Economic Capital Modelling and Basel II Compliance in the Banking Industry

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1 Introduction

It would be a mistake to conclude that the only way to succeed in banking is through ever-greater size and diversity. Indeed, better risk management may be the only truly necessary element of success in banking.
Alan Greenspan, Speach to the American Bankers Association, 10/5/2004.

Risk is an inevitable part of every financial institution, above all banks and insurance companies. Risks are implicitly accepted when such institutions provide their financial services to customers and explicitly when they take risk positions that offer profitable, above-average returns. There is no unique view on risk and usually it is considered in certain sub-classes such as market risk, credit risk and operational risk, also interest rate risk and liquidity risk. Market risk is associated with trading activities; it is defined as the potential loss arising from adverse price changes of a bank's positions in financial markets and encompasses interest rate, foreign exchange, equity and credit-spread risk. Credit risk is defined as potential losses arising from a customer's default or loss of credit rating. Such risks usually include loan default risk, counterparty risk, issuer risk and country risk. Finally, operational risk is due to losses resulting from inadequate or failed internal processes, human errors, technological breakdowns, or from external events.

Moreover, risk can be distinguished by the negative effects and potential hazards it has on different kinds of stakeholders, e.g risks may seriously threaten the firm's market value (shareholders' perspective), create losses to their lenders (debtholders' perspective), or jeopardizing the stability of the financial system (regulators' perspective). Though the individual interests of these groups may be rather diverse, all parties are interested in an continued existence of the institution. Hence, a bank needs a certain amount of capital relative to its risk as a buffer against future potential losses. This capital base must be sufficient so that also very unlikely losses, measured at a high confidence level, can be absorbed.

The growing awareness of risk inherent in banking industry is partially owing to spectacular crunches like the Saving & Loans crisis in the 1970s or the Japanese banking crisis in the 1990s and led to an increasing demand for banking supervision at the international level, finally resulting in the Basel Committee of Banking Supervision under the auspices of the Bank for International Settlement (BIS) in Basel. The basic idea underlying modern banking regulation is pretty simple, namely that banks should quantify their risks and then are required to keep a certain amount of equity capital (the so-called "capital charge") as a buffer against it. For instance, the minimum capital ratio according to the "Basel Accord" should be 8 % of the so-called "risk-weighted assets", although some regulators set different target levels for individual banks, which may be substantially higher than 8 %.

The first important proposal of the Committee was the "1988 Accord", and even though it was primarily dealing with rather crude methods for assessing credit risk, "Basel I" was a major step towards a common framework for calculating minimum capital standards for international banks. In 1996 the Committee then released an amendment to the Basel I Accord where banks were allowed to build sophisticated internal models for calculating capital charges for their market risk exposures.

The new Basel Accord "Basel II" [BII04], which should be fully implemented by year-end 2007, describes a more comprehensive risk measure and minimum standard for capital adequacy and is structured in three Pillars. Pillar I imposes new methodologies of calculating regulatory capital, thereby mainly focusing on credit risk and operational risk. For the latter, banks can then use—similar as it is already the case for market risk—their own internal modelling techniques (commonly referred to as advanced measurement approaches (AMA)) to determine capital charges, and we consider this subject again in section 2.

Pillar II then introduces the so-called Internal Capital Adequacy Assessment Process (ICAAP) and contains guidance to supervisors on how they should review an institution's ICAAP. Besides the treatment of so-called "other" risks that are not covered under Pillar I such as interest rate risk or credit concentration risk, it deals with an institution's overall risk exposure. According to the Committee of European Banking Supervisors [CEBS], banks should calculate an "overall capital number" as an integral part of their ICAAP. This single-number metric should encompass all risks related to different businesses and risk types. Above all, regulators want to understand the extent to which the institution has introduced diversification and correlation effects when aggregating different risk types. A particularly important example of this issue is considered in section 3 where the inter-risk correlation between credit and market risk is investigated.

A milestone in mathematical finance was the idea of dynamic replication introduced in 1973 by Fischer Black, Myron Scholes and Robert C.

Merton [BS73], revolutionizing the theory of pricing and hedging of financial derivatives completely. Then, since the introduction of internal market risk models in 1996, quantitative risk management has become an interesting and fruitful research area for mathematicians and statisticians; cf. Föllmer & Klüppelberg [FK02].

Although our project focussed at the beginning on credit risk problems alone with results documented in Hillebrand [H06], our industry partner was interested in further collaboration in operational risk and aggregation of different risk types, more precisely in aggregation of market and credit risk. As these are new areas with many interesting open problems, we henceforth concentrate on these cutting-edge topics.

Our paper is organised as follows. In section 2 we suggest a novel method for calculating operational risk at a high confidence level by using the new concept of Lévy copulas. Our results can be used as an approximation for operational Value-at-Risk and deliver important insights into extremal dependence modelling in general. In section 3 we then investigate the interaction between a credit portfolio and another risk type, which can be thought of as market risk. Combining Merton-like factor models for credit risk with linear factor models for market risk, we analytically calculate their inter-risk correlation and show how inter-risk correlation bounds can be derived. For known inter-risk correlation the total aggregated credit and market risk can be approximated (cf. (20) below). We conclude with a discussion of possible overlapping risk and indicate the assignment problem of a simple financial instrument to one specific risk like operational, credit or market risk.

2 Analytical Approximation of Operational Risk

One of the determinants of Basel II is Operational Risk, defined as losses resulting from inadequate or failed internal processes, human errors, technological breakdowns, or from external events. Risk in all categories of Basel II is defined as Value-at-Risk (VAR) of the total loss (per year) at a certain confidence level κ near 1. If we denote by S this total loss, then VAR(κ) is the capital amount such that total losses remain below VAR with at least probability κ. This is a rather simplistic risk measure; it only becomes non-trivial because the total loss S is not a straightforward quantity. Below we concentrate on the advanced measurement approach (AMA) and indicate the problems involved for obtaining VAR(κ). It is important to note that the Basel Committee specifies as quantitative standards a confidence level of $\kappa = 0.999$ and only models, which capture potentially severe tail loss events.

2.1 The Loss Distribution Approach

A required feature of AMA for measuring operational risk in the context of Pillar II is that it allows for explicit correlations between different operational risks, usually classified according to an event type/business line matrix consisting of eight business lines and seven loss event types. The core problem here is the multivariate modelling and how the dependence structure between different matrix cells affects a bank's total operational risk. The prototypical loss distribution approach (LDA) assumes that, for each cell $i = 1, \dots, d$, the cumulated operational loss $S_i(t)$ up to time t is described by an aggregate loss process

$$S_i(t) = \sum_{k=1}^{N_i(t)} X_k^i \,, \quad t \geq 0 \,, \tag{1}$$

where for each i the sequence $(X_k^i)_{k\in\mathbb{N}}$ are independent and identically distributed (iid) positive random variables with distribution function F_i describing the magnitude of each loss event (loss severity), and $(N_i(t))_{t\geq 0}$ counts the number of losses in the time interval $[0, t]$ (called frequency), independent of $(X_k^i)_{k\in\mathbb{N}}$. For regulatory capital and economic capital purposes, the time horizon is usually fixed to $t = 1$ year. The bank's total operational risk is then given as

$$S^+(t) := S_1(t) + S_2(t) + \cdots + S_d(t) \,, \quad t \geq 0 \,. \tag{2}$$

The present literature suggests to model dependence between different operational risk cells by means of different concepts, which basically split into models for frequency dependence on the one hand and for severity dependence on the other hand.

