Optimal investment for insurers, when the stock price follows an exponential Lévy process

Radostina Kostadinova*

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Abstract

We consider a stochastic model for the wealth of an insurance company which has the possibility to invest into a risky and a riskless asset under a constant mix strategy. The total claim amount is modeled by a compound Poisson process and the price of the risky asset follows a general exponential Lévy process. We investigate the resulting integrated risk process and the corresponding discounted net loss process. This opens up a way to measure the risk of a negative outcome of the integrated risk process in a stationary way. We provide an approximation of the optimal investment strategy, which maximizes the expected wealth of the insurance company under a risk constraint on the Value-at-Risk. We conclude with some examples.

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*Graduate Program Applied Algorithmic Mathematics, Munich University of Technology, D-85747 Garching, Germany, email: radost@ma.tum.de, http://www.ma.tum.de/gkaam/

1 Introduction

It is well known that the success of an insurance company depends not only on its insurance business, but also on how well the company invests its reserve. This paper applies the theoretical results for an integrated risk process (IRP) of an insurance company investing into bond and stock, obtained in Klüppelberg and Kostadinova [16]. A risk measure frequently used in practice – Value-at-Risk (VaR), is defined in the framework of the integrated risk model. We suggest and compare several methods to find an optimal investment strategy, which maximizes the expected wealth of the insurance company subject to a risk bound. For the risk bound we use the VaR as a risk measure.

We first recall the model under consideration.

For the insurance business, we assume the classical model. The company starts with some initial capital $u > 0$ and it receives premiums at a constant rate $c > 0$. The total claim amount is modeled by a compound Poisson process $S(t) = \sum_{j=1}^{N(t)} Y_j\,,\ t \geq 0\,$. Here $N = (N(t))_{t\geq 0}$ is a homogeneous Poisson process with an intensity $\lambda > 0$, counting the claims, and $(Y_j)_{j\in\mathbb{N}}$ is a sequence, independent of N, of positive iid random variables (rv's) with a distribution function (df) F and a mean $\mu < \infty$, modeling the claim sizes. We denote by Y a generic claim size. Such models are very well studied, see, for example, Asmussen [1], Chapter 3, or Embrechts, Klüppelberg and Mikosch [7], Chapter 1.

For the investment we assume a Black-Scholes type market consisting of a *bond* with a constant interest rate and some *stock* which is modeled by an exponential Lévy process. Their respective prices follow the equations

$$X_0(t) = e^{\delta t} \quad \text{and} \quad X_1(t) = e^{L(t)}\,, \quad t \geq 0.$$

The constant $\delta > 0$ is the *riskless interest rate*. The Lévy process L has *characteristic exponent* Ψ, i.e. $E[\exp(isL(t))] = \exp(t\Psi(s))$, $s \in \mathbb{R}$, $t \geq 0$, where Ψ has a Lévy-Khintchine representation

$$\Psi(s) = is\gamma - \frac{\sigma^2}{2}s^2 + \int_{-\infty}^{+\infty} \left(e^{isx} - 1 - isx1_{\{|x|\leq 1\}}\right)\, \nu(dx)\,,\ \ s \in \mathbb{R}\,,$$

with $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and *Lévy measure* ν satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(dx) < \infty$. The *characteristic triplet* (γ, σ^2, ν) determines completely the distribution of the Lévy process L. For general Lévy process theory we refer to the monographs of Cont and Tankov [4] or Sato [25].

We assume that the insurance company invests by using the so called *constant mix strategy*. Under such a strategy, at each instance of time an initially fixed fraction $\theta \in [0, 1]$ of the wealth is invested in the risky asset and a fraction $1 - \theta$ in the riskless asset; see e.g. Emmer, Klüppelberg and Korn [6] and Emmer and Klüppelberg [5]. This strategy is dynamic in the sense that it requires a rebalancing of the portfolio at any moment of time depending on the corresponding price changes. This approach is based on self-financing portfolios and hence is classical in financial portfolio optimization; see Korn [19], Section 2.1. We call the fraction θ the *investment strategy*.

We introduce the intergrated risk process as the result of the insurance business and the net gains of the investment through a stochastic differential equation (SDE).

Definition 1.1. *For an investment strategy $\theta \in [0,1]$ we call the solution to the SDE*

$$dU_\theta(t) = c\,dt + dS(t) + U_\theta(t-)\left((1-\theta)\delta + \theta d\widehat{L}(t)\right), \quad t > 0\,, \;\; U_\theta(0) = u\,, \tag{1.1}$$

the integrated risk process (IRP)*, where $u > 0$ is the initial capital, $c > 0$ is the constant premium rate, S is the total claim amount process, δ is the riskless interest rate and $\widehat{L}$ is a Lévy process such that* $\exp(L) = \mathcal{E}(\widehat{L})$*.* □

The notation $\mathcal{E}$ in Definition 1.1 stays for the stochastic exponential of a Lévy process, see, e.g. Protter [24], Section 2.8, or Cont and Tankov [4], Section 8.4.2.

Provided that the insurance and the investment processes are independent, by Lemma 1.4 in [16] the solution to the SDE (1.1) is

$$U_\theta(t) = e^{L_\theta(t)}\left(u + \int_0^t e^{-L_\theta(v)}\,(c\,dv - dS(v))\right), \quad t \geq 0\,. \tag{1.2}$$

Here the Lévy process L_θ has characteristic exponent Ψ_θ and characteristic triplet $(\gamma_\theta, \sigma_\theta, \nu_\theta)$ specified in terms of the original process L, see Lemma 1.3 in [16]. Thus, the IRP U_θ fits in the framework of the generalized Ornstein-Uhlenbeck (OU) processes, which have recently attracted a lot of attention, see e.g. Lindner and Maller [20] or Carmona, Petit and Yor [3].

In the insurance framework, similar models have been investigated for example by Paulsen [23] and Kalashnikov and Norberg [14], and, in the special case of a geometric Brownian motion as an investment process, by Gaier and Grandits [9], Gaier, Grandits and Schachermayer [10] and Frolova, Kabanov and Pergamenshchikov [8].

Hipp and Plum [12, 13] analyse a model when the insurance company invests into risky assets, not necessarily financed from the risk reserve. In contrast to that, in our model the trading strategy θ is constant and $\theta \in [0,1]$, i.e. short selling is not allowed and the portfolio is self-financing.

Besides the characteristic exponents Ψ and Ψ_θ of the processes L and L_θ respectively, we shall also use the corresponding Laplace exponents defined by

$$\varphi(s) = \Psi(is) = E[e^{-sL(1)}] \quad \text{and} \quad \varphi_\theta(s) = \Psi_\theta(is) = E[e^{-sL_\theta(1)}]\,, \tag{1.3}$$

provided they exist. As we know from Lemma A1(c) in [16], $\varphi_\theta(s) < \infty$ for all $\theta \in [0,1]$ provided that $\varphi(s) < \infty$.

In the next example, we use the geometric Brownian motion as a model for the risky asset to illustrate the dynamics of the integrated risk model and our futher results.

