Maximize the Sharpe Ratio and Minimize a VaR[1]

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Abstract

In addition to its role as the optimal *ex ante* combination of risky assets for a risk-averse investor, possessing the highest potential return-for-risk tradeoff, the tangency or Maximum Sharpe Ratio portfolio in the Markowitz (1952, 1991) procedure plays an important role in asset management, as it minimizes the probability that a future portfolio return falls below the risk-free or reference rate. This is a kind of Value at Risk (VaR) property of the portfolio. In this paper we demonstrate the way this VaR, and related quantities, vary along the efficient frontier, emphasizing the special role played by the tangency portfolio. The results are illustrated with an analysis of the market crash of October 1987, as an episode of extreme negative market movements, where the tangency portfolio performs best (loses least!) among a variety of portfolios.

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1. Introduction: the "Maximum Sharpe Ratio" and "Tangency" Portfolios

Given a universe of $d \geq 2$ risky assets having raw return vector $\tilde{\boldsymbol{\mu}}$ and excess mean return vector $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} - r\mathbf{i}$ (relative to a reference rate r), and returns covariance matrix $\boldsymbol{\Sigma}$, form a portfolio by taking an allocation, that is, a linear combination with coefficients given by a vector, $\mathbf{x}_p$, say, of the assets.[1] The "reference rate" could be the prevailing risk-free rate, if there is one, or some other benchmark rate, such as the expected market return for the period, etc. Let R_p be the *ex post* excess return achieved from this portfolio, after the portfolio has been in place for a specified, fixed, time period. Suppose R_p has expectation μ_p and variance σ_p^2. Thus

$$\mathrm{E}(R_p) = \mu_p = \mathbf{x}_p'\boldsymbol{\mu} \quad \text{and} \quad \mathrm{Var}(R_p) = \sigma_p^2 = \mathbf{x}_p'\boldsymbol{\Sigma}\mathbf{x}_p \,. \tag{1.1}$$

Assume that, for all such portfolios, the standardized *ex post* excess returns

$$\frac{R_p - \mu_p}{\sigma_p}$$

are identically distributed, having the same distribution as a random variable Z, say, where $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

Define the population Maximum Sharpe Ratio as

$$SR := \max_{\mathbf{i}'\mathbf{x}=1} \left(\frac{\mathbf{x}'\boldsymbol{\mu}}{\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}} \right). \tag{1.2}$$

This quantity has long been used in portfolio theory and practice (Sharpe 1963), either in an *ex ante* fashion, where it can be used to decide on an optimal allocation giving an optimal return-risk tradeoff, or *ex post*, as a portfolio performance evaluation tool. It plays a significant role in

[1] Here $\mathbf{i}$ denotes a *d*-vector, each of whose elements is one, and a prime will denote a vector or matrix transpose.

both discrete and continuous time finance, and is an object of interest in research right up to the present day (see, e.g., Christensen and Platen 2007).

The maximization in (1.2) is over all portfolios satisfying the "total allocation" constraint $\mathbf{i}'\mathbf{x} = 1$, that a unit amount of resources is invested. There is no requirement that the components of the vector $\mathbf{x}$ be nonnegative, so short selling of assets is allowed. The ratio in (1.2) is maximized taking its sign into account, as advocated, e.g. by Sharpe (1994); we are interested in maximizing the actual (risk-adjusted) return – that is, a measure sensitive to losses, as well as to gains.[2]

In this paper, we consider (in Section 2) an optimality property of the "Maximum Sharpe Ratio" portfolio, that is, the portfolio achieving the maximum value in Eq. (1.2), which it possesses with regard to "Value at Risk". The ideas are illustrated with a textbook example in Section 3. We then go on in the fourth section to develop some ideas regarding realized returns on efficient portfolios, which are illustrated with the same textbook data, and in the fifth section, we examine the performance of a spectrum of portfolios calculated from monthly data on US stocks prior to the October 1987 stock market crash, showing how the tangent portfolio, and various other selected portfolios, performed prior to, and on, the day of the crash.

To conclude this section we mention some further facts we will need, concerning the connection between the maximum Sharpe ratio and what we will call the "tangency" portfolio. The quantity SR in Eq. (1.2) is the maximum Sharpe ratio achievable from the d assets. In textbooks, and in applications, the corresponding portfolio is often found or illustrated by drawing a tangent line in the $(\sigma_p, \tilde{\mu}_p)$ plane from the point $(0, r)$ (where r is the risk-free or

[2] In some studies, the quantity in Eq. (1.2) is squared before the maximization is done. While this simplifies the algebra, it unrealistically ignores possible expected losses on the portfolio.

reference rate) to the efficient frontier constructed from $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\Sigma}$. The coordinates of this point, $(\sigma_T, \tilde{\mu}_T)$, say, give the location of the maximum Sharpe ratio portfolio in the $(\sigma_p, \tilde{\mu}_p)$ plane, and the slope of the tangent line gives the maximum Sharpe ratio available for any portfolio constructed from this universe of assets. The corresponding allocation vector $\mathbf{x}_T$ can be calculated from Equation (41) in Merton (1972).

Merton (1972) showed further, however, that this procedure can be misleading or in error, since a tangency point producing a maximum Sharpe ratio need not in fact exist. He gave a necessary and sufficient condition for this to be the case (Theorem II in Merton (1972)). Of course a *maximum value* of the Sharpe ratio still exists (and is finite), but it has to be found by other means; see, e.g., the method outlined in Maller & Turkington (2002). The probability calculation in (2.1) below uses only the existence of the maximum Sharpe ratio portfolio, however calculated; it does not require the existence of a tangent point to the efficient frontier. Nevertheless, we shall continue to refer to the portfolio with maximum Sharpe ratio as the "tangent portfolio" whether or not such exists. For all the data considered in this paper, it turns out that the tangent portfolio does in fact exist, so no confusion should result from this.

