PRIVATE EQUITY INVESTMENTS

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RISK-RETURN PROFILES OF COMPLEX INVESTMENT STRATEGIES

DIPLOMA THESIS

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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

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List of abbreviations

Capex	Capital expenditure
CAPM	Capital asset pricing model
CF	Cash flow
CFaR	Cash flows at risk
CFV	Cash flow value
COGS	Costs of goods sold
DV	Debt value
EBIT	Earnings before interest and tax
EBITDA	Earnings before interest, tax, depreciations and amortisations
EPS	Earnings per share
EV	Enterprise value
FCFE	Free cash flows to equity
FCFF	Free cash flows to the firm
IRR	Internal rate of return
i.i.d.	Independent and identically distributed
IV	Investment value
LBO	Leveraged buy out
NPV	Net present value
Recap	Recapitalisation
SDE	Stochastic differential equation
SG&A	Selling, general and administrative expenses
SML	Security market line (capital asset pricing model)
TV	Terminal value/ exit value
VBA	Virtual basic for applications
WACC	Weighted average cost of capital

Table of Symbols

A_t	Adjustments to the enterprise value at time t
α	Investment stake in the target company
b_t	Expected cash flow add-on at time t
$\hat{b}_t$	Expected multiple add-on at time t
$\beta^{levered}$	Beta factor of a levered company
$\beta^{unlevered}$	Beta factor of unlevered company
C_t	Instantaneous cash flow add-ons at time t
DV_t	Market value of debt at time t
Δc_t	Instantaneous changes in the cash flow process
$\hat{d}_i$	Percentage of debt financing of the i'th jump
δ_i	Sign of the i'th jump
ε_t	Error term for cash flow at time t
$\hat{\varepsilon}_t$	Error term for multiple at time t
E_t	Free cash flow to equity at time t
$E_t^{'}$	Cash flow process composed by recurring cash flows, extracted by the investors, and instantaneous add-ons at time t
E_t^{df}	Earnings level at time t, at which the investment is considered as defaulted
EV_t	Enterprise value at time t
EQV_t	Market value of equity time t
$exp(\lambda)$	Exponential distribution with intensity λ
$\mathfrak{F}_t$	Filtration
γ	Degree of risk aversion
h	Exogenous hurdle rate of return
J_t	Level of the jump process at time t
g	Waiting time for the i'th jump
K	Number of assets
κ	Speed of reversion of multiple process
l_t	Debt to total capital at time t, also referred to as leverage level
λ_t	Growth rate of cash flows at time t
$LN(\mu, \sigma^2)$	Log-normal distribution with mean μ and standard deviation σ
m_t	Multiple or multiplier at time t
$\overline{m}_t$	Expected multiple level at time t
n	Number of jumps
η	Percentage of cash flows that is used for debt redemption
p	Probability of a positive jump sign
μ_i	Expected jump size of the i'th jump
Ω	Probability space
P_0	Enterprise value at time 0

$\mathbb{P}$	Probability measure
$[x, y]_t$	Quadric variation between x and y up to time t $\int_0^t \sigma_s^x \sigma_s^y \, ds$
ρ_{ij}	Correlation of the i'th BROWNIAN motion with the j'th BROWNIAN motion
r_t	Cost of capital at time t
r_e	Cost of equity
r_f	Risk free rate of return
r_m	Market rate of return, derived by a predefined index
σ	Standard deviation
σ_E	Standard deviation of free cash flows to equity
σ_i	Expected standard deviation of the i'th jump
$\sigma_{Industry}$	Industry specific cash flow risk
σ_m	Market specific cash flow risk
$\sigma_{Multiple}$	Standard deviation of the multiple process
Σ_t	Time dependent covariance matrix of BROWNIAN motions
t	Time
T	Maturity
$U(\cdot)$	Utility function
V_t	Cash equity value/ basis at time t
W_t	BROWNIAN motion at time t

„We need to support companies, that create sustainability and have an eye on the interests of their employees, rather than supporting irresponsible grasshopper swarms, that measure success on quarterly basis, suck off assets and let companies fall bankrupt, when they finished grazing.“

Franz Muentefering, SPD[1]

1. Introductory word and methodology

Two worlds, which could not be more different, collided in the Swabian small town Metzingen - the workforce of the fashion label Hugo Boss and the private equity investor Permira. In May 2008 the new owner Permira announced a dividend, including a debt-financed extra dividend, of 500 million Euro in the annual general meeting, decreasing the equity stake from 50% to around 25%. Most of the successful board of directors have already left the company. This kind of innovative financial engineering hits recently the headlines. Private equity firms are blamed for extracting quickly all of a target company's cash, and sometimes for even going further by asking a target company to incur additional debt to be able to pay an additional dividend, and thus driving the target company into bankruptcy. Take, for instance, the history of the automotive company Autoteile Unger (ATU), in which the private equity investors KKR and Doughty Hanson invested. After two mild winters the optimistic sales targets for winter tires could not be fulfilled. This combined with the enormous debt obligations resulted in a failure of the narrowly calculated investment plan. ATU could only be prevented from bankruptcy by massive cash injections.

In this thesis complex investment strategies in terms of risk and return are examined. Cashing out the investment allows the private equity company to quickly achieve returns for their investors. Thus upcoming risks from restructuring programmes and environmental changes can be mitigated by early payments to investors. As financial sponsors are usually judged by the internal rate of return (IRR)

[1] Translation word by word from SPD (2005) „Programmheft I. Tradition und Fortschritt", p. 18

of their investments[2], they prefer risk-free early cash flows to an uncertain value enhancement, whose net present value is low due to a high discount factor.

In order to provide a decision support for different innovative financial engineering strategies[3] we will analyse the investment strategy in terms of discounted cash flows. Especially changes in the capital structure, which affect the risk of investments, are used as levers to map a certain investment strategy to a stochastic model. This stochastic company valuation model is carried out to a multivariate framework, to be able to consider a set of investments – a fund – or different business units of a conglomerate. Within this flexible stochastic model, on the basis of available historic data samples and expectations raised in the Leveraged Buyout (LBO) model Monte-Carlo techniques are used to simulate a large number of sample paths, providing us with an understanding of the resulting density of the net present value. We select different risk measures like Cash Flows at Risk (CFaR), the probability to default, or the probability to fall below a specified target return, to measure the risks of investment strategies or investments.[4] A comparison of different investment strategies within a risk-return profile (refer to figure 1), defined by available historic information up to assessment and future expectations, allows us to evaluate each investment strategy.

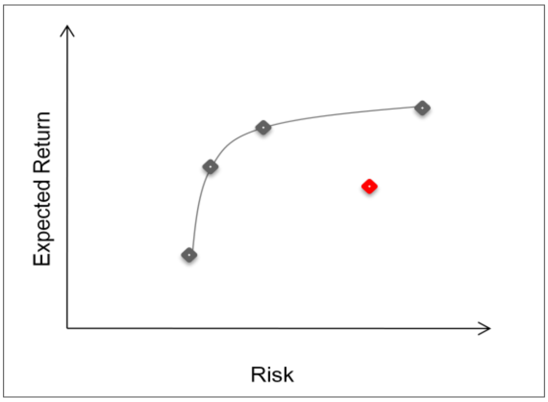

Figure 1: Illustrative risk-return profile, showing a dominated investment strategy (red)

[2] Cf. BERG (2005) What is strategy for buyout associations, p. 42
[3] We refer to JAFFER (2000) An Overview of Alternative Investment Strategies
[4] For a discussion of different risk measures, one is referred to ALBRECHT/ MAURER (2002) Investment- und Risikomanagement, pp. 112-125

Thereby we are able to support financial sponsors in their selection process of exclusive investments as well as in their selection process of investment strategies.

The thesis is structured as follows. Section 2 steps back to the main ideas of business valuation to value a typical investment in a company. Section 3 discusses the value drivers for private equity investments, which are mapped into a discrete time valuation in section 4. The discrete time approach provides the motivation for our new continuous time model defined in Section 5. In this most important part of the thesis we develop an univariate and multivariate setting, that accounts for the risk associated with leverage of the investment as well as for bankruptcy. Section 6 arranges the continuous time model for application software, closing with an univariate case study of different strategies in section 7.

2. Key aspects of a company valuation and overview of previous work

In this short chapter we only want to emphasise the model's underlying techniques instead of repeating and discussing each idea or choice.[5] Further, it is the goal of this chapter to position the model suggested later in this thesis in the context of the existing literature.

Private equity investors value possible investments with the same techniques which are used for portfolio decisions of liquid financial assets: cash flow analyses, IRRs and multiples.[6]

Companies can be valued in different ways. One can think about structural models like asset-based and income-based approaches, or about reduced form models like a market approach.[7] As these approaches are considered as standard knowledge within business valuation we will not describe each approach in detail. We will instead set a hybrid framework of an income approach and a market approach, and afterwards adjust this framework to private equity investments. In doing so, we will restrict ourselves to explain the pillars of our framework by focusing on one share of a company bought at time 0 and held up to time T.

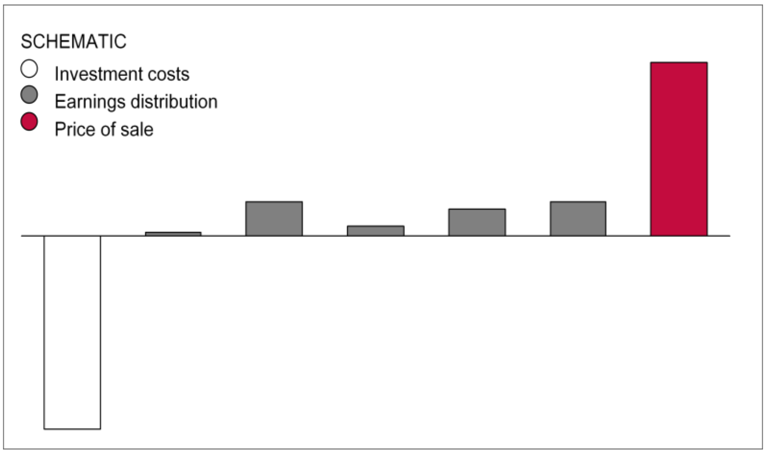

Figure 2: Schematic cash flows over time

[5] For insights and comparisons of different valuation methods, we refer to FERNANDEZ (2002) Valuation Methods and Shareholder Value Creation, Chapter 1

[6] Cf. RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 7

[7] See JARROW (2004) Structural Versus Reduced Form Models: A New Information Based Perspective, p. 2

Generally, referring to figure 2, cash flows over time are typically composed of the initial investment at time 0, the value derived from earnings distribution and the price of sale, which is realised at maturity T – the exit value.

As Cash Flows (CFs) occur at different times, shareholders seek for a comparable decision guidance, which is ensured by discounting cash flows with a risk-adjusted rate of return $r: [0;T] \to \mathbb{R}_+$ to present value.[8] Thus in a discounted setting the Net Present Value (NPV) of the earnings distribution $t = 1, \dots, T$ is given by:

$$\sum_{i=0}^{t} \frac{CF_i}{(1+r)^i}$$

For our setting it is feasible to distinguish between ordinary cash flows and cash flows derived by one-off payments, thus for $t = T$:

$$NPV_T = -Investment\ value_0 + \sum_{i=0}^{T} \frac{CF_i}{(1+r)^i} + \frac{Price\ of\ Sale_T}{(1+r)^T}$$

The price of sale is identified by a comparable transactions analysis, which is a specific market approach. In practice[9] this market approach is measured by a multiplier, referred to as a multiple, times a referent.[10] Hence for us it seems feasible to model the present value for $t = T$:

$$NPV_T = -Investment\ value_0 + \sum_{i=0}^{T} \frac{CF_i}{(1+r)^i} + \frac{referent_T \times multiple_T}{(1+r)^T}$$

The exit value or price of sale, based on a multiple is a simplified pricing procedure and accounts for the market value in comparison to a peer group.[11] Note that the referent at time t can be negative, thus the price of sale at a positive multiple may become negative. This is not a very meaningful feature as the company's share still has a certain net asset value, or just devolves to the creditors without further obligations for the equity investor. We postpone this problem until section 5 and will instead give a short review on existing stochastic company valuation models.

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[8] Note that we operate here on the premises that all capital gains are reinvested, that the money market is complete and frictionless, thus credit and debit interests are equal; these assumptions are obviously not met in real markets. Note that HIRSHLEIFER, DEAN and others found evidence that small market incompleteness implicates that the net present value approach is in general not appropriate to evaluate different investments. Still for our purpose this approach is most suitable as we are interested in net present value probability distributions

[9] According to expert interviews; also refer to IDW (2004A) Item. 154-155 as well as Item. 175-180

[10] E.g. Earnings before Tax, Interest, Depreciations and Amortisations (EBITDA); Earnings per Share (EPS); Net income

[11] See MEYER (2006) Stochastische Unternehmensbewertung. Der Wertbeitrag von Realoptionen, p. 63

Valuing publicly listed companies with stochastic models has not been in the research focus of business valuation for a long time. Since the pioneering work of SCHWARTZ/ MOON[12] various stochastic models have been developed. They all have in common that revenues and costs follow specified stochastic processes. Recent work by BOECKER[13] focuses directly on a bank's earnings process to deduct an adequate economic capital calculation for business risk in the framework of Basel II. In addition, various stochastic models have been developed to answer specific problems, e.g. the MERTON[14] model is deployed to measure credit risk of a company, or real options models are developed to judge the profitability of an investment.[15]

Previous models can only be applied for companies in a stable environment. As we deal within this thesis with private equity investments that involve operational changes or changes in the capital structure, these models do not adequately map the cash flows and risks of private equity investments. We will thus develop an adequate model, which is based upon stochastic cash flows to equity. Thereby the capital structure will be the adjustment screw to map different risks occurring from different investment strategies. Hence, as a first step we have to understand the value of private equity investments, their value drivers, and their associated risks.

[12] SCHWARTZ/ MOON (2000) Rational Pricing of Internet Companies
[13] BOECKER (2008) Modelling and Measuring Business Risk
[14] See MERTON (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, pp. 449-451
[15] Cf. for instance MEYER (2006) Stochastische Unternehmensbewertung: Der Wertbeitrag von Realoptionen

3. Main features of private equity investments

In this section we will provide a brief introduction to private equity investments.[16]

3.1. Structure of private equity investments

The typical set-up of a private equity investment is as follows – cf. to figure 3:

The private equity firm, also referred to as general partner in a limited partnership structure, draws down money from the investors, also known as limited partners. Limited partners only act as financiers without the right to supervise single investments.[17] In contrast, general partners are managing actively the investments. Limited partners compensate the general partners with a two-part fee, which is composed of a fixed management fee and a performance related component.[18]

General partners usually found a new company, referred to as NewCo. General Partners purchase via NewCo a controlling stake in a company from its owners for a limited time, usually financed through a combination of equity and debt.[19] Thereby the new owners discharge the financing structure. The debt to equity ratio is often considered as a lever of performance, by exploiting the leverage effect on the equity return:[20]

$$r_{equity} = r_{total\ capital} + (r_{total\ capital} - r_{debt})\frac{debt}{equity}$$

[16] For basics in private equity we refer to RUDOLPH „Funktionen und Regulierung der Finanzinvestoren“ chapters 1 and 2
[17] See CUMMING/ JOHAN (2007), p. 3222
[18] Cf. RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 2f
[19] See BERG (2005) What is strategy for buyout associations, p. 9
[20] Cf. RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 9

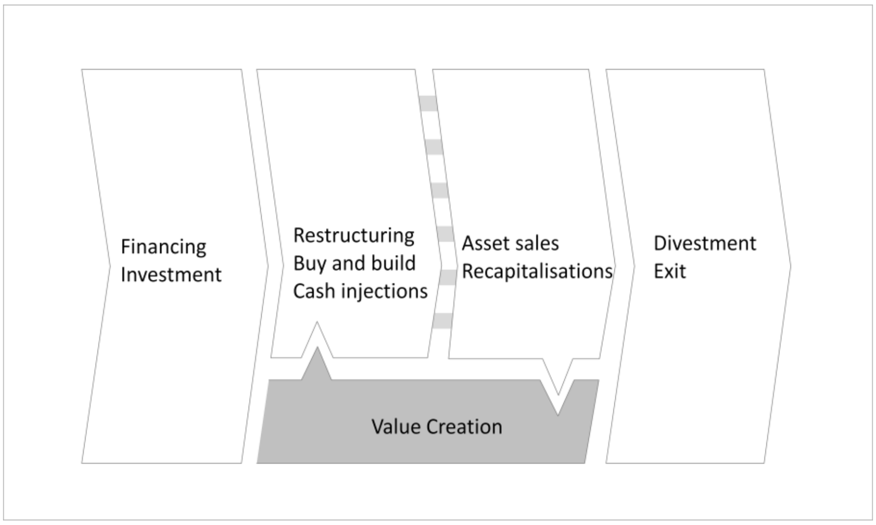

Figure 3: Stages of private equity investments

From this point onwards the private equity firms dynamically determine and implement the optimal investment programme, to fulfil or exceed[21] the announced IRR target[22], which was committed to attract limited partners.[23] Therefore we will also measure the performance of the private equity firm by the cash equity basis of the investment and the implied IRR.[24] Investors have different instruments to improve at any given situation their future investment programme; to give an overview these are operational excellence programmes, asset sales, recapitalisations as well as further cash injections, if the company is close to become bankrupt.[25] Hence, private equity investments are not static investments, as they require a dynamic management throughout the investment horizon.

On this account we will develop a dynamic model, using collected information and expectations for remaining time to maturity. Thus, for any point in time we need to analyse a set of investment programmes or decisions in terms of their risk-return profile, compare figure 4:

[21] If the performance related compensation fee, referred to as carry, may be tied to the announced IRR
[22] IRR on target level net of fees
[23] Refer to the Appendix A1: Financial Times Germany 10.02.2008 and RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 2
[24] In the following we set the implicit IRR on target level gross of fees
[25] Cf. BERG (2005) What is strategy for buyout associations, pp. 99-129

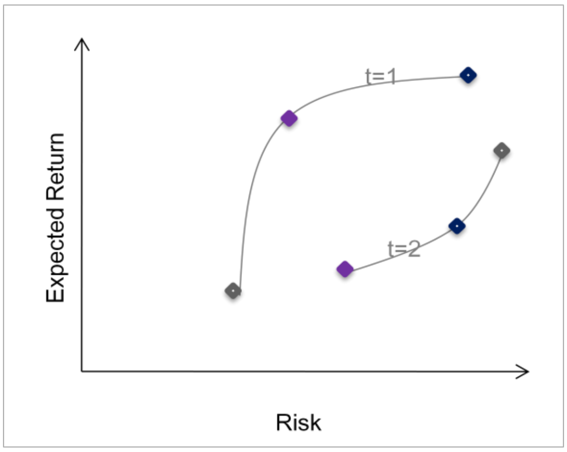

Figure 4: Analysis of investment programmes at different point in times

At different assessment times different investment decisions can be optimal and need not be unique. Thus, before being able to develop a certain risk model, we need to identify and understand the value drivers of the investment in terms of their influence on the cash equity basis and its distribution – the risk of the investment.

3.2. Value drivers

For private equity investors BERG suggests two different ways to achieve high value increments up to the exit:[26]

(1) The first one aims to increase the value of the equity stake of the investment and is referred to as buy-to-sell approach. The investors pursue this strategy by trying to increase the underlying earnings variable, e.g. Free Cash Flows to the Firm (FCFF). At a constant multiple at time T, an increase in FCFF increases the enterprise value (EV). A passive strategy is to cash in on an increase of the referenced multiple, which cannot be affected by the investors. As the amount of debt is constant over an increase in EV, the debt-equity ratio levers an increase in EV only to equity and may thereby lead to excessive returns.[27]

(2) The second approach is to increase the net present value derived by cash flows up to maturity. This can be achieved by pre-drawing cash flows. Later

[26] See BERG (2005) What is strategy for buyout associations, p. 123
[27] We will discuss the irrelevance theorem of capital structure in section 3.2.3 in detail

cash flows suffer from higher discount factors.[28] For example one can pre-draw cash flows by liquidating assets, or incurring additional debt to be able to pay an extra dividend, also referred to as recapitalisation[29] (recap).[30]

In practise, the foregoing approaches are combined to accomplish the highest return. From this background, as one can deduce, there are four main levers to achieve growth in equity value.[31/32] The following figure 5 shows the empirical relevance of each based on a sample of 32 private equity companies:[33]

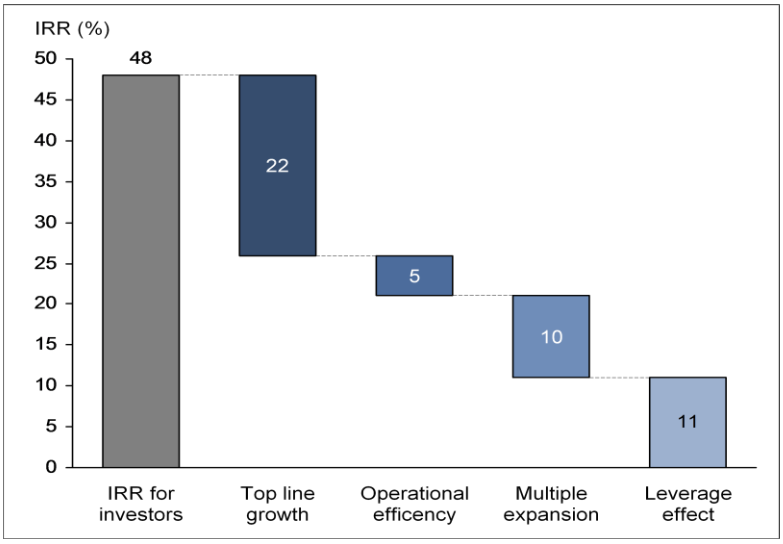

Figure 5: Fundamental value drivers for private equity investments, the terminology can be found in the subsequent sections

[28] Meaning only a time effect; excluding a yield shift in the sense that mature investment bear less risk and thus are discounted at lower risk adjusted discount rates

[29] Here we restrict a recap to this simplification. Usually, recaps are carried out by founding special purpose entities. But as these special purpose vehicles act under the parent trust of NewCo, we will not need a special treatment of cash flows; cash flows from recaps can just be added to the cash flows of NewCo

[30] In an extreme use this will reduce the equity stake in the investment close to zero

[31] According to various expert interviews in the private equity industry. Annotation: Interest rate risk is not decision-relevant as debt is collected by investment banks, and in general transacted as fixed interest payments, and is within the risk scope of lenders. Currency risk is also not taken into account, this is because private equity funds typically invest in the same currency as the fund is raised, and if else currency risks are usually hedged

[32] See also BCG (2008) The Advantage of Persistence, pp. 12-14

[33] Source BCG (2008) The Advantage of Persistence, p. 12 – the analysis is based on financial data from 32 private equity companies in the portfolios of seven European private-equity firms; the analysis compares EVs at the time of purchase with the value realized upon exit. The y-axis shows the contribution of each factor in percentage points of the IRR. We note that the specified values depend on the drawn sample and seem to overestimate the actual average IRR

3.2.1. Top line growth

One way to increase cash flows is to increase revenues at constant gross profit margins. Possible strategies are to increase the market penetration[34], increase regional coverage, offer additional services or products, or to optimise pricing.

