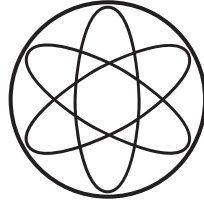


PHYSIK-DEPARTMENT



Two Approaches Towards the Flavour Puzzle:
Dynamical Minimal Flavour Violation
and
Warped Extra Dimensions

Dissertation

von

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Vollständiger Abdruck der von der Fakultät für Physik der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

Vorsitzender: Univ.-Prof. Dr. Stephan Paul

Prüfer der Dissertation: 1. Univ.-Prof. Dr. Thorsten Feldmann
2. Priv.-Doz. Dr. André H. I. Hoang

Die Dissertation wurde am 01.07.2010 bei der Technischen Universität München eingereicht und durch die Fakultät für Physik am 16.08.2010 angenommen.

Gewidmet meiner Mutter Elfriede Albrecht

Abstract

The minimal-flavour-violating (MFV) hypothesis considers the Standard Model (SM) Yukawa matrices as the only source of flavour violation. In this work, we promote their entries to dynamical scalar spurion fields, using an effective field theory approach, such that the maximal flavour symmetry (FS) of the SM gauge sector is formally restored at high energy scales. The non-vanishing vacuum expectation values of the spurions induce a sequence of FS breaking and generate the observed hierarchy in the SM quark masses and mixings. The fact that there exists no explanation for it in the SM is known as the flavour puzzle. Gauging the non-abelian subgroup of the spontaneously broken FS, we interpret the associated Goldstone bosons as the longitudinal degrees of freedom of the corresponding massive gauge bosons. Integrating out the heavy Higgs modes in the Yukawa spurions leads directly to flavour-changing neutral currents (FCNCs) at tree level. The coefficients of the effective four-quark operators, resulting from the exchange of heavy flavoured gauge bosons, strictly follow the MFV principle. On the other hand, the Goldstone bosons associated with the global abelian symmetry group behave as weakly coupled axions which can be used to solve the strong CP problem within a modified Peccei–Quinn formalism.

Models with a warped fifth dimension contain five-dimensional (5D) fermion bulk mass matrices in addition to their 5D Yukawa matrices, which thus represent an additional source of flavour violation beyond MFV. They can address the flavour puzzle since their eigenvalues allow for a different localisation of the fermion zero mode profiles along the extra dimension which leads to a hierarchy in the effective four-dimensional (4D) Yukawa matrices. At the same time, the fermion splitting introduces non-universal fermion couplings to Kaluza–Klein (KK) gauge boson modes, inducing tree-level FCNCs. Within a Randall–Sundrum model with custodial protection (RSc model) we carefully work out the flavour and electroweak (EW) sector, including a derivation of Feynman rules. Moreover, we determine the contributions to the effective Hamiltonian for meson-antimeson mixing due to KK gluon and KK photon exchange.

Kurzfassung

Die Hypothese der minimalen Flavourverletzung (MFV) geht davon aus, dass die Yukawa-Matrizen die einzige Quelle der Flavourverletzung darstellen. Im Rahmen einer effektiven Theorie betrachten wir in dieser Arbeit die Einträge der Yukawa-Matrizen als dynamische skalare Spurionfelder, so dass die maximale Flavoursymmetrie des Standardmodell-(SM)-Eichsektors formal an einer hohen Skala wiederhergestellt wird. Die nicht verschwindenden Vakuumerwartungswerte der Spurionen bewirken eine Sequenz von Flavoursymmetriebrechungen und erzeugen die beobachtbare Hierarchie der SM Quarkmassen und Mischungswinkel. Da es im SM keine Erklärung für diese gibt, spricht man von einem Flavourpuzzle. Wir interpretieren die Goldstonebosonen, die aus der geeichten nicht abelschen Untergruppe der spontan gebrochenen Flavoursymmetrie hervorgehen, als longitudinale Freiheitsgrade der zugehörigen massiven Eichbosonen. Integriert man die schweren Higgsmoden in den Yukawa-Spurionen aus, erhält man direkt flavour-ändernde neutrale Ströme (FCNCs) auf Baumgraphenniveau. Die Koeffizienten der effektiven Vier-Quark-Operatoren, die von dem Austausch eines schweren Flavoureichbosons herrühren, folgen dabei strikt dem MFV-Prinzip. Andererseits verhalten sich die Goldstonebosonen der globalen abelschen Symmetriegruppe wie schwach wechselwirkende Axionen, die benutzt werden können um das starke CP-Problem innerhalb eines modifizierten Peccei–Quinn-Formalismus zu lösen.

Modelle mit einer gekrümmten fünften Raumdimension enthalten neben den fünf-dimensionalen (5D) Yukawa-Matrizen auch 5D Fermionmassenmatrizen. Diese stellen eine weitere Quelle der Flavourverletzung entgegen der MFV-Annahme dar und ermöglichen eine Erklärung des Flavourpuzzles, da ihre Eigenwerte eine unterschiedliche Lokalisation der Fermionnullmoden zulassen, was wiederum zu einer Hierarchie in den effektiven vier-dimensionalen (4D) Yukawa-Matrizen führt. Gleichzeitig bringt das Aufspalten der Fermionen nicht-universelle Fermionkopplungen zu Kaluza–Klein-(KK)-Eichbosonmoden mit sich, die FCNCs auf Baumgraphenniveau verursachen. Wir arbeiten ausführlich den flavour- und elektroschwachen Teil eines Randall–Sundrum-Modells aus, welches eine erweiterte Symmetriegruppe enthält um den elektroschwachen Sektor zu schützen. Dabei leiten wir Feynmandiagramme ab und bestimmen die Beiträge zum effektiven Hamiltonian für Meson-Antimesonmischung, der durch den Austausch von KK-Gluonen oder KK-Photonen zustande kommt.

Contents

1	Introduction and Motivation	1
2	General Aspects of Gauge Theories	7
2.1	Abelian Gauge Symmetry	7
2.2	Non-Abelian Gauge Symmetry	8
2.3	Gauge-Fixing Terms and BRST Transformation	9
2.4	Chiral Fields and Chiral Gauge Symmetry	11
2.5	Chiral Gauge Anomaly	12
3	Brief Review of the Standard Model	15
3.1	Quantum Chromodynamics	15
3.2	Electroweak Sector	16
3.3	Spontaneous Electroweak Symmetry Breaking	18
3.4	Gauge Boson Masses	20
3.5	Custodial Symmetry	22
3.6	R_ξ -Gauge Fixing Terms within the Standard Model	23
3.7	Neutral and Charged Currents	24
3.8	Yukawa Interactions — Fermion Mass Terms	25
3.9	Global Flavour Symmetry of the Standard Model	27
4	Dynamical Minimal Flavour Violation	29
4.1	Basic Concepts of Minimal Flavour Violation	29
4.2	Towards Dynamical Minimal Flavour Violation	32
4.3	Sequential Breaking of the Flavour Symmetry	33
4.3.1	Partly Gauged Flavour Symmetry	34
4.4	Parametrisation of the Yukawa Matrices and Unitary Gauge	38
4.4.1	Solution of the 2-Generation Case	41
4.4.2	Solution of the 3-Generation Case	43
4.5	Simplified Scenario	46
4.5.1	Anomalies of the Local Flavour Symmetry Group $SU(2)_{D_R} \times U(1)_X$	47

4.6	Effective Lagrangian of the Gauged Flavour Subgroup	48
4.6.1	Anomalous $U(1)_X$ Symmetry	48
4.6.2	Local $SU(2)_{D_R}$ Flavour Symmetry	51
4.6.3	Yukawa Spurion Lagrangian	52
4.7	Effective Theories for the Energy Scales $y_b\theta_{13}\Lambda \leq E < y_s\Lambda$	53
4.7.1	Integrating out the Heavy Spurion Field η_{13}	54
4.7.2	Integrating out the $U(1)_X$ Gauge Field	54
4.8	Effective Theories for the Energy Scales $y_s\Lambda \leq E < y_s\theta_{12}\Lambda$	56
4.8.1	Integrating out the Spurion Field η_s	56
4.8.2	Integrating out the Gauge Fields $A_{D_R}^a$	56
4.9	Effective Theory below the Energy Scale $E \leq y_s\theta_{12}\Lambda$	57
4.9.1	Integrating out the Spurion Field η_{12}	57
4.9.2	Global $U(1)_{u_R} \times U(1)_{d_R}$ Flavour Symmetry	58
4.10	The Strong CP Problem	58
4.10.1	Peccei–Quinn Mechanism for $U(1)_{u_R} \times U(1)_{d_R}$	61
4.11	Summary of Chapter 4	64
5	Warped Extra Dimensions	65
5.1	The Randall–Sundrum Model with Custodial Protection	65
5.2	Gauge Sector of the RSc Model	67
5.3	Higgs Sector and Electroweak Symmetry Breaking	72
5.4	Gauge Boson Masses	74
5.5	Analytic Diagonalisation of the EW Gauge Boson Mass Matrices	77
5.6	Fermions in Warped Extra Dimensions	81
5.6.1	Construction of the Warped Fermionic Action	81
5.6.2	KK Decomposition and Localisation of Fermionic Modes	83
5.6.3	Fermion Content of the RSc Model	85
5.7	Flavour Structure	88
5.7.1	Constructing Gauge-Invariant Yukawa Couplings	88
5.7.2	4D Effective Yukawa Coupling Matrices	89
5.7.3	Quark Mass Matrices	90
5.8	Flavour Violation within the RSc Model	92
5.8.1	Fermion-Gauge Boson Interactions	92
5.8.2	Neutral Currents	94
5.8.3	Charged Currents	97
5.8.4	Photonic and Gluonic Currents	98
5.8.5	Tree-Level Contribution of KK Gluons to $\Delta F = 2$ Transitions	98
5.8.6	Tree-Level Contribution of the KK Photon to $\Delta F = 2$ Transitions	101

6	Summary and Outlook	103
A	Addendum to Chapter 4	107
A.1	Differential Forms	107
A.2	Graded Algebra	109
A.3	Usage of the Wess–Zumino Consistency Condition	110
A.4	Diagonalisation of the Spurion Kinetic Terms	112
B	Addendum to Chapter 5	115
B.1	Fermionic Action and RS Spin Connection Term	115
B.2	Bulk Equations of Motion	117
B.3	KK Decomposition — Bulk Profiles of Wave Functions	119
B.4	Zero Mode Profiles	121
B.5	Derivation of the Rayleigh–Schrödinger Formulae	122
B.6	Analytic Diagonalisation of the Hierarchical Matrix	128
B.7	Basis Transformation of the Effective Hamiltonian $\mathcal{H}_{\text{eff}}^{\Delta S=2}$	132
B.8	Effective 4D Feynman Rules: Two Examples	133
	Bibliography	138

Chapter 1

Introduction and Motivation

Following Occam’s razor “*entia non sunt multiplicanda praeter necessitatem*” — the principle that “entities must not be multiplied beyond necessity” and the conclusion thereof, that the simplest explanation or strategy tends to be the best one — elementary particle physicists created the Standard Model (SM) which describes in a satisfactory way the observed strong and electroweak (EW) interactions of the smallest constituents of matter — the elementary particles. The beauty and simplicity of the SM lies in the fact that the SM can be formulated as a consistent quantum field theory based on the *gauge group* $SU(3)_c \times SU(2)_L \times U(1)_Y$.

However, the SM becomes more sophisticated when one tries to explain the existence of gauge boson and fermion masses. A scalar Higgs boson with non-trivial transformation behaviour under the electroweak gauge group is introduced, whose non-vanishing *vacuum expectation value* (VEV) breaks the symmetry spontaneously down to a residual $U(1)_Q$ symmetry. This so-called *Higgs mechanism* gives rise to gauge boson masses corresponding to the broken symmetry generators and, being a doublet under $SU(2)_L$, allows in addition for gauge-invariant fermion mass terms via *chiral Yukawa couplings*. In order to obtain the physical masses belonging to the single fermion flavours, the Yukawa coupling matrices are diagonalised by means of biunitary transformations. The only observable relic of these rotation matrices in the SM takes the form of a single unitary matrix known as *Cabbibo–Kobayashi–Maskawa matrix* (CKM) matrix, which can be parameterised by three real angles and one CP-violating phase.

Despite its success in providing particle masses, the Higgs sector is the origin of two outstanding problems within the SM. The fermion mixings and masses have to be put in by hand and, being hierarchical and not at all of $\mathcal{O}(1)$, they are neither in line with the naturalness principle [1] nor with the aesthetic philosophy of Occam — causing the so-called *flavour puzzle*. The second problem, the *gauge hierarchy problem*, is also related to a particle’s mass, namely the Higgs mass itself. It receives radiative mass contributions from the SM particles that are quadratically divergent such that the Higgs mass is sensitive to the ultimate cutoff scale of the theory. Since the SM does not include a theory of gravity, the highest

possible scale to which the SM is valid is the Planck scale. But assuming that the SM is valid up to the Planck scale would relate the bare Higgs mass of order of the electroweak scale to the Planck scale unless the electroweak scale is saved through an “unnatural” fine-tuning of the bare mass and the quantum-loop corrections. Therefore, it is widely believed that there is some kind of *new physics* (NP) phenomenon becoming relevant already near the EW scale, and in consequence the SM should be interpreted as a low-energy *effective field theory* (EFT).

Interestingly, the Higgs boson is at the same time the only unobserved particle of the SM. Thus, there is much activity in creating NP models concerning this segment of the SM to overcome the above mentioned problems. This has led to the development of many kinds of Higgs models like models with an extended Higgs sector as in *two Higgs doublet models* [2, 3], where an additional fundamental Higgs doublet is added to the particle spectrum. Another possibility is to consider the Higgs as a pseudo-Goldstone boson arising in a spontaneous breakdown of an approximate global symmetry [4, 5] — for example in *little Higgs models* [6–10]. One also tries to live without a fundamental Higgs particle and breaks the EW symmetry dynamically through the VEV of a scalar condensate resulting from a strongly interacting sector as for instance in *technicolour models* [11–17]. An interesting variation of the strong symmetry breaking paradigm that interpolates between simple technicolour theories and the standard Higgs model come to prominence: in *composite Higgs models* a light Higgs boson could emerge as the bound state of a strongly interacting sector. Moreover, if the Higgs arises as a pseudo-Goldstone boson of an enlarged global symmetry of the strong dynamics, it can be naturally light [18–22].

However, as the SM is in extremely good accordance with the high-precision measurements accessible at particle accelerators, the elaboration of NP models is limited. In this context electroweak precision tests (EWPTs) [23–25] should be mentioned and high-precision tests of the CKM matrix coming in particular from B meson and kaon observables (for recent overviews, see for instance [26–29]). Since the SM agrees very well with the flavour observables, NP at the TeV scale requires a highly non-generic flavour sector in order not to be in conflict with present data on rare and CP-violating K and B decays. One efficient possibility to constrain the flavour sector is to impose the concept of *minimal flavour violation* (MFV) [30–34] on the NP models, where the sources of flavour and CP violation are entirely described by the CKM matrix. Restoring the approximate maximal global $U(3)^3$ *flavour symmetry* (FS) present in the SM via promoting the Yukawa matrices to auxiliary spurion fields, the low-energy EFT has to formally respect the flavour symmetry which completely determines its flavour structure.

Flavour symmetries are widely used in various aspects of particle physics. For instance, the QCD Lagrangian has an accidental global flavour symmetry in the limit of vanishing quark masses which is known as the chiral limit. It should be approximately realised for the quarks (m_u, m_d, m_s) since they are much lighter than the QCD scale. However, while hadrons in $SU(3)_V$ representations are observed in the hadronic spectrum according to the

eightfold way [35, 36], the corresponding degenerate multiplets with opposite parity do not exist. Moreover, the fact that the octet of pseudoscalar particles (π , K , η) is very light compared to the other multiplets, suggests to consider them as the pseudo-Goldstone bosons arising from the spontaneous symmetry breakdown $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$. The mass gap in the hadronic spectrum can be used to build an effective field theory which contains only the Goldstone bosons as its dynamical degrees of freedom. Combining the effective theory with the non-linearly realised QCD flavour symmetry [37, 38], leads to a powerful tool of studying the low-energy interactions of the pseudoscalar-meson octet called *chiral perturbation theory* [39–45].

Furthermore, flavour or *family symmetries*, which act horizontally across the three SM generations, provide one path to explain the flavour puzzle. The hierarchical structure of the Yukawa couplings is generated after this new symmetry is spontaneously broken by the VEVs of some set of scalar *flavon fields*, which transform non-trivially under the FS but are singlets with respect to the SM gauge group. In *supersymmetric* (SUSY) theories [46–50], which are perhaps the most famous candidates for solving the gauge hierarchy problem, they can further be used to align the sfermion mass matrices with the mass matrices of their fermion superpartners in order to ameliorate the problem of large flavour violation [51–53]. Since SUSY enables gauge coupling unification, it goes mostly hand in hand with *grand unified theories* (GUTs), where the SM gauge group is embedded into a larger gauge group with one universal gauge coupling constant. In particular $SO(10)$ models have become attractive, since the 16 fermions of one generation, including right-handed singlet neutrinos, can be embedded into a single spinor representation. Thereby, family symmetries offer an elegant solution to point out the special role of the third generation with respect to the lighter two generations [54, 55]. SUSY GUT models with implemented FS, such as the one proposed by Dermisek and Raby [56–58], allow to give a satisfactory description of all quark and lepton masses as well as of the lepton mixing matrix or *Pontecorvo–Maki–Nakagawa–Sakata* (PMNS) matrix [59, 60] and CKM matrix. However, at the same time, it can be challenging to fulfil all constraints coming from *flavour-changing neutral currents* (FCNCs) simultaneously [61, 62]. Altogether, many different models incorporating some sort of FS to explain the fermion masses and mixings exist in the literature, in particular in the context of supersymmetric and/or unified scenarios [63]. While [64–68] also use a global abelian flavour symmetry $U(1)_F$ as in the original *Froggatt–Nielsen* setup [69], there exist models using a non-abelian $U(2)_F$ flavour symmetry [51, 52, 70–75] and $SU(3)_F$ flavour symmetries [76–81], but also models with discrete flavour symmetry groups [82–87].

Inspired by the MFV ansatz, one can promote the Yukawa coupling matrices to dynamical scalar spurion fields [88] that transform non-trivially under the maximal SM quark FS, present in the limit of vanishing Yukawa couplings, and are singlets with respect to the SM gauge group. In the following we will refer to this approach as the *dynamical minimal flavour*

violation (dMFV) model in comparison to the original MFV, where the Yukawa spurion fields are considered as auxiliary fields.

Due to the canonical mass dimension of the spurions, the Yukawa coupling terms have to be interpreted as dimension-5 operators suppressed by a high-scale Λ , which indicates the FS breaking scale of the EFT. The flavour puzzle can be traced back by giving appropriate VEVs to the scalar spurion fields that break the FS spontaneously in a sequential fashion. Goldstone bosons for every broken symmetry generator are introduced and the remaining spurions transform under the residual unbroken part of the non-linearly realised FS [88]. The special role of the top quark Yukawa coupling, being of $\mathcal{O}(1)$, is taken into account by giving the associated spurion a VEV of $\mathcal{O}(\Lambda)$. Thus it is effectively created by a dimension-4 operator whereas the other fermion Yukawa couplings are suppressed by the ratios of two distinct scales.

There is another reason for the dMFV model to be necessarily understood as an EFT. Introducing local flavour symmetries of chiral nature while keeping the SM fermion content, one encounters chiral gauge anomalies. Within an effective theory framework one can formulate a consistent and at least formally gauge-invariant theory, arguing that the existing underlying fundamental is anomaly free [89].

In this work we leave the $U(1)$ groups of the maximal flavour group as global symmetries [90]. First, the global $U(1)$ factor corresponding to baryon number conservation is an accidental global symmetry of the SM since it is respected by the Yukawa interactions. Second, the global $U(1)_{u_R} \times U(1)_{d_R}$ symmetry can be used to resolve the strong CP problem [91, 92] by a modified *Peccei–Quinn mechanism* [93–96]. The almost massless Goldstone bosons can then be identified as axions that are very weakly coupled to the SM fermions in the context of *invisible-axion* scenarios [97, 98]. Gauging the $SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R}$ flavour symmetry allows us to interpret the Goldstone bosons, arising after its breaking, as the longitudinal modes of the massive gauge bosons in the unitary gauge. Integrating out the heavy scalar fields and heavy gauge fields at tree level, generally gives rise to FCNCs. According to our setup, the coefficients of the effective 4-quark operators follow the MFV principle. While the dMFV model proposes an explanation for the flavour puzzle, it does not address the gauge hierarchy problem.

An appealing solution to elucidate both of the two mentioned outstanding questions within the SM is given in *Randall–Sundrum* (RS) models [99]. Augmenting the $3 + 1$ space-time coordinates of daily life by an additional warped spatial coordinate, they are also known as *warped extra dimension* (WED) models. The non-factorisable metric of an *anti-de-Sitter space* (AdS_5) implies an exponential *warp factor*, which relates the mass scale of the fundamental five-dimensional (5D) theory to the physical four-dimensional (4D) mass scale. Localising the Higgs boson on or near the IR brane, the warp factor mediates quite naturally between the Planck scale and the EW scale and thus supplies a geometrical solution to the gauge hierarchy problem [99].

Since we do not have any indication for the existence of extra dimensions (EDIMs) up to now, we consider the extra dimension to be *compactified* via an orbifolding procedure in order to make it finite and small. Denoting the single extra dimension by the coordinate y , the fundamental domain can be represented by an interval $y \in [0, L]$. Its boundaries correspond to 4D subspaces or branes called *ultraviolet* (UV) brane ($y = 0$) and *infrared* (IR) brane ($y = L$) respectively, while the 5D space in between is referred to as the *bulk*. Introducing Dirichlet or Neumann boundary conditions (BCs) for the fields propagating in the 5D bulk, chirality can be implemented in the original non-chiral 5D theory. In addition, compactification allows for an expansion of the 5D fields into Fourier modes or *Kaluza–Klein* (KK) modes [100,101]. This *KK decomposition* expresses the 5D fields as a sum of products of functions depending on the 4D coordinate and on the fifth coordinate, and thus enables us to derive an effective 4D theory from the 5D fundamental theory by performing an integration over the extra dimension. To first approximation, the zero KK modes correspond to the SM fields while the higher KK modes represent new heavy fields with masses depending on the compactification scale.

New ingredients in the 5D theory are the 5D Dirac bulk mass matrices whose eigenvalues determine the localisation of the fermion zero mode profiles along the extra dimension. Since the SM quark masses arise from overlap integrals of the zero quark profiles with the Higgs boson shape function after electroweak symmetry breaking (EWSB), they can account for the observed hierarchy in the SM quark masses and mixing angles by inducing different quark localisations [102,103] (*split fermion mechanism* [104,105]). Furthermore, assuming anarchic 5D Yukawa couplings with $\mathcal{O}(1)$ entries, the flavour puzzle can be solved without fine-tuning in the fundamental parameters as the quark profiles depend exponentially on the slightly different but $\mathcal{O}(1)$ bulk mass parameters.

Despite this appealing picture, the 5D fermion bulk masses reintroduce a number of flavour violating parameters that renders the WED models far from being minimal flavour violating. The fermion splitting causes non-universal fermion couplings to the KK gauge bosons and hence produces new effective operators, contributing to FCNCs already at tree level. While the built-in RS–GIM mechanism [106,107] can at least curtail the excess of these FCNCs, the new chirality flip operator \mathcal{Q}_2^{LR} gives nevertheless sizeable contributions via KK gluon exchange to the CP-violating parameter ε_K , the parameter which measures indirect CP violation in the $K^0 - \bar{K}^0$ mixing. Phenomenological analyses show that it is much more challenging for WED models to fulfil the ε_K bound (RS flavour problem) than the experimental constraints coming from EWPTs [108–115], in particular from the oblique parameter T and the well-measured $Zb_L\bar{b}_L$ coupling [116,117]. In this work we consider a RS model, in which the latter contributions are protected by a simultaneous enlargement of the bulk symmetry group to

$$G_{\text{bulk}} = SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR} \quad (1.1)$$

and of the corresponding fermion symmetry multiplets (see also [118]). Since the protection originates from an unbroken *custodial symmetry* [108,119] in the Higgs sector, the model

is denoted as *custodially protected Randall–Sundrum* (RSc) model. We work out the basic features of the RSc model [120] in order to obtain its Feynman rules, which were used in two detailed phenomenological analyses of $\Delta F = 2$ FCNC processes in the quark sector [121] and those rare K and B decays in which NP contributions enter at tree level [122]. Throughout our analysis we concentrate on the quark sector and truncate the KK expansion already after the first KK mode.

The remainder of the thesis is organised as follows. After having given a brief review of the general aspects of gauge theories in Chapter 2, we adopt the methods in displaying the gauge group and particle content of the SM in Chapter 3. In explaining the flavour puzzle arising in the SM, we promote the Yukawa matrices to dynamical fields in Chapter 4 and design an effective minimal flavour violating theory which includes a partly gauged flavour symmetry in addition to the SM gauge group. Breaking the latter by the development of Yukawa spurion VEVs, renders the corresponding scalars and gauge bosons massive. We integrate out the heavy degrees of freedom and show that the coefficients of the effective operators are in accordance with the MFV assumption. In Chapter 5 we examine the breaking of the RSc gauge group and analytically diagonalise the corresponding gauge boson mass matrices. Subsequently, we introduce the specific fermion content and construct the fermion mass matrices after EWSB. Working out the gauge-fermion couplings, we comment on the two different effects of flavour violation that are characteristic in this model. Finally, in Chapter 6 we give a short summary and outlook. The calculational details to Chapter 4 and Chapter 5 can be found in the two addenda, e.g. the diagonalisation procedure which includes a derivation of the formulae of the Rayleigh–Schrödinger algorithm for the non-degenerate case.

Chapter 2

General Aspects of Gauge Theories

This chapter is dedicated to the general concepts of abelian and non-abelian gauge theories. Thereby, we emphasise the necessity of adding gauge-fixing terms in non-abelian gauge theories in order to remove redundant degrees of freedom. We introduce the enlarged *Becchi–Rouet–Stora–Tyutin* (BRST) transformation under which the new gauge-fixed Lagrangian is invariant. With regard to the arising topic of chiral gauge anomalies in the course of Chapter 4, we introduce the formalism of chiral fields and give the relevant formulae.

2.1 Abelian Gauge Symmetry

The Lagrangian of a free fermion field, which is characterised by the Dirac spinor $\psi(x)$,

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x), \quad (2.1)$$

is invariant under the global $U(1)$ symmetry with the corresponding transformation matrix $U(\theta) = e^{i\theta}$

$$\psi'(x) = U\psi(x) \quad \text{and} \quad \bar{\psi}'(x) = \bar{\psi}(x)U^\dagger. \quad (2.2)$$

In promoting the global symmetry to a local one, θ is replaced by $\theta(x)$ and the transformation matrix $U(\theta(x))$ depends on x . Obviously, the derivative in (2.1) is the reason why \mathcal{L}_0 spoils gauge invariance, as it produces an extra term proportional to $\partial_\mu\theta(x)$. However, gauge invariance can be restored, if one finds a *gauge-covariant derivative* D_μ , such that $D_\mu\psi$ transforms as ψ itself, i.e.

$$(D_\mu\psi(x))' \stackrel{!}{=} UD_\mu\psi(x). \quad (2.3)$$

This can be achieved by a *minimal coupling* with the $U(1)$ coupling constant g of the spinor to a new vector field $A_\mu(x)$, the so-called *gauge field*,

$$\begin{aligned} D_\mu\psi(x) &= (\partial_\mu - igA_\mu(x))\psi(x) \\ A'_\mu(x) &= A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x). \end{aligned} \quad (2.4)$$

In order to make the gauge field dynamical, we also add the following gauge-invariant and renormalisable kinetic term to the Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (2.5)$$

where the abelian field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.6)$$

Putting these two parts \mathcal{L}_0 and \mathcal{L}_A together, the Lagrangian for a $U(1)$ gauge-invariant Lagrangian can be summarised as

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu - ig A_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (2.7)$$

In 1954, *Yang* and *Mills* extended the gauge principle to non-abelian symmetry groups [123], which will be discussed in the following.