2. A Value at Risk Property of the Maximum Sharpe Ratio Portfolio

Let $\mathbf{x}_T$ be the allocation vector corresponding to the portfolio obtained as a result of the maximization in Eq. (1.2). As discussed in the previous section, we will refer to this as the "tangency portfolio". Recall that the excess mean return vector $\boldsymbol{\mu}$ equals $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} - r\mathbf{i}$, where $\tilde{\boldsymbol{\mu}}$ is the mean raw return vector, and r is the reference rate. We can write

$$SR = \frac{\tilde{\mu}_T - r}{\sigma_T} = \frac{\mu_T}{\sigma_T},$$

where $\mu_T = \mathbf{x}'_T \boldsymbol{\mu}$, $\tilde{\mu}_T = \mathbf{x}'_T \tilde{\boldsymbol{\mu}}$ and $\sigma_T = \mathbf{x}'_T \boldsymbol{\Sigma} \mathbf{x}_T$. The maximum Sharpe Ratio portfolio possesses a certain optimality property with respect to VaR, as the following simple calculation shows. For an arbitrary portfolio with allocation $\mathbf{x}_p$, we have

$$SR \geq \left(\frac{\mathbf{x}'_p \boldsymbol{\mu}}{\sqrt{\mathbf{x}'_p \boldsymbol{\Sigma} \mathbf{x}_p}} \right) = \frac{\mu_p}{\sigma_p}.$$

Letting R_T be the excess return on the tangency portfolio, we can calculate

$$\begin{aligned} \mathrm{P}\left(R_p \leq 0\right) &= \mathrm{P}\left(\frac{R_p - \mu_p}{\sigma_p} \leq -\frac{\mu_p}{\sigma_p} \right) \\ &= \mathrm{P}\left(Z \leq -\frac{\mu_p}{\sigma_p} \right) \\ &\geq \mathrm{P}\left(Z \leq -SR\right) \\ &= \mathrm{P}\left(\frac{R_T - \mu_T}{\sigma_T} \leq -\frac{\mu_T}{\sigma_T} \right) \\ &= \mathrm{P}\left(R_T \leq 0\right). \end{aligned} \tag{2.1}$$

It follows that

$$\min_{\mathbf{x}_p} \mathrm{P}\left(R_p \leq 0\right) \geq \mathrm{P}\left(R_T \leq 0\right),$$

and the minimum is achieved for the tangency portfolio. Thus, an allocation of assets according to the tangency portfolio has the lowest probability of the investor receiving a return below the reference rate; in other words, it has the smallest VaR relative to this rate.

As portfolios move away from the maximum Sharpe Ratio allocation, this probability increases. We can illustrate the magnitude of this increase by plotting the probability for portfolios on the efficient frontier, that is, those having expected return and standard deviation $(\tilde{\mu}_p, \sigma_p)$, against σ_p, thus obtaining a representation of the way this VaR changes along the efficient frontier. We have to assume a distribution for Z, the standardized return, and for this we will consider a standard normal, as well as a t-distribution with 4 degrees of freedom. These represent extreme distributions between which returns distributions are likely to lie. While the normal distribution is often assumed for returns, especially over longer periods, it has been long recognized that returns distributions in reality are more heavy tailed and leptokurtic than the normal distribution (Fama, 1965; Embrechts et al. 1997; Platen and Sidorowicz 2007); therefore, we utilize a t-distribution with small degrees of freedom to simulate this feature of the data.

We illustrate the concepts using some data from Ruppert's text book (2004, p.150). There are $d = 3$ assets for which the (raw) mean vector and covariance matrix are

$$\tilde{\boldsymbol{\mu}} = \begin{pmatrix} 0.08 \\ 0.03 \\ 0.05 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} 0.30 & 0.02 & 0.01 \\ 0.02 & 0.15 & 0.03 \\ 0.01 & 0.03 & 0.18 \end{pmatrix}.$$

The efficient frontier for this example is shown on p.155 of that book. In Figure 2.1 we plot the function $\mathrm{P}(R_p \le 0) = \mathrm{P}(Z \le -\mu_p / \sigma_p)$ for portfolios on the efficient frontier, as a function of the portfolio risk, σ_p. As expected, the curves have a minimum at the tangent point, and the curve for the *t*-distribution is higher than for the normal; the probability of a return below the risk-free rate is much higher for the heavier-tailed *t*-distribution.

A "Value at Risk" is usually thought of as a quantile below which a return falls with a specified (low) probability; thus, we should also consider $\mathrm{P}(R_p \le q)$, for values of q not equal to zero. It is not the case in general that this quantity is minimized for the Maximum Sharpe Ratio allocation, but by observation this seems to remain approximately true for q not too far from zero (recall that we are optimizing *excess* returns, relative to a benchmark). In Figure 2.2, the probabilities of efficient portfolio returns lower than q are shown for various values of q. For example, from the lowest curve in Figure 2.2 (left plot) can be read that the probability of a future excess return less than -0.02 for the tangent point portfolio is approximately 0.441. Thus the tangency portfolio is expected to return more than 0.02 below the reference rate at most 44.1% of the time. For the t_4 distribution in the right plot, on the other hand, such a loss happens approximately with probability 0.445.

Although it is not necessarily the case that the minima of the curves in Figure 2.2 should occur at the tangent point (expected for the cases $q = 0$), in fact this happens for this data (and also for the data analyzed in Section 4).