Example 1.2. [Geometric Brownian motion as risky investment process]
Let L be a Brownian motion with drift, i.e. $L(t) = \gamma t + \sigma W(t)$, $t \geq 0$. Then L has characteristic triplet $(\gamma, \sigma^2, 0)$ and Laplace exponent $\varphi(s) = -\gamma s + \frac{\sigma^2}{2} s^2$, $s \in \mathbb{R}$. In this case the SDE (1.1) for U_θ reduces to $U_\theta(0) = u$ and

$$dU_\theta(t) = cdt - dS(t) + U_\theta(t-)\left(((1-\theta)\delta + \theta(\gamma + \frac{\sigma^2}{2}))dt + \theta\sigma dW(t)\right), \quad t > 0\,.$$

This enables the simulation of the IRP U_θ for all investment strategies $\theta \in [0,1]$. Furthermore, the solution to this SDE is given by (1.2), where L_θ is again a Brownian motion with drift with Laplace exponent

$$\varphi_\theta(s) = -\gamma_\theta s + \frac{\sigma_\theta^2}{2} s^2\,, \tag{1.4}$$

where, due to Lemma 1.3 in [16],

$$\gamma_\theta = \theta\gamma + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) \quad \text{and} \quad \sigma_\theta^2 = \theta^2\sigma^2\,. \tag{1.5}$$

However, it has been well observed in numerous empirical studies, that the Brownian motion model is not realistic for various stock prices, as these often exhibit sudden downward jumps and the distribution of the returns has heavier tails than the normal distribution, see e.g. Madan and Seneta [21]. In such cases it would be natural to model the stock prices by a more general exponential Lévy process with jumps. Unfortunately, such models lead to less explicit results, as the jump measure should be taken into account, see the examples in Section 2 of [16]. In particular, the simulation of U_θ is not easy for $\theta < 1$ and the process L_θ in (1.2) typically does not belong to any of the known classes of Lévy processes. □

The paper is organized as follows. In Section 2 we define the risk measure VaR in the framework of the integrated risk model and introduce a portfolio optimization problem. In Section 3 we recall and extend some of the theoretical results on the integrated risk model in [16]. In Section 4 we suggest and compare several methods to approximate the VaR and to solve the optimization problem. We conclude with some examples.

2 Risk measurement

The Value-at-Risk has become a standard risk measure for the insurance and banking industry. It is related to the capital reserve, which the financial institution needs to hold, in order to prevent (at a sufficiently high confidence level) insolvency due to an extremely negative development of the risks in its portfolio. Mathematically, the VaR is defined as some high quantile of the corresponding loss distribution.

In this section we provide a definition of the VaR for the integrated risk model. We aim at a stationary loss distribution. Following long tradition in insurance, we work with discounted losses by introducing the following transformation of the IRP

$$V_\theta(t) = u - e^{-L_\theta(t)} U_\theta(t) = \int_0^t e^{-L_\theta(v)} \left(dS(v) - c\, dv \right), \quad t \geq 0\,, \tag{2.1}$$

which we call the *discounted net loss process* (DNLP), see [16] for details. This process describes the total net loss (both from insurance and investment) of the company, (randomly) discounted to time 0. An important relation between the IRP and the DNLP is

$$P(U_\theta(t) < 0 \,|\, U_\theta(0) = u) = P(V_\theta(t) > u)\,, \;\; t \geq 0\,. \tag{2.2}$$

The advantage of this approach lies in the fact that the DNLP has a natural embedded discrete time skeleton, see Proposition 3.7 of [16]. This allows us to apply standard methods from the theory of stochastic recurrence equations, see for example Kesten [15] and Goldie [11]. The next proposition gives conditions, under which the DNLP defined in (2.1) has an a.s. limit as time tends to infinity. Recall the notations for the Laplace exponents φ and φ_θ in (1.3).

Proposition 2.1. *Let $E[Y] = \mu < \infty$, $0 < E[L(1)] < \infty$, $\delta < \varphi(-1)$ and either $\sigma > 0$ or $\nu((-\infty, 0)) > 0$. Define $\mathcal{V} = \{v \geq 0 : \varphi(v) < \infty\}$ and assume that $\sup \mathcal{V} \notin \mathcal{V}$.*

(a) If $\varphi(1) < \lambda$, then, for every $\theta \in [0, 1]$,

$$V_\theta(t) \overset{\text{a.s.}}{\to} V_\theta^\infty\,, \;\; t \to \infty\,, \tag{2.3}$$

where V_θ^∞ is a finite rv.

(b) If $\varphi(1) \geq \lambda$, then (2.3) holds for every $\theta \in [0, \theta_u)$, where $\theta_u \in (0, 1]$ is the unique strictly positive solution to the equation $\varphi_\theta(1) = \lambda$.

Proof. We apply Theorem 3.8 in [16]. By Lemma 4.1(b) in [16] we have that for a fixed $s > 0$ the function $\varphi_\theta(s)$ is convex in $\theta \in [0, 1]$. Furthermore, $\varphi_0(1) = -\delta < 0 < \lambda$, hence we have two cases. In the first case, if $\varphi_1(1) = \varphi(1) < \lambda$, then $\varphi_\theta(1) < \lambda$ for all $\theta \in [0, 1]$, which proves (a). In the second case, if $\varphi_1(1) = \varphi(1) \geq \lambda$, take θ_u as in (b). Then $\varphi_\theta(1) < \lambda$ for all $\theta \in [0, \theta_u)$, which proves (b). □

Remark 2.2. Note that the conditions in Proposition 2.1 are quite natural. Indeed, $E[Y] < \infty$ is seen as a prerequisite for any insurance, $E[L(1)] > 0$ is a prerequisite for any investment and $\delta < \log E[\exp(L(1))] = \varphi(-1)$ guarantees that the expected value of the risky investment is larger than the riskless investment. Further, any reasonable risky investment model should satisfy also that either $\sigma > 0$ or $\nu((-\infty, 0)) > 0$; otherwise no losses from risky investment are possible. $\square$

The distribution of the a.s. limit V_θ^∞ in Proposition 2.1 is of central interest in our work. In particular, it enables us to measure the risk in a stationary way.

Definition 2.3. *Let the conditions of Proposition 2.1 be satisfied. Denote by $\Theta \subseteq [0,1]$ the non-empty interval of investment strategies θ for which (2.3) holds. For $\theta \in \Theta$ we define*

$$\mathbf{VaR}_\alpha(V_\theta^\infty) = \inf\{x \in \mathbb{R} \,:\, P(V_\theta^\infty > x) \leq \alpha\}\,,$$

where $\alpha \in (0,1)$ is some (typically small) probability. $\square$

For risky assets, for which $\varphi(1) \geq \lambda$, the constant θ_u in Proposition 2.1(b) gives an upper bound for the reasonable investment strategies θ. Above this upper bound we cannot guarantee an a.s. limit of the DNLP and, hence, no reasonable statistical risk assesment is possible. This is illustrated in the following example.

Example 2.4. [Continuation of Example 1.2]
Consider the geometric Brownian motion as a model for the risky asset as in Example 1.2. Recall the Laplace exponent φ_θ in (1.4). Straightforward calculations show that, if the volatility of the risky asset is small enough, i.e. $\sigma^2 < 2(\gamma + \lambda)$, then the set Θ in Definition 2.3 is the whole interval $[0,1]$. Otherwise, if $\sigma^2 \geq 2(\gamma+\lambda)$, then $\Theta = [0, \theta_u)$, where

$$\theta_u = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma + \sigma^2/2 - \delta)^2 + 4\sigma^2(\delta + \lambda)}}{2\sigma^2} \leq 1 \tag{2.4}$$

is the unique strictly positive solution to the equation $\varphi_\theta(1) = \lambda$. Hence, for Brownian motion models with a very large volatility, i.e. $\sigma^2 \geq 2(\gamma+\lambda)$, no investment strategies greater or equal than θ_u given in (2.4) should be allowed. $\square$

Insurance companies usually review their success at predetermined times, e.g. every year or every quarter of a year. Hence, on the one hand, it seems reasonable to choose an investment strategy, which maximizes the wealth at the end of the planing period. On the other hand, there are certain regulatory or financial bounds on the amount of risk, which an insurance company may take on. The following optimization problem is based on these considerations:

$$\max_{\theta \in \Theta} E[U_\theta(t)] \quad \text{subject to} \quad \mathbf{VaR}_\alpha(V_\theta^\infty) \leq C\,, \tag{2.5}$$

for a given constraint $C > 0$, some fixed time period $t > 0$ and a given small probability α. Such problems are typical for the financial industry, see e.g Korn [19]. Our goal is to provide explicit solutions to (2.5).