2.2 Non-Abelian Gauge Symmetry

The main features of a non-abelian gauge symmetry are encoded in its non-trivial generators. In the case of the special unitary group $SU(N)$, these generators can be represented by $n^2 - 1$ traceless and hermitian matrices T^a , which fulfil the group algebra

$$[T^a, T^b] = if^{abc} T^c, \quad (2.8)$$

with f^{abc} denoting the totally antisymmetric structure constants. The normalisation condition, which involves the fundamental representation matrices, is specified by

$$\text{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}. \quad (2.9)$$

In analogy to the abelian case, the covariant derivative

$$D_\mu\psi(x) = (\partial_\mu - igA_\mu^a T^a)\psi(x) \quad (2.10)$$

ensures the gauge invariance of the kinetic fermion term. However, in contrast to the abelian case, the field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (2.11)$$

contains a non-linear term, which is responsible for the self-interactions of the gauge fields.

While the spinor fields transform according to the fundamental representation,

$$\begin{aligned} \psi'(x) &= U(\theta(x))\psi(x) = e^{i\theta^a(x)T^a}\psi(x) \simeq (1 + i\theta^a(x)T^a)\psi(x), \\ \bar{\psi}'(x) &= \psi(x)U^\dagger(\theta(x)) = \bar{\psi}(x)e^{-i\theta^a(x)T^a} \simeq \bar{\psi}(x)(1 - i\theta^a(x)T^a), \end{aligned}$$

the gauge fields show the transformation behaviour of the adjoint representation

$$\begin{aligned} T^a A_\mu^a(x) &= U(\theta(x)) T^a A_\mu^a(x) U^\dagger(\theta(x)) + \frac{i}{g} U(\theta(x)) (\partial_\mu U^\dagger(\theta(x))) \\ &\simeq A_\mu^a(x) T^a + \frac{1}{g} (\partial_\mu \delta^{ac} + g f^{abc} A_\mu^b(x)) \theta^c(x) T^a = (A_\mu^a(x) + \frac{1}{g} D_\mu^{ac} \theta^c(x)) T^a. \end{aligned} \quad (2.12)$$

In the above equations, we also added the infinitesimal versions of the transformations. Moreover, we introduce the abbreviation D_μ^{ac} in (2.12) for the covariant derivative, which acts on a field transforming in the adjoint representation. With the above ingredients, the complete Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\mathcal{D} - m)\psi \quad (2.13)$$

is by construction invariant under the considered non-abelian gauge transformation.

2.3 Gauge-Fixing Terms and BRST Transformation

In quantising gauge theories, one has to get rid of the redundant degrees of freedom, which are present due to the condition of gauge invariance. To do so it is convenient to impose the generalised Lorentz gauge-fixing condition within the path-integral quantisation formalism

$$G^a[A] = \partial^\mu A_\mu^a(x) + \omega^a(x), \quad (2.14)$$

where $\omega^a(x)$ is an arbitrary scalar function which is independent of the gauge field. Following *Faddeev* and *Popov* (FP) [124], we introduce this constraint by incorporating the identity

$$1 = \int \mathcal{D}\theta(x) \delta(G^a[A'] - \omega^a(x)) \text{Det} \left(i \frac{\delta G^a[A']}{\delta \theta} \right) \quad (2.15)$$

into the generating functional

$$Z = \int \mathcal{D}A_\mu \text{Det} \left(i \frac{\delta G^a[A']}{\delta \theta} \right) \delta(G^a[A'] - \omega^a(x)) e^{iS[A]}. \quad (2.16)$$

In (2.16), the gauge-transformed gauge field A' is given by equation (2.4) for abelian gauge theories, and by (2.12) for non-abelian gauge theories. By inserting a constant proportional to $\int \mathcal{D}\omega \exp[-i/(2\xi) \int d^4x \omega^2(x)]$, the generating functional (2.16) changes only by an immaterial normalisation factor, and effectively adds the gauge-fixing term $\mathcal{L}_{\text{gfix}} = -1/(2\xi) (G^a[A])^2$ after having integrated over $\omega^a(x)$.

The functional determinant can be represented as a functional integral over anticommuting scalar fields belonging to the adjoint representation of the gauge group

$$\text{Det} \left(i \frac{\delta G^a[A']}{\delta \theta} \right) = \int \mathcal{D}v \mathcal{D}\bar{v} \exp \left[-i \int d^4x \bar{v}^a \left(\frac{\delta G^a[A']}{\delta \theta^b} \right) v^b \right], \quad (2.17)$$

and contributes the term $\mathcal{L}_{\text{FP}} = -\bar{v}^a (\delta G^a[A']/\delta \theta^b) v^b$ to the Lagrangian. As the above reformulation was proposed by *Faddeev* and *Popov*, the new fields $v = v^a(x) T^a$ ($\bar{v} = \bar{v}^a(x) T^a$)

are called Faddeev–Popov (*anti-*)ghosts. In summary, the restriction of the functional integral to “physically different” gauge field configurations, i.e. those which are not connected by gauge transformations, effectively adds a FP ghost term and a gauge-fixing contribution to the Lagrangian:

$$\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{1}{2\xi} G^a[A]^2 - \bar{v}^a \left(\frac{\delta G^a[A']}{\delta \theta^b} \right) v^b = \mathcal{L} + \mathcal{L}_{\text{gfix}} + \mathcal{L}_{\text{FP}}. \quad (2.18)$$

For example, the effective Lagrangian for a non-abelian gauge theory with the Lorentz gauge condition $G^a = \partial^\mu A_\mu^a$ is explicitly given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 + \frac{1}{g} \bar{v}^a (-\partial^\mu D_\mu^{ac}) v^c. \quad (2.19)$$

Obviously, gauge invariance of the total Lagrangian is lost due to the presence of the gauge-fixing terms. However, *Becchi, Rouet, Stora and Tyutin* found a generalised gauge transformation of the gauge-fixed Lagrangian — the BRST transformation [125,126] — which involves also the ghost fields. For this purpose a *ghost number* (GN) is assigned to each field. While the (anti-)ghost fields carry a GN of +1 (−1), all gauge and fermion fields have a GN of 0. Additionally, a BRST operator δ_v is defined, which acts on the fields with zero GN like the usual infinitesimal gauge transformation with gauge parameter v while it acts on the fields with a non-zero GN such that δ_v is nilpotent, i.e. $\delta_v^2 = 0$. In summary, the BRST transformations read

$$\begin{aligned} \delta_v A_\mu &= \frac{1}{g} D_\mu v = \frac{1}{g} \partial_\mu v - i[A_\mu, v], \\ \delta_v v &= \frac{1}{2} i[v, v] = iv^2, \\ \delta_v \psi &= iv\psi, \end{aligned} \quad (2.20)$$

where the BRST operator δ_v increases the GN by one unit.¹

One can easily verify that the above definitions are in accordance with the nilpotency of the BRST operator,

$$\begin{aligned} \delta_v^2 A_\mu &= \delta_v \left(\frac{1}{g} \partial_\mu v - i[A_\mu, v] \right) = \frac{i}{g} \partial_\mu v^2 - \left[\frac{i}{g} \partial_\mu v, v \right] - [A_\mu, v], v + [A_\mu, v^2] \\ &= -(A_\mu v^2 + v A_\mu v - v A_\mu v - v^2 A_\mu) + A_\mu v^2 - v^2 A_\mu = 0, \\ \delta_v^2 v &= \delta_v(iv^2) = i(iv^2)v - iv(iv^2) = 0, \\ \delta_v^2 \psi &= \delta_v(iv\psi) = i(iv^2)\psi - iv(iv\psi) = 0, \end{aligned} \quad (2.21)$$

where we used the property of the BRST operator to anticommute with the ghost field. In Appendix A.3, the formulae analogous to (2.20), using the language of differential forms, are given.

¹In analogy one can construct an anti-BRST operator $\delta_{\bar{v}}$ which decreases the GN by one unit.

2.4 Chiral Fields and Chiral Gauge Symmetry

In this section, we consider fermion couplings both to vector gauge fields $\mathcal{V}_\mu = \mathcal{V}_\mu^a T^a$ and to axial-vector gauge fields $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$. Focussing on chiral non-abelian gauge theories, the covariant derivative (2.10) of the Lagrangian (2.13) can then be generalised to

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu - i \mathcal{V}_\mu - i \mathcal{A}_\mu \gamma_5) \psi, \quad (2.22)$$

where the coupling constants have been set to unity.

To simplify the notation in the following, we will use the antihermitian group generators \tilde{T}^a , which are related to the hermitian generators T^a via the rescaling

$$\tilde{T}^a = -iT^a. \quad (2.23)$$

The group algebra and normalisation condition with respect to the new basis are then modified to

$$[\tilde{T}^a, \tilde{T}^b] = f^{abc} \tilde{T}^c \quad \text{and} \quad \text{Tr} [\tilde{T}^a \tilde{T}^b] = -\frac{1}{2} \delta^{ab}. \quad (2.24)$$

Defining $\tilde{\mathcal{V}}_\mu = \mathcal{V}_\mu^a \tilde{T}^a$ and $\tilde{\mathcal{A}}_\mu = \mathcal{A}_\mu^a \tilde{T}^a$, the factors i disappear in the rewritten Lagrangian (2.22)

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu + \tilde{\mathcal{V}}_\mu + \tilde{\mathcal{A}}_\mu \gamma_5) \psi. \quad (2.25)$$

As the gauge bosons couple chirally to the Dirac fermions, we introduce the left-handed (L -) and right-handed (R -) projection operators $P_{L,R}$,

$$P_{L,R} = \frac{1 \mp \gamma_5}{2}, \quad \gamma_5 P_{L,R} = \mp P_{L,R}, \quad (2.26)$$

with the usual relations of idempotence, orthogonality and completeness:

$$P_{L,R}^2 = P_{L,R}, \quad P_L P_R = P_R P_L = 0, \quad P_L + P_R = 1. \quad (2.27)$$

The chiral projectors allow to define chiral spinor fields according to

$$\begin{aligned} \psi &= P_L \psi + P_R \psi = \psi_L + \psi_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ i\sigma^2 \xi^* \end{pmatrix}, \\ \bar{\psi} &= \psi^\dagger \gamma^0 = (\psi_L + \psi_R)^\dagger \gamma^0 = \bar{\psi}_L + \bar{\psi}_R = \begin{pmatrix} 0 & \chi^\dagger \end{pmatrix} + \begin{pmatrix} -i\xi^T \sigma^2 & 0 \end{pmatrix}. \end{aligned} \quad (2.28)$$

Thereby, the two Weyl spinors (χ, ξ) correspond to two-dimensional building blocks of the four-component Dirac spinor $\psi = (\chi, i\sigma^2 \xi^*)^T$. Using the projector properties (2.27) and the usual commutator rules for gamma matrices, the Lagrangian (2.22) can finally be reformulated as

$$\mathcal{L}_{L,R} = \bar{\psi} i \left(\not{\partial} + \tilde{\mathcal{V}} + \tilde{\mathcal{A}} \gamma_5 \right) (P_L^2 + P_R^2) \psi = \bar{\psi}_L i (\not{\partial} + \tilde{\mathcal{A}}^L) \psi_L + \bar{\psi}_R i (\not{\partial} + \tilde{\mathcal{A}}^R) \psi_R, \quad (2.29)$$

where we used the following definitions of the R - and L -gauge fields

$$\begin{aligned}\tilde{A}_\mu^R &= \tilde{\mathcal{V}}_\mu + \tilde{\mathcal{A}}_\mu, & \tilde{A}_\mu^L &= \tilde{\mathcal{V}}_\mu - \tilde{\mathcal{A}}_\mu, \\ \tilde{\mathcal{V}}_\mu &= \frac{1}{2}(\tilde{A}_\mu^L + \tilde{A}_\mu^R), & \tilde{\mathcal{A}}_\mu &= \frac{1}{2}(\tilde{A}_\mu^R - \tilde{A}_\mu^L).\end{aligned}\quad (2.30)$$

Having expressed the Lagrangian in terms of chiral fields (2.29), its invariance under a local $SU(N)_L \times SU(N)_R$ symmetry transformation with associated transformation matrices $U(\theta_{L,R}(x)) = \exp[i\theta_{L,R}^a(x)T^a] = \exp[-\theta_{L,R}^a(x)\tilde{T}^a] = U(-\tilde{\theta}_{L,R}(x))$ can easily be observed.

2.5 Chiral Gauge Anomaly

Unless a specific fermion content is chosen, the chiral gauge symmetry $SU(N)_L \times SU(N)_R$ at the level of the classical Lagrangian is spoiled by quantum effects which leads to the so-called *gauge anomaly*. Consequently, the classical conservation laws for the non-abelian L - and R -currents,

$$D_\mu^{ab}[\tilde{A}_\mu^{L,R}]j_{L,R}^{\mu b} = D_\mu^{ab}[\tilde{A}_\mu^{L,R}](\bar{\psi}_{L,R}\gamma^\mu\tilde{T}^b\psi_{L,R}) = 0, \quad (2.31)$$

are no longer valid, but have to be replaced by anomalous conservation laws as given below.

The chiral gauge anomaly can be derived for example by using Fujikawa's path integral method [127, 128] or by the algebraic approach of Zumino and Stora [129, 130]. A comprehensive discussion of anomalies in quantum field theories can also be found in [131]. The result of the anomaly contribution to the L - and R -currents is calculated in [132, 133] to be

$$D_\mu^{ab}[\tilde{A}_\mu^{L,R}]j_{L,R}^{\mu b} = G^a[\tilde{A}_\mu^{L,R}(x)] = \mp \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}[\tilde{T}^a\partial_\mu(\tilde{A}_\nu^{L,R}\partial_\rho\tilde{A}_\sigma^{L,R} + \frac{1}{2}\tilde{A}_\nu^{L,R}\tilde{A}_\rho^{L,R}\tilde{A}_\sigma^{L,R})], \quad (2.32)$$

where $\epsilon^{0123} = 1$.

Since the fermion loop contributions to the anomaly arise from the *effective action* $\Gamma[\tilde{A}_\mu^{L,R}(x)]$, which is defined by the sourceless generating functional for one-particle irreducible Green functions

$$Z[\tilde{A}] = e^{i\Gamma[\tilde{A}^{L,R}]} = \int d\bar{\psi}d\psi \exp\left[i\int d^4x \mathcal{L}(\psi, \bar{\psi}, \tilde{A})\right] = \int d\bar{\psi}d\psi \exp\left[i\int d^4x \bar{\psi}i(\not{\partial} + \tilde{A})\psi\right], \quad (2.33)$$

the anomaly $G^a[\tilde{A}_\mu^{L,R}(x)]$ originates from the gauge-transformed effective action

$$\delta_v\Gamma[\tilde{A}_\mu^{L,R}] = -\int d^4x v^a(x)G^a[\tilde{A}_\mu^{L,R}(x)] = -G(v, \tilde{A}^{L,R}). \quad (2.34)$$

Applying the BRST operator to both sides of the above equation, yields the *Wess–Zumino consistency condition* (WZCC) [134]

$$\delta_v G(v, \tilde{A}^{L,R}) = -\delta_v(\delta_v\Gamma[\tilde{A}_\mu^{L,R}(x)]) = 0, \quad (2.35)$$

which is a direct consequence of the nilpotency of the BRST operator.

The anomaly given in (2.32) represents a non-trivial solution to the WZCC, i.e. it cannot be written as a gauge variation of some local function in the basic fields, for which reason it is called *consistent form*. The freedom of adding trivial anomaly solutions reflects the ambiguity in the regularisation prescriptions of the UV-divergent portion of the fermion loops [135, 136].

The significance of the WZCC results from the non-linearity of the BRST operator. Knowing the first term of the anomaly (2.32), which can be calculated from the triangle diagrams, the WZCC completely determines the second term of the anomaly as explicitly demonstrated in Appendix A.3.

Anomaly Cancellation

In vector-like gauge theories with vanishing axial-vector gauge field $\tilde{\mathcal{A}}$, the fermions couple symmetrically to the chiral gauge fields $\tilde{A}_\mu^L = \tilde{A}_\mu^R = \tilde{A}_\mu$ such that the L -gauge anomalies exactly cancel the R -gauge ones.

Anomaly cancellation may also occur by a specific choice of the fermion content as the trace in the anomaly contribution (2.32) has to be taken over the fermion representations. In the search of an anomaly it is enough to consider only the first part of the anomaly, because if the triangle result vanishes, the higher-loop contributions are automatically absent [137]. Thus the trace

$$\text{Tr} [\tilde{T}^a \{ \tilde{T}^b, \tilde{T}^c \}] \quad (2.36)$$

serves as an indicator to explore the bare existence of an anomaly. The anticommutator in the above trace is a result of the summation of the two triangle diagrams in which the fermions circle in opposite direction.

Another possibility to cancel an anomaly is to add local counterterms with additional fields which transform under the gauge variation such that the anomaly contribution is cancelled. This method will be of importance in Subsection 4.6.1.

Chapter 3

Brief Review of the Standard Model

In the present chapter we recapitulate the basic concepts of the Standard Model (SM), which successfully describes the electroweak and strong interactions of quarks and leptons at energies up to a few hundred GeV. As it is in nearly perfect agreement with all existing experimental data at the moment, every new physics (NP) model should feature a structure similar to the SM in the low-energy limit.

We use the knowledge about abelian and non-abelian gauge theories of the last chapter to construct the SM quantum field theory, which is based on the local gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. While the colour group $SU(3)_c$ remains unbroken in the SM, the non-vanishing Higgs VEV breaks the electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ spontaneously down to the abelian subgroup $U(1)_Q$. Simultaneously, the gauge bosons of the broken gauge symmetry acquire masses through the Higgs mechanism, and fermion masses are generated by their Yukawa couplings to the Higgs boson.

3.1 Quantum Chromodynamics

The $SU(3)_c$ gauge theory is able to describe the strong interactions between the quark fields and the gauge fields called gluons [138–142]. Each quark has three different *colour* degrees of freedom and is represented by a $SU(3)_c$ triplet. The Greek translation of colour — “chroma” — of the internal quantum number gave the theory its name *Quantum Chromodynamics* (QCD).

It is straightforward to adapt the generic non-abelian Lagrangian (2.13) as well as the FP ghost and gauge-fixing terms (2.19) to the case of QCD,

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\text{Tr} [G_{\mu\nu}G^{\mu\nu}] + \bar{\psi} i\not{D}\psi + \mathcal{L}_{\text{gfix}} + \mathcal{L}_{\text{FP}}. \quad (3.1)$$

In the QCD Lagrangian (3.1), $G_{\mu\nu}$ stands for $G_{\mu\nu}^a T^a$ with the gluon field strength tensor

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c, \quad (3.2)$$

which contains the strong coupling constant g_s . The quadratic terms in the field strength tensor lead to gluon self-interactions that are responsible for the *asymptotic freedom* in QCD [143–145].

With the eight *Gell-Mann matrices* $\lambda^a/2$, representing the generators of the fundamental representation, the covariant derivative, which contains interaction terms between gluons and quarks, reads

$$D_\mu \psi = \left(\partial_\mu - ig_s G_\mu^a \frac{\lambda^a}{2} \right) \psi. \quad (3.3)$$

We will argue later in Subsection 4.10 that the QCD Lagrangian should be augmented by a CP-violating term which we also refer to as the θ -term,

$$\mathcal{L}_{\text{CP}} = \frac{\theta g_s^2}{16\pi^2} \text{Tr} [G_{\mu\nu} \tilde{G}^{\mu\nu}] = \frac{\theta g_s^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu}. \quad (3.4)$$

The kinetic term of the gluon in (3.4) is reformulated with the help of the normalisation condition of the colour group generators T^a , and the dual field strength tensor $\tilde{G}^{\mu\nu}$ is defined by

$$\tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}. \quad (3.5)$$

Since the θ -term is neither forbidden by gauge invariance nor by renormalisability, it should be mentioned already here. At first glance one may argue that the term has no physical impact, as it can be written as a total derivative. However, it turns out that this statement is wrong due to the non-trivial topology of the gauge field sector (see Section 4.10).

3.2 Electroweak Sector

The standard theory of the electroweak interactions [106, 146, 147] comprises the gauge group $SU(2)_L \times U(1)_Y$. To take into account the experimental observation that only the left-handed components of the quark and lepton fields couple to the W boson, ψ_L and ψ_R are assigned to different representations of the chiral $SU(2)_L$ gauge group. While the left-handed fields transform as doublets under $SU(2)_L$ and couple to the three $SU(2)_L$ gauge bosons $W_{L,\mu}^a$, the right-handed ones are singlets. Both types of fields have a non-trivial hypercharge quantum number Y , and thus both couple to the $U(1)_Y$ gauge boson B_μ .

In summary, the complete $SU(2)_L \times U(1)_Y$ invariant Lagrangian takes the form

$$\begin{aligned} \mathcal{L}_{\text{EW}} = & \sum_{\psi_L^i} \bar{\psi}_L^i \gamma^\mu i \left(\partial_\mu - ig_L \frac{\sigma^a}{2} W_{L,\mu}^a - ig_Y \frac{Y}{2} B_\mu \right) \psi_L^i + \sum_{\psi_R^i} \bar{\psi}_R^i \gamma^\mu i \left(\partial_\mu - ig_Y \frac{Y}{2} B_\mu \right) \psi_R^i \\ & - \frac{1}{4} L_{\mu\nu}^a L^{a,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \end{aligned} \quad (3.6)$$

with the corresponding field strength tensors

$$\begin{aligned} L_{\mu\nu}^a &= \partial_\mu W_{L,\nu}^a - \partial_\nu W_{L,\mu}^a + g_L \epsilon^{abc} W_{L,\mu}^b W_{L,\nu}^c, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned} \quad (3.7)$$

Lepton	T_L^3	Y	Q	Quark	T_L^3	Y	Q
ν_L^i	$\frac{1}{2}$	-1	0	U_L^i	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$
E_L^i	$-\frac{1}{2}$	-1	-1	D_L^i	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{3}$
				U_R^i	0	$\frac{4}{3}$	$\frac{2}{3}$
E_R^i	0	-2	-1	D_R^i	0	$-\frac{2}{3}$	$-\frac{1}{3}$

Table 3.1: *EW quantum numbers of SM quarks and leptons*

As usual, g_L denotes the $SU(2)_L$ coupling constant and g_Y the $U(1)_Y$ coupling constant. We also used the fact that the $SU(2)$ generators of the fundamental representation are given by the three Pauli matrices $T^a \rightarrow \tau^a = \sigma^a/2$, $a = 1, 2, 3$, and the corresponding structure constants are expressed by the Levi-Civita symbol or epsilon tensor ϵ^{abc} . The sum in (3.6) includes all left-handed fermion doublets $\psi_L^i = Q_L^i, L_L^i$ with $Q_L^i = (U^i, D^i)_L$, $L_L^i = (\nu^i, E^i)_L$ and all right-handed fermion fields $\psi_R^i = U_R^i, D_R^i, E_R^i$. The generation index i takes into account that the SM contains three generations of fermions which share the same gauge quantum numbers. Taking into account the full replication of fermions, the SM fermion content consists of neutrinos $\nu^i = (\nu_e, \nu_\mu, \nu_\tau)$, leptons $E^i = (e, \mu, \tau)$, up-type quarks $U^i = (u, c, t)$ and down-type quarks $D^i = (d, s, b)$. In Table 3.1, we display the quantum numbers of fermions under $SU(2)_L$ and $U(1)_Y$ as well as the electric charge Q (3.20) of the electromagnetic $U(1)_Q$ symmetry to which the EW gauge group is spontaneously broken.

Anomaly Cancellation within the Standard Model

According to our general remarks made in Section 2.5, the triangle diagram involving three gluons is anomaly free with respect to the SM fermion content since the gluons couple identically to the left- and right-handed quarks.

Remembering that the trace (2.36) tests the presence of an anomaly, the analogue pure $SU(2)_L$ contribution is also anomaly free because $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$.

For possible mixed anomalies of the EW gauge group, we find that they vanish for each generation separately

$$\text{Tr}[\lambda^a \lambda^b Y] = 2\delta^{ab} \text{Tr}[Y] = 0, \quad \text{Tr}[\sigma^a \sigma^b Y] = 2\delta^{ab} \text{Tr}[Y] = 0, \quad \text{Tr}[Y^3] = 0, \quad (3.8)$$

and conclude that the SM is free of gauge anomalies.

3.3 Spontaneous Electroweak Symmetry Breaking

Experimentally we know that there are at least three massive gauge bosons Z, W^\pm . However, a bare gauge boson mass term $m_X^2 X_\mu^a X^{a,\mu}$ is forbidden by gauge invariance and one has to find another possibility to render the gauge bosons massive. In the SM, the masses are generated by the *Higgs mechanism*, which relies on the phenomenon of spontaneous *electroweak symmetry breaking* (EWSB). For this purpose, an elementary scalar field — the Higgs field $H(x)$ — is introduced which transforms non-trivially under the EW gauge group. After the Higgs field has received a non-vanishing VEV, the EW gauge symmetry is spontaneously broken to the residual $U(1)_Q$ gauge symmetry of the vacuum state. Being a doublet under $SU(2)_L$, the Higgs doublet also allows for chiral couplings to the fermions fields. These are known as Yukawa couplings and lead to fermion mass terms of the form $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ after EWSB (see Section 3.8).

At first sight, one may suppose that two Higgs doublets are necessary to account for the conservation of hypercharge quantum number, because the up-type quark mass terms ($Y(\bar{U}_L U_R) = 1$) require a Higgs with hypercharge $Y(H) = -1$, whereas the down-type quark mass terms and electron mass terms ($Y(\bar{D}_L D_R) = -1, Y(\bar{E}_L E_R) = -1$) call for a Higgs with hypercharge $Y(H) = 1$. The reason for the fact that there is no need to double the scalar sector relies on the property of the $SU(2)$ group whose representations are real, i.e. there exists a non-singular fixed matrix S for each representation with

$$ST^a S^{-1} = -T^{*a}. \quad (3.9)$$

For the defining representation $T^a = \sigma^a/2$, the above condition can be fulfilled for all a by the choice $S = \sigma^2$ and the relation (3.9) reads

$$\sigma^2 \frac{\sigma^a}{2} \sigma^2 = -\frac{\sigma^{*a}}{2}. \quad (3.10)$$

Using the infinitesimal transformation behaviour of the complex conjugated Higgs doublet $H'^*(x) \simeq (1 - i\theta^a \sigma^{*a}/2)H^*(x)$, the object $\tilde{H}(x) = i\sigma^2 H^*(x)$ with

$$\tilde{H}'(x) = i\sigma^2 \left(1 - i\theta^a \frac{\sigma^{*a}}{2}\right) H^*(x) \stackrel{(3.10)}{=} \left(i\sigma^2 - \theta^a \frac{\sigma^a}{2}\sigma^2\right) H^*(x) = \left(1 + i\theta^a \frac{\sigma^a}{2}\right) \tilde{H}, \quad (3.11)$$

has the same (infinitesimal) transformation behaviour as the Higgs doublet $H(x)$ itself, but with opposite hypercharge. Thus only one complex Higgs doublet with $Y(H) = 1$ has to be added within the SM, whose four real components $H_i, i = 1, 2, 3, 4$, can be parametrised by

$$H(x) = \begin{pmatrix} H_+(x) \\ H_0(x) \end{pmatrix} = \begin{pmatrix} \frac{H_1(x) + iH_2(x)}{\sqrt{2}} \\ \frac{H_3(x) + iH_4(x)}{\sqrt{2}} \end{pmatrix}. \quad (3.12)$$

As we will see later, the subscripts “+ , 0” in (3.12) denote the electric charges after EWSB.