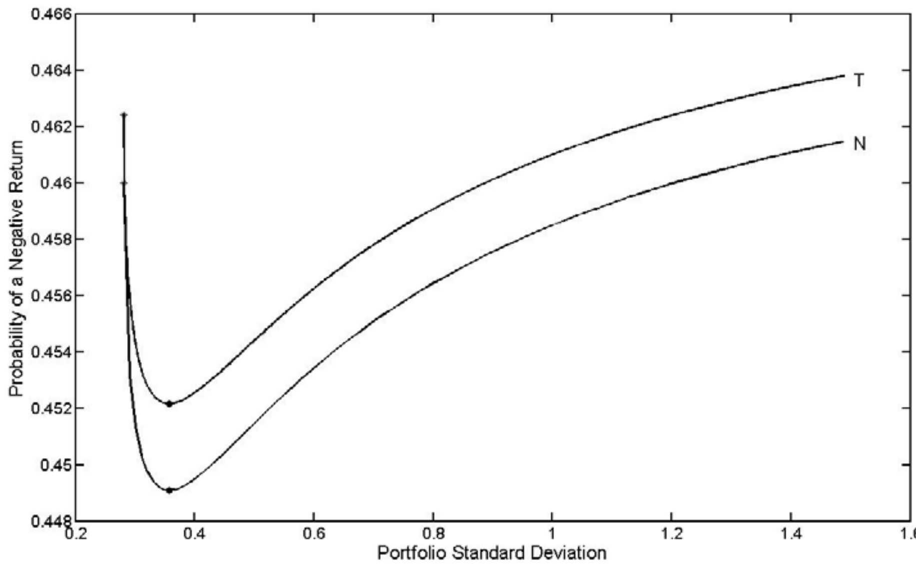

Figure 2.1: Ruppert Data, Normally and t_4 Distributed Returns

The probability of receiving a negative return, as a function of the standard deviation of the efficient portfolio. The curve labeled "N" depicts normally distributed returns and the curve labeled "T" depicts t_4 distributed returns. The curves start at the standard deviation of the minimum variance portfolio on the left, and show the position of the tangent point portfolio (indicated by dot points).

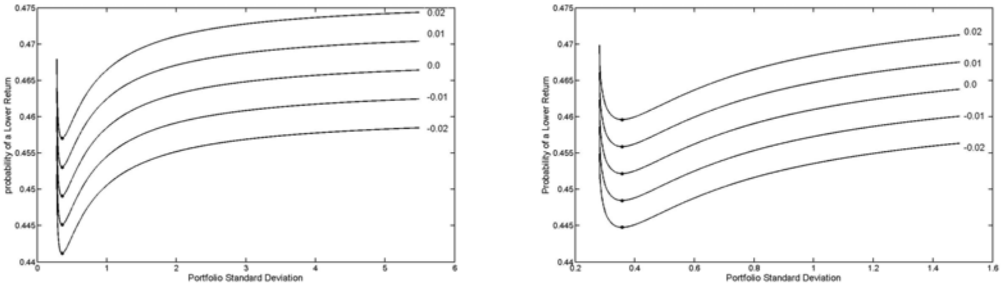

Figure 2.2: Ruppert Data, Values at Risk for Normally and t_4 Distributed Returns

The probability of receiving an excess return lower than q, where q is specified by the numbers at the right hand ends of the curves, as a function of the standard deviation of the efficient portfolio. The left hand diagram depicts normally distributed returns; the right hand diagram t_4 distributed returns. The curves start at the standard deviation of the minimum variance portfolio on the left, and show the position of the tangent point portfolio (indicated by dot points).

3. Efficient Portfolio Returns

To investigate the performances of portfolios on the efficient frontier, we need some facts concerning them. These are derived from Merton (1972). In our notation, the quantities on p.1853 of his paper are:

$$A = \mathbf{i}'\mathbf{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\,,\quad B = \tilde{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\,,\quad C = \mathbf{i}'\mathbf{\Sigma}^{-1}\mathbf{i}\,,\quad D = BC - A^2 > 0\,.$$

(Recall that $\tilde{\boldsymbol{\mu}}$ denotes the raw returns and $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} - r\mathbf{i}$ are the excess returns on the d assets.) We assume that a tangent point exists, so the quantity

$$TPC = \mathbf{i}'\mathbf{\Sigma}^{-1}\boldsymbol{\mu} = A - rC$$

is positive (Merton (1972), p. 1863). The coordinates in the $(\sigma, \tilde{\mu})$ plane of the minimum variance and tangent point portfolios are given by

$$\sigma_m^2 = \frac{1}{C}\,,\quad \tilde{\mu}_m = \frac{A}{C}\,,$$

and

$$\sigma_T^2 = \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\left(\mathbf{i}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)^2}, \quad \tilde{\mu}_T = r + \frac{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}.$$

The corresponding portfolio allocations are

$$\mathbf{x}_m = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{C} \quad \text{and} \quad \mathbf{x}_T = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{i}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}.$$

(*Note*: we have $\boldsymbol{\mu}$, not $\tilde{\boldsymbol{\mu}}$, in σ_T, $\tilde{\mu}_T$ and $\mathbf{x}_T$.)

The equation of the efficient frontier in $(\sigma, \tilde{\mu})$ space is

$$\mathrm{F}(\sigma) = \frac{A + \sqrt{D\left(C\sigma^2 - 1\right)}}{C}. \tag{3.1}$$

(We use the "F" notation for "frontier", rather than Merton's "E" notation, which we reserve for "expectation".) The portfolio allocation corresponding to a portfolio with coordinates $(\sigma, \tilde{\mu})$ on the efficient frontier is given by the vector

$$\mathbf{x} = \frac{\mathrm{F}(\sigma)\left(C\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}} - A\boldsymbol{\Sigma}^{-1}\mathbf{i}\right) + B\boldsymbol{\Sigma}^{-1}\mathbf{i} - A\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}{D} \tag{3.2}$$

(Merton (1972), p.1856 and p.1845).

It is easily checked by differentiation that the curve

$$\frac{\mathrm{F}(\sigma) - r}{\sigma} \tag{3.3}$$

in $(\sigma, \tilde{\mu})$ space has a maximum at the point σ_T which satisfies

$$\sigma_T^2 = \frac{B - 2rA + r^2C}{(A - rC)^2}; \tag{3.4}$$

this of course is the variance of the tangent point portfolio. (Note that the denominator in (3.4) is $(TPC)^2 > 0$.) The function in (3.3) increases for $\sigma < \sigma_T$ and decreases for $\sigma > \sigma_T$. Figure 3.1 shows the curve for the Ruppert data, taking $r = 0.02$ as on p.155 of Ruppert (2004).