The use of $\mathbf{VaR}_\alpha(V_\theta^\infty)$ as a risk measure in the portfolio optimization problem is explained by the fact that this quantity is equal to the capital reserve required to prevent insolvency with a sufficiently high probability $1-\alpha$ over a long time period, see (2.2). Note that in our definition the VaR does not depend on the initial capital and on the time t, but only on the selected investment strategy θ and on the stochastic properties of the insurance and the investment processes. The next lemma shows that the solution to the optimization problem (2.5) is also independent of the time period and of the initial capital.

Lemma 2.5. *Let the IRP U_θ be given by (1.2). Recall that $E[Y] = \mu < \infty$. Assume also that $\varphi(-1) = \log E[\exp(L(1))] < \infty$. Then, for $t \geq 0$, $E\left[U_\theta(t)\right]$ exists and*

$$E\left[U_\theta(t)\right] = uE\left[e^{L_\theta(t)}\right] + (c - \lambda\mu)\int_0^t E\left[e^{L_\theta(v)}\right]\,dv\,,$$

where $E\left[e^{L_\theta(t)}\right] = \exp\left(t\left(\delta + \theta(\varphi(-1) - \delta)\right)\right)$.
Furthermore, if the safety loading condition $c > \lambda\mu$ holds and $\delta < \varphi(-1)$, then $E\left[U_\theta(t)\right]$ is increasing in $\theta \in [0, 1]$.

Proof. First note that due to Lemma A.1(c) in [16], if $\varphi(-1) < \infty$ then also $\varphi_\theta(-1) < \infty$. Then the mean function of U_θ can be obtained from (2.1) and Lemma 3.2(a) in [16]. Also, by Lemma 1.3 in [16] we compute that $\varphi_\theta(-1) = \delta + \theta(\varphi(-1) - \delta)$. Therefore, if $\delta < \varphi(-1)$, the mean function $E[\exp(L_\theta(t))] = \exp(t\varphi_\theta(-1))$ is increasing in θ. Finally, if $c > \lambda\mu$, the mean function of U_θ is increasing in θ. $\square$

From Lemma 2.5 follows that, under natural conditions for the insurance and for the investment process, the expectation of the wealth of the company is an increasing function of the investment strategy θ for every fixed time period $t > 0$ and initial capital $u > 0$. Consequently, the optimization problem (2.5) is equivalent to

$$\max\left\{\theta \in \Theta \,:\, \mathbf{VaR}_\alpha(V_\theta^\infty) \leq C\right\}\,, \tag{2.6}$$

which depends only on the risk measure itself. This is, from a mathematical point of view, no surprise, as the a.s. limit V_θ^∞ of the DNLP, which is independent of u and t, is the basis for the risk measure. For an economic interpretation, recall that the investment strategy takes extreme risks into account during time intervals, where all parameters of the insurance model and the investment model are fixed. Only changes in these parameters would indicate that the investment strategy should be reconsidered.

3 Analytic results

In order to find the solution of the optimization problem (2.6), we need a method to compute $\mathbf{VaR}_\alpha(V_\theta^\infty)$ as a function of θ. As it is hard or even impossible to do this analytically, we approximate the df of V_θ^∞. We are interested in approximating its far out upper tail, as $\mathbf{VaR}_\alpha(V_\theta^\infty)$ is defined as a high $(1-\alpha)$-quantile.

We start with the mean and the variance of V_θ^∞. Recall the notation for the Laplace transforms φ and φ_θ in (1.3).

Lemma 3.1. *Let the conditions of Proposition 2.1 hold.*

(a) If $\varphi(2) < 0$, then for every $\theta \in [0,1]$

$$E\left[V_\theta^\infty\right] = \frac{c - \lambda\mu}{\varphi_\theta(1)} < \infty\,, \tag{3.1}$$

and, provided that $E[Y^2] = \mu_2 < \infty$,

$$\operatorname{var}(V_\theta^\infty) = \frac{2\varphi_\theta(1) - \varphi_\theta(2)}{\varphi_\theta^2(1)\varphi_\theta(2)}(c-\lambda\mu)^2 - \frac{\lambda\mu_2}{\varphi_\theta(2)} < \infty\,. \tag{3.2}$$

(b) If $\varphi(2) \geq 0$ and $\varphi(1) < 0$, then (3.1) holds for every $\theta \in [0,1]$ and (3.2) holds for every $\theta \in [0,\theta_2)$, where θ_2 is the unique positive solution to $\varphi_\theta(2) = 0$.
In this case $0 < \theta_2 \leq 1$.

(c) If $0 \leq \varphi(1) < \lambda$, then (3.1) holds for every $\theta \in [0,\theta_1)$, where θ_1 is the unique positive solution to $\varphi_\theta(1) = 0$ and (3.2) holds for every $\theta \in [0,\theta_2)$, where θ_2 is as in (b).
In this case $0 < \theta_2 < \theta_1 \leq 1$.

(d) If $\varphi(1) \geq \lambda$, then (3.1) holds for every $\theta \in [0,\theta_1)$ and (3.2) holds for every $\theta \in [0,\theta_2)$, where θ_1 and θ_2 are as in (c).
In this case $0 < \theta_2 < \theta_1 < \theta_u \leq 1$, where θ_u is given in Proposition 2.1(b).

Proof. We apply Lemma 3.3 in [16] to get formulae for the moment functions $E[V_\theta(t)]$ and $\operatorname{var}(V_\theta(t))$, $t \geq 0$. Straightforward calculations show that (3.1) holds, whenever $\varphi_\theta(1) < 0$, and (3.2) holds, whenever $\varphi_\theta(2) < 0$. Now recall that by Lemma 4.1(b) in [16], for fixed $s > 0$, the function $\varphi_\theta(s)$ is convex in $\theta \in [0,1]$. Since $\varphi_0(s) = -\delta s < 0$, similar arguments as in the proof of Proposition 2.1(b) imply the required results. $\square$

Remark 3.2. Using Lemma 3.1, a more prudent regulator or insurance company may derive a stricter upper bound for the investment strategies than the upper bound introduced in Proposition 2.1(b), see the comments after Definition 2.3. For instance, let $\varphi(2) \geq 0$, $\varphi(1) < 0$ and $E[Y^2] < \infty$. Then the a.s. limit V_θ^∞ of the DNLP exists and has finite mean for every $\theta \in [0,1]$, see Proposition 2.1 and Lemma 3.1(b). However, V_θ^∞ does not have a finite second moment for investment strategies larger than θ_2 as defined in Lemma 3.1(b). Therefore, a risk averse insurance company may avoid investment strategies $\theta \geq \theta_2$. $\square$

