Behavior of the Quantization Operator for Bandlimited, Nonoversampled Signals

Holger Boche, *Senior Member, IEEE*, and Ullrich J. Mönich, *Student Member, IEEE*

***Abstract*—The process of quantization generates a loss of information, and, thus, the original signal cannot be reconstructed exactly from the quantized samples in general. However, it is desirable to keep the error as small as possible. In this paper, the quantization error is quantified in terms of several distortion measures. All these measures employ the difference between the original signal and the reconstructed signal, which is obtained by bandlimited interpolation of the quantized samples. We assume that the signals are bandlimited and that the samples are taken at Nyquist rate. It is shown that for signals in the Paley–Wiener space $\mathcal{PW}_{\pi}^{1}$, the supremum of the reconstructed signal, and, hence, the quantization error cannot be bounded in the sense that there exists a bounded subset of $\mathcal{PW}_{\pi}^{1}$ on which both quantities can increase unboundedly. This unexpected behavior is due to the nonlinearity of the quantization operator and the slow decay of the sinc function. The nonlinearity is essential for this behavior because every linear operator that fulfills a certain property of the quantization operator would otherwise have to be bounded. Furthermore, it is proven that for a fixed signal the possible quantization error increases as the quantization step size tends to zero. The treatment of the quantization error in this paper is completely deterministic.**

***Index Terms*—Analog-to-digital conversion, bandlimited signal, quantization noise, Shannon sampling series, signal quantization.**

I. INTRODUCTION

MODERN signal processing is done nearly always with digital processors. In order to process real world data, which is represented by continuous-time real-valued signals, it is necessary to convert these signals into a discrete-time signal with only finitely many values. This discretization in time and amplitude can most simply be done by the following two steps. First, the signal is sampled equidistantly at least at Nyquist rate, and, second, the samples are mapped to discrete values, according to a certain quantization law. The whole procedure is called analog-to-digital conversion.

Obviously, some information about the signal is irreversibly lost in the quantization process and an exact recovery of the continuous-time signal is not possible in general. However, it is desirable to have a high accuracy in the analog-to-digital conversion. The accuracy can be measured by the quantization error $f - \overline{f_{\epsilon}}$, where $\overline{f_{\epsilon}}$ is the reconstructed signal, which is obtained from the quantized samples using a reconstruction process, for example the Shannon sampling series.

In this paper, the signals are sampled at Nyquist rate. The operator that maps f to $\overline{f_{\epsilon}}$ is called quantization operator. This operator comprises the analog-to-digital conversion, i.e., sampling at Nyquist rate and quantization, and the subsequent interpolation by the Shannon sampling series. Obviously, the quantization operator maps continuous-time signals to continuous-time signals and is nonlinear. In this paper, we study the properties of this quantization operator. In particular, we analyze whether the quantization operator is a bounded operator or not. It turns out that this depends on the signal space under consideration and the quantization scheme. If the quantization operator is unbounded, the quantization error cannot be bounded on bounded sets of signals. This means there exists a set $\mathcal{S}$ of signals with norm smaller than some specific constant such that for any $K > 0$ there exists a signal $f \in \mathcal{S}$ for which the peak value of the quantization error $\left\| f - \overline{f_{\epsilon}} \right\|_{\infty}$ is larger than K. Moreover, for fixed f, we analyze the behavior of the quantization error as the quantization step size tends to zero. Surprisingly, there are signals in the Paley–Wiener space $\mathcal{PW}_{\pi}^{1}$, for which the quantization error grows arbitrarily large as the quantization step size goes to zero. A definition of the space $\mathcal{PW}_{\pi}^{1}$ follows in Section II.

Normally, the analysis of the quantization operator is done for the space of bandlimited signals with finite energy. This is the same space that Shannon considered when he introduced the sampling theorem [1], [2] in communications. It has to be mentioned that the sampling theorem was earlier published in the mathematical literature by Whittaker [3] and independently of Shannon in the engineering literature by Kotel'nikov [4] and Raabe [5]. Since then several extensions of the sampling theorem have been made. For further details about the history of sampling theory, we would like to refer the reader to the comprehensive paper [6].

Due to its high practical importance, the analysis of the quantization error has gained a lot of attention in research. Often, the quantization operation is modeled as additive white noise [7], [8]. However, it turned out that this noise model is not always satisfactory, because it can lead to false predictions [9]. Experimental results confirm these problems [10]. It is obvious that fundamental properties of analog-to-digital conversion are not fully understood and that further research is necessary for a better understanding.

Since quantization is a deterministic process it is interesting to have a deterministic analysis in addition to the successfully used statistical approaches. The deterministic analysis is difficult because of the nonlinear nature of the quantization operator, but it reveals some properties of the quantization process,

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The authors are with the Technische Universität Berlin, Heinrich-Hertz-Chair for Mobile Communications, Einsteinufer 25, D-10578 Berlin, Germany (e-mail: holger.boche@mk.tu-berlin.de; ullrich.moenich@mk.tu-berlin.de).

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which cannot be analyzed with the additive noise description of the quantization.

There are numerous publications discussing the approximation error of sampling series in the presence of additive noise in the samples [8], [11]. However, these publications do not consider the deterministic nature of the quantization. Only few publications treat the quantization error deterministically. One is [12], where the quantization error is analyzed for absolutely integrable bandlimited signals and certain nonbandlimited signals. Another paper is [13]. There the accuracy of analog to digital converters with oversampling is analyzed for bandlimited signals with finite energy. In [14] and [15], the interpolation problem is analyzed for nonuniform quantized samples and a subset of the bandlimited signals with finite energy. Moreover, oversampled analog to digital conversion in shift-invariant spaces is treated in [16]; [17] discusses the effect of quantization threshold uncertainties in pulse code modulation and $\Sigma\Delta$ modulation analog to digital converters. An extensive account on the history of quantization and the discussion of several developments can be found in [18].

In this paper, we restrict ourselves to uniform sampling at Nyquist rate, and extend the analysis to larger signal spaces, namely the Bernstein spaces $\mathcal{B}_\pi^p$, $1 < p < \infty$, and the Paley–Wiener space $\mathcal{PW}_\pi^1$. The analysis for $\mathcal{PW}_\pi^1$ is important, because this space is larger than the commonly used $\mathcal{PW}_\pi^2$-space of signals with finite energy and because the convergence behavior of sampling series for signals in $\mathcal{PW}_\pi^1$ is closely related to the convergence behavior of sampling series for bandlimited wide-sense stationary stochastic processes [19].