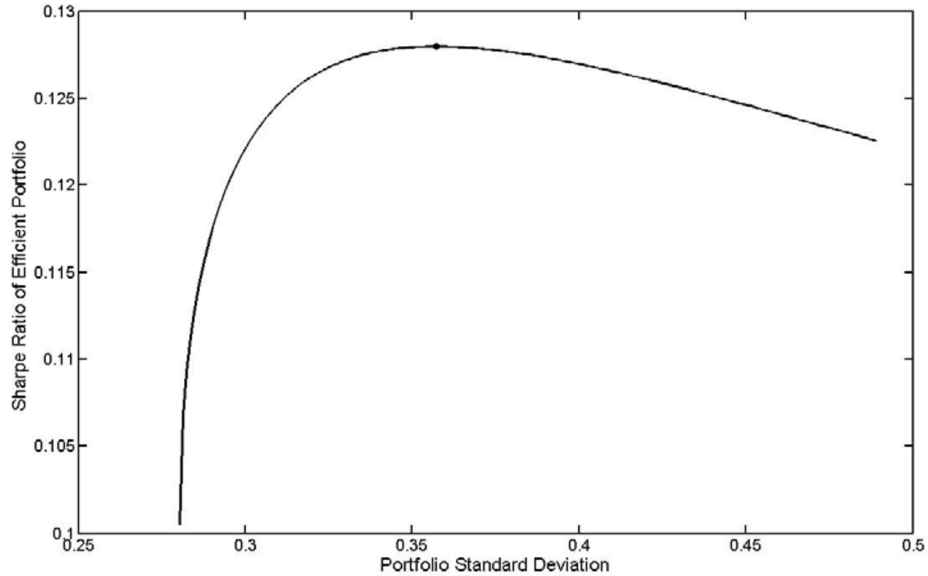

Figure 3.1: Sharpe Ratios for Efficient Portfolios from the Ruppert Data

Plot of the Sharpe ratio (Eq. (3.3)) for portfolios on the efficient frontier, against their standard deviation, Ruppert textbook data. The tangent point portfolio is indicated by a dot.

Now suppose we have a new observation vector, $\tilde{\mathbf{R}}$, on the returns of the d assets. We can think in terms of the efficient portfolio with mean $\mathrm{F}(\sigma)$ and standard deviation σ being put in place at a certain time, then evaluated using the future return $\tilde{\mathbf{R}}$. Using (3.2), and after some algebra, we can write

$$\frac{\tilde{\mathbf{R}}'\mathbf{x}-r}{\sigma}=\frac{\mathrm{F}(\sigma)-r}{\sigma}+\frac{\left(\tilde{\mathbf{R}}-\tilde{\boldsymbol{\mu}}\right)\left((C-A)\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}+(B-A)\boldsymbol{\Sigma}^{-1}\mathbf{i}\right)}{\sigma D}. \tag{3.5}$$

Here $\tilde{\mathbf{R}}'\mathbf{x}$ represents the return on the efficient portfolio corresponding to the returns $\tilde{\mathbf{R}}$ on the d assets, and the quantity on the left of (3.5) is the standardized excess return, i.e., the *ex post* Sharpe ratio for the portfolio. On the right of (3.5) is the population Sharpe ratio for the portfolio plus a random term corresponding to the new return, $\tilde{\mathbf{R}}$. If $\tilde{\mathbf{R}}$ is drawn from the same population as that from which the efficient portfolio was constructed, so that $\mathrm{E}(\tilde{\mathbf{R}})=\tilde{\boldsymbol{\mu}}$ and

$\mathrm{Var}(\tilde{\mathbf{R}}) = \mathbf{\Sigma}$, it is clear that the expectation of the random term in (3.5) is zero, and its variance is one (as can also be checked after some algebra).

Figure 3.2 shows a plot of Eq. (3.5) for 13 returns generated randomly as observations on $N(\tilde{\boldsymbol{\mu}}, \mathbf{\Sigma})$, using Ruppert's values of $\tilde{\boldsymbol{\mu}}$ and $\mathbf{\Sigma}$. (Ruppert does not supply the original returns for which his $\tilde{\boldsymbol{\mu}}$ and $\mathbf{\Sigma}$ were calculated, so we simulated the observations.) We took 13 returns so as to correspond with the 1987 crash data in the next section. It is clear from Figure 3.2 that Eq. (3.5), as a function of σ, need not resemble Eq. (3.3), as shown plotted in Figure 3.1. For this data, the random component in Eq. (3.5), which has a standard deviation of one, overwhelms its expectation, which for this data peaks at about 0.13 (cf. Figure 3.1).