Here we suggest a model based on the new concept of Lévy copulas (see e.g. Cont & Tankov [CT04]), which models dependence in frequency and severity simultaneously, yielding a model with comparably few parameters. Moreover, our model has the same advantage as a distributional copula: the dependence structure between different cells can be separated from the marginal processes S_i for $i = 1, \dots, d$. This approach allows for closed-form approximations for operational VAR (OpVAR).

2.2 Dependent Operational Risks and Lévy Copulas

In accordance with a recent survey of the Basel Committee on Banking Supervision about AMA practices at financial services firms, we assume that the loss frequency processes N_i in (1) follows a homogeneous Poisson process with rate $\lambda_i > 0$. Then the aggregate loss (1) constitutes a compound Poisson process and is therefore a Lévy process .

A key element in the theory of Lévy processes is the notion of the so-called Lévy measure. A Lévy measure controls the jump behaviour of a Lévy process and, therefore, has an intuitive interpretation, in particular in the context of operational risk. The Lévy measure of a single operational risk cell measures the expected number of losses per unit time with a loss amount in a prespecified interval. For our compound Poisson model, the Lévy measure Π_i of the cell process S_i is completely determined by the frequency parameter $\lambda_i > 0$ and the distribution function F_i of the cell's severity: $\Pi_i([0,x)) := \lambda_i P(X^i \le x) = \lambda_i F_i(x)$ for $x \in [0,\infty)$. The corresponding one-dimensional tail integral is defined as

$$\overline{\Pi}_i(x) := \Pi_i([x,\infty)) = \lambda_i P(X^i > x) = \lambda_i \overline{F}_i(x)\,. \tag{3}$$

Our goal is modelling multivariate operational risk. Hence, the question is how different one-dimensional compound Poisson processes $S_i(\cdot) = \sum_{k=1}^{N_i(\cdot)} X_k^i$ can be used to construct a d-dimensional compound Poisson process $S = (S_1, S_2, \ldots, S_d)$ with in general dependent components. It is worthwhile to recall the similar situation in the case of the more restrictive setting of static random variables. It is well-known that the dependence structure of a random vector can be disentangled from its marginals by introducing a distributional copula. Similarly, a multivariate tail integral

$$\overline{\Pi}(x_1,\ldots,x_d) = \Pi([x_1,\infty)\times\cdots\times[x_d,\infty))\,, \quad x \in [0,\infty]^d\,, \tag{4}$$

can be constructed from the marginal tail integrals (3) by means of a Lévy copula. This representation is the content of Sklar's theorem for Lévy processes with positive jumps, which basically says that every multivariate tail integral $\overline{\Pi}$ can be decomposed into its marginal tail integrals and a Lévy copula $\widehat{C}$ according to

$$\overline{\Pi}(x_1,\ldots,x_d) = \widehat{C}(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d))\,, \qquad x \in [0,\infty]^d\,. \tag{5}$$

For a precise formulation of this Theorem we refer to Cont & Tankov [CT04], Theorem 5.6. Now we can define the following prototypical LDA model.

Definition 1. [Multivariate Compound Poisson model]
(1) All aggregate loss processes S_i for $i = 1,\ldots,d$ are compound Poisson processes with tail integral $\overline{\Pi}_i(\cdot) = \lambda_i F_i(\cdot)$.
(2) The dependence between different cells is modelled by a Lévy copula $\widehat{C} : [0,\infty)^d \to [0,\infty)$, i.e. the tail integral of the d-dimensional compound Poisson process $S = (S_1,\ldots,S_d)$ is defined by

$$\overline{\Pi}(x_1,\ldots,x_d) = \widehat{C}(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d)).$$

2.3 The Bivariate Clayton Model

A bivariate model is particularly useful to illustrate how dependence modelling via Lévy copulas works. Therefore, we now focus on two operational risk cells as in Definition 1(1). The dependence structure is modelled by a Clayton Lévy copula, which is similar to the well-known Clayton copula for distribution functions and parameterized by $\vartheta > 0$ (see Cont & Tankov [CT04], Example 5.5):

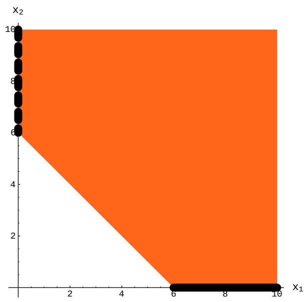

Fig. 1. Decomposition of the domain of the tail integral $\overline{\Pi}^{+}(z)$ for $z = 6$ into a simultaneous loss part $\overline{\Pi}_{\|}^{+}(z)$ (orange area) and independent parts $\overline{\Pi}_{\perp 1}(z)$ (solid black line) and $\overline{\Pi}_{\perp 2}(z)$ (dashed black line).

$$\widehat{C}_{\vartheta}(u,v) = (u^{-\vartheta} + v^{-\vartheta})^{-1/\vartheta}\,, \quad u,v \geq 0\,.$$

This copula covers the whole range of positive dependence. For $\vartheta \to 0$ we obtain independence and then, as we will see below, losses in different cells never occur at the same time. For $\vartheta \to \infty$ we get the complete positive dependence Lévy copula given by $\widehat{C}_{\|}(u,v) = \min(u,v)$. We now decompose the two cells' aggregate loss processes into different components (where the time parameter t is dropped for simplicity),

$$\begin{aligned} S_1 = S_{\perp 1} + S_{\| 1} &= \sum_{k=1}^{N_{\perp 1}} X^1_{\perp k} + \sum_{l=1}^{N_{\|}} X^1_{\| l}\,, \\ S_2 = S_{\perp 2} + S_{\| 2} &= \sum_{m=1}^{N_{\perp 2}} X^2_{\perp m} + \sum_{l=1}^{N_{\|}} X^2_{\| l}\,, \end{aligned} \tag{6}$$

where $S_{\|1}$ and $S_{\|2}$ describe the aggregate losses of cell 1 and 2 that is generated by "common shocks", and $S_{\perp 1}$ and $S_{\perp 2}$ describe aggregate losses of one cell only. Note that apart from $S_{\|1}$ and $S_{\|2}$, all compound Poisson processes on the right-hand side of (6) are mutually independent. The frequency of simultaneous losses is given by

$$\widehat{C}_{\vartheta}(\lambda_1, \lambda_2) = \lim_{x \downarrow 0} \overline{\Pi}_{\|2}(x) = \lim_{x \downarrow 0} \overline{\Pi}_{\|1}(x) = (\lambda_1^{-\vartheta} + \lambda_2^{-\vartheta})^{-1/\vartheta} =: \lambda_{\|}\,,$$

which shows that the number of simultaneous loss events is controlled by the Lévy copula. Obviously, $0 \leq \lambda_{\|} \leq \min(\lambda_1, \lambda_2)$, where the left and right bounds refer to $\vartheta \to 0$ and $\vartheta \to \infty$, respectively. Consequently, in the case of independence, losses never happen at the same instant of time.