To understand why an increase in revenue affects cash flows positively consider a growing company, whose operating position is measured by the economic figure Earnings before Interest, Tax, Depreciations and Amortisations (EBITDA). If the target company is able to increase revenues it usually may exploit economies of scale or economies of scope. Assuming a constant fixed cost basis over time, the total cost per units is, generally, in a long term perspective not increasing. Even at a constant cost level per unit at a constant fix cost basis, economies of scale increase absolute EBITDA.[35] A higher absolute EBITDA will result in a higher free cash flow to equity. Therefore, revenue growth leads to a growth in discounted cash flow value and, if we assume a constant exit multiple, also to an increase in enterprise value.

3.2.2. Operational efficiency

In contrary to top-line growth, operational efficiency strategies try to reduce cost positions of the investment at a constant revenue level. Investors are efficiency seeking along the value chain trying to reduce Selling, General and Administrative Expenses (SG&A) or Costs of Goods Sold (COGS). Their success can also be measured by EBITDA, if we assume a constant revenue level. Possible strategies are, for instance overhead optimisation (SG&A), exploitation of synergy effects (SG&A/COGS)[36], optimisation of processes (COGS), improvement of production technology (COGS), shift of production to low cost countries (COGS), and outsourcing (COGS). These strategies lead to higher net profits and, thereby affect both the value derived from free cash flows as well as the total EV.

[34] RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 4
[35] Also refer to the definition of free cash flows in section 3.2.3
[36] See BERG (2005) What is strategy for buyout associations, p. 14

3.2.3. De-leverage

In Europe 2006, private equity firms have invested more than 80% of raised capital into so called Leveraged Buyouts.[37] Private equity investors draw a high debt-equity ratio upon their investments to profit from the leverage effect. The leverage effect increases the return on equity over an increase in debt-equity ratio, if interest on debt is smaller than return on assets.[38] This mechanism can be followed by recalling the interaction of return on equity, return on debt and return on total assets:[39]

$$r_{equity} = r_{total\ capital} + \left(r_{total\ capital} - r_{debt}\right)\frac{debt}{equity}$$

Another intuition can be derived if we consider a debt repayment. Here, two effects appear:

(1) For the first we bear in mind the relationship between EV, debt value (DV) and equity value (EQV) of a company at time t:

$$EV_t = DV_t + EQV_t$$

The equation shows that debt repayments, at a constant enterprise value, increase the implied equity value.

(2) Debt repayments, due to lower interest payments, increase future free cash flows to equity. To better scrutinise this mechanism consider the following relationship:[40]

+	Earnings before Interest, Tax, Depreciations and Amortisations (EBITDA)
–	Depreciations and Amortisations
–	Cash Interest
–	Cash taxes
+/–	Net working capital changes
+/–	Capital expenditure (Capex)
+/–	Other long-term assets
	Total cash available for debt repayment (Free cash flow to the firm)

The private equity investor will be able to achieve higher cash flows, which affect the present value of the free cash flows positively.[41]

[37] See EUROPEAN CENTRAL BANK (2007), p. 99 and GERMAN PRIVATE EQUITY AND VENTURE CAPITAL ASSOCIATION (2007), p. 110
[38] Cf. RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 9
[39] Cf. DIEM (2004) Finanzierung von Leveraged Buy-outs, p. 5
[40] See COPELAND et al. (1995) Valuation – Measuring and managing the value of companies, p. 172f

Also keep in mind that de-leverage can as well occur the other way round, referred to as re-leverage. General partners raise additional debt and thereby decrease the percentage of the equity stake within total capital. Increasing debt obligations analogously affect future cash flows negatively. However, additional cash provided by debt-financing can be used for recaps.[42] Recaps are mostly accompanied by extra dividends, which constitute a capital pre-drawing. Hence we have two opposite effects; the sign of the total effect is ambiguous. Especially recaps[43] are, besides IRR improvement, often motivated by an effect of risk reduction. This effect occurs from a cut-off of the profit-loss distribution, due to a pre-drawn payment representing a reduction of total equity invested. The different motivations for investment strategies are exemplarily illustrated in figure 6: A buy-low and sell-high strategy (exit scenario) is held against a strategy with a recap scenario in $t = 1$, in terms of their distribution of the aggregated free cash flow to equity. Contrary to the strategy with a recap, the risk effect of the buy-low-sell-high strategy is unambiguous. As information and cash flows are gained, expectations are more precise, and thus the investment loses risk over time.

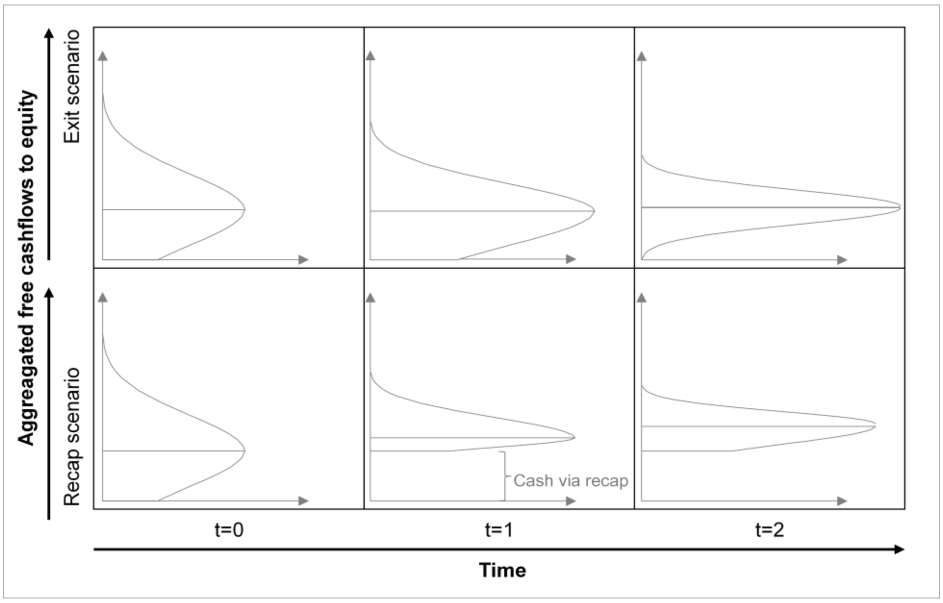

Figure 6: Comparison of cash flow distributions over time of a buy-low and sell-high strategy (exit scenario) and a recap strategy (recap scenario)

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[41] Note that a debt repayment does not increase EBITDA and thus the valuation of the company on basis of $EV = EBITDA \times multiple$. But debt redemption (de-leverage) is decreasing the net debt position and thus increasing the percentage of the equity stake, in other words: earnings contributable to equity increase.

[42] Recap is usually accomplished by an extra dividend that additionally reduces cash available for debt repayment

[43] According to expert interviews

In conclusion, admittedly de-leveraging and re-leveraging affect the risk of an investment. One can derive by the aforementioned leverage formula the risk[44] of the remaining equity stake:

$$\sigma_{equity} = \sigma_{total\ capital}\left(1 + \frac{debt}{equity}\right)$$

A high leverage implies a high risk of the equity stake, assuming that debt is riskless, and hence the investor faces a higher probability to lose the invested capital.[45] As the chance of return and the risk of loss conflict, the irrelevance thesis of the total debt structure seems also to hold in this dynamic structure. If we suppose the assumptions of the irrelevance thesis of MODIGLIANI/ MILLER[46] to hold in this dynamic structure[47], then the market value of the investment cannot be increased by optimising the leverage of the investment.[48/49] Thus, the firm's dividend policy is irrelevant.[50]

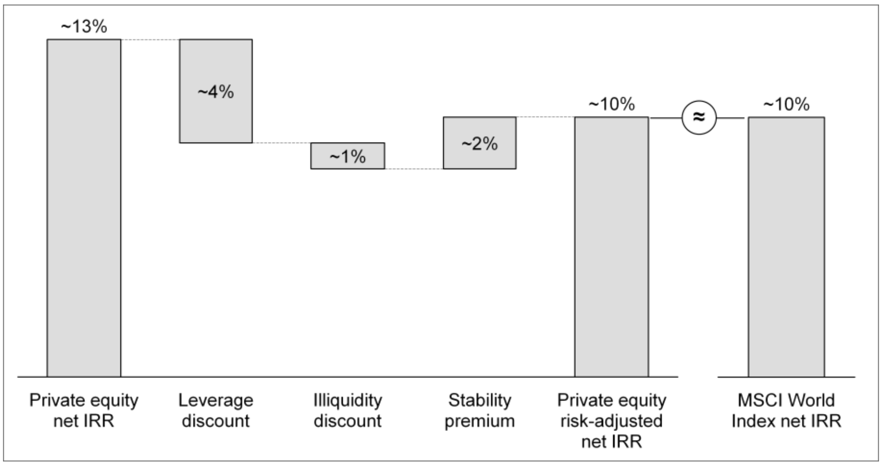

Figure 7: Adjusted for risk, private equity's returns are roughly equivalent to those of the public market[51]

[44] For a definition of possible risk measures see chapter 6.4
[45] German Central Bank (2007) p. 17
[46] See MODIGLIANI/ MILLER (1958) The Cost of Capital, Corporation Finance and the Theory of Investment, pp. 261-265
[47] MODIGLIANI/ MILLER argue in a static setting that the EV is independent of the underlying capital structure at time 0. A debt financed recap at time t is equivalent to a new financing and investment decision at time t accompanied by an extra dividend t. The dividend at time t constitutes equity, that is converted to debt at time t. Thus the MODIGLIANI/ MILLER theorem holds as well at time t, especially $\forall 0 \leq t' \leq t$
[48] Cf. RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 9 and see RUDOLPH (2006) Unternehmensfinanzierung und Kapitalmarkt, pp. 341-350
[49] Cf. to Figure 8
[50] MILLER et al. (1961) Dividend Policy, Growth, and the Valuation of Shares, p. 412
[51] Source BCG (2008) The Advantage of Persistence, p. 14f

As is generally known, capital markets are in reality not complete. The following paragraphs constitute alternative explanations motivated by RUDOLPH, why private equity firms seek to take advantage of the capital structure of an investment.[52]

(1) Consider an edge on information of the invested private equity firms. This market imbalance can be capitalized by arbitrage. Taking the expectation for the leverage formula above we have:

$$\mathbb{E}(r_{equity}) = \mathbb{E}(r_{total\ capital}) + \mathbb{E}(r_{total\ capital} - r_{debt})\frac{debt}{equity}$$

If the financial sponsor is able to implement operative and strategic value drivers, or to select undervalued companies, he can anticipate a gain in return on total assets. Hence, he anticipates an increase in return on equity[53], which enables the investor to increase the expected rate of return on equity by increasing the leverage of the investment. Recaps can thus be regarded as a dynamic instrument to capture an additional edge on information resulting from internal insights.

(2) Another reasoning rests on the interest subsidy, also referred to as tax-shield, which may be used to decrease the tax burdens and thus increases free cash flow to equity.

(3) A further explanation to justify high leverage rates is to benefit from the control function and disciplinarian actions of debt.[54] A high debt financing reduces the agency costs due to a recurrent control of outside creditors.[55]

(4) A last reasoning is discussed in the literature under the terminology "gambling for resurrection". The specific remuneration of financial sponsors is a classical risk incentive problem.[56] Generally the carried interest is only accessible if the general partner manages to beat the hurdle rate. Thus, besides the management fee, the general partner benefits only from an excess to a hurdle rate. This compensation structure is similar to the payoff structure of a call position of an option. The holder of a call option benefits from an increase in the underlying volatility, as the probability of the call to

[52] For following arguments see RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, pp. 10-14

[53] He anticipates a reshaping of the distribution of the net present value

[54] JENSEN (1986) The Agency Costs of Free Cash Flow: Corporate Finance and Takeovers, p. 325

[55] JEPSEN (2007) Die Entlohnung des Managements beim (Leveraged) Management Buy-Out, p. 15

[56] See BREUER (1998) Finanzierungstheorie. Eine systematische Einführung, p. 21

be in the money increases – even at same level of expected earnings.[57] One is referred to figure 8, which shows the Vega[58] of a call option with strike 25, and illustrates the options sensitivity to changes in volatility; one should note that the Vega is always positive, and thus higher volatilities capitalise in a higher present value.

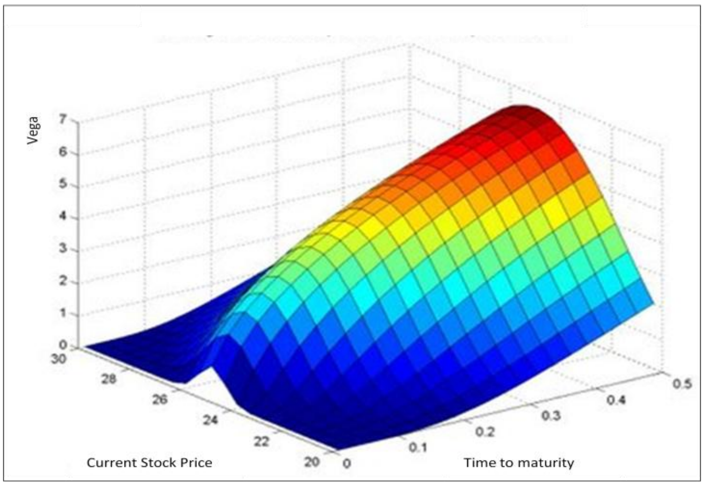

Figure 8: Call Vega versus time to maturity and initial stock price

Assigning this connection to LBOs, we conclude that for general partners it may be desirable to increase the risk of the investment to a critical point.

3.2.4. Multiple expansion

Financial sponsors seek to benefit from an improvement of the multiplier (gap between investment multiple and exit multiple), referred to as multiple expansion. As multiple expansion is influenced by the environment (e.g. industry sector, gross domestic product, population growth) only; the investors are restricted to cash in on smart timing.[59] Multiple expansion can be regarded as a passive value driver, that

[57] Cf. FAN (2001) On the Relationship between Call Price and the Probability of the Call Ending in the Money, p. 3f. Note that FAN proofs that a at the money call's probability to be in the money increases as the volatility increases. For the out of the money case there exists a range to a critical point where the probability to be in the money increases

[58] The Vega of a call option states its value sensitivity to changes in the underlying volatility

[59] See RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 4. This structure is similar to an American call option with the investment multiple as strike and total investment horizon as maturity. In this paper, we will restrict to the finding from the BLACK/ SCHOLES/ MERTON model for American call options. As it is always optimal to exercise an American call option at maturity (if the underlying does not pay any dividends), we conclude that it must be optimal to sell the investment at maturity. As the interaction of multiple, earnings and default is ambiguous and too complex to deal within this work, we assume that the findings from the

cannot be controlled by the general partners. The multiple is supposed to capture what investors actually pay for businesses in the same industries.

WIBEL found evidence that there is very little correlation between a company's earnings growth and share price appreciation[60] $(earnings \times multiple)$. Hence, we assume the investor's ordinary cash flows and the associated multiple are independent.

3.3. Default event

For the private equity firm there is always a trade off between benefiting from a high leverage or high recaps and the associated risk, due to higher debt obligations, which may cause bankruptcy. Within this section we motivate the implementation of a default event within our model.

As private equity investments are to some extent large late-stage venture capital investments[61] - compare figure 9 - we deduce that private equity investments also bear the same risk –albeit smaller.

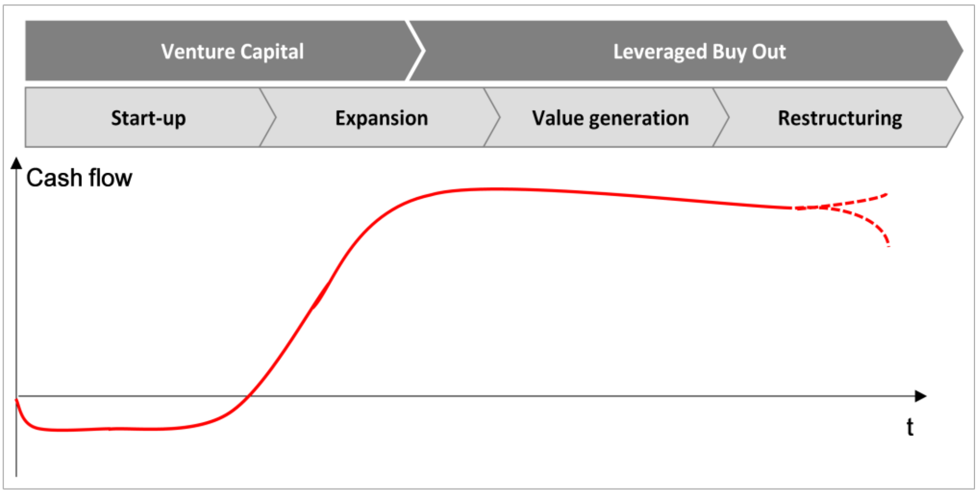

Figure 9: Financing stages[62]

As the private equity industry is not disclosing defaulting investments, and as there are no empirical studies covering single investments of private equity firms, we

BLACK/ SCHOLES/ MERTON model for asset pricing hold. Further research for dealing with this issue and modelling this issue is essential, when it comes to investment pricing

[60] According to WIBEL, investment advisor at Foothills Asset Management. Cf. Appendix A2

[61] BERG (2005) What is strategy for buyout associations, p. 11

[62] Compare to GEORGIEFF/ BIAGOSCH (2005) Finanzierungsinstrumente von Finanzinvestoren, p. 173

access the data published by COCHRANE[63], who provides default and performance data for venture capital investment vehicles:

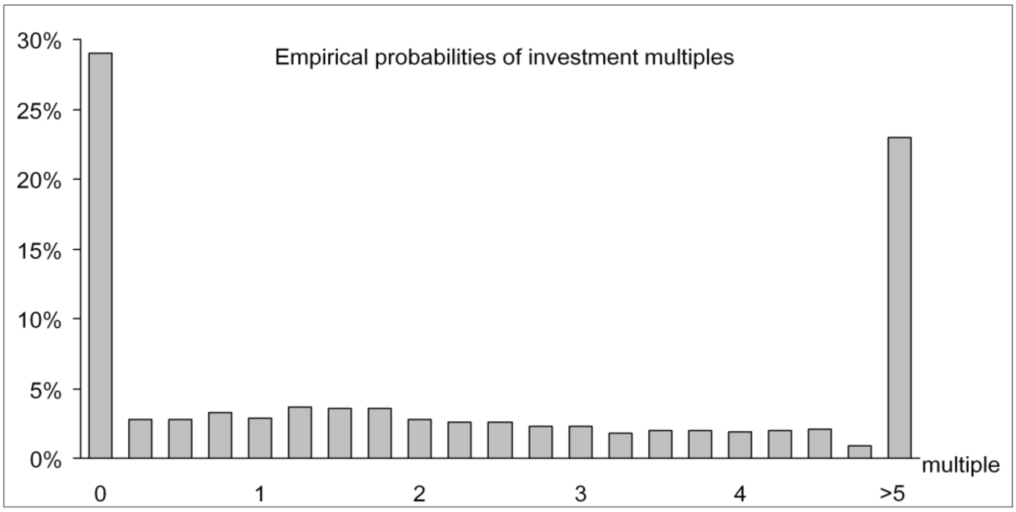

Figure 10: The risk profile of venture capital investment vehicles

Thus, we need to introduce, for any time t, a cash flow level at which the interest payments, occurring from a leveraged buyout and debt-financed recaps, cannot be paid back. Falling below that level causes bankruptcy. The investment defaults, when its market value falls below a certain exogenously given threshold level or the value of its debt.[64]

Having identified the major value drivers, characteristics and strategies of private equity investments, we will now develop a discrete valuation methodology based upon the major framework of company valuation, which was introduced in section 2. Afterwards we will transfer this approach to a stochastic model in continuous time.

[63] See COCHRANE, J.H. (2001) The risk and return of venture capital investments, NBER Working Paper Series No. 8066, p. 38, Table 1

[64] ZHOU (1997) A jump-diffusion approach to modelling credit risk and valuing default able securities, p. 1

4. Discrete time approach

Within this section, we will consider a discrete time model; as we are familiar with business valuation on the discrete time grid $0,1,\dots,T$ we are able to relate the development of our approach step by step. For our purpose we assume a typical private equity investment to be structured as follows:[65]

(1) Financing stage at time 0. Private equity firms raise cash from their limited partners as well as debt from banks. With the resulting capital structure they acquire a certain investment stake $\alpha \in (0;1]$ in a target company. As a controlling stake, $\alpha > 50\%$ usually implies a control premium, and we deduct the control premium from the takeover price to obtain the EV of the investment. Deducting debt we get the equity stake, from which α is the equity stake of the investment.

(2) After the financing stage, we have the structuring stage for $t = 1,..,T$.[66] Equity holders participate in the form of dividends and recaps subject to their investment stake α. If investors invest in distressed companies, they face possible negative cash flows in the form of cash injections, otherwise the investment will fall bankrupt and the investment is terminated.[67]

(3) Lastly, there is the exit stage, in which the controlling stake is sold again. Equity holders receive the exit price minus market value of debt plus potential premiums paid. Thus, the private equity investor can access a stake of α in the exit equity value. The exit value is realised at maturity T by liquidating the investment (referred to as exit).

As a result, the net present value of the cash equity basis V_T is composed of the initial investment value at 0, the net present value of cash flows to equity arising at the discrete time grid $t = 1,\dots,T$, and the exit value at time T.

[65] See BERG (2005) What is strategy for buyout associations, pp. 122-128
[66] We assume that cash flows in T are not omitted
[67] Cf. to Appendix A1

4.1. Investment value

We refer to the actual EV of the target company at time 0 as $P_0 \in \mathbb{R}_+$. Since private equity companies are constrained to achieve a high rate of return, they need to affect the structure of the investment. As the impact depends on the investment stake, investors normally try to take over control of the target,[68] therefore a premium has to be paid. Thus we introduce a premium adjustment referred to as $A_0 \in \mathbb{R}$. [69] The total investment size is given by $P_o + A_o$.

As investors are, in general, interested in taking an equity stake $\alpha \in (0;1]$ in the target company, the investor consortium pays for the investment stake $\alpha P_o + A_o$.[70] As mentioned before[71] we are interested in describing the cash equity basis, thus we aim to derive the initial equity investment value. Denote $l_o \in [0;1[$ the proportion of debt of total capital within the investment stake at time 0, the investment value at time 0 is given by:

$$IV \coloneqq Investment\ value = -[\alpha(1-l_o)P_o + A_o]$$

The quantity l_o marks as well the initial value for a deterministic function $l: [0;T] \rightarrow [0;1[$, which measures the current ratio of debt to total capital at any time t. Thus l_t is a measure for the leverage of the investment at time t.