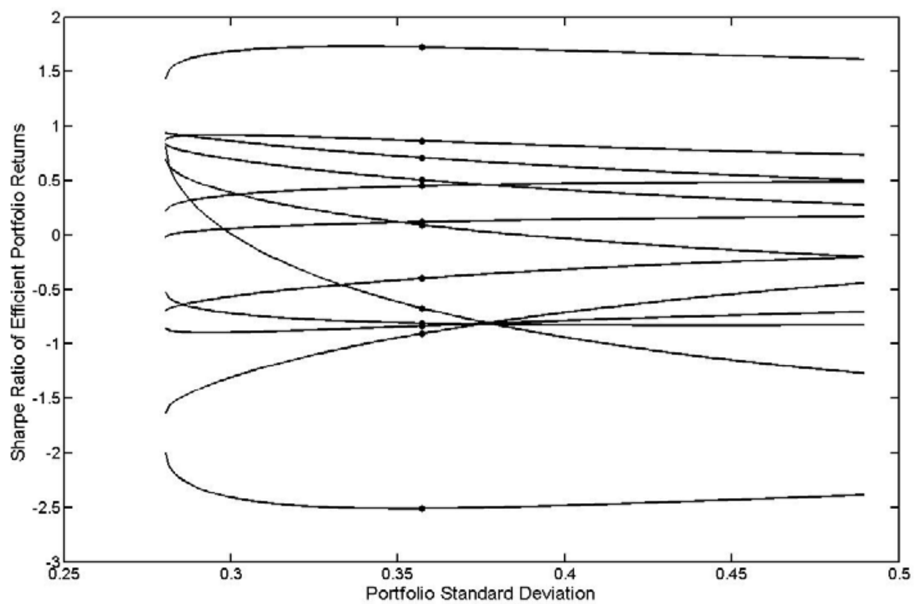

Figure 3.2: Standardized Returns on Efficient Portfolios for Ruppert Data

Ex post Sharpe ratios for returns on portfolios on the efficient frontier, corresponding to a new return, against their standard deviations, Ruppert textbook data. The tangent point portfolio is indicated by a dot.

While the simulated future return curves sometimes peak close to the tangency point, at other times the maximum occurs for much higher risk portfolios, and sometimes the curves are even convex. For such data (and the data in the next section has similar features), unfortunately, investing in the tangency portfolio produces very little benefit for *individual* future returns. Only when averaged over a relatively large number of returns will curves calculated from Eq. (3.5) begin to resemble those from Eq. (3.3).

4. Fama-French Data

In this section we analyze a more realistic example. For population $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\Sigma}$, we take values estimated from monthly data on US stocks for twenty-five value-weighted size and book-to-market portfolios (Fama and French, 1993) downloaded from Ken French's website (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html). This classic set of data has been used in many definitive studies of portfolio and other analyses; see for example Jagannathan and Ma (2003). We refer to these portfolios as the Fama-French portfolios. Choosing an appropriate selection of assets from the real world to demonstrate the VaR minimizing properties of the Sharpe ratio is problematic given the large number of assets from which an investor may choose. The Fama-French portfolios are representative of asset classes that capture factors that appear to be important to investors; it is reasonable that an investor might use these as representative asset classes from which to derive an optimal return to risk trade-off. Figure 4.1 shows the efficient frontier calculated from this data set.

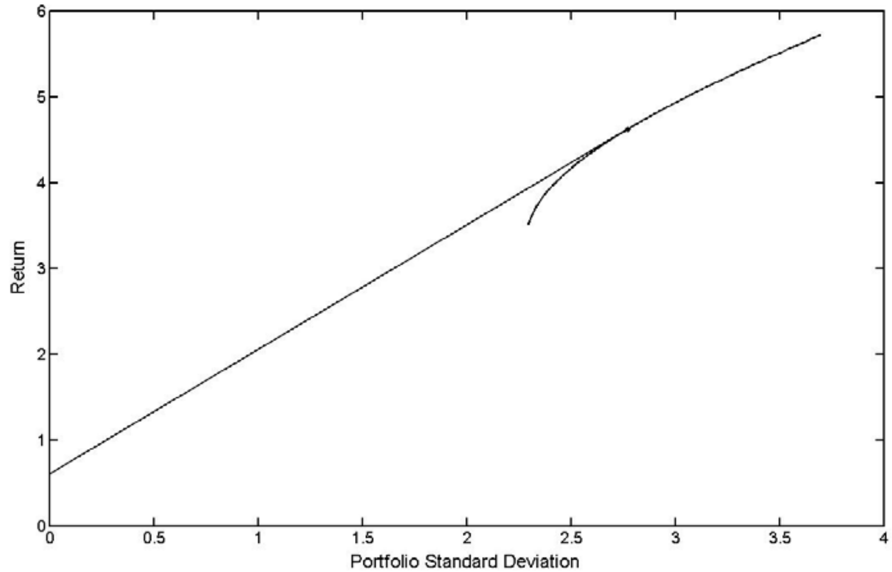

Figure 4.1: E*fficient Frontier constructed from 25 Fama-French portfolios.*
The efficient frontier estimated from monthly returns on the 25 Fama-French portfolios over the period October 1982-September 1987, with the tangent line, and the maximum Sharpe ratio point plotted as a dot.

Figure 4.2 shows the probability of a negative excess return (return less than the risk-free rate, $r = 0.062$) for the normal and t_4-distributions. The minimum probability at the tangent point is clearly apparent, but the minimum is not so well defined as it was for the textbook data (in Figure 2.1). This is a reflection of the fact that the tangent point is not well defined in Figure 4.1; although $TPC > 0$ for this data, TPC is close to zero, and the efficient frontier is practically a straight line to the right of the tangent point. Consequently, the Sharpe ratio is practically the same for all portfolios with higher risk than the tangent point portfolio. The probability of a negative return is of course much higher for the more extreme t_4-distributed returns.