II. Notation and Preliminaries

As usual, $L^p(\mathbb{R})$, $1 \le p < \infty$, denotes the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. Furthermore, l^p, $1 \le p < \infty$, is the space of all sequences such that the p-norm $\|\cdot\|_p$ is finite, and l^∞ denotes the space of bounded sequences with the supremum norm $\|\cdot\|_{l^\infty}$.

Let $\hat{f}$ denote the Fourier transform of a signal f. For $\sigma > 0$, let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \le C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space $\mathcal{B}_\sigma^p$ consists of all signals in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \le p \le \infty$. A signal in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. By the Paley–Wiener–Schwartz theorem, the Fourier transform of a signal bandlimited to σ is supported in $[-\sigma,\sigma]$. For $1 \le p \le 2$, the Fourier transformation is defined in the classical and for $p > 2$ in the distributional sense. It is well known that $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^s$ for $1 \le p \le s \le \infty$. Hence, every signal in $\mathcal{B}_\sigma^p$, $1 \le p \le \infty$, is bounded on the real line.

For $\sigma > 0$ and $1 \le p \le \infty$, we denote by $\mathcal{PW}_\sigma^p$ the Paley–Wiener space of signals f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)\,\mathrm{e}^{iz\omega}\ \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma,\sigma]$. If $f \in \mathcal{PW}_\sigma^p$, then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$, $1 \le p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi)\int_{-\sigma}^{\sigma}|\hat{f}(\omega)|^p\ \mathrm{d}\omega)^{1/p}$.

As a consequence of Parseval's equality, we have $\mathcal{B}_\pi^2 = \mathcal{PW}_\pi^2$. Thus, $\mathcal{PW}_\pi^2$ is nothing else than the space of bandlimited signals with finite energy. Furthermore, the Hausdorff–Young inequality leads to $\mathcal{B}_\pi^q \supset \mathcal{PW}_\pi^p$ for $1 \le p \le 2$, $1/p + 1/q = 1$ and Hölder's inequality to $\mathcal{PW}_\sigma^p \supset \mathcal{PW}_\sigma^s$ for $1 \le p < s \le \infty$. Moreover, it holds $\|f\|_\infty \le \|f\|_{\mathcal{PW}_\pi^1}$. Obviously, every signal in $\mathcal{PW}_\sigma^p$, $1 \le p \le \infty$, is bounded on the real line.

Example 1: The signal space $\mathcal{PW}_\pi^1$ is larger than the space $\mathcal{PW}_\pi^2 = \mathcal{B}_\pi^2$. A signal that is in $\mathcal{PW}_\pi^1$, but not in $\mathcal{PW}_\pi^2 = \mathcal{B}_\pi^2$ is given by $f(z) = 1/(2\pi)\int_{-\pi}^{\pi}\omega^{-1/2}\,\mathrm{e}^{i\omega z}\ \mathrm{d}\omega$.

As we can see from the above definitions, all signals in $\mathcal{B}_\pi^p$, $1 \le p \le \infty$, and in $\mathcal{PW}_\pi^p$, $1 \le p \le \infty$, are defined on the complex plane. However, in practical applications, the signals are usually considered to be a function of a real variable, which often represents the time. Since all signals in the above spaces are analytical, they are uniquely determined by their values on the real line. Therefore, we will not distinguish between signals defined on the complex plane and signals defined on the real axis in the following.

The nomenclature concerning the Bernstein and Paley–Wiener spaces, we introduced so far, is not consistent in the literature. Sometimes the space that we call Bernstein space is called Paley–Wiener space [20]. We adhere to the notation used in [21] by Higgins. For further details about the Bernstein and Paley–Wiener spaces, we would like to refer the reader to [21].

III. Quantization and Signal Reconstruction

In this paper, we consider real-valued signals f that are bandlimited to π and the simple but frequently used uniform midtread quantization, where each sample $f(k)$, $k \in \mathbb{Z}$, is quantized to $f_\epsilon(k)$, depending on the quantization step size $2\epsilon > 0$, according to the following rule:

$$f_\epsilon(k) = \left\lfloor \frac{f(k)}{2\epsilon} + \frac{1}{2} \right\rfloor 2\epsilon, \quad k \in \mathbb{Z}.$$

$\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. In other words, if $f(k) \in [(2l-1)\epsilon, (2l+1)\epsilon)$ for some $l \in \mathbb{Z}$, then $f_\epsilon(k)$ is set to $2l\epsilon$. Furthermore, for $\epsilon > 0$, we denote by $\mathcal{Z}_\epsilon = \{(2l+1)\epsilon : l \in \mathbb{Z}\}$ the set of quantization thresholds. This kind of quantization characteristic is depicted in Fig. 1. In order to determine the quantization error, the quantized samples $\{f_\epsilon(k)\}_{k\in\mathbb{Z}}$ are used to reconstruct an approximation

$$\overline{f_\epsilon}(t) = \sum_{k=-\infty}^{\infty} f_\epsilon(k)\frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)} \tag{1}$$

of the original signal f [9], [16]. It is well known [8], [14] that the series in (1) does not necessarily converge under arbitrary bounded perturbations of the samples. However, the sequence of quantized samples $\{f_\epsilon(k)\}_{k\in\mathbb{Z}}$ cannot be arbitrary, because it is deterministically determined by the signal f. In particular, we will see that only finitely many quantized samples $f_\epsilon(k)$, $k \in \mathbb{Z}$, are different from zero, which implies that, for fixed ϵ and fixed

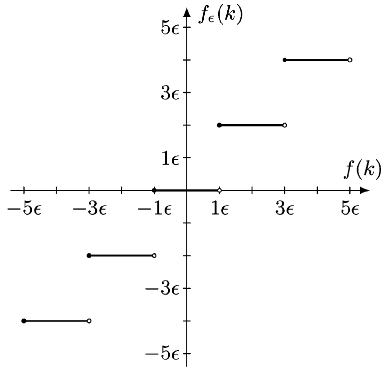

Fig. 1. Quantization characteristic.

f, the series in (1) does always converge for the signals that are considered in this paper.

The reconstruction in (1) corresponds to the case where no oversampling is used. If oversampling is applied with an oversampling factor $a > 1$, we have

$$\overline{f_\epsilon}(t) = \sum_{k=-\infty}^{\infty} f_\epsilon\left(\frac{k}{a}\right)\phi\left(t-\frac{k}{a}\right)$$

as reconstruction formula, where ϕ is a certain reconstruction kernel. Thus, with oversampling, better kernels than the sinc kernel

$$\frac{\sin(\pi(t-k))}{\pi(t-k)}$$

can be used, and certain divergence phenomena such as in Theorems 5 and 6 cannot occur. However, since oversampling is not considered in this paper, the sinc kernel is the only possible kernel.