Example 3.3. [Continuation of Example 1.2]
Consider the geometric Brownian motion as a model for the risky asset as in Example 1.2. For simplicity assume that $E[Y^2] = \mu_2 < \infty$. We apply Lemma 3.1. Straightforward calculations show that we have the following cases.
(a) If $\sigma^2 < \gamma$, then $E[V_\theta^\infty] < \infty$ and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0,1]$.
(b) If $\gamma \le \sigma^2 < 2\gamma$, then $E[V_\theta^\infty] < \infty$ for every $\theta \in [0,1]$ and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0,\theta_2)$, where

$$\theta_2 = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma+\sigma^2/2-\delta)^2 + 6\sigma^2\delta}}{3\sigma^2} \in (0,1]\,. \tag{3.3}$$

(c) If $2\gamma \le \sigma^2 < 2(\gamma+\lambda)$, then $E[V_\theta^\infty] < \infty$ for every $\theta \in [0,\theta_1)$, where

$$\theta_1 = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma+\sigma^2/2-\delta)^2 + 4\sigma^2\delta}}{2\sigma^2}\,, \tag{3.4}$$

and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0,\theta_2)$, where θ_2 is as in (3.3). In this case we have $0 < \theta_2 < \theta_1 \le 1$.
(d) If $\sigma^2 \ge 2(\gamma+\lambda)$, then $E[V_\theta^\infty] < \infty$ for every $\theta \in [0,\theta_1)$ and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0,\theta_2)$, where θ_1 and θ_2 are as in (3.4) and (3.3) respectively. In this case $0 < \theta_2 < \theta_1 < \theta_u \le 1$, where θ_u is as in (2.4). $\square$

Since knowing the mean and the variance of a rv is not sufficient to compute its extreme quantiles, some additional analysis is needed. We make use of the fact that the DNLP has a natural embedded discrete time skeleton, namely the process sampled at the claim arrival times. Using stochastic recurrence equations we achieve two goals, see [16] for details:

(1) explicit and easy to check in practice conditions for the existence of an a.s. limit V_θ^∞ of the DNLP as in Propisition 2.1;

(2) conditions for deriving the tail behaviour of V_θ^∞.

The next theorem combines the results for the behaviour of the right tail of V_θ^∞ in Theorem 4.4 and Theorem 4.7 in [16], which use extensively results for stochastic recurrence equations, see Goldie [11], Konstantinides and Mikosch [18] and Nyrhinen [22]. As elaborated in [16], there are two main regimes to consider, based on the interaction between the insurance claims and the investment process. We clarify this in the next result. Recall the notation for the set Θ in Definition 2.3.

Theorem 3.4. *Let the conditions in Proposition 2.1 hold and denote* $\overline{\theta} = \sup\Theta$. *For* $\theta \in (0,\overline{\theta}]$ *denote by* $\kappa(\theta)$ *the unique strictly positive solution in* s *to* $\varphi_\theta(s) = 0$.

(a) Dangerous investment*: Assume that* Y *has moments of every order. Then, for* $\theta \in \Theta/\{0\}$, *there exists* $C_+(\theta) > 0$, *such that*

$$P(V_\theta^\infty > x) \sim C_+(\theta) x^{-\kappa(\theta)}\,, \;\; x \to \infty\,. \tag{3.5}$$

(b) Dangerous claims*: Assume that $P(Y > x) \sim C_Y x^{-\rho}$, $x \to \infty$, for some constants $C_Y > 0$ and $\rho > 1$. If $\rho < \kappa(\overline{\theta})$, then for $\theta \in \Theta$,*

$$P(V_\theta^\infty > x) \sim \frac{\lambda}{|\varphi_\theta(\rho)|} C_Y x^{-\rho}\,, \quad x \to \infty\,. \tag{3.6}$$

If $\rho > \kappa(\overline{\theta})$, then (3.6) holds for $\theta \in [0, \theta_\rho)$ and (3.5) holds for $\theta \in (\theta_\rho, 1] \cap \Theta \neq \emptyset$, where $\theta_\rho \in (0, \overline{\theta})$ is the unique positive solution in θ to the equation $\varphi_\theta(\rho) = 0$.

Proof. (a) is a direct consequence of Theorem 4.4 in [16]. To show (b), we use Theorem 4.7(a) in [16]. We know that (3.6) holds for every θ such that $\rho < \kappa(\theta)$ and for $\theta = 0$. On the other hand, if $\rho > \kappa(\theta)$, then $E[Y^{\kappa(\theta)+\beta}] < \infty$ for some $\beta > 0$. Hence, if $\rho > \kappa(\theta)$, then (3.5) holds by Theorem 4.4 in [16]. By Lemma 4.1(c) in [16], the function $\kappa(\theta)$ is strictly decreasing in $\theta \in \Theta/\{0\}$. Therefore, if $\rho < \kappa(\overline{\theta})$, then $\rho < \kappa(\theta)$ for every $\theta \in \Theta/\{0\}$.

Let now $\rho > \kappa(\overline{\theta})$. Note that $\varphi_0(\rho) = -\delta\rho < 0$ and that by Lemma 4.1(b) in [16], $\varphi_\theta(\rho)$ is convex in θ. Therefore, to show that $\varphi_\theta(\rho) = 0$ has a unique solution $\theta_\rho \in (0, \overline{\theta})$, it suffices to show that $\varphi_{\overline{\theta}}(\rho) > 0$. There are two cases. First, if we are in case (a) of Proposition 2.1, then $\overline{\theta} = 1$. By definition, $\varphi_1(\kappa(1)) = 0$. Since $\varphi_1(s)$ is convex in s and $\rho > \kappa(1)$, we have $\varphi_1(\rho) > \varphi_1(\kappa(1)) = 0$. Second, assume that we are in case (b) of Proposition 2.1, i.e. $\overline{\theta} = \theta_u$. By definition, $\varphi_{\theta_u}(1) = \lambda > 0$. Since $\varphi_{\theta_u}(s)$ is convex in s and $\rho > 1$, we have $\varphi_{\theta_u}(\rho) > \varphi_{\theta_u}(1) = \lambda > 0$. Finally, note that by definition $\kappa(\theta_\rho) = \rho$. Since $\kappa(\theta)$ is strictly decreasing in $\theta \in \Theta/\{0\}$, we get that $\rho > \kappa(\theta)$ for $\theta > \theta_\rho$ and $\rho < \kappa(\theta)$ for $\theta < \theta_\rho$. This implies the required result. □

Remark 3.5. (i) Note that by Lemma 4.1(c) in [16] the Pareto index $\kappa(\theta)$ is a decreasing function in the investment strategy θ. This means that, whenever (3.5) holds, the more risky investment we choose, the heavier tail of the DNLP we get, which is quite natural.
(ii) The constant $C_+(\theta)$ in (3.5) cannot be computed analytically; see (4.16) in [16].
(iii) As for any fixed $s > 0$ the function $\varphi_\theta(s)$ is convex in θ (see Lemma 4.1(b) in [16]), there exists an investment strategy, minimizing the rhs of (3.6). □

Example 3.6. [Continuation of Example 1.2]
Consider the geometric Brownian motion as a model for the risky asset as in Example 1.2. Recall that in this example the investment process $\exp(L_\theta)$ is again a geometric Brownian motion with drift γ_θ and volatility σ_θ as in (1.5). For simplicity assume that $\sigma^2 < 2(\gamma+\lambda)$, so that $\Theta = [0, 1]$, see Example 2.4. The value $\kappa(\theta)$ is given by

$$\kappa(\theta) = \frac{2\gamma_\theta}{\sigma_\theta^2} = \frac{2}{\sigma^2\theta^2}\left(\gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta)\right)\,.$$

Note that $\kappa(\theta) \to \infty$ as $\theta \to 0$. Therefore, when the claims have moments of every order, and when the fraction invested in the risky asset tends to 0, the tail of V_θ^∞ approaches a tail, which is no longer a Pareto tail. The limit case $\theta = 0$ is treated in detail in [16], Example 3.4; see also Sundt and Teugels [26], and, in the case of Pareto claims, Klüppelberg and Stadtmüller [17].