Unlike to the exit scenario we do not measure the initial investment value by $cash\ flows \times multiple$, since some private equity firms seek to invest in distressed companies with a negative initial cash flow level.[72]

[68] RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 2
[69] Note that A_o can also capture other adjustments to the market price than premiums, thus $A_0$is allowed to be negative
[70] As control premiums have to be paid regardless of the stake or financing
[71] Compare to introductory
[72] When it comes to a continuous time setting a deterministic one off payment is just linear transformation and thus is not influencing the probability distribution of the net present value. One can deal with this issue by modelling the investment value by $revenues \times multiple$, as revenues are always positive

4.2. Cash flow value

Define the earnings process as $E_t \coloneqq FCFE_t$ with $FCFE_t$ being the forecasted values for $t = 1, \dots, T$, in particular $FCFE_t$ are realised between $t-1$ and t.[73] Assuming that the private equity firm holds a stake α, we can deduce that the NPV of cash flows (CFV), that are available for the investor up to time T, is given by:

$$CFV \coloneqq Cash\ flow\ value = \sum_{i=1}^{T} \frac{\alpha E_i}{(1+r)^i}$$

with $r \in \mathbb{R}_+$ being the constant risk adjusted discount rate for all $t = 1, \dots, T$.

4.3. Exit value

In this thesis we want to derive a model that suits to most practical situations. In order to measure the exit value (referred to as terminal value) at time T, we need to select between different approaches that are used to identify the value of a company. One can think of infinite income approaches, asset-based approaches or market-based approaches. Asset-based approaches do not account for future cash flows or the investment's current strategic and operative positioning. As markets are not provided with complete information, market participants have to focus on current and historic data. Companies are analysed with analyst expectations on estimated peer group data;[74] we use a market based approach, in particular a multiple method.

The multiple method is commonly used by practitioners to approximate the EV of the investment and to compare different investments in terms of their valuation. As the focus of this work is to provide a comparable and easily adoptable guidance for decision makers, we will concentrate on this method.

Consequently, we need to define the multiple-process $m: [0; T] \to \mathbb{R}_+$ referring to a cash flow process. We will work with an $\frac{EV}{FCFF}$-multiple, as it is the most accurate multiple measure of the current value of a company.[75] Thus the market value of the investment at time T is given by:

$$EV_T = FCFF_T m_T$$

[73] See BOECKER (2008) Modelling and Measuring Business Risk, p. 4

[74] We suggest a peer to be defined as a company operating in the same industries and bearing the same risks. For a peer company it is desirable to have public information, e.g. a stock traded companies fulfil this issue

[75] JACOBS (2002) Great companies, bad stocks, p. 1

Recalling the relation of FCFF and FCFE:[76]

	FCFE
+	Δdebt (debt repayments – debt issued)
+	Interest
	FCFF

In order to arrive at a model with $E_t = FCFE_t$ and m_t as stochastic processes, we reshuffle the EV at time T as:

$$EV_T = FCFF_T m_T = (E_T + Interest_T + \Delta debt_T)m_T$$

Since

$$(1 - l_T)EV_T = \left(1 - \frac{debt_T}{EV_T}\right)EV_T = \frac{EQV_T}{EV_T}EV_T = EQV_T$$

we get

$$\begin{aligned} EQV_T &= (1 - l_T)EV_T = (1 - l_T)(E_T + Interest_T + \Delta debt_T)m_T \\ &= (1 - l_T)E_T m_T + (1 - l_T)Interest_T m_T + (1 - l_T)\Delta debt_T m_T \end{aligned}$$

As previously the equity of our investment accounts for α of the total equity value, we may write in a discounted time setting

$$TV \coloneqq Terminal\,Value = \frac{\alpha(1 - l_T)E_T m_T + A_T}{(1 + r)^T}$$

where $A_T \epsilon \mathbb{R}$ captures besides $\alpha(1 - l_T)Interest_T m_T$ and $\alpha(1 - l_T)\Delta debt_T m_T$ other deterministic adjustments at time T (for instance control premiums).[77]

[76] See DAMODARAN (2001) Investment Valuation, Chapter 15, p. 1 and Chapter 14, p. 1 and assuming a tax free world on interest payments

[77] One can also account for adjustments like cash & cash equivalents, minority interest, pension underfunding and other deductions, thus A_T is allowed to be negative. Also A_T is assumed to be $\mathfrak{F}_{t-1}$ measurable

4.4. Cash equity basis

We refer to figure 11 to illustrate the undiscounted composition of the cash equity basis.

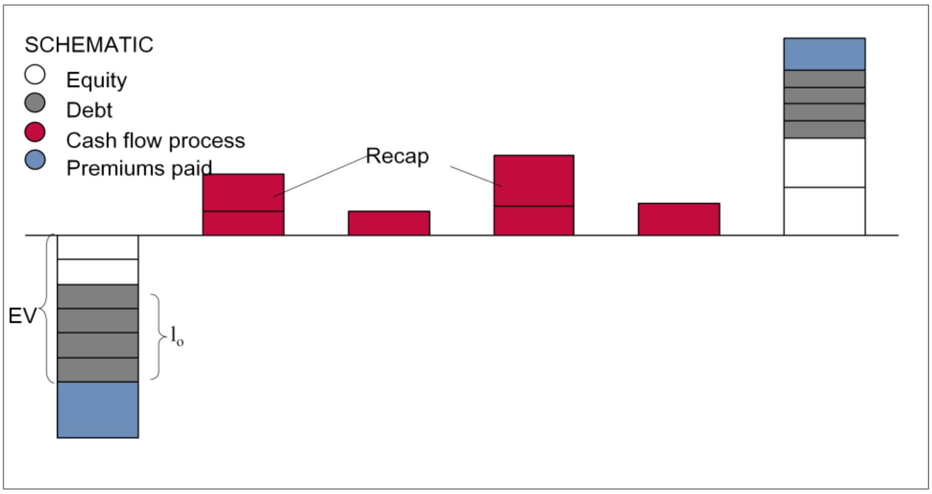

Figure 11: Schematic discrete cash flows

Let us introduce V_T as the discounted cash equity basis up to time T, which is composed by the initial investment value at time 0, by the net present value of cash flows until maturity T and the exit value achievable at maturity:[78]

$$V_T = -[\alpha(1-l_o)P_o + A_o] + \sum_{i=1}^{T} \frac{\alpha E_i}{(1+r)^i} + \frac{\alpha(1-l_T)E_T m_T + A_T}{(1+r)^T}$$

for a fixed discount rate $r \geq 0$.

As a next step one should also take into account that de-leverage and re-leverage strategies may change the risk adjusted discount rate $r \in \mathbb{R}_+$ over time, hence $r: [0;T] \rightarrow \mathbb{R}_+$ is a function over time, where r_t is $\mathfrak{F}_t$ measurable $\forall t = 0, \ldots, T$. Hence in a discrete time setting:

$$V_T = -[\alpha(1-l_o)P_o + A_o] + \sum_{i=1}^{T} \frac{\alpha E_i}{\prod_{j=1}^{i}(1+r_j)} + \frac{\alpha(1-l_T)E_T m_T + A_T}{\prod_{j=1}^{T}(1+r_j)}$$

Before we discuss our model in detail, we want to put on record that the cash flow process E_t is not necessarily restricted to be the company's aggregated cash flow process, one can also think about different cash flow processes derived from different business fields or even the composition of cash flow processes (revenue

[78] Note that we model the FCFE as well as the TV at time T to be contributable for the investors. One can also exclude the last payments (dividends) by introducing a separate exiting time, in which only the TV is contributable to investors. As we will develop a continuous time model, this is no issue with us

minus cost) – one just has to bear in mind possible correlations between the cash flow processes. We will deal with this problem within the multivariate continuous time model in section 5.3.

In order that the upcoming theoretic continuous time approach can be followed easily, we want to complete the discrete time framework. Both the cash flow process E_t as well as the multiple process m_t are considered stochastic. First we set work on the composition of the cash flow process. As we deal with elaborated investment strategies also including recaps, a discrete time model for the cash flow process for $t = 1, \dots, T$ is chosen as:

$$E_t = E_{t-1} + E_{t-1}(b_t + \varepsilon_t)$$
$$E_t^{'} = (1-\eta)E_t + \Delta c_t$$
$$E_0 \in \mathbb{R}$$

where the notation is as follows:

E_t	recurring cash flow process at time t
$E_t^{'}$	cash flow process composed by recurring cash flows, that are extracted by the investor at time t, and instantaneous cash flow add-ons at time t
η	constant rate $\eta \epsilon [0; 1]$ of E_t employed by the investors for repaying debt
b_t	drift at time t
$\Delta c_t = c_t - c_{t-1}$	deterministic function representing changes in cash flow, for instance derived by recaps at time t
ε_t	heteroskedastic and independent error term at time t with $\forall t$: $\mathbb{E}(\varepsilon_t) = 0, \mathbb{V}ar(\varepsilon_t) = \sigma_t^2$

Thereby we account for level-adjusted volatilities and drifts by multiplying with E_{t-1}. As it is not plausible, from a business perspective that future cash flow growth depends on extraordinary events (jumps), which are non-recurrent, we follow the idea of EBIT and EBITDA as comparable measure for a company's ordinary earnings, which can be expected to recur in future. Hence, we adjust the process for level depending drift increments and volatilities with the cash flow level at time t, that reflects the companies ordinary cash flows. Hence, Δc_t is excluded from modelling the ordinary cash flows E_t. Excluding cash flows that are employed to repay debt yields (applying $E_t{}'$ to V_T):

$$V_T = -[\alpha(1-l_o)P_o + A_o] + \sum_{i=1}^{T} \frac{\alpha[(1-\eta)E_i + \Delta c_i]}{\prod_{j=1}^{i}(1+r_j)} + \frac{\alpha(1-l_T)E_T m_T + A_T}{\prod_{j=1}^{T}(1+r_j)}$$

Similar to the cash flow process we will model the multiple process for t$= 1, \dots, T$:

$$m_t = m_{t-1} + \widehat{b_t} + \widehat{\varepsilon_t}$$
$$m_0 \in \mathbb{R}_+{}'$$

with

m_t	multiple process at time t
$\widehat{b_t}$	drift at time t
$\widehat{\varepsilon_t}$	i.i.d. error terms at times $t = 1, .., T$ with $\forall t$: $\mathbb{E}(\widehat{\varepsilon_t}) = 0, \mathbb{V}ar(\widehat{\varepsilon_t}) = \sigma^2$

As we have seen in section 3.2.4 we assume that $\widehat{\varepsilon_t}$ and ε_t are independent, as there is only little correlation between a company's earnings growth and share price appreciations $(multiple \times earnings)$. Further, we indicate the natural filtration by:

$$\forall t \in [0;T]: \mathfrak{F}_t = \sigma(\{\varepsilon_s, \widehat{\varepsilon_s}: 0 \le s \le t\}) = \sigma(\{E_s, m_s: 0 \le s \le t\})$$

and call all $\mathfrak{F}_t$ measurable quantities path dependent.

A crucial issue for a realistic model is the choice of the risk adjusted discount rate; here, we want to put on record that we operate on an equity basis rather than an entity basis, thus there is no need to discount with a Weighted Average Cost of Capital (WACC) discount rate. Discounting free cash flows to equity at cost of equity will yield the value of equity in a business.[79] Working on the premises of the CAPM[80] we recall the security market line (SML), which is given by:

$$r_e = r_f + (r_m - r_f)\beta_{levered}$$

with $\beta_{levered} = \frac{Cov(r_e, r_m)}{Var\,(r_m)}$ denoting the equity beta factor, a measure for the systematic risk of a company's returns, $r_e \in \mathbb{R}_+$ the return on equity, $r_m \in \mathbb{R}_+$ a market rate of return, and $r_f \in \mathbb{R}_+$ the risk free discount rate. $Cov(r_e, r_m)$ is indicated by historic stock returns – cf. section 7.2..

[79] See DAMODARAN (2001) Investment Valuation, Chapter 15, p. 2

[80] Refer to SHARPE (1964) Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, 1964, in: Journal of Finance, pp. 425-442

By:

$$\beta_{levered}^{t} = \beta_{unlevered}\left(1+\frac{debt_t}{equity_t}\right) = \beta_{unlevered}\left(\frac{1}{1-l_t}\right)$$

we can estimate the risk adjusted discount rate $\forall t = 1, \ldots, T$ on the basis of the SML-equation as:

$$r_t := r_e = r_f + r_p\frac{1}{1-l_t}, with\ r_p = \beta_{unlevered}\left(r_m - r_f\right)$$

Based on $\beta_{levered} = \frac{Cov(r_e, r_m)}{Var(r_m)} = \frac{\sigma_e Corr_{e,m}\sigma_m}{\sigma_m^2} = \frac{\sigma_e}{\sigma_m}Corr_{e,m}$ together with $\beta_{levered}^{t} = \beta_{unlevered}\left(\frac{1}{1-l_t}\right)$ we will fragment the appropriate cash flow risk in a systematic risk component and a leverage component:

$$\sigma_e = \frac{\beta_{unlevered}\ \sigma_m}{Corr_{e,,m}}\left(1+\frac{debt}{equity}\right) := \sigma_I\frac{1}{1-l_t}$$

with $\sigma_I := \sigma_{Industry} := \frac{\beta_{unlevered}}{Corr_{e,,m}}\sigma_m$. As leverage l_t depends on time t, the variances σ_t depends on time t – hence one has to keep in mind that changes in the capital structure l_t affect the risk. To provide an intuition consider an increase in debt at a constant enterprise value. An increase in debt will affect risk in two ways. First, the equity share of total capital is reduced, thus changes in a company's revenue hit a lower equity basis, and thus result in a higher volatility of cash flows. Second, future liabilities soar as interest payments are increasing, hence the probability of not being able to repay liabilities augments, and thus the risk to default increases. Hence we can understand the well known result from the Leverage Theory that an increase in leverage increases the variance of the investment and thus σ_t is a function of l_t.[81] Hence, we model σ_t $\forall t = 1, \ldots, T$ by:[82]

$$\sigma_t := \sigma_e = \sigma_I\frac{1}{1-l_t}$$

Where σ_I denotes the appropriate constant industry specific cash flow risk.[83] We take $\forall t = 1, \ldots, T$:

$$l_t = min\left(\frac{debt_t}{EV_t}, 1\right) = min\left(\frac{debt_0 - \sum_{i=1}^{t}\eta E_i + c_t}{(E_t + Interest_t + \Delta debt_t)m_t}, 1\right)$$

[81] TRAUTMANN (2007) Investitionen – Bewertung, Auswahl und Risikomanagement, p. 214: $\sigma\left(r_{equity}\right) = \sigma\left(r_{total\ capital}\right)\left(1+\frac{Debt}{Equity}\right)$

[82] See section 6.3

[83] The industry risk is measured by a non-levered peer group analysis

where $\eta\epsilon[0;1]$ denotes the constant percentage of FCFE, that is employed to repay debt and c_t is the compounded amount of debt-financed recaps at time $t = 1, \dots, T$. So we have that l_t is $\mathfrak{F}_t$ measurable for $t = 1, \dots, T$. As σ_t and r_t depend on l_t we have that σ_t and r_t are path dependent for $t = 1, \dots, T$. Due to the dependence of l_t[84] on $E_0, \dots, E_t, m_t$ $\forall t = 1, \dots, T$ the cash equity basis V_T is path dependent.

The last paragraph of this section is meant to provide a brief outlook on how the model will be realised in practise: Deterministic variables, such as drifts and initial values, can be derived from LBO models and are input variables for simulating the stochastic processes, both the cash flow process as well as the multiple process. On the the basis of different paths we will be able to examine the density[85] of the net present value of the investment, and deduce risk measures to each scenario in an LBO model. Forecasted returns (measured by IRRs) and measured risks can be compared in a risk-return diagram[86] and the appropriate strategies can be evaluated.

Before we turn to the continuous time model, we come back to the important possibility of the investment to default completely.

4.5. Default event

As shown in section 3.3, private equity investments bear significant risk to default. We need to suggest a possibility how to proceed if cash flows at any time $t = 1, .., T$ are not sufficient to repay all payment obligations occurring from debt. We solve this by introducing a predetermined level $E_t^{df} \in \mathbb{R}$ for any $t = 1, .., T$, at which obligations cannot be covered.[87] If a simulated path of the cash flow process E_t strikes at any time $\tau \in \{1, \dots, T\}$ the level E_t^{df}, the investment will default and the exit value at time T as well as future cash flows $E_\tau, \dots, E_T$ are set to be zero.

We define the stopping time $\tau\epsilon\{1, \dots, T\}$ as the time, when the EV strikes the debt value (DV) of the company.[88] Hence the default time τ is defined $\forall t = 1, .., T$ by the equation:

[84] And respectively r_t and l_t
[85] Cf. figure 12
[86] Cf. figure 1
[87] In a continuous time setting we propose E_t^{df} as the cash flow level defined by $E_t^{df} m_t = debt_t$, cf. MERTON (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, pp. 449-454
[88] Cf. MERTON (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, p.449. Note that, we do not allow for a potential recovery, as cash injections are unusual as they dilute the IRR, cf. Appendix A1

$$DV_t = EV_t \Leftrightarrow l_t = 1$$
$$\Longrightarrow \tau := inf\{0 \le t \le T, t \in \mathbb{N}: l_t = 1\}$$
$$\text{with } \tau = inf\emptyset := \infty$$

In particular, E_t^{df} is implicitly defined by $l_t = \frac{debt_0 - \sum_{i=1}^{t} \eta E_i + c_t}{(E_t + Interest_t + \Delta debt_t) m_t} = 1$. Referring to Figure 12, the value process can be written as:

$$V_T = \begin{cases} -IV + \sum_{i=1}^{T} \frac{\alpha[(1-\eta)E_i + \Delta c_i]}{\prod_{j=1}^{i}(1+r_j)} + \frac{\alpha(1-l_T)E_T m_T + A_T}{\prod_{j=1}^{T}(1+r_j)} & \forall t = 1,..,T:\ E_t > E_t^{df} \\ -IV + \sum_{i=1}^{t-1} \frac{\alpha[(1-\eta)E_i + \Delta c_i]}{\prod_{j=1}^{i}(1+r_j)} & \exists t = 1,..,T: E_t \le E_t^{df} \end{cases}$$

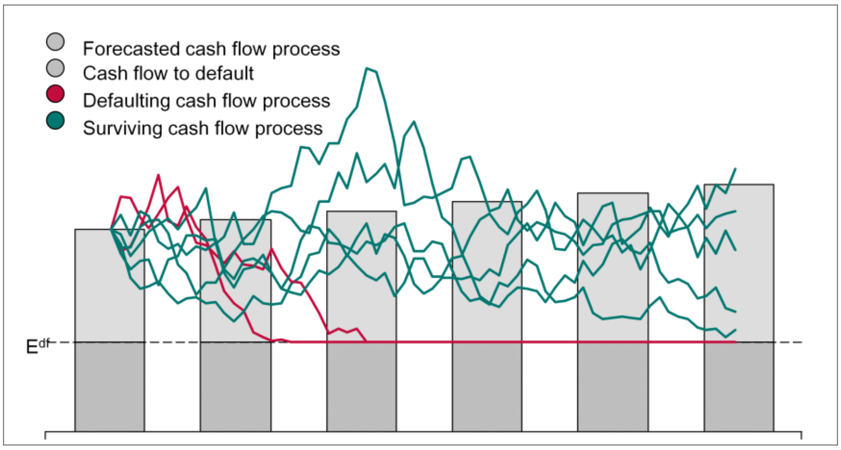

Figure 12: Linear interpolated defaulting cash flow processes without recap scenario

Now we focus on the probability distribution of the aggregated net present value of the equity investment at time T. A schematic illustration can be found in figure 13.

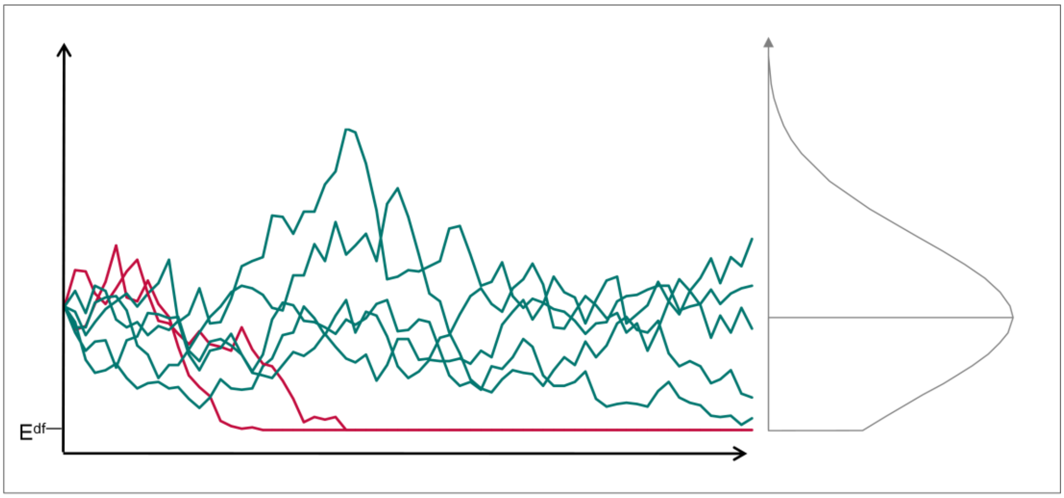

Figure 13: Schematic illustration of the relationship between free cash flow processes and the probability distribution of the aggregated cash equity basis of the investment at time T

For our purpose a continuous time setting will be more feasible, due to three major reasons:

(1) The cash equity basis V_T at time T is path dependent and cannot be simulated straight forwardly, as the distribution of V_T is path dependent with respect to r_t, l_t and σ_t.

(2) We are familiar with processes and their characteristics in continuous time, thus the selection of the distribution of error terms ε_t and $\widehat{\varepsilon_t}$ is not arbitrary.

(3) Investors survey investment not only on a grid (e.g. year, quarter), they are interested in tracking the processes on $t \in [0; T]$. Thus the grid would have to be fine to be able to react in time on adverse developments.

5. Continuous time model

The next step is to transfer the idea, which we have developed in discrete time, to a continuous time setting. In a first approach we restrict our model to a univariate setting, thereby we are able to understand the evolution of the model in detail. In a second step we will tend the model to a multivariate setting to allow for the valuation of different business units or more than one investment, such as funds for instance.

5.1. The univariate setting without default event

We will first deal with the selection of variables that can be reasonably assumed to be deterministic in continuous time, before coming to a specific modelling of the stochastic processes involved. After all we will derive the compounded cash equity basis of the investment.

Throughout this chapter we assume that $(\mathcal{W}_t)_{t\geq 0} = \left(W_t, \widetilde{W}_t\right)_{t\geq 0}$ is a 2-dimensional standard BROWNIAN motion on a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$. We assume $(\mathfrak{F}_t)_{t\geq 0}$ to be the natural filtration generated by $(\mathcal{W}_t)_{t\geq 0}$, which satisfies the usual conditions. Hence for any assessment time expectations are based on all historic information. Thus we are especially able, in dependence on the current situation and information at any time within the investment horizon, to scrutinise possible next strategic steps to take.

