	0.0265	0.0251	0.0161
p-value	**0.1683**	**0.1723**	**0.5519**

	Gaussian	t	Frank
Vuong	0	-2	**2**
Clarke	-2	**2**	0

Possible copula(s):	Selected copula:
N, t, F	F (indifferent)

Table 5.31: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{JPN,BRA|EUR,AUS,SZ}$ of the exchange rate data C-vine

Copula $c_{IN,BRA|EUR,AUS,SZ}$ $(\hat{\tau} = 0.05)$



	Gaussian	t	Clayton	Gumbel	Frank
Cramer-von Mises statistic	0.0209	0.0207	0.0360	0.0220	0.0217
p-value	**0.3761**	**0.3781**	**0.0674**	**0.3641**	**0.2542**

	Gaussian	t	Clayton	Gumbel	Frank	BB1
Vuong	2	-3	2	2	2	-5
Clarke	-1	**5**	0	2	-1	-5

Possible copula(s):	Selected copula:
N, t, F	N (indifferent)

Table 5.32: Pairs-plot, Contourplot, λ-function and goodness-of-fit test of the copula $c_{IN,BRA|EUR,AUS,SZ}$ of the exchange rate data C-vine

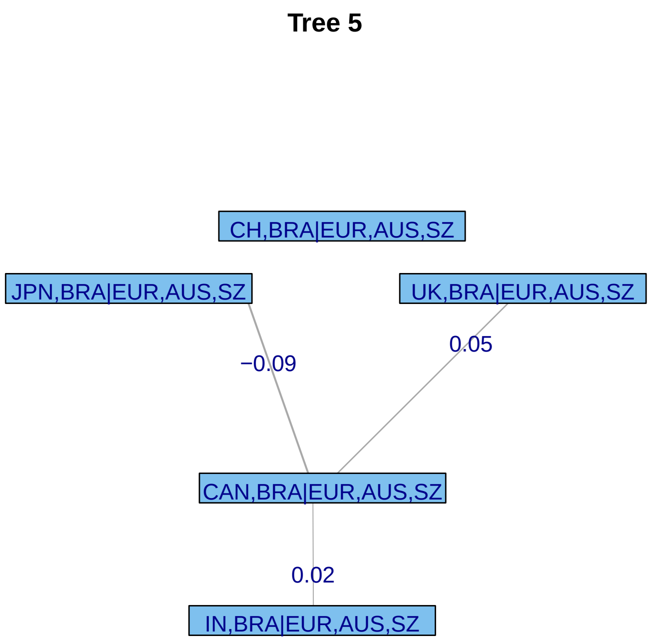


Figure 5.10: Fifth tree of the C-vine of the exchange rate data

For the next trees the Kendall's τ's are very small, i.e. they are nearly independent, and as we saw in the last tree we are indifferent between the copulas to select. Therefore the just draw the plots and choose for all further copulas the Gaussian copula as the underlying copula. One can also choose the t-, or Frank copula, but we think the Gaussian copula is the easiest, best known and well implemented copula.

Copula $c_{UK,CAN|EUR,AUS,BRA,SZ}$ $(\hat{\tau} = 0.05)$

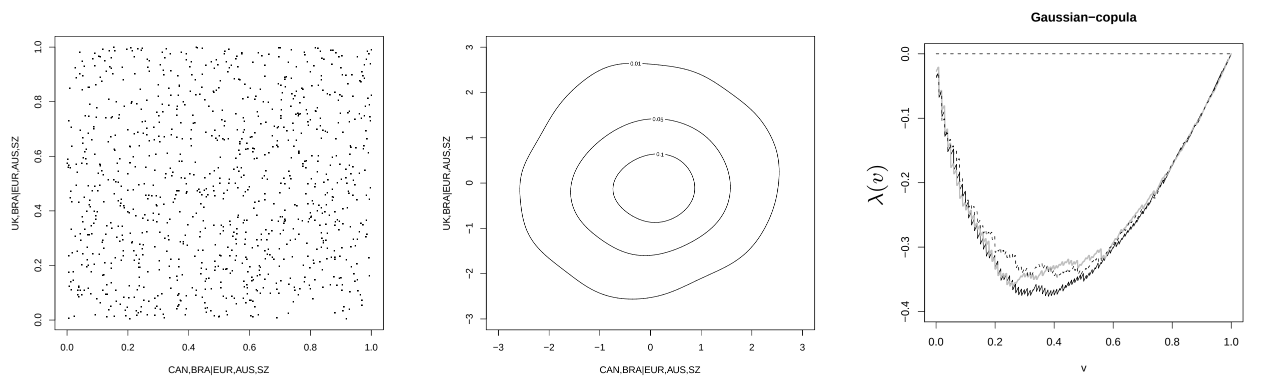


Table 5.33: Pairs-plot, Contourplot and λ-function of the Copula $c_{UK,CAN|EUR,AUS,BRA,SZ}$ of the exchange rate data C-vine

Copula $c_{CH,CAN|EUR,AUS,BRA,SZ}$ $(\hat{\tau} = 0.00)$

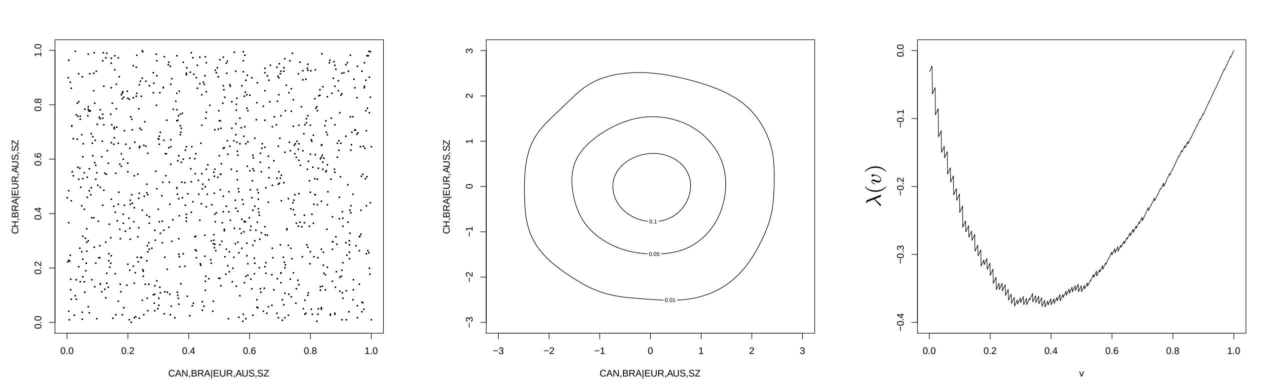


Table 5.34: Pairs-plot, Contourplot and λ-function of the copula $c_{CH,CAN|EUR,AUS,BRA,SZ}$ of the exchange rate data C-vine

Copula $c_{JPN,CAN|EUR,AUS,BRA,SZ}$ $(\hat{\tau} = -0.09)$

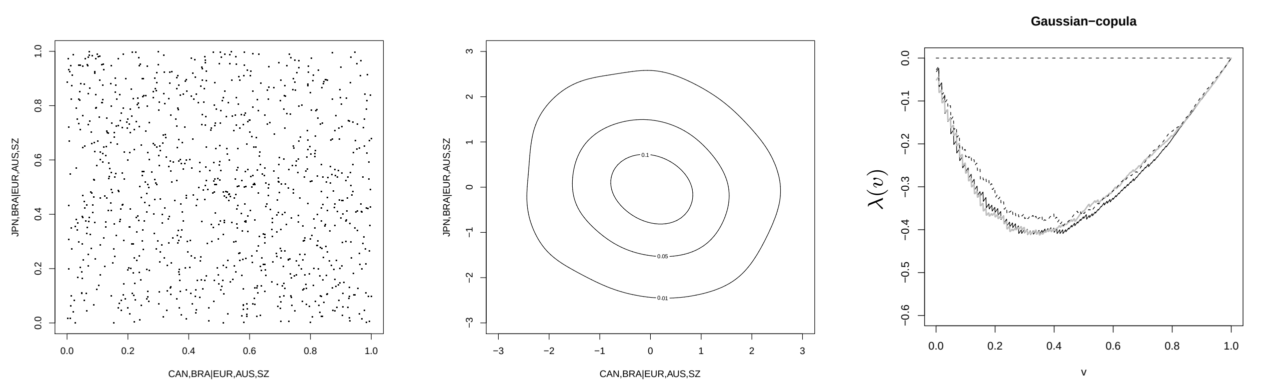


Table 5.35: Pairs-plot, Contourplot and λ-function of the copula $c_{JPN,CAN|EUR,AUS,BRA,SZ}$ of the exchange rate data C-vine

Copula $c_{IN,CAN|EUR,AUS,BRA,SZ}$ $(\hat{\tau} = 0.02)$

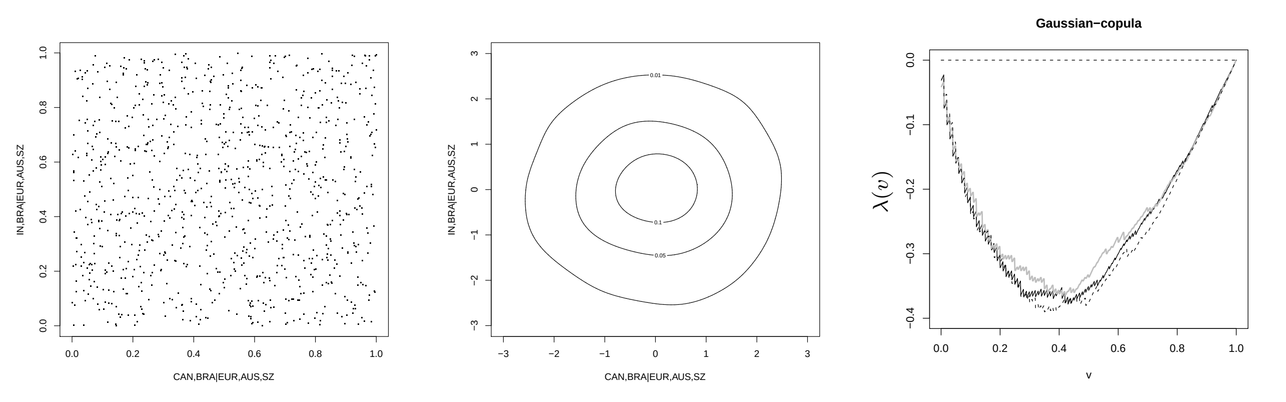


Table 5.36: Pairs-plot, Contourplot and λ-function of the copula $c_{IN,CAN|EUR,AUS,BRA,SZ}$ of the exchange rate data C-vine

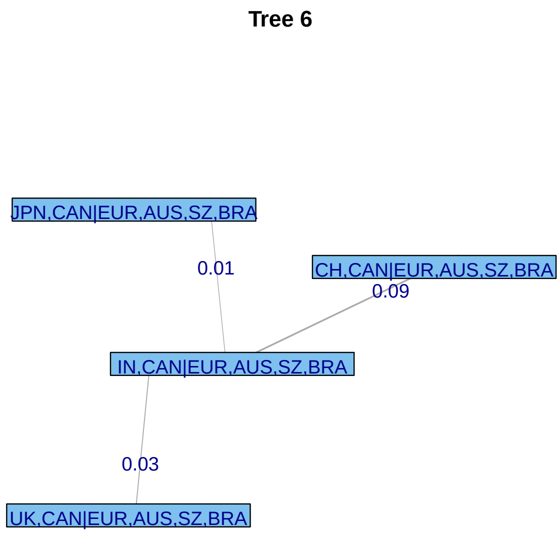


Figure 5.11: Sixth tree of the C-vine of the exchange rate data

Copula $c_{UK,IN|EUR,CAN,AUS,BRA,SZ}$ $(\hat{\tau} = 0.03)$

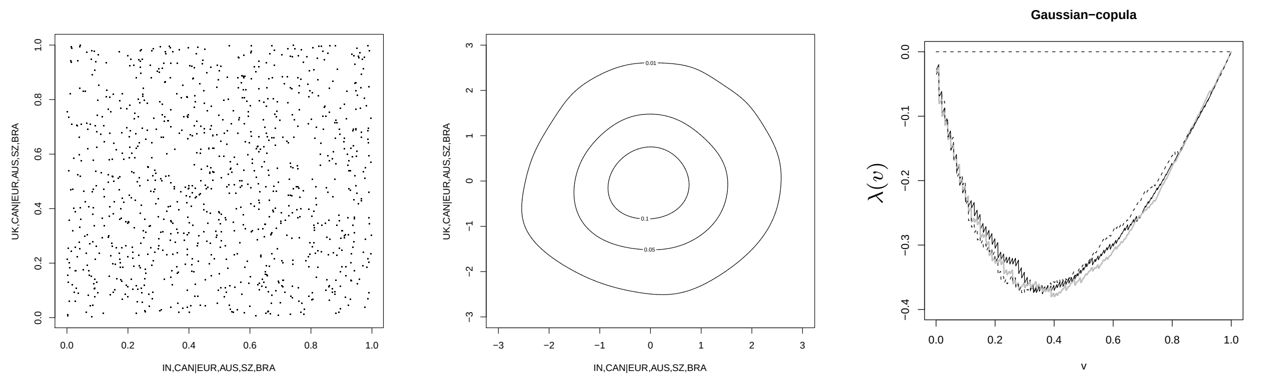


Table 5.37: Pairs-plot, Contourplot and λ-function of the copula $c_{UK,IN|EUR,CAN,AUS,BRA,SZ}$ of the exchange rate data C-vine

Copula $c_{CH,IN|EUR,CAN,AUS,BRA,SZ}$ $(\hat{\tau} = 0.09)$

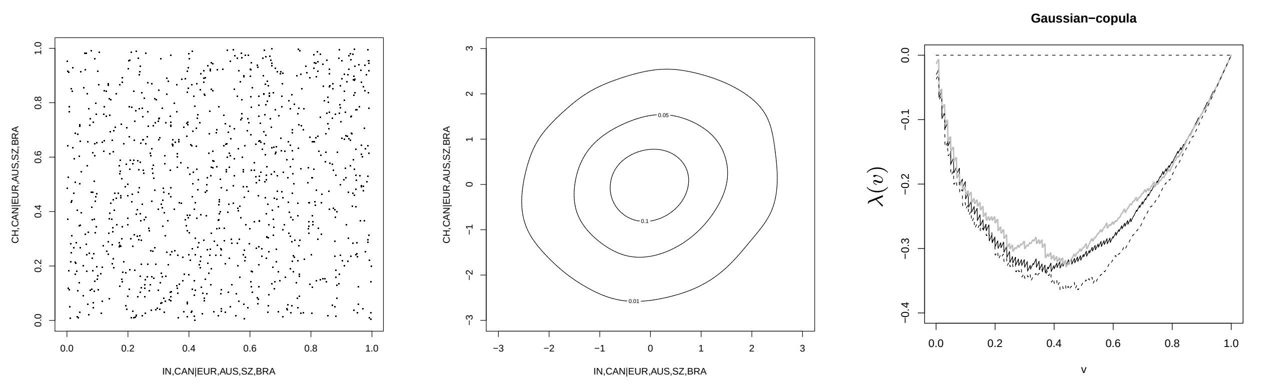


Table 5.38: Pairs-plot, Contourplot and λ-function of the copula $c_{CH,IN|EUR,CAN,AUS,BRA,SZ}$ of the exchange rate data C-vine

Copula $c_{JPN,IN|EUR,CAN,AUS,BRA,SZ}$ $(\hat{\tau} = 0.01)$

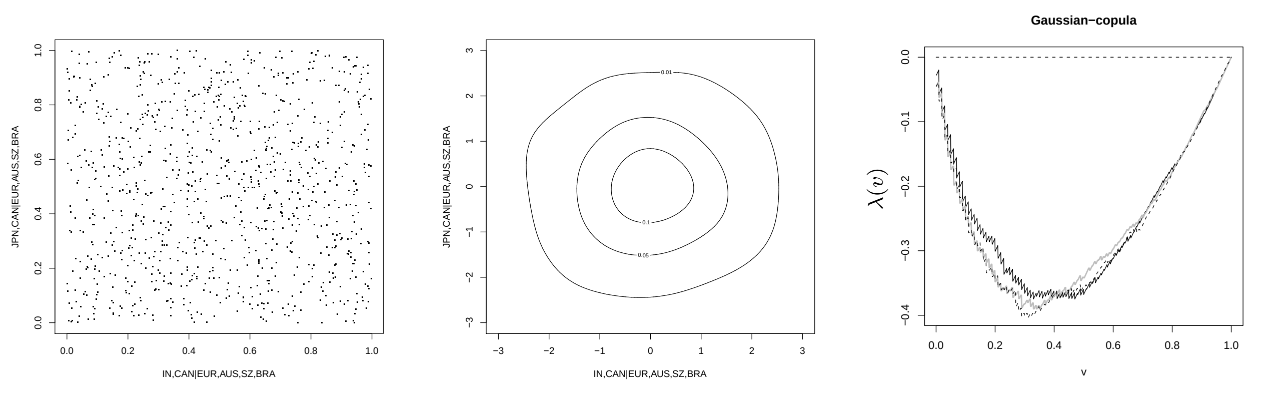


Table 5.39: Pairs-plot, Contourplot and λ-function of the copula $c_{JPN,IN|EUR,CAN,AUS,BRA,SZ}$ of the exchange rate data C-vine

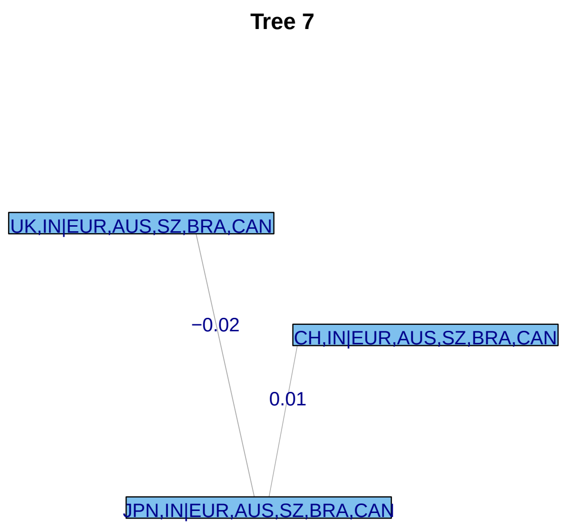


Figure 5.12: Seventh tree of the C-vine of the exchange rate data

Copula $c_{UK,JPN|EUR,CAN,AUS,BRA,SZ,IN}$ $(\hat{\tau} = -0.02)$

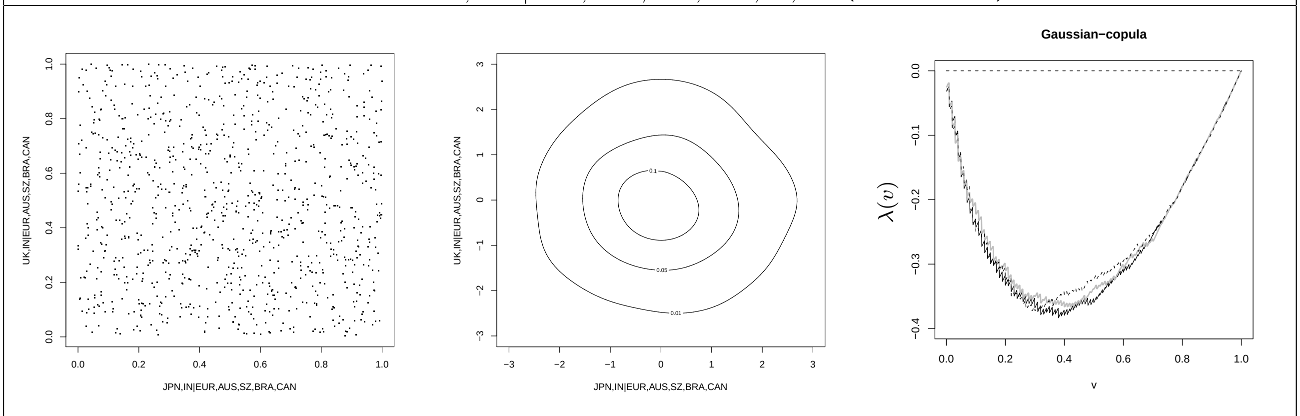


Table 5.40: Pairs-plot, Contourplot and λ-function of the copula $c_{UK,JPN|EUR,CAN,AUS,BRA,SZ,IN}$ of the exchange rate data C-vine

Copula $c_{CH,JPN|EUR,CAN,AUS,BRA,SZ,IN}$ $(\hat{\tau} = 0.01)$

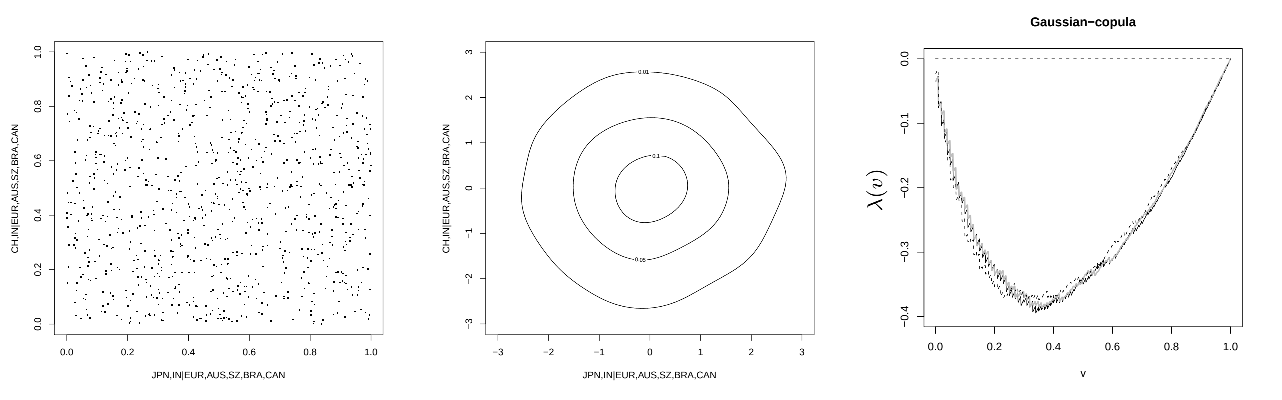


Table 5.41: Pairs-plot, Contourplot and λ-function of the copula $c_{CH,JPN|EUR,CAN,AUS,BRA,SZ,IN}$ of the exchange rate data C-vine

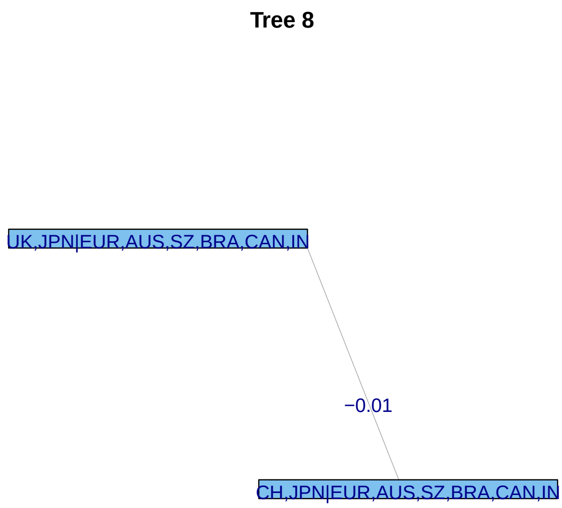


Figure 5.13: Eighth tree of the C-vine of the exchange rate data

Copula $c_{UK,CH|EUR,CAN,AUS,BRA,JPN,SZ,IN}$ $(\hat{\tau} = -0.01)$

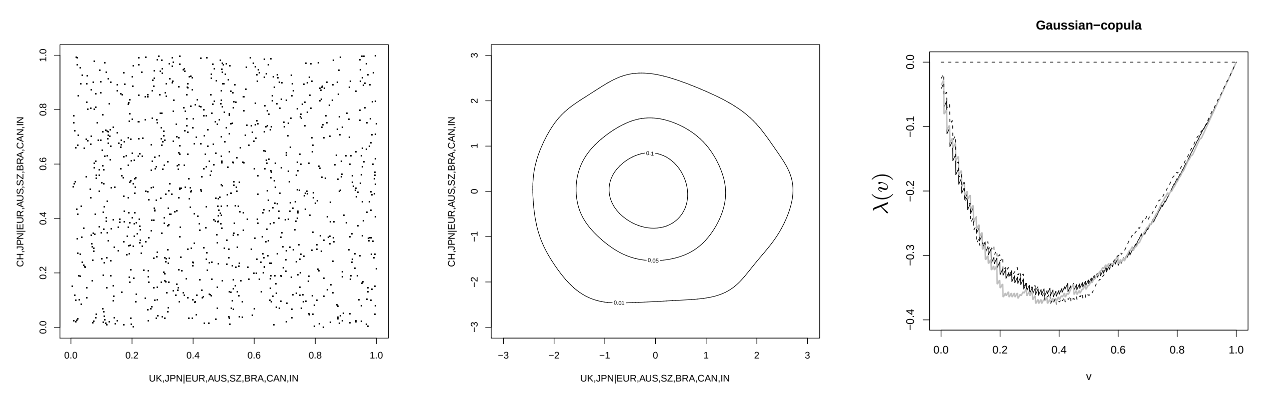


Table 5.42: Pairs-plot, Contourplot and λ-function of the copula $c_{UK,CH|EUR,CAN,AUS,BRA,JPN,SZ,IN}$ of the exchange rate data C-vine

5.2.4 Final model

The final model, called "Mixed C-vine model" (M1), on the basis of our copula and tree selection can be seen in the trees (Figure 5.14 and 5.15). The according parameters can be found in the next section in the Table 5.44.
The copula choice can also be seen in the figures too. The edges of each tree are labeled with Kendall's τ and the copula choice, separated by a comma.
As we mentioned it already above the Kendall's τ of the last 4 trees are very low, i.e. near zero, and we choose the Gaussian copula for them because we are indifferent between the possible copulas and the Gaussian copula is the best known copula. An alternative to this method is to use the independent copula for copulas with very low Kendall's τ. This will be done in "mixed C-vine model with independent copula" (M2). Therefore we test each copula on independence using Kendall's independence test. The results of these test are tabulated in Table 5.43.
The independent copula does not change the other copulas as our test showed (not published in this work). Only the root of the tree before the last tree changes from $JPN, IN|EUR, AUS, SZ, BRA, CAN$ to $UK, IN|EUR, AUS, SZ, BRA, CAN$. But this does not really matter because all copulas of the last two trees are independent copulas (see Table 5.43).
The parameters of the copulas change because of the independent copulas, but not much. Therefore we decided to skip them here.