When the insurance claims have Pareto tail with a small tail index $\rho < 2\gamma/\sigma^2$, then the tail of the a.s. limit V_θ^∞ of the DNLP is similar to that of the claims. In other words, the claims dominate the integrated risk process, regardless of the selected investment strategy. On the other hand, if $\rho > 2\gamma/\sigma^2$, then the claims dominate for the less risky strategies $\theta \in [0, \theta_\rho)$, whereas the investment process dominates for the more risky investment strategies $\theta \in (\theta_\rho, 1]$. We can compute θ_ρ from Theorem 3.4(b) as

$$\theta_\rho = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma + \sigma^2/2 - \delta)^2 + 2\sigma^2\delta(\rho+1)}}{(\rho+1)\sigma^2} \in (0,1)\,.$$

The investment strategy θ_ρ plays the role of a change-point strategy between the dangerous claims regime and the dangerous investment regime. $\square$

4 Examples

Recall the optimization problem (2.6) considering the maximal investment strategy θ, such that a risk constraint is satisfied. To solve this problem we need a method to compute the quantile of V_θ^∞ (the VaR) as a function of the investment strategy θ. Unfortunately, apart from the very few special cases considered in [16], Section 3.2, the distribution of V_θ^∞ is not known. In general we can compute the moments of V_θ^∞, if they exist, see Lemma 3.1. Further, we know from Theorem 3.4 that V_θ^∞ has a Pareto tail. The Pareto index depends on the interaction between the insurance claims and the investment process, see also Example 4.6 and Example 4.8 in [16]. We distinguish between two different regimes.

(a) Dangerous investment: insurance claims have moments of a sufficiently large order; then the Pareto index of V_θ^∞ is determined only by the investment process.

(b) Dangerous claims: insurance claims have a Pareto tail with a sufficiently small Pareto index; then the Pareto index of V_θ^∞ is the same as that of the claims.

4.1 Dangerous investment

First we consider the dangerous investment regime, i.e. when the investment process dominates the integrated risk process. In what follows we assume that there exists an a.s. limit V_θ^∞ of the DNLP with finite mean and variance for all investment strategies $\theta \in [0, 1]$. This is satisfied when $\varphi(2) < 0$ and the insurance claims have finite second moment ($E[Y^2] < \infty$), see Lemma 3.1.

A crude and often used approximation of the $(1-\alpha)$-quantile of V_θ^∞ (the VaR) can be achieved by the $(1-\alpha)$-quantile of a normal rv with the same mean and variance.

Normal approximation algorithm, dangerous investment

Let V_θ^N be a normal rv with mean and variance as those of V_θ^∞ and let $q_\alpha(\theta)$ be its $(1-\alpha)$-quantile. Then we have $P(V_\theta^N > x) = P(E[V_\theta^\infty] + \sqrt{\mathrm{var}(V_\theta^\infty)}N(0,1) > x)$, where $N(0,1)$ is a standard normal rv. Therefore, we obtain that

$$q_\alpha(\theta) = E[V_\theta^\infty] + \Phi^{-1}(1-\alpha)\sqrt{\mathrm{var}(V_\theta^\infty)}\,,$$

where Φ^{-1} is the quantile function of the standard normal distribution. □

Assuming that the df of V_θ^N approximates the df of V_θ^∞ (and hence $q_\alpha(\theta)$ approximates $\mathbf{VaR}_\alpha(V_\theta^\infty)$), we replace the optimization problem (2.6) by

$$\max\left\{\theta \in [0,1] \,:\, q_\alpha(\theta) \le C\right\}\,. \tag{4.1}$$

Note that the moments $E[V_\theta^\infty]$ and $\mathrm{var}(V_\theta^\infty)$ can be computed by Lemma 3.1 and, hence, the optimization problem (4.1) can be solved by numerical methods. However, it is well known that the normal approximation does not take into account interesting properties of the original distribution as skewness or heavy tails. Hence, for our model it will presumably underestimate the risk considerably. We demonstrate this in the following example.

Example 4.1. [Exponential claims]
We consider an insurance model with a premium rate $c = 2.1$, an intensity of the Poisson claim counting process $\lambda = 1$ and exponential claims with a mean $E[Y] = \mu = 2$. We assume also that the price of the risky asset follows a geometric Brownian motion with a drift $\gamma = 0.06$ and a volatility $\sigma = 0.2$. In this example we analyze the pure stock strategy $\theta = 1$ only.

We simulate 10 000 copies of the rv V_1^∞. As we do not know the distribution of V_1^∞, we invoke the forward stochastic recurrence equation corresponding to the discrete time skeleton of the DNLP given in equation (3.5) in [16].

In Figure 1, left plot, the histogram of the simulated data is compared to the corresponding normal density of the rv V_1^N with a mean $E[V_1^\infty] = -2.5$ and a variance $\mathrm{var}(V_1^\infty) = 106.25$, as computed by Lemma 3.1. We see more values close to the mean in the simulated data than the normal approximation suggests.

In Figure 1, right plot, we compare the empirical quantiles of the simulated data to the normal quantiles. We see that in this example, when α is around 2.5%, the normal

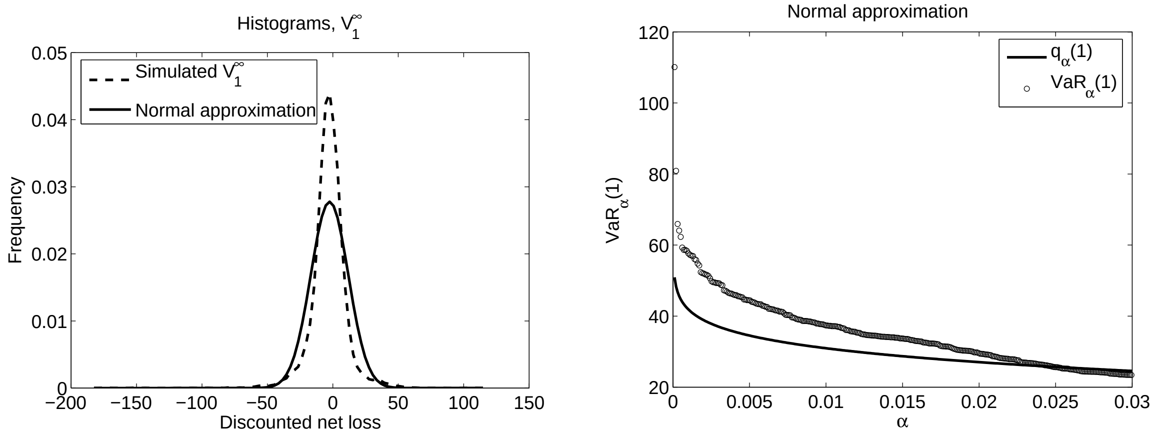

Figure 1: Histograms and quantiles of V_1^∞ for simulated data versus the normal approximation. The parameters are as in Example 4.1. Left plot: histogram of 10 000 simulated copies of V_1^∞ compared to the density of a normal rv with the same mean and variance as V_1^∞. Right plot: empirical quantiles of the simulated data compared to the normal quantiles.

approximation works quite well. However, when we go further in the tail, for α =1% or 0.5%, the normal approximation underestimates the risk significantly. $\square$

Example 4.1 demonstrates that the normal approximation of the quantile of the rv V_1^∞ is not very satisfactory, when one is interested in extreme quantiles. This happens despite the fact, that we have a light-tailed input (i.e. exponentially distributed insurance claims and a geometric Brownian motion for the stock price). In this case Theorem 3.4(a) applies and V_θ^∞ has a Pareto tail, for all $\theta \in (0, 1]$.