In the following sections we assume all functions over time t to be $\mathfrak{F}_t$-measurable at time t and also to be continuous with respect to t. As we deal only with problems on the interval $[0, T]$, we will also assume the usual integrability conditions $\forall 0 \leq t \leq T$ to hold, thus $\int_0^t |\lambda_s| ds < \infty$ and $\int_0^t |\kappa(\overline{m}_s - m_s)| ds < \infty$, $\int_0^t {\sigma_s}^2 ds < \infty$ and $\int_0^t \sigma^2 ds < \infty$ respectivly – the quantities λ_s, $\overline{m}_s$, κ, m_s, σ_s and σ are introduced in the later.

5.1.1. Deterministic and stochastic terms

Within this section we try to simplify our setting in terms of deterministic time add-ons and variables, that do not have an impact on defining the probability distribution of the investment in continuous times. We can incorporate easily their actions by a linear transformation of the modelled cash equity basis.

First, the initial equity stake α is assumed to be constant over time. This is because our considered investments do not include third party strategies, which may cause changes in equity structure, or stage investments. Second, A_o denotes a premium adjustment that is paid to acquire a majority stake in the target company. A_o is a specific premium at time 0, that the private equity investor is willing to pay. Premiums depend on the type of buyer and type of transaction, therefore the premium is not modelled as a stochastic process and is given as a deterministic add-on $A_0, A_T \in \mathbb{R}$.[89] Further A_T captures other deterministic adjustments at time T, that we assume to be deterministic as well. As V_T is linear with respect to α and A_0, respectively A_T, we assume without loss of generality that $\alpha = 1$ and $A_0 = A_T = 0$.

The leverage level l_t is extracted from the LBO model developed by private equity firms and is path dependent. We define accordingly to the discrete time setting for $t \epsilon [0; T]$ l_t by:

$$l_t = min\left(\frac{debt_t}{EV_t}, 1\right) = min\left(\frac{debt_0 - \int_0^t \eta E_s ds + J_t}{(E_t + Interest_t + \Delta debt_t)m_t}, 1\right)$$

where again $\eta \in [0; 1]$ denotes the percentage of free cash flows to equities, that are employed to repay debt. Instantaneous cash flow add-ons J_t are assumed to be debt-financed, thus they increase or decrease current debt level. The exact probabilistic model of J_t is discussed in detail in section 5.1.3.

The risk adjusted discount $r_t > 0$ is assumed to be deterministic; for the sake of simplicity we define r_t to be a path depending variable as it depends on the underlying capital structure, which affect risk-adjusted discount rates, but one could also assume stochastic interest rates. We take r_t for $t \in [0; T]$ to be defined as before in the discrete time model by the CAPM:[90]

[89] In practice it is only of interest to know the initial value A_o and exit value A_t of the premiums paid for the investment. A_t will be an estimate of the private equity company. If for example the private equity company considers an initial public offering as exit strategy it seems most feasible to have $A_t = 0$

[90] For the derivation of this formula compare 4.4

$$r_t = r_f + r_p \frac{1}{1 - l_t}$$

with the constant risk free rate $r_f \in \mathbb{R}_+$ and $r_p = \beta_{unlevered}\left(r_m - r_f\right) \in \mathbb{R}$ denoting the risk premium. Hence we restrict ourselves to focus on the stochastic processes E_t and m_t and their interactions. Similar to the discrete time model, we set the natural filtration $\mathfrak{F}_t \coloneqq \sigma(\{m_s, E_s : s \leq t, 0 \leq t \leq T\})$, and say that all $\mathfrak{F}_t$ measurable quantities are path dependent.

5.1.2. Multiple process

For the multiple process it is crucial that m_t is positive, otherwise the exit value of the investment would not be meaningful. Thus, we use a square-root diffusion process to ensure that almost all sample paths m_t are positive $\forall t \in [0;T]$:[91]

$$dm_t = \kappa(\bar{m} - m_t)dt + \sigma\sqrt{m_t}d\widetilde{W}_t$$
$$m_0 \in \mathbb{R}_+$$

σ is the diffusion of the multiple process m_t, for an illustration of this CIR-process we refer to figure 14:

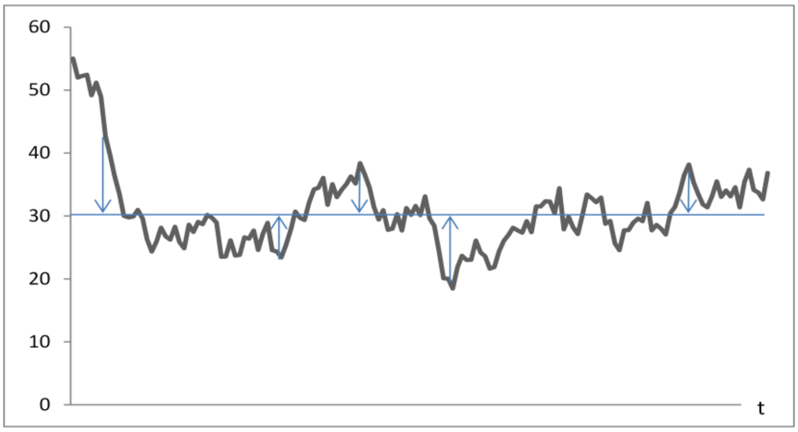

Figure 14: Simulation of a CIR process with long time mean 30

As m_t is supposed to be positive, we assume that the stationary condition holds so that $2\kappa\bar{m} \geq \sigma^2$. The parameter κ indicates the mean-reversion rate to the long time mean $\bar{m}$. We allow for the long time mean to depend on time, hence for $\bar{m}_t \geq 0$ we define m_t by the SDE on $[0;T]$:[92]

[91] Refer to COX/ INGERSOLL/ ROSS (1985) A Theory of the Term Structure of Interest Rates, pp. 386-390
[92] Hence, industry forecast derived by the consensus of broker reports can be included

$$dm_t = \kappa(\overline{m}_t - m_t)dt + \sigma\sqrt{m_t}dW_t$$

$$m_0 \in \mathbb{R}_+$$

Note that we suppose that m_t is independent to the cash flow process E_t. Thus, we only allow the multiple process to account for environmental changes, such as booms, recession, etc. This assumption is based on the findings in section 3.2.4 (Share prices appreciations show only little correlation to earnings[93]).

The following result is well-known for mean reversion to a constant mean $\overline{m}$. We present a proof for the more realistic setting of a time dependent mean $\overline{m}_t$.

Proposition 5.1:

Let $\kappa > 0, m_0 \geq 0,$ and $\forall t \in [0;T]$ hold that $2\kappa\overline{m}_t \geq \sigma^2$, then the SDE

$$dm_t = \kappa(\overline{m}_t - m_t)dt + \sigma\sqrt{m_t}d\widetilde{W}_t$$

has a unique solution, with almost all sample paths being positive.

Proof can be found in MAGHSOODI.[94]

5.1.3. Cash flow process

Now, let us model the cash flow process. First we have to distinguish between two different parts of the cash flows. On the one hand have cash flows derived by operations, which we denote as continuous cash flows E_t. On the other hand there are extra ordinary cash flows, derived by asset sales or recaps. As they occur only as one-off payment, we denote these cash flows as the instantaneous part of the cash flow J_t.

Continuous cash flow

First, we model the continuous part of the cash flow. Our approach to cash flows from operations is based on BOECKER and SCHWARTZ/MOON.[95]

[93] One can also introduce correlation between earnings and multiple process, here we will restrict to a uncorrelated model

[94] MAGHSOODI (1996) Solution of the extended CIR term structure and bond option valution, p. 92 and MAGHSOODI (1993) Solution of the Extended Cox, Ingersol and Ross Model of the Term Structure and Analytical Bond Option Valuation, chapter 3

[95] See SCHWARTZ/ MOON (2000) Rational Pricing of Internet Companies, p. 1 and BOECKER (2008) Modelling and Measuring Business Risk, p. 67

One may consider, according to section 5.1, a mean-reverting process, which is almost surely positive. But as enormous debt obligations increase the company's risk to fall bankrupt, if cash flows decrease, it is not plausible to use a mean-reverting process that pulls the cash flow process up to the forecast scenario or vice versa. Let us introduce λ_t as the expected drift rate of cash flows, which is derived by a functional approximation[96] based on cash flow forecasts in the LBO model. One is referred to figure 15, which compares a constant growth rate λ to a non-constant drift λ_t.

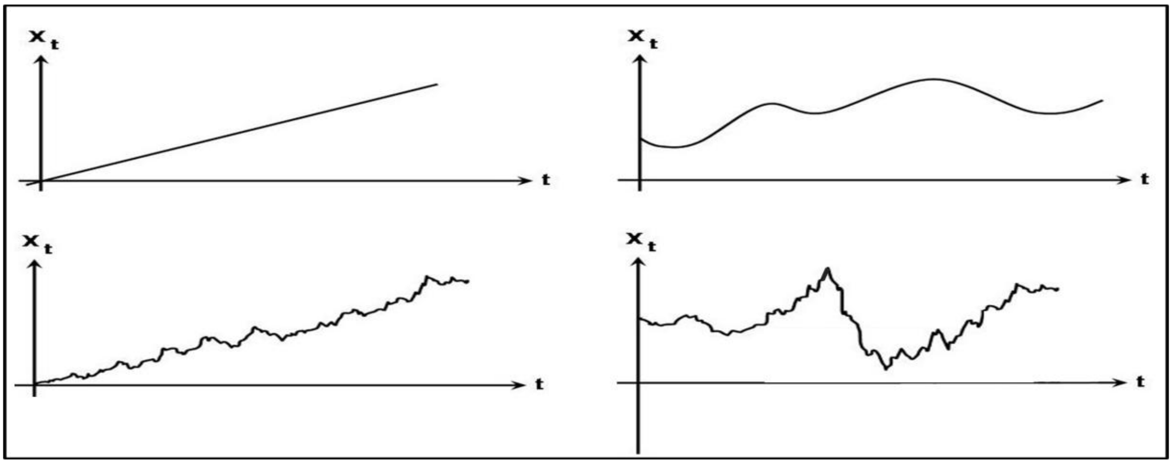

Figure 15: Illustrative comparison of a deterministic (upper) and stochastic (lower) process with constant (left) and functional drift (right)

As companies usually expect different growth rates λ_t over time (for instance due to new products, new technologies, new competition or changes in economic background), we model the continuous cash-flows E_t for $t \in [0; T]$ a geometric BROWNIAN motion given by:

$$dE_t = E_t(\lambda_t dt + \sigma_t dW_t)$$
$$E_0 \in \mathbb{R}_+$$

Thereby we account for level-adjusted volatilities and drifts by multiplying with E_t.[97] The volatility depends on leverage l_t at time t, as a higher debt position increases the sensitivity of FCFE on changes in sales – cf. The discrete time setting of l_t, thus $\forall t \in [0; T]$ we take again:

[96] E.g. by a linear regression.
[97] See BOECKER (2008) Modelling and Measuring Business Risk, p. 16, definition 3.1; also note that the level-adjustments do not depend on cash flow add-ons occurring from recaps, as they are one-off payments

$$\sigma_t = \sigma_I \frac{1}{1 - l_t}$$

The unique solution of the SDE is well known and a consequence of Itô's Lemma:[98]

$$E_t = E_o \exp\left(\int_0^t \sigma_s dW_s + \int_0^t \left(\lambda_s - \frac{1}{2}\sigma_s^2\right) ds\right)$$

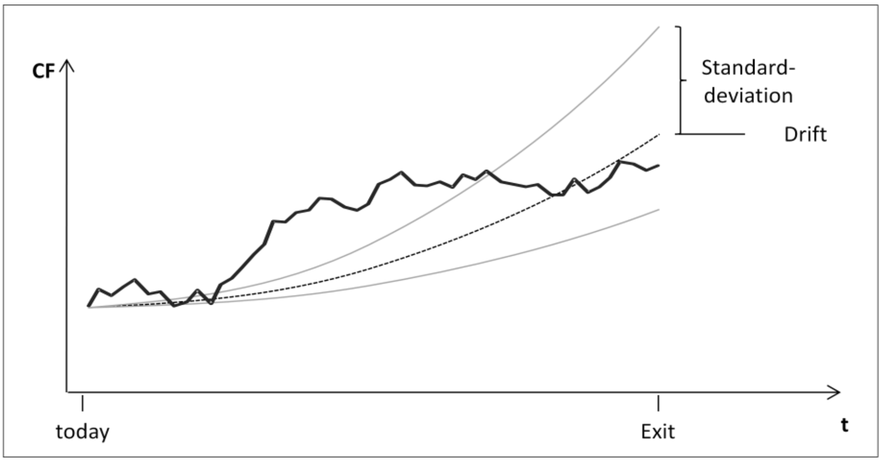

Figure 16: Illustration of a geometric BROWNIAN motion

Taking $E_o \geq 0$ results in $E_t \geq 0$ almost surely. Hence, we restrict our cash flow model only to positive FCFE; no cash-injections are allowed. Together with $m_T \geq 0$ and $E_T \geq 0$ we arrive at a positive exit value. Restricting FCFE to be non negative is a realistic assumption as cash injections are unpopular, as they dilute the IRR.[99] Nevertheless, we will cover cash injections by the model for extraordinary events, which we referred as instantaneous cash flows J_t.

Instantaneous cash flows

To model the instantaneous cash flows we distinguish between deterministic and stochastic extra-ordinary cash flows. There are several deterministic adjustments, such as premiums paid at time 0 or pension obligations at time T, for which the exact times and exact values are known. As there is no stochastic characteristic in this type of instantaneous cash flows we will distinguish these deterministic add-ons from those, which reflect some uncertainty. As we are interested in the density of the present value of the investment, we focus on uncertain events and exclude

[98] One is referred to KARATZAS/ SHREVE (1991) Brownian Motion and Stochastic Calculus, pp. 10-25
[99] Cf. Appendix A1: Financial Times Germany, paragraph 3. By assuming $E_0 > 0$, we exclude solely an adequate modelling of distressed investments

deterministic events from modelling, as it is again just a linear transformation by the appropriately discounted value.

For the stochastic part we introduce a compounded POISSON process with $n \in \mathbb{N}$ jumps. The LBO-Model provides us with information regarding to jump intensity, to the number of jumps and to the values of jump sizes. Let us consider some possible events which lead to stochastic extraordinary events, like recaps, asset-sales, technology purchases or cash injections. For all aforementioned events the time of realisation is uncertain. This relies on the fact that asset-sales, for instance, depend on the retrieval, the due diligence and the agreement of the potential buyer – or a certain technology is not fully developed yet – or even recaps can be pre-drawn or postponed, due to a weak financial position of the company or internal resistance.[100] Thus we consider the waiting times, between two jumps, to follow a specific probability distribution. In this paper we will model the inter jump times G_i by independent exponential random variables $G_i \sim \exp(g)\, i.i.d. \forall i = 1, \dots, n$ for $g > 0$. The associated intensity g defines implicit the average waiting time for the next jump to occur as the mean of the exponential distribution is given by $\frac{1}{g}$.

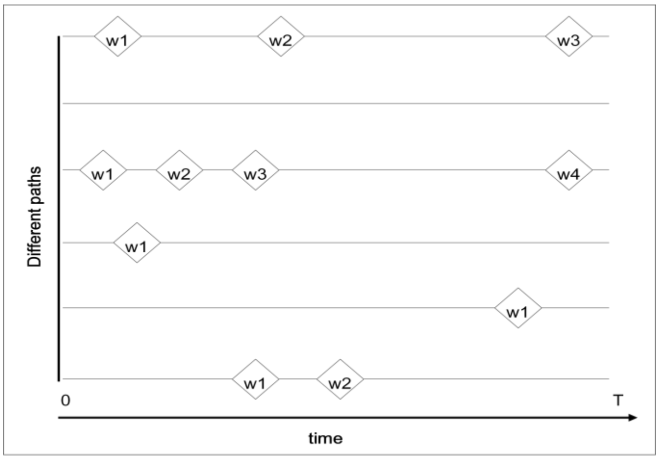

Figure 17: 6 samples paths of a POISSON process with jump times indicated by ✧

[100] Note that a financial position depends heavily on realised cash flows, so one may argue that the jump times and jump heights depend as well on cash flows at this stage of jump. So one wants to install correlation between jump process and jump height, but the interaction of instantaneous adjustments to cash flows are already taken into account in the development of cash flow forecast, we just need to adjust λ_t and σ_t at the jump times. Thus we account at least for the stage of a company's bankruptcy and for all foregoing jumps up to t. A correlation between jump height and cash flow level is missed for simplicity reason

For almost similar reasons the magnitude of the jump at the random time t is random, hence a particular price of sale of an asset is not known in advance. As the sign of the jump is determined by the nature of the event we will use a positive distribution function to model the absolute values of the jumps. The sign of the $i'th$ jump is random, denoted by $\delta_i \epsilon \{-1,1\}$ for $i = 1, \dots, n$ with $P(\delta_i = 1) = p \in [0; 1]$ $i.i.d.$. We will model the distribution of the absolute jump sizes by $C_i \sim \mathcal{LN}(\mu_i, \sigma_i^2)$ $i.i.d.$ for $i = 1, \dots, n$.

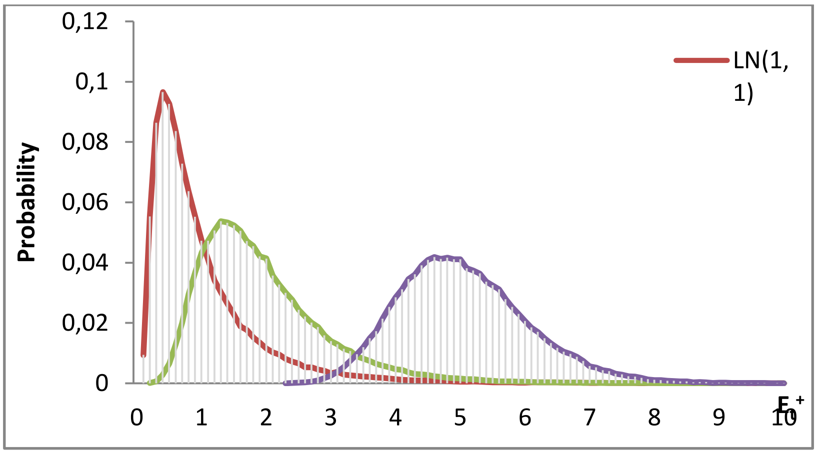

Figure 18: Simulation of a log-normal distribution (n=100.000)

We assume that the LBO Model indicates the sequence of the jumps, the jump process J_t is a compounded POISSON process defined $\forall t \in [0; T]$ by:

$$J_t = \sum_{i=1}^{n} \delta_i C_i 1_{\{\sum_{j=1}^{i} G_i \leq t\}},$$

with the quantities δ_i, C_i, G_i and n defined above.

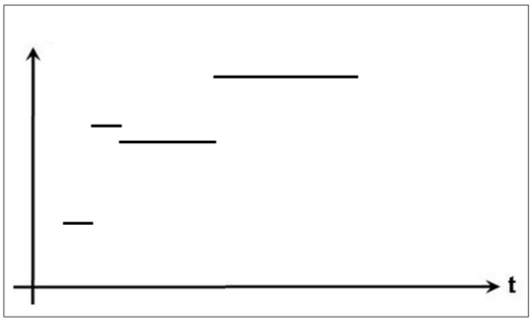

Figure 19: Sample path of the jump process J_t

In case $\sum_{i=1}^{n} G_i > T$, the last events could not be realised within the investment horizon, which is a plausible feature as some delays in the timetable affect events at the end of the investment period, to be deferred out of the investment horizon. Consider for instance milestones of pharmaceutical companies, if $\sum_{i=1}^{n} G_i > T$ some regulatory approvals could, for instance, not be realised within the investment horizon, and thus the associated cash flows have not been contributed to the investors.

Thus the compounded cash flow process $E_t{}'$ available for investors including extraordinary events and which defines the cash flow value of the investment is $\forall t \in [0;T]$ given by:

$$E_t^{'} = (1-\eta)E_t + J_t$$
$$= (1-\eta)E_o exp\left(\int_0^t \sigma_s dW_s + \int_0^t \left(\lambda_s - \frac{1}{2}\sigma_s^2\right) ds\right) + J_t$$

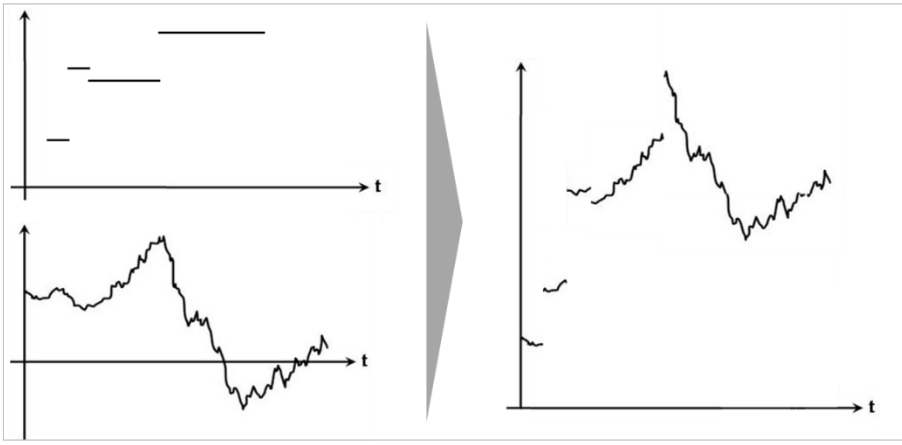

Figure 20: Illustration of the compounded cash flow process E'_t (right). The process is compounded by the ordinary earnings process E_t (lower left) and by the instantaneous part of cash flows J_t(upper left)

5.1.4. Valuation

In this section we will deduce the net present value and the exit value without accounting for default events. In the following we will consider the composition of the exit value and calculate the combined process by using Itô's calculus.

First, we need to introduce continuous time discounting which is provided by $e^{-r_t t}$.[101] r_t is the periodic, usually annualized, interest rate and $t \geq 0$ is the

[101] See HORVATH (1995) Compounding/ discounting in continuous time, pp. 315-325

proportion of the period over which discounting is to be accomplished. This form is based on:[102]

$$e^{-r_t t} = \lim_{k\to\infty}\left(1+\frac{r_t}{k}\right)^{-kt}$$

The net present value of cash flow to equity is simply derived by discounting cash flows contributable to investors by the risk-adjusted rate r_t, hence $\forall t \geq 0$:

$$CFV = \int_0^t exp(-r_s s)\big((1-\eta)E_s + J_s\big)ds$$

In the implementation stage we will approximate this equation by an EULER discretisation.

The exit value is more complicated as it is a one off payment. It is derived by the discounted value of the equity stake, measured by the enterprise value minus the book value of liabilities.[103] As the enterprise value at time T is indicated via $multiple_T \times FCFE_T$ we need to know the enterprise value at any time $t \in [0;T]$, as the cash flow process's volatility and the appropriate discount rate depend $\forall t^{'}: 0 \leq t' \leq t$ on the underlying capital structure $l_{t'}$.