We will analyze the series in (1) for signals f in $\mathcal{B}_\pi^p$, $1 < p < \infty$ and in $\mathcal{PW}_\pi^1$. Throughout the paper, we assume that f is real-valued for simplicity. The results can be extended to the complex-valued case if the quantization is applied separately to the real and imaginary part of f. All these signals have the property that $\lim_{t\to\infty} f(t) = 0$, i.e., for every signal in these spaces and every $\epsilon > 0$ there exists a $t_0 = t_0(\epsilon)$ such that $|f(t)| < \epsilon$ for all $|t| \geq t_0$. As a consequence, we have $f_\epsilon(k) = 0$ for all $|k| \geq t_0$. Hence, the series in (1) has only finitely many summands, which implies that $\overline{f_\epsilon} \in \mathcal{B}_\pi^p$ for all $1 < p \leq \infty$. According to these considerations, we see that (1) defines an operator $Q_\epsilon : f \mapsto \overline{f_\epsilon}$ that maps signals $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$, and signals $f \in \mathcal{PW}_\pi^1$ into signals $\overline{f_\epsilon} \in \mathcal{B}_\pi^\infty$. The operator Q_ϵ is the composition of the analog-to-digital conversion and the subsequent interpolation by the Shannon sampling series. We call Q_ϵ the quantization operator.

The approximation $\overline{f_\epsilon}$ can be decomposed into the original signal f and an additive quantization error signal $n_{\epsilon,f}$, i.e.,

$$\overline{f_\epsilon}(t) = f(t) + n_{\epsilon,f}(t), \quad t \in \mathbb{R}.$$

Of course, $n_{\epsilon,f}$ depends on the signal f and the quantization step size 2ϵ. Often, $n_{\epsilon,f}$ is treated as random noise signal, which implies that its dependence on the signal f is not heeded. In Remark 5, we will discuss why this point of view is not permissible in general.

It is possible to define several distortion measures, which quantify the quantization error $f - \overline{f_\epsilon}$

$$D_p(f,\epsilon) = \|f - \overline{f_\epsilon}\|_p$$

for $1 \leq p \leq \infty$ and $f \in \mathcal{B}_\pi^p$, where $p = \infty$ denotes the usual supremum norm, and

$$D_{\mathcal{PW}_\pi^1}(f,\epsilon) = \|f - \overline{f_\epsilon}\|_{\mathcal{PW}_\pi^1}$$

for $f \in \mathcal{PW}_\pi^1$. These distortion measures can be analyzed in two different ways: The first way is to fix $\epsilon > 0$ and analyze their behavior for varying f. This is done in Section V-C. The second way is to fix f and analyze their behavior for $\epsilon \to 0$, as is done in Section V-D.

IV. Discontinuity of the Quantization Operator

First of all, we analyze the continuity behavior of the quantization operator Q_ϵ. A precise characterization of all points where Q_ϵ is continuous and of all points where Q_ϵ is discontinuous is given in Theorem 1 and 2, respectively. Both theorems express what one would naturally expect: If at least the value of one sample equals a quantization threshold, then the quantization operator is discontinuous. Otherwise, it is continuous.

Theorem 1: The quantization operator $Q_\epsilon : f \mapsto \overline{f_\epsilon}$, which maps $\mathcal{B}_\pi^p \to \mathcal{B}_\pi^\infty$, $1 < p < \infty$, and $\mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$, is continuous for all signals f with $f(k) \notin \mathcal{Z}_\epsilon$ for all $k \in \mathbb{Z}$.

Proof: We only give the proof for the case $Q_\epsilon : \mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$, because the proof for $\mathcal{B}_\pi^p \to \mathcal{B}_\pi^\infty$ is analogous. Let $\epsilon > 0$ be arbitrary but fixed. Furthermore, let $f \in \mathcal{PW}_\pi^1$ and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of signals in $\mathcal{PW}_\pi^1$ such that $\lim_{n\to\infty}\|f - f_n\|_{\mathcal{PW}_\pi^1} = 0$. We have to show that $\lim_{n\to\infty}\|\overline{f_\epsilon} - \overline{(f_n)_\epsilon}\|_\infty = 0$. Since $\|f\|_\infty \leq \|f\|_{\mathcal{PW}_\pi^1}$ for all $f \in \mathcal{PW}_\pi^1$, we have $\lim_{n\to\infty}\|f - f_n\|_\infty = 0$, which implies that there is a $n_0 = n_0(\epsilon)$ such that $|f(t) - f_n(t)| < \epsilon/2$ for all $t \in \mathbb{R}$ and all $n \geq n_0$. Moreover, by the Riemann–Lebesgue lemma there is a $t_0 = t_0(\epsilon)$ such that $|f(t)| < \epsilon/2$ for all $|t| \geq t_0$. Using $|f_n(t)| \leq |f(t)| + |f_n(t) - f(t)|$ we obtain $|f_n(t)| < \epsilon$ for all $|t| \geq t_0$ and $n \geq n_0$. Now, let N_1 be the greatest natural number with $N_1 < t_0$. Then for all $n \geq n_0$ it holds that

$$\overline{(f_n)_\epsilon}(t) = \sum_{k=-N_1}^{N_1} (f_n)_\epsilon(k)\frac{\sin\bigl(\pi(t-k)\bigr)}{\pi(t-k)}$$

and

$$\overline{f_\epsilon}(t) - \overline{(f_n)_\epsilon}(t) = \sum_{k=-N_1}^{N_1} \bigl(f_\epsilon(k) - (f_n)_\epsilon(k)\bigr)\frac{\sin\bigl(\pi(t-k)\bigr)}{\pi(t-k)}.$$

Consequently

$$|\overline{f_\epsilon}(t) - \overline{(f_n)_\epsilon}(t)| \leq \max_{|k|\leq N_1}\bigl|f_\epsilon(k) - (f_n)_\epsilon(k)\bigr| \sum_{k=-N_1}^{N_1}\left|\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|. \quad (2)$$