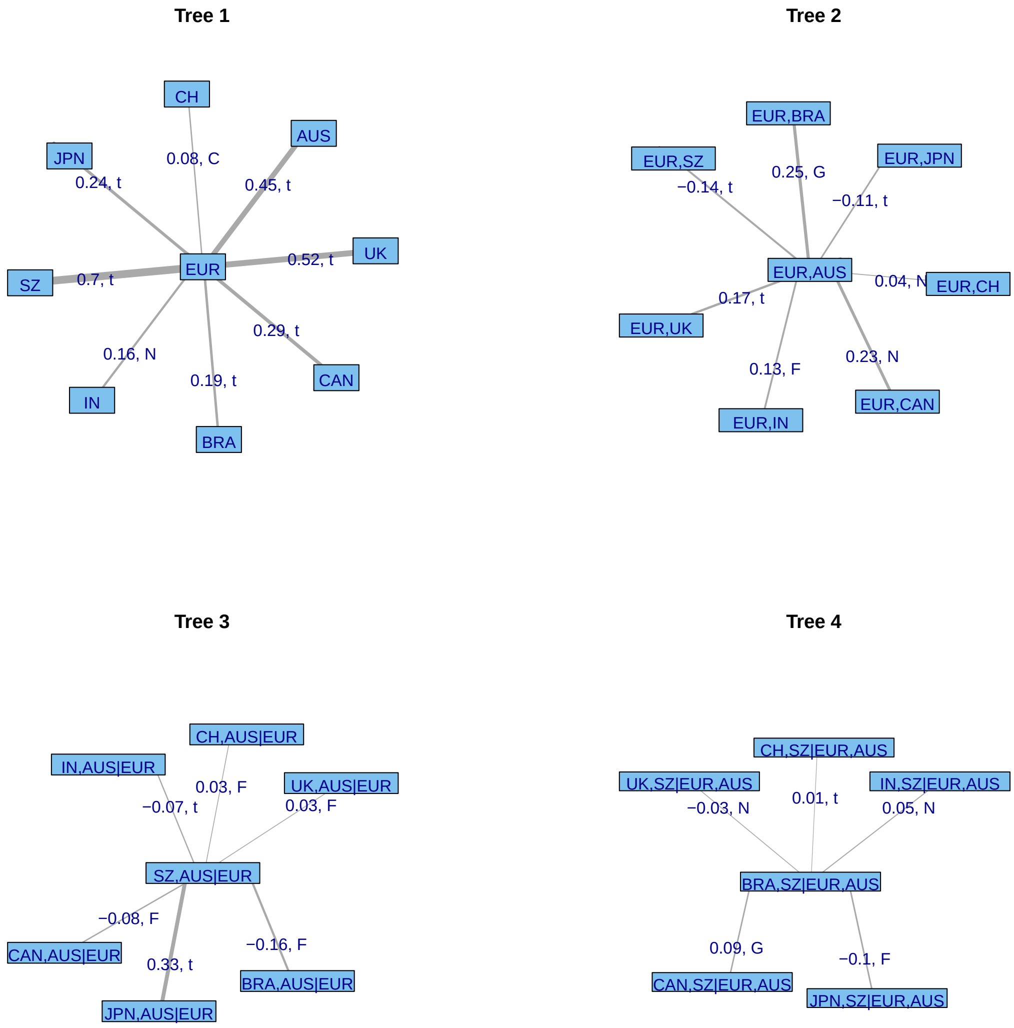


Figure 5.14: Mixed C-vine model (M1) of the exchange rates (part 1)

Figure 5.15: Mixed C-vine model (M1) of the exchange rates (part 2)

Tree	**Copula**	**statistic**	**p-value**
1	EUR, UK	31.0	2.2e-16
	EUR, CAN	14.8	2.2e-16
	EUR, AUS	25.0	2.2e-16
	EUR, BRA	9.29	2.2e-16
	EUR, CH	3.75	0.0001
	EUR, JPN	11.6	2.2e-16
	EUR, SZ	56.3	2.2e-16
	EUR, IN	7.9	7.1e-15
2	$UK, AUS\|EUR$	8.27	4.4e-16
	$CAN, AUS\|EUR$	11.3	2.2e-16
	$BRA, AUS\|EUR$	12.3	2.2e-16
	$CH, AUS\|EUR$	1.70	**0.09**
	$JPN, AUS\|EUR$	-5.15	3.1e-07
	$SZ, AUS\|EUR$	-6.82	1.5e-11
	$IN, AUS\|EUR$	6.26	5.5e-10
3	$UK, SZ\|EUR, AUS$	1.31	**0.19**
	$CAN, SZ\|EUR, AUS$	-4.07	4.9e-05
	$BRA, SZ\|EUR, AUS$	-7.82	1.3e-14
	$CH, SZ\|EUR, AUS$	1.58	**0.11**
	$JPN, SZ\|EUR, AUS$	17.2	2.2e-16
	$IN, SZ\|EUR, AUS$	-3.23	0.001
4	$UK, BRA\|EUR, AUS, SZ$	-1.45	**0.14**
	$CAN, BRA\|EUR, AUS, SZ$	4.54	6.0e-06
	$CH, BRA\|EUR, AUS, SZ$	0.35	**0.72**
	$JPN, BRA\|EUR, AUS, SZ$	-4.48	8.1e-06
	$IN, BRA\|EUR, AUS, SZ$	2.51	0.01
5	$UK, CAN\|EUR, AUS, BRA, SZ$	2.37	0.01
	$CH, CAN\|EUR, AUS, BRA, SZ$	0.25	**0.79**
	$JPN, CAN\|EUR, AUS, BRA, SZ$	-3.99	7.0e-05
	$IN, CAN\|EUR, AUS, BRA, SZ$	0.71	**0.47**
6	$UK, IN\|EUR, CAN, AUS, BRA, SZ$	1.47	**0.14**
	$CH, IN\|EUR, CAN, AUS, BRA, SZ$	4.43	1.0e-05
	$JPN, IN\|EUR, CAN, AUS, BRA, SZ$	0.62	**0.52**
7	$UK, JPN\|EUR, CAN, AUS, BRA, SZ, IN$	-0.51	**0.61**
	$CH, JPN\|EUR, CAN, AUS, BRA, SZ, IN$	-0.84	**0.39**
8	$UK, CH\|EUR, CAN, AUS, BRA, JPN, SZ, IN$	0.15	**0.87**

Table 5.43: Test for independence of the copulas based on Kendall's test statistic

A more compact representation of a vine can be done in two matrices in which the pair-copulas and the copula families for each pair-copula are stored. This representation was primary introduce for regular vines, but because of C-vines are a special form of a regular vine one can adopt this matrices for C-vines too. A detailed definition and explanation how this lower triangular matrix $M \in \{1,\dots,n\}^{n\times n}$, $M = (m_{i,j}|i,j = 1,\dots,n)$ has to be build can be found in the diploma thesis of Dißmann [2010] and in Kurowicka [2009]. The advantage of this representation is that one can see the differences of the two final models (M1) and (M2) quickly. For our data set we have to introduce a short notation for the matrices. Every exchange rate gets a number (EUR = 1, UK = 2, CAN = 3, AUS = 4, BRA = 5, CH = 6, JPN = 7, SZ = 8 and IN = 9). Additionally we use the notation of the copula families we introduced in Chapter 3 for the copula family matrix T, which is similar defined as the matrix M.
Thus we get for (M1) the matrices $M_{(M1)}$ and $T_{(M1)}$ and for the second model (M2) the matrices $M_{(M2)}$ and $T_{(M2)}$, where M_i is the matrix of the copula-pairs and T_i the matrix of the according copula families, $i \in \{(M1),(M2)\}$.

$$M_{(M1)} = \begin{pmatrix} 6 &&&&&&&& \\ 7 & 7 &&&&&&& \\ 2 & 2 & 2 &&&&&& \\ 9 & 9 & 9 & 9 &&&&& \\ 3 & 3 & 3 & 3 & 3 &&&& \\ 5 & 5 & 5 & 5 & 5 & 5 &&& \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 && \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad T_{(M1)} = \begin{pmatrix} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ 1 & 1 & 1 &&&&& \\ 1 & 1 & 1 & 1 &&&& \\ 2 & 5 & 1 & 1 & 4 &&& \\ 5 & 2 & 5 & 2 & 5 & 5 && \\ 1 & 2 & 2 & 5 & 1 & 4 & 2 & \\ 3 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$M_{(M2)} = \begin{pmatrix} 6 &&&&&&&& \\ 2 & 2 &&&&&&& \\ 7 & 7 & 7 &&&&&& \\ 9 & 9 & 9 & 9 &&&&& \\ 3 & 3 & 3 & 3 & 3 &&&& \\ 5 & 5 & 5 & 5 & 5 & 5 &&& \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 && \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad T_{(M2)} = \begin{pmatrix} 0 &&&&&&& \\ 0 & 0 &&&&&& \\ 0 & 0 & 1 &&&&& \\ 0 & 1 & 1 & 0 &&&& \\ 0 & 0 & 5 & 1 & 4 &&& \\ 0 & 0 & 2 & 2 & 5 & 5 && \\ 0 & 2 & 2 & 5 & 1 & 4 & 2 & \\ 3 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

5.2.5 Quality of the selected model

In the following Table 5.44 we compare the results we get by the sequential estimation method, called "Start", and the maximum likelihood estimation (MLE) method. These parameters are the basis for our first quality attribute, Kendall's τ.
As a first result we can say that the parameters we get by the sequential method are close to the MLE parameters, except of the degrees of freedom of the t-copulas.
When we look at the copulas and their estimation with respect to Kendall's τ, we get the following Table 5.45.

Tree	Parameter	Copula	Start	MLE
1	$\rho_{EUR,UK}$	t	0.7254	0.7241
	ν_{EUR_UK}		8.5199	9.2599
	$\rho_{EUR,CAN}$	t	0.4465	0.4575
	$\nu_{EUR,CAN}$		8.4857	8.8068
	$\rho_{EUR,AUS}$	t	0.6434	0.6326
	$\nu_{EUR,AUS}$		10.2029	13.2968
	$\rho_{EUR,BRA}$	t	0.2996	0.3176
	$\nu_{EUR,BRA}$		6.4220	7.5149
	$\theta_{EUR,CH}$	C	0.1707	0.1775
	$\rho_{EUR,JPN}$	t	0.3714	0.3701
	$\nu_{EUR,JPN}$		4.3969	5.8921
	$\rho_{EUR,SZ}$	t	0.8907	0.8862
	$\nu_{EUR,SZ}$		4.0093	4.6286
	$\rho_{EUR,IN}$	N	0.2538	0.2608
2	$\rho_{UK,AUS\|EUR}$	t	0.2658	0.2644
	$\nu_{UK,AUS\|EUR}$		11.9548	12.1880
	$\rho_{CAN,AUS\|EUR}$	N	0.3537	0.3479
	$\theta_{BRA,AUS\|EUR}$	G	1.3249	1.2952
	$\rho_{CH,AUS\|EUR}$	N	0.0566	0.0566
	$\rho_{JPN,AUS\|EUR}$	t	-0.1646	-0.1774
	$\nu_{JPN,AUS\|EUR}$		6.9763	7.7154
	$\rho_{SZ,AUS\|EUR}$	t	-0.2251	-0.2286
	$\nu_{SZ,AUS\|EUR}$		11.4693	14.4395
	$\theta_{IN,AUS\|EUR}$	F	1.1900	1.1931
3	$\theta_{UK,SZ\|EUR,AUS}$	F	0.2600	0.2585
	$\theta_{CAN,SZ\|EUR,AUS}$	F	-0.7600	-0.7808
	$\theta_{BRA,SZ\|EUR,AUS}$	F	-1.4800	-1.5071
	$\theta_{CH,SZ\|EUR,AUS}$	F	0.3100	0.2800
	$\rho_{JPN,SZ\|EUR,AUS}$	t	0.4960	0.4894
	$\nu_{JPN,SZ\|EUR,AUS}$		12.0800	12.3918
	$\rho_{IN,SZ\|EUR,AUS}$	t	-0.1072	-0.1079
	$\nu_{IN,SZ\|EUR,AUS}$		20.0000	22.0165
4	$\rho_{UK,BRA\|EUR,AUS,SZ}$	N	-0.0462	-0.0476
	$\theta_{CAN,BRA\|EUR,AUS,SZ}$	G	1.1042	1.0962
	$\rho_{CH,BRA\|EUR,AUS,SZ}$	t	0.0117	0.0196
	$\nu_{CH,BRA\|EUR,AUS,SZ}$		20.0000	20.4258
	$\theta_{JPN,BRA\|EUR,AUS,SZ}$	F	-0.8700	-0.8835
	$\rho_{IN,BRA\|EUR,AUS,SZ}$	N	0.0800	0.0843
5	$\rho_{UK,CAN\|EUR,AUS,BRA,SZ}$	N	0.0809	0.0867
	$\rho_{CH,CAN\|EUR,AUS,BRA,SZ}$	N	0.0049	0.0113
	$\rho_{JPN,CAN\|EUR,AUS,BRA,SZ}$	N	-0.1304	-0.1199
	$\rho_{IN,CAN\|EUR,AUS,BRA,SZ}$	N	0.0243	0.0463
6	$\rho_{UK,IN\|EUR,CAN,AUS,BRA,SZ}$	N	0.0498	0.0552
	$\rho_{CH,IN\|EUR,CAN,AUS,BRA,SZ}$	N	0.1434	0.1217
	$\rho_{JPN,IN\|EUR,CAN,AUS,BRA,SZ}$	N	0.0198	0.0242
7	$\rho_{UK,JPN\|EUR,CAN,AUS,BRA,SZ,IN}$	N	-0.0272	-0.0197
	$\rho_{CH,JPN\|EUR,CAN,AUS,BRA,SZ,IN}$	N	0.0086	-0.0030
8	$\rho_{UK,CH\|EUR,CAN,AUS,BRA,JPN,SZ,IN}$	N	-0.0154	-0.0049

Table 5.44: Estimated parameters for the C-vine of the exchange rate data

Tree	Parameter	Copula	emp. τ	MLE
1	$\tau_{EUR,UK}$	t	0.52	0.52
	$\tau_{EUR,CAN}$	t	0.29	0.30
	$\tau_{EUR,AUS}$	t	0.45	0.44
	$\tau_{EUR,BRA}$	t	0.19	0.21
	$\tau_{EUR,CH}$	C	0.08	0.08
	$\tau_{EUR,JPN}$	t	0.24	0.24
	$\tau_{EUR,SZ}$	t	0.70	0.69
	$\tau_{EUR,IN}$	N	0.16	0.17
2	$\tau_{UK,AUS\|EUR}$	t	0.17	0.17
	$\tau_{CAN,AUS\|EUR}$	N	0.23	0.23
	$\tau_{BRA,AUS\|EUR}$	G	0.25	0.23
	$\tau_{CH,AUS\|EUR}$	N	0.04	0.04
	$\tau_{JPN,AUS\|EUR}$	t	-0.11	-0.11
	$\tau_{SZ,AUS\|EUR}$	t	-0.14	-0.15
	$\tau_{IN,AUS\|EUR}$	F	0.13	0.13
3	$\tau_{UK,SZ\|EUR,AUS}$	F	0.03	0.03
	$\tau_{CAN,SZ\|EUR,AUS}$	F	-0.08	-0.09
	$\tau_{BRA,SZ\|EUR,AUS}$	F	-0.16	-0.17
	$\tau_{CH,SZ\|EUR,AUS}$	F	0.03	0.03
	$\tau_{JPN,SZ\|EUR,AUS}$	t	0.33	0.33
	$\tau_{IN,SZ\|EUR,AUS}$	t	-0.07	-0.07
4	$\tau_{UK,BRA\|EUR,AUS,SZ}$	N	-0.03	-0.03
	$\tau_{CAN,BRA\|EUR,AUS,SZ}$	G	0.09	0.09
	$\tau_{CH,BRA\|EUR,AUS,SZ}$	t	0.01	0.01
	$\tau_{JPN,BRA\|EUR,AUS,SZ}$	F	-0.09	-0.10
	$\tau_{IN,BRA\|EUR,AUS,SZ}$	N	0.05	0.05
5	$\tau_{UK,CAN\|EUR,AUS,BRA,SZ}$	N	0.05	0.06
	$\tau_{CH,CAN\|EUR,AUS,BRA,SZ}$	N	0.00	0.01
	$\tau_{JPN,CAN\|EUR,AUS,BRA,SZ}$	N	-0.09	-0.08
	$\tau_{IN,CAN\|EUR,AUS,BRA,SZ}$	N	0.02	0.03
6	$\tau_{UK,IN\|EUR,CAN,AUS,BRA,SZ}$	N	0.03	0.04
	$\tau_{CH,IN\|EUR,CAN,AUS,BRA,SZ}$	N	0.09	0.08
	$\tau_{JPN,IN\|EUR,CAN,AUS,BRA,SZ}$	N	0.01	0.01
7	$\tau_{UK,JPN\|EUR,CAN,AUS,BRA,SZ,IN}$	N	-0.02	-0.02
	$\tau_{CH,JPN\|EUR,CAN,AUS,BRA,SZ,IN}$	N	0.01	0.00
8	$\tau_{UK,CH\|EUR,CAN,AUS,BRA,JPN,SZ,IN}$	N	-0.01	0.00

Table 5.45: Estimated conditional Kendall's τ's based on fitted C-vine of the exchange rate data

Remark:
The results of the estimated conditional Kendall's τ's are very good too. The differences between the empirical conditional Kendall's τ's and the estimated conditional Kendall's τ's, which we get from the estimated parameters, are very small.

Since the C-vine gives only close form expressions for conditional Kendall's τ we want now to get a Monte Carlo estimate for the unconditional Kendall's τ using a simulated data set of same size from the fitted C-vine. Therefore we simulate r=100 times a data set on the basis of the fitted parameters. The results are quite good too as we can see in Table 5.46 and 5.47 and Figure 5.16.

	EUR	AUS	SZ	BRA	CAN	IN	JPN	UK	CH
EUR	1.00	0.45	0.70	0.19	0.29	0.16	0.24	0.52	0.08
AUS	0.45	1.00	0.32	0.32	0.36	0.20	0.07	0.41	0.08
SZ	0.70	0.32	1.00	0.08	0.21	0.10	0.37	0.44	0.08
BRA	0.19	0.32	0.08	1.00	0.24	0.14	-0.11	0.17	0.04
CAN	0.29	0.36	0.21	0.24	1.00	0.14	-0.02	0.29	0.05
IN	0.16	0.20	0.10	0.14	0.14	1.00	0.01	0.16	0.11
JPN	0.24	0.07	0.37	-0.11	-0.02	0.01	1.00	0.14	0.03
UK	0.52	0.41	0.44	0.17	0.29	0.16	0.14	1.00	0.06
CH	0.08	0.08	0.08	0.04	0.05	0.11	0.03	0.06	1.00

Table 5.46: The empirical Kendall's τ matrix of the exchange rate data set

	EUR	AUS	SZ	BRA	CAN	IN	JPN	UK	CH
EUR	1.00	0.44	0.69	0.20	0.30	0.16	0.23	0.51	0.08
AUS	0.44	1.00	0.32	0.31	0.35	0.20	0.07	0.41	0.08
SZ	0.69	0.32	1.00	0.08	0.20	0.10	0.36	0.43	0.08
BRA	0.20	0.31	0.08	1.00	0.25	0.15	-0.09	0.16	0.03
CAN	0.30	0.35	0.20	0.25	1.00	0.14	-0.03	0.28	0.04
IN	0.16	0.20	0.10	0.15	0.14	1.00	0.01	0.16	0.11
JPN	0.23	0.07	0.36	-0.09	-0.03	0.01	1.00	0.14	0.05
UK	0.51	0.41	0.43	0.16	0.28	0.16	0.14	1.00	0.06
CH	0.08	0.08	0.08	0.03	0.04	0.11	0.05	0.06	1.00

Table 5.47: Monte Carlo estimate of Kendall's τ on the basis of the estimated parameters (MLE) of the exchange rate data set (n=1000, r=100)

Conclusions:

- As we can see in Figure 5.16 the estimated Kendall's τ's on the basis of our model selection (M1) and MLE function is very good.
- When we look at the log-likelihood of the selected model we get for the starting parameters (Table 5.53), which are based on bivariate estimation, a log-likelihood

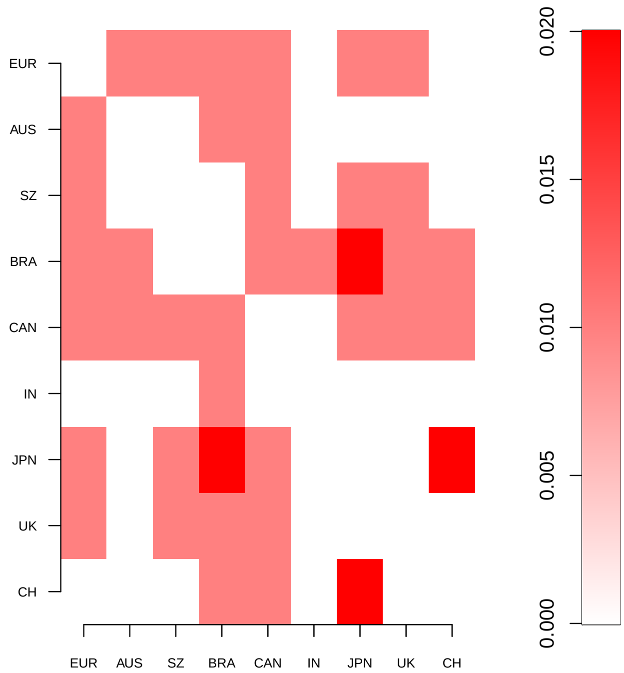


Figure 5.16: Absolute difference between the empirical and the Mont Carlo estimate of Kendall's τ

of 2221.4. With the C-vine model and the MLE function we can improve the log-likelihood value to 2227.3. The difference of just 6.1 is an indicator for the good bivariate estimation of the parameters.
The Vuong-test confirms this assumption with a test statistic of $\nu = -1.6$ and a p-value of 0.11, i.e. non of the models is favored. The Clarke-test even says that the model with the starting values is to be preferred with a test statistic $B = 541$ and a p-value of 0.018. But as we mentioned it in Chapter 2.6.4 the Clarke-test is to be handled cautiously.

5.2.6 Model comparison

One further question is, is the selected model better than other models? A simple approach is to set all copulas even to the Gaussian or t-copula. In the following tables and figures we will test this approach with respect to Kendall's τ and with respect to the Vuong- and Clarke-test. For each C-vine we tabulated the resulting Kendall's τ's (5.49 and 5.50) and in Figure 5.17 and Figure 5.18 we illustrated the difference between the empirical Kendall's τ's and the Kendall's τ's we get with the C-vine Gaussian copula (M3) and C-vine t-copula models (M4) (each with the MLE method).
In the final Tables 5.51ff we tabulated the Vuong- and Clarke-test with BIC correction with their test statistic and p-value and which decision we can deduce from from these tests for the models based on the sequential method and on the MLE method. We made the tests for the combinations "mixed C-vine model (M1)"-"mixed C-vine model with

independent copula (M2)", "mixed C-vine model (M1)"-"C-vine Gaussian copula (M3)", "mixed C-vine model (M1)"-"C-vine t-copula (M4)", "mixed C-vine model with independent copula (M2)"-"C-vine Gaussian copula (M3)", "mixed C-vine model with independent copula (M2)"-"C-vine t-copula (M4)" and "C-vine Gaussian copula (M3)"-"C-vine t-copula (M4)". (see also Table 5.48)

Short	**Model**	**Trees**	**Pair-copulas**
M1	mixed C-vine	Figure 5.14 - 5.15	mixed without independent copula
M2	mixed C-vine with ind. copula	-	mixed with independent copula
M3	C-vine Gaussian copula	Figure 5.14 - 5.15	bivariate Gaussian copula
M4	C-vine t-copula	Figure 5.14 - 5.15	bivariate t-copula

Table 5.48: Compendium of the compared models

	EUR	AUS	SZ	BRA	CAN	IN	JPN	UK	CH
EUR	1.00	0.44	0.68	0.20	0.31	0.17	0.22	0.51	0.09
AUS	0.44	1.00	0.30	0.32	0.36	0.20	0.03	0.41	0.08
SZ	0.68	0.30	1.00	0.08	0.21	0.10	0.36	0.41	0.08
BRA	0.20	0.32	0.08	1.00	0.25	0.15	-0.11	0.17	0.05
CAN	0.31	0.36	0.21	0.25	1.00	0.15	-0.03	0.29	0.06
IN	0.17	0.20	0.10	0.15	0.15	1.00	0.01	0.17	0.10
JPN	0.22	0.03	0.36	-0.11	-0.03	0.01	1.00	0.12	0.04
UK	0.51	0.41	0.41	0.17	0.29	0.17	0.12	1.00	0.07
CH	0.09	0.08	0.08	0.05	0.06	0.10	0.04	0.07	1.00

Table 5.49: Monte Carlo estimate of unconditional Kendall's τ's of the C-vine Gaussian copula model (M3) (n=1000, r=100)

Conclusions:

- The estimation of the Kendall's τ's of both misspecifications (M3) and (M4) are quite good, whereas the C-vine with the t-copulas (M4) is better than the C-vine with the Gaussian copulas (M3).
  If one want just to model the Kendall's τ's the approach to use only Gaussian or t-copula is justified for the exchange rate data.
- In contrast the Vuong- and Clarke-tests suggestive that the mixed C-vine models (M1 or M2) with either the starting parameters or the maximum likelihood parameter is better than the C-vines with Gaussian (M3) or t-copulas (M4). But both tests, especially the Clarke-test, have to be treated cautiously, as some simulation studies by Djunushalieva [2010] show.
- We have also be in mind that many selected copulas are either Gaussian or t-copulas and that the differences are not that big. Especially to the C-vine with the t-copulas.

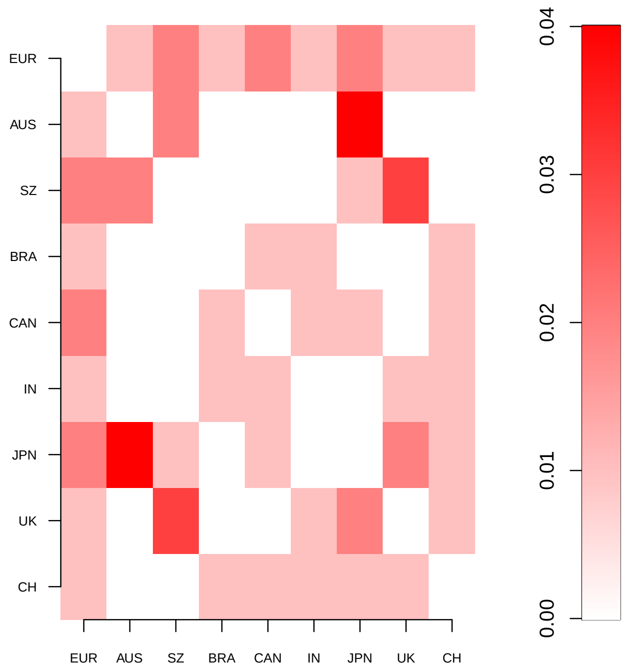


Figure 5.17: Absolute difference between the empirical and the simulated unconditional Kendall's τ's for the C-vine Gaussian copula model (M3)

	EUR	AUS	SZ	BRA	CAN	IN	JPN	UK	CH
EUR	1.00	0.44	0.69	0.20	0.30	0.17	0.24	0.52	0.08
AUS	0.44	1.00	0.31	0.31	0.36	0.20	0.07	0.41	0.07
SZ	0.69	0.31	1.00	0.09	0.21	0.10	0.37	0.43	0.08
BRA	0.20	0.31	0.09	1.00	0.25	0.15	-0.08	0.16	0.04
CAN	0.30	0.36	0.21	0.25	1.00	0.14	-0.01	0.27	0.05
IN	0.17	0.20	0.10	0.15	0.14	1.00	0.02	0.16	0.10
JPN	0.24	0.07	0.37	-0.08	-0.01	0.02	1.00	0.15	0.04
UK	0.52	0.41	0.43	0.16	0.27	0.16	0.15	1.00	0.07
CH	0.08	0.07	0.08	0.04	0.05	0.10	0.04	0.07	1.00

Table 5.50: Monte Carlo estimate of unconditional Kendall's τ's of the C-vine t-copula model (M4) (n=1000, r=100)

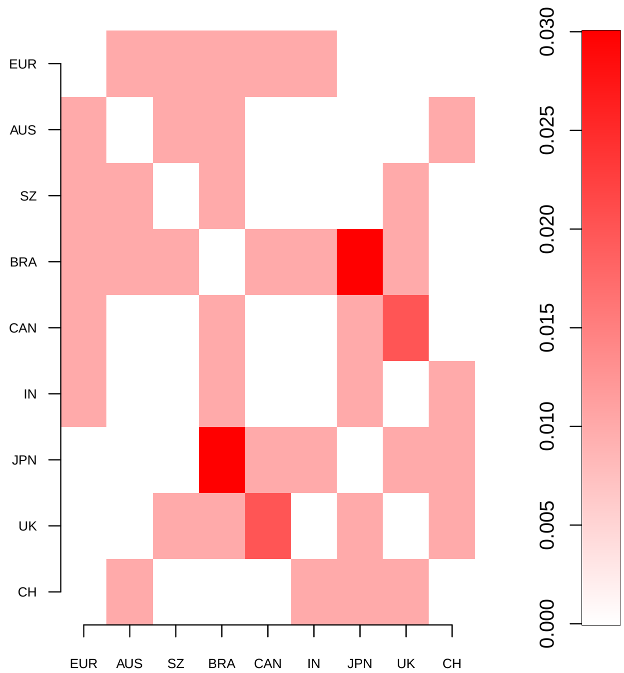


Figure 5.18: Absolute difference between the empirical and the simulated unconditional Kendall's τ's for the C-vine t-copula model (M4)

Sequential estimation

		mixed C-vine with ind. copula (M2)	**C-vine Gaussian copula (M3)**	**C-vine t-copula (M4)**
mixed C-vine (M1)	Vuong stat.	-25.8	11.5	9.39
	Vuong p	0	0	0
	Clarke stat.	208	628	723
	Clarke p	0	0	0
	Decision	**M2**	**M1**	**M1**
C-vine Gauss. copula (M3)	Vuong stat.	-19.2		
	Vuong p	0		
	Clarke stat.	803		
	Clarke p	0		
	Decision	**M2**		
C-vine t-copula (M4)	Vuong stat.	-27	-10.3	
	Vuong p	0	0	
	Clarke stat.	801	497	
	Clarke p	0	0.364	
	Decision	**M2**	**M4***	

* The Vuong-test favors "C-vine t-copula" but the Clarke-test favors non of them.