From now on we assume for simplicity that the claims have finite moments of every order, so that we are in the dangerous investment regime for all $\theta \in (0, 1]$. In the next approximation method we make extensive use of Theorem 3.4(a). Unfortunately, a straightforward approximation of the tail of V_θ^∞ with asymptotic as in (3.5) is not possible, since the constant $C_+(\theta)$ cannot be computed, see Remark 3.5(ii). As a remedy we combine the normal approximation with the Pareto tail behaviour in the following algorithm.

Pareto approximation algorithm, dangerous investment

If (3.5) holds for V_θ^∞, then it also holds for the centered rv:

$$P(V_\theta^\infty - E[V_\theta^\infty] > x) \sim C_+(\theta)(x + E[V_\theta^\infty])^{-\kappa(\theta)} \sim C_+(\theta)x^{-\kappa(\theta)}, \quad x \to \infty. \tag{4.2}$$

Denote by $G_\theta^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} : P(V_\theta^\infty - E[V_\theta^\infty] > x) \leq \alpha\}$, $\alpha \in (0, 1)$, the generalized inverse function of the df of the centered rv. Then, by (4.2) and Theorem 1.5.12 of

Bingham, Goldie and Teugels [2], we have

$$\frac{G_\theta^{\leftarrow}(\alpha)}{G_\theta^{\leftarrow}(\beta)} \sim \left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)}, \quad \alpha < \beta, \quad \alpha \to 0,\ \beta \to 0\,. \tag{4.3}$$

We select some small probability $\beta > \alpha$, where α is the given probability of interest in (2.3). Using (4.3) we approximate

$$G_\theta^{\leftarrow}(\alpha) \approx G_\theta^{\leftarrow}(\beta)\left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)}.$$

Then we apply the normal approximation to $G_\theta^{\leftarrow}(\beta)$ to get

$$G_\theta^{\leftarrow}(\alpha) \approx \Phi^{-1}(1-\beta)\sqrt{\mathrm{var}(V_\theta^\infty)}\left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)}.$$

Since $G_\theta^{\leftarrow}(\alpha) = \mathbf{VaR}_\alpha(V_\theta^\infty) - E[V_\theta^\infty]$, we approximate $\mathbf{VaR}_\alpha(V_\theta^\infty)$ by

$$p_\alpha(\theta) = E[V_\theta^\infty] + \Phi^{-1}(1-\beta)\sqrt{\mathrm{var}(V_\theta^\infty)}\left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)}. \tag{4.4}$$

Note that, when $\alpha = \beta$, then $p_\alpha(\theta) = q_\alpha(\theta)$. □

Using the above algorithm, we replace the optimization problem (2.6) by

$$\max\left\{\theta \in [0,1] \,:\, p_\alpha(\theta) \le C\right\}, \tag{4.5}$$

where $p_\alpha(\theta)$ is as in (4.4). In the next example we investigate the accuracy of the Pareto approximation applied to the model in Example 4.1.

Example 4.2. [Continuation of Example 4.1]
Consider the model with light-tailed input as in Example 4.1. In Figure 2 we compare the normal approximation to the suggested Pareto approximation of $\mathbf{VaR}_\alpha(V_\theta^\infty)$ for two ranges for α. In the left plot we show the VaR for comparatively large probabilities $\alpha \in (0.5\%, 3\%)$ based on $\beta = 0.03$, and in the right plot – for very small probabilities $\alpha < 0.5\%$ based on $\beta = 0.005$. In both cases the Pareto approximation provides a better fit to the empirical quantiles than the normal approximation, in particular for $\alpha \in (0.5\%, 1.5\%)$ in the left plot and for $\alpha < 0.2\%$ in the right plot. Note that the Pareto and the normal approximation are equal when $\alpha = \beta$, which explains the gap between the empirical quantiles and the approximations at $\alpha = \beta = 0.5\%$ in the right plot of Figure 2, see also Figure 1, right plot. □

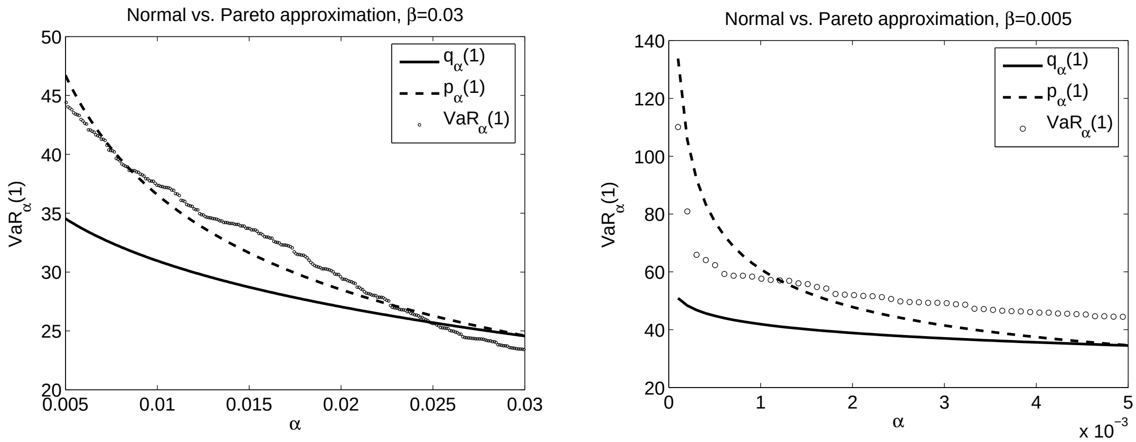

Figure 2: Comparison of the normal approximation, the Pareto approximation and the empirical quantiles of the simulated data. The parameters are as in Example 4.1. Left plot: in the Pareto approximation algorithm we have chosen $\beta = 0.03$ and we are interested in $\alpha \in (0.005, 0.03)$. Right plot: in the Pareto approximation algorithm we have chosen $\beta = 0.005$ and we are interested in $\alpha \in (0, 0.005)$.

4.2 Dangerous insurance claims

We consider the dangerous insurance claims regime, i.e. when the insurance process dominates the integrated risk process. In what follows we assume that there exists an a.s. limit V_θ^∞ of the DNLP with finite mean for all investment strategies $\theta \in [0,1]$. This is satisfied when $\varphi(1) < 0$, see Lemma 3.1. Moreover, we assume that the claims have a Pareto distribution, i.e. for some $\rho > 1$, $l > 0$,

$$P(Y > x) = \left(\frac{l}{l+x}\right)^\rho, \ x > 0 .$$

From now on we assume for simplicity that $\rho < \kappa(1)$ in all considered models for the risky investment. In this case (3.6) holds and V_θ^∞ has Pareto tail with Pareto index ρ for all investment strategies $\theta \in [0,1]$. We suggest the following approximation algorithm:

Pareto approximation algorithm, dangerous claims

Similar to the approximation algorithm in Section 4.1, if (3.6) holds for V_θ^∞, then it also holds for the centered rv:

$$P(V_\theta^\infty - E[V_\theta^\infty] > x) \sim \frac{l^\rho \lambda}{|\varphi_\theta(\rho)|} x^{-\rho}, \ \ x \to \infty .$$