Also the severity distributions of $X^1_{\|}$ and $X^2_{\|}$ as well as their dependence structure are determined by the Lévy copula. To see this, define the joint survival function as

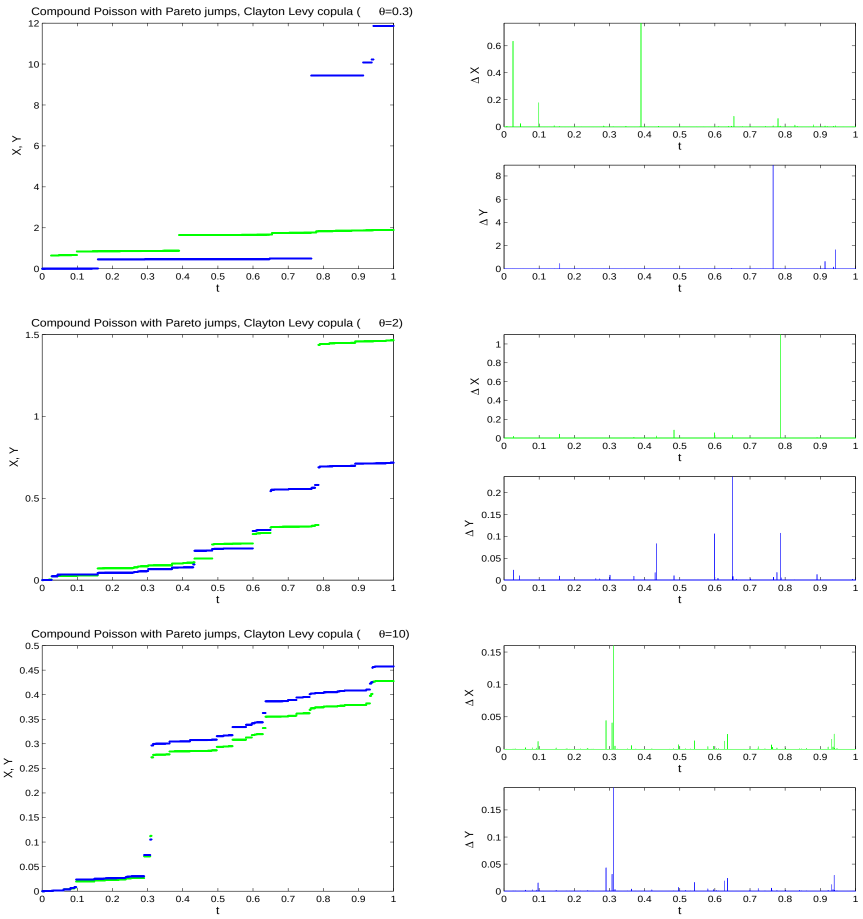

Fig. 2. Two-dimensional LDA Clayton Pareto model (with Pareto tail index $\alpha = 1/2$) for different parameter values.
Left column: compound processes, *right column:* frequencies and severities.
Upper row: $\delta = 0.3$ (low dependence), *middle row:* $\delta = 2$ (medium dependence), *lower row:* $\delta = 10$ (high dependence).

$$\overline{F}_{\parallel}(x_1, x_2) := P(X^1_{\parallel} > x_1, X^2_{\parallel} > x_2) = \frac{1}{\lambda_{\parallel}} \widehat{C}_{\vartheta}(\overline{\Pi}_1(x_1), \overline{\Pi}_2(x_2)) \tag{7}$$

with marginals

$$\overline{F}_{\parallel 1}(x_1) = \lim_{x_2 \downarrow 0} \overline{F}_{\parallel}(x_1, x_2) = \frac{1}{\lambda_{\parallel}} \widehat{C}_{\vartheta}(\overline{\Pi}_1(x_1), \lambda_2) \tag{8}$$

$$\overline{F}_{\parallel 2}(x_2) = \lim_{x_1 \downarrow 0} \overline{F}_{\parallel}(x_1, x_2) = \frac{1}{\lambda_{\parallel}} \widehat{C}_{\vartheta}(\lambda_1, \overline{\Pi}_2(x_2))\,. \tag{9}$$

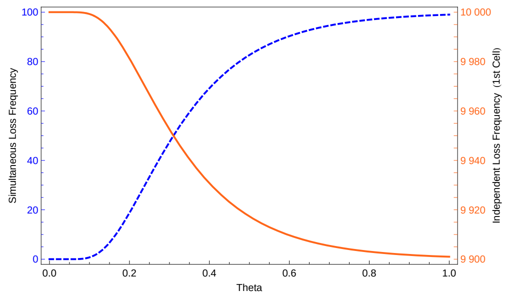

Fig. 3. Visualisation of the cells' loss frequencies controlled by the Clayton Lévy copula for $\lambda_1 = 10\,000$ and $\lambda_2 = 100$. *Left blue axis:* frequency $\lambda_\|$ of the simultaneous loss processes $S_{\|1}$ and $S_{\|2}$ as a function of the Lévy Clayton copula parameter ϑ (blue, dashed line). *Right orange axis:* frequency $\lambda_{\perp 1}$ of the independent loss process $S_{\perp 1}$ of the first cell as a function of the Lévy Clayton copula parameter ϑ (orange, solid line).

In particular, it follows that $F_{\|1}$ and $F_{\|2}$ are different from F_1 and F_2, respectively. To explicitly extract the dependence structure between the severities of simultaneous losses $X_\|^1$ and $X_\|^2$ we use the concept of a distributional survival copula. Using (7)–(9) we see that the survival copula S_ϑ for the tail severity distributions $\overline{F}_{\|1}(\cdot)$ and $\overline{F}_{\|2}(\cdot)$ is the well-known distributional Clayton copula; i.e.

$$S_\vartheta(u,v) = (u^{-\vartheta} + v^{-\vartheta} - 1)^{-1/\vartheta}, \qquad 0 \le u, v \le 1\,.$$

For the tail integrals of the independent loss processes $S_{\perp 1}$ and $S_{\perp 2}$. we obtain for $x_1, x_2 \ge 0$

$$\begin{aligned}\overline{\Pi}_{\perp 1}(x_1) &= \overline{\Pi}_1(x_1) - \overline{\Pi}_{\|1}(x_1) = \overline{\Pi}_1(x_1) - \widehat{C}_\vartheta(\overline{\Pi}_1(x_1), \lambda_2)\,,\\ \overline{\Pi}_{\perp 2}(x_2) &= \overline{\Pi}_2(x_2) - \overline{\Pi}_{\|2}(x_2) = \overline{\Pi}_2(x_2) - \widehat{C}_\vartheta(\lambda_1, \overline{\Pi}_2(x_2))\,,\end{aligned}$$

so that $\lambda_{\perp 1} = \lambda_1 - \lambda_\|\,, \qquad \lambda_{\perp 2} = \lambda_2 - \lambda_\|$.

2.4 Analytical Approximations for Operational VAR

In this section we turn to the quantification of total operational loss encompassing all operational risk cells and, therefore, we focus on the total aggregate loss process S^+ defined in (2). Our goal is to provide some general insight to multivariate operational risk and to find out, how different dependence structures (modelled by Lévy copulas) affect OpVAR, which is the standard metric in operational risk measurement. We need some notation to define it properly.