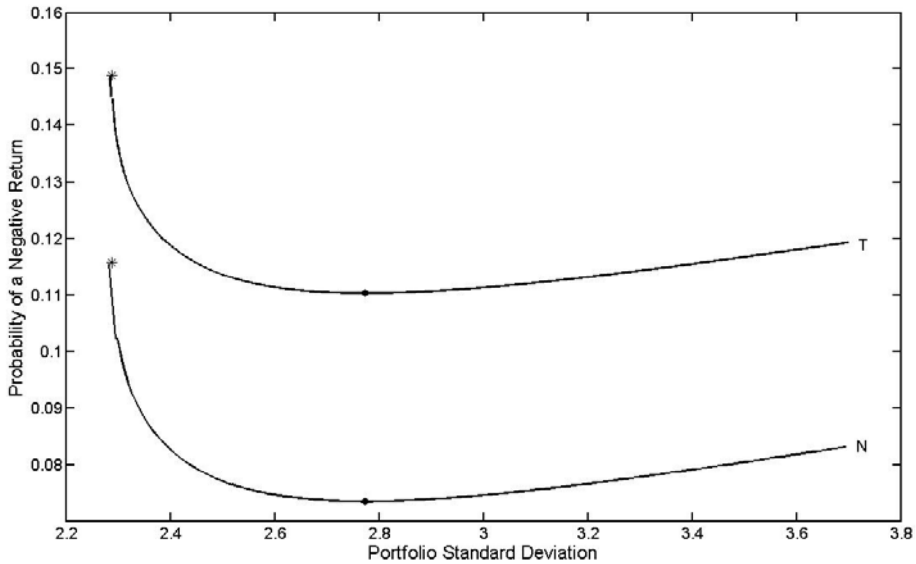

Figure 4.2: Fama-French Monthly Data, t_4 and Normally Distributed Returns

The probability of receiving a negative return, as a function of the standard deviation of the efficient portfolio. The curve labeled "N" depicts normally distributed returns and the curve labeled "T" depicts t_4 distributed returns. Parameters estimated from 60 monthly returns on 25 Fama-French portfolios over the period October 1982-September 1987. The curves start at the standard deviation of the minimum variance portfolio on the left, and show the position of the tangent point portfolio (indicated by dot points).

Figure 4.3 shows the family of curves obtained by plotting $\mathrm{P}(R_p \leq q)$ against σ_p, again for portfolios on the efficient frontier, for various values of q. The same values of $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\Sigma}$ are taken as for Figure 4.1, and Z is $\mathrm{N}(0,1)$. For example, from the lowest curve in Figure 4.3 can be read that the probability of a future excess return less than -0.15 for the tangent point portfolio is approximately 0.055. Thus the tangency portfolio is expected to return less than -0.15 at most 5.5% of the time, for this data, if returns are normally distributed. In other words, the minimum VaR in this data, corresponding to a 5.5% quantile, if a normal distribution is assumed for returns, is a return below -0.15. Since the Fama-French data is in percentage terms, this means a potential loss of 0.15%, on a monthly basis, or about 1.8% p.a.

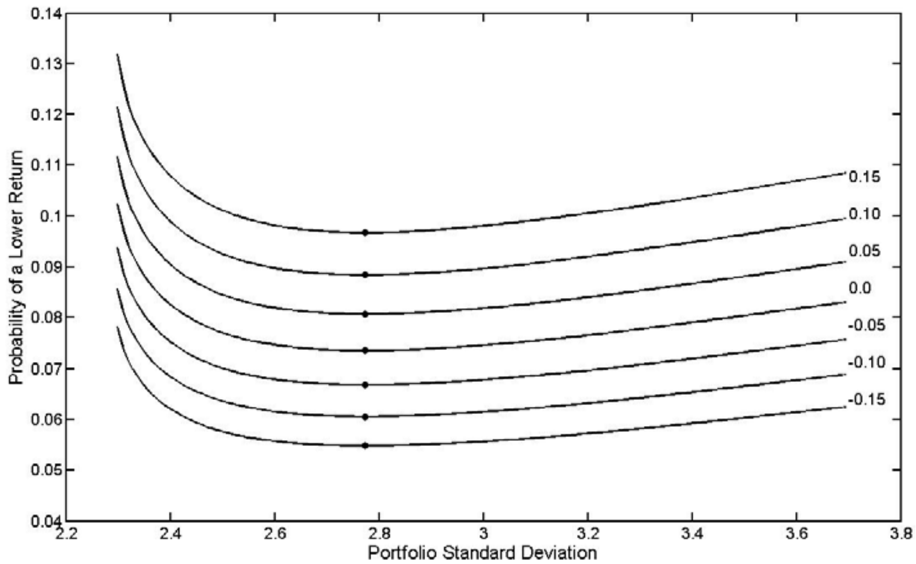

Figure 4.3: Fama-French Monthly Data, Normally Distributed Returns
The probability of receiving an excess return lower than q, where q is specified by the numbers at the right hand ends of the curves, as a function of the standard deviation of the efficient portfolio, assuming normally distributed returns. Parameters are estimated from 60 monthly returns on 25 Fama-French portfolios over the period October 1982-September 1987. The curves start at the standard deviation of the minimum variance portfolio on the left, and show the position of the tangent point portfolio (indicated by dot points).

Figure 4.4 shows similar information as Figure 4.3, but with Z having a t-distribution with 4 degrees of freedom, rather than a normal distribution. The same values of $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\Sigma}$ are taken as for Figure 4.2 and Figure 4.3. With this more extreme distribution for returns, probabilities of large negative returns are much higher than for the normal distribution, and values at risk are correspondingly much higher too. The VaR corresponding to a 5% quantile is about -0.70, thus, a loss of 0.70%, on a monthly basis, or about 8.4% p.a., substantially higher then for a normal distribution, but still woefully inadequate for describing the losses on efficient portfolios on Black Monday, October 19 1987, as we shall see in the next section.

Note that the curves in Figure 4.3 and Figure 4.4 are much shallower than in Figure 2.2, again reflecting the difficulty in locating the tangent point accurately in this data.