Table 5.51: Vuong- and Clarke-tests for the different models (sequential method)

MLE				
		mixed C-vine with ind. copula (M2)	**C-vine Gaussian copula (M3)**	**C-vine t-copula (M4)**
mixed C-vine (M1)	Vuong stat.	-25.4	9.95	9.3
	Vuong p	0	0	0
	Clarke stat.	199	581	699
	Clarke p	0	0	0
	Decision	**M2**	**M1**	**M1**
C-vine Gauss. copula (M3)	Vuong stat.	-19.2		
	Vuong p	0		
	Clarke stat.	807		
	Clarke p	0		
	Decision	**M2**		
C-vine t-copula (M4)	Vuong stat.	-26.5	-8.68	
	Vuong p	0	0	
	Clarke stat.	803	533	
	Clarke p	0	0.031	
	Decision	**M2**	**M4**	

Table 5.52: Vuong- and Clarke-tests for the different models (MLE method)

	M1	M2	M3	M4
Log-likelihood seq.	2221.4	2212.7	2092	2215
Log-likelihood MLE	2227.3	2218.2	2105	2232
# of parameters	48	35	36	72
AIC seq.	-4347.4	-4355.4	-4112	-4286
AIC MLE	-4358.6	-4366.4	-4138	-4320
BIC seq.	-4298.6	-4320.3	-4075.9	-4213.8
BIC MLE	-4310.4	-4331.3	-4101.9	-4247.8

Table 5.53: Log-likelihood, number of parameters, AIC and BIC of the different models

Furthermore we know that the t-copula converges to the Gaussian copula for high degrees of freedom and especially for the last 10 copulas we have high degrees of freedom and low dependence parameter ρ.

- When we look at the log-likelihood, without consideration of the number of parameters, we get for the mixed C-vine model (M1) with the starting parameters a log-likelihood of 2221.4, for the selected model with the MLE parameters a log-likelihood of 2227,3, for the C-vine with Gaussian copulas (M3) a log-likelihood of 2105 and for the C-vine with t-copulas (M4) a log-likelihood of 2232. As we mentioned it above, the log-likelihood of the mixed C-vine models (M1) and (M2) are very close and the log-likelihood of (M4) is even greater. Only the log-likelihood of (M4) is less that high, around 100 points less.
- For (M2) we can make similar conclusions as for (M1). The log-likelihood is in both methods, sequential and MLE, near the log-likelihood of (M1) (see Table 5.53).
- Because of the lower number of parameters of the mixed C-vine model with independent copulas (M2) (35) to the mixed C-vine (M1) (48) and the close log-likelihood it is not surprising that the Vuong- and Clarke-test favor (M2).
- The AIC and BIC values in Table 5.53 approve this conclusions too. As we mentioned it already in Chapter 4 the AIC and the BIC are statistics for nested models and not adapted for non-nested models.

5.3 Economical interpretation

Now we want to interpret the results. The analysis of the data shows that the exchange rate between the US-Dollar and the EURO is the central exchange rate, i.e. that the exchange rates of our data set depend to a greater or lesser extent on the exchange rate of the US-Dollar to the EURO. This is not very surprising in the economical circumstances because the EURO-area and the US-market are the economical strongest markets in the world and the trade between these two economical areas was and still is the main trade route. The exchange rate between two countries or two economical areas depends on the power of the markets and the export and import (trade) between these two countries. The strong correlations between the exchange rate of the EURO to the US-Dollar and the other exchange rates to the US-Dollar can be explained by the power of the different markets and the export and import. The EURO-area as well as the US-market are the main trading partners of the different countries we examined. Especially the strong correlation of the exchange rates of the EURO-area, Switzerland and the United Kingdom can be explained by the neighborhood, the strong cross linking and network between these countries. Australia and Canada are cross linked with the United States (US) (Tables and statistics on WTO [2010]).

A further explanation of the strong correlation may be that the EURO-area, the USA, Canada, Australia, the United Kingdom and Switzerland are so called western countries and high industrialized. The economy of these states depend highly on exports and imports of the same goods, e.g. oil, gas, iron as import and cars, chemical and high technical

good as export goods. If there is an economical or financial crisis as in the years 2008 and 2009 the economies of these states behave similar.
The second root in our analysis/model was EUR,AUS. That means that if the exchange rate of the EURO-zone is given the exchange rate between the US-Dollar and the AUS-Dollar is the next important time series. When we look again at the pairs-plots/Kendall's τ matrix in Figure 5.2 one can detect that the AUS-Dollar is correlated to other economies in a high degree, e.g. AUS-CAN has a Kendall's τ of 0.36 and AUS-BRA a Kendall's τ of 0.32 and the Kendall's τ of AUS-UK is even higher (0.41). The economical interpretation is nearly the same as for the first root (US-Dollar-EURO).
The other countries like Japan, China or India are high industrialized too, but their main trading partners are the USA and countries in the Asian area, whose exchange rates are not in our data set. The same argument may be correct for Brazil and South America too. For more detailed information one needs statistics of the WTO or other worldwide operating economical institutes.
In this sense, the macro economic, our model makes perfect sense and models the economic reality very well.

Chapter 6

Summary and Outlook

In this diploma thesis we dealed with the canonical vine (C-vine) pair-copula construction (PCC). We reviewed how it is defined (see also Aas et al. [2009]) and how one can model dependencies with it (see also Bedford and Cooke [2002]). The main difficulties were to find the C-vine structure and the pair-copula families. For this purpose we first reviewed some copula families and their properties like density, Kendall's τ, tail dependence and λ-function. Based on this properties we developed some methods to identify the pair-copulas using contourplots, the graphical illustration of the λ-function and some goodness-of-fit tests.
The next point was to estimate the copula parameters of a C-vine construction. We explained a simple sequential method based on Kendall's τ, bivariate maximum likelihood estimation and the joint maximum likelihood estimation for parameter estimation of C-vine copulas. Based on the work of Aas et al. [2009] we could implement an algorithm in R for ML estimation.
As application we considered a 9-dimensional data set of exchange rates and explained the model selection tool for structure and for copula families as well as for the final model choice. Furthermore we compared the selected model with models constructed without any selection strategy. We illustrated that the selected model fits the data well and it is better than simple C-vine models with Gaussian or t-copulas with respect to the likelihood and the Vuong- or Clarke-test. We also showed that the reduced model, i.e. the model with independent copulas, is even better than our first selected model because of significant smaller number of parameters and a nearly the same log-likelihood. The log-likelihood is an informal indicator for the goodness-of-fit in applications.
The C-vine is, as well as the D-vine, a special case of the regular vines. A further research point would be to extend the ML estimation and model finding process to regular vines. First steps in this direction are already done in other diploma thesis and research works. In our analysis we observed that the methods we introduced are not optimal for all copula families and copula combinations, especially the detection and fitting of tail dependence should be improved. Therefore one has to introduce and implement more copula families to cover a larger band of possibilities and further identification methods, e.g. further and better goodness-of-fit tests. Berg [2009] and Berg and Quessy [2009] did already some work in this direction by defining some new approaches and comparing a few goodness-of-fit tests in a power comparison study and testing the local power of goodness-of-fit tests

for copulas.
At this state of research the statistician has to select the copula family for each pair-copula. An automation for this step would be helpful. First steps are already done in this thesis and further ideas are in process. One approach is to take the copula with the highest p-value of the goodness-of-fit test. But this method does not consider the error of type 2. Also be interesting would be the Bayesian approach for copula parameter estimation. For the D-vine Ma [2010] and Min and Czado [2010] developed already some algorithms which could be extended to the C-vine. One part for this approach is already be done by defining and implementing a fast computation algorithm for the log-likelihood, see Chapter 3, Figure 3.1 and Example 3.4.
One motivation for canonical vines, or vines in general, is their flexible modeling of dependence structures of financial assets as we mentioned in the introduction. An advantage of these methods is the departure from the normal distribution, both in the marginals and in the dependence. Heinen and Valdesogo [2009] started already an approach to combine C-vines and portfolio theory, mainly CAPM. This thesis may be a help in this direction and research field.

Bibliography

K. Aas, C. Czado, A. Frigessi, and H. Bakken. Pair-copula construction of multiple dependence. *Insurance: Mathematics and Economics*, 44:182–198, 2009.

H. Akaike. A new look at statistical model identification. *IEEE Transaction on Automatic Control*, 19(6):716–723, 1974.

T. Bedford and R.M. Cooke. Vines - a new graphical model for dependent random variables. *Annals of Statistics*, 30:1031–1068, 2002.

D. Berg. Copula goodness-of-fit testing: An overview and power comparison. *The European Journal of Finance*, 15, 2009.

D. Berg and J.-F. Quessy. Local power analysis of goodness-of-fit tests for copulas. *Scandinavian Journal of Statistics*, 36(3), 2009.

T. Bollerslev. Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, 31:307–327, 1986.

P.J. Brockwell and R.A. Davis. *Time series: Theory and Methods*. Springer, New York, 1991.

P.J. Brockwell and R.A. Davis. *Introduction to time series and forecasting*. Springer, New York, 2003.

K.P. Burnham and D.R. Anderson. Multimodel Inference: Understanding AIC and BIC in Model Selection. *Sociological Methods Research*, 33:261–304, 2004.

K.A. Clarke. A Simple Distribution-Free Test for Nonnested Model Selection. *Political Analysis*, 15:347–363, 2007.

J.F. Dißmann. Statistical Inference for Regular Vines and Application. Diplomarbeit, Center of Mathematical Sciences, Munich University of Technology, Garching bei München, 2010.

N. Djunushalieva. Selecting pair copula families for regular vines with application to the multivariate analysis of European stock market indices. Diplomarbeit, Center of Mathematical Sciences, Munich University of Technology, Garching bei München, 2010.

P. Embrechts, F. Lindskog, and A. McNail. Modelling dependence with copulas and applications to risk management. *Handbook of heavy tailed distributions in finance*, 2003.

R.F. Engle. Autoregressive conditional heteroscedasticity with estimates of the variance of uk inflation. *Econometrica*, 50:987–1007, 1982.

V. Erhardt. Verallgemeinerte Poisson und Nullenüberschuß-Regressionsmodelle mit regressiertem Erwartungswert, Dispersions- und Nullenüberschuß-Parameter und eine Anwendung zur Patentmodellierung. Diplomarbeit, Center of Mathematical Sciences, Munich University of Technology, Garching bei München, 2006.

L. Fahrmeir, T. Kneib, and S. Lang. *Regression - Modelle, Methoden und Anwendungen.* Springer, 2008.

J. Franke, W. Härdle, and C. Hafner. *Einführung in die Statistik der Finanzmärkte.* Springer, Berlin, 2001.

C. Genest and A.C. Favre. Everything you always wanted to know about copula modeling but were afraid to ask. *Journal of Hydrologic Engeneering*, 12(4):347–368, 2007.

C. Genest and L.-P. Rivest. Statistical inference procedures for bivariate Archimedean copulas. *J. Amer. Statist. Assoc.*, 88(423):1034–1043, 1993.

C. Genest, J.-F. Quessy, and B. Rémillard. Goodness-of-fit Procedures for Copula Model Based on the Probability Integral Transformation. *Board of the Foundation of the Scandinavian Journal of Statistics*, 32, 2005.

C. Genest, B. Rémillard, and D. Beaudoin. Goodness-of-fit tests for copulas: A review and power study. *Insurance: Mathematics and Economics*, 44 (2009):199–213, 2007.

I.A. Haff, K. Aas, and A. Frigessi. On the simplified pair-copula construction - simply useful or too simplistic? *Journal of Multivariate Analysis*, 101:1296–1310, 2009.

A. Heinen and A. Valdesogo. Asymmetric capm dependence for large dimensions: the canonical vine autoregressive model. 2009. CORE Discussion Papers from Université catholique de Louvain, Center for Operations Research and Econometrics (CORE).

H. Joe. *Multivariate Models and Dependence Concepts.* Chapman und Hall, London, 1997.

M.G. Kendall. A new measure of rank correlation. *Biometrica*, 30:81–93, 1938.

Y.I. Kim. *Essays on Volatility Risk, Asset Returns and Consumption-Based Asset Pricing.* Dissertation, Department of Economics, Graduate School of The Ohio State University, 2008.

D. Kurowicka. Vine truncations, December 2009. Workshop Oslo.

D. Kurowicka and R. Cooke. *Uncertainty Analysis with High Dimensional Dependence Modelling*. John Wiley & Sons Ltd, Chichester, 2006.

P. Lindskog, A. McNail, and U. Schmock. Kendall's tau for elliptical distributions. *Credit Risk: Measurement, Evaluation and Management; Physica-Verlag, Heidelberg*, pages 149–156, 2003.

G.M. Ljung and G.E.P. Box. On a measure of lack of fit in time series models. *Biometrika*, 65:297–303, 1978.

J. Ma. Bayesian inference for D-vine pair-copula constructions based on different bivariate families. Diplomarbeit, Center of Mathematical Sciences, Munich University of Technology, Garching bei München, 2010.

H. Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

A. Min and C. Czado. Bayesian inference for multivariate copulas using pair-copula constructions. *Journal of Financial Econometrics (forthcoming)*, 2010.

R.B. Nelson. *An Introduction to Copulas*. Springer, New York, 2006.

A.K. Nikoloulopoulos, H. Joe, and H. Li. Vine copulas with asymmetric tail dependence and applications to financial return data. *Preprint*, 2010.

World Trade Organization. World trade developments in 2008, April 2010. `http://www.wto.org/english/res_e/statis_e/its2009_e/its09_world_trade_dev_e.htm`.

G.E. Schwarz. Estimating the dimension of a model. *Annals of Statistics*, 6(2):461–464, 1978.

W. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, 19(3):425–442, 1964.

M. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.

P.K. Trivedi and D.M. Zimmer. *Copula modeling: an introduction for practitioners*. Now Publishers, 2007.

Q.H. Vuong. Likelihood Ratio Tests for Model Selection and Non-Nested Hypotheses. *Econometrica*, 57,2:307–333, 1989.

Appendix A

Joe copula

The Joe copula is defined as

$$C(u,v) = 1 - ((1-u)^\delta + (1-v)^\delta - (1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}}$$

for $\delta \geq 1$ as we already mentioned in Chapter 2.1.2.
In Chapter 2.3.2 we calculated the Kendall's τ of the Joe copula as

$$\tau = 1 + \left(\frac{-2 + 2\gamma + 2\ln(2) + \Psi(\frac{1}{\delta}) + \Psi(\frac{1}{2}\frac{2+\delta}{\delta}) + \delta}{-2+\delta} \right)$$

with Euler's constant $\gamma = \lim_{n\to\infty}(\sum_{i=1}^{n} \frac{1}{i} - \ln(n)) \approx 0,57721$ and Digamma-function $\Psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{d}{dx}\Gamma(x)/\Gamma(x)$.
We calculated the last step of the proof of Kendall's τ of the Joe-copula with the math-program MAPLE.
The result we get by the MAPLE-function

int(ln(s)*(1-s)^((2*(1-delta))/delta)*s, s = 0 .. 1)

The h-function and the inverse h-function of the Joe copula we need for the computation of the MLE and for the simulation are:

$$\begin{aligned}
h = \frac{\partial C(u,v)}{\partial v} &= -\frac{1}{\delta}((1-u)^\delta + (1-v)^\delta - (1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}-1} \\
&\cdot \left(\delta(1-v)^{\delta-1}(-1) - (1-u)^\delta\delta(-1)(1-v)^{\delta-1}\right) \\
&= ((1-u)^\delta + (1-v)^\delta -^{(} 1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}}(1-v)^{\delta-1}(1-(1-u)^\delta).
\end{aligned}$$

$$\begin{aligned}
h' = \frac{\partial h}{\partial v} &= (\frac{1}{\delta} - 1)((1-u)^\delta + (1-v)^\delta - (1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}-2} \\
&\cdot \left(\delta(1-v)^{\delta-1}(-1) - (1-u)^\delta\delta(-1)(1-v)^{\delta-1}\right) \\
&+ ((1-u)^\delta + (1-v)^\delta - (1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}-1}(\delta-1)(1-v)^{\delta-2}(-1)(1-(1-u)^\delta) \\
&= (1-\delta)((1-u)^\delta + (1-v)^\delta - (1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}-2}(1-v)^{\delta-1}(1-(1-u)^\delta) \\
&- (\delta-1)((1-u)^\delta + (1-v)^\delta - (1-u)^\delta(1-v)^\delta)^{\frac{1}{\delta}-1}(1-v)^{\delta-2}(1-(1-u)^\delta).
\end{aligned}$$

Appendix B

Data set

To find the best time series model for our data set, the exchange rates, we tried several models with different innovations: Gaussian, t and skew-t innovations. For each innovation group we tried the GARCH(1,1), the AR(1)+GARCH(1,1) and ARMA(1,1)+GARCH(1,1) model. As an indication for a good fit we looked at the Ljung-Box-test (s. Table 5.2) and the QQ-Plots (Figure B.1 - B.3). The following tables and figures are the appropriate results.

Country	μ	ar1	ma1	ω	α_1	β_1
Euro	0.00	1.00	0.01	0.00	0.04	0.96
p-value	0.07	0.00	0.78		0.00	0.00
UK	0.00	0.99	0.00	0.00	0.06	0.94
p-value	0.00	0.00	0.95		0.00	0.00
Canada	0.01	0.99	0.03	0.00	0.07	0.93
p-value	0.00	0.00	0.38		0.00	0.00
Australia	0.00	1.00	0.01	0.00	0.10	0.90
p-value	0.04	0.00	0.74		0.00	0.00
Brazil	0.00	1.00	-0.01	0.00	0.21	0.80
p-value	0.81	0.00	0.79		0.00	0.00
China	0.00	1.00	-0.17	0.00	0.08	0.92
p-value	0.00	0.00	0.00	0.97	0.00	0.00
Japan	0.19	1.00	0.01	0.01	0.04	0.95
p-value	0.06	0.00	0.82		0.00	0.00
Switzerland	0.00	1.00	-0.02	0.00	0.03	0.96
p-value	0.00	0.00	0.50		0.00	0.00
India	0.07	1.00	-0.08	0.00	0.12	0.88
p-value	0.00	0.00	0.00		0.00	0.00

Table B.1: Estimates and p-values for the exchange rates in the ARMA(1,1)+GARCH(1,1)-model with Gaussian innovations

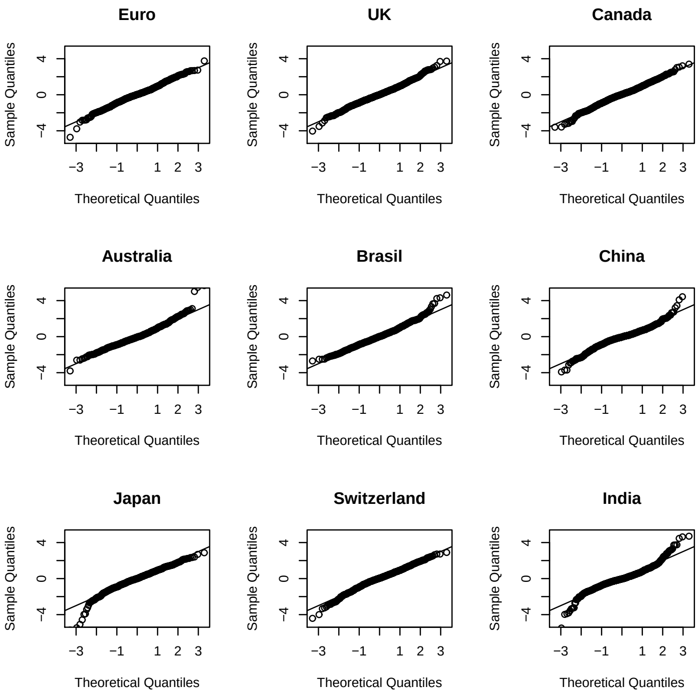


Figure B.1: QQ-Plots of the std. residuals with Gaussian innovations for ARMA(1,1) + GARCH(1,1)

Country	μ	ar1	ma1	ω	α_1	β_1	shape
EURO	0.00	1.00	-0.01	0.00	0.04	0.96	9.23
p-value	0.31	0.00	0.77		0.00	0.00	0.00
UK	0.00	1.00	0.00	0.00	0.05	0.95	8.16
p-value	0.04	0.00	0.95		0.00	0.00	0.00
Canada	0.01	1.00	0.04	0.00	0.06	0.94	9.16
p-value	0.00	0.00	0.24		0.00	0.00	0.00
Australia	0.00	1.00	0.00	0.00	0.08	0.91	6.81
p-value	0.40	0.00	0.97		0.00	0.00	0.00
Brazil	0.00	1.00	-0.00	0.00	0.20	0.80	9.43
p-value	0.43	0.00	0.98		0.00	0.00	0.00
China	0.00	1.00	-0.18	0.00	0.26	0.82	3.26
p-value	0.00	0.00	0.00	0.97	0.00	0.00	0.00
Japan	0.12	1.00	-0.03	0.01	0.04	0.95	6.28
p-value	0.67	0.00	0.42			0.00	0.00
Switzerland	0.00	1.00	-0.04	0.00	0.03	0.96	8.22
p-value	0.19	0.00	0.23				0.00
India	0.01	1.00	-0.11	0.00	0.25	0.82	2.92
p-value	0.30	0.00	0.00			0.00	0.00

Table B.2: Estimates and p-values for the exchange rates in the ARMA(1,1)+GARCH(1,1)-model with t-innovations

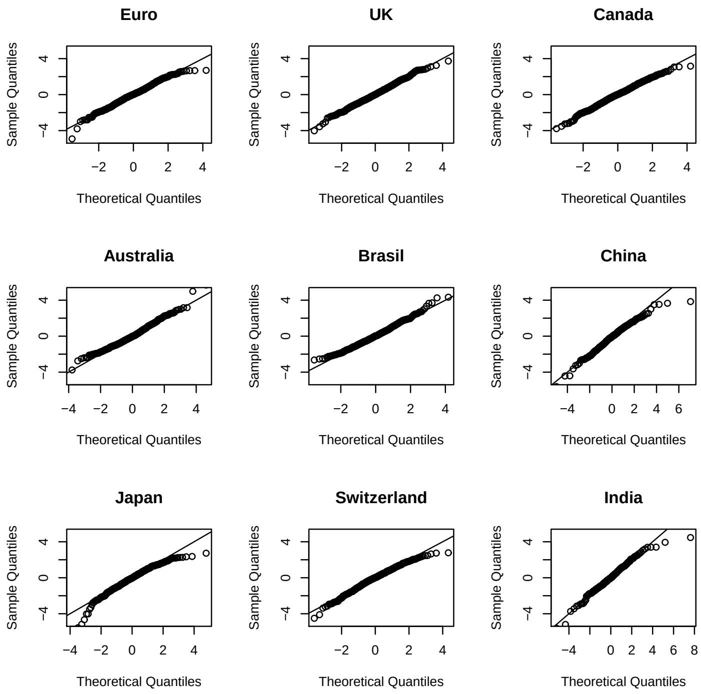


Figure B.2: QQ-Plots of the std. residuals with t innovations for ARMA(1,1) + GARCH(1,1)

Country	μ	ar1	ma1	ω	α_1	β_1	skew	shape
EURO	0.00	1.00	-0.01	0.00	0.04	0.96	0.98	9.23
p-value	0.34	0.00	0.78		0.00	0.00	0.00	0.00
UK	0.00	1.00	0.00	0.00	0.05	0.95	1.05	8.19
p-value	0.11	0.00	0.98			0.00	0.00	0.00
Canada	0.01	1.00	0.04	0.00	0.06	0.94	0.95	9.30
p-value	0.10	0.00	0.21		0.00	0.00	0.00	0.00
Australia	0.00	1.00	-0.02	0.00	0.07	0.92	1.23	7.37
p-value	0.50	0.00	0.48		0.00	0.00	0.00	0.00
Brazil	0.00	1.00	-0.02	0.00	0.18	0.81	1.14	10.00
p-value	0.40	0.00	0.52		0.00	0.00	0.00	0.00
China	0.00	1.00	-0.18	0.00	0.27	0.82	1.02	3.22
p-value	0.00	0.00	0.00	0.97	0.00	0.00	0.00	0.00
Japan	0.14	1.00	-0.03	0.01	0.04	0.95	0.88	7.08
p-value	0.52	0.00	0.32				0.00	0.00
Switzerland	0.00	1.00	-0.04	0.00	0.03	0.96	0.88	9.02
p-value	0.25	0.00	0.17			0.00	0.00	0.00
India	0.02	1.00	-0.11	0.00	0.26	0.82	1.07	2.93
p-value	0.18	0.00	0.00		0.00	0.00	0.00	0.00

Table B.3: Estimates and p-values for the exchange rates in the ARMA(1,1)+GARCH(1,1)-model with skew-t-innovations

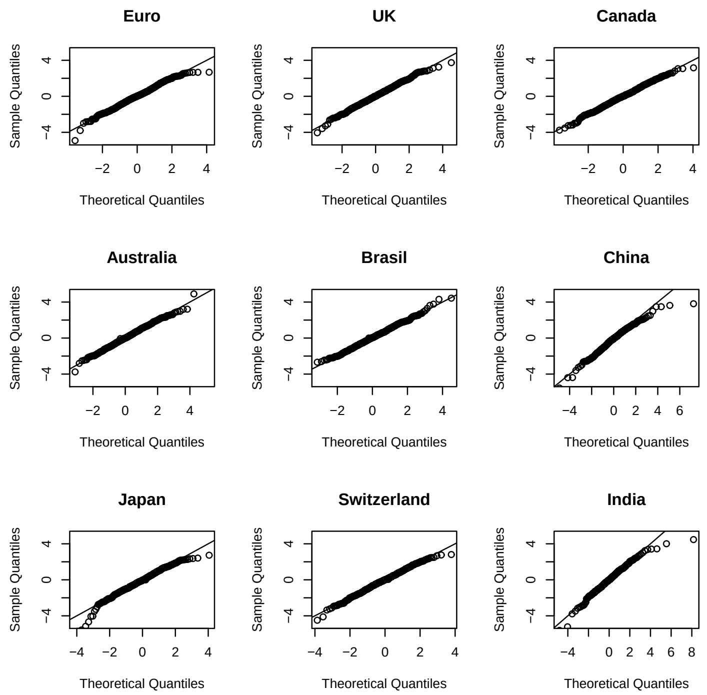


Figure B.3: QQ-Plots of the std. residuals with skewed-t innovations for ARMA(1,1) + GARCH(1,1)

Appendix C

Stability and robustness tests of the R-package "VineMLE"

We made a few stability and robustness tests of the R-package "VineMLE" with different scenarios. In each scenario we tested the VineMLE-function in four different combinations of Kendall's τ. As a common setup we chose a 4 dimensional C-vine with $n = 500$ observations and $r = 100$ replications. The combinations of each scenario are tabulated in Table 3.3 in Chapter 4, where we made the conclusions of this tests.
In Table 3.4 one can find the notation and definitions of the test-statistics for the MLE test, like mean, variance, bias and MSE.
In Table C.1 - C.35 we tested the MLE-algorithm in a C-vine, where all six pair-copulas have the same copula family, e.g. in Table C.1 we set all copulas equal to the Gaussian copula. For the BB1- and BB7-copula we made some additional tests because these two copulas are more difficult to implement and we had some difficulties and instabilities. Furthermore, the BB1 and BB7-copula are two-parameter copulas and more combinations are possible (Table C.8 - C.35).

BB1

For our test scenarios we derived the following four equations (for BB1):

- $\tau = 0.2$ and $\lambda^L = \lambda^U \Rightarrow \theta = 0.2910,\ \delta = 1.0912$
- $\tau = 0.8$ and $\lambda^L = \lambda^U \Rightarrow \theta = 0.8048,\ \delta = 3.5652$
- $\tau = 0.8$ and $2 * \lambda^L = \lambda^U \Rightarrow \theta = 0.1730,\ \delta = 4.6018$
- $\tau = 0.8$ and $0.5 * \lambda^L = \lambda^U \Rightarrow \theta = 4.3067,\ \delta = 1.5856.$

Remark:
The combination $\tau = 0.2$ and $2 * \lambda^L = \lambda^U$ and the combination $\tau = 0.2$ and $0.5 * \lambda^L = \lambda^U$ result in values of θ and δ which are not allowed, i.e. not in the definition area.