Corresponding to the chosen quantum numbers under the $SU(2)_L \times U(1)_Y$ local symmetry group, the Higgs is coupled to the EW gauge fields through the covariant derivative

$$D_\mu H(x) = \left(\partial_\mu - ig_L \frac{\sigma^a}{2} W_{L,\mu}^a - ig_Y \frac{Y}{2} B_\mu \right) H(x), \quad (3.13)$$

which enters the scalar Higgs Lagrangian

$$\mathcal{L}_{\text{Higgs}} = (D_\mu H)^\dagger (D^\mu H) - V(H). \quad (3.14)$$

The most general renormalisable potential $V(H)$ for a complex scalar field with mass dimension $[H] = 1$ can be described by

$$V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 = \lambda \left(H^\dagger H - \frac{\mu^2}{2\lambda} \right)^2 - \frac{\mu^4}{4\lambda}. \quad (3.15)$$

In the following we suppose that μ^2 and λ are real and positive parameters, such that the Higgs potential has the shape of a Mexican hat which allows for a stable, non-vanishing ground state. Inserting the Higgs as parametrised by (3.12), the Higgs potential exhibits a whole set of degenerate minima, when

$$H^\dagger H = \frac{1}{2} (H_1^2 + H_2^2 + H_3^2 + H_4^2) = \frac{\mu^2}{2\lambda}. \quad (3.16)$$

Without loss of generality, we can choose the minimum corresponding to the VEVs

$$\langle H_i \rangle = 0, \quad i = 1, 2, 4, \quad \langle H_3 \rangle = \sqrt{\mu^2/\lambda} \equiv v. \quad (3.17)$$

Inserting the VEVs of the various real Higgs components into the Higgs doublet (3.12), its VEV is given by

$$\langle H(x) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (3.18)$$

To explore the remaining symmetry of the vacuum, we impose the condition

$$\langle H'(x) \rangle = e^{i\theta^a(x) \frac{\sigma^a}{2}} e^{i\beta(x) \frac{Y}{2}} \langle H \rangle \simeq \langle H \rangle + i \left(\theta^a(x) \frac{\sigma^a}{2} + \beta(x) \frac{Y}{2} \right) \langle H \rangle \stackrel{!}{=} \langle H \rangle. \quad (3.19)$$

The fact that $\sigma^a \langle H \rangle \neq 0$ and $Y \langle H \rangle \neq 0$ for the Higgs VEV given in (3.18) demonstrates that $SU(2)_L$ and $U(1)_Y$ are indeed broken separately. Only a gauge transformation with $\theta^1(x) = \theta^2(x) = 0$ and $\theta^3(x) = \beta(x)$ leaves the vacuum invariant. This particular combination of generators,

$$Q = \frac{\sigma^3}{2} + \frac{Y}{2} = T_L^3 + \frac{Y}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.20)$$

corresponds to the generator of the unbroken residual electromagnetic $U(1)_Q$ symmetry.

Alternatively, the most general complex-valued two-component scalar field can be formulated by

$$H(x) = e^{i \frac{\pi^a(x) \sigma^a}{v}} \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}, \quad H^\dagger(x) = \begin{pmatrix} 0 & \frac{v+h(x)}{\sqrt{2}} \end{pmatrix} e^{-i \frac{\pi^a(x) \sigma^a}{v}}. \quad (3.21)$$

This special form suggests to “gauge away” the three scalar fields $\pi^a(x)$ by means of a $SU(2)_L$ gauge transformation with the specific choice of the gauge parameter $\alpha_{\text{gfix}}^a(x) = \pi^a(x)/v$,

$$H'(x) = U_{\text{gfix}}(x)H(x) = e^{-i\frac{\pi^a(x)}{v}\sigma^a}H(x) = \begin{pmatrix} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{pmatrix}. \quad (3.22)$$

Since they disappear from the Lagrangian, these scalar fields are called *would-be-Goldstone bosons*. The gauge, in which those unphysical scalar degrees of freedom are absent and the particle content of the theory is obvious, is called *physical* or *unitary gauge*. Actually, the Goldstone degrees of freedom become the longitudinal polarisation states of the new massive gauge bosons. In Section 3.6, we will discuss another class of gauge choices where the Goldstone bosons are not eliminated explicitly. According to the Goldstone theorem [148], the number of Goldstone bosons is equal to the number of symmetry generators that are spontaneously broken. Thus, by extracting the broken symmetry generators in the parameterisation (3.21), the effects of spontaneous symmetry breaking can be made more transparent. In the case of a continuous global symmetry, such an elimination mechanism of the massless Goldstone bosons is not possible.

The real fluctuation $h(x)$ around the non-trivial vacuum v then represents the only physical scalar degree of freedom — the famous scalar *Higgs boson*. In the unitary gauge, the scalar potential takes the form

$$V(h(x)) = -\mu^2 \frac{(v+h(x))^2}{2} + \lambda \frac{(v+h(x))^4}{4}, \quad (3.23)$$

which contains a mass term for the Higgs field

$$V(h(x)) \supset -\frac{\mu^2}{2}h(x)^2 \stackrel{(3.17)}{=} \lambda v^2 h(x)^2 = -\frac{1}{2}m_h^2 h^2. \quad (3.24)$$

In the next section, we show that three gauge bosons Z , W^\pm become massive through the Higgs mechanism, while one gauge boson A , corresponding to the unbroken $U(1)_Q$ symmetry, remains massless.

3.4 Gauge Boson Masses

After the Higgs has received a non-vanishing VEV, the masses for the gauge bosons arise from the Higgs kinetic term

$$\begin{aligned} (D_\mu H)^\dagger (D^\mu H) &\supset \langle H \rangle^\dagger \left(-ig_L \frac{\sigma^a}{2} W_{L,\mu}^a - ig_Y \frac{Y}{2} B_\mu \right) \left(ig_L \frac{\sigma^a}{2} W_{L,\mu}^a + ig_Y \frac{Y}{2} B_\mu \right) \langle H \rangle \\ &= \frac{1}{8}v^2 (g_L W_{L,\mu}^3 - g_Y B_\mu)^2 + \frac{1}{4}v^2 g_L^2 W_\mu^+ W^{-\mu} \\ &= M_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu^2 + \frac{1}{2}M_A^2 A_\mu^2, \end{aligned} \quad (3.25)$$

where the electrically charged gauge bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_{L,\mu}^1 \mp iW_{L,\mu}^2) \quad (3.26)$$

are given with respect to the basis

$$\sigma^\pm = \frac{(\sigma^1 \pm i\sigma^2)}{\sqrt{2}}. \quad (3.27)$$

The analytic diagonalisation of the real and symmetric mass matrix $\mathcal{M}_{\text{neutral}}$ for the neutral states by means of the orthogonal transformation matrix \mathcal{G}_Z ,

$$\mathcal{G}_Z \mathcal{M}_{\text{neutral}}^2 \mathcal{G}_Z^T = \frac{v^2}{4} \mathcal{G}_Z \begin{pmatrix} g_L^2 & -g_L g_Y \\ -g_L g_Y & g_Y^2 \end{pmatrix} \mathcal{G}_Z^T = \text{Diag}(M_Z^2, M_A^2), \quad (3.28)$$

yields the mass eigenvalues

$$M_Z^2 = \frac{v^2(g_L^2 + g_Y^2)}{4} \quad \text{and} \quad M_A^2 = 0. \quad (3.29)$$

Introducing the usual definition of the *weak mixing angle* or *Weinberg angle* θ_W , the explicit expression of the transformation matrix is found to be

$$\mathcal{G}_Z = \frac{1}{\sqrt{g_L^2 + g_Y^2}} \begin{pmatrix} g_L & -g_Y \\ g_Y & g_L \end{pmatrix} = \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix}. \quad (3.30)$$

It rotates the gauge bosons from the gauge eigenstate basis $(W_{L,\mu}^3, B_\mu)$ into the normalised neutral mass eigenstates (Z_μ, A_μ) ,

$$Z_\mu = \cos \theta_W W_{L,\mu}^3 - \sin \theta_W B_\mu, \quad A_\mu = \sin \theta_W W_{L,\mu}^3 + \cos \theta_W B_\mu. \quad (3.31)$$

In summary, the spectrum of EW gauge bosons contains three different mass eigenvalues

$$M_W = \frac{1}{2}v g_L, \quad M_Z = \frac{1}{2}v (g_L^2 + g_Y^2)^{1/2}, \quad M_A = 0, \quad (3.32)$$

where $v = 246$ GeV, and only one gauge boson — the photon — remains massless due to the residual $U(1)_Q$ symmetry. The other gauge bosons become massive with a mass relation that is captured by the so-called ρ -parameter,

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1. \quad (3.33)$$

The fact that the ρ -parameter is equal to one in the SM at tree level is a consequence of an accidental global $SO(4)$ symmetry of the Higgs sector, which is called the *custodial symmetry* [119].

3.5 Custodial Symmetry

As it can be seen from the expression of HH^\dagger given in (3.16), the Higgs potential has an accidental global $SO(4)$ symmetry, which is isomorphic to the simple product group $SU(2)_L \times SU(2)_R$. Besides the global symmetry $SU(2)_L$, which is gauged in the SM, the global $SU(2)_R$ symmetry can be interpreted as a symmetry that mixes H and $\tilde{H} = i\sigma^2 H^*$ as the two doublets are equivalent with respect to the $SU(2)_L$ transformation.

Intuitively, we may then represent the Higgs as a 2×2 matrix

$$\mathcal{H} = (i\sigma_2 H^* \ H) = \begin{pmatrix} H_0^* & H_+ \\ -H_+^* & H_0 \end{pmatrix}, \quad (3.34)$$

which transforms as a bidoublet under the global $SU(2)_L \times SU(2)_R$ symmetry

$$\mathcal{H} \rightarrow U_L \mathcal{H} U_R^\dagger. \quad (3.35)$$

While the $SU(2)_L$ symmetry group acts vertically from the left, the global $SU(2)_R$ group acts horizontally on this matrix from the right. The invariance of the Higgs potential under global $SU(2)_L \times SU(2)_R$ transformations can again be made explicit by noting that

$$\text{Tr} [\mathcal{H}^\dagger \mathcal{H}]' = \text{Tr} [U_R \mathcal{H}^\dagger U_L^\dagger U_L \mathcal{H} U_R^\dagger] = \text{Tr} [\mathcal{H}^\dagger \mathcal{H}] = H^\dagger H \text{Tr} [\mathbb{1}] = 2H^\dagger H. \quad (3.36)$$

Rewriting the Higgs Lagrangian in terms of the Higgs matrix field (3.34),

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2} \text{Tr} [(D_\mu \mathcal{H})^\dagger D^\mu \mathcal{H}] - \frac{\mu^2}{2} \text{Tr} [\mathcal{H}^\dagger \mathcal{H}] + \frac{\lambda}{4} (\text{Tr} [\mathcal{H}^\dagger \mathcal{H}])^2, \quad (3.37)$$

one has to introduce the Pauli matrix σ^3 in the hypercharge coupling,

$$D_\mu \mathcal{H} = \partial_\mu \mathcal{H} - ig_L \frac{\sigma^a}{2} W_{L,\mu}^a \mathcal{H} - ig_Y B_\mu \mathcal{H} \frac{\sigma^3}{2}, \quad (3.38)$$

in order to ensure opposite hypercharges for H and $i\sigma_2 H^*$. While $\text{Tr} [(D_\mu \mathcal{H})^\dagger D^\mu \mathcal{H}]$ is invariant under a global $SU(2)_L$, it is not under a global $SU(2)_R$ rotation. The $SU(2)_R$ symmetry is only exact in the limit of a vanishing hypercharge coupling. During EWSB the Higgs VEV $\langle \mathcal{H} \rangle = 1/\sqrt{2} \text{Diag}(v, v)$ breaks the global approximate $SU(2)_L \times SU(2)_R$ symmetry down to the diagonal subgroup $SU(2)_V$ with transformation matrices $U_L = U_R$. This left-over $SU(2)_V$ symmetry is called custodial symmetry [119] and is crucial for the understanding of the EW sector.

Especially, the above global symmetry breaking pattern can also be formulated by a non-linear σ -model, and thus the custodial symmetry can be considered in a more general — Higgs independent — framework. Therefore, we replace the Higgs matrix field \mathcal{H} by a Goldstone-boson matrix field Σ ,

$$\mathcal{H} \rightarrow \frac{v}{\sqrt{2}} \Sigma, \quad \Sigma = e^{i \frac{\pi^a \sigma^a}{v}}, \quad (3.39)$$

which contains the three Goldstone bosons arising in the breakdown of the global symmetry according to $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$. In the unitary gauge, $\Sigma = 1$, the effective operator $v^2/4 \text{Tr} [(D_\mu \Sigma)^\dagger D^\mu \Sigma]$ produces the masses for the gauged subgroup, analogous to those in (3.25), and predicts the ρ -parameter to be one at tree level.

3.6 R_ξ -Gauge Fixing Terms within the Standard Model

When not working in the unitary gauge, the Higgs kinetic term induces mixings between vector bosons and scalar fields which makes the interpretation of the mass terms given in (3.32) less clear. Hence in spontaneously broken gauge theories it is convenient to cancel these mixings, which can be achieved by adding R_ξ -gauge fixing terms [149], as discussed in the following. Splitting the Higgs into its VEV and fluctuations around the VEV,

$$H(x) = \langle H(x) \rangle + \delta H(x), \quad (3.40)$$

the following mixing terms of the SM EW gauge group arise:

$$(D_\mu H)^\dagger (D^\mu H) \supset -ig_L W_{L,\mu}^a \partial^\mu \left((\delta H)^\dagger \frac{\sigma^a}{2} \langle H \rangle - \text{h.c.} \right) - ig_Y \frac{B_\mu}{2} \partial^\mu \left((\delta H)^\dagger \langle H \rangle - \text{h.c.} \right). \quad (3.41)$$

Inserting the Higgs representation of (3.21), the mixing terms have the form

$$(D_\mu H)^\dagger (D^\mu H) \supset v \left(g_Y B_\mu \partial^\mu \pi^3 - g_L (W_{L,\mu}^1 \partial^\mu \pi^1 + W_{L,\mu}^2 \partial^\mu \pi^2 + W_{L,\mu}^3 \partial^\mu \pi^3) \right), \quad (3.42)$$

from which we can observe that the scalar fields that mix with the SM EW gauge bosons correspond exactly to the three Goldstone bosons $\pi^a(x)$. Moreover, the Goldstone boson $\pi^3(x)$ will become the longitudinal degree of freedom of the linear combination of neutral gauge boson fields proportional to $(g_Y B_\mu - g_L W_{L,\mu}^3)$. The latter corresponds to the Z boson after normalisation which is massive after EWSB. Choosing the gauge-fixing function G_1^a for $SU(2)_L$,

$$G_1^a = \partial^\mu W_{L,\mu}^a + ig_L \xi \left(\delta H^\dagger \frac{\sigma^a}{2} \langle H \rangle - \text{h.c.} \right), \quad (3.43)$$

and the gauge-fixing function G_2 for $U(1)_Y$,

$$G_2 = \partial^\mu B_\mu + ig_Y \frac{\xi}{2} \left(\delta H^\dagger \langle H \rangle - \text{h.c.} \right), \quad (3.44)$$

the gauge-fixing terms

$$\mathcal{L}_{\text{gfix}} = -\frac{1}{2\xi} \left(\partial^\mu W_{L,\mu}^a + ig_L \xi \left(\delta H^\dagger \frac{\sigma^a}{2} \langle H \rangle - \text{h.c.} \right) \right)^2 - \frac{1}{2\xi} \left(\partial^\mu B_\mu + ig_Y \frac{\xi}{2} \left(\delta H^\dagger \langle H \rangle - \text{h.c.} \right) \right)^2 \quad (3.45)$$

cancel the mixing terms in (3.41) without transforming away the $\pi^a(x)$ s. Apart from the ξ -independent mixing terms, the ξ -dependent term $1/\xi (\partial^\mu A_\mu^a)^2$ ensures a good high-energy behaviour of the gauge boson propagator for finite ξ , and makes the renormalisability of the theory more transparent [150, 151]. Moreover, Goldstone mass terms proportional to ξ arise. The gauge parameter ξ can vary continuously from 0 to ∞ . In the limit $\xi \rightarrow \infty$ one obtains the unitary gauge since the unphysical particles decouple from the theory.

3.7 Neutral and Charged Currents

In order to make the residual unbroken $U(1)_Q$ gauge symmetry manifest, we restate the covariant derivative (3.6) of the fermions in terms of the physical gauge boson fields W^\pm , Z and A :

$$\begin{aligned} D_\mu \psi_L &= \left(\partial_\mu - i \frac{g_L}{\sqrt{2}} \left(\frac{\sigma^+}{2} W_\mu^+ + \frac{\sigma^-}{2} W_\mu^- \right) - i \frac{g_L}{\cos \theta_W} \left(\frac{\sigma^3}{2} - \sin^2 \theta_W Q \right) Z_\mu - ie Q A_\mu \right) \psi_L, \\ D_\mu \psi_R &= \left(\partial_\mu + i \frac{g_L}{\cos \theta_W} \left(\sin^2 \theta_W Q \right) Z_\mu - ie Q A_\mu \right) \psi_R. \end{aligned} \quad (3.46)$$

In deriving the above result, we have identified the coefficient of the electromagnetic interaction with the electron charge e

$$e = \frac{g_L g_Y}{\sqrt{g_L^2 + g_Y^2}} \stackrel{(3.30)}{=} g_L \sin \theta_W = g_Y \cos \theta_W. \quad (3.47)$$

Inserting the quantum numbers of the various fermions, as given in Table 3.1, we obtain the EW Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{EW}} &= \sum_i \bar{Q}_L^i (i \not{\partial}) Q_L^i + \bar{E}_L^i (i \not{\partial}) E_L^i + \bar{U}_R^i (i \not{\partial}) U_R^i + \bar{D}_R^i (i \not{\partial}) D_R^i + \bar{E}_R^i (i \not{\partial}) E_R^i \\ &\quad + g_L \left(W_\mu^+ J_W^{\mu+} + W_\mu^- J_W^{\mu-} + Z_\mu J_Z^\mu \right) + e A_\mu J_Q^\mu - \frac{1}{4} L_{\mu\nu}^a L^{a,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \end{aligned} \quad (3.48)$$

where we have found for the weak currents

$$\begin{aligned} J_W^{\mu+} &= \frac{1}{\sqrt{2}} \left(\bar{U}_L^i \gamma^\mu D_L^i + \bar{\nu}_L^i \gamma^\mu E_L^i \right), \\ J_W^{\mu-} &= \frac{1}{\sqrt{2}} \left(\bar{D}_L^i \gamma^\mu U_L^i + \bar{E}_L^i \gamma^\mu \nu_L^i \right), \\ J_Z^\mu &= \frac{1}{\cos \theta_W} \left(\bar{U}_L^i \gamma^\mu \left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_W \right) U_L^i + \bar{U}_R^i \gamma^\mu \left(-\frac{2}{3} \sin^2 \theta_W \right) U_R^i \right. \\ &\quad \left. + \bar{D}_L^i \gamma^\mu \left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_W \right) D_L^i + \bar{D}_R^i \gamma^\mu \left(\frac{1}{3} \sin^2 \theta_W \right) D_R^i \right. \\ &\quad \left. + \bar{\nu}_L^i \gamma^\mu \left(\frac{1}{2} \right) \nu_L^i + \bar{E}_L^i \gamma^\mu \left(-\frac{1}{2} + \sin^2 \theta_W \right) E_L^i + \bar{E}_R^i \gamma^\mu \left(\sin^2 \theta_W \right) E_R^i \right), \\ J_Q^\mu &= \bar{U}_L^i \gamma^\mu \left(\frac{2}{3} \right) U_L^i + \bar{U}_R^i \gamma^\mu \left(\frac{2}{3} \right) U_R^i + \bar{D}_L^i \gamma^\mu \left(-\frac{1}{3} \right) D_L^i + \bar{D}_R^i \gamma^\mu \left(-\frac{1}{3} \right) D_R^i \\ &\quad + \bar{E}_L^i \gamma^\mu (-1) E_L^i + \bar{E}_R^i \gamma^\mu (-1) E_R^i. \end{aligned} \quad (3.49)$$

The fermions are still given in their gauge eigenstate basis. The origin of their masses, as well as the transformation from the gauge eigenstate basis into the mass eigenstate basis, will be the subject of the next section.

3.8 Yukawa Interactions — Fermion Mass Terms

We have indicated in Section 3.3 that the introduction of the Higgs boson allows for the construction of fermion-Higgs coupling terms, which are known as Yukawa couplings. The most general renormalisable and gauge-invariant Yukawa Lagrangian, coupling the quark fields to the scalar doublets H, \tilde{H} reads

$$-\mathcal{L}_{\text{Yuk}} = \overline{Q}_L^i \tilde{H} Y_U^{ij} U_R^j + \overline{Q}_L^i H Y_D^{ij} D_R^j + \text{h.c.} \quad (3.50)$$

The SM Yukawa couplings Y_U and Y_D are generic complex 3×3 matrices and thus are described by 18 real parameters (R) and 18 complex phases (P). After EWSB the VEV of the Higgs boson gives rise to the fermion mass terms

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}} \overline{U}_L Y_U^{ij} U_R^j - \frac{v}{\sqrt{2}} \overline{D}_L Y_D^{ij} D_R^j + \text{h.c.}, \quad (3.51)$$

where $M_U^{ij} = v/\sqrt{2} Y_U^{ij}$ contains the masses for the up-type quarks, and $M_D^{ij} = v/\sqrt{2} Y_D^{ij}$ supplies the masses for the down-type quarks.

By means of biunitary transformations the Yukawa matrices can be diagonalised,

$$\mathcal{L}_{\text{mass}} = -\frac{v}{\sqrt{2}} \overline{U}_L V_{U_L} \text{Diag}(Y_U) V_{U_R}^\dagger U_R - \frac{v}{\sqrt{2}} \overline{D}_L V_{D_L} \text{Diag}(Y_D) V_{D_R}^\dagger D_R + \text{h.c.}, \quad (3.52)$$

where the rotation matrices $V_{U_{L,R}}^\dagger, V_{D_{L,R}}^\dagger$ transform the quark gauge eigenstates into their mass eigenstates, which we indicate by a prime

$$U'_{L,R} = V_{U_{L,R}}^\dagger U_{L,R}, \quad D'_{L,R} = V_{D_{L,R}}^\dagger D_{L,R}. \quad (3.53)$$

Rewriting the above Lagrangian (3.52) in terms of quark mass eigenstates, we obtain

$$\mathcal{L}_{\text{mass}} = -\overline{U}'_L M_U U'_R - \overline{D}'_L M_D D'_R + \text{h.c.}, \quad (3.54)$$

where $M_U = v/\sqrt{2} \text{Diag}(Y_U)$ and $M_D = v/\sqrt{2} \text{Diag}(Y_D)$ are the diagonal quark mass matrices.

Transforming the quarks into their mass eigenbases, we recognise that the rotation matrices only occur in the charged weak currents $J_W^{\mu\pm}$, e.g.

$$J_W^{\mu+} \supset \frac{1}{\sqrt{2}} \overline{U}_L \gamma^\mu D_L = \frac{1}{\sqrt{2}} \overline{U}'_L V_{U_L}^\dagger \gamma^\mu V_{D_L} D'_L. \quad (3.55)$$

This is due to the mismatch of the left-handed rotation matrices V_{U_L} and V_{D_L} , which is described by the unitary quark mixing matrix or *Cabbibo–Kobayashi–Maskawa* (CKM) matrix [152, 153],

$$V_{\text{CKM}} \equiv V_{U_L}^\dagger V_{D_L}. \quad (3.56)$$

Parameterisation of Unitary Matrices

A generic unitary 3×3 matrix can be expressed by three real parameters and six complex phases. Amongst all mathematically equivalent parameterisation possibilities, it is convenient to express the matrix by a product of a matrix \tilde{R} (with 3 rotational angles and 3 complex phases) and a diagonal phase matrix $F_\phi = \text{Diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$. The diagonal phase matrix corresponds to an exponentiated linear combination of the diagonal Gell-Mann matrices λ_3, λ_8 with the unit matrix $\mathbb{1}$. The matrix \tilde{R} originates from a product of three complex rotation matrices $\tilde{R}_{12}, \tilde{R}_{13}, \tilde{R}_{23}$, where \tilde{R}_{12} is specified by

$$\tilde{R}_{12} = e^{i\tilde{\theta}_{12}} = e^{i\theta_{12}(\sin\delta\lambda^1 + \cos\delta\lambda^2)} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12}e^{i\delta} & 0 \\ -\sin\theta_{12}e^{-i\delta} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{i\delta/2\lambda^3} e^{i\theta_{12}\lambda^2} e^{-i\delta/2\lambda^3}, \quad (3.57)$$

and the others are defined in full analogy. According to a possible classification of the parameterisation of the orthogonal 3×3 matrices R_{ij} in [154], which equal \tilde{R}_{ij} with $\delta = 0$, there are six different forms of the type $R = R_{ij}R_{kl}R_{ij}$ and another six of the type $R = R_{ij}R_{kl}R_{mn}$.

Parameterisation of the CKM Matrix

By describing the unitary CKM matrix, not all of the 3R+6P parameters are physically relevant. As one can see from (3.54), there is still the freedom to redefine the mass eigenstates by a diagonal phase matrix according to $\bar{U}'_L \rightarrow \bar{U}'_L F_{\phi_U}$ and $U'_R \rightarrow F_{\phi_U}^{-1} U'_R = F_{\phi_U}^\dagger U'_R$, and analogously in the down sector with the phase matrix F_{ϕ_D} . For the special case of $F_{\phi_U} \equiv F_{\phi_D}$ and all the phases are identical, the diagonal phase matrices are proportional to the unit matrix and thus commute with the CKM matrix in (3.55) (this case corresponds to an accidental unbroken global baryon number symmetry $U(1)_B$). Thus in summary, five of the six phases in the CKM matrix can be removed by field redefinitions. Of the many possible parameterisations, the following has become the ‘‘standard parameterisation’’ [155]

$$V_{\text{CKM}} = \begin{pmatrix} c_{12} c_{13} & c_{13} s_{12} & s_{13} e^{-i\delta} \\ -c_{12} s_{23} s_{13} - c_{23} s_{12} e^{-i\delta} & -s_{23} s_{13} s_{12} + c_{23} c_{12} e^{-i\delta} & c_{13} s_{23} \\ -c_{23} c_{12} s_{13} + s_{23} s_{12} e^{-i\delta} & -c_{23} s_{13} s_{12} - c_{12} s_{23} e^{-i\delta} & c_{23} c_{13} \end{pmatrix}, \quad (3.58)$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. The phase δ is the only CP-violating source in the SM. In terms of the rotation matrices introduced above, the standard parameterisation can be presented by

$$\text{Diag}(1, e^{i\delta}, e^{i\delta}) R_{23} \hat{R}_{31} R_{12} \text{Diag}(1, 1, e^{-i\delta}), \quad (3.59)$$

where \hat{R}_{31} equals R_{31} with the replacement $1 \rightarrow e^{-i\delta}$ on the diagonal element.