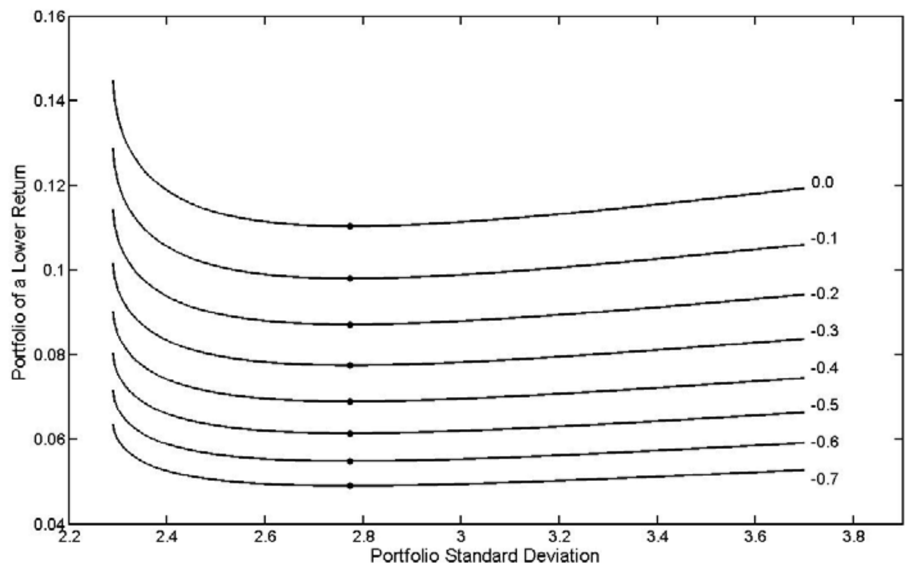

Figure 4.4: Fama-French Monthly Data, t_4 Distributed Returns

The probability of receiving a return lower than q, where q is specified by the numbers at the right hand ends of the curves, as a function of the standard deviation of the efficient portfolio, assuming t_4 distributed returns. Parameters estimated from 60 monthly returns on 25 Fama-French portfolios over the period October 1982-September 1987. The curves start at the standard deviation of the minimum variance portfolio on the left, and show the position of the tangent point portfolio (indicated by dot points).

5. 1987 Crash Performance

In this section we make some evaluations of the performance of a spectrum of portfolios using real data sets for illustration. Minimizing VaR matters most when prices are falling;

therefore, we examine the performance of tangent portfolios on Black Monday, October 19, 1987, when the S&P 500 index fell by 20.5%.

An advantage of using the Black Monday crash is that this is perhaps the only case where the *reality* of an observed event is more extreme than something that might reasonably have been simulated! As in Section 4, we take monthly data on the Fama-French portfolios over a period of 60 months prior to the 1987 crash, and use it to set up a spectrum of portfolios, each of whose performance (excess return, risk-adjusted) is then evaluated a day after the crash. The spectrum of portfolios consists of those on the efficient frontier, including the minimum variance and tangency portfolios, together with selected portfolios such as the equally weighted portfolio.

To evaluate the performances of the portfolios, we consider the case of an investor whose strategy involves putting in place one of the above-mentioned portfolios; for example she may maximize the *ex ante* Sharpe ratio of her portfolio. Of course with the benefit of hindsight, our investor would have shorted the entire market, but to keep the analysis realistic we assume she reviews and rebalances her portfolio regularly, without foresight, and that rebalancing takes some nonzero time. Introducing such a friction is not unreasonable, though it may mean that we are erring on the side of conservativism.[3]

Thus, to summarize, the investor makes her allocations at the beginning of October 1987, using a portfolio constructed from the Fama-French portfolios, and the information on their returns over the preceding 60 months (that is, using the monthly data from October 1982

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[3] We also confine the illustration to the direct use of the historic data, again, erring on the side of conservativism; for example, using a adjustment, such as that analyzed in Jorion (1996) or Ledoit and Wolf (2004), might result in better estimates to the optimization and, consequentially, improved outcomes for the tangent portfolio

to September, 1987 used in the analysis in the preceding section). As benchmark reference rate for the calculation of the tangency portfolio we take the expected risk-free rate at the end of October, 1987.[4] The crash takes place on October 19, 1987, and we evaluate the return on each portfolio at the close of business on that day. Table 5.1 shows the absolute returns on some selected portfolios (not all of which are efficient), on that day.

Portfolio	Return on Black Monday
Tangent	-11.1709
Minimum variance	-12.4610
Equal weighted portfolio of the 25 Fama-French portfolios	-13.9984
S&P 500 index (equal-weighted)	-18.4222
S&P 500 index	-20.4669

Table 5.1: Black Monday Returns

Returns (in percentages) on Black Monday for selected portfolios constructed from 60 monthly returns on 25 Fama-French portfolios over the period October 1982-September 1987.

The returns in Table 5.1 are far below anything that could be expected from Figure 4.3 and Figure 4.4. Even under a t-distribution with four degrees of freedom, the probability of receiving a return below -10.0 is only 0.0003. Nonetheless, our investor would undoubtedly have been very pleased with the *relatively* large return (smaller loss) she achieved!

[4] The risk-free rate at the end of October 1987 was also downloaded from Kenneth French's data library. Using the end-of-October risk-free rate assumes that our analyst had perfect expectations about this aspect of the return calculation.

Figure 5.1 shows a plot of the expected Sharpe ratio, that is, the function in Eq. (3.3), for this data. The maximum occurs at the tangent point, as it should, and is reasonably well defined.

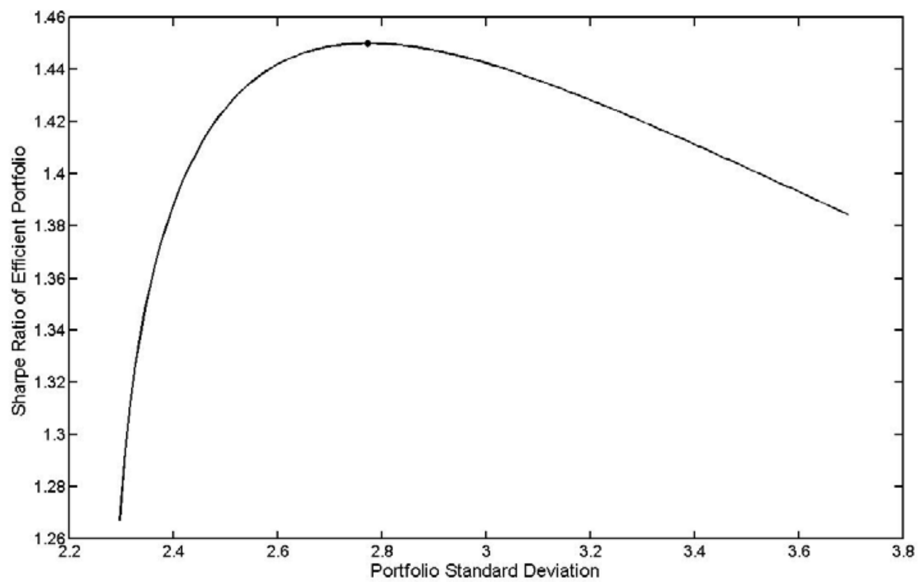

Figure 5.1: Sharpe Ratios for Efficient Portfolios from the Fama-French Data
A plot of the Sharpe ratio for portfolios on the efficient frontier, against their standard deviations. The tangent point portfolio is indicated by a dot.