<u>BB7</u>

And for the BB7-copulas we derived the following equations:

- $\tau = 0.2$ and $\lambda^L = \lambda^U \Rightarrow \theta = 1.1237,\ \delta = 0.3614$
- $\tau = 0.8$ and $\lambda^L = \lambda^U \Rightarrow \theta = 5.9265,\ \delta = 5.2324$
- $\tau = 0.2$ and $2 * \lambda^L = \lambda^U \Rightarrow \theta = 1.1779,\ \delta = 0.3$
- $\tau = 0.8$ and $2 * \lambda^L = \lambda^U \Rightarrow \theta = 8.0799,\ \delta = 0.88.$

Finally we made some tests with combinations of different copula families.
The scenarios are:

- "N","C","G","F","J","t3"
- "t3","G","J","F","C","N"
- "C","t5","F","G","N","J"
- "G","t5","J","C","t10","F"
- "N","C","BB1","BB7","t10","F"
- "BB1","t5","J","BB7","t10","F"
- "BB7","t5","F","N","t10","BB1"
- "BB1","N","BB7","BB1","t5","BB7",

which can be found in Table C.36 - C.47.

		Copula parameter						Kendall's τ				
τ		ρ	$\bar{\rho}$	$\bar{\rho}_{0.05}$	$\hat{b}(\bar{\rho})$	$s^2(\bar{\rho})$	$\widehat{mse}(\bar{\rho})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	ρ_{12}	0.9511	0.9504	0.9504	-0.0007	0.0000	0.0000	0.7987	0.7987	-0.0013	0.0000	0.0000
0.8	ρ_{13}	0.9511	0.9502	0.9502	-0.0009	0.0000	0.0000	0.7983	0.7984	-0.0017	0.0000	0.0000
0.8	ρ_{14}	0.9511	0.9502	0.9502	-0.0009	0.0000	0.0000	0.7983	0.7983	-0.0017	0.0000	0.0000
0.8	$\rho_{23\|1}$	0.9511	0.9514	0.9515	0.0004	0.0000	0.0000	0.8010	0.8011	0.0010	0.0000	0.0000
0.8	$\rho_{24\|1}$	0.9511	0.9511	0.9511	0.0000	0.0000	0.0000	0.8002	0.8002	0.0002	0.0000	0.0000
0.8	$\rho_{34\|12}$	0.9511	0.9519	0.9518	0.0008	0.0000	0.0000	0.8019	0.8017	0.0019	0.0000	0.0000
0.2	ρ_{12}	0.3090	0.3116	0.3124	0.0026	0.0003	0.0003	0.2019	0.2024	0.0019	0.0001	0.0001
0.2	ρ_{13}	0.3090	0.3095	0.3091	0.0005	0.0003	0.0003	0.2005	0.2001	0.0005	0.0001	0.0001
0.2	ρ_{14}	0.3090	0.3031	0.3026	-0.0059	0.0003	0.0003	0.1962	0.1958	-0.0038	0.0001	0.0001
0.2	$\rho_{23\|1}$	0.3090	0.3177	0.3185	0.0087	0.0003	0.0004	0.2060	0.2065	0.0060	0.0001	0.0002
0.2	$\rho_{24\|1}$	0.3090	0.3089	0.3094	-0.0001	0.0003	0.0003	0.2001	0.2003	0.0001	0.0001	0.0001
0.2	$\rho_{34\|12}$	0.3090	0.3016	0.3015	-0.0074	0.0003	0.0003	0.1952	0.1951	-0.0048	0.0001	0.0002
0.8	ρ_{12}	0.9511	0.9513	0.9514	0.0003	0.0000	0.0000	0.8007	0.8008	0.0007	0.0000	0.0000
0.8	ρ_{13}	0.9511	0.9504	0.9505	-0.0006	0.0000	0.0000	0.7989	0.7990	-0.0011	0.0000	0.0000
0.8	ρ_{14}	0.9511	0.9511	0.9512	0.0000	0.0000	0.0000	0.8002	0.8004	0.0002	0.0000	0.0000
0.2	$\rho_{23\|1}$	0.3090	0.3036	0.3030	-0.0054	0.0003	0.0004	0.1966	0.1961	-0.0034	0.0002	0.0002
0.2	$\rho_{24\|1}$	0.3090	0.3093	0.3090	0.0003	0.0003	0.0003	0.2004	0.2001	0.0004	0.0001	0.0001
0.2	$\rho_{34\|12}$	0.3090	0.3182	0.3195	0.0092	0.0003	0.0004	0.2064	0.2071	0.0064	0.0001	0.0002
0.2	ρ_{12}	0.3090	0.3109	0.3105	0.0018	0.0004	0.0004	0.2014	0.2012	0.0014	0.0002	0.0002
0.2	ρ_{13}	0.3090	0.3104	0.3101	0.0014	0.0004	0.0004	0.2012	0.2009	0.0012	0.0002	0.0002
0.2	ρ_{14}	0.3090	0.3104	0.3101	0.0014	0.0004	0.0004	0.2012	0.2008	0.0012	0.0002	0.0002
0.8	$\rho_{23\|1}$	0.9511	0.9501	0.9502	-0.0010	0.0000	0.0000	0.7982	0.7983	-0.0018	0.0000	0.0000
0.8	$\rho_{24\|1}$	0.9511	0.9502	0.9503	-0.0008	0.0000	0.0000	0.7984	0.7985	-0.0016	0.0000	0.0000
0.8	$\rho_{34\|12}$	0.9511	0.9506	0.9508	-0.0004	0.0000	0.0000	0.7994	0.7996	-0.0006	0.0000	0.0000

Table C.1: Results of the stability and robustness tests of the Gaussian copula in different scenarios (n=500, r=100).

		Copula parameters						**Kendall's τ**				
τ		θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	ρ_{12}	0.9511	0.9498	0.9499	-0.0013	0.0000	0.0000	0.7976	0.7977	-0.0024	0.0000	0.0000
	ν_{12}	5.0000	5.4424	5.2654	0.4424	0.6790	0.8747					
0.8	ρ_{13}	0.9511	0.9500	0.9501	-0.0010	0.0000	0.0000	0.7981	0.7981	-0.0019	0.0000	0.0000
	ν_{13}	5.0000	5.4154	5.2440	0.4154	0.6352	0.8078					
0.8	ρ_{14}	0.9511	0.9500	0.9500	-0.0011	0.0000	0.0000	0.7979	0.7980	-0.0021	0.0000	0.0000
	ν_{14}	5.0000	5.4308	5.2151	0.4308	0.7419	0.9275					
0.8	$\rho_{23\|1}$	0.9511	0.9511	0.9510	0.0000	0.0000	0.0000	0.8003	0.8000	0.0003	0.0000	0.0000
	$\nu_{23\|1}$	5.0000	5.6381	5.2633	0.6381	1.4493	1.8565					
0.8	$\rho_{24\|1}$	0.9511	0.9510	0.9510	-0.0000	0.0000	0.0000	0.8002	0.8000	0.0002	0.0000	0.0000
	$\nu_{24\|1}$	5.0000	5.4415	5.1873	0.4415	0.9784	1.1733					
0.8	$\rho_{34\|12}$	0.9511	0.9516	0.9516	0.0006	0.0000	0.0000	0.8014	0.8013	0.0014	0.0000	0.0000
	$\nu_{34\|12}$	5.0000	5.6334	5.4759	0.6334	0.7380	1.1392					
0.2	ρ_{12}	0.3090	0.3148	0.3151	0.0058	0.0004	0.0004	0.2041	0.2042	0.0041	0.0002	0.0002
	ν_{12}	5.0000	5.6162	5.4727	0.6162	0.5679	0.9476					
0.2	ρ_{13}	0.3090	0.3106	0.3102	0.0016	0.0004	0.0004	0.2013	0.2009	0.0013	0.0002	0.0002
	ν_{13}	5.0000	5.5741	5.2519	0.5741	1.6040	1.9336					
0.2	ρ_{14}	0.3090	0.3095	0.3100	0.0005	0.0004	0.0004	0.2005	0.2008	0.0005	0.0002	0.0002
	ν_{14}	5.0000	5.6403	5.3121	0.6403	1.4074	1.8175					
0.2	$\rho_{23\|1}$	0.3090	0.3044	0.3056	-0.0046	0.0004	0.0004	0.1971	0.1979	-0.0029	0.0002	0.0002
	$\nu_{23\|1}$	5.0000	5.2365	5.0958	0.2365	0.5275	0.5834					
0.2	$\rho_{24\|1}$	0.3090	0.3144	0.3145	0.0054	0.0004	0.0004	0.2038	0.2038	0.0038	0.0002	0.0002
	$\nu_{24\|1}$	5.0000	5.3899	5.0762	0.3899	1.4381	1.5901					
0.2	$\rho_{34\|12}$	0.3090	0.3094	0.3102	0.0003	0.0004	0.0004	0.2005	0.2009	0.0005	0.0002	0.0002
	$\nu_{34\|12}$	5.0000	5.4070	5.1754	0.4070	0.8147	0.9804					

Table C.2: Results of the stability and robustness tests of the t-copula with five degrees of freedom in different scenarios (n=500, r=100). (part 1)

		Copula parameters						**Kendall's τ**				
τ		θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	ρ_{12}	0.9511	0.9512	0.9513	0.0001	0.0000	0.0000	0.8004	0.8007	0.0004	0.0000	0.0000
	ν_{12}	5.0000	5.6283	5.4112	0.6283	0.8695	1.2642					
0.8	ρ_{13}	0.9511	0.9509	0.9509	-0.0002	0.0000	0.0000	0.7998	0.7999	-0.0002	0.0000	0.0000
	ν_{13}	5.0000	5.6205	5.3623	0.6205	0.9597	1.3447					
0.8	ρ_{14}	0.9511	0.9513	0.9513	0.0003	0.0000	0.0000	0.8007	0.8007	0.0007	0.0000	0.0000
	ν_{14}	5.0000	5.5637	5.3056	0.5637	1.1386	1.4564					
0.2	$\rho_{23\|1}$	0.3090	0.3050	0.3044	-0.0040	0.0004	0.0004	0.1976	0.1971	-0.0024	0.0002	0.0002
	$\nu_{23\|1}$	5.0000	5.9626	5.5783	0.9626	1.8470	2.7736					
0.2	$\rho_{24\|1}$	0.3090	0.3070	0.3064	-0.0020	0.0004	0.0004	0.1989	0.1984	-0.0011	0.0002	0.0002
	$\nu_{24\|1}$	5.0000	5.7435	5.5010	0.7435	0.9078	1.4605					
0.2	$\rho_{34\|12}$	0.3090	0.3034	0.3029	-0.0056	0.0004	0.0004	0.1965	0.1960	-0.0035	0.0002	0.0002
	$\nu_{34\|12}$	5.0000	5.5346	5.2812	0.5346	1.0019	1.2877					
0.2	ρ_{12}	0.3090	0.3124	0.3116	0.0034	0.0005	0.0005	0.2025	0.2019	0.0025	0.0002	0.0002
	ν_{12}	5.0000	5.4938	5.3611	0.4938	0.5505	0.7943					
0.2	ρ_{13}	0.3090	0.3132	0.3122	0.0042	0.0004	0.0004	0.2031	0.2023	0.0031	0.0002	0.0002
	ν_{13}	5.0000	5.5792	5.4922	0.5792	0.5151	0.8506					
0.2	ρ_{14}	0.3090	0.3132	0.3123	0.0041	0.0004	0.0004	0.2030	0.2023	0.0030	0.0002	0.0002
	ν_{14}	5.0000	5.5539	5.4884	0.5539	0.4725	0.7793					
0.8	$\rho_{23\|1}$	0.9511	0.9513	0.9514	0.0003	0.0000	0.0000	0.8008	0.8009	0.0008	0.0000	0.0000
	$\nu_{23\|1}$	5.0000	6.1138	5.6833	1.1138	1.8530	3.0936					
0.8	$\rho_{24\|1}$	0.9511	0.9515	0.9515	0.0004	0.0000	0.0000	0.8011	0.8011	0.0011	0.0000	0.0000
	$\nu_{24\|1}$	5.0000	5.8139	5.5052	0.8139	1.2349	1.8973					
0.8	$\rho_{34\|12}$	0.9511	0.9502	0.9503	-0.0008	0.0000	0.0000	0.7986	0.7987	-0.0014	0.0000	0.0000
	$\nu_{34\|12}$	5.0000	5.1683	5.0151	0.1683	0.4970	0.5254					

Table C.3: Results of the stability and robustness tests of the t-copula with five degrees of freedom in different scenarios (n=500, r=100). (part 2)

		Copula parameter						**Kendall's τ**				
τ		θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	θ_{12}	8.0000	8.0636	8.0495	0.0636	0.0213	0.0254	0.8011	0.8009	0.0011	0.0000	0.0000
0.8	θ_{13}	8.0000	8.0643	8.0486	0.0643	0.0208	0.0249	0.8011	0.8009	0.0011	0.0000	0.0000
0.8	θ_{14}	8.0000	8.0650	8.0496	0.0650	0.0209	0.0251	0.8011	0.8009	0.0011	0.0000	0.0000
0.8	$\theta_{23\|1}$	8.0000	7.9989	7.9953	-0.0011	0.0365	0.0365	0.7996	0.7997	-0.0004	0.0000	0.0000
0.8	$\theta_{24\|1}$	8.0000	7.9931	7.9907	-0.0069	0.0329	0.0330	0.7995	0.7996	-0.0005	0.0000	0.0000
0.8	$\theta_{34\|12}$	8.0000	7.9719	7.9769	-0.0281	0.0333	0.0341	0.7991	0.7993	-0.0009	0.0000	0.0000
0.2	θ_{12}	0.5000	0.4996	0.4992	-0.0004	0.0007	0.0007	0.1994	0.1995	-0.0006	0.0001	0.0001
0.2	θ_{13}	0.5000	0.5064	0.5048	0.0064	0.0008	0.0009	0.2015	0.2012	0.0015	0.0001	0.0001
0.2	θ_{14}	0.5000	0.5248	0.5246	0.0248	0.0009	0.0015	0.2073	0.2075	0.0073	0.0001	0.0001
0.2	$\theta_{23\|1}$	0.5000	0.4942	0.4925	-0.0058	0.0010	0.0011	0.1975	0.1972	-0.0025	0.0001	0.0001
0.2	$\theta_{24\|1}$	0.5000	0.4909	0.4911	-0.0091	0.0012	0.0013	0.1963	0.1966	-0.0037	0.0001	0.0001
0.2	$\theta_{34\|12}$	0.5000	0.5063	0.5047	0.0063	0.0012	0.0012	0.2013	0.2010	0.0013	0.0001	0.0001
0.8	θ_{12}	8.0000	8.0247	8.0248	0.0247	0.0190	0.0197	0.8003	0.8004	0.0003	0.0000	0.0000
0.8	θ_{13}	8.0000	7.9899	7.9846	-0.0101	0.0263	0.0264	0.7995	0.7995	-0.0005	0.0000	0.0000
0.8	θ_{14}	8.0000	8.0277	8.0196	0.0277	0.0232	0.0239	0.8003	0.8002	0.0003	0.0000	0.0000
0.2	$\theta_{23\|1}$	0.5000	0.5138	0.5131	0.0138	0.0008	0.0010	0.2039	0.2038	0.0039	0.0001	0.0001
0.2	$\theta_{24\|1}$	0.5000	0.5026	0.5014	0.0026	0.0009	0.0009	0.2002	0.2001	0.0002	0.0001	0.0001
0.2	$\theta_{34\|12}$	0.5000	0.5067	0.5060	0.0067	0.0011	0.0012	0.2014	0.2014	0.0014	0.0001	0.0001
0.2	θ_{12}	0.5000	0.4971	0.4979	-0.0029	0.0004	0.0005	0.1988	0.1991	-0.0012	0.0000	0.0000
0.2	θ_{13}	0.5000	0.4980	0.4986	-0.0020	0.0004	0.0004	0.1991	0.1994	-0.0009	0.0000	0.0000
0.2	θ_{14}	0.5000	0.4976	0.4982	-0.0024	0.0004	0.0004	0.1989	0.1992	-0.0011	0.0000	0.0000
0.8	$\theta_{23\|1}$	8.0000	8.0709	8.0659	0.0709	0.0180	0.0230	0.8012	0.8012	0.0012	0.0000	0.0000
0.8	$\theta_{24\|1}$	8.0000	8.0943	8.0871	0.0943	0.0196	0.0285	0.8017	0.8016	0.0017	0.0000	0.0000
0.8	$\theta_{34\|12}$	8.0000	7.9967	7.9957	-0.0033	0.0395	0.0396	0.7995	0.7997	-0.0005	0.0000	0.0000

Table C.4: Results of the stability and robustness tests of the Clayton copula in different scenarios (n=500, r=100).

		Copula parameter						Kendall's τ				
τ		θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	θ_{12}	5.0000	5.0199	5.0178	0.0199	0.0050	0.0054	0.8006	0.8006	0.0006	0.0000	0.0000
0.8	θ_{13}	5.0000	5.0209	5.0194	0.0209	0.0051	0.0055	0.8006	0.8006	0.0006	0.0000	0.0000
0.8	θ_{14}	5.0000	5.0185	5.0166	0.0185	0.0051	0.0054	0.8005	0.8005	0.0005	0.0000	0.0000
0.8	$\theta_{23\|1}$	5.0000	4.9973	5.0066	-0.0027	0.0138	0.0138	0.7993	0.8000	-0.0007	0.0000	0.0000
0.8	$\theta_{24\|1}$	5.0000	4.9974	5.0045	-0.0026	0.0139	0.0139	0.7993	0.7999	-0.0007	0.0000	0.0000
0.8	$\theta_{34\|12}$	5.0000	5.0090	4.9956	0.0090	0.0171	0.0172	0.7997	0.7995	-0.0003	0.0000	0.0000
0.2	θ_{12}	1.2500	1.2547	1.2539	0.0047	0.0002	0.0003	0.2024	0.2021	0.0024	0.0001	0.0001
0.2	θ_{13}	1.2500	1.2583	1.2576	0.0083	0.0003	0.0004	0.2046	0.2044	0.0046	0.0001	0.0001
0.2	θ_{14}	1.2500	1.2533	1.2531	0.0033	0.0002	0.0002	0.2015	0.2016	0.0015	0.0001	0.0001
0.2	$\theta_{23\|1}$	1.2500	1.2469	1.2464	-0.0031	0.0004	0.0004	0.1971	0.1971	-0.0029	0.0001	0.0002
0.2	$\theta_{24\|1}$	1.2500	1.2441	1.2439	-0.0059	0.0004	0.0004	0.1953	0.1955	-0.0047	0.0001	0.0002
0.2	$\theta_{34\|12}$	1.2500	1.2530	1.2523	0.0030	0.0005	0.0005	0.2007	0.2007	0.0007	0.0002	0.0002
0.8	θ_{12}	5.0000	5.0066	5.0092	0.0066	0.0052	0.0053	0.8001	0.8002	0.0001	0.0000	0.0000
0.8	θ_{13}	5.0000	4.9992	4.9941	-0.0008	0.0069	0.0069	0.7997	0.7996	-0.0003	0.0000	0.0000
0.8	θ_{14}	5.0000	4.9731	4.9731	-0.0269	0.0063	0.0070	0.7987	0.7988	-0.0013	0.0000	0.0000
0.2	$\theta_{23\|1}$	1.2500	1.2499	1.2502	-0.0001	0.0004	0.0004	0.1989	0.1994	-0.0011	0.0002	0.0002
0.2	$\theta_{24\|1}$	1.2500	1.2497	1.2494	-0.0003	0.0004	0.0004	0.1988	0.1990	-0.0012	0.0002	0.0002
0.2	$\theta_{34\|12}$	1.2500	1.2542	1.2529	0.0042	0.0003	0.0004	0.2018	0.2013	0.0018	0.0001	0.0001
0.2	θ_{12}	1.2500	1.2512	1.2507	0.0012	0.0002	0.0002	0.2002	0.2001	0.0002	0.0001	0.0001
0.2	θ_{13}	1.2500	1.2522	1.2515	0.0022	0.0002	0.0002	0.2009	0.2007	0.0009	0.0001	0.0001
0.2	θ_{14}	1.2500	1.2525	1.2518	0.0025	0.0002	0.0002	0.2010	0.2008	0.0010	0.0001	0.0001
0.8	$\theta_{23\|1}$	5.0000	4.9998	5.0047	-0.0002	0.0084	0.0084	0.7996	0.8000	-0.0004	0.0000	0.0000
0.8	$\theta_{24\|1}$	5.0000	4.9934	4.9945	-0.0066	0.0086	0.0087	0.7994	0.7996	-0.0006	0.0000	0.0000
0.8	$\theta_{34\|12}$	5.0000	5.0289	5.0293	0.0289	0.0133	0.0141	0.8006	0.8009	0.0006	0.0000	0.0000

Table C.5: Results of the stability and robustness tests of the Gumbel copula in different scenarios (n=500, r=100).

		Copula parameter						Kendall's τ				
τ		θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	θ_{12}	18.1000	18.1528	18.1352	0.0528	0.0958	0.0986	0.7994	0.7993	-0.0006	0.0000	0.0000
0.8	θ_{13}	18.1000	18.1556	18.1287	0.0556	0.0799	0.0830	0.7995	0.7993	-0.0005	0.0000	0.0000
0.8	θ_{14}	18.1000	18.1261	18.0985	0.0261	0.0750	0.0757	0.7992	0.7990	-0.0008	0.0000	0.0000
0.8	$\theta_{23\|1}$	18.1000	18.1774	18.1707	0.0774	0.1029	0.1089	0.7996	0.7996	-0.0004	0.0000	0.0000
0.8	$\theta_{24\|1}$	18.1000	18.0909	18.0914	-0.0091	0.0992	0.0993	0.7988	0.7989	-0.0012	0.0000	0.0000
0.8	$\theta_{34\|12}$	18.1000	18.0619	18.0652	-0.0381	0.1187	0.1202	0.7984	0.7986	-0.0016	0.0000	0.0000
0.2	θ_{12}	1.8600	1.8868	1.8868	0.0268	0.0154	0.0161	0.2022	0.2023	0.0022	0.0002	0.0002
0.2	θ_{13}	1.8600	1.9044	1.9025	0.0444	0.0170	0.0190	0.2039	0.2039	0.0039	0.0002	0.0002
0.2	θ_{14}	1.8600	1.8534	1.8533	-0.0066	0.0129	0.0130	0.1989	0.1990	-0.0011	0.0001	0.0001
0.2	$\theta_{23\|1}$	1.8600	1.8525	1.8511	-0.0075	0.0128	0.0128	0.1988	0.1988	-0.0012	0.0001	0.0001
0.2	$\theta_{24\|1}$	1.8600	1.8561	1.8638	-0.0039	0.0133	0.0133	0.1992	0.2001	-0.0008	0.0001	0.0001
0.2	$\theta_{34\|12}$	1.8600	1.8752	1.8740	0.0152	0.0135	0.0137	0.2011	0.2011	0.0011	0.0001	0.0001
0.8	θ_{12}	18.1000	18.1551	18.1682	0.0551	0.0978	0.1008	0.7994	0.7996	-0.0006	0.0000	0.0000
0.8	θ_{13}	18.1000	18.0997	18.0903	-0.0003	0.1305	0.1305	0.7988	0.7988	-0.0012	0.0000	0.0000
0.8	θ_{14}	18.1000	18.0682	18.0595	-0.0318	0.1019	0.1029	0.7985	0.7985	-0.0015	0.0000	0.0000
0.2	$\theta_{23\|1}$	1.8600	1.8814	1.8838	0.0214	0.0154	0.0159	0.2017	0.2021	0.0017	0.0002	0.0002
0.2	$\theta_{24\|1}$	1.8600	1.8734	1.8743	0.0134	0.0117	0.0119	0.2010	0.2011	0.0010	0.0001	0.0001
0.2	$\theta_{34\|12}$	1.8600	1.9269	1.9254	0.0669	0.0119	0.0164	0.2063	0.2063	0.0063	0.0001	0.0002
0.2	θ_{12}	1.8600	1.9150	1.9127	0.0550	0.0073	0.0103	0.2052	0.2051	0.0052	0.0001	0.0001
0.2	θ_{13}	1.8600	1.9065	1.9048	0.0465	0.0062	0.0084	0.2044	0.2043	0.0044	0.0001	0.0001
0.2	θ_{14}	1.8600	1.9074	1.9043	0.0474	0.0066	0.0089	0.2045	0.2043	0.0045	0.0001	0.0001
0.8	$\theta_{23\|1}$	18.1000	18.2090	18.1906	0.1090	0.1028	0.1146	0.7999	0.7998	-0.0001	0.0000	0.0000
0.8	$\theta_{24\|1}$	18.1000	18.2359	18.1955	0.1359	0.1117	0.1302	0.8002	0.7999	0.0002	0.0000	0.0000
0.8	$\theta_{34\|12}$	18.1000	17.9797	17.9549	-0.1203	0.1370	0.1515	0.7975	0.7974	-0.0025	0.0000	0.0000

Table C.6: Results of the stability and robustness tests of the Frank copula in different scenarios (n=500, r=100).

		Copula parameter						Kendall's τ				
τ		θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	θ_{12}	8.7000	8.7144	8.7107	0.0144	0.0138	0.0141	0.7988	0.7988	-0.0012	0.0000	0.0000
0.8	θ_{13}	8.7000	8.7223	8.7186	0.0223	0.0145	0.0150	0.7989	0.7989	-0.0011	0.0000	0.0000
0.8	θ_{14}	8.7000	8.7226	8.7191	0.0226	0.0145	0.0150	0.7989	0.7989	-0.0011	0.0000	0.0000
0.8	$\theta_{23\|1}$	8.7000	8.7247	8.7363	0.0247	0.0284	0.0290	0.7988	0.7991	-0.0012	0.0000	0.0000
0.8	$\theta_{24\|1}$	8.7000	8.7223	8.7292	0.0223	0.0293	0.0298	0.7988	0.7990	-0.0012	0.0000	0.0000
0.8	$\theta_{34\|12}$	8.7000	8.7197	8.7338	0.0197	0.0460	0.0464	0.7985	0.7991	-0.0015	0.0000	0.0000
0.2	θ_{12}	1.4400	1.4370	1.4364	-0.0030	0.0008	0.0008	0.1970	0.1970	-0.0030	0.0001	0.0001
0.2	θ_{13}	1.4400	1.4378	1.4358	-0.0022	0.0006	0.0006	0.1974	0.1969	-0.0026	0.0001	0.0001
0.2	θ_{14}	1.4400	1.4312	1.4304	-0.0088	0.0006	0.0007	0.1950	0.1949	-0.0050	0.0001	0.0001
0.2	$\theta_{23\|1}$	1.4400	1.4353	1.4349	-0.0047	0.0008	0.0008	0.1964	0.1965	-0.0036	0.0001	0.0001
0.2	$\theta_{24\|1}$	1.4400	1.4430	1.4404	0.0030	0.0010	0.0010	0.1989	0.1983	-0.0011	0.0001	0.0001
0.2	$\theta_{34\|12}$	1.4400	1.4400	1.4392	-0.0000	0.0009	0.0009	0.1979	0.1979	-0.0021	0.0001	0.0001
0.8	θ_{12}	8.7000	8.7281	8.7256	0.0281	0.0170	0.0177	0.7990	0.7990	-0.0010	0.0000	0.0000
0.8	θ_{13}	8.7000	8.7073	8.7158	0.0073	0.0243	0.0243	0.7985	0.7988	-0.0015	0.0000	0.0000
0.8	θ_{14}	8.7000	8.6560	8.6598	-0.0440	0.0199	0.0218	0.7975	0.7977	-0.0025	0.0000	0.0000
0.2	$\theta_{23\|1}$	1.4400	1.4330	1.4328	-0.0070	0.0009	0.0009	0.1954	0.1956	-0.0046	0.0001	0.0001
0.2	$\theta_{24\|1}$	1.4400	1.4370	1.4362	-0.0030	0.0011	0.0011	0.1967	0.1967	-0.0033	0.0001	0.0001
0.2	$\theta_{34\|12}$	1.4400	1.4454	1.4456	0.0054	0.0009	0.0010	0.1998	0.2001	-0.0002	0.0001	0.0001
0.2	θ_{12}	1.4400	1.4464	1.4470	0.0064	0.0004	0.0004	0.2006	0.2009	0.0006	0.0000	0.0000
0.2	θ_{13}	1.4400	1.4471	1.4475	0.0071	0.0004	0.0004	0.2009	0.2011	0.0009	0.0000	0.0000
0.2	θ_{14}	1.4400	1.4472	1.4474	0.0072	0.0004	0.0004	0.2009	0.2011	0.0009	0.0000	0.0000
0.8	$\theta_{23\|1}$	8.7000	8.6671	8.6815	-0.0329	0.0265	0.0276	0.7976	0.7981	-0.0024	0.0000	0.0000
0.8	$\theta_{24\|1}$	8.7000	8.6744	8.6881	-0.0256	0.0265	0.0271	0.7978	0.7983	-0.0022	0.0000	0.0000
0.8	$\theta_{34\|12}$	8.7000	8.6957	8.6970	-0.0043	0.0497	0.0497	0.7980	0.7983	-0.0020	0.0000	0.0000

Table C.7: Results of the stability and robustness tests of the Joe copula in different scenarios (n=500, r=100).