3.9 Global Flavour Symmetry of the Standard Model

While the Higgs sector itself incorporates a global custodial symmetry, the Higgs couplings to the fermions via the Yukawa interactions destroy another global symmetry present in the SM Lagrangian. The maximal global SM flavour symmetry group is the largest group of unitary field transformations that commutes with the SM gauge group. It is given by

$$U(3)^5 = U(3)_{Q_L} \times U(3)_{U_R} \times U(3)_{D_R} \times U(3)_{L_L} \times U(3)_{E_R}. \quad (3.60)$$

Decomposing the unitary groups $U(N)$ into a semidirect product of a special unitary group $SU(N)$ and an abelian group $U(1)$, the chiral quark flavour symmetry group can be rewritten as

$$G_F^{\max} = SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R} \times U(1)_{Q_L} \times U(1)_{U_R} \times U(1)_{D_R}. \quad (3.61)$$

The quark fields transform as fundamentals

$$Q_L \sim (3, 1, 1), \quad U_R \sim (1, 3, 1), \quad D_R \sim (1, 1, 3) \quad (3.62)$$

under the special unitary groups.

In the SM, the Higgs field is a singlet under the flavour group and the Yukawa coupling matrices are constant parameters. It is then evident that the Yukawa Lagrangian (3.50) is in general not invariant under the global quark transformations (3.62) with the exception of a global $U(1)_B$ symmetry which corresponds to the limit of equal rotation matrices. It will be useful to define the quark flavour group as exactly that part which is broken by the Yukawas (see also [156, 157])

$$G_F = U(3)^3 / U(1)_B = SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R} \times U(1)_{U_R} \times U(1)_{D_R}, \quad (3.63)$$

where we chose to leave the abelian subgroup of the right-handed quark rotations as independent linear combinations besides $U(1)_B$.

Chapter 4

Dynamical Minimal Flavour Violation

4.1 Basic Concepts of Minimal Flavour Violation

Although the SM is a very successful model, there are several open questions, which are related to the — so far — undiscovered scalar Higgs particle. At higher energy scales, the SM Higgs boson mass receives large UV-sensitive loop corrections from the SM particles, especially from the top quark, and only a precise adjustment of parameters can keep the Higgs VEV around the weak scale. This, however, is quite unnatural and is the origin of the so-called fine-tuning problem. For this reason, it is widely believed that there exist new physics contributions which stabilise the weak scale and resolve this hierarchy problem. In consequence, the SM should be interpreted as a low-energy effective field theory with an unspecified cutoff scale Λ , in which the NP particles with masses heavier than the EW scale appear through higher-dimensional operators. Assuming that the new interactions/particles arise already at the TeV scale, the flavour sector of the NP models is highly constrained and non-generic, because all present data on rare and CP-violating K and B decays are in very good accordance with the SM predictions [26–29]. In order to mimic the SM flavour sector and its phenomenological outcomes, one may assume that the NP models follow the concept of minimal flavour violation (MFV). It postulates that flavour transitions and CP violation are solely induced by the Yukawa matrices in such a way that the low-scale effective field theory (EFT) is completely determined by their structure [30–32]. Pushing the NP scale well above the TeV scale, the hypothesis of MFV can be softened and more generic flavour transitions in higher-dimensional NP operators are allowed to appear. Settled in between these two scenarios lie the models that are referred to as models with next-to-minimal flavour violation [158, 159]. Focusing on models incorporating the MFV assumption, one can further distinguish between a linear representation of the FS [30, 32] and a non-linear one [88, 157, 160], as discussed in more detail below. A nice overview of this topic can also be found in [161].

The common idea of all MFV models is to promote the Yukawa matrices to dimensionless

auxiliary spurion fields,

$$Y_U \sim (3, \bar{3}, 1) \quad \text{and} \quad Y_D \sim (3, 1, \bar{3}), \quad (4.1)$$

which are bifundamentals under the maximal global quark FS group $U(3)_{Q_L} \times U(3)_{U_R} \times U(3)_{D_R}$ present in the SM (3.61). Having a formally restored flavour symmetry at scales above Λ , it is always possible to rotate the background values of the spurions to a basis, so that either $\langle Y_U \rangle$ or $\langle Y_D \rangle$ is diagonal. Moreover, if there are no other fields transforming under the FS, the background values of the Yukawa spurions are the only sources responsible for the breakdown of the FS as shown in Figure 4.1.

$$\begin{array}{c}
 G_F^{\text{max}} = U(3)_{Q_L} \times U(3)_{U_R} \times U(3)_{D_R} \\
 \begin{array}{ccc}
 \uparrow & \swarrow & \uparrow \\
 Y_U \sim (3, \bar{3}, 1) & & Y_D \sim (3, 1, \bar{3})
 \end{array}
 \end{array}$$

Figure 4.1: *The breaking of the SM FS by the background values of the Yukawa matrices.*

Linear Realisation of Minimal Flavour Violation

In the linear realisation of the MFV approach it is assumed that the full FS is broken at a single scale Λ [30], where the Yukawa couplings are “frozen” to their background values. Promoting the Yukawas to non-trivially transforming objects under the flavour group, the counting of parameters can be based on symmetrical grounds rather than on redefining the fields as done in Section 3.8. According to the number of broken flavour group generators, $3 \times 8 + 2 = 26$ out of the 36 real spurion fields play the role of Goldstone fields. The residual $36 - 26 = 10$ parameters are physically relevant, representing the six quark masses and the four CKM parameters. Below the scale Λ , the effective MFV theory contains all higher-dimensional operators, constructed from SM and Yukawa fields, which are invariant under CP and formally under the global flavour symmetry [30]. There exist the trivial operators with no spurion insertion

$$\bar{Q}_L Q_L, \quad \bar{U}_R U_R, \quad \bar{D}_R D_R, \quad (4.2)$$

followed by the formally invariant left-right coupling operators with a single insertion of spurions

$$\bar{Q}_L Y_U U_R + \text{h.c.}, \quad \bar{Q}_L Y_D D_R + \text{h.c.} \quad (4.3)$$

At the level of two spurion insertions the possible operators

$$\bar{Q}_L Y_U Y_U^\dagger Q_L + \text{h.c.}, \quad \bar{Q}_L Y_D Y_D^\dagger Q_L + \text{h.c.} \quad (4.4)$$

may arise, where the first one is dominant due to the enhancement of the top-quark Yukawa coupling y_t . Bringing the various MFV NP operators into the quark mass eigenbasis according to (3.53), the list of operators is given by

$$\bar{U}_L U_L = \bar{U}'_L U'_L, \quad \bar{D}_L D_L = \bar{D}'_L D'_L, \quad (4.5)$$

$$\bar{U}_L D_L = \bar{U}'_L V_{\text{CKM}} D'_L \quad (4.6)$$

$$\bar{U}_L Y_U U_R = \bar{U}'_L \text{Diag}(Y_U) U'_R \quad (4.7)$$

$$\bar{D}_L Y_U U_R = \bar{D}'_L V_{\text{CKM}}^\dagger \text{Diag}(Y_U) U'_R \quad (4.8)$$

$$\bar{U}_L Y_D D_R = \bar{U}'_L V_{\text{CKM}} \text{Diag}(Y_D) D'_R \quad (4.9)$$

$$\bar{D}_L Y_D D_R = \bar{D}'_L \text{Diag}(Y_D) D'_R \quad (4.10)$$

$$\bar{U}_L Y_U Y_U^\dagger U_L = \bar{U}'_L (\text{Diag}(Y_U))^2 U'_L \quad (4.11)$$

$$\bar{D}_L Y_U Y_U^\dagger D_L = \bar{D}'_L V_{\text{CKM}}^\dagger (\text{Diag}(Y_U))^2 V_{\text{CKM}} D'_L \quad (4.12)$$

$$\bar{U}_L Y_D Y_D^\dagger U_L = \bar{U}'_L V_{\text{CKM}} (\text{Diag}(Y_D))^2 V_{\text{CKM}}^\dagger U'_L \quad (4.13)$$

$$\bar{D}_L Y_D Y_D^\dagger D_L = \bar{D}'_L (\text{Diag}(Y_D))^2 D'_L, \quad (4.14)$$

where the expressions for the corresponding hermitian conjugate operators can be obtained analogously. Apart from the flavour diagonal operators ((4.5), (4.7), (4.10), (4.11), (4.14)), we observe that the operators which involve at least one CKM element contain the flavour structures which induce flavour transitions ((4.6), (4.8), (4.9), (4.12), (4.13)). By construction, only the CKM matrix and the diagonal Yukawa coupling elements, which correspond to the quark masses up to the proportionality factor $v/\sqrt{2}$, appear in the NP operators. To complete the list of flavour changing operators with a minimal number of spurion insertions [158], we use the replacements in Table 4.1 in order to be able to deduce the effective operators involving right-handed quarks out of the purely left-handed basic bilinear operators.

$D'_L \rightarrow \text{Diag}(Y_D) D'_R$	$\bar{D}'_L \rightarrow \bar{D}'_R \text{Diag}(Y_D)$
$U'_L \rightarrow \text{Diag}(Y_U) U'_R$	$\bar{U}'_L \rightarrow \bar{U}'_R \text{Diag}(Y_U)$

Table 4.1: *Replacement rules for obtaining the corresponding right-handed effective operators.*

Note that for left-handed FCNCs at least two spurion insertions are needed, while for right-handed FCNCs even four have to be included due to the additional chiral mass suppression factors given in Table 4.1. With the exception of the charged $t \rightarrow b$ transitions, involving the $\mathcal{O}(1)$ CKM element $|V_{tb}|$ and/or $\text{Diag}(Y_D)_{33} = y_t = \sqrt{2}m_t/v$, the coefficients of operators arising from higher number of spurion insertions become smaller and allow for a systematic power expansion.

Non-Linear Realisation of Minimal Flavour Violation

As indicated, the large top quark mass makes the convenient power-counting argument less transparent in the linear MFV formulation. An attempt to take the special role of the top Yukawa coupling into account is to represent the breaking of the flavour group non-linearly via giving it a large FS breaking VEV at the UV scale Λ [88]. The breakdown of the FS group G_F to the subgroup $G_F^{(1)}$ can then be realised by a non-linear σ -model-like parameterisation of the Yukawas [160]. The Goldstone fields, living in the coset space $G_F/G_F^{(1)}$, are factored out of the matrix which still contains the spurion fields transforming under the residual $G_F^{(1)}$ symmetry [37, 38]. After initialising the breaking of the FS through the top-quark Yukawa coupling, one can proceed further and break the FS in a step-wise fashion until all Yukawa coupling entries have been generated.

In Section 4.2 we will promote the auxiliary Yukawa spurions to dynamical scalar fields and demonstrate in Section 4.3 that a sequential breaking of the FS via appropriate chosen spurion VEVs can account for a hierarchy in the Yukawa matrices and thus for a hierarchy in the quark masses and mixings. In Subsection 4.3.1 we will take the FS even more seriously and consider a part of the FS as a local symmetry.

4.2 Towards Dynamical Minimal Flavour Violation

Besides the fact that the Yukawa matrices are objects transforming non-trivially under the flavour symmetry, we want to “revive” them in supplying the auxiliary spurions with a mass dimension and thus allow for kinetic terms.

This “Higgsing” of the Yukawa matrices resembles the model proposed in [162], where the Yukawa couplings have a Higgs-dependent structure of the form

$$Y_{ij}^{u,d} = c_{ij}^{u,d} \left(\frac{H^\dagger H}{M^2} \right)^{n_{ij}^{u,d}}, \quad (4.15)$$

where i, j are generation indices. The hierarchy in fermion masses is generated by the integer numbers $n_{ij}^{u,d}$, which count the number of Higgs insertions $H^\dagger H$. The NP scale M is around 1–2 TeV — a scale which is also favoured by hierarchy problem considerations. The coefficients $c_{ij}^{u,d}$ of $\mathcal{O}(1)$ cause a non-hermitian structure of the Yukawas.

In contrast to the above ansatz (4.15), where no additional scalar fields besides the SM Higgs field are introduced, we follow [88, 157] and treat the Yukawa matrices as independent new scalar degrees of freedom. Taking a multi-Higgs ansatz with an appropriate potential that supplies non-vanishing VEVs for the spurion fields, the hierarchical masses and mixing angles arise “naturally” according to the different breaking scales $\Lambda \gg \Lambda^{(1)} \gg \Lambda^{(2)} \gg \dots$ of the spontaneously broken flavour symmetry. The explicit form of the potential can be worked out by constructing several invariants under the flavour group, which consist of monomials of $Y_U(x)$ and $Y_D(x)$ with a definite canonical dimension (see [157] for further details).

4.3 Sequential Breaking of the Flavour Symmetry

In our setup the canonical dimension of the Yukawa matrices requires that the SM Yukawa couplings appear as dimension-five operators of an effective theory

$$-\mathcal{L}_{\text{Yuk}} = \frac{1}{\Lambda}(\overline{Q}_L \tilde{H})Y_U U_R + \frac{1}{\Lambda}(\overline{Q}_L H)Y_D D_R + \text{h.c.} \quad (4.16)$$

Only the spurion generating the top quark Yukawa coupling with $y_t \sim \mathcal{O}(1)$, which initialises the breaking chain at the UV scale Λ , gets a VEV of the same order of magnitude:

$$\langle Y_U \rangle \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_t \Lambda \end{pmatrix}. \quad (4.17)$$

Hence, the top quark Yukawa coupling effectively originates from a dimension-four operator. The VEVs of the other spurion fields occur at scales $\Lambda^{(n)} \ll \Lambda$ and thus, stemming effectively from dimension-five operators in (4.16), can naturally reproduce the smallness of the residual quark masses [88].

As we only have accessible information about the eigenvalues of the Yukawa matrices and the CKM matrix elements, one has to make further assumptions to be able to implement a hierarchical ranking of the Yukawa matrix entries. Since the right-handed rotation matrices are unobservable, we choose a basis in which they are equal to the unit matrix. Knowing also that there exists a basis in which the up-type Yukawa matrix is diagonal and one in which the down-type Yukawa matrix is diagonal, the CKM matrix then rotates between these two bases. Restricting ourselves to left-handed rotations matrices which scale in the same manner as the standard power counting for the CKM matrix with the Wolfenstein parameter $\lambda \sim 0.2 \ll 1$,

$$V_{\text{CKM}} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}, \quad (4.18)$$

we obtain the following hierarchy in the Yukawa matrices,

$$\begin{aligned} \langle Y_U \rangle &\sim V_{\text{CKM}} \text{Diag}(Y_U) \Lambda \sim \begin{pmatrix} \lambda^{n_u} & \lambda^{1+n_c} & \lambda^3 \\ \lambda^{1+n_u} & \lambda^{n_c} & \lambda^2 \\ \lambda^{3+n_u} & \lambda^{2+n_c} & 1 \end{pmatrix} \Lambda, \\ \langle Y_D \rangle &\sim V_{\text{CKM}} \text{Diag}(Y_D) \Lambda \sim \begin{pmatrix} \lambda^{n_d} & \lambda^{1+n_s} & \lambda^{3+n_b} \\ \lambda^{1+n_d} & \lambda^{n_s} & \lambda^{2+n_b} \\ \lambda^{3+n_d} & \lambda^{2+n_s} & \lambda^{n_b} \end{pmatrix} \Lambda. \end{aligned} \quad (4.19)$$

Thereby, the diagonal matrices $\text{Diag}(Y_{U,D})$ contain the quark masses expressed in powers of the Wolfenstein parameter $y_q \sim \lambda^{n_q}$. The scaling can be constrained from the phenomenological information on the quark masses.

4.3.1 Partly Gauged Flavour Symmetry

In the following, we will take the global FS group G_F even more seriously and will consider a specific scenario where the three $SU(3)$ factors are promoted to local symmetries

$$G_F = [SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R}] \times U(1)_{U_R} \times U(1)_{D_R}. \quad (4.20)$$

To distinguish the global from the local parts of the flavour symmetry, we indicate the gauged ones by squared brackets.

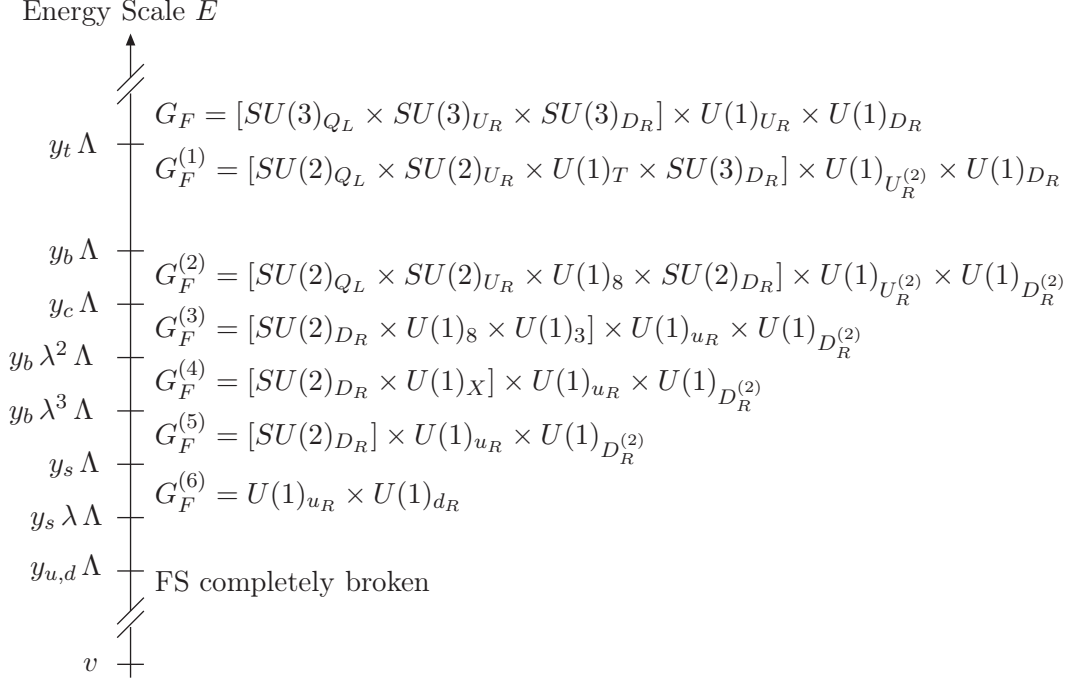
In the course of systematically breaking the gauged FS group, the gauge bosons will become massive and the Goldstone modes of the broken symmetry generators can be identified as the longitudinal modes of the gauge bosons in the unitary gauge. Our choice to leave the two $U(1)$ factors in G_F as global symmetries is motivated two-fold. On the one hand, the $U(1)_B$ symmetry which is respected by the Yukawa coupling terms is considered as global anyway. On the other hand, as the $U(1)$ Goldstone modes have anomalous couplings to the SM gauge fields, we expect that they will contribute to the effective θ -parameter in QCD. Identifying at least one linear combination of them as an axion with a finite mass, that is generated by anomalous couplings to QCD instantons, a potential solution to the strong CP problem is provided (see Section 4.10).

In the course of spontaneous FS breaking the scalar spurions contained in the Yukawa matrices Y_U and Y_D acquire a VEV. Without loss of generality we work in the basis where the VEV of the up-type Yukawa matrix is diagonal. In the following, we choose a possible scenario of sequential flavour symmetry breaking, where $n_t = 0 < n_b \simeq 2 < n_c \simeq 3 < n_b + 2 \simeq 4 < n_b + 3 \simeq 5 < n_s \simeq 6 < n_{u,d} \simeq 8$ as proposed in [157]. This implies an uniform separation of the breaking scales $\Lambda^{(n)} = \lambda^{(n+1)}\Lambda$ for n greater than one. Corresponding to the specific sequence, the spurion VEVs arising in the (i)-th breaking step can be illustrated by

$$\langle Y_U \rangle \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & (3) & 0 \\ 0 & 0 & (1) \end{pmatrix}, \quad (4.21)$$

$$\langle Y_D \rangle \sim \begin{pmatrix} 0 & 0 & (5) \\ 0 & (6) & (4) \\ 0 & 0 & (2) \end{pmatrix} \hat{=} \exp \left[-i \begin{pmatrix} 0 & 0 & (5) \\ 0 & 0 & (4) \\ (5) & (4) & 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & (6) & 0 \\ 0 & 0 & (2) \end{pmatrix}. \quad (4.22)$$

After the last breaking step (6), in which the strange quark Yukawa coupling is produced, only the global flavour symmetry $G_F^{(6)} = U(1)_{u_R} \times U(1)_{d_R}$ is left. In the discussion below, the VEVs of the Yukawa matrices $\langle Y_{U,D} \rangle$ have to be understood as the ones already produced in the (i)-th breaking step of (4.21) and (4.22). We represent the breaking steps, which are shown in Figure 4.2, in detail in the following because in contrast to [157], we are not allowed to consider linear combinations with the remaining global symmetry generators.

Figure 4.2: *The considered FS breaking pattern.***First Breaking Step (1):**

To determine the residual symmetry which is unbroken by the VEV (4.17), we consider the infinitesimal transformation behaviour of the up-type Yukawa matrix concerning the local part of G_F ,

$$Y'_U = e^{i\alpha_L^a(x) T_L^a} Y_U e^{-i\alpha_{U_R}^a(x) T_{U_R}^a} \simeq Y_U + i\alpha_L^a(x) T_L^a Y_U - i\alpha_{U_R}^a(x) Y_U T_{U_R}^a. \quad (4.23)$$

To simplify the notation we replace Q_L by L , as the quark flavour symmetry acting on the left-handed fermion doublets does not differentiate between the up and the down sector. The unbroken generators, that leave the VEV $\langle Y_U \rangle$ invariant, can be derived from the condition

$$\alpha_L^a(x) T_L^a \langle Y_U \rangle - \alpha_{U_R}^a(x) \langle Y_U \rangle T_{U_R}^a \stackrel{!}{=} 0, \quad (4.24)$$

which leads to two possibilities. First, if both terms vanish separately for the generators $a = 1, 2, 3$, the first summand implies an unbroken $SU(2)_{Q_L}$ symmetry while the second summand corresponds to an unbroken $SU(2)_{U_R}$ symmetry. Second, for the special case of $\alpha_L^8(x) = \alpha_{U_R}^8(x) = \alpha_T(x)$, an extra abelian group $U(1)_T$ remains

$$U(1)_T = e^{i\alpha_T(x)(T_L^8 + T_{U_R}^8)}. \quad (4.25)$$

Further, the symmetry generator of a possible residual global symmetry follows from the condition

$$\alpha_L^a T_L^a \langle Y_U \rangle - \alpha_{U_R}^a \langle Y_U \rangle T_{U_R}^a - \langle Y_U \rangle \theta_{U_R} \mathbb{1} \stackrel{!}{=} 0. \quad (4.26)$$

In addition to the global version of the local $SU(2)_{Q_L} \times SU(2)_{U_R}$ symmetry, an appropriate adjustment of the coefficients $\alpha_L^8, \alpha_{U_R}^8$ with θ_{U_R} leads to a conservation of a global $U(1)_{U_R^{(2)}}$ symmetry. It acts on the two-dimensional flavour subspace, where the original right-handed triplet $U_R = (u_R, c_R, t_R)$ is restricted to the flavour doublet $U_R^{(2)} = (u_R, c_R)$.

Finally, the flavour symmetry is broken by the first scalar spurion VEV to

$$G_F^{(1)} = [SU(2)_{Q_L} \times SU(2)_{U_R} \times U(1)_T \times SU(3)_{D_R}] \times U(1)_{U_R^{(2)}} \times U(1)_{D_R}, \quad (4.27)$$

and the number of broken generators ($16 - 7 = 9$) equals the number of Goldstone bosons.

Second Breaking Step (2):

The next spurion receiving a non-vanishing VEV is the one which generates the bottom quark Yukawa coupling y_b . Taking into account that the term $\alpha_L^a(x) T^a \langle Y_D \rangle$ vanishes independently for $a = 1, 2, 3$ with respect to the infinitesimal gauge transformation

$$\langle Y_D \rangle' \simeq \langle Y_D \rangle + i\alpha_T(x) T_L^8 \langle Y_D \rangle + i\alpha_L^a(x) T_L^a \langle Y_D \rangle - i\langle Y_D \rangle \alpha_{D_R}^c(x) T_{D_R}^c, \quad (4.28)$$

the $SU(2)_{Q_L}$ symmetry is not affected and still remains intact after the breaking. Furthermore, the $SU(3)_{D_R}$ symmetry is broken down to a $SU(2)_{D_R}$ symmetry and for $\alpha_T(x) = \alpha_{D_R}^8(x) = \alpha^8(x)$ the former $U(1)_T$ gauge symmetry changes into the new local abelian symmetry

$$U(1)_8 = e^{i\alpha^8(x)(T_L^8 + T_{U_R}^8 + T_{D_R}^8)}. \quad (4.29)$$

In analogy to the previous breaking step in the up-sector, the global $U(1)_{D_R}$ is reduced to a $U(1)_{D_R^{(2)}}$ symmetry of the two-dimensional subspace.

In summary, the development of the bottom Yukawa coupling breaks the flavour symmetry group $G_F^{(1)}$ down to

$$G_F^{(2)} = [SU(2)_{Q_L} \times SU(2)_{U_R} \times U(1)_8 \times SU(2)_{D_R}] \times U(1)_{U_R^{(2)}} \times U(1)_{D_R^{(2)}}, \quad (4.30)$$

corresponding to five new Goldstone fields.

Third Breaking Step (3):

The third VEV produces the charm Yukawa coupling. Since diagonal matrices commute with themselves (in this case T_L^8 and $T_{U_R}^8$ commute with $\langle Y_U \rangle$), the $U(1)_8$ symmetry remains unbroken and the condition for the unbroken generators simplifies to

$$i\alpha_L^a(x) T_L^a \langle Y_U \rangle - i\langle Y_U \rangle \alpha_{U_R}^a(x) T_{U_R}^a - i\langle Y_U \rangle \theta_{U_R} \mathbb{1} \stackrel{!}{=} 0. \quad (4.31)$$

For $\alpha_L^3(x) = \alpha_{U_R}^3(x) = \alpha^3(x)$ an additional local abelian group arises

$$U(1)_3 = e^{i\alpha^3(x)(T_L^3 + T_{U_R}^3)}, \quad (4.32)$$

which involves the diagonal generators T^3 of $SU(3)_{Q_L}$ and $SU(3)_{U_R}$.

The corresponding orthogonal component with $\alpha_L^3(x) = -\alpha_{U_R}^3(x)$ is broken and produces a new Goldstone boson $\phi^3(x)$. The global residual symmetry of $U(1)_{U_R^{(2)}}$ is reduced to $U(1)_{U_R^{(1)}} = U(1)_{u_R}$. The FS valid at scales below $y_c\Lambda$ is found to be

$$G_F^{(3)} = [SU(2)_{D_R} \times U(1)_8 \times U(1)_3] \times U(1)_{u_R} \times U(1)_{D_R^{(2)}}, \quad (4.33)$$

and altogether five Goldstone degrees of freedom are added to the theory. Thus there are 19 Goldstone and 14 spurion degrees of freedom, combining with the three real VEVs to the 36 degrees of freedom of the two Yukawa matrices.