The tail integral associated with S^+ is given by

$$\overline{\Pi}^+(z) = \Pi(\{(x_1,\ldots,x_d) \in [0,\infty)^d : \sum_{i=1}^d x_i \ge z\})\,, \quad z \ge 0\,. \tag{10}$$

For $d = 2$ we can write

$$\overline{\Pi}^{+}(z) = \overline{\Pi}_{\perp 1}(z) + \overline{\Pi}_{\perp 2}(z) + \overline{\Pi}_{\parallel}^{+}(z) \,, \qquad z \geq 0 \,, \tag{11}$$

where $\overline{\Pi}_{\perp 1}(\cdot)$ and $\overline{\Pi}_{\perp 2}(\cdot)$ are the independent jump parts and

$$\overline{\Pi}_{\parallel}^{+}(z) = \Pi(\{(x_1, x_2) \in (0,\infty)^2 : x_1 + x_2 \geq z\}) \,, \qquad z \geq 0 \,,$$

describes the dependent part due to simultaneous loss events.

Since for every compound Poisson process with intensity $\lambda > 0$ and positive jumps with distribution function F, the tail integral is given by $\overline{\Pi}(\cdot) = \lambda \overline{F}(\cdot)$, it follows from (11) that the total aggregate loss process S^{+} is again compound Poisson with frequency parameter and severity distribution

$$\lambda^{+} = \lim_{z \downarrow 0} \overline{\Pi}^{+}(z) \quad \text{and} \quad F^{+}(z) = 1 - \overline{F}^{+}(z) = 1 - \frac{\overline{\Pi}^{+}(z)}{\lambda^{+}} \,, \quad z \geq 0 \,. \tag{12}$$

This result proves now useful to determine a bank's total operational risk consisting of several cells. Before doing that, recall the definition of OpVAR for a single operational risk cell (henceforth called stand-alone OpVAR.) For each cell, stand-alone OpVAR at confidence level $\kappa \in (0,1)$ and time horizon t is the κ-quantile of the aggregate loss distribution, i.e.

$$\mathrm{VAR}_t(\kappa) = G_t^{\leftarrow}(\kappa) = \inf\{x \in \mathbb{R} : \ P(S(t) \leq x) \geq \kappa\} \,. \tag{13}$$

In Böcker & Klüppelberg [BK05, BK06, BK07a, BK07b] it was shown that OpVAR at high confidence level can be approximated by a closed-form expression, if the loss severity is subexponential, i.e. heavy-tailed. As this is common believe we consider in the sequel this approximation, which can be written as

$$\mathrm{VAR}_t(\kappa) \sim F^{\leftarrow}\left(1 - \frac{1-\kappa}{EN(t)}\right), \quad \kappa \uparrow 1 \,, \tag{14}$$

where the symbol "$\sim$" means that the ratio of left and right hand side converges to 1. Moreover, $EN(t)$ is the cell's expected number of losses in the time interval $[0,t]$. Important examples for subexponential distributions are lognormal, Weibull, and Pareto. We want to emphasize already here that such first order asymptotics work extremely well for heavy-tailed Pareto-like tails, which are realistic in operational risk. Since the loss frequencies only enter as their mean $EN(t)$, any sophisticated modelling of the loss number process is superfluous, see Böcker & Klüppelberg [BK06] for more details. Instead all effort should be directed into a more accurate modelling of the loss severity distribution.

Here, we extend the idea of an asymptotic OpVAR approximation to the multivariate problem. In doing so, we exploit the fact that S^{+} is a compound Poisson process with parameters as in (12). In particular, if F^{+} is subexponential, we can apply (14) to estimate total OpVAR. Consequently, if we are able to specify the asymptotic behaviour of $\overline{F}^{+}(x)$ as $x \to \infty$ we have automatically an approximation of $\mathrm{VAR}_t(\kappa)$ as $\kappa \uparrow 1$.

To make more precise statements about OpVAR, we focus our analysis on Pareto distributed severities with distribution function

$$\overline{F}(x) = \left(1 + \frac{x}{\theta}\right)^{-\alpha}, \quad x > 0\,,$$

with shape parameters $\theta > 0$ and tail parameter $\alpha > 0$. Pareto's law is the prototypical parametric example for a heavy-tailed distribution and suitable for operational risk modelling. As a simple consequence of (14), in the case of a compound Poisson model with Pareto severities (Pareto-Poisson model) analytic OpVAR is given by

$$\mathrm{VAR}_t(\kappa) \sim \theta \left[\left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha} - 1\right] \sim \theta \left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha}, \quad \kappa \uparrow 1\,. \tag{15}$$

To demonstrate the kind of results we obtain by such approximation methods we consider a Pareto-Poisson model, where the severity distributions F_i of the first (say) $b \le d$ cells are tail equivalent with tail parameter $\alpha > 0$ and dominant to all other cells, i.e.

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \left(\frac{\theta_i}{\theta_1}\right)^{\alpha}, \; i = 1, \ldots, b\,, \qquad \lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = 0\,, \; i = b+1, \ldots, d\,. \tag{16}$$

In the important cases of complete positive dependence and independence, closed-form results can be found and may serve as extreme cases concerning the dependence structure of the model.

Theorem 1. *Consider a compound Poisson model with cell processes* $S_1, \ldots, S_d$ *with Pareto distributed severities satisfying* (16). *Let* $\mathrm{VAR}_t^i(\cdot)$ *be the stand-alone* OpVAR *of cell* i.

(1) *If all cells are completely dependent with the same frequency* λ *for all cells, then* S^+ *is compound Poisson with parameters*

$$\lambda^+ = \lambda \qquad \textit{and} \qquad \overline{F}^{+}(z) \sim \left(\sum_{i=1}^{b} \theta_i\right)^{\alpha} z^{-\alpha}, \quad z \to \infty\,,$$

and total OpVAR *is asymptotically given by*

$$\mathrm{VAR}_{\|t}^{+}(\kappa) \sim \sum_{i=1}^{b} \mathrm{VAR}_t^i(\kappa), \qquad \kappa \uparrow 1\,. \tag{17}$$

(2) *If all cells are independent, then* S^+ *is compound Poisson with parameters*

$$\lambda^+ = \lambda_1 + \cdots + \lambda_d \quad \textit{and} \quad \overline{F}^{+}(z) \sim \frac{1}{\lambda^+} \sum_{i=1}^{b} \left(\frac{\theta_i}{z}\right)^{\alpha} \lambda_i\,, \quad z \to \infty\,, \tag{18}$$

and total OpVAR *is asymptotically given by*

$$\mathrm{VAR}^{+}_{\perp t}(\kappa) \sim \left[\sum_{i=1}^{b} \left(\mathrm{VAR}^{i}_{t}(\kappa)\right)^{\alpha}\right]^{1/\alpha}, \qquad \kappa \uparrow 1\,. \tag{19}$$

Theorem 1 states that for the completely dependent Pareto-Poisson model, total asymptotic OpVAR is simply the sum of the dominating cell's asymptotic stand-alone OpVARs. Recall that this is similar to the new proposals of Basel II, where the standard procedure for calculating capital charges for operational risk is just the simple-sum VAR. To put it another way, regulators implicitly assume complete dependence between different cells, meaning that losses within different business lines or risk categories always happen at the same instants of time.