By assumption, $f(k) \notin \mathcal{Z}_\epsilon$ for all $k \in \mathbb{Z}$, i.e., there is a $\delta > 0$ such that $|f(k) - (2l+1)\epsilon| > \delta$ for all $l \in \mathbb{Z}$ and all $k \in \mathbb{Z}$, but since there is a $n_1 = n_1(\delta)$ such that $|f(k) - f_n(k)| < \delta/2$ for all $n \geq n_1$ and all $k \in \mathbb{Z}$, we have $|f_\epsilon(k) - (f_n)_\epsilon(k)| = 0$ for all $n > n_1$ and all $k \in \mathbb{Z}$. It follows together with (2) that there is a $n_2 = \max(n_0, n_1)$ such that $|\overline{f_\epsilon}(t) - \overline{(f_n)_\epsilon}(t)| = 0$ for all $t \in \mathbb{R}$ and $n \geq n_2$, and consequently $\|\overline{f_\epsilon} - \overline{(f_n)_\epsilon}\|_\infty = 0$ for all $n \geq n_2$. ∎

Theorem 2: The quantization operator $Q_\epsilon : f \mapsto \overline{f_\epsilon}$, which maps $\mathcal{B}_\pi^p \to \mathcal{B}_\pi^\infty$, $1 < p < \infty$, and $\mathcal{PW}_\pi^1 \to \mathcal{B}_\pi^\infty$, is discontinuous for all signals f with $f(k) \in \mathcal{Z}_\epsilon$ for some $k \in \mathbb{Z}$.

Proof: We only consider the example where f is given by $f(t) = \epsilon \sin(\pi t)/(\pi t)$, with $\overline{f_\epsilon}(t) = 2\epsilon \sin(\pi t)/(\pi t)$ and where f_n is given by $f_n(t) = (\epsilon - 1/n)\sin(\pi t)/(\pi t)$, $n \in \mathbb{N}$, with $\overline{(f_n)_\epsilon}(t) = 0$, $t \in \mathbb{R}$, for all $n > 1/(2\epsilon)$. Thus, we have $\lim_{n\to\infty} \|\overline{(f_n)_\epsilon} - \overline{f_\epsilon}\|_\infty = 2\epsilon$, which shows that Q_ϵ is discontinuous at the point f. This example shows how the proof has to be done for general f. ∎

Remark 1: The discontinuity of the quantization operator does not depend on the specific choice of the quantization law. For example, it is irrelevant whether signal values in $[(2l-1)\epsilon, (2l+1)\epsilon)$ or signal values in $((2l-1)\epsilon, (2l+1)\epsilon]$, $l \in \mathbb{Z}$, are mapped to $2l\epsilon$.

V. Further Properties of the Quantization Operator

A very useful tool in the convergence analysis of the Shannon sampling series is the Plancherel–Pólya Theorem [22], [23].

Theorem 3 (Plancherel–Pólya): Let $1 < p < \infty$. Then there are two constants $C_1(p) > 0$ and $C_2(p) > 0$, depending only on p, such that

$$C_1(p)\left(\sum_{k=-\infty}^{\infty} |f(k)|^p\right)^{1/p} \leq \left(\int_{-\infty}^{\infty} |f(t)|^p \,\mathrm{d}t\right)^{1/p} \leq C_2(p)\left(\sum_{k=-\infty}^{\infty} |f(k)|^p\right)^{1/p} \quad (3)$$

for all $f \in B_\pi^p$.

We will use this theorem in the following analysis of the quantization operator.

A. Boundedness of the Quantization Operator for $\mathcal{B}_\pi^p$, $1 < p < \infty$

In this section, we analyze the quantization operator for the space $\mathcal{B}_\pi^p$, $1 < p < \infty$. We do not consider a single quantization rule but a whole class of quantization rules. The quantization scheme that was defined in Section III is included in this class.

In particular, we assume that the quantization thresholds $\{\Lambda_r\}_{r\in\mathbb{Z}, r\neq 0}$ fulfill

$$\cdots < \Lambda_{-3} < \Lambda_{-2} < \Lambda_{-1} < 0 < \Lambda_1 < \Lambda_2 < \Lambda_3 < \cdots.$$

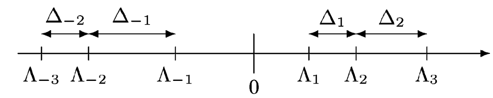

Fig. 2. Nonuniform quantization.

Except for the quantization level $\lambda_0 = 0$, to which all quantizer inputs in the interval $[\Lambda_{-1}, \Lambda_1)$ are mapped, the quantization levels $\{\lambda_r\}_{r\in\mathbb{Z}, r\neq 0}$ can lie arbitrarily within the quantization intervals, i.e.,

$$\begin{aligned}\cdots &\leq \Lambda_{-3} < \lambda_{-2} \leq \Lambda_{-2} < \lambda_{-1} \leq \Lambda_{-1} < \lambda_0 \\ &= 0 < \Lambda_1 < \lambda_1 \leq \Lambda_2 < \lambda_2 \leq \Lambda_3 < \cdots.\end{aligned}$$

For $k \in \mathbb{Z}$, we define the operator

$$(\Upsilon f)(k) = \begin{cases} 0, & \text{if } f(k) \in [\Lambda_{-1}, \Lambda_1) \\ \lambda_r, & \text{if } f(k) \in [\Lambda_r, \Lambda_{r+1}), \text{ for some } r \in \mathbb{Z}, r \geq 1 \\ \lambda_r, & \text{if } f(k) \in [\Lambda_{r-1}, \Lambda_r), \text{ for some } r \in \mathbb{Z}, r \leq -1. \end{cases}$$

Furthermore, for $r \in \mathbb{Z}$, $r \neq 0$, we need the lengths

$$\Delta_r = \begin{cases} \Lambda_{r+1} - \Lambda_r, & r \geq 1 \\ \Lambda_r - \Lambda_{r-1}, & r \leq 1 \end{cases}$$

of the quantization intervals and the sets

$$\mathcal{D}_r = \begin{cases} \{k \in \mathbb{Z} : f(k) \in [\Lambda_r, \Lambda_{r+1})\}, & r \geq 1 \\ \{k \in \mathbb{Z} : f(k) \in [\Lambda_{r-1}, \Lambda_r)\}, & r \leq -1. \end{cases}$$

Fig. 2 illustrates this quantization scheme. Using this general quantization rule, the quantization operator Q is given by

$$(Qf)(t) = \sum_{k=-\infty}^{\infty} (\Upsilon f)(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \quad (4)$$

Since the sum in (4) only has finitely many summands, it follows that $Qf \in \mathcal{B}_\pi^p$ for any signal $f \in \mathcal{B}_\pi^p$, $1 < p < \infty$.