Copopula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	6.4800	5.7422	5.8152	-0.7378	0.0465	0.5908
δ_{12}	1.1792	1.2995	1.2913	0.1202	0.0014	0.0159
θ_{13}	6.4800	5.7660	5.8391	-0.7140	0.0428	0.5526
δ_{13}	1.1792	1.2971	1.2896	0.1179	0.0011	0.0150
θ_{14}	6.4800	5.7662	5.8399	-0.7138	0.0438	0.5533
δ_{14}	1.1792	1.2975	1.2898	0.1183	0.0012	0.0151
$\theta_{23\|1}$	6.4800	5.7900	5.8737	-0.6900	0.0486	0.5247
$\delta_{23\|1}$	1.1792	1.2600	1.2619	0.0807	0.0011	0.0076
$\theta_{24\|1}$	6.4800	5.7933	5.8759	-0.6867	0.0476	0.5192
$\delta_{24\|1}$	1.1792	1.2559	1.2576	0.0767	0.0013	0.0071
$\theta_{34\|12}$	6.4800	5.7506	5.8268	-0.7293	0.0492	0.5811
$\delta_{34\|12}$	1.1792	1.2951	1.2737	0.1158	0.0079	0.0213
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.8003	0.8005	0.0003	0.0000	0.0000
τ_{13}	0.8000	0.8006	0.8009	0.0006	0.0000	0.0000
τ_{14}	0.8000	0.8007	0.8009	0.0007	0.0000	0.0000
$\tau_{23\|1}$	0.8000	0.7942	0.7979	-0.0058	0.0001	0.0002
$\tau_{24\|1}$	0.8000	0.7936	0.7972	-0.0064	0.0001	0.0002
$\tau_{34\|12}$	0.8000	0.7970	0.7965	-0.0030	0.0001	0.0001
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.2911	0.2872	0.0911	0.0005	0.0088
λ^U_{13}	0.2000	0.2901	0.2864	0.0901	0.0005	0.0086
λ^U_{14}	0.2000	0.2903	0.2864	0.0903	0.0005	0.0086
$\lambda^U_{23\|1}$	0.2000	0.2621	0.2659	0.0621	0.0007	0.0046
$\lambda^U_{24\|1}$	0.2000	0.2583	0.2624	0.0583	0.0008	0.0042
$\lambda^U_{34\|12}$	0.2000	0.2745	0.2707	0.0745	0.0019	0.0075

Table C.8: Results of the stability and robustness tests of the BB1-copula in different scenarios. (n=500, r=100) (part 1)

Copopula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.1200	0.1308	0.1290	0.0108	0.0008	0.0009
δ_{12}	1.1792	1.1786	1.1783	-0.0006	0.0006	0.0006
θ_{13}	0.1200	0.1306	0.1265	0.0106	0.0012	0.0014
δ_{13}	1.1792	1.1820	1.1828	0.0028	0.0004	0.0004
θ_{14}	0.1200	0.1244	0.1230	0.0044	0.0009	0.0009
δ_{14}	1.1792	1.1782	1.1777	-0.0010	0.0004	0.0004
$\theta_{23\|1}$	0.1200	0.1107	0.1066	-0.0093	0.0012	0.0013
$\delta_{23\|1}$	1.1792	1.1829	1.1837	0.0036	0.0005	0.0005
$\theta_{24\|1}$	0.1200	0.1088	0.1056	-0.0112	0.0010	0.0011
$\delta_{24\|1}$	1.1792	1.1896	1.1869	0.0104	0.0006	0.0007
$\theta_{34\|12}$	0.1200	0.1298	0.1285	0.0098	0.0010	0.0011
$\delta_{34\|12}$	1.1792	1.1728	1.1724	-0.0065	0.0005	0.0005
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2020	0.2018	0.0020	0.0002	0.0002
τ_{13}	0.2000	0.2044	0.2041	0.0044	0.0001	0.0001
τ_{14}	0.2000	0.1994	0.1995	-0.0006	0.0002	0.0002
$\tau_{23\|1}$	0.2000	0.1972	0.1974	-0.0028	0.0001	0.0001
$\tau_{24\|1}$	0.2000	0.2009	0.2004	0.0009	0.0002	0.0002
$\tau_{34\|12}$	0.2000	0.1977	0.1976	-0.0023	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.1967	0.1976	-0.0033	0.0005	0.0005
λ^U_{13}	0.2000	0.2005	0.2020	0.0005	0.0003	0.0003
λ^U_{14}	0.2000	0.1970	0.1973	-0.0030	0.0003	0.0004
$\lambda^U_{23\|1}$	0.2000	0.2007	0.2024	0.0007	0.0004	0.0004
$\lambda^U_{24\|1}$	0.2000	0.2065	0.2052	0.0065	0.0004	0.0005
$\lambda^U_{34\|12}$	0.2000	0.1917	0.1921	-0.0083	0.0004	0.0005

Table C.9: Results of the stability and robustness tests of the BB1-copula in different scenarios. (n=500, r=100) (part 2)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	6.4800	5.7971	5.8429	-0.6828	0.0233	0.4896
δ_{12}	1.1792	1.2728	1.2698	0.0935	0.0009	0.0096
θ_{13}	6.4800	5.6535	5.7351	-0.8265	0.0619	0.7450
δ_{13}	1.1792	1.2899	1.2868	0.1106	0.0010	0.0133
θ_{14}	6.4800	5.6268	5.7416	-0.8532	0.1031	0.8311
δ_{14}	1.1792	1.2912	1.2803	0.1120	0.0023	0.0148
$\theta_{23\|1}$	0.1200	0.1188	0.1166	-0.0012	0.0010	0.0010
$\delta_{23\|1}$	1.1792	1.1844	1.1836	0.0052	0.0006	0.0006
$\theta_{24\|1}$	0.1200	0.1297	0.1290	0.0097	0.0012	0.0013
$\delta_{24\|1}$	1.1792	1.1787	1.1793	-0.0005	0.0006	0.0006
$\theta_{34\|12}$	0.1200	0.1449	0.1398	0.0249	0.0020	0.0027
$\delta_{34\|12}$	1.1792	1.1839	1.1832	0.0047	0.0006	0.0007
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7978	0.7983	-0.0022	0.0000	0.0000
τ_{13}	0.8000	0.7961	0.7975	-0.0039	0.0000	0.0001
τ_{14}	0.8000	0.7947	0.7961	-0.0053	0.0000	0.0001
$\tau_{23\|1}$	0.2000	0.2012	0.2012	0.0012	0.0002	0.0002
$\tau_{24\|1}$	0.2000	0.2014	0.2018	0.0014	0.0002	0.0002
$\tau_{34\|12}$	0.2000	0.2100	0.2093	0.0100	0.0002	0.0003
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.2729	0.2719	0.0729	0.0005	0.0058
λ^U_{13}	0.2000	0.2851	0.2843	0.0851	0.0005	0.0077
λ^U_{14}	0.2000	0.2825	0.2780	0.0825	0.0009	0.0077
$\lambda^U_{23\|1}$	0.2000	0.2017	0.2023	0.0017	0.0005	0.0005
$\lambda^U_{24\|1}$	0.2000	0.1965	0.1981	-0.0035	0.0005	0.0005
$\lambda^U_{34\|12}$	0.2000	0.2011	0.2015	0.0011	0.0005	0.0005

Table C.10: Results of the stability and robustness tests of the BB1-copula in different scenarios. (n=500, r=100) (part 3)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.1200	0.1451	0.1390	0.0251	0.0011	0.0017
δ_{12}	1.1792	1.1675	1.1693	-0.0118	0.0004	0.0006
θ_{13}	0.1200	0.1433	0.1377	0.0233	0.0010	0.0015
δ_{13}	1.1792	1.1681	1.1700	-0.0112	0.0004	0.0006
θ_{14}	0.1200	0.1433	0.1376	0.0233	0.0010	0.0015
δ_{14}	1.1792	1.1681	1.1701	-0.0111	0.0004	0.0006
$\theta_{23\|1}$	6.4800	5.7981	5.8577	-0.6819	0.0256	0.4906
$\delta_{23\|1}$	1.1792	1.2974	1.2927	0.1181	0.0010	0.0150
$\theta_{24\|1}$	6.4800	5.7833	5.8465	-0.6966	0.0285	0.5138
$\delta_{24\|1}$	1.1792	1.2992	1.2937	0.1199	0.0011	0.0155
$\theta_{34\|12}$	6.4800	5.8350	5.9029	-0.6450	0.0309	0.4469
$\delta_{34\|12}$	1.1792	1.2563	1.2504	0.0770	0.0014	0.0073
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.1999	0.2000	-0.0001	0.0001	0.0001
τ_{13}	0.2000	0.1997	0.1997	-0.0003	0.0001	0.0001
τ_{14}	0.2000	0.1997	0.1998	-0.0003	0.0001	0.0001
$\tau_{23\|1}$	0.8000	0.8016	0.8016	0.0016	0.0000	0.0000
$\tau_{24\|1}$	0.8000	0.8015	0.8015	0.0015	0.0000	0.0000
$\tau_{34\|12}$	0.8000	0.7956	0.7971	-0.0044	0.0000	0.0001
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.1870	0.1900	-0.0130	0.0004	0.0006
λ^U_{13}	0.2000	0.1875	0.1907	-0.0125	0.0004	0.0006
λ^U_{14}	0.2000	0.1875	0.1908	-0.0125	0.0004	0.0006
$\lambda^U_{23\|1}$	0.2000	0.2905	0.2886	0.0905	0.0005	0.0086
$\lambda^U_{24\|1}$	0.2000	0.2915	0.2893	0.0915	0.0005	0.0089
$\lambda^U_{34\|12}$	0.2000	0.2591	0.2575	0.0591	0.0006	0.0041

Table C.11: Results of the stability and robustness tests of the BB1-copula in different scenarios. (n=500,r=100) (part 4)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.2910	0.2807	0.2796	-0.0103	0.0012	0.0013
δ_{12}	1.0912	1.0973	1.0972	0.0061	0.0004	0.0004
θ_{13}	0.2910	0.2898	0.2915	-0.0012	0.0011	0.0011
δ_{13}	1.0912	1.0976	1.0975	0.0064	0.0003	0.0004
θ_{14}	0.2910	0.2980	0.2972	0.0070	0.0017	0.0017
δ_{14}	1.0912	1.0939	1.0922	0.0027	0.0005	0.0005
$\theta_{23\|1}$	0.2910	0.3067	0.3066	0.0157	0.0017	0.0019
$\delta_{23\|1}$	1.0912	1.0854	1.0845	-0.0058	0.0003	0.0004
$\theta_{24\|1}$	0.2910	0.2929	0.2901	0.0019	0.0016	0.0016
$\delta_{24\|1}$	1.0912	1.0992	1.0986	0.0080	0.0003	0.0004
$\theta_{34\|12}$	0.2910	0.2964	0.2954	0.0054	0.0016	0.0017
$\delta_{34\|12}$	1.0912	1.0943	1.0933	0.0031	0.0004	0.0004
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.1993	0.1996	-0.0007	0.0002	0.0002
τ_{13}	0.2000	0.2028	0.2032	0.0028	0.0001	0.0002
τ_{14}	0.2000	0.2026	0.2029	0.0026	0.0001	0.0001
$\tau_{23\|1}$	0.2000	0.1995	0.1991	-0.0005	0.0001	0.0001
$\tau_{24\|1}$	0.2000	0.2048	0.2047	0.0048	0.0001	0.0002
$\tau_{34\|12}$	0.2000	0.2024	0.2027	0.0024	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1126	0.1169	0.1176	0.0043	0.0004	0.0004
λ^U_{13}	0.1126	0.1174	0.1179	0.0048	0.0004	0.0004
λ^U_{14}	0.1126	0.1125	0.1118	-0.0001	0.0005	0.0005
$\lambda^U_{23\|1}$	0.1126	0.1038	0.1037	-0.0088	0.0004	0.0005
$\lambda^U_{24\|1}$	0.1126	0.1190	0.1193	0.0064	0.0004	0.0004
$\lambda^U_{34\|12}$	0.1126	0.1134	0.1133	0.0008	0.0005	0.0005

Table C.12: Results of the stability and robustness tests of the BB1-copula in different scenarios. ($\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copopula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.8048	0.7607	0.7781	-0.0441	0.0055	0.0074
δ_{12}	3.5652	3.6140	3.6131	0.0488	0.0083	0.0107
θ_{13}	0.8048	0.7584	0.7757	-0.0464	0.0052	0.0074
δ_{13}	3.5652	3.6213	3.6207	0.0561	0.0084	0.0116
θ_{14}	0.8048	0.7561	0.7732	-0.0487	0.0053	0.0076
δ_{14}	3.5652	3.6234	3.6229	0.0582	0.0085	0.0119
$\theta_{23\|1}$	0.8048	0.7829	0.7947	-0.0219	0.0067	0.0072
$\delta_{23\|1}$	3.5652	3.6068	3.6101	0.0416	0.0106	0.0123
$\theta_{24\|1}$	0.8048	0.7862	0.7964	-0.0186	0.0072	0.0076
$\delta_{24\|1}$	3.5652	3.6095	3.6149	0.0443	0.0118	0.0138
$\theta_{34\|12}$	0.8048	0.7973	0.7856	-0.0075	0.0135	0.0136
$\delta_{34\|12}$	3.5652	3.6080	3.6274	0.0428	0.0191	0.0209
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7986	0.7993	-0.0014	0.0000	0.0000
τ_{13}	0.8000	0.7989	0.7994	-0.0011	0.0000	0.0000
τ_{14}	0.8000	0.7988	0.7994	-0.0012	0.0000	0.0000
$\tau_{23\|1}$	0.8000	0.7996	0.7999	-0.0004	0.0000	0.0000
$\tau_{24\|1}$	0.8000	0.7998	0.8003	-0.0002	0.0000	0.0000
$\tau_{34\|12}$	0.8000	0.7997	0.7995	-0.0003	0.0000	0.0000
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.7854	0.7878	0.7879	0.0024	0.0000	0.0000
λ^U_{13}	0.7854	0.7882	0.7884	0.0028	0.0000	0.0000
λ^U_{14}	0.7854	0.7884	0.7885	0.0030	0.0000	0.0000
$\lambda^U_{23\|1}$	0.7854	0.7871	0.7876	0.0017	0.0000	0.0000
$\lambda^U_{24\|1}$	0.7854	0.7871	0.7878	0.0017	0.0001	0.0001
$\lambda^U_{34\|12}$	0.7854	0.7861	0.7882	0.0007	0.0001	0.0001

Table C.13: Results of the stability and robustness tests of the BB1-copula in different scenarios. ($\tau = 0.8$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.1730	0.3745	0.3836	0.2015	0.0024	0.0429
δ_{12}	4.6018	3.9823	3.9949	-0.6195	0.0020	0.3858
θ_{13}	0.1730	0.3753	0.3835	0.2023	0.0022	0.0431
δ_{13}	4.6018	3.9860	3.9971	-0.6158	0.0020	0.3812
θ_{14}	0.1730	0.3739	0.3838	0.2009	0.0020	0.0424
δ_{14}	4.6018	3.9859	3.9970	-0.6159	0.0020	0.3813
$\theta_{23\|1}$	0.1730	0.3420	0.3472	0.1690	0.0035	0.0321
$\delta_{23\|1}$	4.6018	3.9879	3.9998	-0.6139	0.0020	0.3789
$\theta_{24\|1}$	0.1730	0.3432	0.3441	0.1702	0.0039	0.0329
$\delta_{24\|1}$	4.6018	3.9836	3.9986	-0.6182	0.0022	0.3844
$\theta_{34\|12}$	0.1730	0.3925	0.3718	0.2195	0.0080	0.0562
$\delta_{34\|12}$	4.6018	3.9900	4.0000	-0.6118	0.0020	0.3763
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7879	0.7892	-0.0121	0.0000	0.0002
τ_{13}	0.8000	0.7882	0.7895	-0.0118	0.0000	0.0002
τ_{14}	0.8000	0.7881	0.7896	-0.0119	0.0000	0.0002
$\tau_{23\|1}$	0.8000	0.7850	0.7860	-0.0150	0.0000	0.0003
$\tau_{24\|1}$	0.8000	0.7849	0.7856	-0.0151	0.0000	0.0003
$\tau_{34\|12}$	0.8000	0.7889	0.7888	-0.0111	0.0001	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.8374	0.8097	0.8105	-0.0277	0.0000	0.0008
λ^U_{13}	0.8374	0.8099	0.8106	-0.0275	0.0000	0.0008
λ^U_{14}	0.8374	0.8099	0.8106	-0.0275	0.0000	0.0008
$\lambda^U_{23\|1}$	0.8374	0.8100	0.8108	-0.0274	0.0000	0.0008
$\lambda^U_{24\|1}$	0.8374	0.8097	0.8107	-0.0277	0.0000	0.0008
$\lambda^U_{34\|12}$	0.8374	0.8101	0.8108	-0.0273	0.0000	0.0008

Table C.14: Results of the stability and robustness tests of the BB1-copula in different scenarios. ($\tau = 0.8$ and $2 * \lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	4.3067	4.1361	4.2060	-0.1706	0.0912	0.1203
δ_{12}	1.5856	1.6312	1.6094	0.0456	0.0066	0.0087
θ_{13}	4.3067	4.1449	4.2150	-0.1618	0.0916	0.1178
δ_{13}	1.5856	1.6316	1.6087	0.0460	0.0069	0.0091
θ_{14}	4.3067	4.1460	4.2170	-0.1607	0.0910	0.1168
δ_{14}	1.5856	1.6314	1.6085	0.0458	0.0069	0.0090
$\theta_{23\|1}$	4.3067	4.2701	4.2636	-0.0366	0.0840	0.0853
$\delta_{23\|1}$	1.5856	1.6043	1.6024	0.0187	0.0039	0.0043
$\theta_{24\|1}$	4.3067	4.2558	4.2473	-0.0509	0.0907	0.0932
$\delta_{24\|1}$	1.5856	1.6074	1.6032	0.0218	0.0044	0.0049
$\theta_{34\|12}$	4.3067	4.3229	4.3110	0.0162	0.1266	0.1269
$\delta_{34\|12}$	1.5856	1.5756	1.5762	-0.0100	0.0065	0.0066
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7972	0.7996	-0.0028	0.0000	0.0001
τ_{13}	0.8000	0.7975	0.7999	-0.0025	0.0000	0.0001
τ_{14}	0.8000	0.7975	0.7999	-0.0025	0.0000	0.0001
$\tau_{23\|1}$	0.8000	0.7983	0.8002	-0.0017	0.0001	0.0001
$\tau_{24\|1}$	0.8000	0.7980	0.8000	-0.0020	0.0001	0.0001
$\tau_{34\|12}$	0.8000	0.7952	0.7957	-0.0048	0.0001	0.0001
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.4517	0.4632	0.4595	0.0115	0.0006	0.0008
λ^U_{13}	0.4517	0.4630	0.4591	0.0113	0.0007	0.0008
λ^U_{14}	0.4517	0.4630	0.4590	0.0113	0.0007	0.0008
$\lambda^U_{23\|1}$	0.4517	0.4534	0.4552	0.0017	0.0007	0.0007
$\lambda^U_{24\|1}$	0.4517	0.4541	0.4553	0.0024	0.0008	0.0008
$\lambda^U_{34\|12}$	0.4517	0.4357	0.4407	-0.0160	0.0015	0.0017

Table C.15: Results of the stability and robustness tests of the BB1-copula in different scenarios. ($\tau = 0.8$ and $0.5 * \lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.2910	0.2826	0.2816	-0.0084	0.0016	0.0017
δ_{12}	1.0912	1.0928	1.0917	0.0016	0.0005	0.0005
θ_{13}	0.2910	0.2840	0.2828	-0.0070	0.0015	0.0015
δ_{13}	1.0912	1.0913	1.0898	0.0001	0.0005	0.0005
θ_{14}	0.2910	0.2839	0.2826	-0.0071	0.0015	0.0015
δ_{14}	1.0912	1.0918	1.0903	0.0006	0.0005	0.0005
$\theta_{23\|1}$	0.8048	0.7830	0.7941	-0.0218	0.0057	0.0062
$\delta_{23\|1}$	3.5652	3.6056	3.6077	0.0404	0.0085	0.0102
$\theta_{24\|1}$	0.8048	0.7810	0.7920	-0.0238	0.0058	0.0063
$\delta_{24\|1}$	3.5652	3.6076	3.6103	0.0424	0.0095	0.0113
$\theta_{34\|12}$	0.8048	0.7798	0.7888	-0.0250	0.0062	0.0068
$\delta_{34\|12}$	3.5652	3.5980	3.6030	0.0328	0.0124	0.0134
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.1962	0.1967	-0.0038	0.0002	0.0002
τ_{13}	0.2000	0.1956	0.1958	-0.0044	0.0002	0.0002
τ_{14}	0.2000	0.1960	0.1961	-0.0040	0.0002	0.0002
$\tau_{23\|1}$	0.8000	0.7998	0.8002	-0.0002	0.0000	0.0000
$\tau_{24\|1}$	0.8000	0.7997	0.8000	-0.0003	0.0000	0.0000
$\tau_{34\|12}$	0.8000	0.7989	0.7992	-0.0011	0.0000	0.0000
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1126	0.1110	0.1110	-0.0016	0.0006	0.0006
λ^U_{13}	0.1126	0.1094	0.1090	-0.0032	0.0006	0.0006
λ^U_{14}	0.1126	0.1099	0.1094	-0.0027	0.0006	0.0006
$\lambda^U_{23\|1}$	0.7854	0.7872	0.7876	0.0018	0.0000	0.0000
$\lambda^U_{24\|1}$	0.7854	0.7872	0.7877	0.0018	0.0000	0.0000
$\lambda^U_{34\|12}$	0.7854	0.7863	0.7870	0.0009	0.0001	0.0001

Table C.16: Results of the stability and robustness tests of the BB1-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.8$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.8048	0.8012	0.7981	-0.0036	0.0046	0.0046
δ_{12}	3.5652	3.5759	3.5839	0.0107	0.0107	0.0108
θ_{13}	0.8048	0.7744	0.7857	-0.0304	0.0064	0.0073
δ_{13}	3.5652	3.5815	3.5816	0.0163	0.0096	0.0098
θ_{14}	0.8048	0.7821	0.7923	-0.0227	0.0059	0.0064
δ_{14}	3.5652	3.5751	3.5748	0.0099	0.0092	0.0093
$\theta_{23\|1}$	0.2910	0.2925	0.2915	0.0015	0.0020	0.0020
$\delta_{23\|1}$	1.0912	1.0950	1.0936	0.0038	0.0005	0.0005
$\theta_{24\|1}$	0.2910	0.2894	0.2854	-0.0016	0.0015	0.0015
$\delta_{24\|1}$	1.0912	1.0988	1.0967	0.0076	0.0005	0.0005
$\theta_{34\|12}$	0.2910	0.3106	0.3089	0.0196	0.0023	0.0027
$\delta_{34\|12}$	1.0912	1.0892	1.0879	-0.0020	0.0004	0.0004
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7995	0.7997	-0.0005	0.0000	0.0000
τ_{13}	0.8000	0.7977	0.7982	-0.0023	0.0000	0.0000
τ_{14}	0.8000	0.7980	0.7984	-0.0020	0.0000	0.0000
$\tau_{23\|1}$	0.2000	0.2012	0.2014	0.0012	0.0002	0.0002
$\tau_{24\|1}$	0.2000	0.2031	0.2029	0.0031	0.0002	0.0002
$\tau_{34\|12}$	0.2000	0.2033	0.2024	0.0033	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.7854	0.7850	0.7860	-0.0004	0.0001	0.0001
λ^U_{13}	0.7854	0.7855	0.7858	0.0001	0.0000	0.0000
λ^U_{14}	0.7854	0.7851	0.7854	-0.0003	0.0000	0.0000
$\lambda^U_{23\|1}$	0.1126	0.1135	0.1132	0.0009	0.0006	0.0006
$\lambda^U_{24\|1}$	0.1126	0.1179	0.1169	0.0053	0.0005	0.0005
$\lambda^U_{34\|12}$	0.1126	0.1078	0.1073	-0.0048	0.0005	0.0005

Table C.17: Results of the stability and robustness tests of the BB1-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.8$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.2910	0.2223	0.2212	-0.0687	0.0018	0.0065
δ_{12}	1.0912	1.1674	1.1681	0.0762	0.0006	0.0064
θ_{13}	0.2910	0.2150	0.2141	-0.0760	0.0018	0.0075
δ_{13}	1.0912	1.1721	1.1732	0.0809	0.0006	0.0072
θ_{14}	0.2910	0.2136	0.2124	-0.0774	0.0017	0.0077
δ_{14}	1.0912	1.1726	1.1737	0.0814	0.0006	0.0073
$\theta_{23\|1}$	0.1730	0.3825	0.3935	0.2095	0.0024	0.0463
$\delta_{23\|1}$	4.6018	3.9893	3.9945	-0.6125	0.0002	0.3754
$\theta_{24\|1}$	0.1730	0.3804	0.3921	0.2074	0.0023	0.0453
$\delta_{24\|1}$	4.6018	3.9952	3.9975	-0.6066	0.0000	0.3680
$\theta_{34\|12}$	0.1730	0.3310	0.3312	0.1580	0.0016	0.0265
$\delta_{34\|12}$	4.6018	3.9975	4.0000	-0.6043	0.0001	0.3653
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2270	0.2275	0.0270	0.0002	0.0009
τ_{13}	0.2000	0.2275	0.2279	0.0275	0.0002	0.0009
τ_{14}	0.2000	0.2274	0.2279	0.0274	0.0002	0.0009
$\tau_{23\|1}$	0.8000	0.7891	0.7903	-0.0109	0.0000	0.0001
$\tau_{24\|1}$	0.8000	0.7892	0.7906	-0.0108	0.0000	0.0001
$\tau_{34\|12}$	0.8000	0.7850	0.7852	-0.0150	0.0000	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1126	0.1859	0.1877	0.0733	0.0006	0.0059
λ^U_{13}	0.1126	0.1902	0.1923	0.0776	0.0006	0.0066
λ^U_{14}	0.1126	0.1907	0.1927	0.0781	0.0006	0.0067
$\lambda^U_{23\|1}$	0.8374	0.8102	0.8105	-0.0272	0.0000	0.0007
$\lambda^U_{24\|1}$	0.8374	0.8105	0.8107	-0.0269	0.0000	0.0007
$\lambda^U_{34\|12}$	0.8374	0.8107	0.8108	-0.0267	0.0000	0.0007

Table C.18: Results of the stability and robustness tests of the BB1-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.8$ and $2 * \lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.1730	0.3299	0.3293	0.1569	0.0007	0.0254
δ_{12}	4.6018	3.9973	4.0000	-0.6045	0.0001	0.3655
θ_{13}	0.1730	0.3258	0.3259	0.1528	0.0007	0.0241
δ_{13}	4.6018	3.9995	4.0000	-0.6023	0.0000	0.3627
θ_{14}	0.1730	0.3209	0.3210	0.1479	0.0011	0.0230
δ_{14}	4.6018	3.9985	4.0000	-0.6033	0.0000	0.3641
$\theta_{23\|1}$	0.2910	0.3035	0.3020	0.0125	0.0023	0.0024
$\delta_{23\|1}$	1.0912	1.1247	1.1239	0.0335	0.0007	0.0018
$\theta_{24\|1}$	0.2910	0.2723	0.2739	-0.0187	0.0016	0.0019
$\delta_{24\|1}$	1.0912	1.1312	1.1293	0.0400	0.0006	0.0022
$\theta_{34\|12}$	0.2910	0.2874	0.2839	-0.0036	0.0018	0.0018
$\delta_{34\|12}$	1.0912	1.1117	1.1099	0.0205	0.0006	0.0010
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7851	0.7851	-0.0149	0.0000	0.0002
τ_{13}	0.8000	0.7849	0.7849	-0.0151	0.0000	0.0002
τ_{14}	0.8000	0.7842	0.7844	-0.0158	0.0000	0.0003
$\tau_{23\|1}$	0.2000	0.2257	0.2258	0.0257	0.0001	0.0008
$\tau_{24\|1}$	0.2000	0.2199	0.2204	0.0199	0.0002	0.0006
$\tau_{34\|12}$	0.2000	0.2113	0.2113	0.0113	0.0002	0.0003
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.8374	0.8106	0.8108	-0.0268	0.0000	0.0007
λ^U_{13}	0.8374	0.8108	0.8108	-0.0266	0.0000	0.0007
λ^U_{14}	0.8374	0.8107	0.8108	-0.0267	0.0000	0.0007
$\lambda^U_{23\|1}$	0.1126	0.1440	0.1444	0.0314	0.0007	0.0017
$\lambda^U_{24\|1}$	0.1126	0.1512	0.1503	0.0386	0.0006	0.0021
$\lambda^U_{34\|12}$	0.1126	0.1308	0.1306	0.0182	0.0006	0.0010