Very often, the simple-sum OpVAR (17) is considered to be the worst case scenario and, hence, as an upper bound for total OpVAR in general, which in the heavy-tailed case can be grossly misleading. To see this, assume the same frequency λ in all cells also for the independent model, and denote by $\mathrm{VAR}^{+}_{\|}(\kappa)$ and $\mathrm{VAR}^{+}_{\perp}(\kappa)$ completely dependent and independent total OpVAR, respectively. Then, as explained in detail in [BK06] for heavy-tailed severity data with $\overline{F}_i(x_i) \sim (x_i/\theta_i)^{-\alpha}$ as $x_i \to \infty$, subadditivity of OpVAR is violated because the sum of stand-alone OpVARs is smaller than independent total OpVAR. The following table, taken from [RK99], illustrates this.

α	$\mathrm{VAR}^{+}_{\|}$	$\mathrm{VAR}^{+}_{\perp}$
1.2		178.2
1.1		187.8
1.0	200.0	200.0
0.9		216.0
0.8		237.8
0.7		269.2

Table 1. Comparison of total OpVaR for two operational risk cells (each with stand alone VaR of 100 million) in the case of complete dependence ($\|$) and independence ($\perp$) for different values of α.

More general dependence structures can be investigated within the framework of multivariate regular variation. For homogeneous models, in particular for the Clayton Lévy copula, precise results have been derived in Klüppelberg and Resnick [KR07] and applied to find OpVAR approximations in Böcker and Klüppelberg [BK06].

3 Inter-Risk Correlation of Market and Credit Risk

3.1 The Necessity for Risk Aggregation

A core element of modern risk control is the calculation of an aggregated group-wide risk figure, which is used to evaluate the capital adequacy of a financial institution. Until now no standard procedure for risk aggregation has emerged, but a widespread approach in the banking industry is "aggregation across risk types", where in a first step marginal, institution-wide loss distributions for all relevant risk types are calculated. These marginal risk figures describe the group-wide, pre-aggregated risk of a given risk type encompassing different legal entities, divisions, regions etc. Then, in a second step, the dependence structure between these pre-aggregated risk-type figures is modelled and finally the total risk can be calculated.

The easiest way of aggregating risks is simply to add up all pre-aggregated risk-type figures (cf. Theorem 1(1) in the case of different operational risk figures). Problems with this procedure have been indicated after Theorem 1 and made transparent in Table 1). Consequently, this yields only a very rough estimate of the bank-wide total risk. Furthermore, banks usually try to reduce overall risk by accounting for diversification between different risk types–measured by correlation–because this allows them to reduce expensive equity capital. Hence, advanced approaches for risk aggregation begin with an analysis of the dependence structure between different risk types.

Important measures of dependence in the context of risk-type aggregation are correlation (which models linear dependence); possible non-linear dependence is often modeled by means of copulas. In practise, a widespread approach for aggregating different risk types is the so-called *square-root-formula approach* or *variance-covariance approach*. Though mathematically justified only in the case of elliptically distributed risk types (with the multivariate normal or t distributions as prominent examples), this approach is very often used as a first approximation because total aggregated capital can then be calculated explicitly without expensive simulations. If $X^T = (X_1, \ldots, X_m)$ is the vector of pre-aggregated risk figures (e.g. economic capital X_i for risk-types $i = 1, \ldots, m$), and R the inter-risk correlation matrix, then total aggregated risk X_{tot} is for elliptically distributed X given by

$$X_{tot} = \sqrt{X^T R X}\,. \tag{20}$$

Hence, a typical problem of risk aggregation is the estimation of the inter-risk correlation matrix R.

In the sequel we concentrate on the two-dimensional problem consisting of credit risk together with another risk type, which henceforth is referred to as market risk. Credit risk can be more than six times as large as the classical market risk associated with trading activities, and it is clear that in this case total risk (20) is mainly dominated by credit risk alone and, in particular, it is only little affected by inter-risk correlation. However, the exposures of other market-like risk types like financial investment risk, real estate risk, or business risk, which are often measured by banks in the context of economic capital and Basel II compliance, are comparable in volume to overall credit risk, and the question regarding correct modelling of inter-risk correlation again becomes important.

We combine a Merton-like factor model for credit risk with a linear factor model for market risk. Both models are driven by a set of (macroeconomic) factors $Y = (Y_1, \ldots, Y_K)$ where the factor weights are allowed to be zero so that a risk type may only depend on a subset of Y. This section is based on [BH07].

3.2 Modelling Credit and Market Risk

Normal Factor Model for Credit Risk

To describe credit portfolio loss, we choose a classical structural model as it can be found e.g. in Bluhm, Overbeck & Wagner [BOW02]. Within these models, a borrower's credit quality is driven by a so-called "ability-to-pay" process. Consider a portfolio of n loans. Then, default of an individual obligor $i \in \{1, \ldots, n\}$ is described by a Bernoulli random variable L_i with $\mathbb{P}(L_i = 1) = p_i = 1 - \mathbb{P}(L_i = 0)$ where p_i is the obligor's probability of default within time period $[0, T]$ for fixed $T > 0$. Following Merton's idea, counterparty i defaults if its asset value log-return A_i falls below some threshold D_i, sometimes referred to as default point, i.e.

$$L_i = \mathbb{1}_{\{A_i < D_i\}}\,, \qquad i = 1, \ldots, n\,. \tag{21}$$

If we denote the exposure at default (perhaps enriched by discounting factors and/or net of recovery rates) of an individual obligor by e_i, portfolio loss is given by

$$L^{(n)} = \sum_{i=1}^{n} e_i\, L_i\,. \tag{22}$$

In a factor-model approach, the asset values A_i are linked to a set of macroeconomic factors $Y_1, \ldots, Y_K$, which are assumed to be normally distributed and the vector $(Y_1, \ldots, Y_K)$ has been transformed to standard normal.

Definition 2. [Normal factor model for credit risk] *Let $Y = (Y_1, \ldots, Y_K)$ be a random vector of (macroeconomic) factors with multivariate standard normal distribution. We assume that each of the asset value log-returns A_i for $i = 1, \ldots, n$ linearly depends on Y as well as on a standard normally distributed idiosyncratic factor ε_i (which models the performance of firm i) independent of Y, i.e.*

$$A_i = \sum_{k=1}^{K} \beta_{ik} Y_k + \sqrt{1 - \sum_{k=1}^{K} \beta_{ik}^2}\; \varepsilon_i\,, \qquad i = 1, \ldots, n\,, \tag{23}$$

with factor loadings β_{ik} *satisfying* $R_i^2 := \sum_{k=1}^K \beta_{ik}^2 \in [0,1]$, *which is that part of the variance of* A_i *which can be explained by the systematic factor vector* Y. *Then* $L^{(n)}$ *as given in* (22) *is called normal factor model for credit risk.*

Equation (23) implies that log-returns $A_1, \ldots, A_n$ are standard normally distributed, but dependent with correlations

$$\rho_{ij} := \operatorname{corr}(A_i, A_j) = \sum_{k=1}^{K} \beta_{ik}\beta_{jk}\,, \qquad i,j = 1,\ldots,n\,. \tag{24}$$

Owing to the normal factor structure of the model, the default point D_i of every obligor is related to its default probability p_i by

$$D_i = \Phi^{-1}(p_i)\,, \qquad i = 1,\ldots,n\,, \tag{25}$$

where Φ is the standard normal distribution function. Moreover, the joint default probability of two obligors is given by

$$p_{ij} := \mathbb{P}(A_i \le D_i, A_j \le D_j) = \begin{cases} \Phi_{\rho_{ij}}(D_i, D_j)\,, & i \ne j\,, \\ p_i\,, & i = j\,, \end{cases} \tag{26}$$

where $\Phi_{\rho_{ij}}$ denotes the bivariate normal distribution function with standardized marginals and correlation ρ_{ij} given by (24). Finally, the default correlation between two different obligors is given by

$$\operatorname{corr}(L_i, L_j) = \frac{p_{ij} - p_i\,p_j}{\sqrt{p_i(1-p_i)\,p_j(1-p_j)}}\,, \qquad i,j = 1,\ldots,n\,. \tag{27}$$

Factor Models for Market Risk

We assume that market risk is already pre-aggregated and can be approximated by a one-dimensional random variable Z, representing the aggregated profit and loss (P/L) distribution due to changes in some market variables, such as interest rates or equity prices.