Hence, for the generalized inverse function $G_\theta^{\leftarrow}(\alpha)$ of the df of the centered rv holds

$$G_\theta^{\leftarrow}(\alpha) \sim \alpha^{-1/\rho} \left(\frac{\lambda l^\rho}{|\varphi_\theta(\rho)|}\right)^{1/\rho}, \quad \alpha \to 0 . \tag{4.6}$$

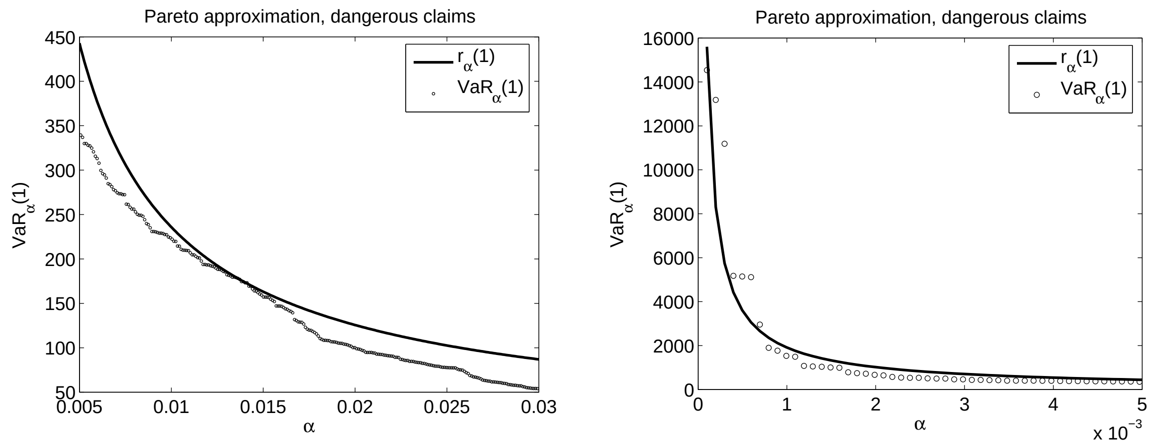

Figure 3: Comparison of the Pareto approximation and the empirical quantiles of the simulated data. The parameters are from Example 4.3. Left plot: we are interested in $\alpha \in (0.005, 0.03)$ Right plot: we are interested in extreme quantiles for $\alpha < 0.005$.

Using (4.6) and the fact that $G_\theta^{\leftarrow}(\alpha) = \mathbf{VaR}_\alpha(V_\theta^\infty) - E[V_\theta^\infty]$, we approximate $\mathbf{VaR}_\alpha(V_\theta^\infty)$, for a small α, by

$$r_\alpha(\theta) = E[V_\theta^\infty] + \alpha^{-1/\rho} \left(\frac{\lambda l^\rho}{|\varphi_\theta(\rho)|} \right)^{1/\rho} . \tag{4.7}$$

□

Using the above algorithm, we replace the optimization problem (2.6) by

$$\max \{ \theta \in [0,1] \, : \, r_\alpha(\theta) \leq C \} \ , \tag{4.8}$$

where $r_\alpha(\theta)$ is as in (4.7). We investigate the accuracy of the suggested Pareto approximation for the dangerous claims regime in the next example.

Example 4.3. [Pareto claims]
We consider an insurance model with a premium rate $c = 2.1$, an intensity of the claim counting process $\lambda = 1$ and Pareto claims with $\rho = 1.1$ and $l = 0.2$. The parameters of the investment model are the same as in Example 4.1. In this example we analyze the pure stock strategy $\theta = 1$ only.

We simulate 10 000 copies of the rv V_1^∞ using the same method as in Example 4.1.

In Figure 3 we compare the suggested Pareto approximation to the empirical quantiles of the simulated data. We see that in the tail, i.e. for probabilities less than 2%, the suggested approximation is quite accurate. □

4.3 Comparison of the models

In the previous two sections we have discussed methods to approximate the VaR in the dangerous investment and in the dangerous claims regime. This enables us to find in each of the two regimes an (approximate) solution to the optimization problem (2.6) by numerical methods. Up to now for simplicity we have considered only examples where the risky asset is modeled by a geometric Brownian motion. In this section we focus on the impact of different models for both the insurance claims and the risky asset on the optimal investment strategy. From one side we discuss the difference between the optimal investment strategy in the dangerous claims regime and in the dangerous investment regime. From another side, we compare the magnitude of the influence of different models for the risky asset on the optimal investment strategy within each of the regimes.

Example 4.4. [Comparison of the models]
We compare the Brownian motion model from Example 4.1 to a variance gamma (VG) model for the risky asset. The VG model, sugested by Madan and Seneta [21], is a normal mixture model and is given by

$$L(t) = \xi t + W_{a,b}(S_\Gamma(t)) \,,\ t \geq 0 \,,$$

where $\xi > 0$, $W_{a,b}$ is a Brownian motion with drift $a < 0$ and volatility $b > 0$, independent of the gamma Lévy process $(S_\Gamma(t))_{t\geq 0}$, $S_\Gamma(1) \stackrel{d}{=} \Gamma(\eta, r)$, $r, \eta > 0$. Thus the VG process is a pure jump process of infinite activity (with infinitely many jumps in every compact interval) with drift. We consider a VG process with paramethers $\xi = 0.16$, $a = -0.1$, $b^2 = 0.04 - a^2$ and $\eta = r = 1$.

In order to allow for a comparison, we have selected the parameters of the Brownian motion and of the VG process in such a way, that the mean and the variance of $L(1)$, i.e. of the log returns of the stock price, coincide in both models. However, the VG model has jumps, taken here with a negative mean, corresponding to (downward) jumps of the stock price. Hence, one would consider it as a more 'dangerous' investment model.

For the insurance business we use the parameters from Example 4.1 (dangerous investment regime) and Example 4.3 (dangerous claims regime). We set the riskless interest rate to $\delta = 0.01$. We solve (4.5) for the dangerous investment regime and (4.8) for the dangerous claims regime, using a number of risk constraints C.

Let us first fix the regime (i.e. the distribution of the claim sizes) and compare the impact of the stock price model on the VaR and on the optimal investment strategy.

In Figure 4, left plot, we show the optimal investment strategy in the dangerous investment regime (i.e. the solution to the optimization problem (4.5)), vs. the risk constraint C for the VaR. We observe that the more risky VG model affects significantly the VaR, in particular for more risky investment strategies $\theta > 80\%$. For a fixed risk constraint C,

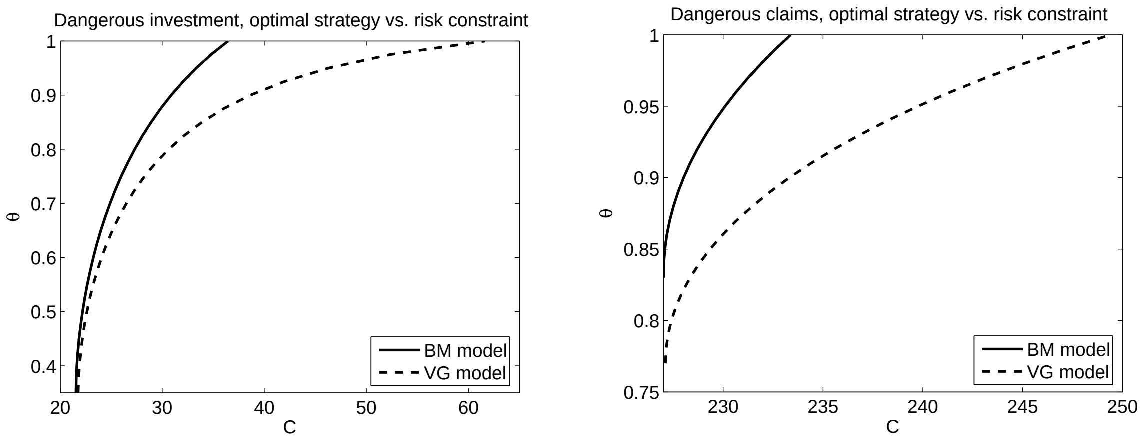

Figure 4: The solution to optimization problem (2.6) for different risk constraints. The parameters are from Example 4.4. Left plot: Dagerous investment regime. The claims are modeled as in Example 4.1. Right plot: Dangerous claims regime. The claims are modeled as in Example 4.3

it allows for less investment in the risky asset than the Brownian motion model. Hence, we may conclude that the more risky model for the stock price leads to more conservative investment strategies.