Table C.19: Results of the stability and robustness tests of the BB1-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.8$ and $2 * \lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	0.2910	0.3093	0.3092	0.0183	0.0011	0.0014
δ_{12}	1.0912	1.0902	1.0895	-0.0010	0.0004	0.0004
θ_{13}	0.2910	0.3097	0.3097	0.0187	0.0011	0.0014
δ_{13}	1.0912	1.0912	1.0908	-0.0000	0.0004	0.0004
θ_{14}	0.2910	0.3092	0.3092	0.0182	0.0011	0.0014
δ_{14}	1.0912	1.0913	1.0908	0.0001	0.0004	0.0004
$\theta_{23\|1}$	4.3067	4.1312	4.1671	-0.1755	0.0621	0.0928
$\delta_{23\|1}$	1.5856	1.6391	1.6272	0.0535	0.0049	0.0078
$\theta_{24\|1}$	4.3067	4.1160	4.1477	-0.1907	0.0634	0.0998
$\delta_{24\|1}$	1.5856	1.6407	1.6294	0.0551	0.0051	0.0081
$\theta_{34\|12}$	4.3067	4.3572	4.3475	0.0505	0.0970	0.0996
$\delta_{34\|12}$	1.5856	1.5518	1.5506	-0.0338	0.0055	0.0066
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2042	0.2043	0.0042	0.0001	0.0001
τ_{13}	0.2000	0.2050	0.2053	0.0050	0.0001	0.0001
τ_{14}	0.2000	0.2049	0.2051	0.0049	0.0001	0.0001
$\tau_{23\|1}$	0.8000	0.7990	0.7996	-0.0010	0.0000	0.0000
$\tau_{24\|1}$	0.8000	0.7987	0.7992	-0.0013	0.0000	0.0000
$\tau_{34\|12}$	0.8000	0.7942	0.7951	-0.0058	0.0001	0.0001
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1126	0.1089	0.1092	-0.0037	0.0005	0.0005
λ^U_{13}	0.1126	0.1100	0.1106	-0.0026	0.0005	0.0005
λ^U_{14}	0.1126	0.1101	0.1107	-0.0025	0.0005	0.0005
$\lambda^U_{23\|1}$	0.4517	0.4672	0.4652	0.0155	0.0006	0.0009
$\lambda^U_{24\|1}$	0.4517	0.4676	0.4659	0.0159	0.0007	0.0009
$\lambda^U_{34\|12}$	0.4517	0.4267	0.4306	-0.0250	0.0013	0.0019

Table C.20: Results of the stability and robustness tests of the BB1-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.8$ and $0.5 * \lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	4.3067	4.2157	4.2292	-0.0910	0.0526	0.0609
δ_{12}	1.5856	1.6126	1.6103	0.0270	0.0031	0.0038
θ_{13}	4.3067	4.1178	4.1662	-0.1889	0.1029	0.1386
δ_{13}	1.5856	1.6348	1.6211	0.0492	0.0074	0.0098
θ_{14}	4.3067	3.9528	4.0020	-0.3539	0.1237	0.2489
δ_{14}	1.5856	1.6760	1.6636	0.0904	0.0080	0.0162
$\theta_{23\|1}$	0.2910	0.2995	0.2972	0.0085	0.0019	0.0020
$\delta_{23\|1}$	1.0912	1.0878	1.0877	-0.0034	0.0004	0.0004
$\theta_{24\|1}$	0.2910	0.2869	0.2862	-0.0041	0.0021	0.0022
$\delta_{24\|1}$	1.0912	1.0960	1.0964	0.0048	0.0004	0.0005
$\theta_{34\|12}$	0.2910	0.3022	0.2964	0.0112	0.0021	0.0022
$\delta_{34\|12}$	1.0912	1.0945	1.0906	0.0033	0.0004	0.0004
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7990	0.7992	-0.0010	0.0000	0.0000
τ_{13}	0.8000	0.7968	0.7979	-0.0032	0.0000	0.0000
τ_{14}	0.8000	0.7957	0.7969	-0.0043	0.0000	0.0001
$\tau_{23\|1}$	0.2000	0.1985	0.1993	-0.0015	0.0002	0.0002
$\tau_{24\|1}$	0.2000	0.1999	0.2010	-0.0001	0.0002	0.0002
$\tau_{34\|12}$	0.2000	0.2041	0.2027	0.0041	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.4517	0.4582	0.4594	0.0065	0.0005	0.0006
λ^U_{13}	0.4517	0.4628	0.4621	0.0111	0.0009	0.0010
λ^U_{14}	0.4517	0.4779	0.4780	0.0262	0.0010	0.0017
$\lambda^U_{23\|1}$	0.1126	0.1063	0.1071	-0.0063	0.0005	0.0005
$\lambda^U_{24\|1}$	0.1126	0.1149	0.1162	0.0023	0.0005	0.0005
$\lambda^U_{34\|12}$	0.1126	0.1135	0.1106	0.0009	0.0004	0.0004

Table C.21: Results of the stability and robustness tests of the BB1-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.8$ and $0.5 * \lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1792	1.1795	1.1727	0.0002	0.0029	0.0029
δ_{12}	8.4000	8.3632	8.3913	-0.0368	0.0543	0.0557
θ_{13}	1.1792	1.1815	1.1755	0.0022	0.0027	0.0027
δ_{13}	8.4000	8.3621	8.3897	-0.0379	0.0531	0.0545
θ_{14}	1.1792	1.1809	1.1750	0.0016	0.0027	0.0027
δ_{14}	8.4000	8.3627	8.3897	-0.0373	0.0535	0.0549
$\theta_{23\|1}$	1.1792	1.2023	1.1992	0.0230	0.0025	0.0031
$\delta_{23\|1}$	8.4000	8.3365	8.3692	-0.0635	0.0692	0.0732
$\theta_{24\|1}$	1.1792	1.2005	1.1980	0.0213	0.0026	0.0030
$\delta_{24\|1}$	8.4000	8.3369	8.3612	-0.0631	0.0760	0.0800
$\theta_{34\|12}$	1.1792	1.2317	1.1806	0.0525	0.0394	0.0422
$\delta_{34\|12}$	8.4000	8.4941	8.5083	0.0941	0.0956	0.1045
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7976	0.7985	-0.0024	0.0000	0.0000
τ_{13}	0.8000	0.7974	0.7984	-0.0026	0.0000	0.0000
τ_{14}	0.8000	0.7975	0.7984	-0.0025	0.0000	0.0000
$\tau_{23\|1}$	0.8000	0.7957	0.7963	-0.0043	0.0000	0.0000
$\tau_{24\|1}$	0.8000	0.7958	0.7963	-0.0042	0.0000	0.0000
$\tau_{34\|12}$	0.8000	0.7997	0.8000	-0.0003	0.0000	0.0000
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.1871	0.1860	-0.0129	0.0021	0.0023
λ^U_{13}	0.2000	0.1897	0.1892	-0.0103	0.0019	0.0020
λ^U_{14}	0.2000	0.1895	0.1890	-0.0105	0.0019	0.0020
$\lambda^U_{23\|1}$	0.2000	0.2086	0.2096	0.0086	0.0019	0.0019
$\lambda^U_{24\|1}$	0.2000	0.2066	0.2082	0.0066	0.0020	0.0021
$\lambda^U_{34\|12}$	0.2000	0.1950	0.1863	-0.0050	0.0041	0.0041

Table C.22: Results of the stability and robustness tests of the BB7-copula in different scenarios. (n=500, r=100) (part 1)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1792	1.1788	1.1779	-0.0004	0.0007	0.0007
δ_{12}	0.2900	0.2988	0.2990	0.0088	0.0011	0.0012
θ_{13}	1.1792	1.1840	1.1821	0.0048	0.0007	0.0007
δ_{13}	0.2900	0.2842	0.2853	-0.0058	0.0010	0.0011
θ_{14}	1.1792	1.1813	1.1791	0.0021	0.0008	0.0008
δ_{14}	0.2900	0.2846	0.2853	-0.0054	0.0009	0.0010
$\theta_{23\|1}$	1.1792	1.1778	1.1776	-0.0015	0.0006	0.0006
$\delta_{23\|1}$	0.2900	0.2913	0.2895	0.0013	0.0011	0.0011
$\theta_{24\|1}$	1.1792	1.1824	1.1819	0.0032	0.0008	0.0008
$\delta_{24\|1}$	0.2900	0.2893	0.2876	-0.0007	0.0010	0.0010
$\theta_{34\|12}$	1.1792	1.1802	1.1795	0.0010	0.0009	0.0009
$\delta_{34\|12}$	0.2900	0.2890	0.2890	-0.0010	0.0010	0.0010
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.1990	0.1988	-0.0010	0.0001	0.0001
τ_{13}	0.2000	0.1963	0.1958	-0.0037	0.0001	0.0001
τ_{14}	0.2000	0.1954	0.1952	-0.0046	0.0001	0.0001
$\tau_{23\|1}$	0.2000	0.1961	0.1964	-0.0039	0.0002	0.0002
$\tau_{24\|1}$	0.2000	0.1972	0.1973	-0.0028	0.0001	0.0001
$\tau_{34\|12}$	0.2000	0.1961	0.1967	-0.0039	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.1960	0.1968	-0.0040	0.0006	0.0006
λ^U_{13}	0.2000	0.2010	0.2009	0.0010	0.0005	0.0005
λ^U_{14}	0.2000	0.1978	0.1974	-0.0022	0.0007	0.0007
$\lambda^U_{23\|1}$	0.2000	0.1958	0.1964	-0.0042	0.0005	0.0005
$\lambda^U_{24\|1}$	0.2000	0.1992	0.2000	-0.0008	0.0006	0.0006
$\lambda^U_{34\|12}$	0.2000	0.1965	0.1971	-0.0035	0.0007	0.0007

Table C.23: Results of the stability and robustness tests of the BB7-copula in different scenarios. (n=500, r=100) (part 2)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1792	1.2058	1.1995	0.0266	0.0036	0.0043
δ_{12}	8.4000	8.4609	8.4554	0.0609	0.0401	0.0438
θ_{13}	1.1792	1.1981	1.1906	0.0188	0.0032	0.0036
δ_{13}	8.4000	8.3368	8.3821	-0.0632	0.0762	0.0802
θ_{14}	1.1792	1.1929	1.1838	0.0136	0.0035	0.0037
δ_{14}	8.4000	8.3701	8.4406	-0.0299	0.1089	0.1097
$\theta_{23\|1}$	1.1792	1.1852	1.1836	0.0059	0.0008	0.0008
$\delta_{23\|1}$	0.2900	0.2860	0.2862	-0.0040	0.0012	0.0012
$\theta_{24\|1}$	1.1792	1.1861	1.1859	0.0068	0.0009	0.0009
$\delta_{24\|1}$	0.2900	0.2895	0.2882	-0.0005	0.0014	0.0014
$\theta_{34\|12}$	1.1792	1.1911	1.1877	0.0118	0.0009	0.0010
$\delta_{34\|12}$	0.2900	0.3076	0.2992	0.0176	0.0024	0.0027
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7983	0.7985	-0.0017	0.0000	0.0000
τ_{13}	0.8000	0.7959	0.7968	-0.0041	0.0000	0.0000
τ_{14}	0.8000	0.7964	0.7983	-0.0036	0.0000	0.0001
$\tau_{23\|1}$	0.2000	0.1971	0.1966	-0.0029	0.0001	0.0002
$\tau_{24\|1}$	0.2000	0.1986	0.1988	-0.0014	0.0001	0.0001
$\tau_{34\|12}$	0.2000	0.2048	0.2027	0.0048	0.0003	0.0003
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.2075	0.2071	0.0075	0.0025	0.0026
λ^U_{13}	0.2000	0.2026	0.2015	0.0026	0.0022	0.0022
λ^U_{14}	0.2000	0.1964	0.1942	-0.0036	0.0025	0.0025
$\lambda^U_{23\|1}$	0.2000	0.2018	0.2018	0.0018	0.0006	0.0006
$\lambda^U_{24\|1}$	0.2000	0.2017	0.2034	0.0017	0.0007	0.0007
$\lambda^U_{34\|12}$	0.2000	0.2066	0.2054	0.0066	0.0006	0.0007

Table C.24: Results of the stability and robustness tests of the BB7-copula in different scenarios. (n=500,r=100) (part 3)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1792	1.1750	1.1739	-0.0043	0.0004	0.0005
δ_{12}	0.2900	0.3028	0.2983	0.0128	0.0007	0.0009
θ_{13}	1.1792	1.1757	1.1748	-0.0036	0.0004	0.0004
δ_{13}	0.2900	0.3005	0.2965	0.0105	0.0006	0.0007
θ_{14}	1.1792	1.1764	1.1755	-0.0029	0.0004	0.0004
δ_{14}	0.2900	0.3006	0.2967	0.0106	0.0006	0.0007
$\theta_{23\|1}$	1.1792	1.1869	1.1818	0.0077	0.0027	0.0027
$\delta_{23\|1}$	8.4000	8.4462	8.4339	0.0462	0.0558	0.0579
$\theta_{24\|1}$	1.1792	1.1837	1.1780	0.0044	0.0029	0.0029
$\delta_{24\|1}$	8.4000	8.4214	8.4099	0.0214	0.0560	0.0564
$\theta_{34\|12}$	1.1792	1.2055	1.2003	0.0262	0.0040	0.0047
$\delta_{34\|12}$	8.4000	8.5041	8.4929	0.1041	0.0754	0.0862
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.1989	0.1981	-0.0011	0.0001	0.0001
τ_{13}	0.2000	0.1986	0.1979	-0.0014	0.0001	0.0001
τ_{14}	0.2000	0.1988	0.1982	-0.0012	0.0001	0.0001
$\tau_{23\|1}$	0.8000	0.7986	0.7989	-0.0014	0.0000	0.0000
$\tau_{24\|1}$	0.8000	0.7984	0.7986	-0.0016	0.0000	0.0000
$\tau_{34\|12}$	0.8000	0.7988	0.7988	-0.0012	0.0000	0.0000
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.2000	0.1940	0.1938	-0.0060	0.0004	0.0004
λ^U_{13}	0.2000	0.1949	0.1948	-0.0051	0.0003	0.0003
λ^U_{14}	0.2000	0.1955	0.1954	-0.0045	0.0003	0.0003
$\lambda^U_{23\|1}$	0.2000	0.1944	0.1941	-0.0056	0.0020	0.0020
$\lambda^U_{24\|1}$	0.2000	0.1904	0.1895	-0.0096	0.0022	0.0023
$\lambda^U_{34\|12}$	0.2000	0.2046	0.2044	0.0046	0.0030	0.0031

Table C.25: Results of the stability and robustness tests of the BB7-copula in different scenarios. (n=500, r=100) (part 4)

Copupla parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1237	1.1258	1.1257	0.0021	0.0006	0.0006
δ_{12}	0.3614	0.3708	0.3694	0.0094	0.0011	0.0011
θ_{13}	1.1237	1.1294	1.1288	0.0057	0.0007	0.0007
δ_{13}	0.3614	0.3607	0.3588	-0.0007	0.0013	0.0013
θ_{14}	1.1237	1.1335	1.1314	0.0098	0.0006	0.0007
δ_{14}	0.3614	0.3478	0.3457	-0.0136	0.0009	0.0011
$\theta_{23\|1}$	1.1237	1.1178	1.1171	-0.0059	0.0006	0.0006
$\delta_{23\|1}$	0.3614	0.3586	0.3580	-0.0028	0.0012	0.0012
$\theta_{24\|1}$	1.1237	1.1235	1.1227	-0.0002	0.0006	0.0006
$\delta_{24\|1}$	0.3614	0.3549	0.3551	-0.0065	0.0014	0.0014
$\theta_{34\|12}$	1.1237	1.1287	1.1266	0.0050	0.0006	0.0006
$\delta_{34\|12}$	0.3614	0.3784	0.3773	0.0170	0.0016	0.0019
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2028	0.2032	0.0028	0.0001	0.0001
τ_{13}	0.2000	0.2010	0.2010	0.0010	0.0001	0.0001
τ_{14}	0.2000	0.1982	0.1980	-0.0018	0.0001	0.0001
$\tau_{23\|1}$	0.2000	0.1960	0.1965	-0.0040	0.0001	0.0002
$\tau_{24\|1}$	0.2000	0.1969	0.1978	-0.0031	0.0001	0.0002
$\tau_{34\|12}$	0.2000	0.2060	0.2066	0.0060	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1469	0.1458	0.1467	-0.0011	0.0006	0.0006
λ^U_{13}	0.1469	0.1489	0.1498	0.0020	0.0007	0.0007
λ^U_{14}	0.1469	0.1532	0.1525	0.0063	0.0006	0.0006
$\lambda^U_{23\|1}$	0.1469	0.1373	0.1379	-0.0096	0.0006	0.0007
$\lambda^U_{24\|1}$	0.1469	0.1432	0.1435	-0.0037	0.0006	0.0006
$\lambda^U_{34\|12}$	0.1469	0.1486	0.1476	0.0017	0.0006	0.0006

Table C.26: Results of the stability and robustness tests of the BB7-copula in different scenarios. ($\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	5.9265	5.5317	5.5322	-0.3948	0.0799	0.2358
δ_{12}	5.2324	2.3594	2.3117	-2.8730	0.5124	8.7667
θ_{13}	5.9265	5.6276	5.6176	-0.2989	0.0702	0.1596
δ_{13}	5.2324	2.1686	2.1000	-3.0638	0.5504	9.9375
θ_{14}	5.9265	5.6147	5.6050	-0.3118	0.0707	0.1679
δ_{14}	5.2324	2.1448	2.0738	-3.0876	0.5462	10.0798
$\theta_{23\|1}$	5.9265	6.0276	5.9600	0.1011	0.0999	0.1101
$\delta_{23\|1}$	5.2324	3.5223	3.5637	-1.7101	0.7750	3.6994
$\theta_{24\|1}$	5.9265	6.0717	5.9952	0.1452	0.1001	0.1212
$\delta_{24\|1}$	5.2324	2.9221	2.8968	-2.3103	0.7904	6.1277
$\theta_{34\|12}$	5.9265	6.2443	6.1989	0.3178	0.0984	0.1994
$\delta_{34\|12}$	5.2324	3.2623	3.2617	-1.9701	1.0942	4.9756
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7527	0.7538	-0.0473	0.0002	0.0025
τ_{13}	0.8000	0.7516	0.7524	-0.0484	0.0003	0.0026
τ_{14}	0.8000	0.7508	0.7516	-0.0492	0.0003	0.0027
$\tau_{23\|1}$	0.8000	0.7805	0.7832	-0.0195	0.0002	0.0006
$\tau_{24\|1}$	0.8000	0.7745	0.7764	-0.0255	0.0002	0.0008
$\tau_{34\|12}$	0.8000	0.7815	0.7832	-0.0185	0.0002	0.0005
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.8759	0.8647	0.8660	-0.0112	0.0001	0.0002
λ^U_{13}	0.8759	0.8675	0.8683	-0.0084	0.0000	0.0001
λ^U_{14}	0.8759	0.8672	0.8680	-0.0087	0.0000	0.0001
$\lambda^U_{23\|1}$	0.8759	0.8766	0.8761	0.0007	0.0000	0.0000
$\lambda^U_{24\|1}$	0.8759	0.8776	0.8768	0.0017	0.0000	0.0000
$\lambda^U_{34\|12}$	0.8759	0.8811	0.8809	0.0052	0.0000	0.0001

Table C.27: Results of the stability and robustness tests of the BB7-copula in different scenarios. ($\tau = 0.8$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copupla parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1779	1.1840	1.1858	0.0061	0.0007	0.0008
δ_{12}	0.3000	0.3089	0.3092	0.0089	0.0011	0.0012
θ_{13}	1.1779	1.1941	1.1927	0.0162	0.0009	0.0012
δ_{13}	0.3000	0.2980	0.2982	-0.0020	0.0013	0.0013
θ_{14}	1.1779	1.1846	1.1837	0.0067	0.0009	0.0009
δ_{14}	0.3000	0.3017	0.3005	0.0017	0.0013	0.0013
$\theta_{23\|1}$	1.1779	1.1758	1.1770	-0.0021	0.0006	0.0006
$\delta_{23\|1}$	0.3000	0.3074	0.3066	0.0074	0.0010	0.0011
$\theta_{24\|1}$	1.1779	1.1870	1.1856	0.0091	0.0010	0.0010
$\delta_{24\|1}$	0.3000	0.2997	0.2988	-0.0003	0.0011	0.0011
$\theta_{34\|12}$	1.1779	1.1803	1.1815	0.0024	0.0008	0.0008
$\delta_{34\|12}$	0.3000	0.3013	0.3011	0.0013	0.0009	0.0009
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2040	0.2038	0.0040	0.0001	0.0001
τ_{13}	0.2000	0.2040	0.2041	0.0040	0.0002	0.0002
τ_{14}	0.2000	0.2018	0.2018	0.0018	0.0001	0.0002
$\tau_{23\|1}$	0.2000	0.2010	0.2008	0.0010	0.0001	0.0001
$\tau_{24\|1}$	0.2000	0.2023	0.2023	0.0023	0.0001	0.0001
$\tau_{34\|12}$	0.2000	0.2002	0.2007	0.0002	0.0001	0.0001
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1987	0.2005	0.2035	0.0018	0.0006	0.0006
λ^U_{13}	0.1987	0.2088	0.2093	0.0101	0.0007	0.0008
λ^U_{14}	0.1987	0.2005	0.2014	0.0018	0.0007	0.0007
$\lambda^U_{23\|1}$	0.1987	0.1937	0.1958	-0.0050	0.0005	0.0006
$\lambda^U_{24\|1}$	0.1987	0.2025	0.2023	0.0038	0.0007	0.0007
$\lambda^U_{34\|12}$	0.1987	0.1970	0.2000	-0.0017	0.0007	0.0007

Table C.28: Results of the stability and robustness tests of the BB7-copula in different scenarios. ($\tau = 0.2$ and $2 * \lambda^L = \lambda^U$, n=500,r=100)

Copopula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	8.0799	6.1284	6.0927	-1.9515	0.6333	4.4417
δ_{12}	0.8800	0.0258	0.0122	-0.8542	0.0010	0.7306
θ_{13}	8.0799	6.2280	6.1964	-1.8519	0.5976	4.0273
δ_{13}	0.8800	0.0043	0.0009	-0.8757	0.0001	0.7669
θ_{14}	8.0799	6.2354	6.2048	-1.8445	0.5972	3.9995
δ_{14}	0.8800	0.0018	0.0003	-0.8782	0.0000	0.7712
$\theta_{23\|1}$	8.0799	7.0988	7.0267	-0.9811	0.3496	1.3122
$\delta_{23\|1}$	0.8800	0.1918	0.1108	-0.6882	0.0441	0.5177
$\theta_{24\|1}$	8.0799	7.1268	7.0520	-0.9531	0.3415	1.2499
$\delta_{24\|1}$	0.8800	0.1154	0.0230	-0.7646	0.0582	0.6428
$\theta_{34\|12}$	8.0799	7.3153	7.2786	-0.7646	0.4276	1.0122
$\delta_{34\|12}$	0.8800	0.4884	0.3100	-0.3916	0.2062	0.3596
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7108	0.7161	-0.0892	0.0012	0.0091
τ_{13}	0.8000	0.7156	0.7205	-0.0844	0.0010	0.0081
τ_{14}	0.8000	0.7158	0.7208	-0.0842	0.0010	0.0081
$\tau_{23\|1}$	0.8000	0.7587	0.7579	-0.0413	0.0002	0.0019
$\tau_{24\|1}$	0.8000	0.7573	0.7566	-0.0427	0.0002	0.0021
$\tau_{34\|12}$	0.8000	0.7670	0.7668	-0.0330	0.0003	0.0014
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.9104	0.8673	0.8717	-0.0431	0.0005	0.0023
λ^U_{13}	0.9104	0.8709	0.8748	-0.0395	0.0004	0.0019
λ^U_{14}	0.9104	0.8710	0.8750	-0.0394	0.0004	0.0019
$\lambda^U_{23\|1}$	0.9104	0.8941	0.8938	-0.0163	0.0001	0.0003
$\lambda^U_{24\|1}$	0.9104	0.8946	0.8942	-0.0158	0.0001	0.0003
$\lambda^U_{34\|12}$	0.9104	0.8966	0.8970	-0.0138	0.0001	0.0003

Table C.29: Results of the stability and robustness tests of the BB7-copula in different scenarios. ($\tau = 0.8$ and $2 * \lambda^L = \lambda^U$, n=500,r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1237	1.2823	1.2676	0.1586	0.0142	0.0393
δ_{12}	0.3614	0.4630	0.4617	0.1016	0.0061	0.0164
θ_{13}	1.1237	1.2924	1.2771	0.1687	0.0150	0.0435
δ_{13}	0.3614	0.4547	0.4526	0.0933	0.0064	0.0151
θ_{14}	1.1237	1.2928	1.2771	0.1691	0.0151	0.0437
δ_{14}	0.3614	0.4530	0.4512	0.0916	0.0062	0.0146
$\theta_{23\|1}$	5.9265	4.9026	4.9260	-1.0239	0.0651	1.1134
$\delta_{23\|1}$	5.2324	2.5323	2.5411	-2.7001	0.4065	7.6971
$\theta_{24\|1}$	5.9265	4.9261	4.9512	-1.0004	0.0668	1.0677
$\delta_{24\|1}$	5.2324	2.3901	2.3837	-2.8423	0.4275	8.5064
$\theta_{34\|12}$	5.9265	5.8263	5.8531	-0.1002	0.2126	0.2227
$\delta_{34\|12}$	5.2324	3.5381	3.5079	-1.6943	0.8714	3.7420
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2654	0.2653	0.0654	0.0017	0.0060
τ_{13}	0.2000	0.2649	0.2647	0.0649	0.0019	0.0061
τ_{14}	0.2000	0.2646	0.2644	0.0646	0.0019	0.0060
$\tau_{23\|1}$	0.8000	0.7397	0.7404	-0.0603	0.0001	0.0038
$\tau_{24\|1}$	0.8000	0.7370	0.7377	-0.0630	0.0002	0.0041
$\tau_{34\|12}$	0.8000	0.7732	0.7768	-0.0268	0.0003	0.0011
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1469	0.2376	0.2340	0.0907	0.0062	0.0144
λ^U_{13}	0.1469	0.2440	0.2403	0.0971	0.0061	0.0156
λ^U_{14}	0.1469	0.2443	0.2404	0.0974	0.0061	0.0156
$\lambda^U_{23\|1}$	0.8759	0.8455	0.8471	-0.0304	0.0001	0.0010
$\lambda^U_{24\|1}$	0.8759	0.8462	0.8480	-0.0297	0.0001	0.0010
$\lambda^U_{34\|12}$	0.8759	0.8684	0.8707	-0.0075	0.0002	0.0002