Theorem 4: Let $Q : \mathcal{B}_\pi^p \to \mathcal{B}_\pi^p$, $1 < p < \infty$. If

$$\sup_{\substack{r\in\mathbb{Z} \\ r\neq 0}} \left(\frac{\Delta_r}{|\Lambda_r|}\right) < \infty \quad (5)$$

then there exists a constant $C(p)$ that only depends on p such that

$$\|Qf\|_{\mathcal{B}_\pi^p} \leq C(p)\, \|f\|_{\mathcal{B}_\pi^p} \quad (6)$$

for all $f \in \mathcal{B}_\pi^p$.

Remark 2:
Note that the condition (5) is equivalent to the condition that

$$\sup_{\substack{r\in\mathbb{Z} \\ r\geq 1}} \left(\frac{\Lambda_{r+1}}{\Lambda_r}\right) < \infty \quad \text{and} \quad \sup_{\substack{r\in\mathbb{Z} \\ r\leq 1}} \left(\frac{\Lambda_{r-1}}{\Lambda_r}\right) < \infty.$$

Proof: Let $1 < p < \infty$ and $f \in \mathcal{B}_\pi^p$ be arbitrary but fixed. We bound $\|Qf\|_p$ from above. According to the

Plancherel–Pólya inequality (3), which is applicable because $Qf \in \mathcal{B}_\pi^p$, we have

$$\|Qf\|_p \leq C_2(p)\left(\sum_{k=-\infty}^{\infty}|(\Upsilon f)(k)|^p\right)^{1/p} = C_2(p)\left(\sum_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\sum_{k\in\mathcal{D}_r}|(\Upsilon f)(k)|^p\right)^{1/p}. \quad (7)$$

Since $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for all $1<p<\infty$ and $a,b>0$, we obtain for $r\in\mathbb{Z}$, $r\neq 0$

$$\begin{aligned}\sum_{k\in\mathcal{D}_r}|(\Upsilon f)(k)|^p &\leq \sum_{k\in\mathcal{D}_r}(|f(k)|+\Delta_r)^p\\ &\leq 2^{p-1}\left(\sum_{k\in\mathcal{D}_r}|f(k)|^p+\sum_{k\in\mathcal{D}_r}\Delta_r^p\right)\\ &\leq 2^{p-1}\left(\sum_{k\in\mathcal{D}_r}|f(k)|^p+\Delta_r^p\sum_{k\in\mathcal{D}_r}\left(\frac{|f(k)|}{|\Lambda_r|}\right)^p\right)\\ &= 2^{p-1}\left(1+\left(\frac{\Delta_r}{|\Lambda_r|}\right)^p\right)\sum_{k\in\mathcal{D}_r}|f(k)|^p. \quad (8)\end{aligned}$$

Furthermore, (8) leads to

$$\begin{aligned}&\sum_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\sum_{k\in\mathcal{D}_r}|(\Upsilon f)(k)|^p\\ &\leq 2^{p-1}\left(1+\sup_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\left(\frac{\Delta_r}{|\Lambda_r|}\right)^p\right)\sum_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\sum_{k\in\mathcal{D}_r}|f(k)|^p\\ &\leq 2^{p-1}\left(1+\sup_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\left(\frac{\Delta_r}{|\Lambda_r|}\right)^p\right)\sum_{k=-\infty}^{\infty}|f(k)|^p\\ &\leq 2^{p-1}\left(1+\sup_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\left(\frac{\Delta_r}{|\Lambda_r|}\right)^p\right)\left(\frac{\|f\|_p}{C_1(p)}\right)^p \quad (9)\end{aligned}$$

where (3) was used again. Inserting (9) into (7) gives

$$\|Qf\|_p \leq \frac{C_2(p)}{C_1(p)}2^{1-1/p}\|f\|_p\left(1+\sup_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\left(\frac{\Delta_r}{|\Lambda_r|}\right)^p\right)^{1/p}.$$

■

Remark 3: For the proof of Theorem 4 it was essential that $p>1$, because the application of the Plancherel–Pólya inequality requires $p>1$.

Remark 4: Theorem 4 shows that $Q:\mathcal{B}_\pi^p\to\mathcal{B}_\pi^p, 1<p<\infty$ is a bounded operator if (5) is fulfilled, because (6) implies that every bounded set is mapped into a bounded set. Since we have $\|f\|_{\mathcal{B}_\pi^\infty}\leq \tilde{C}(p)\|f\|_{\mathcal{B}_\pi^p}$, $1<p<\infty$, for some constant $\tilde{C}(p)$, it follows that $Q:\mathcal{B}_\pi^p\to\mathcal{B}_\pi^\infty$, $1<p<\infty$, is also a bounded operator if (5) is fulfilled.

Corollary 1: For every $\epsilon>0$, the quantization operator $Q_\epsilon:\mathcal{B}_\pi^p\to\mathcal{B}_\pi^p$, $1<p<\infty$, that was introduced in Section III is bounded.

Proof: Let $1<p<\infty$ be arbitrary but fixed. For the uniform quantization with quantization step size 2ϵ that was introduced in Section III, we have $\Delta_r=2\epsilon$, $r\in\mathbb{Z}$, $r\neq 0$, and $\Lambda_r=(2r-1)\epsilon$, $r\in\mathbb{Z}$, $r\geq 1$, as well as $\Lambda_r=(2r+1)\epsilon$, $r\in\mathbb{Z}$, $r\leq -1$. Therefore

$$\sup_{\substack{r\in\mathbb{Z}\\ r\neq 0}}\left(\frac{\Delta_r}{|\Lambda_r|}\right)=2<\infty$$

and Theorem 4 implies that Q_ϵ is bounded. ■

B. *Unboundedness of the Quantization Operator for $\mathcal{PW}_\pi^1$*

In Section V-A, it has been shown that the quantization operator $Q_\epsilon:\mathcal{B}_\pi^p\to\mathcal{B}_\pi^p$, $1<p<\infty$, is a bounded operator. Interestingly, this is not true for $f\in\mathcal{PW}_\pi^1$: Although only finitely many $f_\epsilon(k)$, $k\in\mathbb{Z}$, are different from zero, $\|\overline{f_\epsilon}\|_\infty$, and, hence, $\|\overline{f_\epsilon}\|_{\mathcal{PW}_\pi^1}$ can become arbitrarily large on the set $\{f\in\mathcal{PW}_\pi^1:\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon\}$. This fact is expressed by Theorem 5. The dependence of $\|\overline{f_\epsilon}\|_\infty$ on ϵ is analyzed in Section V-C.