Figure 5.2 shows the risk-adjusted excess returns from efficient portfolios calculated from the Fama-French data, plotted against the standard deviations of the portfolios, for the first 14 trading days in October 1987. Forewarned by Figure 3.2 and the analysis of the Ruppert data, we expect high variation around the population Sharpe ratio of 1.45. The curves in the top part of Figure 5.2 cover the first 13 trading days in October. In this period some of the curves tend to show a maximum near the tangent point portfolio, but the curvature is very slight and the maxima are not at all well-defined. This feature disappears as October goes on, and the curves become monotone increasing. The lowest curve is for October 19. The return on the tangent portfolio is not the highest on this day. There is a monotonic increase in risk-adjusted excess returns as we increase the risk of the portfolio along the efficient frontier, from the minimum possible risk, on. The least losses would have been given by taking positions at extreme risks. Of course the crash period represents an extreme situation whereas our theory assumes that returns have a constant distribution. This was almost certainly not the case on October 19, 1987. We discuss the implications further in the next section.

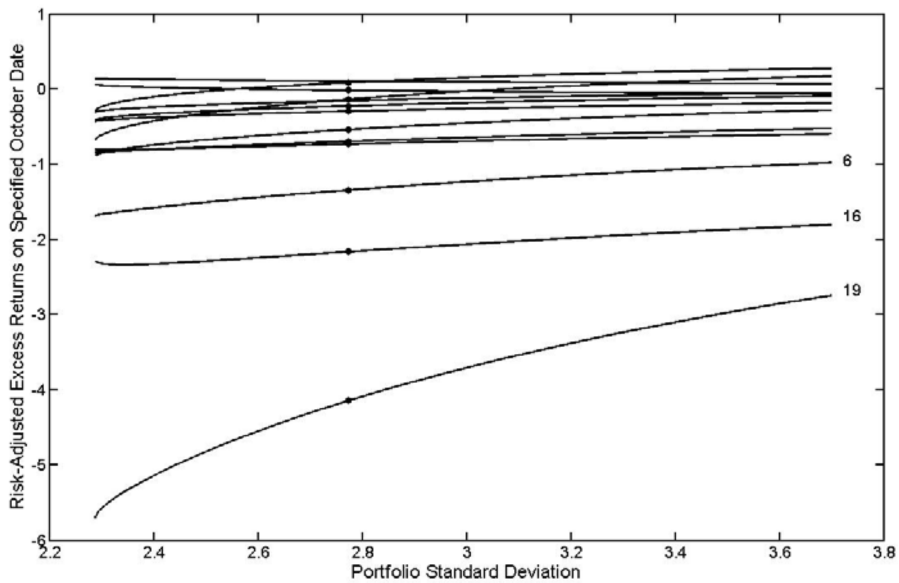

Figure 5.2: Returns for efficient portfolios from 25 Fama-French portfolios, first 14 trading days in October

The risk-adjusted returns on efficient portfolios calculated from monthly returns on the 25 Fama-French portfolios over the period October 1982-September 1987, evaluated on the first 14 trading days in October 1987, plotted against the standard deviation of the portfolios. The number at the right hand end of a curve indicates the date on which the portfolio was evaluated. The lowest curve corresponds to Black Monday, October 19, 1987. The minimum variance portfolio is at the left end of each curve and the tangent portfolio is indicated by a dot point. Excess returns are adjusted for risk by dividing by the standard deviation of the original portfolio.

6. Discussion and Conclusion

Maximizing the return to risk trade-off through investing in the tangency portfolio is very well-known and understood by educated investors. Rational investors, especially investors whose trustees focus only on returns, will want to guard at all cost against the possibility that their portfolio will earn less than the risk-free, or reference, rate. Our work demonstrates the way that maximizing the expected Sharpe ratio through selecting the tangency portfolio minimizes the chances, not only of a return lower than the reference rate, but of even lower returns as well, across the range of efficient portfolios. These VaR minimizing properties of the tangency portfolio have not, to our knowledge, been implemented in a practical situation, and, as a result, the very desirable consequences of implementing a simple “black-box” approach to portfolio selection have not been thoroughly explored.

The *ex ante* allocation of assets to a portfolio is always based on imperfect foresight. In using the crash of 1987 to illustrate the VaR minimizing properties of the tangency portfolio, we have chosen a particularly extreme example where investors cared desperately about the downside risk of their portfolios. By constructing a portfolio based on five-years of monthly data before the crash, we have put ourselves in the situation of an investor following a reasonably realistic tangency-portfolio strategy. The in-sample estimates of expected loss presented in Figure 4.2 and Figure 4.3 demonstrate that, should the distributions have remained stable, our investor's expected losses would have been minimized by holding the tangency portfolio. However, in the extreme circumstances of October 1987, the assumption that past returns distributions would remain the same was almost certainly violated. But after the damage had been done, it turns out that the tangency portfolio performed best (lost least!) out of the range of benchmarks presented in Table 5.1. Perhaps this kind of damage control is the most we can hope for with this kind of event. If any bonuses were awarded at the end of 1987, our tangent-portfolio investor would have deserved hers.

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