Proposition 5.2:

Consider $\forall t \in [0;T]$ the two processes with $E_0, m_0 \in \mathbb{R}_+$:

$$dE_t = E_t(\lambda_t dt + \sigma(l_t)dW_t)$$

$$dm_t = \kappa(\overline{m}_t - m_t)dt + \sigma\sqrt{m_t}d\widetilde{W}_t$$

then the exit value at time $t \in [0;T]$of the investment is given by:

$$(Em)_t = (Em)_o + \int_0^t m_s dE_s + \int_0^t E_s dm_s + [m,E]_t$$

where $[m,E]_t$ denotes the quadric variation up to time t.

Proof:

(1) We will apply Itô's Lemma thus we need:

$f \in \mathcal{C}^{1,2}: [0;T] \times \mathbb{R} \to \mathbb{R}$ which is ensured by taking $f(x) = x^2$

[102] Refer to BIERMAN/ SMIDT (1986) Application of the Capital Asset Pricing Model to Multi-period Investments, p. 328 and EMERY/ FINNERTY (1991) Principles of finance with corporate applications pp. 820-823 as well as WESTON/COPELAND (1992) Financial Theory and Corporate Policy, p. 66

[103] As the company is not bankrupt, market value of debt is equal to book value of debt

(2) $m_t^2 = m_o^2 + \int_0^t 2m_s dm_s + \frac{1}{2}\int_0^t 2d[m,m]_s = m_o^2 + 2\int_0^t m_s dm_s + \sigma^2 \int_0^t m_s ds$

(3) $E_t^2 = E_0^2 + \int_0^t 2E_s dE_s + \frac{1}{2}\int_0^t 2d[E,E]_s = E_0^2 + 2\int_0^t E_s dE_s + \int_0^t (E_s\sigma_s)^2 ds$

(4) $(E+m)_t^2 =$

$(E+m)_0^2 + \int_0^t 2(E+m)_s(dE_s + dm_s) + \frac{1}{2}\int_0^t 2d[E+m, E+m]_s =$

E_t and m_t are independent processes, hence

$(E+m)_t^2 =$

$= E_0^2 + 2E_0 m_0 + m_o^2 + 2\int_0^t (E+m)_s(dE_s + dm_s) + \int_0^t \left(\sigma\sqrt{m_s} + E_s\sigma_s\right)^2 ds$

(5) $(Em)_t = E_t m_t = \frac{1}{2}[(E+m)_t^2 - E_t^2 - m_t^2]$

$= E_0 m_0 + \int_0^t m_s dE_s + \int_0^t E_s dm_s + \int_0^t \sigma\sigma_s\sqrt{m_s}E_s ds$

$= (Em)_0 + \int_0^t m_s dE_s + \int_0^t E_s dm_s + [m,E]_t$

$q.e.d.$

Some reshuffling and employing of original processes yields:

Corollary 5.3:

$$(Em)_t$$

$$= E_0 m_0 + \int_0^t E_s m_s \lambda_s ds + \int_0^t \kappa E_s\,(\overline{m}_s - m_s) ds + \int_0^t E_s\,\sigma\sigma_s\sqrt{m_s} ds$$

$$+ \int_0^t E_s m_s \sigma_s dW_s + \int_0^t E_s \sigma\sqrt{m_s}\, d\widetilde{W}_s$$

As we can see in corollary 5.3, the deterministic part of the enterprise value of the company at time t depends on the initial value $E_0 m_0$ at time 0, but also on the expected growth in cash flows $\lambda_s E_s m_s$, which reflects the value driver "top line growth" of the company. It also depends on the multiple $\kappa E_s(\overline{m}_s - m_s)$, which can be interpreted as "multiple expansion", as the exit value is pushed with mean reverting speed $\kappa > 0$ towards the projected multiple $\overline{m}_s$ – this can be either positive or negative, depending on if we have $m_s < \overline{m}_s$ or $m_s > \overline{m}_s$. But it also depends on the term $\sigma\sigma_s E_s\sqrt{m_s}$, which accounts for risk premiums; a higher risk in the industry (σ) or a riskier investment (σ_s) precipitates in an add-on to the enterprise value at time t (growth of return), which is what we expect and see on equity markets. Investors in riskier assets demand higher returns as a risk compensation.

Corollary 5.4:

The net present value of the exit value is given by:

$$TV = (1 - l_T)e^{-\int_0^T r_s ds}\left\{(Em)_o + \int_0^T m_s dE_s + \int_0^T E_s dm_s + [m, E]_T\right\}$$

Proof:

As we assume that investments are only realised at maturity, we are not interested in discounting changes in exit value at any time $t \leq T$, as the focus is on the value at time T. As one can see from the discrete time setting at time T, the amount $(1 - l_T)E_T m_T$ is attributable to the limited partners.

Now the choice of the discount rate is crucial. We employ a geometric approach.[104]

It is well known that the arithmetic mean $\overline{x_a}$ and the geometric mean $\overline{x_g}$ follow:

$$ln(\overline{x_g}) = \overline{ln(x_a)}$$

Thus by applying the arithmetic value of a function r_s between time 0 and time t, denoted by $\overline{r_t}$:

$$\overline{r_t} = \frac{1}{t}\int_0^t r_s ds$$

Hence, we can derive the geometric mean risk adjusted discount rate up to time t:

$$\lim_{k \to \infty}\left(1 + \frac{\overline{r_t}}{k}\right)^{-kt} = e^{-\overline{r_t}t} = e^{-\int_0^t r_s ds}$$

So, the present value of the exit value at time T is given by:

$$e^{-\int_0^T r_s ds}[(1 - l_T)Em]_T \overset{5.3}{=} (1 - l_T)e^{-\int_0^T r_s ds}\left\{(Em)_o + \int_0^T m_s dE_s + \int_0^T E_s dm_s + [m, E]_T\right\}$$

$q.e.d.$

[104] One may argue, that discounting with the risk adjusted rate r_o or the risk adjusted rate r_T is more suitable as the investment is highly illiquid until maturity. But there is a strong reason that supports a geometric mean calculation: As the investment strategy, with all its risk changes between 0 and T, is known in advance, it is wrong to work only with risk adjusted rate time 0 or T respectively. As risk can change dramatically we thus need to establish the average (geometric mean) risk adjusted rate as discount rate

5.2. Univariate setting with default event

To identify whether an investment falls bankrupt, we introduce a stopping time $\tau \in [0;T]$. We define τ as the time when the enterprise value strikes the DV of the company.[105] This is captured by the equation $\forall 0 \leq t \leq T: DV_t = EV_t \Leftrightarrow l_t = 1$. Hence the default time τ is defined by:

$$\Rightarrow \tau := inf\{0 \leq t \leq T, t \in \mathbb{N}: l_t = 1\}$$

$$\tau = inf\emptyset := \infty$$

Note that τ is a stopping time, as we only need information up to time t, hence $\forall t \epsilon \mathbb{R}^+: \{\tau \leq t\} \in \mathfrak{F}_t$.

If it holds that $\tau \leq T$ then the company enters in the insolvency proceedings and so $\forall t \in [\tau; T]\ E_t = 0$, since the company devolves to the creditors[106] in τ. Hence we can define the stopped processes:

The cash flow process:[107] $\forall 0 \leq t \leq T: E_t^{df} = 1_{\{t<\tau\}} E_t$

The multiple process: $\forall 0 \leq t \leq T: m_t^{df} = 1_{\{t<\tau\}} m_t$

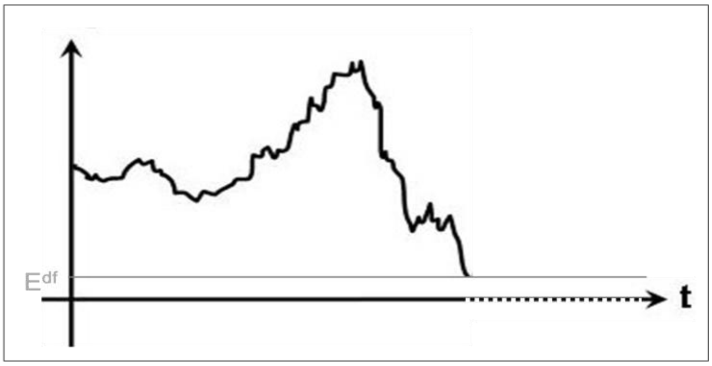

Figure 21: Illustration of a stopped cash flow process with constant $\boldsymbol{E_t^{df}}$

Bringing together the results, we can derive the process of the cash equity basis of a univariate investment:

[105] Cf. MERTON (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates, p. 1

[106] Refer to Appendix A1 and the article of the Financial Times Germany 10.02.2008: "additional injection of capital are usually not used by private equity investments – it is more occurring that the company is sold"

[107] Note that, we exclude the cash flows at time τ, as the company is already assumed to be bankrupt and in the insolvency proceedings

Theorem 5.5:

The cash equity basis of the univariate investment is given by:

$$V_T \coloneqq Cash\ equity\ basis(T) = -IV + CFV + TV =$$

$$-[(1-l_o)P_o]$$

$$+(1-\eta)\int_0^{T\wedge\tau}\exp(-r_s s)E_s ds + \int_0^{T\wedge\tau} exp(-r_s s)J_s ds +$$

$$+(1-l_{T\wedge\tau})e^{-\int_0^{T\wedge\tau} r_s ds}\left[(Em)_o + \int_0^T m_s dE_s + \int_0^T E_s dm_s + [m,E]_t\right]1_{\{T<\tau\}}$$

With $E_t = E_o \exp\left(\int_0^t \sigma_s dW_s + \int_0^t \left(\lambda_s - \frac{1}{2}\sigma_s^2\right)ds\right)$

Proof:

(1) The cash equity basis of the investment up to time t is given by:

$$V_T = -IV + CFV + TV$$

(2) $IV = [(1-l_o)P_o]$, note that we assumed $\alpha = 1$ and $A_0 = A_T = 0$

(3) Cash flow value at time t, without the default event is given by:

$$CFV_t =$$

$$\int_0^t \exp(-r_s s)((1-\eta)E_o \exp\left(\int_0^t \sigma_u dW_u + \int_0^t \left(\lambda_u - \frac{1}{2}\sigma_u^2\right)du\right) + J_s)ds$$

Accounting for stopping time $\tau \coloneqq inf\{0 \le t \le T, t \in \mathbb{N}: l_t = 1\}$ yields with $J_t = \sum_{i=1}^n C_i 1_{\left\{\sum_{j=1}^i G_i \le t\right\}}$ to $J_{t\wedge\tau} = \sum_{i=1}^n C_i 1_{\left\{\sum_{j=1}^i G_i \le t\wedge\tau\right\}}$, as the stopped cash flow process as well as the stopped jumps are set to be zero after τ, hence:

$$CFV_{t\wedge\tau} = \int_0^{t\wedge\tau}\exp(-r_s s)(1-\eta)E_0 \exp\left(\int_0^t \sigma_u dW_u + \int_0^t \left(\lambda_u - \frac{1}{2}\sigma_u^2\right)du\right)ds + \int_0^{t\wedge\tau}\exp(-r_s s)J_s ds$$

(4) Corollary 5.4 yields by introducing the stopping time τ and by definition of E_τ and m_τ:

$$TV =$$

$$(1-l_{T\wedge\tau})e^{-\int_0^{T\wedge\tau} r_s ds}\left\{\left[(Em)_o + \int_0^T m_s dE_s + \int_0^T E_s dm_s + [m,E]_T\right]1_{\{T<\tau\}}\right\}$$

(5) Employing $E_t = E_o \exp\left(\int_0^t \sigma_s dW_s + \int_0^t \left(\lambda_s - \frac{1}{2}\sigma_s^2\right)ds\right)$

(6) Introduction of stopping time τ yields

$$V_T = IV + CFV(t\wedge\tau) + TV$$

(7) (2)-(6) in (1) establishes the claim

$q.e.d.$

5.3. Multivariate setting with default event

In this section we enrich the univariate model by providing the opportunity to analyse the interaction of different investments. Thus a multivariate setting, including correlations, enables us to analyse a portfolio of investments such as funds or analyse for instance multinational conglomerates with different business units.

Thus in the following, we suggest a K-dimensional model driven by $(\mathcal{W}_t)_{t\geq 0} = (W_t^1, \dots W_t^{2K})_{t\geq 0} = \left(W_t^1, \dots W_t^K, \widetilde{W}_t^1, \dots \widetilde{W}_t^K\right)_{t\geq 0}$ a 2K-dimensional BROWNIAN motion on a filtered probability space $(\Omega; \mathfrak{F}; (\mathfrak{F}_t)_{t\geq 0}; \mathbb{P})$ satisfying the usual conditions.

The aggregated value of different business units or investments, denoted as asset component, is given by the sum of each cash equity basis $\mathrm{V_T^i}$, thus:

$$V_T = \sum_{i=1}^{K} \mathrm{V_T^i}$$

Thus, following the intuition of the univariate setting, we model the multivariate setting such that each asset component becomes:

The cash flow process $i \in [1; K]$ is given $\forall 0 \leq t \leq T$ by:

$$dE_t^i = E_t^i\left(\lambda_t^i dt + \sigma_t^i dW_t^i\right)$$

The thereof independent multiple process $j \in [1; K]$ is given $\forall 0 \leq t \leq T$ by:

$$dm_t^j = \kappa^j\left(\overline{m}_t^j - m_t^j\right)dt + \sqrt{m_t^j}\sigma^j d\widetilde{W}_t^j$$

$\forall j \in [1; K]$ m_t^i is the corresponding multiple process for the cash flow process j. σ_t^i is the instantaneous time dependent volatility of the $i'th$ cash flow process E_t^i, whereas σ^j, the volatility of the $j'th$ multiple process, is analogous to the univariate setting defined as time independent. The multidimensional BROWNIAN motion $(\mathcal{W}_t)_{t\geq 0} = (W_t^1, \dots W_t^{2K})_{t\geq 0}$ is a martingal, that is, each component is a martingal, and satisfies $\forall i = 1,..,2K$ $\forall j = 1,..,2K$ $\forall 0 \leq t \leq T$ following properties:[108]

$$\mathbb{E}\left(W_t^i\right) = 0$$

$$d\left[W^i, W^j\right]_t = \rho^{ij} dt$$

[108] One may note that this is equivalent to the notation: $dE_t^i = E_t^i\left(\lambda_t^i dt + \sum_{j=1}^{k} \tilde{\sigma}_t^{ij} d\widehat{W}_t^j\right)$ and $m_t^i = \kappa^i\left(\overline{m}_t^i - m_t^i\right)dt + \sqrt{m_t^i}\sum_{j=m+1}^{2k} \tilde{\sigma}_t^{ij} d\widehat{W}_t^j$ where the Wiener processes $\widehat{W}^i$ are all independent, by setting $\sigma_t^i = \sqrt{\sum_{j=1}^{2m} \tilde{\sigma}_t^{ij} \tilde{\sigma}_t^{ij}}$ and $\rho^{ij} = \frac{\sum_{k=1}^{2k} \tilde{\sigma}_t^{ik} \tilde{\sigma}_t^{jk}}{\sigma_t^i \sigma_t^j}$

with

$$\rho^{ij} = \begin{cases} 0 & i = 1, \dots, K \; and \; j = K+1, \dots, 2K \\ 1 & i = j \\ \hat{\rho}^{ij} & else \end{cases}$$

Where $[\cdot,\cdot]_t$ represents the quadric variation up to time t and $\hat{\rho}^{ij} \in [-1; 1]$ the constant instantaneous correlation between W^i and W^j. Here we will set the multiple processes as well as the cash flow processes as two independent processes, as already discussed in the univariate case, thus ρ^{ij} =0 for $i \in [1; k]$, $j \in [k+1; 2k]$. The time dependent covariance matrix evolving according to the dynamics of the time-dependent volatilities and the constant correlation among the asset components is denoted by:

$$\forall i, j \in [1; 2K] \; \forall 0 \leq t \leq T : \; \Sigma_t^{i,j} = \rho^{ij} \, \sigma_t^i \sigma_t^j$$

Further, we assume the integrability conditions of the processes to hold. Thus $\forall i \in [1; K]$ and $\forall 0 \leq t \leq T$ it holds that $\int_0^t |\lambda_s^i| ds < \infty$, $\int_0^t |\kappa^i (\overline{m}_s^i - m_s^i)| ds < \infty$, $\int_0^t (\sigma_s^i)^2 ds < \infty$ and $\int_0^t (\sigma^i)^2 ds < \infty$ respectivly.

The solution of the cash flow process is well known by multivariate BLACK SCHOLES approaches:[109]

$$E_t^i = E_0^i \exp\left(\int_0^t \sigma_s^i dW_s^i + \int_0^t \left(\lambda_s^i - \frac{1}{2} {\sigma_s^i}^2 \right) ds \right)$$

Note that the quantity $\int_0^t \left(\frac{{\sigma_s^i}^2}{t} \right) ds$ is the total volatility of the $i'th$ asset component. The solution is a multidimensional geometric BROWNIAN motion, in the sense that it can be obtained applying Itô's Lemma to $E_t^i = f(X_t^i) = exp(X_t^i)$ with X_t^i the $i'th$ component of the multi-dimensional BROWNIAN motion with drift λ_t^i and the $i'th$ diffusion $\int_0^t \left(\frac{{\sigma_s^i}^2}{2} \right) ds$.

Next, each jump process refers to events such as asset-sales or special investment decision that solely depend on the underlying structure, we do not introduce any dependency between jump times or jump heights of different jump processes. Thus, we have for $i \in [1; K]$ in style of the univariate setting the $j'th$ jump waiting time of the $i'th$ jump process is exponential distributed with parameter

[109] See PIERGIACOMO/ SABINO (2007) Monte Carlo Methods and Path-Generation techniques for Pricing Multi-asset Path-dependent Options, p. 6

$i = 1, \ldots K\ j = 1, \ldots, n_i\ g_i > 0$: $G_j^i \sim exp(g_i)\ i.i.d$ and $j'th$ jump heights of the $i'th$ jump process is log-normal distributed $i = 1, \ldots K\ j = 1, \ldots, n_i$ $C_j^i \sim \mathcal{LN}\left(\mu_j^i, {\sigma_j^i}^2\right)\ i.i.d.$ with $n_i \epsilon \mathbb{N}$ denoting the number of jumps of the $i'th$ jump process. Hence, by defining the sign of the $j'th$ jump of the $i'th$ asset component by $i = 1, \ldots K\ j = 1, \ldots, n_i\ \delta_j^i \epsilon \{-1,1\}$: $P\left(\delta_j^i = 1\right) = p_i \in [0; 1]\ i.i.d.$, the multivariate POISSON process is $\forall t \in [0; T]$ and $\forall i = 1, \ldots, K$ with independent components C_j^i, δ_j^i and G_j^i given by:

$$J_t^i = \sum_{j=1}^{n_i} \delta_j^i C_j^i 1_{\left\{\sum_{k=1}^{j} G_k^i \leq t\right\}}$$

Instead of executing and repeating each milestone of the univariate setting for the multivariate setting again, we will only carry out the calculation of the aggregated cash equity basis, without once more reviewing the economic intuition. The multivariate setting differs from the univariate setting only by the interaction, due to correlations, of different components or multiples, leading to diversification effects, well known from portfolio management.

The stopping time of the $i'th$ cash flow process, is analogously defined as in the univariate model, hence $i = 1, \ldots, K$:[110]

$$\tau^i := inf\left\{0 \leq t \leq T : l_t^i = 1\right\}$$

with $\tau^i = inf\emptyset := \infty$

Again we have that τ^i is a stopping time, as we only need information up to time t, hence $\forall t \epsilon \mathbb{R}^+ : \left\{\tau^i \leq t\right\} \in \mathfrak{F}_t$. Once more we set the defaulting cash flow process and the resulting defaulting multiple process accordingly $\forall i = 1, \ldots, K\ \forall 0 \leq t \leq T$:

$$E_t^{i,df} = 1_{\{t < \tau^i\}} E_t^i$$

$$m_t^{i,df} = 1_{\{t < \tau^i\}} m_t^i$$

With l_t^i, r_t^i and η^i being defined adequately to the univariate setting for the $i'th$ asset component, we conclude:

[110] Note the correlations (implied volatility) between different cash flow processes, say i and j, introduces a desired dependency between τ^i and τ^j. If, for instance, the cash flow process i is positively correlated with the cash flow process j and is close to default, lower cash flow of process i will also lower the cash flow j and thus implicate that the stopping time τ^j will be smaller

Theorem 5.6:

The aggregated cash equity basis V_T for K assets including default events for each investment is given by:

$$
\begin{aligned}
\mathrm{V_T} = &-\sum_{i=1}^{K}\left[\left(1-l_0^i\right)P_o^i\right] \\
&+\sum_{i=1}^{K}\left(1-\eta^i\right)\int_0^{T\wedge\tau^i}\exp\left(-r_s^i s\right)E_s^i ds+\sum_{i=1}^{K}\int_0^{T\wedge\tau^i}\exp\left(-r_s^i s\right)J_s^i ds \\
&+\sum_{i=1}^{K}\left(1-l_{T\wedge\tau^i}^i\right)e^{-\int_0^T r_s^i ds}\left[(Em)_T^i\right]1_{\{T<\tau^i\}}
\end{aligned}
$$

with

$E_t^i := E_0^i \exp\left(\int_0^t \sigma_s^i dW_s^i + \int_0^t \left(\lambda_s^i - \frac{1}{2}{\sigma_s^i}^2\right) ds\right)$

Proof:

(1) The cumulated cash flow process E'^i_t available for investors $\forall i = 1,..,K$ given by:

$$E'^i_t = \left(1-\eta^i\right)E_t^i + J_t^i$$

with

$$E_t^i = E_0^i \exp\left(\int_0^t \sigma_s^i dW_s^i + \int_0^t \left(\lambda_s^i - \frac{1}{2}{\sigma_s^i}^2\right) ds\right)$$

thus the discounted cash flow to equity value $\mathrm{CFV_t^i}$ of the $i'th$ asset component is given by:

$$
\begin{aligned}
\mathrm{CFV^i} &= \int_0^{T\wedge\tau^i}\exp\left(-r_s^i s\right)E'^i_s ds \\
&= \left(1-\eta^i\right)\int_0^{T\wedge\tau^i}\exp\left(-r_s^i s\right)E_s^i ds + \int_0^{T\wedge\tau^i}\exp\left(-r_s^i s\right)J_s^i\, ds
\end{aligned}
$$

(2) Next we derive the discounted exit value $\mathrm{TV^i}$ for the asset component $i = 1,..,K$:

$$(Em)_T^i = (Em)_0^i + \int_0^T m_s^i dE_s^i + \int_0^T E_s^i dm_s^i + \left[m^i, E^i\right]_T$$

Since $d\left[W^i, W^i\right]_t = dt$

$\Rightarrow d\left[E^i, E^i\right]_t = \left(E_t^i\sigma_t^i\right)^2 dt$ and

$\Rightarrow d\left[m^i, m^i\right]_t = m_t^i\left(\sigma^{k+i}\right)^2 dt$

As E_t^i and m_t^i are independent

$$\Rightarrow d\left[(E+m)^i,(E+m)^i\right]_t = \left(E_t^i\sigma_t^i + \sqrt{m_t^i}\left(\sigma^{k+i}\right)\right)^2 dt$$

The risk adjusted discount rate for the $i^{'}th$ asset component is given by:

$$e^{-\int_0^T r_s^i ds}$$

Including default events we have for the terminal value

$$\mathrm{TV^i} = \left(1 - l_{T\wedge\tau^i}^i\right)e^{-\int_0^T r_s^i ds}\left[(Em)_T^i\right]1_{\{T<\tau^i\}}$$

(3) The investment value in the $i^{'}th$ asset is $\forall i = 1,..,K$ given by:

$$\mathrm{IV^i} = \left[\left(1 - l_0^i\right)P_o^i\right]$$

(4) Take to establish the claim:

$$\mathrm{V_T} = \sum_{i=1}^{K} -\mathrm{IV^i} + \mathrm{CFV^i} + \mathrm{TV^i}$$

$q.e.d.$

So V_T can be written as the sums of the individual investment values, of the ordinary cash flow values, of exceptional cash flow values and of the exit values. Let us put on record that this is not contradictory to the diversification effect, well known from portfolio theory, as diversification is incorporated in each valuation of each component since $\forall i,j = 1, \ldots, 2K\ \hat{\rho}^{ij} \in [-1;1]$.