Theorem 5: Let $0<\epsilon<1$ be arbitrary. Then we have $\|\overline{f_\epsilon}\|_\infty<\infty$ for all $f\in\mathcal{PW}_\pi^1$ and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon}\|\overline{f_\epsilon}\|_\infty=\infty.$$

Corollary 2: Let $0<\epsilon<1$ be arbitrary. Then we have

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon}D_\infty(f,\epsilon)=\infty \quad (10)$$

as well as

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon}\|\overline{f_\epsilon}\|_{\mathcal{PW}_\pi^1}=\infty \quad (11)$$

and

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon}D_{\mathcal{PW}_\pi^1}(f,\epsilon)=\infty. \quad (12)$$

Proof of Corollary 2: Equation (10) follows directly from Theorem 5 because for $\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon$ we have

$$\|f-\overline{f_\epsilon}\|_\infty\geq\|\overline{f_\epsilon}\|_\infty-\|f\|_\infty\geq\|\overline{f_\epsilon}\|_\infty-9\epsilon.$$

Equation (11) and (12) follow from Theorem 5 and equation (10), respectively, by using $\|f\|_\infty\leq\|f\|_{\mathcal{PW}_\pi^1}$. ■

Remark 5: Corollary 2 shows that the peak value of the quantization error can increase unboundedly for all $\epsilon>0$ even if $\|f\|_{\mathcal{PW}_\pi^1}$ is bounded. From Theorem 5 and $\|\overline{f_\epsilon}\|_\infty-\|f\|_\infty\leq\|n_{\epsilon,f}\|_\infty\leq\|n_{\epsilon,f}\|_{\mathcal{PW}_\pi^1}$, we see that the peak value of the quantization noise $n_{\epsilon,f}$ as well as the L^1-norm of its spectrum, i.e., the $\mathcal{PW}_\pi^1$-norm of $n_{\epsilon,f}$, is unbounded, despite $\|f\|_{\mathcal{PW}_\pi^1}\leq 9\epsilon$. Thus, $n_{\epsilon,f}$ cannot be modeled as random noise signal, which is independent of f.

Proof of Theorem 5: If $f \in \mathcal{PW}_\pi^1$ it follows directly from the Riemann–Lebesgue lemma that $\overline{f_\epsilon} \in \mathcal{PW}_\pi^2$, because only finitely many values of $f_\epsilon(k)$, $k \in \mathbb{Z}$, are different from zero. Since $\overline{f_\epsilon} \in \mathcal{PW}_\pi^2$ it follows that $\|\overline{f_\epsilon}\|_\infty < \infty$.

Let $N \in \mathbb{N}$, $N \geq 3$, be arbitrary but fixed. For the rest of the proof, we need to define the function

$$h_N(t) = \sum_{k=-\infty}^{\infty} h_N(k) \frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)} \tag{13}$$

where $h_N(k) = (-1)^k g_N(k)$ and

$$g_N(k) = \begin{cases} 1, & |k| \leq N \\ 2\left(1 - \frac{|k|}{2N}\right), & N < |k| < 2N \\ 0, & |k| \geq 2N. \end{cases}$$

h_N has the property that $\|h_N\|_{\mathcal{PW}_\pi^1} \leq 3$. The proof thereof can be found in the Appendix .

Let $0 < \epsilon < 1$ be arbitrary. Next, we choose $\mu \in \mathbb{R}$ such that $1 \geq \mu > \epsilon > \mu(N-1)/N$. Then the quantized samples of $q_N(t) = \mu h_N(t)$ are given by

$$(q_N)_\epsilon(k) = \begin{cases} 2\epsilon(-1)^k & |k| \leq N \\ 0 & |k| > N \end{cases}$$

because

$$|q_N(k)| \leq \mu 2 \left(1 - \frac{N+1}{2N}\right) = \mu \frac{N-1}{N} < \epsilon$$

for all $|k| > N$. Note that $\|q_N\|_{\mathcal{PW}_\pi^1} \leq 3\mu \leq 9\epsilon$ because $N \geq 3$. Furthermore, we obtain

$$\overline{(q_N)_\epsilon}(t) = \sum_{k=-N}^{N} (q_N)_\epsilon(k) \frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)}$$

and

$$\begin{aligned} \left|\overline{(q_N)_\epsilon}\left(N + \frac{1}{2}\right)\right| &= \left|\sum_{k=-N}^{N} (q_N)_\epsilon(k) \frac{(-1)^k}{\pi(N + \frac{1}{2} - k)}\right| \\ &\geq \frac{2\epsilon}{\pi} \left|\sum_{k=-N}^{N} \frac{1}{\pi(N + \frac{1}{2} - k)}\right| \\ &\geq \frac{2\epsilon}{\pi} \log(N) \end{aligned}$$

which implies that $\|\overline{(q_N)_\epsilon}\|_\infty \geq 2\epsilon \log(N)/\pi$.

Since $N \in \mathbb{N}$, $N \geq 3$, was arbitrary, $\|\overline{(q_N)_\epsilon}\|_\infty$ can become arbitrarily large independently of ϵ, and it follows that

$$\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 9\epsilon} \|\overline{f_\epsilon}\|_\infty = \infty$$

for all $0 < \epsilon < 1$. ■

We have shown that $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 9\epsilon} D_\infty(f,\epsilon) = \infty$ and $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 9\epsilon} D_{\mathcal{PW}_\pi^1}(f,\epsilon) = \infty$ for all $0 < \epsilon < 1$, whereas $\sup_{\|f\|_{\mathcal{B}_\pi^p} \leq 9\epsilon} D_\infty(f,\epsilon) < \infty$ and $\sup_{\|f\|_{\mathcal{B}_\pi^p} \leq 9\epsilon} D_p(f,\epsilon) < \infty$ for $1 < p < \infty$ and all $0 < \epsilon < 1$.

Obviously, for every $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} < \epsilon$, we have $\|f\|_\infty \leq \|f\|_{\mathcal{PW}_\pi^1} < \epsilon$, and as a consequence $\|\overline{f_\epsilon}\|_\infty = 0$ and $\sup_{\|f\|_{\mathcal{PW}_\pi^1} < \epsilon} \|\overline{f_\epsilon}\|_\infty = 0 < \infty$. Therefore, it is only interesting to analyze $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq \beta\epsilon} \|\overline{f_\epsilon}\|_\infty$ with $\beta > 1$. We conjecture that for all $\beta > 1$ fixed, $\sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq \beta\epsilon} \|\overline{f_\epsilon}\|_\infty = \infty$ holds, i.e., that 9ϵ in Theorem 5 can be replaced by $\beta\epsilon$, $\beta > 1$ arbitrary but fixed.