As in the credit risk model of Definition 2, we explain fluctuations of the P/L random variable Z by means of (macroeconomic) factors $Y = (Y_1, \ldots, Y_K)$. We use the same macroeconomic factors for credit and market risk, where independence of risk from such a factor is indicated by a loading factor 0.

As we want to add market and credit risk quantities, we use the convention that losses correspond to positive values of Z. One can think of Y as a vector describing the healthiness of the economy in the sense that positive (negative) values of the Y_k correspond to a good (bad) economy, implying a decreasing (increasing) market risk.

Definition 3. [Normal factor model for market risk] *Let $Y = (Y_1, \dots, Y_K)$ be a random vector of (macroeconomic) factors with multivariate standard normal distribution. Then, the normal factor model for the pre-aggregated market risk P/L is given by*

$$Z = -\sigma \left(\sum_{k=1}^{K} \gamma_k Y_k + \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2}\, \eta \right) \tag{28}$$

with factor loadings satisfying $\sum_{k=1}^{K} \gamma_k^2 \in [0,1]$, which is that part of the variance of Z which can be explained by the systematic factor Y. Furthermore, η is a standard normally distributed idiosyncratic factor, independent of Y. Finally, σ is the standard deviation of Z.

Definition 4. [Normal factor model for credit and market risk] *Let $Y = (Y_1, \dots, Y_K)$ be a random vector of (macroeconomic) factors with multivariate standard normal distribution. Let the credit portfolio loss $L^{(n)}$ be given by* (22) *and the asset value log-returns A_i for $i = 1, \dots, n$ are modeled by the normal factor model* (23)*. Let Z be the pre-aggregated market risk P/L modeled by the normal factor model* (28)*. When the credit model's idiosyncratic factors ε_i for $i = 1, \dots, n$ are independent of η, then we call $(L^{(n)}, Z)$ the normal factor model for credit and market risk.*

In order to account for possible heavy tails for Z we introduce the following global shock approach.

Definition 5. [Shock model for market risk] *Let $Y = (Y_1, \dots, Y_K)$ be a random vector of (macroeconomic) factors with multivariate standard normal distribution and let η be the standard normally distributed idiosyncratic factor, independent of Y. Further, let W be a positive random variable, independent of Y and η. Then the shock model for the pre-aggregated market risk P/L is given by the normal mixture model*

$$\widetilde{Z} = -\sigma W \left(\sum_{k=1}^{K} \gamma_k Y_k + \sqrt{1 - \sum_{k=1}^{K} \gamma_k^2}\, \eta \right), \tag{29}$$

where σ is a scaling factor. If $W = \sqrt{\nu / S_\nu}$ and S_ν is a χ^2_ν distributed random variable with ν degrees of freedom, then we call $\widetilde{Z}$ a t_ν-model for the pre-aggregated market risk P/L.

The mixing variable W can be interpreted as a "global shock" driving the variance of all factors. Such an overarching shock may occur from political distress, severe economic recession or some natural disaster.

3.3 Inter-Risk Correlation

We now investigate the correlation between credit risk $L^{(n)}$ and market risk Z, which is defined as

$$\operatorname{corr}(L^{(n)}, Z) = \frac{\operatorname{cov}(L^{(n)}, Z)}{\sqrt{\operatorname{var}(L^{(n)})}\sqrt{\operatorname{var}(Z)}} \,. \tag{30}$$

Within our modelling framework, we are able to analytically investigate inter-risk correlation yielding closed-form results.

First we assume that both market and credit risk have a normally distributed factor structure.

Theorem 2 (Inter-risk correlation for the normal factor model). *Suppose that credit portfolio loss $L^{(n)}$ and market risk Z are described by the normal factor model of Definition 4. Then correlation between $L^{(n)}$ and Z is given by*

$$\operatorname{corr}(L^{(n)}, Z) = \frac{\sum_{i=1}^{n} r_i\, e_i \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi \operatorname{var}(L^{(n)})}} \,, \tag{31}$$

where D_i is the default point (25)

$$r_i := \operatorname{corr}(A_i, Z) = \sum_{k=1}^{K} \beta_{ik}\gamma_k \,, \qquad i = 1, \dots, n, \tag{32}$$

and

$$\operatorname{var}(L^{(n)}) = \sum_{i,j=1}^{n} e_i\, e_j\, (p_{ij} - p_i\, p_j), \tag{33}$$

where p_{ij} the joint default probability (26).

Proof. Using $\mathbb{E}(Z) = 0$ and that η in (28) is independent of Y (and thus of L_i), the covariance between $L^{(n)}$ and Z is

$$\operatorname{cov}(L^{(n)}, Z) = \mathbb{E}(Z L^{(n)}) = -\sigma \sum_{i=1}^{n} e_i \sum_{k=1}^{K} \gamma_k\, \mathbb{E}(Y_k L_i) \,. \tag{34}$$

Recall the definition of L_i in (21) with A_i as in (23), and define for $k \in \{1, \dots, K\}$

$$A_i^{(-k)} = \sum_{\substack{l=1 \\ l \neq k}}^{K} \beta_{il}\, Y_l + \sqrt{1 - \sum_{j=1}^{K} \beta_{ij}^2}\; \varepsilon_i \,.$$

Conditioning on Y_k yields for the expectation

$$
\begin{aligned}
\mathbb{E}(Y_k\, L_i) &= \mathbb{E}\big(Y_k\, \mathbb{E}(\mathbb{1}_{\{A_i<D_i\}} \mid Y_k)\big) \\
&= \mathbb{E}\big(Y_k\, \mathbb{P}(A_i^{(-k)} \le D_i - \beta_{ik} Y_k)\big) \\
&= \mathbb{E}\left(Y_k\, \Phi\Big(\frac{D_i - \beta_{ik}\, Y_k}{\sqrt{1-\beta_{ik}^2}}\Big)\right),
\end{aligned}
$$

where we have used that $A_i^{(-k)}$ is normally distributed with variance $1-\beta_{ik}^2$. By partial integration and the fact that for the density φ of the standard normal distribution $y\,\varphi(y)$ has antiderivative $\varphi(y)$, we obtain

$$
\begin{aligned}
\mathbb{E}(Y_k\, L_i) &= \int_{-\infty}^{\infty} y\, \Phi\Big(\frac{D_i - \beta_{ik}\, y}{\sqrt{1-\beta_{ik}^2}}\Big) \varphi(y)\, dy \\
&= -\frac{\beta_{ik}}{\sqrt{1-\beta_{ik}^2}} \int_{-\infty}^{\infty} \varphi\left(\frac{D_i - \beta_{ik}\, y}{\sqrt{1-\beta_{ik}^2}}\right) \varphi(y)\, dy\,.
\end{aligned}
$$