Note that exceptional cash flows, such as recaps or asset sales, are the levers for the private equity investor to change return and risk of the investment – in particular the risk to default of a single investment.

5.4. Risk measures

In this section we suggest different risk measures, which are of interest for private equity investors. As the success of private equity firms is measured by the announced IRR that was committed to attract limited partners,[111] the investors may be interested in the risk of falling below that IRR. Thus in accordance with FISHBURN[112] risk is associated with outcomes falling below some specified target level, a hurdle h.

$$Risk(f) = \int_{-\infty}^{h} \varphi(h-x)dF(x) = \int_{-\infty}^{h} \varphi(h-x)f(x)dx$$

Where $\varphi(\cdot)$ is a nonnegative and non-decreasing function, and $F(x)$ is the probability distribution function of outcomes, e.g. $F(x)$ gives the probability of getting an IRR of less than or equal to x.[113] For instance we choose $\varphi(\cdot)$ so that

$$Risk(f) = \int_{-\infty}^{h} (h-x)^{\gamma} dF(x)$$

FISHBURN has shown congruence between this model and the expected utility model in which the utility function is

$$U(x) = \begin{cases} x & x > h \\ x - k(h-x)^{\gamma} & x \leq h \end{cases}$$

where k and γ are positive constants. The decision maker, here general partner, may display various degrees of risk aversion or preference for outcomes below h depending on the value of γ, but he is risk neutral for outcomes above h. After surveying a number of empirical studies of utility functions, FISHBURN concludes "that most individuals in the investment context do indeed exhibit a target return – which can be above, at, or below the point of no gain and no loss – at which there is a pronounced change in the shape of their utility functions, and that the given utility function can provide a reasonably good fit to most of these curves in the below-target region".[114]

[111] Cf. to BERG (2005) What is strategy for buyout associations, p. 42 and RUDOLPH (2008) Funktionen und Regulierungen der Finanzinvestoren, p. 2
[112] FISHBURN (1977) Mean Risk Analysis with Risk Associated Below-Target Returns, p. 116-120
[113] We can whether an IRR of x could be fulfilled, if we discount a cash equity basis with x instead of r_t. If we have $C(T) < 0$ then the associated path implied an IRR of less than x
[114] HOLTHAUSEN (1981): A Risk-Return Model with Risk and Return Measured as Deviations from a Target Return, pp. 182-185

Since investors are in general interested in a successful track record in order to attract further limited partners, they try to turnaround distressed investments through cash injections.[115] Along with a hit on the headlines, investors try to preclude investments with a significant risk to default, unvalued the chance to achieve an above average return. Hence another adequate risk measure for private equity investors is the probability to default.

We will close this section with a more theoretical risk measurement. The cash flow at risk (CFaR), which we define similarly to the business value at risk defined by BOECKER.[116] Denote, for $t \geq 0$, F_T as the distribution function of the cash equity basis V_T according to section 5.2 and 5.3, with mean $\mathbb{E}V_T < \infty$. Then we define the cash flow at risk at exit T and the confidence interval $\beta \epsilon (0,1)$ by:[117]

$$CFaR_{\beta}(T) := \mathbb{E}V_T - F_T^{\leftarrow}(1-\beta)$$

where $F_T^{\leftarrow}(\beta) = inf\{x \in \mathbb{R}: F_T(x) \geq \beta\}$ being the generalized inverse of F_T. If $F_T$is strictly increasing and continuous, we have $F_T^{\leftarrow} = F_T^{-1}$.

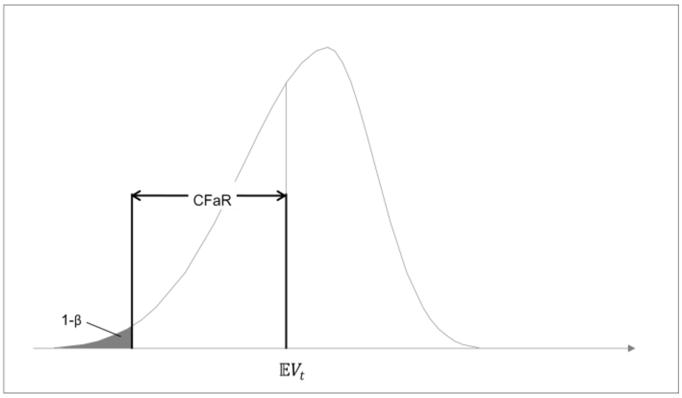

Figure 22: Schematic description of cash flow at risk following BOECKER[118]

CFaR is the maximum loss of cash flows not exceeded with a given probability β defined as the confidence level, over the investment interval. We can thus approximate the CFaR by the outcomes of Monte Carlo simulations on the value process V_T. Having finished the theoretic framework of the stochastic part of the cash equity basis, we will now turn to a hands-on approach on application software.

[115] See Appendix A1: Financial Times Deutschland 10.02.2008:
[116] See BOECKER (2008) Modelling and Measuring Business Risk, p. 8
[117] See EMMER, KLÜPPELBERG, KORN (2000) Optimal portfolios with bounded downside risks, pp. 4-10
[118] See BOECKER (2008) Modelling and Measuring Business Risk, p. 8

6. Implementation on application software

6.1. Selecting the Software Package

We will draw our simulation upon expectations and appraisements captured by investors in an LBO model. As we are dealing with a hands-on approach which aims at usability, we want to avoid changes in the application software. Preferably we want to fall back on linking the LBO forecasts directly to our tool. Hence, as investors normally work with Microsoft's spreadsheet programme Excel, we will rely on Visual Basic for Applications (VBA)[119] to solve the problem of the probability distribution of V_T with Monte Carlo Simulation.

We illustrate the methodology for valuing private equity investments by applying it to one mid cap investment, which is hold for 3 periods. The basic data are given in figure 23, showing the input sheet of our simulation tool.

Inputs from LBO Models						**Simulation setting**	
1. Dataset						**Other variables**	
	t=0	t=1	t=2	t=3			
FCF	1.627,4	1.829,0	2.715,4	4.237,2		EV/FCF t=0	10,3x
Debt	11.370,0	11.370,0	11.370,0	11.370,0		EV/FCF t=3	10,5x
Equity	13.630,0	13.630,0	13.630,0	13.630,0		Multiple Vola (%)	4,6%
Recovery	-	-	-	-		Multiple Reversion (%)	100,0%
Deterministic Adjustments	(1.870,0)					Stake size (%)	100,0%
Expected jump size		4.000,0				Initial value	13630,0
Vola of Add-on (%)		15%				Premium paid	-
						Implied IRR	78,8%
						Equity Contribution (%)	100,0%
						Market Vola (%)	14,8%
						Return on Debt (%)	7,0%
						Markt rate of return (%)	8,0%
						Hurdle rate of return(%)	30%
						Beta unlevered	0,99

Figure 23: Anonymised inputs from the LBO Model[120]

We describe the parameters of the model in figure 24 and give some suggestions about how to estimate them:

[119] The VBA codes can be found in Appendix A4-A6

[120] Note that $(1.870,0) := -1.870,0$. Also note that we adjust for financing in $t = 0$, as it is a non-recurring cash flow

Parameter	**Notation**	**Proposed Estimation Procedure**	**Numerical example figure 23**
Maturity	T	Observable from LBO model	3,0
Initial FCF	E_0	Observable from current cash flow statement	1.627,4
Stake size	$\propto$	Observable from LBO model	100%
Initial multiple	m_0	Estimated from the stock data of a public listed peer group	10,3
Initial Debt	D_0	Observable from current balance sheet	13.630,0
Initial equity	EQ_0	Observable from current balance sheet	11.370,0
Recovery at default	R^{df}	Random/ Asset-based company valuation	0
Leverage at time 0	l_0	Estimated by Debt and Equity at time 0	$\frac{D_0}{D_0 + EQ_0} = 54{,}52\%$
Exit multiple	m_T	Investor's future projections on peer group data	10,5
Standard deviation of multiple	σ	Investor's future projections	4,6%
Initial standard deviation of cash flow process	σ_I	Inferred from volatility of stock price	14,8%
Speed of adjustment of the multiple process	K	Estimated from assumptions about the half life of process $\overline{m}_t$	100%
Deterministic Adjustments at time t	A_t	Random	$A_0 = -1.870{,}0$
Stochastic Add-on at time t			
Maximum number of jumps	n	Random/ Investor's future projections	1
Intensity	g	Investor's future projections	1
Associated jump height	$\hat{\mu}_i$	Investor's future projections	4.000
Standard deviation of jump height	$\hat{\sigma}_i$	Investor's future projections	$15\% \cdot 4500 = 600$
Probability of a positive jump	p	Investor's future projections	100%
Degree of equity contribution	η	Investor's strategy characteristics	100%
Growth rate of cash flows to equity	λ_t	From current cash flow statement and investors future projections at time t, e.g. linear interpolation of FCF: $\lambda_t = \frac{E_t - E_{t-1}}{E_{t-1}}$	$\lambda_1 = 12{,}39\%$ $\lambda_2 = 48{,}46\%$ $\lambda_3 = 75{,}93\%$
Risk free rate of return	r_f	Government bond	7,0%
Market rate of return	r_m	Estimated by applying CAPM to the public listed peer group	8,0%
Beta unlevered	β	Estimated by applying CAPM to the public listed peer group	0,99
Implied internal rate of return	IRR	Investor's investment projection	78,8%
Hurdle rate of return	h	Assumption/ random	30%

Figure 24: Key parameters of the model

We restrict our case study in chapter 7 to the first jump, in order to study the effects of a single action isolated from side effects. Hence $\forall t \in [0;T]\, J_t$ becomes:

$$J_t = \delta_G C_G 1_{\{G \leq t\}}$$

The intensity $g > 0$ of $G \sim exp(g)$ is indicated by the presumed jump time t_i from the LBO model, with $g = \frac{1}{t_i}$. The simulation tool, however, can deal with more than one jump.

6.2. Simulation of relevant distributions

For our Monte Carlo simulation we need to generate random error terms in the diffusion processes, which are normally distributed. Further, we need to draw an exponentially distributed random variable for the inter jump time and a log normally distributed random variable for the jump size. The subsequent sections provide us with techniques to achieve the desired randomisation – the VBA codes can be found in Appendix A4.

6.2.1. Exponential distribution

Exponentially distributed random variables, which are needed for simulating the inter jump times of the instantaneous cash flows, can be obtained by applying the simulation lemma – which is also referred to as inverse method.

Lemma 6.1: [121]

Let $U \sim U(0,1)$ and let F be a distribution function. For $0 < p < 1$ define

$$F^{-1}(p) = inf\{x: F(x) \geq p\}$$

Then $X = F^{-1}(U) \sim F$

Thereby, on the basis of the uniformly distributed random variables on $[0,1]$ we are able to generate every invertible distribution function F. The distribution function of an exponential distributed random variable $G \sim exp(g), g > 0$, is given by:

$$P(G \leq x) = F(x) = 1 - e^{-gx}, x \geq 0$$

Then

$$G = -\frac{1}{g} ln(1 - U) \text{ for } U \sim U(0; 1)$$

For $U \sim U(0; 1)$ also $V \coloneqq 1 - U \sim U(0; 1)$ hence

$$-\frac{lnV}{g} \sim exp(g)$$

[121] GLASSERMANN (2004) Monte Carlo Methods in Financial Engineering, pp. 54-56

6.2.2. Normal distribution

As the distribution function of a one dimensional normal distributed random variable $\varepsilon \sim N(0,1)$ is not invertible, we need other techniques than the simulation lemma. Perhaps the simplest method to implement is the BOX-MULLER algorithm:[122] The algorithm is based on the following properties:

(1) Assuming $\varepsilon_1, \varepsilon_2 \sim N(0;1)\ i.i.d.$ it holds that $R = \varepsilon_1^2 + \varepsilon_2^2 \sim \exp\left(\frac{1}{2}\right)$

(2) Given R the normal vector $(\varepsilon_1, \varepsilon_2)$ is uniformly distributed on the circle of radius $\sqrt{R}$. Thus, we take the angle B with $B \coloneqq 2\pi U_1$ and $U_1 \sim U(0;1)$ uniformly distributed on $[0;2\pi]$

(3) We have seen in section 6.2.1 that we get R by $R \coloneqq -2lnU_2$ with $U_2 \sim U(0;1)$ independent from U_1

(4) Ascribing the coordinates $\left(\sqrt{R}cosB; \sqrt{R}sinB\right)$ on the corresponding point on the circle with radius $\sqrt{R}$ to $(\varepsilon_1, \varepsilon_2)$ yields:

$$\varepsilon_1, \varepsilon_2 \sim N(0,1)\ i.i.d.$$

where $B \coloneqq 2\pi U_1$, $R \coloneqq -2lnU_2$ and $U_1, U_2 \sim U(0;1)\ i.i.d.$.

Within this thesis we will only deal with an implementation of different strategies on univariate, single investment level. Nevertheless, we will also briefly raise the main ideas for the multivariate setting. In the multivariate model, discussed in section 5.3, we introduced correlated BROWNIAN motions, wherefore we need to find a way to generate correlated normal distributed random variables.

For all points t_i $(i = 1, .., n \in \mathbb{N})$ in the time grid $0 = t_0 < t_1 < \cdots < t_n = T$ we will use the instantaneous covariance matrix at time t_i, defined by:

$$\Sigma_{t_i}^{ij} = \rho_{ij}\sigma_{t_i}^{i}\sigma_{t_i}^{j}$$

with ρ_{ij} the constant correlation of the underlying processes of asset i with asset j, for $i, j = 1, \dots, 2K$, where K denotes the number of assets. All information is carried out by n, the granularity of the time grid, time-varying $2K \times 2K$ block matrices. For any time $t_i \epsilon [0;T]$ we will use the Cholesky decomposition of Σ_{t_i}.[123]

$$\Sigma_{t_i} = A_{t_i}A_{t_i}^T$$

[122] GLASSERMANN (2004) Monte Carlo Methods in Financial Engineering, pp. 65-67

[123] PIERGIACOMO (2007) Monte Carlo Methods and Path-Generation techniques for Pricing Multi-asset Path-dependent Options, pp. 9-16

The CHOLESKY decomposition is unique and A is a lower triangular matrix with strictly positive diagonal entries and A^T denotes the transpose of A.

Multiplying A to a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{2K})$ independent normally distributed random variables $i = 1, \dots, 2K$ $\varepsilon_i \sim N(0,1)$ we obtain a correlated normal distributed vector $\check{\varepsilon}$.

$$A\varepsilon \coloneqq \check{\varepsilon}$$

which can be easily seen, from

$$[A\varepsilon]^T[A\varepsilon] = \varepsilon^T A^T A\varepsilon = \varepsilon^T \Sigma \varepsilon$$

together with $\mathbb{E}(\varepsilon) = 0$.

6.2.3. Lognormal distribution

Finally, we need to find an approach for modelling the jump size C, which we model as log-normally distributed. Mean and variance of the log-normal distribution $C \sim LN(\mu, \sigma^2)$ are:[124]

$$\hat{\mu} \coloneqq \mathbb{E}(C) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

$$\hat{\sigma}^2 \coloneqq VAR(C) = \mathrm{e}^{(2\mu+\sigma)}\left(\mathrm{e}^{\sigma^2} - 1\right)$$

As we rely on the appraisements of the investors in terms of mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$, hence solving for μ and σ^2 yields:

$$\sigma^2 = ln\left(\frac{\hat{\sigma}^2}{\hat{\mu}^2} + 1\right)$$

$$\mu = ln(\hat{\mu}) - \frac{\sigma^2}{2}$$

Finally on the basis of a normally distributed $\varepsilon \sim N(0,1)$, the transformation $e^{\varepsilon\sigma+\mu}$ yields a random variable, with mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$:[125]

$$e^{\varepsilon\sigma+\mu} = e^{\left(\varepsilon - \frac{1}{2}\right) ln\left(\frac{\hat{\sigma}^2}{\hat{\mu}^2} + 1\right) + ln(\hat{\mu})} \sim LN(\mu, \sigma^2)$$

[124] Cf. GLASSERMANN (2004) Monte Carlo Methods in Financial Engineering, p. 95
[125] Cf. GLASSERMANN (2004) Monte Carlo Methods in Financial Engineering, p. 63

6.3. EULER Scheme

The model developed in the previous section is path dependent. The EV at any time, which determines when bankruptcy is triggered, depends on the whole history of past cash flows and multiples. Similarly, stochastic adjustments from recaps or asset sales, are also path dependent. These path dependencies can easily be taken into account by using Monte Carlo Simulation[126] to solve for the risk of a private equity investment. The associated calculation of the first two moments for the cash equity basis is, due to the composition of the terminal value (TV), not straight forward, thus we make a step back and discretise the continuous time model with the EULER approximation. Hence, for the implementation of the simulation, of the univariate setting, we apply the EULER scheme for the cash flow process and the multiple process with time-dependent drift and diffusion:[127]

$$E_{t+\Delta t} - E_t = \Delta E_t \approx E_t \lambda_t \Delta t + E_t \sigma_t \Delta W_t$$

$$m_{t+\Delta t} - m_t = \Delta m_t \approx \kappa(\overline{m}_t - m_t)\Delta t + \sigma\sqrt{m_t}\Delta\widetilde{W}_t$$

For $n \in \mathbb{N}$ with the partition of $[0;T]$ $0 = t_0 < t_1 < \cdots < t_n = T$ at a constant step size $\frac{T}{n} = t_{i+1} - t_i = \Delta t$, and $\Delta W_t \sim N\left(0, \sqrt{\Delta t}\right)$ as well as the thereof independent $\Delta\widetilde{W}_t \sim N\left(0, \sqrt{\Delta t}\right)$. Hence, with $E_0, m_0 > 0$:

$$E_{t+\Delta t} \approx E_t + E_t \lambda_t \Delta t + E_t \sigma_t \Delta W_t$$

$$m_{t+\Delta t} \approx m_t + \kappa(\overline{m}_t - m_t)\Delta t + \sigma\sqrt{m_t}\Delta\widetilde{W}_t$$

Because BROWNIAN motion has independent and stationary, normally distributed increments, simulation of a single BROWNIAN motion of the grid $t_1, \dots, t_n$ is straightforward with $\varepsilon \sim N(0,1)$.[128]

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t}\varepsilon$$

[126] Cf. SCHWARTZ/ MOON (2000) Rational Pricing Of Internet Companies, p. 3

[127] Cf. GLASSERMANN (2004) Monte Carlo Methods in Financial Engineering, p. 81 and pp. 340-342

[128] See PIERGIACOMO (2007) Monte Carlo Methods and Path-Generation techniques for Pricing Multi-asset Path-dependent Options, p. 11

6.4. Interpolation of time-dependent Functions

We have already seen in section 6.1 that we will set up our simulation via the uniform steps $0 = t_0 < t_1 < \dots < t_n = T$ (usually years or quarters) of the LBO model, which we denote as major time grid. In order to improve approximation we introduce a minor time grid, which is derived by splitting up one interval of the LBO model, for instance $[t_1, t_2]$ into m smaller subintervals $t_1 = t_{10} < t_{11} < \dots < t_{1m} = t_2$ of equal length. Hence, the minor time grid is given by $0 = t_0 = t_{00} < \dots < t_{0m} < t_{11} < \dots < t_{1m} < \dots < t_{(n-1)1} < \dots < t_{(n-1)m} = t_n = T$.

We take the information of the major time grid and derive the values of the time dependent functions at the minor time grid by a linear interpolation.[129] In particular, we will interpolate the expectations to the time dependent drift rate of the cash flows as well as mean expectations of the multiple – cf. figures 23 and 24. We rely on the growth rates of the FCFE forecast $\hat{E}_{t_0}, \dots, \hat{E}_{t_n}$ in the LBO model, and thus have for the major time grid t_i for $i = 0,1,\dots,n$ by a linear interpolation:

$$\lambda_{t_i} := \frac{\hat{E}_{t_{i+1}} - \hat{E}_{t_i}}{\hat{E}_{t_i}}$$

We want to put on record, that we do not rely on the instantaneous cash flow forecast $\hat{E}_{t_i}$. We assign the major time with grid growth rate λ_{t_i} for $i = 0,1,\dots,n-1$ to the minor time grid t_{ij} for $i = 0,1,\dots,n-1$ and $j = 0,1,\dots,m$:

$$\lambda_{t_{ij}} := \lambda_{t_i}$$

Similar, we assign the mean expectation of the multiple $(\hat{\hat{m}}_0, \hat{\hat{m}}_T)$ to the major time grid t_i with $\forall i = 0,1,\dots,n$.

$$\hat{\hat{m}}_{t_i} := \hat{\hat{m}}_0 + \frac{i}{n}(\hat{\hat{m}}_T - \hat{\hat{m}}_0)$$

Finally linear interpolation to the minor time grid t_{ij} for $i = 0,1,\dots,n-1$ and $j = 0,1,\dots,m$ yields:

$$\hat{\hat{m}}_{t_{ij}} := \hat{\hat{m}}_0 + \frac{im + j}{nm}(\hat{\hat{m}}_T - \hat{\hat{m}}_0)$$

[129] One can also establish interpolations of higher order

7. Case Study

Before working through the case study, we want to put on record, that some dimensions, e.g. the structure of outside investors of buyout funds or the revenue model's characteristic, is handled confidentially.[130] For the accessed investment data a non-disclosure agreement (NDA) was signed, so we needed to anonymous information can only show selected or adjusted data.[131]

7.1. Specification of examined strategies

Within this thesis we will scrutinise three main strategic configurations. The first one is the equity contribution η of the investment. Investors determine the percentage of free cash flows to equity that are reinvested to repay debt obligations, and thus decrease the risk of the investment. Remaining cash is contributed to investors. In terms of equity contribution to the investor we use the following configurations: $\eta =$ 0%, 10%, 20%, 30%, 40%, 50%, 60%, 70%, 80%, 90%, 100%.