Fourth Breaking Step (4):

The first non-diagonal spurion VEV arises at the energy scale $E \sim y_b\lambda^2\Lambda$ and produces the CKM rotation angle $\theta_{23} \sim \lambda^2$ (see the linear representation of the CKM matrix in (4.63)). The physical scalar fluctuation $\eta_{23}(x)$ around the angle θ_{23} induces FCNCs at tree level through effective 4-quark operators. Note that the VEV breaks the abelian product group $U(1)_8 \times U(1)_3$ down to one residual local $U(1)$ group

$$U(1)_X = e^{i\theta_X(x)\left(\frac{1}{\sqrt{3}}(T_{Q_L}^8 + T_{U_R}^8 + T_{D_R}^8) + (T_{Q_L}^3 + T_{U_R}^3)\right)}, \quad (4.34)$$

such that at the end of the day

$$G_F^{(4)} = [SU(2)_{D_R} \times U(1)_X] \times U(1)_{u_R} \times U(1)_{D_R^{(2)}}, \quad (4.35)$$

and only one additional Goldstone boson arises in the breaking $G_F^{(3)} \rightarrow G_F^{(4)}$.

Fifth Breaking Step (5):

The spurion fluctuation $\eta_{13}(x)$ around the VEV of the second non-diagonal spurion field of Y_D are related to fluctuations around the CKM angle $\theta_{13} \sim \lambda^3$. In Subsection 4.7.1 we will show explicitly that integrating out the heavy spurion $\eta_{13}(x)$ results in the effective 4-fermion interactions (4.116). As the VEV spontaneously breaks the $U(1)_X$ symmetry, we have to deal with one more Goldstone boson and

$$G_F^{(5)} = [SU(2)_{D_R}] \times U(1)_{u_R} \times U(1)_{D_R^{(2)}} \quad (4.36)$$

represents the remaining flavour symmetry after the fourth breaking step.

Sixth Breaking Step (6):

Finally, with the creation of the strange-quark Yukawa coupling y_s according to (4.22), also the local flavour symmetry of the right-handed down-type quarks $SU(2)_{D_R}$ gets broken, creating three more Goldstone bosons.

In Section 4.10 we will use the residual global symmetry

$$G_F^{(6)} = U(1)_{u_R} \times U(1)_{d_R}, \quad (4.37)$$

acting on the right-handed up quark u_R and down quark d_R , to propose a possible solution to the strong CP problem.

A counting of the Goldstone fields arising in the above discussed breaking steps yields $9 + 5 + 5 + 1 + 1 + 3 = 24$, which is obviously in accordance with the number of generators corresponding to the broken local flavour group $SU(3)^3$.

It is important to notice that the last non-diagonal spurion field χ^{12} is a singlet under the flavour symmetry. Its complex VEV generates the CKM angle θ_{12} and in addition the CP-violating phase δ . Again the off-diagonal fluctuation $\eta_{12}(x)$ will supply a tree-level contribution to FCNCs.

4.4 Parametrisation of the Yukawa Matrices and Unitary Gauge

The aim of this section is to parameterise the Yukawa matrices in such a way that the physical scalar fluctuations are separated from the Goldstone degrees of freedom. The Goldstone bosons, corresponding to the broken generators of the local FS group, become the longitudinal modes of the massive gauge bosons and disappear in the unitary gauge. The discussion of the Goldstone bosons of the global FS is postponed to Section 4.10.

According to the breaking scenario shown in (4.21) and (4.22), with the CP-violating phase δ appearing in the mixing between the first and second generation, we will parameterise the CKM matrix as

$$V_{\text{CKM}} = e^{2i\theta_{23} T^7} e^{2i\theta_{13} T^5} e^{i\delta T^3} e^{2i\theta_{12} T^2} e^{-i\delta T^3}. \quad (4.38)$$

Apart from a redefinition of the CP phase, this representation corresponds to the standard parametrisation of the CKM matrix (see (3.57)–(3.59)) and the power counting of the CKM angles is given by $\theta_{12} \sim \lambda$, $\theta_{23} \sim \lambda^2$ and $\theta_{13} \sim \lambda^3$.

Having fixed the Yukawa VEVs which are responsible for the spontaneous breakdown of the local flavour symmetry

$$\langle Y_U \rangle = \text{Diag}(y_u e^{-i\pi_u}, y_c, y_t), \quad \langle Y_D \rangle = V_{\text{CKM}} \text{Diag}(y_d e^{-i\pi_d}, y_s, y_b), \quad (4.39)$$

where we have made the phases related to the two remaining global symmetries explicit, the next goal is to find a proper parameterisation of the Yukawa matrices. Generally for n quark generations, the $3(n^2 - 1)$ independent generators of the broken local flavour symmetry $SU(n)^3$ are identified with the Goldstone bosons $\phi_{Q_L, U_R, D_R}^a(x)$, $a = 1, \dots, n^2 - 1$. Apart from the two explicit phases for the global $U(1)_{u_R} \times U(1)_{d_R}$ symmetry, the remaining degrees of freedom of the $4n^2$ fluctuations of the two complex $n \times n$ Yukawa matrices correspond to the $(n^2 + 1)$ fluctuations $\eta_i(x)$ of the physical masses and mixing parameters.

Inspired by a singular value decomposition of the Yukawa matrices

$$Y_U(x) = V_{U_L}(x) D_U(x) V_{U_R}^\dagger(x), \quad Y_D(x) = V_{D_L}(x) D_D(x) V_{D_R}^\dagger(x), \quad (4.40)$$

we make an ansatz which actually contains more parameters than scalar degrees of freedom:

$$\begin{aligned} Y_U(x) &= \Sigma_{Q_L}(x) \cdot \Xi_{U_L}(x) D_U(x) \Xi_{U_R}^\dagger(x) \cdot \Sigma_{U_R}(x), \\ Y_D(x) &= \Sigma_{Q_L}(x) \cdot V_{\text{CKM}} \cdot \Xi_{D_L}(x) D_D(x) \Xi_{D_R}^\dagger(x) \cdot \Sigma_{D_R}(x). \end{aligned} \quad (4.41)$$

However, it is consistent with the standard parameterisation proposed in [37, 38] since the broken symmetry generators, representing the Goldstone bosons of $SU(3)^3$,

$$\Sigma_X(x) = \exp[i\phi_X^a(x) T^a], \quad (X = Q_L, U_R, D_R), \quad (4.42)$$

are ‘‘factored’’ out of the residual matrix products $\Xi_{U_L} D_U \Xi_{U_R}^\dagger$ and $\Xi_{D_L} D_D \Xi_{D_R}^\dagger$ which are supposed to contain the physically relevant scalar fields or Higgs modes.

The fluctuations around the VEVs of the quark Yukawa couplings $y_q(x) = y_q + \eta_q(x)/\sqrt{2}$ are contained in the diagonal matrices

$$\begin{aligned} D_U(x) &= \text{Diag}(y_u(x) e^{-i\pi_u(x)}, y_c(x), y_t(x)), \\ D_D(x) &= \text{Diag}(y_d(x) e^{-i\pi_d(x)}, y_s(x), y_b(x)), \end{aligned} \quad (4.43)$$

where the normalisation factor is chosen such that the spurion kinetic terms of the η_q s will come out canonically in (4.45). The non-diagonal physical scalar fluctuations around the VEVs of the CKM angles and phases arise from the matrix field

$$\Xi_X(x) = \exp[i\xi_X^a(x) T^a], \quad (X = U_L, D_L, U_R, D_R; a \neq 3, 8). \quad (4.44)$$

We exclude the diagonal generators T^3 and T^8 in the CKM fluctuations (4.44), as they would reintroduce a complex phase into the spurion fields $y_c(x)$, $y_t(x)$, $y_s(x)$, $y_b(x)$ in (4.43). Thus, the fluctuations along their direction in group space have to be identified as Goldstone-like degrees of freedom, which we have already parameterised by the matrix fields $\Sigma_X(x)$.

To further disentangle the degrees of freedom introduced in the ansatz (4.41), we search for a guideline that tells us which fluctuations have to be assigned to the Goldstone matrix field $\Sigma_X(x)$ and which ones to the matrix field of the physically fluctuations in $\Xi_X(x)$ and $D_{U,D}(x)$, respectively. As pointed out in Section 3.6, the scalar degrees of freedom that have mixings with the gauge fields have to be interpreted as Goldstone fields which will disappear when the physical or unitary gauge $\Sigma_X(x) \rightarrow \mathbb{1}$ is chosen. Equivalently, the scalar fluctuations which do not mix with the $SU(3)^3$ flavour gauge fields $A_{X,\mu}^a(x)$ in the gauge-invariant kinetic terms

$$\Lambda^2 \text{Tr}[(D_\mu Y_U^\dagger)(D^\mu Y_U)] + \Lambda^2 \text{Tr}[(D_\mu Y_D^\dagger)(D^\mu Y_D)], \quad (4.45)$$

where the covariant derivatives stand for

$$\begin{aligned} D_\mu Y_U(x) &= \partial_\mu Y_U(x) - ig_{Q_L} A_{Q_L,\mu}^a(x) T^a Y_U(x) + ig_{U_R} A_{U_R,\mu}^a(x) Y_U(x) T^a, \\ D_\mu Y_D(x) &= \partial_\mu Y_D(x) - ig_{Q_L} A_{Q_L,\mu}^a(x) T^a Y_D(x) + ig_{D_R} A_{D_R,\mu}^a(x) Y_D(x) T^a, \end{aligned} \quad (4.46)$$

are the physical ones. Thus, the ambiguity can be resolved by requiring that our ansatz does not generate any mixing terms between scalar fields and gauge fields if we work in the unitary gauge. Separating the Yukawa matrices into their VEVs and fluctuations around the latter

$$\begin{aligned} Y_U(x) &= \langle Y_U \rangle + \delta Y_U(x), \\ Y_D(x) &= \langle Y_D \rangle + \delta Y_D(x), \end{aligned} \quad (4.47)$$

the mixing terms with the $SU(3)_{Q_L}$ gauge fields get contributions from the kinetic terms both of the up-type Yukawa matrix and of the down-type Yukawa matrix

$$\mathcal{L}_{\text{kin}} \supset \Lambda^2 ig_{Q_L} A_{Q_L,\mu}^c \partial^\mu \text{Tr} [T^c (\delta Y_U(x) \langle Y_U^\dagger \rangle + \delta Y_D(x) \langle Y_D^\dagger \rangle - \text{h.c.})]. \quad (4.48)$$

The mixing terms involving the $SU(3)_{U_R}$ gauge bosons are solely devoted to the up-Yukawa matrix

$$\Lambda^2 ig_{U_R} A_{U_R,\mu}^c \partial^\mu \text{Tr} [T^c (\delta Y_U^\dagger(x) \langle Y_U \rangle - \langle Y_U^\dagger \rangle \delta Y_U(x))], \quad (4.49)$$

while the $SU(3)_{D_R}$ mixing terms only get contributions from the kinetic term of the down-Yukawa matrix

$$\Lambda^2 ig_{D_R} A_{D_R,\mu}^c \partial^\mu \text{Tr} [T^c (\delta Y_D^\dagger(x) \langle Y_D \rangle - \langle Y_D^\dagger \rangle \delta Y_D(x))]. \quad (4.50)$$

Inserting our Yukawa parameterisation ansatz (4.41), we identify the fluctuations according to (4.47),

$$\begin{aligned} \delta Y_U(x) &= Y_U(x) - \langle Y_U \rangle = \Xi_{U_L}(x) D_U(x) \Xi_{U_R}^\dagger(x) - \langle D_U \rangle, \\ \delta Y_D(x) &= Y_D(x) - \langle Y_D \rangle = V_{\text{CKM}} \cdot \Xi_{D_L}(x) D_D(x) \Xi_{D_R}^\dagger(x) - V_{\text{CKM}} \langle D_D \rangle. \end{aligned} \quad (4.51)$$

In the following it is enough to concentrate only on the linear fluctuations which are given by

$$\begin{aligned} \delta Y_U(x) &= (D_U(x) - \langle D_U \rangle) + i \xi_{U_L}^a(x) T^a \langle D_U \rangle - i \langle D_U \rangle \xi_{U_R}^a(x) T^a, \\ \delta Y_D(x) &= V_{\text{CKM}} (D_D(x) - \langle D_D \rangle) + V_{\text{CKM}} (i \xi_{D_L}^a(x) T^a \langle D_D \rangle - i \langle D_D \rangle \xi_{D_R}^a(x) T^a). \end{aligned} \quad (4.52)$$

Moreover, we can focus on the mixing terms between the scalar fluctuations $\xi_i(x)$ and the gauge fields, as the fluctuations around the quark Yukawa couplings

$$\begin{aligned} D_U(x) - \langle D_U \rangle &= \frac{1}{\sqrt{2}} \text{Diag} (\eta_u(x), \eta_c(x), \eta_t(x)), \\ D_D(x) - \langle D_D \rangle &= \frac{1}{\sqrt{2}} \text{Diag} (\eta_d(x), \eta_s(x), \eta_b(x)), \end{aligned} \quad (4.53)$$

do not mix with the $SU(3)^3$ gauge bosons.

In the linear approximation, the requirement that the mixing terms (4.48)–(4.50) should vanish is implemented by imposing

$$\begin{aligned}
A_{U_R}^\mu &: \text{Tr} [T^c (\langle D_U^\dagger \rangle \xi_{U_L}^a T^a \langle D_U \rangle - \langle |D_U|^2 \rangle \xi_{U_R}^a T^a + \text{h.c.})] \stackrel{!}{=} 0, \\
A_{D_R}^\mu &: \text{Tr} [T^c (\langle D_D^\dagger \rangle V_{\text{CKM}}^\dagger \xi_{D_L}^a T^a \langle D_D \rangle - \langle D_D^\dagger \rangle V_{\text{CKM}}^\dagger \langle D_D \rangle \xi_{D_R}^a T^a + \text{h.c.})] \stackrel{!}{=} 0, \\
A_{Q_L}^\mu &: \text{Tr} [T^c (\xi_{U_L}^a T^a \langle |D_U|^2 \rangle - \langle D_U \rangle \xi_{U_R}^a T^a \langle D_U^\dagger \rangle + \text{h.c.}) \\
&\quad + T^c (\xi_{D_L}^a T^a \langle |D_D|^2 \rangle V_{\text{CKM}}^\dagger - \langle D_D \rangle \xi_{D_R}^a T^a \langle D_D^\dagger \rangle V_{\text{CKM}}^\dagger + \text{h.c.})] \stackrel{!}{=} 0. \quad (4.54)
\end{aligned}$$

For demonstration, we will first give the solution to the above conditions (4.54) in the simpler 2-generation case, as there is only one physical spurion $\tilde{\eta}_{12}(x)$, which describes the fluctuation around the Cabibbo angle. In the 3-generation case three physical spurions according to the three CKM angles are involved, which will have kinetic mixings amongst themselves.

4.4.1 Solution of the 2-Generation Case

In the case of two generations, the above conditions (4.54) can be easily solved for the various ξ_X -fields

$$\begin{aligned}
\xi_{U_L}^1(x) &\rightarrow 0, & \xi_{U_L}^2(x) &\rightarrow -F_{12}^2 \frac{y_u^2 + y_c^2}{(y_u^2 - y_c^2)^2} \tilde{\eta}_{12}(x) \simeq -\frac{F_{12}^2}{y_c^2} \tilde{\eta}_{12}(x), \\
\xi_{D_L}^1(x) &\rightarrow 0, & \xi_{D_L}^2(x) &\rightarrow -F_{12}^2 \frac{y_d^2 + y_s^2}{(y_d^2 - y_s^2)^2} \tilde{\eta}_{12}(x) \simeq -\frac{F_{12}^2}{y_s^2} \tilde{\eta}_{12}(x), \\
\xi_{U_R}^1(x) &\rightarrow -F_{12}^2 \frac{2y_u y_c \sin \pi_u}{(y_u^2 - y_c^2)^2} \tilde{\eta}_{12}(x) \simeq 0, & \xi_{U_R}^2(x) &\rightarrow -F_{12}^2 \frac{2y_u y_c \cos \pi_u}{(y_u^2 - y_c^2)^2} \tilde{\eta}_{12}(x) \simeq 0, \\
\xi_{D_R}^1(x) &\rightarrow F_{12}^2 \frac{2y_d y_s \sin \pi_d}{(y_d^2 - y_s^2)^2} \tilde{\eta}_{12}(x) \simeq 0, & \xi_{D_R}^2(x) &\rightarrow F_{12}^2 \frac{2y_d y_s \cos \pi_d}{(y_d^2 - y_s^2)^2} \tilde{\eta}_{12}(x) \simeq 0. \quad (4.55)
\end{aligned}$$

To simplify the notation we have introduced the factor F_{12}^2 ,

$$F_{12}^2 = \frac{2(y_u^2 - y_c^2)^2 (y_d^2 - y_s^2)^2}{(y_u^2 - y_c^2)^2 (y_d^2 + y_s^2) + (y_d^2 - y_s^2)^2 (y_u^2 + y_c^2)}, \quad (4.56)$$

which enters the kinetic term of $\tilde{\eta}_{12}(x)$,

$$\mathcal{L}_{\text{kin}}^{\tilde{\eta}_{12}} = \frac{1}{2} F_{12}^2 \Lambda^2 (\partial^\mu \tilde{\eta}_{12}(x))^2. \quad (4.57)$$

The normalised spurion $\eta_{12}(x)$ is then given by $\eta_{12}(x) = F_{12} \Lambda \tilde{\eta}_{12}(x)$. Note, that we use the tilde to distinguish the unnormalised spurion fields from the normalised ones.

It is important to notice that the fluctuation $\tilde{\eta}_{12}(x)$ occurs symmetrically in the up- and down-quark sector (4.55), despite the fact that our original ansatz assigned the CKM matrix solely to the down-quark Yukawa matrix. Thus the naive replacement $\theta_{12} \rightarrow \theta_{12} + \frac{F_{12}^2 \tilde{\eta}_{12}(x)}{\sqrt{2}}$ in the parameterisation of the CKM matrix (4.38) would have induced spurion couplings solely to the down-type quarks (4.39), and would have led to an incorrect result. In particular this

ansatz would have induced FCNCs only in the down-quark sector — and vice versa only in the up-quark sector, if we had chosen a basis in which the down-type Yukawa is diagonal and the up-type one contains the CKM matrix. Obviously a contradiction would have emerged as the physical observables should not depend on an arbitrarily chosen parametrisation of the Yukawa VEVs. Still, as one can see from the limit $y_{u,d} \ll y_s \ll y_c$, which is added to the exact solutions in (4.55), the coupling of $\tilde{\eta}_{12}(x)$ to d_L and s_R will dominate. One can further observe that this limit would have also been obtained by making the rather minimalistic ansatz

$$\begin{aligned} Y_U(x) &= e^{i\xi_{U_L}^2(x)T^2} D_U(x), \\ Y_D(x) &= V_{\text{CKM}} \cdot e^{i\xi_{D_L}^2 T^2(x)} D_D(x). \end{aligned} \quad (4.58)$$

This corresponds to the case in which the matrices $V_{U_R}^\dagger(x)$, $V_{D_R}^\dagger(x)$ in the singular decomposition ansatz (4.40) are identified — from the early beginning — with the Goldstone degrees of freedom. These Goldstones can then be gauged away independently by the right-handed flavour symmetry group $SU(3)_{U_R} \times SU(3)_{D_R}$. Note again that, besides this “decoupling” of the right-handed flavour symmetry, one cannot use the left-handed $SU(3)_{Q_L}$ symmetry to interpret either $V_{U_L}(x)$ or $V_{D_L}(x)$ as pure Goldstone fields, but has to carefully work out the linear combinations that mix with the gauge fields.

Restricting ourselves to the linear order in the fluctuation $\tilde{\eta}_{12}(x)$ (and omitting the diagonal fluctuations), the Yukawa couplings in the unitary gauge for the 2-generation case read

$$\begin{aligned} Y_U^{\text{u.g.}}(x) &= \text{Diag}(y_u e^{i\pi_u}, y_c) + \begin{pmatrix} 0 & -\frac{y_c}{2(y_c^2 - y_u^2)} \\ -\frac{y_u e^{i\pi_u}}{2(y_c^2 - y_u^2)} & 0 \end{pmatrix} F_{12}^2 \tilde{\eta}_{12}(x), \\ Y_D^{\text{u.g.}}(x) &= V_{\text{CKM}} \cdot \left\{ \text{Diag}(y_d e^{i\pi_d}, y_s) + \begin{pmatrix} 0 & \frac{y_s}{2(y_s^2 - y_d^2)} \\ \frac{y_d e^{i\pi_d}}{2(y_s^2 - y_d^2)} & 0 \end{pmatrix} F_{12}^2 \tilde{\eta}_{12}(x) \right\}. \end{aligned} \quad (4.59)$$

The coupling matrices of the $\tilde{\eta}_{12}(x)$ field can be expressed entirely in terms of the VEVs of the Yukawa matrices, e.g. for the up-type Yukawa,

$$\begin{aligned} \delta Y_U|_{\tilde{\eta}_{12}} &\equiv F_{12}^2 \begin{pmatrix} 0 & -\frac{y_c}{2(y_c^2 - y_u^2)} \\ -\frac{y_u e^{i\pi_u}}{2(y_c^2 - y_u^2)} & 0 \end{pmatrix} \\ &= \frac{F_{12}^2}{y_c^2 - y_u^2} \left\{ -\frac{y_c^2 y_s^2 \tan \theta_{12} + y_c^2 y_d^2 \cot \theta_{12} - y_u^2 y_s^2 \cot \theta_{12} - y_u^2 y_d^2 \tan \theta_{12}}{2(y_c^2 - y_u^2)(y_d^2 - y_s^2)} \langle Y_U \rangle \right. \\ &\quad \left. + \frac{\cot 2\theta_{12}}{y_c^2 - y_u^2} \langle Y_U Y_U^\dagger Y_U \rangle - \frac{\csc 2\theta_{12}}{y_s^2 - y_d^2} \langle Y_D Y_D^\dagger Y_U \rangle \right\} \\ &\simeq \frac{y_s^2 \tan \theta_{12}}{y_c^2} \langle Y_U \rangle + \frac{2y_s^2 \cot 2\theta_{12}}{y_c^4} \langle Y_U Y_U^\dagger Y_U \rangle - \frac{2 \csc 2\theta_{12}}{y_c^2} \langle Y_D Y_D^\dagger Y_U \rangle, \end{aligned} \quad (4.60)$$

and an analogous relation for $\delta Y_D|_{\tilde{\eta}_{12}}$. Here, the first identity in (4.60) holds in the basis where $\langle Y_U \rangle$ is diagonal and $\langle Y_D \rangle = V_{\text{CKM}} D_D$, while the second and third line are basis independent. Obviously, when inserted into $Q_L \dots U_R$, the three different structures $\langle Y_U \rangle$, $\langle Y_U Y_U^\dagger Y_U \rangle$ and

$\langle Y_D Y_D^\dagger Y_U \rangle$ of spurion insertions correspond to gauge-invariant combinations of the flavour symmetry group. We have already encountered the first operator with one spurion insertion in (4.3). At the level of two spurion insertions there is no possible left-right coupling, so that the three spurion operators in (4.60) are really the next-to-minimal ones. From the MFV perspective, we thus expect the coefficients in front of the three individual flavour structures to be of $\mathcal{O}(1)$ or smaller. Taking into account that $\tilde{\eta}_{12}(x)$ scales as θ_{12} , we obtain in the approximation $y_c \sim \mathcal{O}(1)$, $y_c \gg y_s \gg y_{u,d}$

$$\frac{y_s^2 \tan \theta_{12}}{y_c^2} \tilde{\eta}_{12} \sim y_s^2 \theta_{12}^2 \ll 1, \quad \frac{2y_s^2 \cot 2\theta_{12}}{y_c^4} \tilde{\eta}_{12} \sim y_s^2 \ll 1, \quad \frac{2 \csc 2\theta_{12}}{y_c^2} \tilde{\eta}_{12} \sim 1, \quad (4.61)$$

and find that the third coefficient in the above expansion is the dominant one. It is interesting to note that

$$\text{Tr} [\langle Y_U^\dagger \rangle \delta Y_U |_{\tilde{\eta}_{12}}] = \text{Tr} [\langle Y_U^\dagger Y_U Y_U^\dagger \rangle \delta Y_U |_{\tilde{\eta}_{12}}] = 0, \quad (4.62)$$

which shows that our construction for $\tilde{\eta}_{12}(x)$ indeed involves a variation that is orthogonal to the VEV of Y_U . An analogous statement holds for δY_D and $\langle Y_D \rangle$.

4.4.2 Solution of the 3-Generation Case

In the 3-family case, the situation is more complicated since the generators for the 3 CKM rotations do not commute anymore, and the kinetic terms for the related spurion fields $\tilde{\eta}_{12}(x)$, $\tilde{\eta}_{13}(x)$ and $\tilde{\eta}_{23}(x)$ will also mix. This mixing is controlled by the CKM matrix and the ratios of quark Yukawa couplings. We can identify the leading effects by expanding (4.38) to first order in the off-diagonal CKM elements $V_{i \neq j}$

$$V_{\text{CKM}} = \begin{pmatrix} 1 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 1 & \theta_{23} \\ -\theta_{13} & -\theta_{23} & 1 \end{pmatrix}. \quad (4.63)$$

If we further set $y_u = y_d = 0$ and neglect the CP phase, the ansatz reduces to

$$\Xi_X(x) = \exp [i \xi_X^a(x) T^a], \quad a = 2, 5, 7, \quad X = U_L, D_L, U_R, D_R. \quad (4.64)$$

With these approximations, the conditions (4.54) lead to the following fluctuations around the CKM angles

$$\begin{aligned} \delta Y_U^{\text{u.g.}}(x) \supset & \frac{1}{2} \begin{pmatrix} 0 & -\frac{y_s^2}{y_c} & \frac{\theta_{23} y_s^2}{y_t} \\ 0 & 0 & \frac{\theta_{13} y_s^2 y_t}{y_c^2 - y_t^2} \\ 0 & \frac{\theta_{13} y_c y_s^2}{y_c^2 - y_t^2} & 0 \end{pmatrix} \tilde{\eta}_{12}(x) + \frac{1}{2} \begin{pmatrix} 0 & -\frac{\theta_{23} y_b^2}{y_c} & -\frac{y_b^2}{y_t} \\ 0 & 0 & \frac{\theta_{12} y_b^2 y_t}{(y_t^2 - y_c^2)} \\ 0 & -\frac{\theta_{12} y_b^2 y_c}{(y_c^2 - y_t^2)} & 0 \end{pmatrix} \tilde{\eta}_{13}(x) \\ & + \frac{1}{2} \begin{pmatrix} 0 & \frac{\theta_{13} (y_b^2 - y_s^2)^2}{y_c (y_b^2 + y_s^2)} & -\frac{\theta_{12} (y_b^2 - y_s^2)^2}{y_t (y_b^2 + y_s^2)} \\ 0 & 0 & \frac{y_t (y_b^2 - y_s^2)^2}{(y_b^2 + y_s^2)(y_c^2 - y_t^2)} \\ 0 & \frac{y_c (y_b^2 - y_s^2)^2}{(y_b^2 + y_s^2)(y_c^2 - y_t^2)} & 0 \end{pmatrix} \tilde{\eta}_{23}(x), \end{aligned} \quad (4.65)$$

and

$$\begin{aligned}
V_{\text{CKM}}^\dagger \delta Y_D^{\text{u.g.}}(x) \supset & \begin{pmatrix} 0 & \frac{y_s}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{\eta}_{12}(x) + \begin{pmatrix} 0 & 0 & \frac{y_b}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{\eta}_{13}(x) \\
& + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{y_b(y_b^2 - y_s^2)}{2(y_b^2 + y_s^2)} \\ 0 & \frac{y_s(y_b^2 - y_s^2)}{2(y_b^2 + y_s^2)} & 0 \end{pmatrix} \tilde{\eta}_{23}(x). \tag{4.66}
\end{aligned}$$

For simplicity we again omit the diagonal spurion fluctuations and restrict ourselves to the linear contributions. As already mentioned in the beginning, inserting the spurion fluctuations (4.65)–(4.66) into the spurion kinetic terms (4.45) induces mixings in the kinetic terms of the various spurions which will be the topic of the next subsection.