The right-hand side is $-\beta_{ik}$ times the density of a random variable $U = \sqrt{1-\beta_{ik}^2}X + \beta_{ik}Y$ for standard normal iid X, Y at point D_i. Since U is then again standard normal, we obtain

$$
\mathbb{E}(Y_k\, L_i) = -\beta_{ik}\varphi(D_i) \;=\; -\frac{\beta_{ik}}{\sqrt{2\pi}}\, e^{-\frac{D_i^2}{2}}\,. \tag{35}
$$

Plugging this into (34) with r_i as in (32) this yields

$$
\mathrm{cov}(L^{(n)}, Z) = \frac{\sigma}{\sqrt{2\pi}} \sum_{i=1}^{n} e_i\, r_i\, e^{-\frac{D_i^2}{2}}\,.
$$

Furthermore, from (22) we calculate

$$
\begin{aligned}
\mathrm{var}(L^{(n)}) &= \sum_{i,j=1}^{n} e_i\, e_j\, \big(\mathbb{E}(L_i L_j) - \mathbb{E}(L_i)\mathbb{E}(L_j)\big) \\
&= \sum_{i,j=1}^{n} e_i\, e_j\, (p_{ij} - p_i\, p_j)\,,
\end{aligned}
$$

where p_{ij} is the joint default probability (26). □

Note that r_i may become negative if (some) factor weights β_{ik} and γ_k have different signs. Therefore, in principal, also negative inter-risk correlations can occur between the credit and market portfolio. Typical values for the inter-risk correlation lie in a range between 10 % and 60 % and vary significantly within the banking sector. A similar result can be obtained for the shock model o Definition 5.

Theorem 3 (Inter-risk correlation for the t_ν factor model). *Suppose that credit portfolio loss $L^{(n)}$ is described by the normal factor model of Definition 2. Denote by Z and $\widetilde{Z}$ the market risk described by the normal factor and by the shock model of Definition 3 and Definition 5, respectively. If W has finite second moment, then*

$$\operatorname{corr}(L^{(n)}, \widetilde{Z}) = \frac{\mathbb{E}(W)}{\sqrt{\mathbb{E}(W^2)}} \operatorname{corr}(L^{(n)}, Z)\,. \tag{36}$$

For the t_ν model with $\nu > 2$ we get

$$\operatorname{corr}(L^{(n)}, \widetilde{Z}) = f(\nu) \operatorname{corr}(L^{(n)}, Z) \tag{37}$$

with

$$f(\nu) := \sqrt{\frac{\nu - 2}{2}}\, \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\,. \tag{38}$$

Proof. Since $\mathbb{E}(Z) = 0$, we obtain with

$$\operatorname{cov}(L^{(n)}, \widetilde{Z}) = \mathbb{E}(W) \operatorname{cov}(L^{(n)}, Z) \quad \text{and} \quad \operatorname{var}(\widetilde{Z}) = \mathbb{E}(W^2) \operatorname{var}(Z)$$

that

$$\operatorname{corr}(L^{(n)}, \widetilde{Z}) = \frac{\mathbb{E}(W)}{\sqrt{\mathbb{E}(W^2)}} \operatorname{corr}(L^{(n)}, Z)\,.$$

For the t_ν model with $\nu > 0$ we have $W = \sqrt{\nu/S}$, where S is χ^2_ν distributed with density

$$f_\nu(s) = \frac{2^{-\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)}\, e^{-s/2}\, s^{\nu/2-1}\,, \qquad s \geq 0\,.$$

It follows for $\nu > 1$ that

$$\mathbb{E}\left(\frac{1}{\sqrt{S}}\right) = \frac{2^{-\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty e^{-s/2}\, s^{\nu/2-3/2}\, ds = \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{\nu}{2}\right)}\,.$$

Analogously, for $\nu > 2$ we calculate $\mathbb{E}\left(\frac{1}{S}\right) = \left(\frac{1}{\nu-2}\right)$. Plugging this into (36) gives formula (37). □

Remark 1. Since $\mathbb{E}(W) > 0$, by the Cauchy-Schwarz inequality,

$$0 < \frac{\mathbb{E}(W)}{\sqrt{\mathbb{E}(W^2)}} \leq 1\,.$$

As a consequence thereof, given a positive inter-risk correlation $\operatorname{corr}(L^{(n)}, Z) \in (0, 1]$ for normally distributed market risk, introducing a shock into the model results in a smaller inter-risk correlation (36). For the t_ν model this situation is depicted in Figure 4. □

The fact that $\mathrm{corr}(L^{(n)}, Z)$ linearly depends on the correlations r_i and thus on the factor loadings γ_k implies the following Proposition, which can be used to estimate upper bounds for the inter-risk correlation, when no specific information about market risk is available.

Proposition 1 (Inter-risk correlation bounds). *Suppose that credit portfolio loss $L^{(n)}$ and market risk Z are described by the normal factor model of Definition 4. Assume that the market model factor loadings γ_k for $k = 1, \ldots, K$ are unknown. Then correlation between $L^{(n)}$ and Z is bounded by*

$$|\mathrm{corr}(L^{(n)}, Z)| \leq \frac{\sum_{i=1}^{n} e_i \sqrt{\sum_{k=1}^{K} \beta_{ik}^2}\, \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi\, \mathrm{var}(L^{(n)})}} \leq 1\,, \tag{39}$$

where $\mathrm{var}(L^{(n)})$ *is given in* (33).

Proof. Since the obligor's exposures e_i are assumed to be positive, it follows from (31) that

$$|\mathrm{corr}(L^{(n)}, Z)| \leq \frac{\sum_i e_i\, |r_i| \exp\left(-\frac{1}{2} D_i^2\right)}{\sqrt{2\pi \sum_{ij} e_i\, e_j\, (p_{ij} - p_i\, p_j)}}\,.$$

From $\sum_{k=1}^{K} \gamma_k^2 \leq 1$ it follows by the Cauchy-Schwartz inequality that

$$|r_i| = \left| \sum_{k=1}^{K} \beta_{ik} \gamma_k \right| \leq \left(\sum_{k=1}^{K} \beta_{ik}^2 \right)^{1/2} \left(\sum_{k=1}^{K} \gamma_k^2 \right)^{1/2} \leq \left(\sum_{k=1}^{K} \beta_{ik}^2 \right)^{1/2}.$$

The right-hand side is bounded by one, since $\sum_{k=1}^{K} \gamma_k^2 = 1$ corresponds to the correlation of the degenerate case of model (28). □

Therefore, solely based on the parametrization of the normal credit factor model and the assumption of a normally distributed, pre-aggregated market risk, bounds for the inter-risk correlation can be derived. Moreover, from the explicit form of (37) in Theorem 3 it is clear that a similar result holds also for the t_ν distributed market risk.