The second configuration is an enhancement of the 100% equity contribution strategy: a debt financed recap, which increases the net debt position of the investment, but also constitutes an extra dividend. We will compare different executions relating to the extent and due date of the recap. We restrict the recap configuration to a maximum of 5.000 which is due to creditor issues arising from recaps exceeding 5.000.

The third configuration is the initial leverage l_0. We will take a brief look on empirical averages to obtain a meaningful range. Back in the 1980s, a typical buyout was put with only 10% of the value of a deal in equity.[132] By the late 1990s, however, the required equity stake had widened to 20% of a deal's value.[133] Now, with big banks' balance sheets stretched, the buyout specialists are finding that they have to

[130] See BERG (2005) What is Strategy for Buyout Associations, p. 86
[131] Thus we do not show calculations, currency, cash flows of the investment
[132] BAKER/ SMITH (1998) The New Financial Capitalists: KKR and the Creation of Corporate Value, p. 201
[133] CLOW/ SMITH (2002) Scands Help Break the Deal-Drought: Life Has Come Back to a Moribund Sector, p. 1 as well as HARDYMON et al. (2003) Between a Rock and a Hard Place: Valuation and Distribution in Private Equity, p. 3

put up around 40% in equity.[134] Hence, we will analyse following configurations of financing: l_0 =95%, 85%, 75%, 65%, 55%, 45%, 35%.

7.2. Parameter estimation

Besides expectations of cash flows and debt which can be derived from the LBO model, we will need to estimate the standard deviation of the associated processes as well as the risk-adjusted discount rate.

First, we recall the security market line (SML), which is given by:

$$r_e = r_f + (r_m - r_f)\beta_{levered}$$

with $\beta_{levered} = \frac{Cov(r_e, r_m)}{Var(r_m)}$ denoting the equity beta factor, $r_e \in \mathbb{R}_+$ the return on equity, $r_m \in \mathbb{R}_+$ a market rate of return, and $r_f \in \mathbb{R}_+$ the risk free discount rate. We will estimate the investments $\beta_{levered}$ as follows:[135]

- First we will identify the business in which the target operates
- Second we find other publicly traded firms in the same sector, the peer group, and obtain their regression betas. Where r_m are the returns of an embracing index, e.g. the Morgan Stanley Capital International (MSCI) index, and r_e are the stock returns of each peer with r_f denoting the risk free rate of return approximated by government bonds
- We estimate the unlevered beta for each company i in the peer group, by unlevering the beta for the firm by their average debt to equity ratio. Assuming a tax free world on dividends we have:

$$\beta^i_{unlevered} = \frac{\beta^i_{levered}}{1 + \frac{debt_i}{equity_i}}$$

- To estimate the unlevered beta for the firm that we are analysing, we take the mean of the unlevered betas of their peers.
- Finally, we estimate the current market values of debt and equity of the firm and use this debt to equity ratio to estimate a levered beta

The betas estimated using this process are called bottom-up betas.

[134] The Economist (2003) The Charms of the Discreet Deal, p. 60
[135] See DAMODARAN (2001) Investment Valuation, Chapter 8, p. 23f

We also base the estimation of the free cash flow volatility upon these bottom-up betas. We have:

$$\beta_{levered} = \frac{\sigma_e}{\sigma_m} Corr_{e,m}$$

Thus, we will estimate the appropriate cash flow risk by:

$$\sigma_e = \frac{\beta_{unlevered}\ \sigma_m}{Corr_{e,,m}} \left(1 + \frac{debt}{equity}\right) \coloneqq \sigma_I \frac{1}{1 - l_t}$$

with $\sigma_I \coloneqq \sigma_{Industry} \coloneqq \frac{\beta_{unlevered}}{Corr_{e,,m}} \sigma_m$. Accordingly to the procedure of bottom-up betas we will thus use the average $\sigma_{industry}$ of the peer group, which we will then re-lever, by taking the investments' actual market value of debt and equity, in order to arrive at the appropriate cash flow risk to equity investors.

Finally, we need to estimate the standard deviation of the multiple. This is straightforward: We take, from the peer group universe, the average standard deviation of historic analyst expectations. We will thus capture the industry specific risk occurring from dependencies on business cycle or technology jumps.

Our estimates for the scrutinised target's industry are $\beta_{unlevered} = 0{,}99$; $\beta_{levered} = 1{,}07$; $\sigma_m = 12{,}02\%$; $\sigma_{indu\ stry} = 14{,}79\%$; $\sigma_{multiple} = 4{,}6\%$ - cf. figure 23.[136]

[136] Data for the last 250 days provided by Thomson Financial Banker

7.3. Risk-Return profiles

Within this section we are dealing with an investor whose risk aversion is infinite for outcomes below 30% and who is risk neutral for outcomes above 30%.[137] We will determine the optimal configuration of the financing strategy in three steps. First, we determine the optimal leverage level. Second, given the selected leverage level, we scrutinise the degree of equity contribution. And third, we will examine, whether the chosen strategy could be improved by a debt-financed recap.[138]

Drawing a leverage level of 35% on the investment, we arrive at a targeted IRR of 29%. A de-leveraging of the transaction leads to a higher equity investment and translates into a lower expected return from a successful investment.[139]

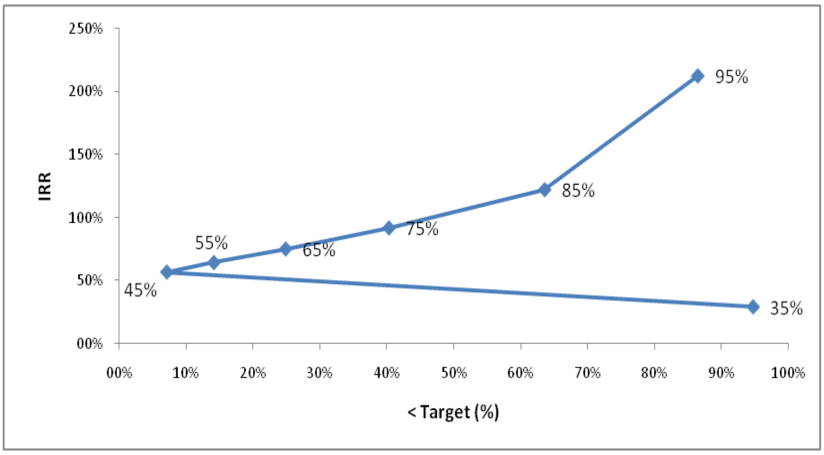

Figure 25: Risk return profile for the leverage level, showing the probability to fall below the targeted return of 30% in dependence on the actual leverage level (n=50.000)

Figure 25 shows the risk-return profile in terms of targeted IRR and the associated risk, measured by the probability to fall below the target hurdle rate of 30%.[140]

Additionally, we establish that the investor should prefer a leverage level of 45%, as it dominates the strategy with a leverage of 35% and all other strategies bear more risk to fall below the targeted 30%.

[137] Hence, within this case study, we take the probability to fall below the target return level of 30% as the adequate risk measure

[138] Note that this approach examines each strategic event on its own, without taking interdependencies into account. As we provide no analytical solution to our model, we are restricted to this procedure

[139] PEACOCK/ COOPER (2000) Private Equity: Implications for Financial Efficiency and Stability, p. 71 and HARDYMON et. al (2003) Between a Rock and a Hard Place: Valuation and Distribution in Private Equity, p. 3 and cf. to section 3.2.3.

[140] Cf. Appendix A3 table 1. Configuration of the simulation $l_0 = 35\%, 45\%, 55\%, 65\%, 75\%, 85\%, 95\%;\ \eta = 0\%;\ n = 0$

Taking a configuration of 45% leverage, figure 26 shows the outcomes in terms of different equity contributions.[141] We can see from figure 26, that a different equity contribution does not change significantly both the implicit IRR and the associated risk measured by the probability to fall below the target of 30%. Nevertheless the strategy with a 100% distribution to equity dominates all other strategies in terms of risk and return.

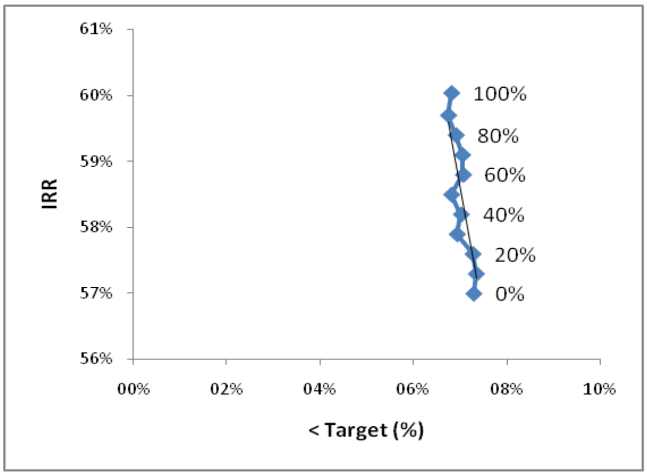

Figure 26: Risk return profile, showing the probability to default in dependence on the actual contribution to equity (n=50.000)

Finally, we will analyse, whether the configuration of a 100% equity contribution and a 45% leverage, can be improved by a debt financed recap. Figure 27 and Figure 28 show the results from different recap scenarios.[142]

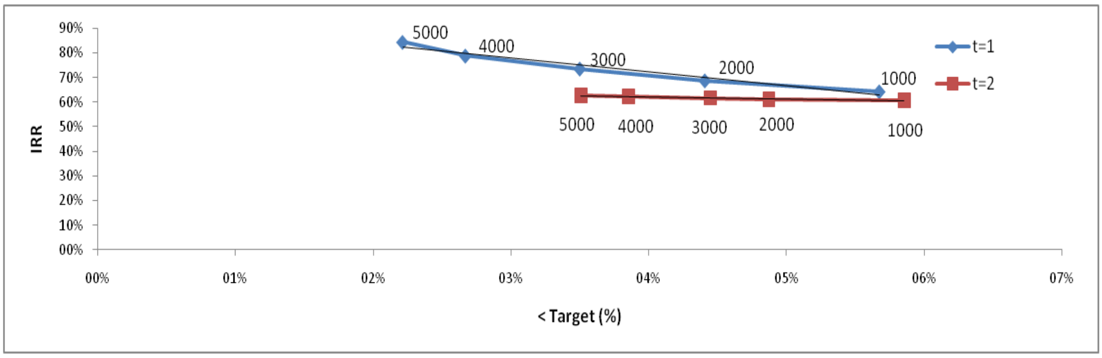

Figure 27: Risk return profile, showing the probability to fall below the target return of 30% in dependence on different recap scenarios (n=50.000)

[141] Refer to Appendix A3, table 2. $l_0 = 45\%;\ \eta = 0\%, 20\%, 40\%, 60\%, 80\%, 100\%;\ n = 0$

[142] Refer to Appendix A3, table 3 $l_0 = 45\%;\ \eta = \ 100\%;\ n = 1;\ \hat{\mu}_1 = 1.000, 2.000, 3.000, 4.000, 5.000;\ \hat{\sigma}_1 = 0{,}15\hat{\mu}_i;\ g = 1{,}2$

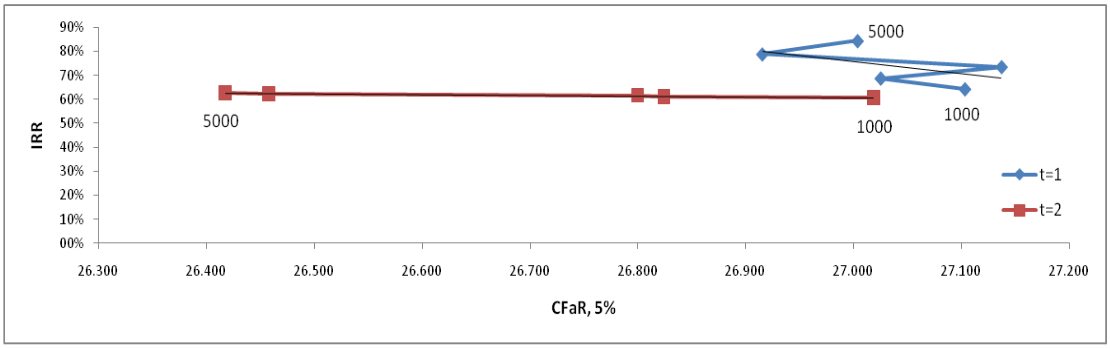

Figure 28: Risk return profile, showing CFaR to the 5% quantile in dependence on different recap scenarios (n=50.000)

Both figures support a recap, both in $t = 1$ and $t = 2$, as it dominates the strategy without a recap scenario.[143] As one can see from figure 28 a recap decreases cash flows at risk. An investor, taking the expectations from $t = 0$, that values a strategy according to risk and return, in terms of IRR and the probability to fall below the exogenously given target level of 30%, should cash out this investment.

We close this case study with figure 29 constituting the probability distribution of the chosen strategy, with 45% leverage, 100% equity contribution and a recap of 5.000 in $t = 2$.

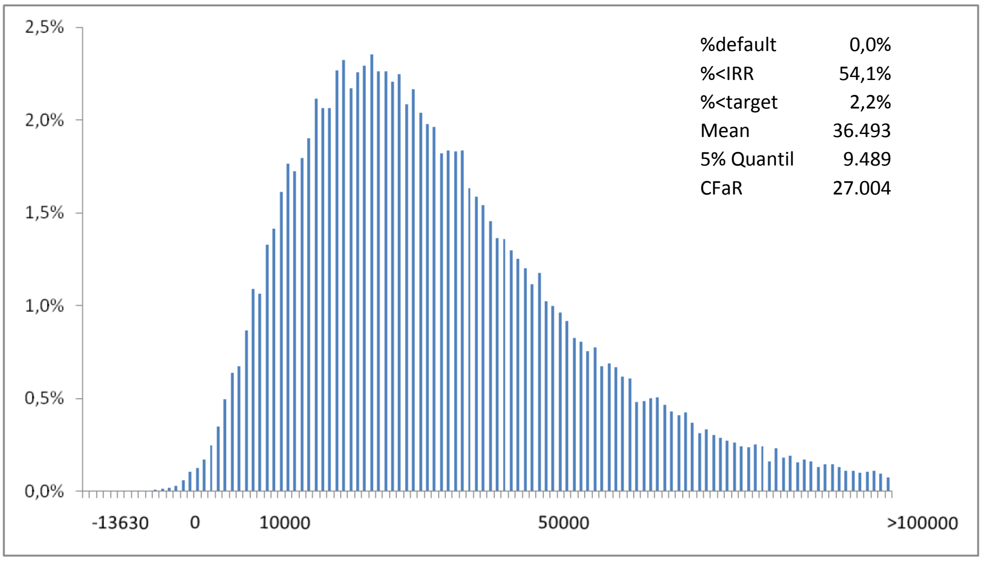

Figure 29: Probability distribution of the net cash equity basis of the investment with a 45% leverage, 100% equity contribution and a recap value of 5.000 in t=1 (n=50.000)

[143] The appropriate values for the strategy without risk can be found in table 2 Appendix A3

8. Conclusion

Within this thesis we developed a stochastic model for cash flows to private equity investors based on an ordinary DCF approach. Thereby we have been able to analyse risks of a mid cap investment in terms of different investment strategies. Our model compromises the four fundamental value drivers identified in section 3.2: The active value drivers top line growth and operational efficiency are accounted by the cash flow drift. The passive value driver multiple expansion is integrated by assuming a stochastic process to the multiple evolution, which accounts for market expectations. Finally, the leverage effect is mapped by the underlying capital structure, affecting cash flow risk, the risk adjusted discount rate, and risk premiums that are captured in the expected exit value.[144]

Furthermore, the empirical analysis by COCHRANE could be incorporated, which can be seen from figure 30. Figure 30 shows the probability distribution of the investment analysed in section 7 incurred with a 75% leverage and a 0% equity contribution.[145]

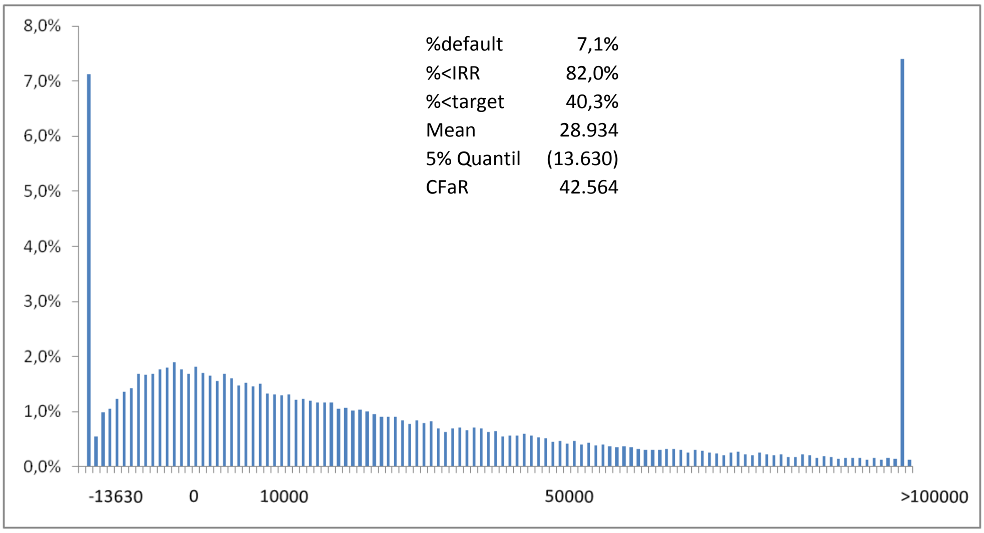

Figure 30: Probability distribution of the net cash equity basis of the investment with a 75% leverage, 0% equity contribution (n=50.000)

Moreover, we want to check whether our model maps our expectations in terms of the dependence of leverage and default adequately. We expect that the more debt

[144] Cf. to corollary 5.4
[145] Cf. to figure 10

investors incur on a company, the more likely default will be; this is what we can see from figure 31:

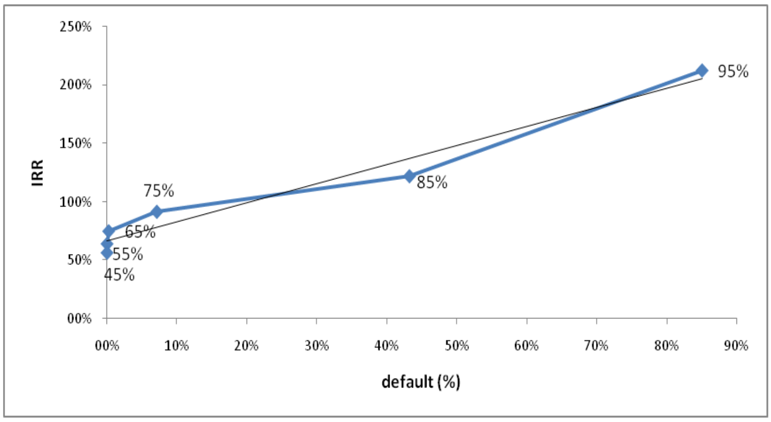

Figure 31: Risk return profile, showing the probability to default in dependence on the actual leverage level (n=50.000)

We can also see that incurring debt up to 65% of total capital does not increase the probability to default, whereas the IRR increases significantly. After 65% the probability to default starts to soar and to conflict with a higher IRR.

Finally, we will bring up the shortcomings of our model. We worked on the premises that multiple and cash flow processes are independent. An extended version of the model provided in this work could be set up without this assumption. Further, a risk neutral valuation of the investment strategies was not provided; this could be a topic for further research in these fields. Also, a flexible investment horizon similar to the exercise time of American options, adapting to the dynamics of the cash flows, could be interesting for further research. Nevertheless, it was the goal of this thesis to develop an easy-to-use tool to measure the risk of private equity investments. The paper has shown a first approach to solve this request, by taking up several ideas and assumptions made by practitioners.

We have seen in the case study in section 7 that it can be optimal for investors to reduce cash flows at risk by cashing out the investment via recaps. Thus at least for the studied investment, our model supports on the one hand the image of grasshoppers[146] discussed in the media. But on the other hand it has been shown that investment risk can be reduced by cashing out the investment. Hence, we raise the question whether the analysed risks are also crucial for social welfare or just crucial for the private equity investor to fulfil a target return level.

146 Cf. the quotation of Franz Muentefering at the beginning of this work

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Appendix A1– Financial times article

Financial Times Deutschland 10.02.2008:
KKR schießt bei ATU Millionen nach
Der US-Finanzinvestor KKR will sein kriselndes Investment in die Werkstattkette ATU Auto-Teile-Unger mit einer Kapitalspritze retten. "Wir stehen zu unserem Investment und werden den Banken ein Finanzierungspaket vorlegen, das auch die Zuführung von Eigenkapital vorsieht", sagte ein KKR-Sprecher.

Wie viel KKR nachschießen wird, wollte er nicht sagen. Laut Finanzkreisen dürfte der Betrag im unteren dreistelligen Millionenbereich liegen. ATU ist seit dem Kauf durch KKR im Jahr 2004 für 1,45 Mrd. Euro hoch verschuldet. Der zweite sehr milde Winter und die durch den Investor beschleunigte Expansion haben die Werkstattkette mit 15.000 Mitarbeitern in die Enge getrieben.

Der Fall von ATU ist der größte in der deutschen Unternehmensgeschichte, in dem eine Private-Equity-Gesellschaft ihre Firma per Nachschuss rettet. Nachträgliche Kapitalspritzen sind bei den Investoren unbeliebt und sehr selten, da sie die Rendite drastisch verwässern. 2002 stand der britische Finanzinvestor Apax bei der Bundesdruckerei ebenfalls vor einem Nachschuss, übertrug den Geldnotendrucker aber dann für 1 Euro einem Treuhänder. Der Zusammenbruch der Bundesdruckerei hatte Apax' Ruf über Jahre schwer geschädigt.

Nach dem Ende der Kredithausse der vergangenen Jahre könnte eine Krise wie bei ATU 2008 auch anderen Firmen in der Hand von Finanzinvestoren drohen. Als Kandidat nennen Banker den Folienhersteller Klöckner Pentaplast, den Blackstone 2007 für 1,3 Mrd. Euro gekauft hatte. Wie ATU hat auch Klöckner die Geschäftspläne verfehlt, was die Firmen in Konflikt mit den Kreditgebern bringt.

ATU hatte vergangene Woche den Gläubigern verheerende Zahlen gemeldet: 2007 brach der Gewinn vor Zinsen, Steuern und Abschreibungen (Ebitda) laut Finanzkreisen um 35 Prozent auf 105,5 Mio. Euro ein. Dies lag um mehr als 30 Mio. Euro unter den schon revidierten Erwartungen. Der ursprüngliche Plan hatte 210 Mio. Euro vorgesehen.