Diagonalisation of the Spurion Kinetic Terms

In order to obtain the usual field normalisation and basis, we will diagonalise and rescale the fields such that they are canonically normalised afterwards and the full information about the flavour structure is encoded in the Yukawa matrices [163].

As we do not specify the spurion potential, we can use the freedom to carry out a unitary transformation U on an already normalised and diagonalised kinetic term

$$\left(\partial_\mu \tilde{\eta}_{12}, \partial_\mu \tilde{\eta}_{13}, \partial_\mu \tilde{\eta}_{23} \right) \mathbb{1} \begin{pmatrix} \partial^\mu \tilde{\eta}_{12} \\ \partial^\mu \tilde{\eta}_{13} \\ \partial^\mu \tilde{\eta}_{23} \end{pmatrix} \equiv \left(\partial_\mu \tilde{\eta}_{12}, \partial_\mu \tilde{\eta}_{13}, \partial_\mu \tilde{\eta}_{23} \right) U^\dagger U \begin{pmatrix} \partial^\mu \tilde{\eta}_{12} \\ \partial^\mu \tilde{\eta}_{13} \\ \partial^\mu \tilde{\eta}_{23} \end{pmatrix} \tag{4.67}$$

to diagonalise $\text{Tr}[(\partial_\mu Y_U)(\partial^\mu Y_U^\dagger)]$ and $\text{Tr}[(\partial_\mu Y_D)(\partial^\mu Y_D^\dagger)]$ simultaneously.

Therefore, one can start with a diagonalisation of the spurion mixing terms stemming from $\text{Tr}[(\partial_\mu Y_U)(\partial^\mu Y_U^\dagger)]$, followed by an appropriate normalisation of the new states. The above mentioned freedom then ensures that diagonalising the mixing terms from the down-Yukawa spurion matrix $\text{Tr}[(\partial_\mu Y_D)(\partial^\mu Y_D^\dagger)]$ will not reintroduce mixing terms in the up sector. The calculational details about how to obtain the symmetric distribution of the kinetic terms from the up- and down-quark sector,

$$\begin{aligned}
\text{Tr}[(\partial_\mu Y_U)(\partial^\mu Y_U^\dagger)] &= \frac{1}{y_c^2} (\partial_\mu \tilde{\eta}_{12}(x))^2 + \frac{1}{y_t^2} (\partial_\mu \tilde{\eta}_{13}(x))^2 + \frac{(y_s^2 - y_b^2)^2}{(y_s^2 + y_b^2)} (\partial_\mu \tilde{\eta}_{23}(x))^2, \\
\text{Tr}[(\partial_\mu Y_D)(\partial^\mu Y_D^\dagger)] &= \frac{1}{y_s^2} (\partial_\mu \tilde{\eta}_{12}(x))^2 + \frac{1}{y_b^2} (\partial_\mu \tilde{\eta}_{13}(x))^2 + \frac{(y_c^2 - y_t^2)^2}{(y_c^2 + y_t^2)} (\partial_\mu \tilde{\eta}_{23}(x))^2, \tag{4.68}
\end{aligned}$$

can be found in the Appendix A.4. In (4.68) we reintroduced the tilde-notation after the various field redefinitions as a reminder that the spurion kinetic terms are accompanied by

the normalisation factors F_{ij} , which are defined as before,

$$F_{ij}^2 = \frac{2(y_{U_i}^2 - y_{U_j}^2)^2 (y_{D_i}^2 - y_{D_j}^2)^2}{(y_{U_i}^2 - y_{U_j}^2)^2 (y_{D_i}^2 + y_{D_j}^2) + (y_{D_i}^2 - y_{D_j}^2)^2 (y_{U_i}^2 + y_{U_j}^2)}. \quad (4.69)$$

Finally, the Yukawa matrices in the new basis change into

$$\begin{aligned} Y_U^{\text{u.g.}}(x) &\simeq \text{Diag}(0, y_c, y_t) + \begin{pmatrix} 0 & \frac{1}{y_c} & -\frac{\theta_{23} y_b^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & \frac{\theta_{13} y_b^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & \frac{\theta_{13} y_b^2 y_c}{y_b^2 y_c^2 - y_s^2 y_t^2} & 0 \end{pmatrix} \frac{1}{2} F_{12}^2 \tilde{\eta}_{12}(x) \\ &+ \begin{pmatrix} 0 & \frac{\theta_{23} y_b^2 y_c}{y_b^2 y_c^2 - y_s^2 y_t^2} & \frac{1}{y_t} \\ 0 & 0 & \frac{\theta_{12} y_s^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2} F_{13}^2 \tilde{\eta}_{13}(x) + \begin{pmatrix} 0 & -\frac{\theta_{13} y_b^2 y_c}{y_b^2 y_c^2 - y_s^2 y_t^2} & -\frac{\theta_{12} y_s^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & \frac{1}{y_t} \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2} F_{23}^2 \tilde{\eta}_{23}(x), \end{aligned} \quad (4.70)$$

and

$$\begin{aligned} V_{\text{CKM}}^\dagger Y_D^{\text{u.g.}}(x) &\simeq \text{Diag}(0, y_s, y_b) + \begin{pmatrix} 0 & -\frac{1}{y_s} & \frac{\theta_{23} y_b y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & -\frac{\theta_{13} y_b y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & -\frac{\theta_{13} y_s y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} & 0 \end{pmatrix} \frac{1}{2} F_{12}^2 \tilde{\eta}_{12}(x) \\ &+ \begin{pmatrix} 0 & -\frac{\theta_{23} y_s y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} & -\frac{1}{y_b} \\ 0 & 0 & -\frac{\theta_{12} y_b y_c^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2} F_{13}^2 \tilde{\eta}_{13}(x) + \begin{pmatrix} 0 & \frac{\theta_{13} y_s y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} & \frac{\theta_{12} y_b y_c^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & -\frac{1}{y_b} \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2} F_{23}^2 \tilde{\eta}_{23}(x), \end{aligned} \quad (4.71)$$

where we have assumed $y_s \ll y_b$ and $y_c \ll y_t$, and have omitted the fluctuations around the quark Yukawa couplings.

As in the 2-family case, in this special limit the same result can be obtained with the minimalistic ansatz

$$\begin{aligned} Y_U(x) &= e^{i\xi_{U_L}^2(x)T^2} e^{i\xi_{U_L}^5(x)T^5} e^{i\xi_{U_L}^7(x)T^7} D_U(x), \\ Y_D(x) &= V_{\text{CKM}} \cdot e^{i\xi_{D_L}^2(x)T^2} e^{i\xi_{D_L}^5(x)T^5} e^{i\xi_{D_L}^7(x)T^7} D_D(x), \end{aligned} \quad (4.72)$$

which implies that the linear combinations of the physical fluctuations consist only of left-handed fields

$$\begin{aligned} \xi_{U_L}^2(x) &\rightarrow \frac{-y_s^2 \xi_{D_L}^2(x) - \theta_{23} y_b^2 \xi_{D_L}^5(x) + \theta_{13} y_b^2 \xi_{D_L}^7(x)}{y_c^2}, \\ \xi_{U_L}^5(x) &\rightarrow \frac{-y_b^2 \xi_{D_L}^5(x) + \theta_{23} y_s^2 \xi_{D_L}^2(x) - \theta_{12} y_b^2 \xi_{D_L}^7(x)}{y_t^2}, \\ \xi_{U_L}^7(x) &\rightarrow \frac{-y_b^2 \xi_{D_L}^7(x) - \theta_{13} y_s^2 \xi_{D_L}^2(x) + \theta_{12} y_b^2 \xi_{D_L}^5(x)}{y_t^2}. \end{aligned} \quad (4.73)$$

With the identification

$$\xi_{D_L}^2(x) \equiv \tilde{\eta}_{12}(x), \quad \xi_{D_L}^5(x) \equiv \tilde{\eta}_{13}(x), \quad \xi_{D_L}^7(x) \equiv \tilde{\eta}_{23}(x), \quad (4.74)$$

we are able to reproduce the ‘‘asymmetric result’’ given in (4.65)–(4.66) in the limit $y_s \ll y_b$ and $y_c \ll y_t$.

Again, the fluctuations are orthogonal to the Yukawa VEVs

$$\text{Tr}[\langle Y_U^\dagger \rangle \delta Y_U] = \text{Tr}[\langle Y_U^\dagger Y_U Y_U^\dagger \rangle \delta Y_U] = \text{Tr}[\langle Y_U^\dagger Y_U Y_U^\dagger Y_U Y_U^\dagger \rangle \delta Y_U] = 0, \quad (4.75)$$

and accordingly for the down Yukawa fields. Furthermore, the contributions to the invariants $\text{Tr}[(Y_U^\dagger Y_U)^n]$ as well as $\text{Tr}[(Y_D^\dagger Y_D)^n]$, appearing in the spurion potential [157], are diagonal in the spurion fields $\tilde{\eta}_{ij}(x)$.

In addition to the subleading terms in the kinetic mixing terms, further corrections would be induced by radiative corrections involving the Yukawa couplings. By construction, we expect these effects to follow the MFV principle, in a similar way as we have discussed for the 2-family example. A precise calculation of these terms is beyond the scope of this work.

4.5 Simplified Scenario

To keep the following discussion transparent, we will actually consider the smaller flavour symmetry group which appears in the fourth intermediate breaking step in the sequence of flavour symmetry breaking, already derived in Section 4.3

$$G_F^{(4)} = [SU(2)_{D_R} \times U(1)_X] \times U(1)_{u_R} \times U(1)_{D_R^{(2)}}. \quad (4.76)$$

Since the masses of the scalar and gauge fields are closely related to the breaking scale, the higher-dimensional operators originating from the highest breaking scales will be more suppressed as the ones arising from the lower breaking steps. Hence this approximation still comprehends the most severe bounds to FCNCs for a future phenomenological analysis.

In $G_F^{(4)}$ the two $U(1)$ factors are global and while $D_R^{(2)} = (d_R, s_R)$ is restricted to a right-handed flavour doublet of down-type quarks, the $U(1)_{u_R}$ symmetry acts solely on the right-handed up-quark. At this stage, the VEVs of the Yukawa matrices contain the three eigenvalues y_t, y_b, y_c and the mixing angle between the 2^{nd} and 3^{rd} generation

$$\langle Y_U \rangle \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \bullet \end{pmatrix}, \quad \langle Y_D \rangle \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bullet \\ 0 & 0 & \bullet \end{pmatrix} \hat{=} \exp \left[-i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bullet \\ 0 & \bullet & 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \bullet \end{pmatrix}. \quad (4.77)$$

Apart from an intact local $SU(2)_{D_R}$ symmetry, these vacua respect the local abelian $U(1)_X$ symmetry (4.34), whose generator is closely related to the previous breaking cascade of the

ψ_L	:	$Q_L^{(1)}$	$Q_L^{(2)}$	$Q_L^{(3)}$	u_R^c	c_R^c	t_R^c	D_R^c	b_R^c
q_X	:	$+\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$+\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{6}$	$+\frac{1}{3}$

Table 4.2: q_X quantum numbers of quarks

flavour group — in detail it is given in terms of the diagonal symmetry generators of the original FS group G_F by

$$Q_X = \frac{1}{\sqrt{3}} (T_{Q_L}^8 + T_{U_R}^8 + T_{D_R}^8) + (T_{Q_L}^3 + T_{U_R}^3). \quad (4.78)$$

In this context, we want to emphasise that the asymmetric treatment of the up- and down-type quark sector in the definition of the charge operator Q_X is correlated with the particular choice for the parameterisation of the Yukawa matrices, where $\langle Y_U \rangle$ is diagonal, while $\langle Y_D \rangle$ contains the CKM rotations.

In the next section we will identify the gauge anomalies of the $G_F^{(4)}$ FS group under the SM fermion content in order to construct an effective theory in Section 4.6 that is at least formally gauge invariant under the $G_F^{(4)}$ FS group. We will proceed in Section 4.7 to derive the effective theory of $G_F^{(5)}$, and in Section 4.8 the corresponding one of $G_F^{(6)} = U(1)_{u_R} \times U(1)_{d_R}$, via integrating out the spurions and gauge fields that receive masses from the Higgs mechanisms at the breaking steps.

4.5.1 Anomalies of the Local Flavour Symmetry Group $SU(2)_{D_R} \times U(1)_X$

In Section 2.5 we gave a short introduction into the comprehensive topic of chiral gauge anomalies and showed in Section 3.2 that the SM fermion content is non-anomalous with respect to the SM gauge group G_{SM} . In our simplified setup we augment the SM gauge group with a partly gauged flavour symmetry group, such that the total local gauge group corresponds to the direct product group $G_F^{(4)}|_{\text{local}} \times G_{\text{SM}}$. Keeping in mind that all representations of the $SU(2)$ group are real and anomaly free, we thus have to check whether we encounter $U(1)_X$ gauge anomalies or mixed anomalies with the hypercharge and colour group. To do so, we summarise the various $U(1)_X$ charges q_X in Table 4.2, which denote the eigenvalues of the charge operator Q_X concerning the left-handed fermions ψ_L . Note that the fermions that transform as irreducible representations of $G_F^{(4)} \times G_{\text{SM}}$ are the same as in the SM.

It turns out that the representation ψ_L of $U(1)_X$ is anomalous,

$$\text{Tr}[Q_X^3] = \frac{3}{4} \neq 0. \quad (4.79)$$

Due to the fact that the generator Q_X is a linear combination of $SU(3)^3$ generators, its trace $\text{Tr}[Q_X]$ vanishes individually for every SM gauge multiplet, and therefore we will not

encounter mixed anomalies with $SU(N)$ generators, where

$$\text{Tr} [\{T^a, T^b\} Q_X] \propto \delta^{ab} \text{Tr} [Q_X] = 0. \quad (4.80)$$

While the triangle diagram contributions involving two hypercharge gauge fields and one $U(1)_X$ gauge field vanish as well, there is a mixed anomaly with the SM hypercharge group from triangle diagrams involving two $U(1)_X$ gauge fields and one hypercharge gauge field,

$$\text{Tr} [Q_X^2 Y] = -1 \neq 0. \quad (4.81)$$

The assignment of hypercharge quantum numbers has already been given in Table 3.1, but for the sake of completeness let us repeat that they are normalised according to $Y(Q_L) = 1/3$, $Y(U_R) = 4/3$ and $Y(D_R) = -2/3$.

4.6 Effective Lagrangian of the Gauged Flavour Subgroup

In deriving a consistent formulation of an effective field theory including an anomalous gauge symmetry, we will closely follow the formalism given in [89]. The usual requirement that chiral gauge anomalies must cancel in order to avoid a breakdown of gauge invariance at the quantum level, severely restricts the representation content of the fermions transforming under the gauge group.

However, from an effective field theory point of view, it is legitimate to assume that there exists an underlying fundamental theory which is anomaly free because it contains new fermions that contribute to the anomaly coefficient. When the gauge symmetry is spontaneously broken by a Higgs mechanism, these fermions, as well as the Higgs boson that induces the breaking, become heavy and thus are “integrated out” of the theory. It is then quite natural that they leave an uncancelled anomaly contribution of the remaining light fermions in the effective theory. It is possible that the gauge bosons receive only small masses from the Higgs mechanism, if the corresponding gauge coupling is weak. The fact that the gauge bosons nevertheless acquire a small mass ensures unitarity and Lorentz invariance of the theory.

Thus we deal with a consistent non-renormalisable effective field theory of massive gauge fields coupled to fermions in an anomalous representation. It is convenient to make the theory at least formally gauge invariant by introducing so-called Wess–Zumino counterterms. They couple pure-gauge dynamical scalar fields to the gauge fields in order to compensate the fermion anomaly contributions.

4.6.1 Anomalous $U(1)_X$ Symmetry

The above mentioned procedure is now applied to the case of the anomalous $U(1)_X$ symmetry whose generator Q_X (4.78) acts on the chiral SM fermion representations. We showed already that there is also a mixed anomaly with the gauged $U(1)_Y$ symmetry. The classical part of

the Lagrangian involving the $U(1)_X$ gauge field reads

$$\begin{aligned}\mathcal{L}_X &= -\frac{1}{4} X_{\mu\nu}(x) X^{\mu\nu}(x), \\ \mathcal{L}_\psi &= \bar{\psi}_L(x) i \not{D} \psi_L(x) \quad \text{and} \quad \mathcal{L}_{\text{spurion}},\end{aligned}\tag{4.82}$$

where the covariant derivative $D_\mu = \partial_\mu - ig_X q_X X_\mu + \dots$ contains the couplings of the various fermion species to the $U(1)_X$ gauge boson. While the classical Lagrangian is invariant under the $U(1)_X$ gauge transformation

$$X'_\mu(x) = X_\mu(x) + \frac{1}{g_X} \partial_\mu \omega_X(x), \quad \psi'_L(x) = e^{i\omega_X(x) q_X} \psi_L(x),\tag{4.83}$$

the quantum effective action changes since the fermion representation is anomalous with the non-vanishing anomaly coefficients (4.79) and (4.81) presented in the previous section.

The general formula of the abelian anomaly can be derived from the formula of the non-abelian anomaly given in (2.32) by replacing the generator $\tilde{T}^a \rightarrow Q$, and by taking into account that the second contribution disappears due to the commutativity of the abelian gauge fields

$$\partial_\mu j_L^\mu = G[A_\mu^L(x)] = -\frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} [Q^3 \partial_\mu A_\nu^L \partial_\rho A_\sigma^L] = -\frac{g^2}{48\pi^2} \text{Tr} [Q^3] F_{\mu\nu} \tilde{F}^{\mu\nu}.\tag{4.84}$$

In the last step we made the coupling constants explicit and introduced the abelian dual field-strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.\tag{4.85}$$

Under the abelian gauge transformation

$$A'_\mu(x) = A_\mu(x) + \frac{1}{g} \partial_\mu \theta(x), \quad \psi'_L(x) = e^{i\theta(x) Q} \psi_L(x),\tag{4.86}$$

the effective action Γ changes according to (2.34)

$$\delta_v \Gamma[A_\mu^L(x)] = - \int d^4x v(x) G[A_\mu^L(x)] = \frac{g^2}{48\pi^2} \text{Tr} [Q^3] \int d^4x v(x) F_{\mu\nu} \tilde{F}^{\mu\nu}.\tag{4.87}$$

Adapting the above formula to local $U(1)_X$ and $U(1)_Y$ gauge transformations, implies the following total change of the effective action

$$\begin{aligned}\delta_{\omega_X} \Gamma &= \text{Tr} [Q_X^3] \frac{g_X^2}{48\pi^2} \int d^4x \omega_X X_{\mu\nu} \tilde{X}^{\mu\nu} \\ &\quad + c_1 \text{Tr} [Q_X^2 Y] \frac{g_Y g_X}{48\pi^2} \int d^4x \omega_X X_{\mu\nu} \tilde{Y}^{\mu\nu}, \\ \delta_{\omega_Y} \Gamma &= (1 - c_1) \text{Tr} [Q_X^2 Y] \frac{g_X^2}{48\pi^2} \int d^4x \omega_Y X_{\mu\nu} \tilde{X}^{\mu\nu}.\end{aligned}\tag{4.88}$$

The coefficient c_1 arises from the freedom to add an appropriate local counterterm [89],

$$\Gamma_{\text{c.t.}} = c_1 \text{Tr} [Q_X^2 Y] \frac{g_X^2 g_Y}{24\pi^2} \int d^4x \epsilon_{\mu\nu\lambda\sigma} Y^\mu X^\nu \partial^\lambda X^\sigma,\tag{4.89}$$

which will allow to attach the mixed anomaly completely to the $U(1)_X$ gauge transformation, as discussed below. Using the infinitesimal form of the abelian gauge transformation (2.20),

$$\delta_{\omega_X} X_\mu = \frac{1}{g_X} \partial_\mu \omega_X, \quad \delta_{\omega_Y} Y_\mu = \frac{1}{g_Y} \partial_\mu \omega_Y, \quad (4.90)$$

the local counterterm changes according to

$$\begin{aligned} \delta_{\omega_Y} \Gamma_{\text{c.t.}} &= c_1 \text{Tr} [Q_X^2 Y] \frac{g_X^2}{48\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} (\partial_\mu \omega_Y) X_\nu \partial_\rho X_\sigma \\ &= -c_1 \text{Tr} [Q_X^2 Y] \frac{g_X^2}{48\pi^2} \int d^4x \omega_Y(x) X_{\mu\nu} \tilde{X}^{\mu\nu}, \\ \delta_{\omega_X} \Gamma_{\text{c.t.}} &= c_1 \text{Tr} [Q_X^2 Y] \frac{g_X g_Y}{48\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} Y_\mu (\partial_\nu \omega_X) \partial_\rho X_\sigma \\ &\stackrel{(\text{IBP}, \mu \leftrightarrow \nu)}{=} c_1 \text{Tr} [Q_X^2 Y] \frac{g_X g_Y}{48\pi^2} \int d^4x \omega_X(x) X_{\mu\nu} \tilde{Y}^{\mu\nu}. \end{aligned} \quad (4.91)$$

Choosing the renormalisation condition $c_1 = 1$, one is able to obtain a manifestly non-anomalous $U(1)_Y$ gauge symmetry with $\delta_{\omega_Y} \Gamma = 0$. Thus only the $U(1)_X$ gauge symmetry is spoiled by anomalous contributions to the effective action

$$\delta_{\omega_X} \Gamma = \frac{1}{48\pi^2} \int d^4x \omega_X \left(\text{Tr} [Q_X^3] g_X^2 X_{\mu\nu} \tilde{X}^{\mu\nu} + \text{Tr} [Q_X^2 Y] g_X g_Y X_{\mu\nu} \tilde{Y}^{\mu\nu} \right). \quad (4.92)$$

It is essential to keep the $U(1)_Y$ gauge symmetry exact until it is spontaneously broken by the usual Higgs mechanism at the electroweak scale since otherwise the $U(1)_Y$ gauge boson would get anomaly contributions to its mass and no light SM Z boson would emerge.

Still, the local gauge invariance of the anomalous $U(1)_X$ symmetry can be formally restored by exploiting the behaviour of the Goldstone field $\pi_X(x)$ under a gauge transformation,

$$\pi'_X(x) = \pi_X(x) + \omega_X(x). \quad (4.93)$$

Adding the following term to the effective Lagrangian,

$$\Delta \mathcal{L}_\pi = - \frac{\pi_X(x)}{48\pi^2} \left(\text{Tr} [Q_X^3] g_X^2 X_{\mu\nu}(x) \tilde{X}^{\mu\nu}(x) + \text{Tr} [Q_X^2 Y] g_X g_Y X_{\mu\nu}(x) \tilde{Y}^{\mu\nu}(x) \right), \quad (4.94)$$

obviously compensates the change in Γ from the fermion measure in (4.92). On the quantum level, loop corrections involving the anomalous couplings of the Goldstone mode π_X to the gauge fields in (4.94) will lead to a mass term for the $U(1)_X$ gauge boson. In addition, the latter will also receive a mass contribution (4.107) from the spurion VEVs, originating from the ‘‘Higgsing’’ of the Yukawa matrices. Defining the total mass of the $U(1)_X$ gauge boson as

$$M_X \equiv g_X F_X, \quad (4.95)$$

the diagrams from the anomaly contributions to the mass are quadratically divergent and contribute to F_X as

$$F_X \supset \frac{g_X^2 \text{Tr} [Q_X^3]}{64\pi^3} \Lambda, \quad \text{and} \quad \frac{g_Y g_X \text{Tr} [Y Q_X^2]}{64\pi^3} \Lambda, \quad \text{respectively.} \quad (4.96)$$

F_X is a dimensional constant such that $(F_X \pi_X)$ in the quadratic term of the effective Lagrangian

$$\mathcal{L}_\pi = \frac{F_X^2}{2} (\partial_\mu \pi_X(x) - g_X X_\mu(x))^2 \quad (4.97)$$

has canonical dimensions and a correctly normalised kinetic term.

Covariant gauges can be introduced via the gauge-fixing term

$$\mathcal{L}_{\text{gfix}} = -\frac{1}{2\xi_X} (\partial_\mu X^\mu(x) - \xi_X g_X F_X^2 \pi_X(x))^2, \quad (4.98)$$

which removes the mixing term between $X^\mu(x)$ and $\pi_X(x)$ in \mathcal{L}_π .

In summary we are left with the quadratic terms

$$\frac{M_X^2}{2} (X_\mu)^2 - \frac{1}{2\xi_X} (\partial_\mu X^\mu)^2 - \xi_X \frac{M_X^2}{2} (F_X \pi_X)^2 + \frac{1}{2} (F_X \partial_\mu \pi_X)^2. \quad (4.99)$$

In the 't Hooft-Landau gauge the gauge parameter ξ_X vanishes and the π_X field is massless. In unitary gauge, corresponding to $\xi_X \rightarrow \infty$, the Goldstone field decouples and disappears from the theory.

4.6.2 Local $SU(2)_{D_R}$ Flavour Symmetry

Up to now, we focused on the contributions to the effective Lagrangian due to the local abelian anomalous $U(1)_X$ symmetry. Since the $SU(2)_{D_R}$ symmetry group is not anomalous, the discussion is somewhat simpler than in the previous case. The effective Lagrangian contains a kinetic term for the $SU(2)_{D_R}$ gauge fields

$$\mathcal{L}_{A_{D_R}} = -\frac{1}{4} F_{D_R, \mu\nu}^a(x) F_{D_R}^{a, \mu\nu}(x), \quad F_{D_R, \mu\nu}^a(x) = \partial_\mu A_{D_R, \nu}^a - \partial_\nu A_{D_R, \mu}^a + g_{D_R} \epsilon^{abc} A_{D_R, \mu}^b A_{D_R, \nu}^c, \quad (4.100)$$

as well as couplings to the SM fermions

$$\mathcal{L}_\psi = \bar{\psi}_L(x) i \not{D} \psi_L(x) \supset g_{D_R} A_{D_R, \mu}^a [\bar{D}_R \gamma^\mu T^a D_R] \equiv g_{D_R} A_{D_R, \mu}^a (J_{A_{D_R}}^\mu)^a. \quad (4.101)$$

In analogy to (4.95) we define the mass of the gauge bosons by

$$m_{A_{D_R}}^2 \equiv g_{D_R}^2 F_{D_R}^2, \quad (4.102)$$

where this time the value of F_{D_R} is solely determined by the couplings of the gauge bosons to the spurion Yukawas. As in the previous case, after spontaneous breaking of the flavour symmetry, appropriate gauge-fixing terms for the $SU(2)_{D_R}$ gauge fields can be added to cancel the arising mixing terms with the Goldstone fields.