In Figure 4, right plot, we show the optimal investment strategy in the dangerous claims regime (i.e. the solution to the optimization problem (4.8)), vs. the risk constraint C for the VaR. Again, we observe that the more risky VG model implies more conservative investment strategies. However, in the dangerous claims regime this impact is weaker compared to the dangerous investment regime; compare the right and the left plot of Figure 4 and note the difference in the scales of the horizontal axes. For instance, at investment strategy $\theta = 1$, changing the investment model from a Brownian motion to a VG leads to about 90% increase of the VaR in the dangerous investment regime, whereas this change leads to only 10% increase of the VaR in the dangerous claims regime.

Let us fix the model of the stock price and investigate the impact of the claim size distribution. Note that in the dangerous claims and in the dangerous investment regime we have the same interest rate, premium rate, claim arrival intensity and mean claim size. This implies that, holding the stock price model fixed, for every fixed investment strategy $\theta \in [0, 1]$, the mean of the a.s. limit V_θ^∞ of the DNLP in both regimes is the same. However, for both stock price models, the Pareto claims from Example 4.3 lead to higher risk compared to the exponential claims from Example 4.1. Indeed, in the dangerous claims regime, the risk constraint C has to be set much higher than in the dangerous investment regime (almost 10 times), in order to obtain a solution to the optimization problem at all, notice again the difference in the scales of the horizontal axes in Figure 4. Furthermore, the difference between the light- and the heavy-tailed claims model is much more severe than between the two stock price models. For instance, at investment strategy $\theta = 1$, changing the insurance claims model from exponential to Pareto leads to almost 9 times increase of the VaR in the Brownian motion case (compare the two plots in Figure 4), while changing the investment model from a Brownian motion to a VG leads to only about 90% increase of the VaR in the exponential claims case (Figure 4, left plot).

Recall that, in the dangerous claims regime, the Pareto index of V_θ^∞ is the same as that of the insurance claims. In contrast to that, in Example 4.1, the investment process determines the Pareto index $\kappa(\theta)$ of V_θ^∞ regardless of the insurance process. Therefore, the choice of the investment strategy is much more important in the case of Example 4.1 than in Example 4.3. In other words, the VaR is less sensitive to the investment strategy in the dangerous claims regime than it is in the dangerous investment regime. For instance, within the Brownian motion model, increasing the investment strategy from 0.85 to 1 leads to an increase of about 3% of the VaR in the dangerous claims regime (Figure 4, right plot), while the same change in the investment strategy leads to about 24% increase of the VaR in the dangerous investment regime (Figure 4, left plot). Similar observations can be made for the VG model for the stock price. $\square$

Aknowledgements

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References

[1] Asmussen, S. (2000) *Ruin Probabilities.* World Scientific, London.

[2] Bingham, N.H., Goldie, C.M and Teugels, J.L. (1987) *Regular Variation.* Cambridge University Press, Cambridge.

[3] Carmona, P., Petit, F. and Yor, M. (1997) On the distribution and asymptotic results for exponential functionals of Lévy processes. In: Yor, M. (Ed.) *Exponential functionals and principal values related to Brownian motion.* pp. 73-126. Biblioteca de la Revista Matematica Ibero-Amerikana.

[4] Cont, R. and Tankov, P. (2004) *Financial Modelling With Jump Processes.* Chapman &Hall/CRC.

[5] Emmer, S. and Klüppelberg, C. (2004) Optimal portfolios when stock prices follow an exponential Lévy process. *Finance and Stochastics* **8** (1), 17-44.

[6] Emmer, S., Klüppelberg, C. and Korn, K. (2001) Optimal portfolios with bounded Capital-at-Risk. *Math. Finance* **11**, 365-384.

[7] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997) *Modelling Extremal Events for Insurance and Finance.* Springer, Berlin.

[8] Frolova, A., Kabanov, Y. and Pergamenchtchikov, S. (2002) In the insurance business risky investments are dangerous. *Finance and Stochastics* **6**, 227-235.

[9] Gaier, J. and Grandits, P. (2002) Ruin probabilities in the presence of regularly varying tails and optimal investment. *Insurance: Mathematics and Economics* **30**, 211-217.

[10] Gaier, J., Grandits, P. and Schachermayer, W. (2003) Asymptotic ruin probabilities and optimal investment. *Ann. Appl. Probab.* **13**, no. 3, 1054-1076.

[11] Goldie, C.M. (1991) Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126-166.

[12] Hipp, C. and Plum, M. (2000) Optimal investment for insurers. *Insurance: Mathematics and Economics* **27**, 215-228.

[13] Hipp, C. and Plum, M. (2003) Optimal investment for investors with state dependent income, and for insurers. *Finance and Stochastics* **7**, 299-321.

[14] Kalashnikov, V. and Norberg, R. (2002) Power tailed ruin probabilities in the presence of risky investments. *Stoch. Proc. Appl.* **98**, 211-228.

[15] Kesten, H. (1973) Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207-248.

[16] Klüppelberg, C. and Kostadinova, R. (2005) Integrated insurance risk models with exponential Lévy investment. Available at http://www-m4.ma.tum.de/Papers/.

[17] Klüppelberg, C. and Stadtmüller, U. (1998) Ruin Probabilities in teh Presence of Heavy-Tails and Interest Rates. *Scand. Actuarial J.* **1**: 49-58.

[18] Konstantinides, D.G. and Mikosch, T. (2005) Large deviations and ruin probabilities for solutions to stochastic recurrence equations with heavy-tailed innovations. *Ann. Probab.* **33**, no. 5, 1992-2035.

[19] Korn, K. (1997) *Optimal Portfolios.* World Scientific. Singapore.

[20] Lindner, A. and Maller, R. (2005) Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. *Stoch. Proc. Appl.* **115** no. 10, 1701-1722.

[21] Madan, D. and Seneta, E. (1990) The variance gamma (VG) model for share market returns. *Journal of Business* **63**, 511-524.

[22] Nyrhinen, H. (2001) Finite and infinite time ruin probabilities in a stochastic economic environment. *Stoch. Proc. Appl.* **92**, 265-285.

[23] Paulsen, J. (2002) On Cramér-like asymptotics for risk processes with stochastic return on investement. *Ann. Appl. Prob.* **12**, 1247-1260.

[24] Protter, P. (1990) *Stochastic Integration and Differential Equations.* Springer, New York.

[25] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[26] Sundt, B. and Teugels, J.L. (1995) Ruin estimates under interest force. *Insurance: Math. Econ.* **8**, 279-285.