In Finanzkreisen wird damit gerechnet, dass ATU die im vergangenen Sommer erst gelockerten Kreditbedingungen gebrochen hat. Damit hätten die Banken das Recht gehabt, die Kredite im Volumen von etwa 800 Mio. Euro zu kündigen. ATU ist schon zum Zielobjekt für Hedge-Fonds geworden: "Die ATU-Kredite wurden in den vergangenen Monaten gut gehandelt", sagte ein Händler. So seien Blue Bay und Silver Point eingestiegen. Die Hedge-Fonds hatten beim Autozulieferer Kiekert per Kreditkauf die Kontrolle übernommen und die Beteiligungsfirma Permira herausgedrängt.

Ein solches Szenario hätte auch ATU gedroht - was KKR aus Imagegründen verhindern will. "Wenn sie ATU an die Wand fahren, haben sie in Deutschland ein Problem", heißt es in der Private-Equity-Branche. KKR hat für ATUs Sanierung die US-Investmentbank Goldman Sachs und die Beratung Roland Berger mandatiert. Vor wenigen Tagen kam die auf schwierige Fälle spezialisierte US-Bank Houlihan Lokey hinzu. Das neue Finanzierungspaket soll laut Finanzkreisen im März abgeschlossen werden.

Appendix A2– Keith Wibel column

Source: http://bigpicture.typepad.com/comments/2005/08/earnings_or_mul.html

Earnings or Multiple Expansion?

There's a fascinating analysis (in Barron's), looking at S&P500 earnings in a very different way than our prior discussions of using year-over-year S&P500 earnings changes as a buy signal.

Keith Wibel, an investment adviser at Foothills Asset Management, observes that:

"Over 10-year periods, the major determinant of stock-price returns isn't growth in corporate profits, but rather *changes in price-earnings multiples*. The bull market of the 1980s represented a period when multiples in the stock market doubled- then they doubled again in the 1990s. Though earnings of the underlying businesses climbed about 6% per year, stock prices appreciated nearly 14% annually."

I've seen other analyses that show well over half, and as much as 80% of the gains of the 1982-2000 Bull market may be attributable to P/E multiple expansion.

Wibel's piece in Barron's lends some more weight to this theory that *"rising price-earnings multiples are the key driver of stock-price gains, and further, the decline in P/Es since the 1990s bodes ill for equity investors."*

Here's the Historical Data:

	S&P 500 Annual Change		**P/E Ratio**	
Decade	**EPS**	**Index**	**Beginning**	**Ending**
1950s	3.9%	13.6%	7.2	17.7
1960s	5.5	5.1	17.7	15.9
1970s	9.9	1.6	15.9	7.3
1980s	4.4	12.6	7.3	15.4
1990s	7.7	15.3	15.4	30.5
2000s*	4.1	-3.8	30.5	20.7
Average	6.1%	8.1%	7.2	16.4

Projected Figures For
S&P 500 In 2014

	Average	High	Low
EPS	$105.85	$131.16	$81.02
P/E	16.4	23.4	9.4
Level	1735.94	3069.14	761.59
10-Year Growth Rate**	3.5%	9.5%	-4.7%
Dividend Yield	1.7%	1.7%	1.7%
Annual Gain***	5.2%	11.2%	-3.0%

*Through Dec. 31, 2004
**Compound rate
***From S&P 500's level of 1234.18 on July 31, 2005

Even after the multiple compression during the 2000's from 30 to 20, we are still at relatively high P/Es, at least when compared to prior early Bull market stages. That's yet another factor which argues against this being anything other than a cyclical Bull market within a secular Bear. Or in plain English, this is not the early stages of a decade plus of market growth.

Here's the Ubiq-cerpt:™

"Conventional wisdom states that share prices follow earnings. Over very long periods, this statement is correct. However, the time necessary to validate this assertion is much longer than is relevant to most investors.

In order to test the conventional wisdom, we examined the growth in earnings in each decade, beginning with the 1950s. We chose 10-year periods because they're long enough to allow the cyclical peaks and valleys to offset each other, yet short enough to be a reasonable planning horizon for most investors. The results of the study are shown in one of the accompanying tables.

There is very little correlation between earnings growth and share-price appreciation. During the 1950s, earnings grew less than 4% a year, yet that was one of the best decades for stock-price performance. The 1970s saw the fastest earnings growth in the past 55 years, but that was the worst decade for investors in the stock market. (Fortunately, the book is still open on the 2000s.)

The average rate of earnings growth clusters around 6% a year, reflecting growth in the economy which tends to average 3% to 4% per year. Add 2% to 3% annually for inflation and one is back to approximately 5% to 7% growth in nominal gross domestic product and the growth in profits for the companies in the S&P 500 Index.">

Appendix A3– risk return tables

Optimal Leverage							
Return Figures							
	95%	85%	75%	65%	55%	45%	35%
IRR	212,5%	122,1%	91,7%	75,1%	64,3%	56,7%	29,1%
Mean	114.843	27.638	28.934	30.906	32.520	34.461	22.442
Mean/ Initial	8,4x	2,0x	2,1x	2,3x	2,4x	2,5x	1,6x
Risk Measures							
	95%	85%	75%	65%	55%	45%	35%
default (%)	85,1%	43,2%	7,1%	0,2%	0,0%	0,0%	0,0%
< IRR (%)	90,4%	87,9%	82,0%	72,3%	58,0%	41,0%	28,3%
< Target (%)	86,4%	63,6%	40,3%	24,8%	14,1%	7,1%	94,7%
CFaR, 5%	128.472,6	41.268,1	42.563,6	35.069,4	30.257,6	27.178,3	17.343,0

Table 1: Risk return table in terms of leverage level (n=50.000)

Equity Contribution											
Return Figures											
	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
IRR	60,0%	59,7%	59,4%	59,1%	58,8%	58,5%	58,2%	57,9%	57,6%	57,3%	57,0%
Mean	33.895	33.910	33.780	33.537	33.598	33.764	33.587	33.700	33.684	34.026	34.369
Mean/ Initial	2,5x	2,5x	2,5x	2,5x	2,5x	2,5x	2,5x	2,5x	2,5x	2,5x	2,5x
Risk Measures											
	100%	90%	80%	70%	60%	50%	40%	30%	20%	10%	0%
default (%)	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
< IRR (%)	40,8%	41,1%	40,9%	41,3%	41,5%	41,2%	41,5%	41,5%	42,0%	42,0%	41,7%
< Target (%)	6,8%	6,7%	6,9%	7,0%	7,1%	6,8%	7,0%	6,9%	7,3%	7,3%	7,3%
CFaR, 5%	27.321,5	27.237,4	27.112,6	26.885,8	26.825,2	26.770,9	26.692,5	26.678,5	26.614,1	27.023,4	27.217,6

Table 2: Risk return table in terms of equity contribution (n=50.000)

Recap Scenarios										
Return Figures										
	1000 in t=1	2000 in t=1	3000 in t=1	4000 in t=1	5000 in t=1	1000 in t=2	2000 in t=2	3000 in t=2	4000 in t=2	5000 in t=2
IRR	64,2%	68,7%	73,5%	78,8%	84,4%	60,6%	61,1%	61,6%	62,2%	62,7%
Mean	34.228	34.841	35.533	36.186	36.493	34.038	34.448	34.624	34.900	35.315
Mean/ Initial	2,5x	2,6x	2,6x	2,7x	2,7x	2,5x	2,5x	2,5x	2,6x	2,6x
Risk Measures										
	1000 in t=1	2000 in t=1	3000 in t=1	4000 in t=1	5000 in t=1	1000 in t=2	2000 in t=2	3000 in t=2	4000 in t=2	5000 in t=2
default (%)	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
< IRR (%)	43,4%	46,0%	48,4%	51,2%	54,1%	38,8%	36,5%	34,5%	32,4%	30,1%
< Target (%)	5,7%	4,4%	3,5%	2,7%	2,2%	5,9%	4,9%	4,4%	3,9%	3,5%
CFaR, 5%	27.103,0	27.025,1	27.136,8	26.915,6	27.003,7	27.018,6	26.824,1	26.799,7	26.458,3	26.417,6

Table 3: Risk return table in terms of different recap scenarios (n=50.000)

Appendix A4 – Generating random variables

```
'*********************************************************************************************
'*  Return random numbers from Standard Normal Distribution with Box-Muller-Transformation   *
'*********************************************************************************************
Function gauss()
Dim fac As Double, r As Double, V1 As Double, V2 As Double
1  V1 = 2 * Rnd - 1
   V2 = 2 * Rnd - 1
   r = V1 ^ 2 + V2 ^ 2
   If (r >= 1) Then GoTo 1
   fac = Sqr(-2 * Log(r) / r)
   gauss = V2 * fac
End Function
```

```
'*********************************************************************************************
'*      Return random numbers from Exponential Distribution with Invers Method               *
'*********************************************************************************************
Function exprdn(lamda)
   exprdn = -Log(Rnd) / lamda
End Function
```

```
'*********************************************************************************************
'*      Return random numbers from Log Normal Distribution by Transformation                 *
'*********************************************************************************************
Function lnorm(ma, sa)
Dim fac As Double, r As Double, V1 As Double, V2 As Double, V3 As Double
   'drawing a N(0,1) random variable
1  V1 = 2 * Rnd - 1
   V2 = 2 * Rnd - 1
   r = V1 ^ 2 + V2 ^ 2
   If (r >= 1) Then GoTo 1
   fac = Sqr(-2 * Log(r) / r)
   'adjusting to LN(ma,sa^2) random variable
   V3 = (V2 * fac) * sa + ma
   lnorm = Exp(V3)
End Function
```

Appendix A5 – EULER discretisation

```
'*************************************************************************************************
'*          Simulation of the Cash Equity Basis with EULERdiscretisation                          *
'*************************************************************************************************
Sub StartButton()
  'Fast programming
  Application.Calculation = xlCalculationManual
  Application.ScreenUpdating = False

  'Number of Simulations
  Number = Cells(Range("maturity").Row - 1, Range("maturity").Column).Value

For y = 1 To Number
  'Copy Data to Simulation Data
  Jumprow = 0
  Range("B4:X17").Select
  Selection.Copy
  Range("B401").Select
  Selection.PasteSpecial Paste:=xlPasteValues, Operation:=xlNone, SkipBlanks :=False, Transpose:=False

  'Clear old Data
  Range("G407:Q409").Select
  Selection.ClearContents
  Range("A405").Select
  Range("returnrate") = "Discount Rate (%)"
  Range("NPV") = "NPV"

  'Determination of simulation horizon
  Endcolumn = Range("maturity").Value + 7
  Delta = Range("Horizon").Value
  simulationpoints = Range("maturity").Value * Delta
  fidelity = Cells(Range("Horizon").Row + 1, Range("horizon").Column)

  'Initiate Variables
  Wait = 0
  Totalwait = 0

  'Generation of the processes
  mul_simulated = Range("in_mul").Value
  mul_expected = Range("in_mul").Value

    For i = 1 To simulationpoints

      'FCF Process
      FCF = Cells(Range("data").Row, Range("Start").Column + i - 1)
      dFCF = 0
      Drift = (Cells(Range("FCF").Row, Range("Start").Column + i) - Cells(Range("FCF").Row, Range("Start").Column + i - 1)) /
             Cells(Range("FCF").Row, Range("Start").Column + i - 1)
      Vola_Levered = Range("Cashflow_Vola").Value * (1 + Cells(Range("data").Row + 1, Range("data").Column + i - 1) /
                     Cells(Range("data").Row + 2, Range("data").Column + i - 1))
      Vola = Vola_Levered
      For w = 1 To fidelity
        Brownian1 = gauss() / ((Delta * fidelity) ^ 0.5)
        dFCF = FCF * (Drift * Delta / fidelity + Vola * Brownian1)
        FCF = FCF + dFCF
      Next w
      Cells(Range("Data").Row, Range("Data").Column + i) = FCF

      'Multiple Process
      Reversion = Range("Mul_Rev").Value
      mean = 1 + (Range("out_mul").Value - Range("in_mul").Value) / (Range("in_mul").Value)
      mean = mean ^ (1 / (simulationpoints * fidelity))
      dmul = 0
      Mul_Sigma = Range("mul_vola").Value
      For w = 1 To fidelity
        Brownian2 = gauss() / (Delta * fidelity) ^ 0.5
        mul_expected = mul_expected * mean
        dmul = Reversion * (mul_expected - mul_simulated) * Delta / fidelity + Mul_Sigma * (mul_simulated) ^ 0.5 * Brownian2
        mul_simulated = mul_simulated + dmul
```

```
Next w

'Interest rate calculation
 Cells(Range("returnrate").Row + 1, Range("Start").Column + i) = Cells(Range("data").Row + 1, Range("data").Column + i) *
                                                              Range("Discount_Rate").Value

'Default Check
EV = mul_simulated * (Cells(Range("Data").Row, Range("Data").Column + i) + Cells(Range("returnrate").Row + 1,
     Range("Start").Column + i))
Debt = Cells(Range("data").Row + 1, Range("data").Column + i)
If EV < Debt Then
  Bankrupt = 1
  Cells(Range("Data").Row, Range("Data").Column + i) = 0
  FCF = 0
Else
  Bankrupt = 0
End If

'Debt Redemption
Contribution = Range("Contribution").Value
olddebt = Cells(Range("data").Row + 1, Range("data").Column + i - 1)
For j = i To simulationpoints
  Cells(Range("data").Row + 1, Range("data").Column + j) = olddebt - Cells(Range("Data").Row, Range("Data").Column + i) * (1 -
                                                         Contribution)
  If Cells(Range("data").Row + 1, Range("data").Column + j) < 0 Then
    Cells(Range("data").Row + 1, Range("data").Column + j) = 0
    Contribution = 1
  End If
Next j

'Stochastic Adjustments
If Totalwait < simulationpoints Then
  If Cells(Range("Stochastic").Row, Range("Start").Column + i) <> "" Then

    'Simulation of Waiting time
    Wait = i - Wait
    lam = 1 / Wait
    Wait = exprdn(lam)
    Wait = WorksheetFunction.RoundUp(Wait, 0)
    Totalwait = Totalwait + Wait
    If Totalwait <= simulationpoints Then

      'Simulation of Jump Size
      expvalue = Cells(Range("Stochastic").Row, Range("Start").Column + i)
      Sign = 1
      If expvalue < 0 Then
        expvalue = Abs(expvalue)
        Sign = -1
      End If
      varvalue = Cells(Range("Varvalue").Row, Range("Start").Column + i) *
                 Cells(Range("Stochastic").Row, Range("Start").Column + i)
      varlnvalue = (varvalue ^ 2) / (expvalue ^ 2) + 1
      varlnvalue = Log(varlnvalue)
      explnvalue = Log(expvalue) - varlnvalue / 2
      varlnvalue = varlnvalue ^ 0.5
      Assovalue = lnorm(explnvalue, varlnvalue) * Sign

      'Debtadjustment
        Debtadjustments = Assovalue * Cells(Range("debt_per").Row, Range("start").Column + i)
        Debtadjustments = Abs(Debtadjustments)
      For j = 0 To simulationpoints - Totalwait
        Cells(Range("data").Row + 1, Range("data").Column + Totalwait + j) = Cells(Range("data").Row + 1, Range("data").Column +
                                                                      Totalwait + j) + Debtadjustments
      Next j
      Cells(Range("data").Row + 5, Range("data").Column + Totalwait) = Assovalue
      Cells(Range("data").Row + 6, Range("data").Column + Totalwait) = Cells(Range("Varvalue").Row, Range("Start").Column + i)
      Cells(Range("data").Row + 7, Range("data").Column + Totalwait) = Cells(Range("debt_per").Row, Range("start").Column + i)
    End If
  End If
End If

'Discount Rate
Debt_Return = Range("Discount_rate").Value
```

```
    Total_Return = Range("Total_Return").Value
    Equity_Return = Total_Return + (Total_Return - Debt_Return) * Range("Beta").Value * (1 + Cells(Range("data").Row + 1,
                    Range("data").Column + i) / Cells(Range("data").Row + 2, Range("data").Column + i))
    If i > 1 Then
      Cells(Range("returnrate").Row, Range("Start").Column + i) = 1 / (1 + Equity_Return) * Cells(Range("returnrate").Row,
                                                                  Range("Start").Column + i - 1)
    Else
      Cells(Range("returnrate").Row, Range("Start").Column + i) = 1 / (1 + Equity_Return)
    End If

     'Interest Rate recalculation
     Cells(Range("returnrate").Row + 1, Range("Start").Column + i) = Cells(Range("data").Row + 1, Range("data").Column + i) *
                                                                     Range("Discount_Rate").Value

  Next i
  Range("Sim_Mul") = mul_simulated

  'NPV Calculation
  NePV = -Range("investment").Value
  For i = 1 To simulationpoints
    UnFCF = Cells(Range("Data").Row, Range("Data").Column + i)
    FCF = UnFCF + Cells(Range("Data").Row + 1, Range("Data").Column + i) - Cells(Range("Data").Row + 1, Range("Data").Column + i - 1)
    Det_Add = Cells(Range("Data").Row + 4, Range("Data").Column + i)
    Sto_Add = Cells(Range("Data").Row + 5, Range("Data").Column + i)
    Disc = Cells(Range("Returnrate").Row, Range("Data").Column + i)
    NePV = (FCF + Det_Add + Sto_Add) * Disc + NePV
  Next i
  interest = Cells(Range("returnrate").Row + 1, Range("Start").Column + i - 1)
  EV = (UnFCF + interest) * Range("Sim_mul").Value
  Equityvalue = EV - Cells(Range("Data").Row + 1, Range("Data").Column + simulationpoints)
  Disc_Eqvalue = Equityvalue * Disc
  If Disc_Eqvalue > 0 Then
    NePV = NePV + Disc_Eqvalue
  End If
  Cells(Range("NPV").Row, Range("Data").Column) = Range("Stake").Value * NePV

  '% < IRR
  NePV_IRR = -Range("investment").Value
  For i = 1 To simulationpoints
    UnFCF = Cells(Range("Data").Row, Range("Data").Column + i)
    FCF = UnFCF + Cells(Range("Data").Row + 1, Range("Data").Column + i) - Cells(Range("Data").Row + 1, Range("Data").Column + i - 1)
    Det_Add = Cells(Range("Data").Row + 4, Range("Data").Column + i)
    Sto_Add = Cells(Range("Data").Row + 5, Range("Data").Column + i)
    Disc = Range("IRR").Value
    Disc = 1 / (1 + Disc) ^ i
    NePV_IRR = (FCF + Det_Add + Sto_Add) * Disc + NePV_IRR
  Next i
  EV = (UnFCF + interest) * Range("Sim_mul").Value
  Equityvalue = EV - Cells(Range("Data").Row + 1, Range("Data").Column + simulationpoints)
  Disc_Eqvalue = Equityvalue * Disc
  If Disc_Eqvalue > 0 Then
    NePV_IRR = NePV_IRR + Disc_Eqvalue
  End If
  If NePV_IRR > 0 Then
    NePV_IRR = 0
  Else
    NePV_IRR = 1
  End If

  '% < Hurdle
  NePV_Hurdle = -Range("investment").Value
  For i = 1 To simulationpoints
    UnFCF = Cells(Range("Data").Row, Range("Data").Column + i)
    FCF = UnFCF + Cells(Range("Data").Row + 1, Range("Data").Column + i) - Cells(Range("Data").Row + 1, Range("Data").Column + i - 1)
    Det_Add = Cells(Range("Data").Row + 4, Range("Data").Column + i)
    Sto_Add = Cells(Range("Data").Row + 5, Range("Data").Column + i)
    Disc = Range("Hurdle").Value
    Disc = 1 / (1 + Disc) ^ i
    NePV_Hurdle = (FCF + Det_Add + Sto_Add) * Disc + NePV_Hurdle
  Next i
  EV = (UnFCF + interest) * Range("Sim_mul").Value
  Equityvalue = EV - Cells(Range("Data").Row + 1, Range("Data").Column + simulationpoints)
  Disc_Eqvalue = Equityvalue * Disc
```

```
    If Disc_Eqvalue > 0 Then
      NePV_Hurdle = NePV_Hurdle + Disc_Eqvalue
    End If
    If NePV_Hurdle > 0 Then
      NePV_Hurdle = 0
    Else
      NePV_Hurdle = 1
    End If

  'Risk measurement
  Cells(y, 100) = NePV * Range("Stake").Value
  Cells(y, 101) = Bankrupt
  Cells(y, 102) = NePV_IRR
  Cells(y, 103) = NePV_Hurdle
Next y
  Run Reporting()
End Sub
```

Appendix A6 – VBA code for probability distribution

```
'*****************************************************************************************
'*                    Updates the Reporting on old Simulation Data                       *
'*****************************************************************************************
Function Reporting()
  'Assign Fidelity
  Grain_Size = Cells(Range("Horizon").Row + 2, Range("horizon").Column)
  Cells(1, 99) = -Range("initial").Value
  Points = (Range("In_mul").Value * 10 * Cells(Range("start").Row + 1, Range("start").Column) / Range("initial").Value)
  Points = Points / Grain_Size * Range("initial").Value
  Points = WorksheetFunction.RoundUp(Points, 0)
  For j = 1 To Points
    Cells(1 + j, 99) = Cells(j, 99) + Grain_Size
  Next j

  'Assign length of the value array
  numbers = Cells(Range("maturity").Row - 1, Range("maturity").Column).Value

  'Counting the number of occurrences for each of the bins
  For i = 1 To numbers
    Oldvalue = -100000000
    For j = 1 To Points
      Newvalue = Cells(j, 99)
      If Cells(i, 100) <= Newvalue And Cells(i, 100) > Oldvalue Then
        Cells(j, 98) = Cells(j, 98) + 1
        GoTo 1
      End If
      Oldvalue = Newvalue
  1   Next j
  Next i

  For i = 1 To numbers
    If Cells(i, 100) > Cells(Points, 99) Then
      Cells(Points + 1, 98) = Cells(Points + 1, 98) + 1
    End If
  Next i

  For i = 1 To Points + 1
    Cells(i, 98) = Cells(i, 98) / numbers
  Next i

  '% default
  Default_Per = 0
  For i = 1 To numbers
    Default_Per = Default_Per + Cells(i, 101)
  Next i
  Default_Per = Default_Per / numbers
  Range("Default_Per") = Default_Per

  '% < IRR
  IRR_Per = 0
  For i = 1 To numbers
    IRR_Per = IRR_Per + Cells(i, 102)
  Next i
  IRR_Per = IRR_Per / numbers
  Range("IRR_Per") = IRR_Per

  '% < Hurdle
  Hurdle_per = 0
  For i = 1 To numbers
    Hurdle_per = Hurdle_per + Cells(i, 103)
  Next i
  Hurdle_per = Hurdle_per / numbers
  Range("Hurdle_out") = Hurdle_per
End Function
```