One-Factor Approximations

Instructive examples regarding the inter-risk correlation and its bounds can be obtained for one-factor models and they are useful to explain general characteristics of inter-risk correlation. As shown in Böcker & Hillebrand [BH07], section 4.1, such a common one-factor framework for both credit and market risk can be defined consistently, and in the sequel we want to summarize some of their results.

Within the one-factor framework, the credit portfolio is assumed to be homogenous; i.e. for $i = 1, \ldots, n$ exposure $e_i = e$, default probability $p_i = p$,

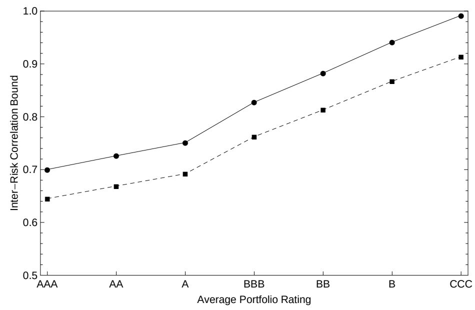

Fig. 4. LHP approximations of the inter-risk correlation bound as a function of the average portfolio rating according to (41). The solid line corresponds to the normal factor model (formally $\nu \to \infty$) and the dashed line to the shock model with $\nu = 5$. The uniform asset correlation is assumed to be $\rho = 10$ %.

and factor loadings $\beta_{ik} = \beta_k$ for $k = 1, \dots, K$, i.e. these quantities are the same for all credits of the portfolio. and both market and credit risk are systematically explained only by one single factor $\tilde{Y} := \frac{1}{\sqrt{\rho}} \sum_{k=1}^{K} \beta_k Y_k$, which is a compound of all Y_k for $k = 1, \dots, K$, where $\rho := \sum_{k=1}^{K} \beta_k^2$ is the uniform asset correlation of the credit portfolio; i.e. for any two asset value log-returns A_i, A_j the correlation is equal to ρ. The situation simplifies further in the case of a sufficiently large portfolio, where we consider $n \to \infty$, resulting in the so-called large homogenous portfolio (LHP) approximation (see also Bluhm, Overbeck & Wagner [BOW02], section 2.5.1.)

$$\frac{L^{(n)}}{n\,e} \overset{\text{a.s.}}{\to} \Phi\left(\frac{D - \sqrt{\rho}\,\tilde{Y}}{\sqrt{1-\rho}}\right) =: L\,, \qquad n \to \infty\,,$$

where $D = \Phi^{-1}(p)$ and ne is the total exposure of the credit portfolio. The LHP approximation plays an important role in the context of credit portfolio modelling; e.g. it is the underlying assumption in the calculation formula for regulatory capital charges in the *internal-ratings-based* (IRB) approach of Basel II.

Adopting the LHP approximation for the t_ν market model with the normal model as formal limit model with $\lim_{\nu\to\infty} f(\nu) = 1$, inter-risk correlation simplifies considerably. From (26) we get the joint default probability $p_{12} = \Phi_\rho(D, D)$ for two arbitray firms in the portfolio, and from (32) we see that $r = \sum_{k=1}^{K} \beta_k \gamma_k$. Then

$$\operatorname{corr}(L, \widetilde{Z}) = f(\nu)\, \frac{r\, e^{-D^2/2}}{\sqrt{2\pi(p_{12} - p^2)}}\,, \tag{40}$$

which is $\operatorname{corr}(L, Z)$ for the normal model with $f(\nu) = 1$. The bound (39) simplifies to

$$|\mathrm{corr}(L, \widetilde{Z})| \leq f(\nu)\, \frac{\sqrt{\rho}\, e^{-D^2/2}}{\sqrt{2\pi(p_{12} - p^2)}} \,. \tag{41}$$

According to equations (40) and (41), inter-risk correlation and its bound are functions of the homogeneous asset correlation ρ and the average default probability p and thus on the average rating structure of the credit portfolio. This is depicted in Figure 4 where LHP approximations of the inter-risk correlation bound are plotted as a function of the average portfolio rating.

A crucial point in the above approximation is the homogeneity of the credit portfolio. Even if actual credit portfolios are rarely exactly homogenous, the derived LHP approximations is a useful approximation in practice for the upper inter-risk correlation bound. Let us consider the normal factor model and so equation (41). For a loss distribution of a general credit portfolio (obtained for instance by Monte Carlo simulation) with expected loss μ, standard deviation ς, and total exposure e_{tot}, estimators $\hat{p}$ and $\hat{\rho}$ for p and ρ, respectively, can be found by moment matching; i.e. by comparing the expected loss and the variance of the simulated portfolio with those of an LHP:

$$\hat{\mu} = e_{\mathrm{tot}}\, \hat{p} \tag{42}$$

$$\hat{\varsigma}^2 = e_{\mathrm{tot}}^2\, (\hat{p}_{12} - \hat{p}^2) \;=\; e_{\mathrm{tot}}^2\, \left[\Phi_{\hat{\rho}}(\Phi^{-1}(\hat{p}), \Phi^{-1}(\hat{p})) - \hat{p}^2\right]. \tag{43}$$

From (41) we then obtain the following moment estimator for the upper inter-risk correlation bound

$$\widehat{B}_{\mathrm{LHP}}(\hat{p}, \hat{\rho}) = f(\hat{\nu})\, \frac{e_{\mathrm{tot}}}{\hat{\varsigma}}\, \frac{\sqrt{\hat{\rho}}\, \exp\left[-\frac{1}{2}\left(\Phi^{-1}(\hat{p})\right)^2\right]}{\sqrt{2\pi}} \,. \tag{44}$$

Conclusion

In this paper we suggested separate models for operational risk, credit risk and market risk, aiming at an integrated model quantifying the overall risk of a financial institution. In doing so, we adopted the common idea that "risk" of a financial position or even an entire bank can be separated into different risk types.

In general, however, such a silo approach often causes problems when risk-type definitions are overlapping, or the classification into risk types is unrealistic or even not possible. We want to present a simple but convincing example.

Consider a fixed-rate corporate bond, where the investor receives fixed, regular interest payments (with a rate set at the time the bond is issued) until the bond matures, called the coupon rate. On one hand, such an investment bears market risk, in particular interest rate risk: If market interest rates rise, then the market price of the bond will fall, because new bonds are expected to be issued with higher coupon rates, making old bonds less attractive. On the other hand, the bond also has credit risk, since the coupon rate of a bond also depends on the financial health of the issuer; i.e. on the credit rating of the company. The higher the company's default probability is, the less likely is that it will be able to pay the interest on the bond and to pay-off the bond at maturity. In this example (and of course also for more complex financial instruments) it does not make sense to distinguish market from credit risk, the only threat for the trader is a decrease in the market value of the bond.

Furthermore, professional trading of financial instruments requires a complex IT-infrastructure, and so also bears a significant fraction of operational risk. However, even in our simple example of the coupon bond, the question regarding its operational VAR remains unsolved. Similar problems arise in the context of other Pillar II risk types such as business and strategic risk, which are currently only poorly considered within a firm's enterprise risk management process. For a novel approach to this particular risk see Böcker [B07].

In accordance with Alan Greenspan we belief that a reliable and functioning risk management system is the basis for success in banking. Therefore, future research has to tackle the problem of how total risk (beyond that of market, credit and operational risk) can be measured and managed properly. To achieve such a "grand unified theory" of risk, a more holistic view on risk instead of the widespread silo approach is called for.

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