Remark 6: In the proof of Theorem 5, the slow decay of the sinc function was important to create the divergence. However, this is not the only reason for divergence. The nonlinearity of the quantization operator Q_ϵ was also essential to construct the specific sequence of quantized samples $\{(q_N)_\epsilon(k)\}_{k \in \mathbb{Z}}$, which was used to prove the divergence. The following discussion further illustrates why nonlinearity is important for the unboundedness of the quantization operator. Let $0 < \delta < \epsilon$ and $f \in \mathcal{PW}_\pi^1$ be arbitrary and consider the difference between f and $Q_\epsilon f = \overline{f_\epsilon}$. Since $\overline{f_\epsilon} \in \mathcal{PW}_\pi^1$, it follows that the operator $Id - Q_\epsilon$, where Id denoted the identity operator, maps $\mathcal{PW}_\pi^1$ into $\mathcal{PW}_\pi^1$. Further, for all $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} < \epsilon$ we have $(Id - Q_\epsilon)f = f$, because $Q_\epsilon f \equiv 0$. Thus

$$\|(Id - Q_\epsilon)f\|_{\mathcal{PW}_\pi^1} = \|f\|_{\mathcal{PW}_\pi^1} < \epsilon \tag{14}$$

for all $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} < \epsilon$. A simple scaling argument shows that every linear operator $T : \mathcal{PW}_\pi^1 \to \mathcal{PW}_\pi^1$ that fulfills (14), i.e., $\|(Id - T)f\|_{\mathcal{PW}_\pi^1} < \epsilon$ for all $f \in \mathcal{PW}_\pi^1$ with $\|f\|_{\mathcal{PW}_\pi^1} < \epsilon$, is a bounded operator. Thus, nonlinearity is a necessary condition for an operator fulfilling (14) to be unbounded.

C. Behavior of the Quantization Operator for $\mathcal{PW}_\pi^1$ and $\epsilon \to 0$

Certainly, for all $f \in \mathcal{PW}_\pi^1$ and all $\epsilon > 0$, we have $D_\infty(f,\epsilon) < \infty$. On the other hand it is not obvious what happens if ϵ tends to zero. Next, we will analyze how $\|\overline{f_\epsilon}\|_\infty$ and $D_\infty(f,\epsilon) = \|f - \overline{f_\epsilon}\|_\infty$ behave for fixed $f \in \mathcal{PW}_\pi^1$ when ϵ tends to zero. One could guess that the error $\|f - \overline{f_\epsilon}\|_\infty$ gets smaller for $\epsilon \to 0$, because a finer quantization should give a better representation of the signal. However, the quantization operator $Q_\epsilon : f \mapsto \overline{f_\epsilon}$ is nonlinear, which will lead to a unexpected behavior.

This fact is expressed by the following theorem.

Theorem 6: There is a signal $f \in \mathcal{PW}_\pi^1$, such that

$$\limsup_{\epsilon \to 0} \|\overline{f_\epsilon}\|_\infty = \infty.$$

Corollary 3: There is a signal $f \in \mathcal{PW}_\pi^1$, such that

$$\limsup_{\epsilon \to 0} D_\infty(f,\epsilon) = \infty$$

as well as

$$\limsup_{\epsilon \to 0} \|\overline{f_\epsilon}\|_{\mathcal{PW}_\pi^1} = \infty$$

and

$$\limsup_{\epsilon \to 0} D_{\mathcal{PW}_\pi^1}(f,\epsilon) = \infty.$$

Remark 7: From Theorem 6 and $\|\overline{f_\epsilon}\|_\infty - \|f\|_\infty \leq \|n_{\epsilon,f}\|_\infty \leq \|n_{\epsilon,f}\|_{\mathcal{PW}_\pi^1}$, we see that the peak value of the additive noise signal $n_{\epsilon,f}$ and the L^1-norm of its spectrum can increase unboundedly as ϵ tends to zero. This result is in contrast to the common model, where the quantization error decreases as ϵ tends to zero [7].

Proof of Theorem 6: In order to construct the signal f, we use the functions h_N, which were defined in (13), as basic building blocks. Next, a sequence $\{n_l\}_{l\in\mathbb{N}}$ of natural numbers is inductively constructed. Let $n_1=1$ and $N(k)=2^{(k^3)}$, $k\in\mathbb{N}$. Furthermore, let n_{l+1} be the smallest natural number that is larger than n_l and fulfills

$$\frac{1}{n_l^2}\frac{N(n_l)-1}{N(n_l)}+\frac{1}{n_{l+1}-1}<\frac{1}{n_l^2}. \tag{15}$$

We define the signal

$$f(t)=\sum_{l=1}^{\infty}\frac{1}{n_l^2}h_{N(n_l)}(t).$$

First, note that $f\in\mathcal{PW}_\pi^1$, because

$$\|f\|_{\mathcal{PW}_\pi^1}\le\sum_{l=1}^{\infty}\frac{1}{n_l^2}\|h_{N(l)}\|_{\mathcal{PW}_\pi^1}\le 3\sum_{l=1}^{\infty}\frac{1}{n_l^2}<\infty.$$

Next, $f(k)$, $k\in\mathbb{Z}$, is analyzed. Let $r\in\mathbb{N}$ be arbitrary and $|k|\ge N(n_r)+1$. For $m<r$ we have $n_r\ge n_m+1$ and consequently $n_r^3\ge n_m^3+1$. It follows that $N(n_r)=2^{(n_r^3)}\ge 2^{(n_m^3+1)}=2\cdot 2^{(n_m^3)}=2N(n_m)$, which implies that $|k|\ge 2N(n_m)$. Thus, for $m<r$, $h_{N(n_m)}(k)=0$ and

$$\begin{aligned}|f(k)|&=\left|\sum_{l=r}^{\infty}\frac{1}{n_l^2}h_{N(n_l)}(k)\right|\\&=\left|(-1)^k\sum_{l=r}^{\infty}\frac{1}{n_l^2}g_{N(n_l)}(k)\right|\\&=\frac{1}{n_r^2}g_{N(n_r)}(k)+\sum_{l=r+1}^{\infty}\frac{1}{n_l^2}g_{N(n_l)}(k)\\&\le\frac{2}{n_r^2}\left(1-\frac{N(n_r)+1}{2N(n_r)}\right)+\sum_{l=r+1}^{\infty}\frac{1}{n_l^2}\\&\le\frac{1}{n_r^2}\frac{N(n_r)-1}{N(n_r)}+\sum_{l=n_{r+1}}^{\infty}\frac{1}{l^2}\\&<\frac{1}{n_r^2}\frac{N(n_r)-1}{N(n_r)}+\frac{1}{n_{r+1}-1}.\end{aligned}$$