Table C.30: Results of the stability and robustness tests of the BB7-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.8$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1237	5.0305	5.0612	3.9068	0.1486	15.4115
δ_{12}	5.2324	2.4958	2.4524	-2.7366	0.8909	8.3797
θ_{13}	1.1237	4.9630	4.9852	3.8393	0.1417	14.8822
δ_{13}	5.2324	2.4400	2.3841	-2.7924	0.8840	8.6815
θ_{14}	1.1237	4.9789	5.0044	3.8552	0.1552	15.0177
δ_{14}	5.2324	2.3691	2.3097	-2.8633	0.8557	9.0544
$\theta_{23\|1}$	5.9265	1.2844	1.2570	-4.6421	0.0113	21.5605
$\delta_{23\|1}$	0.3614	1.0576	0.9883	0.6962	0.1325	0.6172
$\theta_{24\|1}$	5.9265	1.2989	1.2762	-4.6276	0.0107	21.4255
$\delta_{24\|1}$	0.3614	1.0295	0.9689	0.6681	0.1277	0.5740
$\theta_{34\|12}$	5.9265	1.2323	1.2255	-4.6942	0.0024	22.0380
$\delta_{34\|12}$	0.3614	0.4764	0.4669	0.1150	0.0056	0.0188
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.7212	0.7269	-0.0788	0.0013	0.0075
τ_{13}	0.8000	0.7181	0.7231	-0.0819	0.0013	0.0080
τ_{14}	0.8000	0.7169	0.7220	-0.0831	0.0014	0.0083
$\tau_{23\|1}$	0.2000	0.3563	0.3507	0.1563	0.0040	0.0285
$\tau_{24\|1}$	0.2000	0.3542	0.3502	0.1542	0.0041	0.0279
$\tau_{34\|12}$	0.2000	0.2639	0.2628	0.0639	0.0005	0.0046
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.8759	0.8461	0.8491	-0.0298	0.0002	0.0011
λ^U_{13}	0.8759	0.8442	0.8469	-0.0317	0.0002	0.0012
λ^U_{14}	0.8759	0.8442	0.8470	-0.0317	0.0002	0.0013
$\lambda^U_{23\|1}$	0.1469	0.2532	0.2468	0.1063	0.0039	0.0152
$\lambda^U_{24\|1}$	0.1469	0.2649	0.2601	0.1180	0.0038	0.0177
$\lambda^U_{34\|12}$	0.1469	0.2355	0.2335	0.0886	0.0014	0.0093

Table C.31: Results of the stability and robustness tests of the BB7-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.8$ and $\lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1237	1.1180	1.1180	-0.0057	0.0006	0.0006
δ_{12}	0.3614	0.3757	0.3747	0.0143	0.0010	0.0012
θ_{13}	1.1237	1.1196	1.1187	-0.0041	0.0005	0.0006
δ_{13}	0.3614	0.3659	0.3679	0.0045	0.0011	0.0011
θ_{14}	1.1237	1.1271	1.1270	0.0034	0.0005	0.0005
δ_{14}	0.3614	0.3658	0.3642	0.0044	0.0009	0.0009
$\theta_{23\|1}$	1.1779	1.1909	1.1879	0.0130	0.0009	0.0010
$\delta_{23\|1}$	0.3000	0.3146	0.3133	0.0146	0.0011	0.0013
$\theta_{24\|1}$	1.1779	1.1876	1.1843	0.0097	0.0007	0.0008
$\delta_{24\|1}$	0.3000	0.3065	0.3066	0.0065	0.0013	0.0013
$\theta_{34\|12}$	1.1779	1.1689	1.1707	-0.0090	0.0006	0.0007
$\delta_{34\|12}$	0.3000	0.2963	0.2967	-0.0037	0.0010	0.0010
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2018	0.2021	0.0018	0.0001	0.0001
τ_{13}	0.2000	0.1990	0.1996	-0.0010	0.0001	0.0001
τ_{14}	0.2000	0.2018	0.2017	0.0018	0.0001	0.0001
$\tau_{23\|1}$	0.2000	0.2083	0.2084	0.0083	0.0001	0.0002
$\tau_{24\|1}$	0.2000	0.2044	0.2042	0.0044	0.0001	0.0002
$\tau_{34\|12}$	0.2000	0.1947	0.1952	-0.0053	0.0001	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1469	0.1374	0.1387	-0.0095	0.0007	0.0008
λ^U_{13}	0.1469	0.1395	0.1399	-0.0074	0.0006	0.0006
λ^U_{14}	0.1469	0.1475	0.1486	0.0006	0.0005	0.0005
$\lambda^U_{23\|1}$	0.1987	0.2064	0.2059	0.0077	0.0006	0.0007
$\lambda^U_{24\|1}$	0.1987	0.2041	0.2026	0.0054	0.0005	0.0006
$\lambda^U_{34\|12}$	0.1987	0.1874	0.1903	-0.0113	0.0006	0.0007

Table C.32: Results of the stability and robustness tests of the BB7-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $2 * \lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1779	1.1864	1.1858	0.0085	0.0007	0.0007
δ_{12}	0.3000	0.3075	0.3072	0.0075	0.0011	0.0012
θ_{13}	1.1779	1.1804	1.1793	0.0025	0.0007	0.0007
δ_{13}	0.3000	0.3070	0.3067	0.0070	0.0010	0.0011
θ_{14}	1.1779	1.1869	1.1851	0.0090	0.0009	0.0010
δ_{14}	0.3000	0.3048	0.3041	0.0048	0.0011	0.0012
$\theta_{23\|1}$	1.1237	1.1203	1.1194	-0.0034	0.0006	0.0006
$\delta_{23\|1}$	0.3614	0.3548	0.3552	-0.0066	0.0010	0.0011
$\theta_{24\|1}$	1.1237	1.1225	1.1203	-0.0012	0.0007	0.0007
$\delta_{24\|1}$	0.3614	0.3540	0.3548	-0.0074	0.0010	0.0011
$\theta_{34\|12}$	1.1237	1.1275	1.1265	0.0038	0.0006	0.0007
$\delta_{34\|12}$	0.3614	0.3672	0.3671	0.0058	0.0011	0.0012
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.2044	0.2045	0.0044	0.0001	0.0001
τ_{13}	0.2000	0.2023	0.2016	0.0023	0.0001	0.0001
τ_{14}	0.2000	0.2038	0.2037	0.0038	0.0001	0.0001
$\tau_{23\|1}$	0.2000	0.1958	0.1961	-0.0042	0.0001	0.0001
$\tau_{24\|1}$	0.2000	0.1964	0.1959	-0.0036	0.0001	0.0001
$\tau_{34\|12}$	0.2000	0.2021	0.2019	0.0021	0.0002	0.0002
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1987	0.2032	0.2040	0.0045	0.0005	0.0006
λ^U_{13}	0.1987	0.1979	0.1983	-0.0008	0.0005	0.0005
λ^U_{14}	0.1987	0.2027	0.2027	0.0040	0.0007	0.0007
$\lambda^U_{23\|1}$	0.1469	0.1401	0.1407	-0.0068	0.0006	0.0006
$\lambda^U_{24\|1}$	0.1469	0.1418	0.1410	-0.0051	0.0007	0.0007
$\lambda^U_{34\|12}$	0.1469	0.1470	0.1474	0.0001	0.0007	0.0007

Table C.33: Results of the stability and robustness tests of the BB7-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$ and $2 * \lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	1.1237	1.2202	1.1956	0.0965	0.0133	0.0226
δ_{12}	0.3614	0.1099	0.1001	-0.2515	0.0017	0.0650
θ_{13}	1.1237	1.2548	1.2283	0.1311	0.0171	0.0343
δ_{13}	0.3614	0.0525	0.0460	-0.3089	0.0006	0.0960
θ_{14}	1.1237	1.2551	1.2284	0.1314	0.0171	0.0344
δ_{14}	0.3614	0.0496	0.0430	-0.3118	0.0006	0.0979
$\theta_{23\|1}$	8.0799	4.7753	4.8000	-3.3046	0.0979	11.0184
$\delta_{23\|1}$	0.8800	0.0136	0.0022	-0.8664	0.0007	0.7513
$\theta_{24\|1}$	8.0799	4.8448	4.8766	-3.2351	0.0992	10.5653
$\delta_{24\|1}$	0.8800	0.0033	0.0001	-0.8767	0.0001	0.7687
$\theta_{34\|12}$	8.0799	6.9107	6.8954	-1.1692	0.4926	1.8597
$\delta_{34\|12}$	0.8800	0.2602	0.1666	-0.6198	0.0643	0.4484
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.2000	0.1372	0.1300	-0.0628	0.0020	0.0060
τ_{13}	0.2000	0.1300	0.1233	-0.0700	0.0022	0.0071
τ_{14}	0.2000	0.1292	0.1227	-0.0708	0.0021	0.0072
$\tau_{23\|1}$	0.8000	0.6603	0.6636	-0.1397	0.0004	0.0199
$\tau_{24\|1}$	0.8000	0.6637	0.6672	-0.1363	0.0004	0.0190
$\tau_{34\|12}$	0.8000	0.7513	0.7531	-0.0487	0.0005	0.0028
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.1469	0.1896	0.1799	0.0427	0.0059	0.0077
λ^U_{13}	0.1469	0.2101	0.2011	0.0632	0.0066	0.0105
λ^U_{14}	0.1469	0.2104	0.2014	0.0635	0.0065	0.0106
$\lambda^U_{23\|1}$	0.9104	0.8393	0.8418	-0.0711	0.0002	0.0052
$\lambda^U_{24\|1}$	0.9104	0.8418	0.8444	-0.0686	0.0002	0.0049
$\lambda^U_{34\|12}$	0.9104	0.8877	0.8895	-0.0227	0.0002	0.0007

Table C.34: Results of the stability and robustness tests of the BB7-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.8$ and $2 * \lambda^L = \lambda^U$, n=500, r=100)

Copula parameters						
	$\alpha = (\theta, \delta)$	$\bar{\alpha}$	$\bar{\alpha}_{0.05}$	$\hat{b}(\bar{\alpha})$	$s^2(\bar{\alpha})$	$\widehat{mse}(\bar{\alpha})$
θ_{12}	8.0799	4.5318	4.5563	-3.5481	0.2230	12.8117
δ_{12}	0.8800	0.0012	0.0002	-0.8788	0.0000	0.7723
θ_{13}	8.0799	4.4616	4.4727	-3.6183	0.2377	13.3297
δ_{13}	0.8800	0.0024	0.0004	-0.8776	0.0000	0.7702
θ_{14}	8.0799	4.4121	4.4348	-3.6678	0.2135	13.6662
δ_{14}	0.8800	0.0006	0.0001	-0.8794	0.0000	0.7734
$\theta_{23\|1}$	1.1237	2.0855	1.9965	0.9618	0.1252	1.0503
$\delta_{23\|1}$	0.3614	0.9381	0.9435	0.5767	0.0216	0.3542
$\theta_{24\|1}$	1.1237	2.1616	2.0724	1.0379	0.1235	1.2007
$\delta_{24\|1}$	0.3614	0.8746	0.8683	0.5132	0.0227	0.2861
$\theta_{34\|12}$	1.1237	1.2631	1.2613	0.1394	0.0030	0.0224
$\delta_{34\|12}$	0.3614	0.6403	0.6281	0.2789	0.0117	0.0895
Kendall's τ						
	τ	$\bar{\tau}$	$\bar{\tau}_{0.05}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
τ_{12}	0.8000	0.6354	0.6439	-0.1646	0.0014	0.0285
τ_{13}	0.8000	0.6299	0.6380	-0.1701	0.0014	0.0304
τ_{14}	0.8000	0.6276	0.6359	-0.1724	0.0014	0.0311
$\tau_{23\|1}$	0.2000	0.4827	0.4811	0.2827	0.0019	0.0818
$\tau_{24\|1}$	0.2000	0.4863	0.4830	0.2863	0.0017	0.0837
$\tau_{34\|12}$	0.2000	0.3119	0.3115	0.1119	0.0005	0.0130
Upper tail dependence λ^U						
	λ^U	$\overline{\lambda^U}$	$\overline{\lambda^U}_{0.05}$	$\hat{b}(\overline{\lambda^U})$	$s^2(\overline{\lambda^U})$	$\widehat{mse}(\overline{\lambda^U})$
λ^U_{12}	0.9104	0.8206	0.8289	-0.0898	0.0008	0.0088
λ^U_{13}	0.9104	0.8168	0.8249	-0.0936	0.0008	0.0096
λ^U_{14}	0.9104	0.8155	0.8236	-0.0949	0.0008	0.0098
$\lambda^U_{23\|1}$	0.1469	0.5412	0.5446	0.3943	0.0055	0.1610
$\lambda^U_{24\|1}$	0.1469	0.5673	0.5709	0.4204	0.0043	0.1811
$\lambda^U_{34\|12}$	0.1469	0.2572	0.2598	0.1103	0.0018	0.0140

Table C.35: Results of the stability and robustness tests of the BB7-copula in different scenarios. (copulas $c_{12} - c_{14}$: $\tau = 0.8$ and $2 * \lambda^L = \lambda^U$, copulas $c_{23|1} - c_{34|12}$: $\tau = 0.2$ and $\lambda^L = \lambda^U$, n=500, r=100)

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	N	ρ_{12}	0.9511	0.9510	0.9509	-0.0001	0.0000	0.0000	0.7998	0.7997	-0.0002	0.0000	0.0000
0.8	C	θ_{13}	8.0000	8.0197	8.0084	0.0197	0.0198	0.0202	0.8002	0.8001	0.0002	0.0000	0.0000
0.8	G	θ_{14}	5.0000	5.0011	4.9968	0.0011	0.0034	0.0034	0.7999	0.7998	-0.0001	0.0000	0.0000
0.8	F	$\theta_{23\|1}$	18.1000	17.9578	17.9674	-0.1422	0.1549	0.1751	0.7973	0.7975	-0.0027	0.0000	0.0000
0.8	J	$\theta_{24\|1}$	8.7000	8.6544	8.6583	-0.0456	0.0381	0.0401	0.7973	0.7975	-0.0027	0.0000	0.0000
0.8	t3	$\rho_{34\|12}$	0.9511	0.9518	0.9520	0.0007	0.0000	0.0000	0.8020	0.8023	0.0020	0.0000	0.0000
		$\nu_{34\|12}$	3.0000	4.8626	4.1787	1.8626	2.9613	6.4306					
0.2	N	ρ_{12}	0.3090	0.3040	0.3040	-0.0050	0.0002	0.0003	0.1968	0.1967	-0.0032	0.0001	0.0001
0.2	C	θ_{13}	0.5000	0.5035	0.5032	0.0035	0.0008	0.0008	0.2006	0.2007	0.0006	0.0001	0.0001
0.2	G	θ_{14}	1.2500	1.2496	1.2493	-0.0004	0.0003	0.0003	0.1991	0.1991	-0.0009	0.0001	0.0001
0.2	F	$\theta_{23\|1}$	1.8600	1.7957	1.7936	-0.0643	0.0115	0.0157	0.1931	0.1930	-0.0069	0.0001	0.0002
0.2	J	$\theta_{24\|1}$	1.4400	1.4428	1.4432	0.0028	0.0008	0.0008	0.1990	0.1994	-0.0010	0.0001	0.0001
0.2	t3	$\rho_{34\|12}$	0.3090	0.3025	0.3033	-0.0066	0.0004	0.0005	0.1958	0.1963	-0.0042	0.0002	0.0002
		$\nu_{34\|12}$	3.0000	3.0829	3.0417	0.0829	0.0701	0.0770					
0.8	N	ρ_{12}	0.9511	0.9508	0.9509	-0.0002	0.0000	0.0000	0.7996	0.7996	-0.0004	0.0000	0.0000
0.8	C	θ_{13}	8.0000	8.0097	8.0022	0.0097	0.0289	0.0290	0.7999	0.7999	-0.0001	0.0000	0.0000
0.8	G	θ_{14}	5.0000	5.0266	5.0228	0.0266	0.0068	0.0075	0.8008	0.8007	0.0008	0.0000	0.0000
0.2	F	$\theta_{23\|1}$	1.8600	1.8402	1.8361	-0.0198	0.0162	0.0166	0.1975	0.1972	-0.0025	0.0002	0.0002
0.2	J	$\theta_{24\|1}$	1.4400	1.4416	1.4408	0.0016	0.0010	0.0011	0.1984	0.1984	-0.0016	0.0001	0.0001
0.2	t3	$\rho_{34\|12}$	0.3090	0.3080	0.3089	-0.0010	0.0005	0.0005	0.1996	0.2001	-0.0004	0.0002	0.0002
		$\nu_{34\|12}$	3.0000	3.1067	3.0489	0.1067	0.1057	0.1171					
0.2	N	ρ_{12}	0.3090	0.3092	0.3092	0.0002	0.0000	0.0000	0.2001	0.2001	0.0001	0.0000	0.0000
0.2	C	θ_{13}	0.5000	0.4999	0.5002	-0.0001	0.0000	0.0000	0.2000	0.2001	-0.0000	0.0000	0.0000
0.2	G	θ_{14}	1.2500	1.2502	1.2503	0.0002	0.0000	0.0000	0.2001	0.2002	0.0001	0.0000	0.0000
0.8	F	$\theta_{23\|1}$	18.1000	18.0846	18.0807	-0.0155	0.0721	0.0723	0.7988	0.7988	-0.0012	0.0000	0.0000
0.8	J	$\theta_{24\|1}$	8.7000	8.7053	8.7013	0.0053	0.0146	0.0147	0.7986	0.7985	-0.0014	0.0000	0.0000
0.8	t3	$\rho_{34\|12}$	0.9511	0.9513	0.9515	0.0002	0.0000	0.0000	0.8007	0.8010	0.0007	0.0000	0.0000
		$\nu_{34\|12}$	3.0000	3.2303	3.1227	0.2303	0.2091	0.2621					

Table C.36: Results of the stability and robustness tests of the copula combination N,C,G,F,J,t3 in different scenarios (n=500, r=100).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	t3	ρ_{12}	0.9511	0.9511	0.9511	0.0000	0.0000	0.0000	0.8001	0.8000	0.0001	0.0000	0.0000
		ν_{12}	3.0000	3.0114	3.0084	0.0114	0.0021	0.0022					
0.8	G	θ_{13}	5.0000	5.0130	5.0114	0.0129	0.0011	0.0012	0.8005	0.8004	0.0005	0.0000	0.0000
0.8	J	θ_{14}	8.7000	8.7124	8.7080	0.0124	0.0022	0.0023	0.7989	0.7988	-0.0011	0.0000	0.0000
0.8	F	$\theta_{23\|1}$	18.1000	18.0666	18.0565	-0.0334	0.1588	0.1600	0.7984	0.7984	-0.0016	0.0000	0.0000
0.8	C	$\theta_{24\|1}$	8.0000	7.9204	7.9149	-0.0796	0.0289	0.0352	0.7981	0.7981	-0.0019	0.0000	0.0000
0.8	N	$\rho_{34\|12}$	0.9511	0.9477	0.9510	-0.0033	0.0001	0.0001	0.7957	0.8001	-0.0043	0.0002	0.0002
0.2	t3	ρ_{12}	0.3090	0.3092	0.3098	0.0002	0.0004	0.0004	0.2004	0.2007	0.0004	0.0002	0.0002
		ν_{12}	3.0000	3.0543	3.0239	0.0543	0.0554	0.0584					
0.2	G	θ_{13}	1.2500	1.2484	1.2469	-0.0016	0.0003	0.0003	0.1982	0.1975	-0.0018	0.0001	0.0001
0.2	J	θ_{14}	1.4400	1.4470	1.4453	0.0070	0.0007	0.0007	0.2006	0.2002	0.0006	0.0001	0.0001
0.2	F	$\theta_{23\|1}$	1.8600	1.8770	1.8776	0.0170	0.0149	0.0152	0.2012	0.2015	0.0012	0.0002	0.0002
0.2	C	$\theta_{24\|1}$	0.5000	0.4972	0.4981	-0.0028	0.0012	0.0012	0.1983	0.1989	-0.0017	0.0001	0.0001
0.2	N	$\rho_{34\|12}$	0.3090	0.3103	0.3105	0.0013	0.0003	0.0003	0.2010	0.2011	0.0010	0.0002	0.0002
0.8	t3	ρ_{12}	0.9511	0.9511	0.9513	0.0001	0.0000	0.0000	0.8003	0.8006	0.0003	0.0000	0.0000
		ν_{12}	3.0000	3.1872	3.1615	0.1872	0.0881	0.1231					
0.8	G	θ_{13}	5.0000	5.0126	5.0148	0.0126	0.0066	0.0068	0.8002	0.8004	0.0002	0.0000	0.0000
0.8	J	θ_{14}	8.7000	8.7352	8.7434	0.0352	0.0322	0.0334	0.7990	0.7993	-0.0010	0.0000	0.0000
0.2	F	$\theta_{23\|1}$	1.8600	1.7800	1.7796	-0.0800	0.0133	0.0197	0.1915	0.1916	-0.0085	0.0001	0.0002
0.2	C	$\theta_{24\|1}$	0.5000	0.4999	0.4991	-0.0001	0.0010	0.0010	0.1993	0.1993	-0.0007	0.0001	0.0001
0.2	N	$\rho_{34\|12}$	0.3090	0.3144	0.3153	0.0054	0.0004	0.0004	0.2038	0.2043	0.0038	0.0002	0.0002
0.2	t3	ρ_{12}	0.3090	0.3088	0.3086	-0.0002	0.0000	0.0000	0.1999	0.1997	-0.0001	0.0000	0.0000
		ν_{12}	3.0000	2.9967	2.9987	-0.0033	0.0016	0.0016					
0.2	G	θ_{13}	1.2500	1.2498	1.2496	-0.0002	0.0000	0.0000	0.1998	0.1997	-0.0002	0.0000	0.0000
0.2	J	θ_{14}	1.4400	1.4397	1.4391	-0.0003	0.0001	0.0001	0.1985	0.1983	-0.0015	0.0000	0.0000
0.8	F	$\theta_{23\|1}$	18.1000	18.1115	18.1548	0.0115	0.1078	0.1079	0.7989	0.7995	-0.0011	0.0000	0.0000
0.8	C	$\theta_{24\|1}$	8.0000	7.9842	8.0141	-0.0158	0.0254	0.0256	0.7994	0.8002	-0.0006	0.0000	0.0000
0.8	N	$\rho_{34\|12}$	0.9511	0.9496	0.9514	-0.0015	0.0001	0.0001	0.7984	0.8009	-0.0016	0.0001	0.0001

Table C.37: Results of the stability and robustness tests of the copula combination t3,G,J,G,C,N in different scenarios (n=500, r=100).

			Copula parameter(s)						Kendall's τ				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	C	θ_{12}	8.0000	8.0606	8.0496	0.0606	0.0153	0.0190	0.8011	0.8009	0.0011	0.0000	0.0000
0.8	t5	ρ_{13}	0.9511	0.9516	0.9515	0.0005	0.0000	0.0000	0.8011	0.8010	0.0011	0.0000	0.0000
		ν_{13}	5.0000	5.0745	5.0322	0.0745	0.0445	0.0501					
0.8	F	θ_{14}	18.1000	18.2414	18.2231	0.1414	0.0678	0.0878	0.8003	0.8002	0.0003	0.0000	0.0000
0.8	G	$\theta_{23\|1}$	5.0000	5.1222	5.1223	0.1222	0.0165	0.0315	0.8042	0.8044	0.0042	0.0000	0.0000
0.8	N	$\theta_{24\|1}$	0.9511	0.9525	0.9527	0.0015	0.0000	0.0000	0.8033	0.8036	0.0033	0.0000	0.0000
0.8	J	$\theta_{34\|12}$	8.7000	7.8713	7.8827	-0.8287	0.0958	0.7826	0.7788	0.7796	-0.0212	0.0001	0.0005
0.2	C	θ_{12}	0.5000	0.5022	0.5023	0.0022	0.0008	0.0008	0.2002	0.2004	0.0002	0.0001	0.0001
0.2	t5	ρ_{13}	0.3090	0.3084	0.3099	-0.0006	0.0003	0.0003	0.1998	0.2007	-0.0002	0.0001	0.0001
		ν_{13}	5.0000	5.2706	5.1995	0.2706	0.3065	0.3797					
0.2	F	θ_{14}	1.8600	1.8885	1.8914	0.0285	0.0146	0.0154	0.2024	0.2028	0.0024	0.0001	0.0002
0.2	G	$\theta_{23\|1}$	1.2500	1.2545	1.2542	0.0045	0.0003	0.0003	0.2021	0.2022	0.0021	0.0001	0.0001
0.2	N	$\theta_{24\|1}$	0.3090	0.3125	0.3131	0.0034	0.0003	0.0003	0.2025	0.2028	0.0025	0.0001	0.0001
0.2	J	$\theta_{34\|12}$	1.4400	1.4427	1.4420	0.0027	0.0008	0.0008	0.1989	0.1989	-0.0011	0.0001	0.0001
0.8	C	θ_{12}	8.0000	8.0127	8.0176	0.0127	0.0184	0.0186	0.8001	0.8002	0.0001	0.0000	0.0000
0.8	t5	ρ_{13}	0.9511	0.9511	0.9512	0.0001	0.0000	0.0000	0.8003	0.8004	0.0003	0.0000	0.0000
		ν_{13}	5.0000	5.4910	5.3648	0.4910	0.5297	0.7707					
0.8	F	θ_{14}	18.1000	18.1501	18.1236	0.0501	0.1152	0.1177	0.7993	0.7992	-0.0007	0.0000	0.0000
0.2	G	$\theta_{23\|1}$	1.2500	1.2574	1.2575	0.0074	0.0003	0.0003	0.2041	0.2044	0.0041	0.0001	0.0001
0.2	N	$\theta_{24\|1}$	0.3090	0.3036	0.3034	-0.0055	0.0003	0.0003	0.1965	0.1964	-0.0035	0.0001	0.0002
0.2	J $\theta_{34\|12}$	1.4400	1.4384	1.4355	-0.0016	0.0011	0.0011	0.1972	0.1965	-0.0028	0.0001	0.0001	
0.2	C	θ_{12}	0.5000	0.5004	0.5000	0.0004	0.0001	0.0001	0.2001	0.2000	0.0001	0.0000	0.0000
0.2	t5	ρ_{13}	0.3090	0.3084	0.3084	-0.0007	0.0000	0.0000	0.1996	0.1996	-0.0004	0.0000	0.0000
		ν_{13}	5.0000	5.0048	5.0014	0.0048	0.0032	0.0033					
0.2	F	θ_{14}	1.8600	1.8569	1.8569	-0.0031	0.0017	0.0018	0.1996	0.1996	-0.0004	0.0000	0.0000
0.8	G	$\theta_{23\|1}$	5.0000	5.0668	5.0647	0.0668	0.0072	0.0117	0.8024	0.8024	0.0024	0.0000	0.0000
0.8	N	$\theta_{24\|1}$	0.9511	0.9519	0.9520	0.0009	0.0000	0.0000	0.8019	0.8021	0.0019	0.0000	0.0000
0.8	J	$\theta_{34\|12}$	8.7000	8.4670	8.4787	-0.2330	0.0709	0.1252	0.7929	0.7935	-0.0071	0.0000	0.0001

Table C.38: Results of the stability and robustness tests of the copula combination C,t5,F,G,N,J in different scenarios (n=500, r=100).

τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	G	θ_{12}	5.0000	5.0664	5.0560	0.0664	0.0043	0.0087	0.8025	0.8021	0.0025	0.0000	0.0000
0.8	t5	ρ_{13}	0.9511	0.9523	0.9521	0.0013	0.0000	0.0000	0.8028	0.8023	0.0028	0.0000	0.0000
		ν_{13}	5.0000	5.2950	5.0848	0.2950	0.3139	0.4009					
0.8	J	θ_{14}	8.7000	8.8360	8.8128	0.1360	0.0197	0.0382	0.8012	0.8008	0.0012	0.0000	0.0000
0.8	C	$\theta_{23\|1}$	8.0000	8.0686	8.0673	0.0686	0.0680	0.0727	0.8007	0.8009	0.0007	0.0000	0.0000
0.8	t10	$\rho_{24\|1}$	0.9511	0.9513	0.9514	0.0003	0.0000	0.0000	0.8008	0.8010	0.0008	0.0000	0.0000
		$\nu_{24\|1}$	10.0000	11.6403	11.5439	1.6403	4.2493	6.9400					
0.8	F	$\theta_{34\|12}$	18.1000	14.8705	14.8991	-3.2295	0.6463	11.0761	0.7580	0.7595	-0.0420	0.0001	0.0019
0.2	G	θ_{12}	1.2500	1.2536	1.2525	0.0036	0.0003	0.0003	0.2016	0.2012	0.0016	0.0001	0.0001
0.2	t5	ρ_{13}	0.3090	0.3103	0.3099	0.0013	0.0003	0.0003	0.2011	0.2007	0.0011	0.0001	0.0001
		ν_{13}	5.0000	5.5225	5.2933	0.5225	0.8419	1.1149					
0.2	J	θ_{14}	1.4400	1.4481	1.4492	0.0081	0.0007	0.0008	0.2009	0.2015	0.0009	0.0001	0.0001
0.2	C	$\theta_{23\|1}$	0.5000	0.4936	0.4919	-0.0064	0.0013	0.0013	0.1971	0.1969	-0.0029	0.0001	0.0001
0.2	t10	$\rho_{24\|1}$	0.3090	0.3098	0.3102	0.0007	0.0004	0.0004	0.2007	0.2010	0.0007	0.0002	0.0002
		$\nu_{24\|1}$	10.0000	15.5108	11.8590	5.5108	72.9256	103.2948					
0.2	F	$\theta_{34\|12}$	1.8600	1.8422	1.8429	-0.0178	0.0145	0.0148	0.1978	0.1980	-0.0022	0.0001	0.0002
0.8	G	θ_{12}	5.0000	5.0129	5.0129	0.0129	0.0076	0.0077	0.8002	0.8003	0.0002	0.0000	0.0000
0.8	t5	ρ_{13}	0.9511	0.9505	0.9506	-0.0005	0.0000	0.0000	0.7990	0.7992	-0.0010	0.0000	0.0000
		ν_{13}	5.0000	5.8479	5.5723	0.8479	1.0876	1.8064					
0.8	J	θ_{14}	8.7000	8.7019	8.6957	0.0019	0.0217	0.0218	0.7984	0.7984	-0.0016	0.0000	0.0000
0.2	C	$\theta_{23\|1}$	0.5000	0.5134	0.5144	0.0134	0.0011	0.0012	0.2036	0.2041	0.0036	0.0001	0.0001
0.2	t10	$\rho_{24\|1}$	0.3090	0.3138	0.3133	0.0048	0.0004	0.0004	0.2034	0.2030	0.0034	0.0002	0.0002
		$\nu_{24\|1}$	10.0000	12.7023	12.7206	2.7023	6.0088	13.3111					
0.2	F	$\theta_{34\|12}$	1.8600	1.8808	1.8818	0.0208	0.0121	0.0125	0.2017	0.2019	0.0017	0.0001	0.0001
0.2	G	θ_{12}	1.2500	1.2471	1.2468	-0.0029	0.0000	0.0000	0.1981	0.1979	-0.0019	0.0000	0.0000
0.2	t5	ρ_{13}	0.3090	0.3067	0.3065	-0.0023	0.0000	0.0000	0.1985	0.1983	-0.0015	0.0000	0.0000
		ν_{13}	5.0000	5.0022	5.0024	0.0022	0.0025	0.0025					
0.2	J	θ_{14}	1.4400	1.4368	1.4365	-0.0032	0.0001	0.0001	0.1975	0.1974	-0.0025	0.0000	0.0000
0.8	C	$\theta_{23\|1}$	8.0000	8.0575	8.0583	0.0575	0.0187	0.0220	0.8010	0.8010	0.0010	0.0000	0.0000
0.8	t10	$\rho_{24\|1}$	0.9511	0.9511	0.9511	0.0001	0.0000	0.0000	0.8003	0.8003	0.0003	0.0000	0.0000
		$\nu_{24\|1}$	10.0000	11.8053	11.6825	1.8053	4.1145	7.3737					
0.8	F	$\theta_{34\|12}$	18.1000	18.1199	18.1283	0.0199	0.1755	0.1759	0.7989	0.7991	-0.0011	0.0000	0.0000

Table C.39: Results of the stability and robustness tests of the copula combination G,t5,J,C,t10,F in different scenarios (n=500,

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	N	ρ_{12}	0.9511	0.9489	0.9486	-0.0021	0.0000	0.0000	0.7958	0.7950	-0.0042	0.0000	0.0000
0.8	C	θ_{13}	8.0000	7.7614	7.7020	-0.2386	0.0434	0.1004	0.7947	0.7938	-0.0053	0.0000	0.0000
0.8	BB1	θ_{14}	6.4800	5.9621	5.9890	-0.5179	0.0060	0.2742	0.7944	0.7933	-0.0056	0.0000	0.0000
		δ_{14}	1.1792	1.2267	1.2137	0.0475	0.0019	0.0041					
0.8	BB7	$\theta_{23\|1}$	1.1792	1.1951	1.1828	0.0159	0.0049	0.0052	0.8054	0.8082	0.0054	0.0001	0.0001
		$\delta_{23\|1}$	8.4000	8.9703	9.0933	0.5703	0.2615	0.5866					
0.8	t10	$\rho_{24\|1}$	0.9511	0.9562	0.9573	0.0052	0.0000	0.0000	0.8116	0.8136	0.0116	0.0001	0.0002
		$\nu_{24\|1}$	10.0000	14.3357	14.4154	4.3357	4.6261	23.4246					
0.8	F	$\theta_{34\|12}$	18.1000	11.6565	11.5974	-6.4435	0.7918	42.3099	0.6994	0.7003	-0.1006	0.0004	0.0105
0.2	N	ρ_{12}	0.3090	0.3006	0.3005	-0.0084	0.0003	0.0003	0.1945	0.1944	-0.0055	0.0001	0.0001
0.2	C	θ_{13}	0.5000	0.5068	0.5064	0.0068	0.0008	0.0009	0.2016	0.2018	0.0016	0.0001	0.0001
0.2	BB1	θ_{14}	0.1200	0.1308	0.1300	0.0108	0.0010	0.0011	0.1944	0.1945	-0.0056	0.0001	0.0002
		δ_{14}	1.1792	1.1672	1.1666	-0.0120	0.0004	0.0005					
0.2	BB7	$\theta_{23\|1}$	1.1792	1.1729	1.1716	-0.0064	0.0008	0.0009	0.1966	0.1963	-0.0034	0.0001	0.0002
		$\delta_{23\|1}$	0.2900	0.2981	0.2964	0.0081	0.0013	0.0013					
0.2	t10	$\rho_{24\|1}$	0.3090	0.3049	0.3044	-0.0041	0.0002	0.0003	0.1974	0.1970	-0.0026	0.0001	0.0001
		$\nu_{24\|1}$	10.0000	19.2011	15.5936	9.2011	106.8133	191.4737					
0.2	F	$\theta_{34\|12}$	1.8600	1.8576	1.8572	-0.0024	0.0172	0.0172	0.1992	0.1993	-0.0008	0.0002	0.0002

Table C.40: Results of the stability and robustness tests of the copula combination N,C,BB1,BB7,t10,F in different scenarios (n=500, r=100) (part 1).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	N	ρ_{12}	0.9511	0.9510	0.9510	-0.0000	0.0000	0.0000	0.8001	0.8000	0.0001	0.0000	0.0000
0.8	C	θ_{13}	8.0000	7.9944	7.9829	-0.0056	0.0287	0.0287	0.7996	0.7995	-0.0004	0.0000	0.0000
0.8	BB1	θ_{14}	6.4800	5.8646	5.9031	-0.6154	0.0139	0.3926	0.7974	0.7977	-0.0026	0.0000	0.0000
		δ_{14}	1.1792	1.2581	1.2556	0.0788	0.0007	0.0069					
0.2	BB7	$\theta_{23\|1}$	1.1792	1.1964	1.1961	0.0172	0.0009	0.0012	0.2014	0.2015	0.0014	0.0002	0.0002
		$\delta_{23\|1}$	0.2900	0.2875	0.2867	-0.0025	0.0012	0.0012					
0.2	t10	$\rho_{24\|1}$	0.3090	0.3223	0.3237	0.0132	0.0004	0.0006	0.2091	0.2100	0.0091	0.0002	0.0003
		$\nu_{24\|1}$	10.0000	12.6069	12.5888	2.6069	5.3612	12.1571					
0.2	F	$\theta_{34\|12}$	1.8600	1.8341	1.8332	-0.0259	0.0195	0.0201	0.1968	0.1969	-0.0032	0.0002	0.0002
0.2	N	ρ_{12}	0.3090	0.3095	0.3095	0.0005	0.0000	0.0000	0.2003	0.2003	0.0003	0.0000	0.0000
0.2	C	θ_{13}	0.5000	0.5008	0.5004	0.0008	0.0000	0.0000	0.2002	0.2001	0.0002	0.0000	0.0000
0.2	BB1	θ_{14}	0.1200	0.1192	0.1191	-0.0008	0.0000	0.0000	0.2002	0.2001	0.0002	0.0000	0.0000
		δ_{14}	1.1792	1.1800	1.1801	0.0008	0.0000	0.0000					
0.8	BB7	$\theta_{23\|1}$	1.1792	1.1689	1.1692	-0.0104	0.0007	0.0008	0.7989	0.7991	-0.0011	0.0000	0.0000
		$\delta_{23\|1}$	8.4000	8.4148	8.4075	0.0148	0.0311	0.0314					
0.8	t10	$\rho_{24\|1}$	0.9511	0.9513	0.9514	0.0002	0.0000	0.0000	0.8006	0.8008	0.0006	0.0000	0.0000
		$\nu_{24\|1}$	10.0000	11.5244	11.3553	1.5244	3.5140	5.8378					
0.8	F	$\theta_{34\|12}$	18.1000	18.0118	18.0129	-0.0882	0.2385	0.2462	0.7976	0.7978	-0.0024	0.0000	0.0000

Table C.41: Results of the stability and robustness tests of the copula combination N,C,BB1,BB7,t10,F in different scenarios (n=500, r=100) (part 2).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	BB1	θ_{12}	6.4800	5.9997	6.0000	-0.4803	0.0000	0.2307	0.7952	0.7946	-0.0048	0.0000	0.0000
		δ_{12}	1.1792	1.2218	1.2176	0.0426	0.0003	0.0021					
0.8	t5	ρ_{13}	0.9511	0.9480	0.9477	-0.0031	0.0000	0.0000	0.7938	0.7932	-0.0062	0.0000	0.0000
		ν_{13}	5.0000	5.0968	4.9466	0.0968	0.1694	0.1788					
0.8	J	θ_{14}	8.7000	8.4710	8.4364	-0.2290	0.0258	0.0782	0.7936	0.7930	-0.0064	0.0000	0.0001
0.8	BB7	$\theta_{23\|1}$	1.1792	1.2238	1.2236	0.0446	0.0027	0.0046	0.8053	0.8070	0.0053	0.0001	0.0001
		$\delta_{23\|1}$	8.4000	9.0280	9.1236	0.6280	0.2148	0.6091					
0.8	t10	$\rho_{24\|1}$	0.9511	0.9550	0.9560	0.0039	0.0000	0.0000	0.8089	0.8107	0.0089	0.0001	0.0001
		$\nu_{24\|1}$	10.0000	13.4794	13.6079	3.4794	6.7994	18.9055					
0.8	F	$\theta_{34\|12}$	18.1000	12.8069	12.7341	-5.2931	0.2305	28.2477	0.7265	0.7258	-0.0735	0.0001	0.0055
0.2	BB1	θ_{12}	0.1200	0.1249	0.1235	0.0049	0.0010	0.0010	0.1980	0.1983	-0.0020	0.0001	0.0002
		δ_{12}	1.1792	1.1757	1.1756	-0.0036	0.0004	0.0004					
0.2	t5	ρ_{13}	0.3090	0.3143	0.3148	0.0053	0.0004	0.0004	0.2037	0.2040	0.0037	0.0002	0.0002
		ν_{13}	5.0000	5.7502	5.4761	0.7502	1.2447	1.8074					
0.2	J	θ_{14}	1.4400	1.4510	1.4499	0.0110	0.0006	0.0008	0.2020	0.2018	0.0020	0.0001	0.0001
0.2	BB7	$\theta_{23\|1}$	1.1792	1.1892	1.1872	0.0099	0.0008	0.0009	0.1991	0.1986	-0.0009	0.0001	0.0001
		$\delta_{23\|1}$	0.2900	0.2873	0.2867	-0.0027	0.0008	0.0008					
0.2	t10	$\rho_{24\|1}$	0.3090	0.3022	0.3016	-0.0069	0.0004	0.0004	0.1956	0.1952	-0.0044	0.0002	0.0002
		$\nu_{24\|1}$	10.0000	17.7561	13.8784	7.7561	109.0161	169.1736					
0.2	F	$\theta_{34\|12}$	1.8600	1.9183	1.9155	0.0583	0.0113	0.0147	0.2055	0.2053	0.0055	0.0001	0.0001

Table C.42: Results of the stability and robustness tests of the copula combination BB1,t5,J,BB7,t10,F in different scenarios (n=500, r=100) (part 1).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	BB1	θ_{12}	6.4800	5.8790	5.9313	-0.6010	0.0197	0.3809	0.7991	0.7995	-0.0009	0.0000	0.0000
		δ_{12}	1.1792	1.2675	1.2641	0.0883	0.0008	0.0086					
0.8	t5	ρ_{13}	0.9511	0.9503	0.9504	-0.0008	0.0000	0.0000	0.7986	0.7987	-0.0014	0.0000	0.0000
		ν_{13}	5.0000	5.3333	5.1497	0.3333	0.5601	0.6712					
0.8	J	θ_{14}	8.7000	8.7340	8.7572	0.0340	0.0383	0.0394	0.7989	0.7996	-0.0011	0.0000	0.0000
0.2	BB7	$\theta_{23\|1}$	1.1792	1.1899	1.1884	0.0107	0.0006	0.0008	0.1989	0.1984	-0.0011	0.0001	0.0002
		$\delta_{23\|1}$	0.2900	0.2864	0.2848	-0.0036	0.0010	0.0010					
0.2	t10	$\rho_{24\|1}$	0.3090	0.3159	0.3168	0.0068	0.0004	0.0004	0.2048	0.2053	0.0048	0.0002	0.0002
		$\nu_{24\|1}$	10.0000	12.5528	12.5356	2.5528	4.8885	11.4055					
0.2	F	$\theta_{34\|12}$	1.8600	1.8551	1.8579	-0.0049	0.0176	0.0176	0.1990	0.1995	-0.0010	0.0002	0.0002
0.2	BB1	θ_{12}	0.1200	0.1176	0.1186	-0.0024	0.0000	0.0001	0.2016	0.2012	0.0016	0.0000	0.0000
		δ_{12}	1.1792	1.1831	1.1819	0.0039	0.0001	0.0001					
0.2	t5	ρ_{13}	0.3090	0.3109	0.3104	0.0019	0.0000	0.0000	0.2013	0.2009	0.0013	0.0000	0.0000
		ν_{13}	5.0000	5.1618	5.0462	0.1618	0.2616	0.2877					
0.2	J	θ_{14}	1.4400	1.4434	1.4420	0.0034	0.0001	0.0001	0.1998	0.1993	-0.0002	0.0000	0.0000
0.8	BB7	$\theta_{23\|1}$	1.1792	1.1887	1.1873	0.0094	0.0009	0.0010	0.8002	0.8002	0.0002	0.0000	0.0000
		$\delta_{23\|1}$	8.4000	8.5534	8.5420	0.1534	0.0658	0.0893					
0.8	t10	$\rho_{24\|1}$	0.9511	0.9519	0.9519	0.0008	0.0000	0.0000	0.8018	0.8019	0.0018	0.0000	0.0000
		$\nu_{24\|1}$	10.0000	11.8849	11.8156	1.8849	4.4577	8.0107					
0.8	F	$\theta_{34\|12}$	18.1000	16.7209	16.6962	-1.3791	0.3955	2.2974	0.7831	0.7832	-0.0169	0.0001	0.0003

Table C.43: Results of the stability and robustness tests of the copula combination BB1,t5,J,BB7,t10,F in different scenarios (n=500, r=100) (part 2).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	BB7	θ_{12}	1.1792	1.1944	1.1935	0.0151	0.0017	0.0020	0.7969	0.7972	-0.0031	0.0000	0.0000
		δ_{12}	8.4000	8.3919	8.3808	-0.0081	0.0843	0.0843					
0.8	t5	ρ_{13}	0.9511	0.9505	0.9503	-0.0005	0.0000	0.0000	0.7992	0.7986	-0.0008	0.0000	0.0000
		ν_{13}	5.0000	5.2090	5.0896	0.2090	0.2231	0.2667					
0.8	F	θ_{14}	18.1000	18.1041	17.9987	0.0041	0.2806	0.2806	0.7985	0.7978	-0.0015	0.0000	0.0000
0.8	N	$\rho_{23\mid 1}$	0.9511	0.9528	0.9544	0.0018	0.0000	0.0000	0.8048	0.8072	0.0048	0.0001	0.0001
0.8	t10	$\rho_{24\mid 1}$	0.9511	0.9523	0.9542	0.0013	0.0000	0.0000	0.8040	0.8068	0.0040	0.0001	0.0001
		$\nu_{24\mid 1}$	10.0000	19.9062	19.9971	9.9062	0.1799	98.3117					
0.8	BB1	$\theta_{34\mid 12}$	6.4800	2.8972	2.8455	-3.5828	0.1168	0.0000	0.7824	0.7824	-0.0176	0.0001	0.0004
	$\delta_{34\mid 12}$	1.1792	1.9355	1.9287	0.7562	0.0215	0.5933						
0.2	BB7	θ_{12}	1.1792	1.1800	1.1776	0.0008	0.0005	0.0005	0.1986	0.1984	-0.0014	0.0001	0.0001
		δ_{12}	0.2900	0.2961	0.2951	0.0061	0.0009	0.0009					
0.2	t5	ρ_{13}	0.3090	0.3017	0.3013	-0.0073	0.0003	0.0004	0.1953	0.1949	-0.0047	0.0001	0.0002
		ν_{13}	5.0000	5.4373	5.2136	0.4373	0.7622	0.9534					
0.2	F	θ_{14}	1.8600	1.8525	1.8527	-0.0075	0.0135	0.0136	0.1988	0.1989	-0.0012	0.0001	0.0001
0.2	N	$\rho_{23\mid 1}$	0.3090	0.3086	0.3079	-0.0005	0.0003	0.0003	0.1999	0.1994	-0.0001	0.0001	0.0001
0.2	t10	$\rho_{24\mid 1}$	0.3090	0.3120	0.3128	0.0030	0.0004	0.0004	0.2022	0.2026	0.0022	0.0002	0.0002
		$\nu_{24\mid 1}$	10.0000	14.1871	11.8580	4.1871	42.2095	59.7411					
0.2	BB1	$\theta_{34\mid 12}$	0.1200	0.1241	0.1229	0.0041	0.0011	0.0004	0.1993	0.1992	-0.0007	0.0002	0.0002
		$\delta_{34\mid 12}$	1.1792	1.1783	1.1775	-0.0010	0.0004	0.0005					

Table C.44: Results of the stability and robustness tests of the copula combination BB7,t5,F,N,t10,BB1 in different scenarios (n=500, r=100) (part 1).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	BB7	θ_{12}	1.1792	1.1907	1.1825	0.0114	0.0039	0.0040	0.7981	0.7981	-0.0019	0.0000	0.0000
		δ_{12}	8.4000	8.4111	8.4080	0.0111	0.0440	0.0441					
0.8	t5	ρ_{13}	0.9511	0.9511	0.9511	0.0000	0.0000	0.0000	0.8001	0.8001	0.0001	0.0000	0.0000
		ν_{13}	5.0000	5.6408	5.4402	0.6408	0.7711	1.1818					
0.8	F	θ_{14}	18.1000	18.2255	18.2107	0.1255	0.1109	0.1267	0.8001	0.8000	0.0001	0.0000	0.0000
0.2	N	$\rho_{23\|1}$	0.3090	0.3051	0.3055	-0.0039	0.0004	0.0004	0.1976	0.1977	-0.0024	0.0002	0.0002
0.2	t10	$\rho_{24\|1}$	0.3090	0.3115	0.3112	0.0024	0.0004	0.0004	0.2019	0.2016	0.0019	0.0002	0.0002
		$\nu_{24\|1}$	10.0000	14.8910	15.1357	4.8910	6.0762	29.9984					
0.2	BB1	$\theta_{34\|12}$	0.1200	0.1268	0.1241	0.0068	0.0011	0.0004	0.2001	0.2000	0.0001	0.0002	0.0002
		$\delta_{34\|12}$	1.1792	1.1781	1.1763	-0.0011	0.0005	0.0005					
0.2	BB7	θ_{12}	1.1792	1.1847	1.1835	0.0055	0.0001	0.0002	0.2010	0.2000	0.0010	0.0001	0.0001
		δ_{12}	0.2900	0.2971	0.2946	0.0071	0.0002	0.0003					
0.2	t5	ρ_{13}	0.3090	0.3143	0.3130	0.0053	0.0001	0.0002	0.2036	0.2027	0.0036	0.0001	0.0001
		ν_{13}	5.0000	5.1453	5.0344	0.1453	0.2461	0.2673					
0.2	F	θ_{14}	1.8600	1.8955	1.8873	0.0355	0.0049	0.0062	0.2033	0.2026	0.0033	0.0000	0.0001
0.8	N	$\rho_{23\|1}$	0.9511	0.9521	0.9523	0.0010	0.0000	0.0000	0.8023	0.8026	0.0023	0.0000	0.0000
0.8	t10	$\rho_{24\|1}$	0.9511	0.9522	0.9524	0.0011	0.0000	0.0000	0.8025	0.8030	0.0025	0.0000	0.0000
	t10	$\nu_{24\|1}$	10.0000	12.6636	12.5848	2.6636	4.1689	11.2637					
0.8	BB1	$\theta_{34\|12}$	6.4800	4.3989	4.3479	-2.0810	0.0623	0.0000	0.7943	0.7946	-0.0057	0.0000	0.0001
		$\delta_{34\|12}$	1.1792	1.5326	1.5391	0.3533	0.0033	0.1281					

Table C.45: Results of the stability and robustness tests of the copula combination BB7,t5,F,N,t10,BB1 in different scenarios (n=500, r=100) (part 2).

			Copula parameter(s)						**Kendall's τ**				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	BB1	θ_{12}	6.4800	5.9948	5.9996	-0.4852	0.0001	0.2356	0.8005	0.8008	0.0005	0.0000	0.0000
		δ_{12}	1.1792	1.2555	1.2564	0.0763	0.0004	0.0062					
0.8	N	ρ_{13}	0.9511	0.9504	0.9505	-0.0007	0.0000	0.0000	0.7986	0.7989	-0.0014	0.0000	0.0000
0.8	BB7	θ_{14}	1.1792	1.1852	1.1844	0.0060	0.0004	0.0004	0.7966	0.7968	-0.0034	0.0000	0.0000
		δ_{14}	8.4000	8.3366	8.3422	-0.0634	0.0418	0.0458					
0.8	BB1	$\theta_{23\|1}$	6.4800	5.2729	5.2856	-1.2071	0.0558	1.5129	0.8141	0.8140	0.0141	0.0001	0.0003
		$\delta_{23\|1}$	1.1792	1.5067	1.4976	0.3275	0.0100	0.1173					
0.8	t5	$\rho_{24\|1}$	0.9511	0.9551	0.9552	0.0040	0.0000	0.0000	0.8092	0.8091	0.0092	0.0001	0.0001
		$\nu_{24\|1}$	5.0000	17.2834	17.7397	12.2834	4.2144	155.0969					
0.8	BB7	$\theta_{34\|12}$	1.1792	1.3061	1.2703	0.1268	0.0239	0.0400	0.6579	0.6584	-0.1421	0.0005	0.0207
		$\delta_{34\|12}$	8.4000	3.8846	3.8644	-4.5154	0.2090	20.5980					
0.2	BB1	θ_{12}	0.1200	0.1323	0.1296	0.0124	0.0011	0.0012	0.2000	0.1994	-0.0000	0.0002	0.0002
		δ_{12}	1.1792	1.1747	1.1744	-0.0046	0.0004	0.0004					
0.2	N	ρ_{13}	0.3090	0.3058	0.3064	-0.0032	0.0003	0.0003	0.1980	0.1984	-0.0020	0.0001	0.0001
0.2	BB7	θ_{14}	1.1792	1.1847	1.1841	0.0055	0.0007	0.0007	0.1997	0.1997	-0.0003	0.0001	0.0001
		δ_{14}	0.2900	0.2940	0.2913	0.0040	0.0011	0.0011					
0.2	BB1	$\theta_{23\|1}$	0.1200	0.1438	0.1403	0.0238	0.0013	0.0019	0.2042	0.2041	0.0042	0.0002	0.0002
		$\delta_{23\|1}$	1.1792	1.1753	1.1741	-0.0040	0.0007	0.0007					
0.2	t5	$\rho_{24\|1}$	0.3090	0.3114	0.3122	0.0024	0.0004	0.0004	0.2018	0.2023	0.0018	0.0002	0.0002
		$\nu_{24\|1}$	5.0000	5.6233	5.3799	0.6233	0.9434	1.3319					
0.2	BB7	$\theta_{34\|12}$	1.1792	1.1858	1.1854	0.0066	0.0007	0.0008	0.2004	0.1999	0.0004	0.0001	0.0001
		$\delta_{34\|12}$	0.2900	0.2955	0.2934	0.0055	0.0011	0.0011					

Table C.46: Results of the stability and robustness tests of the copula combination BB1,N,BB7,BB1,t5,BB7 in different scenarios (n=500, r=100) (part 1).

			Copula parameter(s)						Kendall's τ				
τ			θ	$\bar{\theta}$	$\bar{\theta}_{0.05}$	$\hat{b}(\bar{\theta})$	$s^2(\bar{\theta})$	$\widehat{mse}(\bar{\theta})$	$\bar{\tau}$	$\bar{\tau}$	$\hat{b}(\bar{\tau})$	$s^2(\bar{\tau})$	$\widehat{mse}(\bar{\tau})$
0.8	BB1	θ_{12}	6.4800	5.8356	5.9037	-0.6444	0.0313	0.4465	0.7957	0.7973	-0.0043	0.0000	0.0001
		δ_{12}	1.1792	1.2563	1.2533	0.0770	0.0012	0.0071					
0.8	N	ρ_{13}	0.9511	0.9480	0.9498	-0.0031	0.0000	0.0000	0.7945	0.7975	-0.0055	0.0001	0.0001
0.8	BB7	θ_{14}	1.1792	1.1641	1.1571	-0.0152	0.0034	0.0036	0.7912	0.7945	-0.0088	0.0001	0.0002
		δ_{14}	8.4000	8.0879	8.1911	-0.3121	0.1929	0.2903					
0.2	BB1	$\theta_{23\|1}$	0.1200	0.1236	0.1189	0.0036	0.0014	0.0014	0.2088	0.2066	0.0088	0.0003	0.0004
		$\delta_{23\|1}$	1.1792	1.1940	1.1917	0.0147	0.0006	0.0009					
0.2	t5	$\rho_{24\|1}$	0.3090	0.3237	0.3208	0.0147	0.0006	0.0008	0.2103	0.2081	0.0103	0.0003	0.0004
		$\nu_{24\|1}$	5.0000	5.6173	5.2599	0.6173	1.5309	1.9120					
0.2	BB7	$\theta_{34\|12}$	1.1792	1.2008	1.1956	0.0215	0.0011	0.0016	0.2068	0.2047	0.0068	0.0002	0.0003
		$\delta_{34\|12}$	0.2900	0.3016	0.2969	0.0116	0.0014	0.0016					
0.2	BB1	θ_{12}	0.1200	0.1184	0.1183	-0.0016	0.0000	0.0000	0.2027	0.2025	0.0027	0.0000	0.0000
		δ_{12}	1.1792	1.1843	1.1837	0.0051	0.0000	0.0001					
0.2	N	ρ_{13}	0.3090	0.3114	0.3113	0.0024	0.0000	0.0000	0.2016	0.2015	0.0016	0.0000	0.0000
0.2	BB7	θ_{14}	1.1792	1.1802	1.1801	0.0009	0.0000	0.0000	0.1981	0.1980	-0.0019	0.0000	0.0000
		δ_{14}	0.2900	0.2921	0.2916	0.0021	0.0000	0.0000					
0.8	BB1	$\theta_{23\|1}$	6.4800	5.9990	6.0000	-0.4810	0.0000	0.2314	0.7975	0.7971	-0.0025	0.0000	0.0000
	BB1	$\delta_{23\|1}$	1.1792	1.2369	1.2338	0.0577	0.0006	0.0040					
0.8	t5	$\rho_{24\|1}$	0.9511	0.9492	0.9490	-0.0019	0.0000	0.0000	0.7964	0.7960	-0.0036	0.0000	0.0000
		$\nu_{24\|1}$	5.0000	5.5289	5.3477	0.5289	0.4286	0.7084					
0.8	BB7	$\theta_{34\|12}$	1.1792	1.1314	1.1095	-0.0479	0.0061	0.0083	0.7891	0.7910	-0.0109	0.0001	0.0002
		$\delta_{34\|12}$	8.4000	7.8995	7.9470	-0.5005	0.2365	0.4870					

Table C.47: Results of the stability and robustness tests of the copula combination BB1,N,BB7,BB1,t5,BB7 in different scenarios (n=500, r=100) (part 2).