4.6.3 Yukawa Spurion Lagrangian

Apart from the new interactions to the flavour gauge bosons, the fermion couplings to the SM gauge fields maintain their standard form. The SM Lagrangian remains unchanged except for the Yukawa sector where the Yukawa coupling matrices have to be replaced by dynamical fields. Starting from the general discussion in Section 4.4, we can drop all Goldstone modes and spurion fields related to the breaking $G_F \rightarrow G_F^{(4)}$, and obtain

$$\begin{aligned} Y_U &\rightarrow Y_U(x) = e^{i\pi_X(x)(T^3+T^8/\sqrt{3})} \cdot Y_U^{\text{u.g.}}(x) \cdot e^{-i\pi_X(x)(T^3+T^8/\sqrt{3})}, \\ Y_D &\rightarrow Y_D(x) = e^{i\pi_X(x)(T^3+T^8/\sqrt{3})} \cdot Y_D^{\text{u.g.}}(x) \cdot e^{-i\sum_{a=1}^3 \pi_{D_R}^a(x)T^a} e^{-i\pi_X(x)T^8/\sqrt{3}}, \end{aligned} \quad (4.103)$$

where $Y_{U,D}^{\text{u.g.}}(x)$ are given by (4.70) and (4.71) with $\tilde{\eta}_{23}(x)$ set to zero.

The Lagrangian is supplemented by spurion kinetic terms and a $G_F^{(4)}$ -invariant potential term $V(Y_U, Y_D)$

$$\mathcal{L}_{\text{spurion}} = \Lambda^2 \text{Tr} [(D^\mu Y_U^\dagger)(D_\mu Y_U)] + \Lambda^2 \text{Tr} [(D^\mu Y_D^\dagger)(D_\mu Y_D)] - V(Y_U, Y_D) \quad (4.104)$$

with the adopted covariant derivatives containing only the flavour gauge fields of the residual flavour symmetry group $G_F^{(4)}$,

$$\begin{aligned} D_\mu Y_U(x) &= \partial_\mu Y_U(x) - ig_X X_\mu(x) [T^3 + T^8/\sqrt{3}, Y_U(x)], \\ D_\mu Y_D(x) &= \partial_\mu Y_D(x) + ig_{D_R} \sum_{a=1}^3 A_{D_R, \mu}^a(x) Y_D(x) T_{D_R}^a \\ &\quad - ig_X X_\mu(x) (T^3 + T^8/\sqrt{3}) Y_D(x) + ig_X X_\mu(x) Y_D(x) T^8/\sqrt{3}. \end{aligned} \quad (4.105)$$

Inserting the above expressions (4.103), expanding in the gauge and spurion fields, and using again the approximations in terms of small Yukawa couplings and CKM angles, we identify the kinetic terms of the spurion fields, as well as the mass terms of the $SU(2)_{D_R} \times U(1)_X$ gauge bosons induced by the VEVs θ_{13} , θ_{12} and y_s as:

$$\mathcal{L}_{\text{kin}} = \Lambda^2 (\partial_\mu y_s(x))^2 + \frac{\Lambda^2}{2} F_{12}^2 (\partial_\mu \tilde{\eta}_{12}(x))^2 + \frac{\Lambda^2}{2} F_{13}^2 (\partial_\mu \tilde{\eta}_{13}(x))^2, \quad (4.106)$$

$$\begin{aligned} \mathcal{L}_{\text{mass}} &\simeq \Lambda^2 y_b^2 \theta_{13}^2 (\partial_\mu \pi_X - g_X X_\mu)^2 \\ &\quad + \frac{\Lambda^2}{4} (y_s(x))^2 [2\mathcal{A}_\mu^+ \mathcal{A}_\mu^- + (\partial_\mu \pi_X - g_X X_\mu - \mathcal{A}_\mu^3)^2]. \end{aligned} \quad (4.107)$$

Here we have introduced the gauge-invariant combinations

$$\mathcal{A}_\mu^a = -2 \text{Tr} [e^{-i\pi_{D_R}} (i\partial_\mu + g_{D_R} A_{D_R, \mu}) e^{i\pi_{D_R}} T^a] \simeq \partial_\mu \pi_{D_R}^a - g_{D_R} A_{D_R, \mu}^a \quad (4.108)$$

and

$$\mathcal{A}_\mu^\pm = \frac{1}{\sqrt{2}} (\mathcal{A}_\mu^1 \pm i\mathcal{A}_\mu^2). \quad (4.109)$$

4.7 Effective Theories for the Energy Scales $y_b\theta_{13}\Lambda \leq E < y_s\Lambda$

In the standard scenario for the sequence of flavour symmetry breaking discussed in Section 4.3, the next spurion to get a VEV is $\eta_{13}(x) \equiv \Lambda F_{13} \tilde{\eta}_{13}(x)$ which is related to fluctuations around the CKM angle $\theta_{13} \sim \lambda^3$. The spurion VEV induces the spontaneous symmetry breakdown of the FS group $G_F^{(4)}$ to

$$G_F^{(5)} = [SU(2)_{D_R}] \times U(1)_{u_R} \times U(1)_{D_R^{(2)}}, \quad (4.110)$$

and produces a $U(1)_X$ gauge boson mass via contributing to F_X according to $\mathcal{L}_{\text{mass}}$ (4.107),

$$F_X^2 \supset 2y_b^2\theta_{13}^2\Lambda^2. \quad (4.111)$$

We assume that the spurion potential will generate a mass term for η_{13} with a generic size of order

$$m_{13}^2 \sim y_b^2\theta_{13}^2\Lambda^2. \quad (4.112)$$

Assuming further that the spurion contribution to F_X in (4.111) is dominating over the anomaly contributions (4.96), such that

$$M_X^2 \equiv g_X^2 F_X^2 \sim 2g_X^2 y_b^2 \theta_{13}^2 \Lambda^2, \quad (4.113)$$

the following relation of scales holds

$$\Lambda \gg m_{13} \sim F_X > M_X. \quad (4.114)$$

Otherwise we would have to integrate out the gauge boson X_μ before integrating out the scalar field $\eta_{13}(x)$ (see Figure 4.3 for the order of integrating out the various particles).

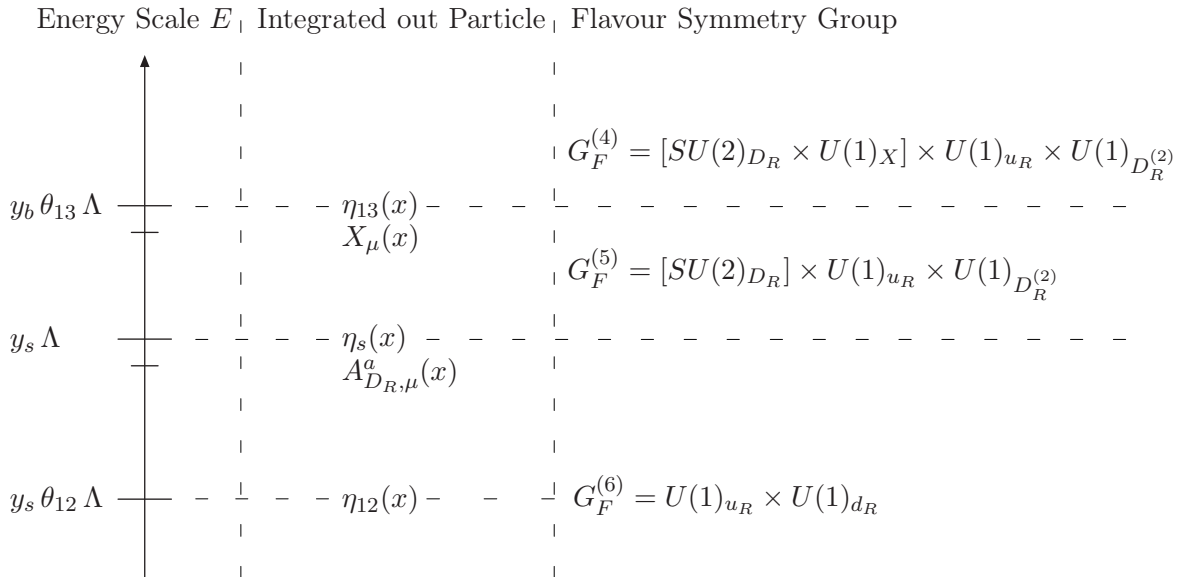


Figure 4.3: *Particles that are integrated out at the different energy scales.*

4.7.1 Integrating out the Heavy Spurion Field η_{13}

To obtain the higher-dimensional operators of the effective theory which is valid below the energy scale of the order F_X , we will restrict ourselves to the leading tree-level effects by solving the equations of motion (EOM) for η_{13} . Using the approximate form of the Yukawa spurion couplings to the fermions in (4.70) and (4.71), the relevant effective Lagrangian involving η_{13} reads

$$\mathcal{L}_{13} = \frac{1}{2} (\partial^\mu \eta_{13})^2 - \frac{1}{2} m_{13}^2 \eta_{13}^2 - (J_U^{13} + \bar{J}_U^{13} + J_D^{13} + \bar{J}_D^{13}) \eta_{13}, \quad (4.115)$$

with

$$J_U^{13} \simeq \frac{F_{13}}{2} (\bar{U}'_L, \bar{D}'_L V_{\text{CKM}}^\dagger) \frac{\tilde{H}}{\Lambda} \begin{pmatrix} 0 & \frac{\theta_{23} y_b^2 y_c}{y_b^2 y_c^2 - y_s^2 y_t^2} & \frac{1}{y_t} \\ 0 & 0 & \frac{\theta_{12} y_s^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & 0 \end{pmatrix} U_R,$$

$$J_D^{13} \simeq \frac{F_{13}}{2} (\bar{U}'_L V_{\text{CKM}}, \bar{D}'_L) \frac{H}{\Lambda} \begin{pmatrix} 0 & -\frac{\theta_{23} y_s y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} & -\frac{1}{y_b} \\ 0 & 0 & -\frac{\theta_{12} y_b y_c^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & 0 \end{pmatrix} D_R. \quad (4.116)$$

In (4.116) we used a symbolic notation concerning the SM Higgs field. Its VEV selects \bar{U}'_L in the current J_U^{13} and \bar{D}'_L in J_D^{13} , respectively, where the primed fields denote the mass eigenstates of the quarks. In the limit $m_{13} \rightarrow \infty$, one obtains the effective 4-quark interactions

$$\frac{1}{2m_{13}^2} \left(J_U^{13} + \bar{J}_U^{13} + J_D^{13} + \bar{J}_D^{13} \right)^2 \quad (4.117)$$

that induce flavour transitions with an overall suppression factor v^2/Λ^2 when the SM Higgs field H has developed its VEV. The individual coefficients of the specific flavour transitions follow from the matrix structure in J_U^{13} and J_D^{13} . However, we observe that the η_{13} spurion predominantly induces transitions between left-handed quarks from the first generation and right-handed quarks from the second or third generation. Of course, more operators — which may also include additional gauge fields — will be in general generated by radiative corrections and higher-dimensional operators from $\mathcal{L}_{\text{spurion}}$ in (4.104).

4.7.2 Integrating out the $U(1)_X$ Gauge Field

Below the scale $M_X = g_X F_X$, we may integrate out the heavy gauge boson of the $U(1)_X$ flavour symmetry. Focusing on the leading terms in (4.107), and considering unitary gauge ($\pi_X = 0$), we may again solve the classical EOM following from

$$\mathcal{L}_X \simeq \mathcal{L}_{\text{mass}} + g_X X_\mu J_X^\mu, \quad J_X^\mu \equiv [\bar{\psi}_L \gamma^\mu Q_X \psi_L], \quad (4.118)$$

in the limit $F_X \rightarrow \infty$, where $\mathcal{L}_{\text{mass}}$ is defined in (4.107) and F_X in (4.113). Again, this induces effective 4-quark operators of the form

$$-\frac{1}{2F_X^2} [\bar{\psi}_L \gamma_\mu Q_X \psi_L] [\bar{\psi}_L \gamma^\mu Q_X \psi_L], \quad (4.119)$$

where ψ_L denotes the set of left-handed fermion fields which are illustrated in Table 4.2 together with the corresponding $U(1)_X$ charges of the diagonal charge operator Q_X . As the up-type Yukawa matrix is already diagonal, the above operator does not induce FCNCs between up-type quarks. On the other hand, rotating the down-type quarks into the mass eigenbasis, one obtains

$$J_X^\mu \Big|_{\text{down}} = (\bar{d}'_L, \bar{s}'_L, \bar{b}'_L) \gamma^\mu X'_{DL} \begin{pmatrix} d'_L \\ s'_L \\ b'_L \end{pmatrix},$$

$$X'_{DL} = V_{\text{CKM}}^\dagger \text{Diag} \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3} \right) V_{\text{CKM}} \simeq \begin{pmatrix} \frac{2}{3} & \theta_{12} & \theta_{13} \\ \theta_{12} & -\frac{1}{3} & 0 \\ \theta_{13} & 0 & -\frac{1}{3} \end{pmatrix} + \mathcal{O}(\theta_{ij}^2), \quad (4.120)$$

containing FCNCs between d_L and s_L or b_L , which are suppressed by the SM CKM angles. The phenomenology induced by these subleading effects is qualitatively similar to Z' models with non-universal flavour couplings [164], where interesting new flavour effects have been identified in the context of present puzzles in flavour observables (see e.g. [165–169] for recent applications). However, compared to the commonly favoured Z' scenarios, our case displays a number of important modifications:

- Typical Z' scenarios are motivated by electroweak physics and consider Z' masses in the TeV range. In this case, precision flavour observables in the kaon sector already disfavour non-universal flavour couplings of the first and second generation. In our case, the $U(1)_X$ gauge boson is naturally allowed to be much heavier. At the same time, the non-universal effects are precisely between the first and second (or third) generation, and therefore kaon observables essentially will provide a lower bound on the scale F_X .
- The $U(1)_X$ gauge boson does not couple to leptons, and thus constraints from lepton-flavour violating observables do not apply to our case.

Taking into account the subleading effects proportional to y_s^2 in (4.107), the mixing between the gauge boson X_μ with the $SU(2)_{DR}$ gauge field \mathcal{A}_μ^3 induces an additional effective operator, such that finally

$$\mathcal{L}_{\text{mass}} + g_X X_\mu J_X^\mu \rightarrow \frac{y_s^2 \Lambda^2}{4} [2\mathcal{A}_\mu^+ \mathcal{A}_\mu^- + (\mathcal{A}_\mu^3)^2] - \frac{1}{2F_X^2} [J_X^\mu]^2 - \frac{y_s^2}{4\theta_{13}^2 y_b^2} J_X^\mu \mathcal{A}_\mu^3. \quad (4.121)$$

4.8 Effective Theories for the Energy Scales $y_s \Lambda \leq E < y_s \theta_{12} \Lambda$

Again, we assume that through an appropriate spurion potential, the spurion field $y_s(x) = y_s + \eta_s(x)/\sqrt{2}$ obtains a non-vanishing VEV at a scale of the order of its mass $m_{\eta_s} \sim y_s \Lambda$. According to $\mathcal{L}_{\text{mass}}$ in (4.107), the VEV supplies also a mass term for the $SU(2)_{D_R}$ gauge bosons,

$$\mathcal{L}_{\text{mass}} \supset \frac{1}{2} m_{A_{D_R}}^2 A_{D_R, \mu}^a A_{D_R}^{a, \mu} = \frac{1}{2} g_{D_R}^2 F_{D_R}^2 A_{D_R, \mu}^a A_{D_R}^{a, \mu} \stackrel{(4.107)}{\simeq} \frac{\Lambda^2}{4} g_{D_R}^2 y_s^2 A_{D_R, \mu}^a A_{D_R}^{a, \mu}. \quad (4.122)$$

We are now going to integrate out the heavy spurion and the gauge fields corresponding to the symmetry breakdown $G_F^{(5)} \rightarrow G_F^{(6)}$ in order to extract the leading effective higher-dimensional operators.

4.8.1 Integrating out the Spurion Field η_s

Integrating out the spurion fluctuation $\eta_s(x)$, the Yukawa coupling to the down-type quarks induces an effective 4-quark operator,

$$\frac{1}{4m_{\eta_s}^2} J_s^2, \quad J_s = \frac{1}{\Lambda} [(\bar{u}'_L V_{us} + \bar{c}'_L V_{cs}, \bar{s}'_L) H s_R + \text{h.c.}], \quad (4.123)$$

where we have expressed the quarks in the mass eigenbasis. As expected, the fluctuation $\eta_s(x)$ around the Yukawa eigenvalue y_s do not induce flavour transitions, once the SM Higgs is replaced by its VEV.

4.8.2 Integrating out the Gauge Fields $A_{D_R}^a$

Summarising the terms involving the $SU(2)_{D_R}$ gauge fields $A_{D_R, \mu}^a$, i.e. the kinetic terms (4.100), the couplings to the fermions (4.101), the mass terms as given in (4.122), and the leading term from the mixing between X_μ and \mathcal{A}_μ^3 (4.121) yields

$$-\frac{1}{4} F_{D_R, \mu\nu}^a(x) F_{D_R}^{a, \mu\nu}(x) + g_{D_R} A_{D_R, \mu}^a (J_{A_{D_R}}^\mu)^a + \frac{1}{2} g_{D_R}^2 F_{D_R}^2 A_{D_R, \mu}^a A_{D_R}^{a, \mu} + \frac{g_{D_R} y_s^2}{4\theta_{13}^2 y_b^2} J_X^\mu A_{D_R, \mu}^3. \quad (4.124)$$

Using the EOMs, we integrate out the $SU(2)_{D_R}$ gauge fields $A_{D_R, \mu}^a$ and obtain the effective 4-quark operators

$$-\frac{1}{2F_{D_R}^2} \left([(J_{A_{D_R}}^\mu)^a]^2 + \frac{y_s^2}{2\theta_{13}^2 y_b^2} [(J_{A_{D_R}}^\mu)^3 J_{X, \mu}] \right). \quad (4.125)$$

Inserting the Fierz identity for the Pauli matrices

$$\sigma_{ij}^a \sigma_{kl}^a = 2\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}, \quad (4.126)$$

the operator $[(J_{A_{D_R}}^\mu)^a]^2$ can be rewritten as

$$\frac{1}{4} \left(2 [(\bar{D}_R)_l \gamma^\mu (D_R)_k] [(\bar{D}_R)_k \gamma_\mu (D_R)_l] - [(\bar{D}_R)_k \gamma^\mu (D_R)_k] [(\bar{D}_R)_l \gamma_\mu (D_R)_l] \right). \quad (4.127)$$

Utilising the Fierz identity (B.142) to rearrange the right-handed fermion bilinears, we finally find that only flavour-diagonal currents $(\bar{d}_R \gamma_\mu d_R)$ and $(\bar{s}_R \gamma_\mu s_R)$, but with different colour structure are involved

$$[(J_{A_{D_R}}^\mu)^a]^2 = \frac{1}{4} \left(2 [(\bar{D}_R)_l^\alpha \gamma^\mu (D_R)_l^\beta] [(\bar{D}_R)_k^\alpha \gamma_\mu (D_R)_k^\beta] - [(\bar{D}_R)_k^\alpha \gamma^\mu (D_R)_k^\alpha] [(\bar{D}_R)_l^\beta \gamma_\mu (D_R)_l^\beta] \right). \quad (4.128)$$

In the second term in (4.125), flavour transitions appear as before, as soon as the current J_X is written in the mass eigenbasis (4.120).

4.9 Effective Theory below the Energy Scale $E \leq y_s \theta_{12} \Lambda$

We mentioned in Section 4.3.1 that the residual non-diagonal spurion $\chi_{12}(x)$ is a singlet under the residual FS group $G_F^{(6)} = U(1)_{u_R} \times U(1)_{d_R}$ such that its VEV accounts for the CP-violating phase δ in addition to the creation of the CKM angle θ_{12} . However, we will neglect the effects coming from the fluctuations around δ in the following discussion, in accordance with our derivation of the Yukawa matrix parameterisations in (4.70) and (4.71).

4.9.1 Integrating out the Spurion Field η_{12}

Finally, we may integrate out the field η_{12} which we assume to have a generic mass of order

$$m_{12}^2 \sim y_s^2 \theta_{12}^2 \Lambda^2. \quad (4.129)$$

Notice that the corresponding contributions to the $SU(2)_{D_R}$ gauge boson masses are subleading, and have been neglected in the above analysis.¹

In complete analogy to the case of η_{13} (4.117), we obtain the effective 4-quark operators

$$\frac{1}{2m_{12}^2} \left(J_U^{12} + \bar{J}_U^{12} + J_D^{12} + \bar{J}_D^{12} \right)^2, \quad (4.130)$$

with the currents given by

$$\begin{aligned} J_U^{12} &\simeq \frac{F_{12}}{2} (\bar{U}'_L, \bar{D}'_L V_{\text{CKM}}^\dagger) \frac{\tilde{H}}{\Lambda} \begin{pmatrix} 0 & \frac{1}{y_c} & -\frac{\theta_{23} y_b^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & \frac{\theta_{13} y_b^2 y_t}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & \frac{\theta_{13} y_b^2 y_c}{y_b^2 y_c^2 - y_s^2 y_t^2} & 0 \end{pmatrix} U_R, \\ J_D^{12} &\simeq \frac{F_{12}}{2} (\bar{U}'_L V_{\text{CKM}}, \bar{D}'_L) \frac{H}{\Lambda} \begin{pmatrix} 0 & -\frac{1}{y_s} & \frac{\theta_{23} y_b y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & 0 & -\frac{\theta_{13} y_b y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} \\ 0 & -\frac{\theta_{13} y_s y_t^2}{y_b^2 y_c^2 - y_s^2 y_t^2} & 0 \end{pmatrix} D_R. \end{aligned} \quad (4.131)$$

¹They lead however to small mass splittings between \mathcal{A}_μ^\pm and \mathcal{A}_μ^3 .

4.9.2 Global $U(1)_{u_R} \times U(1)_{d_R}$ Flavour Symmetry

Having integrated out the spurion fluctuations and gauge fields according to the spontaneous symmetry breaking $G_F \rightarrow G_F^{(6)} = U(1)_{u_R} \times U(1)_{d_R}$, the Yukawa matrices

$$Y_U(x) = \begin{pmatrix} y_u(x) e^{-i\pi_u(x)} & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \quad Y_D(x) = V_{\text{CKM}} \begin{pmatrix} y_d(x) e^{-i\pi_d(x)} & 0 & 0 \\ 0 & y_s & 0 \\ 0 & 0 & y_b \end{pmatrix}, \quad (4.132)$$

contain the two leftover complex spurion fields

$$Y_U^{(1)}(x) = y_u(x) \cdot e^{-i\pi_u(x)}, \quad Y_D^{(1)}(x) = y_d(x) \cdot e^{-i\pi_d(x)}. \quad (4.133)$$

Their VEVs will spontaneously break the $U(1)_{u_R} \times U(1)_{d_R}$ symmetry group at the scale $\Lambda^{(6)} = y_{u,d}\Lambda \sim \lambda^8\Lambda$, and will give rise to the masses of the lightest quarks m_u and m_d . The global $U(1)_{u_R} \times U(1)_{d_R}$ flavour symmetry in the effective theory below the scale $y_s\theta_{12}\Lambda$ acts on the right-handed quarks of the first generation,

$$u_R \rightarrow e^{i\theta_u} u_R, \quad d_R \rightarrow e^{i\theta_d} d_R, \quad (4.134)$$

and transforms the Goldstone fields in the exponentials of $Y_U^{(1)}(x)$ and $Y_D^{(1)}(x)$ as

$$\pi'_u = \pi_u + \theta_u, \quad \pi'_d = \pi_d + \theta_d. \quad (4.135)$$

Due to the above shift symmetry of the Goldstone modes, the (classical) scalar potential only depends on $y_u(x)$ and $y_d(x)$,

$$V_0 = V_0(y_u, y_d). \quad (4.136)$$

4.10 The Strong CP Problem

Global Flavour Symmetry of the QCD Lagrangian

The QCD Lagrangian in the limit of vanishing quark masses, as given in (3.1), possesses a large global symmetry. To display the full symmetry, we rewrite the Lagrangian in terms of chiral quark fields

$$\mathcal{L}_{\text{QCD}} = \bar{q}_L i \not{D} q_R + \bar{q}_R i \not{D} q_L - \frac{1}{2} \text{Tr} [G_{\mu\nu} G^{\mu\nu}] + \mathcal{L}_{\text{gfix}}, \quad (4.137)$$

where $q_{L,R} = (u_{L,R}, c_{L,R}, t_{L,R}, d_{L,R}, s_{L,R}, b_{L,R})$ contains all the six different quark flavours, i.e. the number of flavours $N_f = 6$. Obviously, the above Lagrangian is invariant under the chiral unitary group transformations

$$q_L \rightarrow U_L q_L, \quad q_R \rightarrow U_R q_R, \quad (4.138)$$

corresponding to two independent rotations of q_L and q_R in the 6-dimensional flavour space. Thus, there exists a $U(N_f)_L \times U(N_f)_R$ global flavour symmetry in the *chiral limit* $m_q \rightarrow 0$ of the QCD Lagrangian.

However, as the quarks are not massless, this symmetry can only be an approximate symmetry for quark masses m_q that are much lower than the hadronic mass scale Λ_{QCD} . Hence the symmetry is almost exact for the up and down quark and more approximate for the strange quark. For this reason we will work with $N_f = 3$. Combining the corresponding Noether currents of the approximate $U(3)_L \times U(3)_R$ chiral symmetry into vector- and axial-vector currents, we obtain

$$SU(3)_V \times U(1)_V : J_\mu^a = \bar{q}\gamma_\mu \frac{\lambda^a}{2} q, \quad J_\mu = \bar{q}\gamma_\mu q, \quad (4.139)$$

$$SU(3)_A \times U(1)_A : J_{5\mu}^a = \bar{q}\gamma_\mu \gamma_5 \frac{\lambda^a}{2} q, \quad J_{5\mu} = \bar{q}\gamma_\mu \gamma_5 q. \quad (4.140)$$

This form of the currents is convenient, as the $SU(3)_V$ symmetry has its manifestation in the hadronic spectrum, which contains flavour multiplets that are approximately degenerate in mass, e.g. a baryon decuplet and baryon octet (see the eightfold way [35, 36]). The $U(1)_V$ symmetry corresponds to the baryon number symmetry restricted to the N_f -flavour case. As the axial symmetries have not a similar pendant in the particle spectrum, it is assumed that they are not only broken explicitly by the non-zero quark masses, but also spontaneously by the QCD vacuum. This breaking can be realised by $\bar{q}q$ scalar condensates which acquire non-zero VEVs, $\langle 0|\bar{q}q|0\rangle \neq 0$. The existence of the light pseudoscalar octet further confirms this assumption, as they can be interpreted as the Goldstone bosons corresponding to the broken generators of the spontaneously broken $SU(3)_A$. However, the absence of a light ninth isoscalar, pseudoscalar Goldstone boson in the particle spectrum — the η' seems to be too heavy ($m_{\eta'}^2 \gg m_\pi^2$) to be a suitable candidate — evokes the so-called $U(1)_A$ problem (originally considered in [170] for the two-flavour symmetry case corresponding to the chiral limit $m_u = m_d = 0$) which will be discussed in the following subsection.

The Resolution to the $U(1)_A$ Problem

In Section 3.1 we argued that the colour group $SU(3)_c$ is free of chiral gauge anomalies, as the vector gauge fields couple equally to left-handed and right-handed quarks and their contributions to the triangle diagrams cancel properly.