For ϵ_r with

$$\frac{1}{n_r^2}\frac{N(n_r)-1}{N(n_r)}+\frac{1}{n_{r+1}-1}<\epsilon_r<\frac{1}{n_r^2}$$

we have $f_{\epsilon_r}(k)=0$ for $|k|\ge N(n_r)+1$. For $|k|\le N(n_r)$ and k even, we have

$$f(k)=\sum_{l=1}^{\infty}\frac{1}{n_l^2}g_{N(n_l)}(k)\ge\frac{1}{n_r^2}g_{N(n_r)}(k)=\frac{1}{n_r^2}$$

and

$$f_{\epsilon_r}(k)\ge 2\epsilon_r\ge\frac{1}{n_r^2}$$

because $\epsilon_r\ge 1/(2n_r^2)$. Similarly it can be shown that for $|k|\le N(n_r)$ and k odd

$$f_{\epsilon_r}(k)\le-\frac{1}{n_r^2}$$

holds. This implies that

$$\overline{f_{\epsilon_r}}(t)=\sum_{k=-N(n_r)}^{N(n_r)}f_{\epsilon_r}(k)\frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)}$$

and

$$\begin{aligned}\left|\overline{f_{\epsilon_r}}\left(N(n_r)+\frac{1}{2}\right)\right|&=\left|\sum_{k=-N(n_r)}^{N(n_r)}f_{\epsilon_r}(k)\frac{(-1)^k}{\pi(N(n_r)+\frac{1}{2}-k)}\right|\\&\ge\frac{1}{\pi n_r^2}\sum_{k=-N(n_r)}^{N(n_r)}\frac{1}{N(n_r)+\frac{1}{2}-k}\\&\ge\frac{1}{\pi n_r^2}\log(N(n_r))\\&=\frac{n_r}{\pi}\log(2).\end{aligned}$$

Therefore, $\|\overline{f_{\epsilon_r}}\|_\infty\ge n_r\log(2)/\pi$ and $\limsup_{\epsilon\to 0}\|\overline{f_\epsilon}\|_\infty\ge\limsup_{r\to\infty}\|\overline{f_{\epsilon_r}}\|_\infty=\infty$. ■

APPENDIX
PROOF OF $\|h_N\|_{\mathcal{PW}_\pi^1}\le 3$

The Fourier coefficients $F_N(k)$, $k\in\mathbb{Z}$, of the Fejér kernel

$$\hat{F}_N(\omega)=\frac{1}{N+1}\frac{\sin^2\big((N+1)\frac{\omega}{2}\big)}{\sin^2\big(\frac{\omega}{2}\big)}$$

are given by

$$F_N(k)=\begin{cases}1-\frac{|k|}{N}, & |k|<N\\ 0, & |k|\ge N.\end{cases}$$

Thus, $g_N(k)=2F_{2N}(k)-F_N(k)$, $k\in\mathbb{Z}$, and the Fourier transformation of

$$h_N(t)=\sum_{k=-2N}^{2N}(-1)^k g_N(k)\frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)}$$

is

$$\hat{h}_N(\omega)=2\hat{F}_{2N}(\pi-\omega)-\hat{F}_N(\pi-\omega),\quad |\omega|\le\pi.$$

As a consequence, we obtain

$$\begin{aligned}\|h_N\|_{\mathcal{PW}_\pi^1}&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|2\hat{F}_{2N}(\pi-\omega)-\hat{F}_N(\pi-\omega)\right|\,\mathrm{d}\omega\\&\le\frac{1}{2\pi}\int_{-\pi}^{\pi}2\hat{F}_{2N}(\omega)+\hat{F}_N(\omega)\,\mathrm{d}\omega=3\end{aligned}$$

because $\hat{F}_{2N}$ and $\hat{F}_N$ are 2π-periodic and non-negative.

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Holger Boche (M’04–SM’07) received the Dipl.-Ing. and Dr.-Ing. degrees in electrical engineering from the Technische Universität Dresden, Germany, in 1990 and 1994, respectively.

In 1992, he graduated in mathematics from the Technische Universität Dresden, and in 1998, he received the Dr.rer.nat. degree in pure mathematics from the Technische Universität Berlin, Berlin, Germany. From 1994 to 1997, he did postgraduate studies in mathematics at the Friedrich–Schiller Universität Jena, Germany. In 1997, he joined the Heinrich-Hertz-Institut (HHI) für Nachrichtentechnik Berlin. Since 2002, he has been a Full Professor for Mobile Communication Networks at the Technische Universität Berlin at the Institute for Communications Systems. In 2003, he became the Director of the Fraunhofer German-Sino Lab for Mobile Communications, Berlin, and since 2004, he has also been the Director of the Fraunhofer Institute for Telecommunications (HHI), Berlin. He was a Visiting Professor at the ETH Zurich, Switzerland, during the winter 2004 and 2006 terms and at KTH Stockholm, Sweden, during the summer 2005 term.

Prof. Boche received the Research Award “Technische Kommunikation” from the Alcatel SEL Foundation in October 2003, the “Innovation Award” from the Vodafone Foundation in June 2006, and the Gottfried Wilhelm Leibniz Prize from the Deutsche Forschungsgemeinschaft (German Research Foundation) in 2008. He was a corecipient of the 2006 IEEE Signal Processing Society Best Paper Award and the recipient of the 2007 IEEE Signal Processing Society Best Paper Award. He is a member of the SPTM Technical Committee. He was elected a member of the German Academy of Sciences (Leopoldina) in 2008.

Ullrich J. Mönich (S’06) received the Dipl.-Ing. degree in electrical engineering from the Technische Universität Berlin, Berlin, Germany, in 2005, where he is currently pursuing the Ph.D. degree.

During the winter 2003 term, he was a Visiting Researcher at the University of California, Santa Barbara. His research activities comprise sampling theory, ultrawideband mobile communications, and information theory.