However, there exists a second kind of anomaly, which is related to global chiral transformations [171]. The fermion path-integral measure changes under an axial transformation and thus the corresponding axial $U(1)_A$ fermion current is not conserved. Its non-zero divergence is given by the *Adler–Bell–Jackiw* (ABJ) anomaly [172, 173], resulting from the triangle graphs which connect the axial current $J_{5\mu}$ to two gluon fields

$$\partial^\mu J_{5\mu} = \partial^\mu (\bar{\psi}\gamma_\mu \gamma_5 \psi) = N_f \frac{1}{8\pi^2} \text{Tr} [G_{\mu\nu} \tilde{G}^{\mu\nu}] = N_f \frac{1}{16\pi^2} G_{\mu\nu}^a \tilde{G}^{a,\mu\nu}. \quad (4.141)$$

At first sight, the presence of the anomaly seems to solve the $U(1)_A$ problem as the chiral anomaly affects the action $\delta\Gamma \sim \int d^4x \partial^\mu J_{5\mu}$. However, the right-hand side of (4.141) can be reformulated as a total divergence $G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} = \partial_\mu K^\mu$ [174] so that $\delta\Gamma$ is proportional to a pure surface integral $\int d\sigma_\mu K^\mu$. Using the naive boundary condition that the gluon gauge fields G_μ^a vanish at spatial infinity, the integral vanishes and $U(1)_A$ appears again to be an unbroken symmetry of QCD.

The final resolution of the $U(1)_A$ problem was given by 't Hooft [175, 176], who realised that the QCD vacuum has a non-trivial structure and the correct boundary condition is to require that G_μ^a are *pure gauge* fields at spatial infinity. Apart from setting $G_\mu^a = 0$, one has to include also the gauge-transformed version of the condition $G_\mu^a = 0$, i.e.

$$G_\mu^a = U G_\mu^a U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \Big|_{G_\mu^a=0} = \frac{i}{g} U \partial_\mu U^\dagger. \quad (4.142)$$

It turns out that with the choice of these boundary conditions there is indeed an anomaly contribution and $U(1)_A$ is not a true symmetry of QCD, even though it is an apparent symmetry of the QCD Lagrangian in the limit of vanishing quark masses. The non-trivial topological properties of the QCD gauge configurations imply a more complicated QCD vacuum state beyond perturbation theory. The true vacuum or θ -vacuum consists of a suitable superposition of distinct degenerate QCD vacuum states that are labelled by their topological quantum number or *winding number* n . Quantum tunnelling can occur between different vacua which can be expressed by the vacuum-to-vacuum transition amplitude in the θ -vacuum

$$\langle 0|0 \rangle_\theta = \sum_{n=-\infty}^{\infty} \int (\mathcal{D}A_\mu)_n \int \mathcal{D}\phi \exp[i n \theta] \exp \left[i \int d^4x \mathcal{L}(A, \phi) \right], \quad (4.143)$$

with ϕ denoting generic matter fields. For a given topological sector n , the functional integration is restricted to the QCD gauge-potential² configurations $(\mathcal{D}A_\mu)_n$ which satisfy

$$n = \frac{g_s^2}{32\pi^2} \int d^4x G_{\mu\nu}^a \tilde{G}^{a\mu\nu}. \quad (4.144)$$

Thus the complicated structure of the QCD vacuum effectively adds the θ -term (see (3.4))

$$\mathcal{L}_{\text{QCD}} \rightarrow \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{CP}} = \mathcal{L}_{\text{QCD}} + \theta \frac{g_s^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} \quad (4.145)$$

to the QCD Lagrangian. However, while solving the $U(1)_A$ problem, this term violates CP for non-vanishing θ . Since the strong interactions are found to respect CP to very high accuracy, as required from the strong bound on the neutron electric dipole moment, the problem turns into a new problem called *strong CP problem*. Formulated differently one can ask why is CP not badly broken in QCD, or analogously, why is the angle θ so small? Unless there is a symmetry that can explain why θ approximately vanishes, this outcome is in conflict with the naturalness argument and produces a fine-tuning problem.

²In the following, we concentrate on the strong-interaction gauge sector, and suppress the weak-interaction effects in the notation.

The Resolution to the Strong CP Problem

Including also the weak interactions, the complex generic mass matrices of the up-type and down-type quarks have to be diagonalised in order to transform the quarks into their mass eigenbases. This is achieved by performing a global chiral transformation under the assumption that the QCD Lagrangian is invariant except for the axial $U(1)$ transformation which, owing to the anomaly, changes the value of θ to

$$\tilde{\theta} = \theta + \theta_{\text{EW}}. \quad (4.146)$$

From this point of view, the strong CP problem can be formulated as the question: why should the values of those unrelated contributions to $\tilde{\theta}$ be such tuned that they cancel so accurately?

An explanation has been proposed by Peccei and Quinn [91–93]. Clearly the mass term in the Lagrangian of the form $\bar{\psi}H\psi$ is not invariant under axial rotations involving only the fermion fields. However, invariance can be restored if the theory obeys an enlarged axial symmetry $U(1)_{\text{PQ}}$ that includes also the Higgs field. While this transformation does not influence the already diagonalised mass terms nor the other terms in the classical Lagrangian, it has an effect on the QCD vacuum and can be used to rotate $\tilde{\theta}$ to zero. In the standard *Peccei–Quinn* (PQ) mechanism two different Higgs doublets are necessary to ensure $U(1)_{\text{PQ}}$ invariance. The Goldstone boson of the spontaneously broken $U(1)_{\text{PQ}}$ symmetry is called axion.

4.10.1 Peccei–Quinn Mechanism for $U(1)_{u_R} \times U(1)_{d_R}$

In our setup, the two residual Yukawa spurions $Y_U^{(1)}(x)$ and $Y_D^{(1)}(x)$ ensure that the Yukawa coupling terms

$$-\mathcal{L}_{\text{Yuk}} = \bar{U}_L(x) H Y_U(x) U_R(x) + \bar{D}_L(x) H Y_D(x) D_R(x) + \text{h.c.} \quad (4.147)$$

respect the global symmetry when transforming the fermions under the chiral global $U(1)_{u_R} \times U(1)_{d_R}$ flavour symmetry, and thus they inherit the task of the two Higgs doublets within the standard Peccei–Quinn mechanism.³ Here we recall that the effective action (in the QCD gauge sector) changes under chiral rotations as

$$\Gamma \rightarrow \Gamma + n(\theta_u + \theta_d), \quad (4.148)$$

which is equivalent to a change in the QCD θ -parameter,

$$\tilde{\theta} \rightarrow \tilde{\theta} - \theta_u - \theta_d. \quad (4.149)$$

³The connection between flavour symmetries and PQ symmetries has been discussed before, see e.g. [177–179]

Assuming $\langle y_q \rangle > 0$, the fermion mass terms get their canonical form, after a chiral transformation of u_R and d_R with the corresponding phases set by $\langle \pi_u(x) \rangle$ and $\langle \pi_d(x) \rangle$, respectively. To avoid the strong CP problem, one thus has to require that

$$\langle \theta_{\text{eff}} \rangle \equiv \tilde{\theta} - \langle \pi_u(x) + \pi_d(x) \rangle \stackrel{!}{=} 0. \quad (4.150)$$

This can be achieved by examining the effective potential in the non-trivial QCD θ -vacuum which can be obtained from an expansion in small Yukawa couplings, with the leading term coming from the $n = \pm 1$ sectors (see appendices in [90] and [93]), leading to

$$V_\theta = V_0 - K v^6 \text{Re} [\text{Det} Y_U \text{Det} Y_D e^{i\tilde{\theta}}] + \dots, \quad (4.151)$$

where $K > 0$ is a positive constant. Using the explicit form of the Yukawa matrices (4.132),

$$\text{Det}(Y_U) = y_u(x) y_c y_t e^{-i\pi_u(x)}, \quad \text{Det}(Y_D) = y_d(x) y_s y_b e^{-i\pi_d(x)}, \quad (4.152)$$

the potential can then be rewritten as

$$V_\theta = V_0 - K v^6 y_c y_t y_s y_b y_u(x) y_d(x) \cos[\pi_u(x) + \pi_d(x) - \tilde{\theta}] + \dots. \quad (4.153)$$

Thus the potential (4.153) breaks the original shift symmetry for the Goldstone fields. Its minimum is given by $\langle \pi_u + \pi_d \rangle = \tilde{\theta}$, and therefore $\langle \theta_{\text{eff}} \rangle \equiv 0$, as required. Notice that the potential only depends on the combination $\pi_u(x) + \pi_d(x)$, such that we identify the PQ axion field as the linear combination

$$a(x) \equiv f_a (\pi_u(x) + \pi_d(x)), \quad (4.154)$$

where the dimensional normalisation constant f_a ensures a canonical axion mass dimension. The corresponding PQ symmetry is defined such that the axion transforms as

$$a(x) \rightarrow a(x) + f_a \theta_{\text{PQ}}, \quad (4.155)$$

where

$$\theta_{\text{PQ}} = \theta_u + \theta_d. \quad (4.156)$$

In order to determine the normalisation constant f_a and to find the linear combination of $\pi_u(x)$ and $\pi_d(x)$ orthogonal to $a(x)$, denoted by $b(x)$, we consider the flavour-invariant kinetic terms and require

$$\Lambda^2 \partial_\mu Y_U^{(1)} \partial^\mu Y_U^{(1)\dagger} + \Lambda^2 \partial_\mu Y_D^{(1)} \partial^\mu Y_D^{(1)\dagger} \Big|_{y_{u,d} \rightarrow \langle y_{u,d} \rangle} \stackrel{!}{=} \frac{1}{2} (\partial_\mu a(x))^2 + \frac{1}{2} (\partial_\mu b(x))^2. \quad (4.157)$$

Making the ansatz $b(x) = b_1 \pi_u(x) - b_2 \pi_d(x)$, this condition reduces to

$$\begin{aligned} \Lambda^2 (\langle y_u \rangle^2 (\partial_\mu \pi_u)^2 + \langle y_d \rangle^2 (\partial_\mu \pi_d)^2) &\stackrel{!}{=} \frac{1}{2} (f_a^2 + b_1^2) (\partial_\mu \pi_u)^2 + \frac{1}{2} (f_a^2 + b_2^2) (\partial_\mu \pi_d)^2 \\ &\quad + (f_a^2 - b_1 b_2) (\partial_\mu \pi_u) (\partial^\mu \pi_d). \end{aligned} \quad (4.158)$$

Comparing the coefficient of the third term yields $f_a^2 = b_1 b_2$, such that we obtain

$$\Lambda^2 (\langle y_u \rangle^2 (\partial_\mu \pi_u)^2 + \langle y_d \rangle^2 (\partial_\mu \pi_d)^2) \stackrel{!}{=} \frac{1}{2} (b_1 b_2 + b_1^2) (\partial_\mu \pi_u)^2 + \frac{1}{2} (b_1 b_2 + b_2^2) (\partial_\mu \pi_d)^2 \quad (4.159)$$

which can be finally solved by

$$b_1 = \frac{\sqrt{2} \Lambda \langle y_u \rangle^2}{\sqrt{\langle y_d \rangle^2 + \langle y_u \rangle^2}}, \quad b_2 = \frac{\sqrt{2} \Lambda \langle y_d \rangle^2}{\sqrt{\langle y_d \rangle^2 + \langle y_u \rangle^2}}. \quad (4.160)$$

Here we restrict ourselves to the positive solution (doing otherwise would only result in an overall change of sign), as we also do in the case of f_a

$$f_a = \sqrt{2} \Lambda \left\langle \frac{y_d y_u}{\sqrt{y_d^2 + y_u^2}} \right\rangle, \quad b(x) = f_a \left(\left\langle \frac{y_u}{y_d} \right\rangle \pi_u(x) - \left\langle \frac{y_d}{y_u} \right\rangle \pi_d(x) \right). \quad (4.161)$$

We also define the corresponding linear combination of $U(1)$ charge,

$$\theta_{\text{diff}} = \left\langle \frac{y_u}{y_d} \right\rangle \theta_u - \left\langle \frac{y_d}{y_u} \right\rangle \theta_d, \quad (4.162)$$

such that the orthogonal combination of Goldstone bosons transforms as

$$b(x) \rightarrow b(x) + f_a \theta_{\text{diff}}. \quad (4.163)$$

In terms of $a(x)$ and $b(x)$, the up- and down-quark Yukawa couplings can be expressed as

$$\begin{aligned} Y_U^{(1)}(x) &= \exp \left[-i \left\langle \frac{y_d^2}{y_u^2 + y_d^2} \right\rangle \frac{a(x)}{f_a} \right] \exp \left[-i \left\langle \frac{y_u y_d}{y_u^2 + y_d^2} \right\rangle \frac{b(x)}{f_a} \right] y_u(x), \\ Y_D^{(1)}(x) &= \exp \left[-i \left\langle \frac{y_u^2}{y_u^2 + y_d^2} \right\rangle \frac{a(x)}{f_a} \right] \exp \left[+i \left\langle \frac{y_u y_d}{y_u^2 + y_d^2} \right\rangle \frac{b(x)}{f_a} \right] y_d(x). \end{aligned} \quad (4.164)$$

Note that $b(x)$ remains massless, apart from anomalous contributions from the electroweak vacuum. We may or may not remove $b(x)$ by gauging the remaining $U(1)_{\text{diff}}$ symmetry and subsequently integrating out the corresponding massive gauge boson.

The axion field $a(x)$ remains in the physical spectrum of the theory. However, compared to the original Peccei–Quinn axion, its couplings are now determined by the scale Λ of the Yukawa fields and not by the electroweak VEV of the two Higgs fields. In particular, the scale Λ has to be chosen well above the electroweak scale, in which case the axion couplings become very small, since they scale as $1/f_a$. Thus the phenomenology of this model will be similar to the phenomenology of invisible axion models [97, 98].

4.11 Summary of Chapter 4

In this chapter, we have discussed a MFV scenario, where the entries of the Yukawa matrices are promoted to scalar fields. They become dynamical at high scales and are subject to an appropriately chosen scalar potential. Transforming as bifundamentals under the partly gauged SM FS, which is broken by the fermion Yukawa couplings, a cascade of scalar VEVs generates the hierarchy in the fermion masses and mixing angles in the SM. While the gauge bosons, corresponding to the local part of the FS group, become massive by “eating” the Goldstone bosons within the usual Higgs mechanism, the global chiral $U(1)$ factors can serve as a Peccei–Quinn symmetry for a possible resolution of the strong CP problem.⁴

Our scenario necessarily has to be understood in the context of an effective theory approximation of a more fundamental underlying theory. The canonical dimension of the Yukawa spurions implies that the Yukawa interactions are described by dimension-5 operators. Moreover, considering the usual SM fermion representations, we encounter gauge anomalies due to the chiral nature of the SM FS group. Appropriate higher-dimensional operators involving the Goldstone fields have to be added to formally restore the local symmetry. In this way, we have constructed a consistent non-renormalisable effective theory of a smaller FS group, which arises at an intermediate step in the sequential FS breaking.

According to the chosen breaking pattern, the masses of the new heavy gauge bosons as well as of the new physical Higgs modes are hierarchically ordered. They determine the sequence of integrating out the heavy degrees of freedom by using the equations of motion, and thus specify the validity scales of the series of effective field theories.

Though we have not included a detailed phenomenological analysis, a few remarks about the general structure of the obtained effective 4-quark operators could be made. In order to be in line with the experimental constraints from precision measurements in the K and B sector, the induced flavour transitions of the new states have to be suppressed by sufficiently large masses. Thus the most stringent constraint will be set from the the spurion field $\eta_{12}(x)$ which receives the lightest mass out of the spontaneous breakdown of the local flavour symmetry.

⁴If the concept of MFV is applied to discrete subgroups of the FS as discussed in [180], no Goldstone bosons arise from its breaking.

Chapter 5

Warped Extra Dimensions

5.1 The Randall–Sundrum Model with Custodial Protection

As motivated in the introduction, we consider a Randall–Sundrum (RS) model in which the usual infinite space-time coordinates $x^\mu = x$ are augmented by a single warped extra dimension restricted to an interval $y \in [0, L]$. At the same time we implement a custodial protection for the RS model (RSc model) due to the choice of a $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}$ symmetry group in the 5D bulk. The 5D bulk is limited by two four-dimensional branes that are called UV brane ($y = 0$) and IR brane ($y = L$). While the SM gauge and fermion fields are allowed to propagate in the bulk, we will show below that the Higgs field has to be localised on or near the IR brane in order to solve the hierarchy problem.

The RS Metric

The basic ingredient of the model under consideration is the RS metric [99] defined by the line element

$$ds^2 = G_{NM}(x, y) dx^N dx^M = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (5.1)$$

which corresponds to a slice of 5D *anti-de-Sitter spacetime* (AdS_5) and preserves 4D Poincaré invariance. Due to the AdS/CFT correspondence [181], the 5D AdS space is related to a 4D *conformal field theory* (CFT). This dual description, linking a 5D weakly coupled theory with a 4D strongly interacting one, offers the possibility to determine certain quantities, e.g. the Higgs potential, in composite Higgs models [111, 112, 182].

The exponential factor, multiplying the 4D Minkowski metric tensor, indicates a non-factorisable metric and is known as the *warp factor*. This factor depends explicitly on the fifth-dimensional coordinate and on the parameter k , which is assumed to be of order of the Planck scale $M_{\text{Pl}} \simeq 10^{19}$ GeV. Using the following signature of the 5D Minkowski metric tensor of the flat space

$$\eta_{AB} = \text{Diag}(1, -1, -1, -1, -1), \quad (5.2)$$

the metric tensors of the warped metric $G_{NM}(x, y)$ and its inverse $G^{NM}(x, y)$ are specified by

$$G_{NM}(x, y) = \begin{cases} -1 & \text{for } N = M = 5, \\ e^{-2ky} & \text{for } N = M = \mu, \\ 0 & \text{otherwise,} \end{cases} \quad G^{NM}(x, y) = \begin{cases} -1 & \text{for } N = M = 5, \\ e^{2ky} & \text{for } N = M = \mu, \\ 0 & \text{otherwise.} \end{cases} \quad (5.3)$$

Generally we are using A, B, \dots for the indices of the tangent space and M, N, \dots for the curved space. The warped metric tensors fulfil the condition

$$G_{NM}G^{MP} = \delta_N^P, \quad (5.4)$$

such that $G_{MN}G^{MN}$ is equal to the dimensionality $D = 5$ of the space-time manifold. For convenience we introduce abbreviations for the determinants

$$G = \text{Det}(G_{MN}) = e^{-8ky} \quad \text{and} \quad G^{-1} = \text{Det}(G^{MN}) = e^{8ky}. \quad (5.5)$$

The Hierarchy Problem

In order to demonstrate that the non-factorisable metric supplies a solution to the hierarchy problem [99], we consider a fundamental 5D Higgs field located at the IR brane which is described by the action

$$S_{\text{IR}}^{\text{Higgs}} = \int d^4x \sqrt{G_{\text{IR}}} \left(G_{\text{IR}}^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda (H^\dagger H - v_0^2)^2 \right). \quad (5.6)$$

Inserting the metric factors

$$G_{\text{IR}} = G|_{y=L} \stackrel{(5.5)}{=} e^{-8kL} \quad \text{and} \quad G_{\text{IR}}^{\mu\nu} = G^{\mu\nu}|_{y=L} \stackrel{(5.3)}{=} e^{2kL} \eta^{\mu\nu}, \quad (5.7)$$

the action can be reformulated as

$$S_{\text{IR}}^{\text{Higgs}} = \int d^4x \left(e^{-2kL} \eta^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda e^{-4kL} (H^\dagger H - v_0^2)^2 \right). \quad (5.8)$$

To obtain a canonically normalised Higgs, the rescaling of the Higgs according to $H \rightarrow e^{kL} H$

$$S_{\text{IR}}^{\text{Higgs}} \supset \int d^4x \left(\eta^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda (H^\dagger H - e^{-2kL} v_0^2)^2 \right), \quad (5.9)$$

induces the relation $v \equiv e^{-kL} v_0$ between the physically relevant 4D effective mass scale v and the breaking scale v_0 of the fundamental 5D theory. Assuming that the 5D fundamental scale is of order of the Planck scale, the warp factor generates an effective energy scale $M_{\text{Pl}} e^{-kL}$ on the IR brane. In the following we will choose $e^{-kL} \simeq 10^{-16}$, which corresponds to $kL \sim 36$, such that the effective mass scale on the IR brane is of order of the TeV scale. Indicating the two effective energy scales, the UV brane is also called *Planck brane* and the IR brane is referred to as the *TeV brane* (see also Figure 5.1).

Variation of the 5D Action

The aim of the next chapters will be to shed some light on the various components of the 5D fundamental action

$$S = \int d^4x \int_0^L dy \mathcal{L} = \int d^4x \int_0^L dy (\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yuk}}). \quad (5.10)$$

Starting point for the derivation of the equations of motion for the fields Φ living in the bulk is the variation principle of the five-dimensional action

$$\delta S = \int d^4x \int_0^L dy \left(\frac{\partial \mathcal{L}}{\partial \Phi} \delta \Phi + \frac{\partial \mathcal{L}}{\partial (\partial_M \Phi)} \delta (\partial_M \Phi) \right) \stackrel{!}{=} 0. \quad (5.11)$$

Performing an integration by parts over the ordinary 4D coordinates, we require that the fields vanish at infinity. Thus the boundary terms disappear and (5.11) can be rewritten as

$$\delta S = \int d^4x \int_0^L dy \left(\frac{\partial \mathcal{L}}{\partial \Phi} \delta \Phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \delta \Phi + \frac{\partial \mathcal{L}}{\partial (\partial_5 \Phi)} \delta (\partial_5 \Phi) \right) \stackrel{!}{=} 0. \quad (5.12)$$

However, in order to produce a generalised 5D version of the 4D equations of motion, we have to perform an integration by parts with respect to the extra dimension as well. In this case one has to keep the finite boundary terms and (5.12) splits into two pieces

$$\delta S = \int d^4x \int_0^L dy \left[\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_M \left(\frac{\partial \mathcal{L}}{\partial (\partial_M \Phi)} \right) \right] \delta \Phi + \left[\int d^4x \left(\frac{\partial \mathcal{L}}{\partial (\partial_5 \Phi)} \right) \delta \Phi \right]_0^L \stackrel{!}{=} 0. \quad (5.13)$$

Thus, in addition to the equations of motions corresponding to the first term in (5.13), the full action is only minimised if the second term, which is evaluated at the two boundaries, also vanishes. From this requirement a set of consistent boundary conditions (BCs) results, which can be of Neumann ($\partial_5 \Phi|_{0,L} = 0$) or Dirichlet ($\Phi|_{0,L} = 0$) kind, or a mixture of both.

5.2 Gauge Sector of the RSc Model

To begin with, we focus on the gauge sector of the 5D action (5.10). We discuss the various constituents of the bulk symmetry group

$$G_{\text{bulk}} = SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_X \times P_{LR}, \quad (5.14)$$

and give a very brief overview of gauge-fixing terms in 5D theories. Furthermore, we illustrate the breaking of the bulk gauge group on the UV brane through an appropriate choice of BCs. In Section 5.3 we will continue with the discussion of the bulk gauge-group breaking via EWSB.

Gauge Boson Content of the RSc Model

Corresponding to the local part of the bulk symmetry group, the Lagrangian consists of four different field strength tensors

$$\mathcal{L}_{\text{gauge}} = \sqrt{G} \left(-\frac{1}{4} G_{MN}^A G^{A,MN} - \frac{1}{4} L_{MN}^a L^{a,MN} - \frac{1}{4} R_{MN}^\alpha R^{\alpha,MN} - \frac{1}{4} X_{MN} X^{MN} \right), \quad (5.15)$$

where the factor of \sqrt{G} ensures an invariant integration measure. In detail, the $SU(3)_c$ field strength tensor with 5D strong coupling constant g_s reads

$$G_{MN}^A = \partial_M G_N^A - \partial_N G_M^A - g_s f^{ABC} G_M^B G_N^C \quad (A = 1, \dots, 8), \quad (5.16)$$

where the $SU(3)_c$ indices are denoted by capital Latin letters A, B, \dots , but are usually made implicit in order to simplify the notation.¹

The discrete symmetry P_{LR} , joining the bulk gauge group, describes the interchange between the two $SU(2)$ groups of the electroweak sector. It implies the equality of the 5D gauge couplings ($g_L = g_R = g$), such that the $SU(2)_L$ and $SU(2)_R$ non-abelian field strength tensors are given by

$$\begin{aligned} L_{MN}^a &= \partial_M W_{L,N}^a - \partial_N W_{L,M}^a - g \varepsilon^{abc} W_{L,M}^b W_{L,N}^c \quad (a, b, c = 1, 2, 3), \\ R_{MN}^\alpha &= \partial_M W_{R,N}^\alpha - \partial_N W_{R,M}^\alpha - g \varepsilon^{\alpha\beta\gamma} W_{R,M}^\beta W_{R,N}^\gamma \quad (\alpha, \beta, \gamma = 1, 2, 3). \end{aligned} \quad (5.17)$$

In order to distinguish between the two $SU(2)$ groups, we denote the $SU(2)_L$ indices by lower-case Latin letters a, b, \dots and the $SU(2)_R$ indices by lower-case Greek letters α, β, \dots . The abelian $U(1)_X$ gauge boson with the corresponding field strength tensor

$$X_{MN} = \partial_M X_N - \partial_N X_M \quad (5.18)$$

couples with the 5D coupling constant g_X . Note that the sign of the 5D gauge coupling constants throughout this chapter are opposite to the definition in (2.11). Moreover, the coupling constants have mass dimension $-1/2$, reflecting the non-renormalisability of the 5D theory.

As already mentioned in the introduction, the *KK decomposition* allows to separate the 5D bulk fields into *KK modes* or *KK excitations* $\phi^{(n)}(x)$, which depend only on the 4D coordinate x , and the *KK profiles* or *shape functions* $f^{(n)}(y)$, depending only on the extra-dimensional coordinate y

$$\Phi(x, y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} \phi^{(n)}(x) f^{(n)}(y). \quad (5.19)$$

The KK decomposition is an essential ingredient in deriving an effective 4D theory from the 5D one, since it enables to perform the integration over the fifth dimension $\int_0^L dy \mathcal{L} \rightarrow \mathcal{L}_{\text{eff}}^{4D}$.

¹Without the colour indices, the distinction between the metric tensor and the field strength tensor has to be derived from the context.

Gauge-Fixing Terms

In general, a 5D gauge field consists of a 4D gauge field V_μ and a 4D scalar V_5 which corresponds to its fifth component. Thus, the 4D gauge field does not only mix with the usual scalar Higgs modes, but also with V_5 through the pure gauge kinetic terms. In order to eliminate those mixing terms one has to add appropriate R_ξ -gauge fixing terms. As indicated in [183], it is not advantageous to add a covariant 5D gauge-fixing term of the form $\mathcal{L}_{\text{gfix}}^{5\text{D}} = -1/(2\xi)(\partial_M V^M)^2$, which one would naively suggest in an $SO(1,4)$ invariant 5D theory. Instead — as compactification in general breaks $SO(1,4)$ invariance anyway — one can choose a non-covariant 5D generalised gauge-fixing condition of the form $\mathcal{L}_{\text{gfix}}^{5\text{D}} = -1/(2\xi)(\partial_\mu V^\mu - \xi\partial_5 V_5)^2$. After performing the KK decomposition, one obtains the usual propagators for the 4D gauge fields within the covariant 4D R_ξ -gauges. Following [184], additional boundary gauge-fixing terms have to be introduced in order to eliminate the mixing terms arising at the boundaries as well.

However, one can follow a different strategy and carry out the KK expansion first, then apply the integral over the fifth dimension and add the generalised 4D R_ξ -gauge fixing Lagrangian $\mathcal{L}_{\text{gfix}}^{4\text{D}}$ in the effective 4D theory [185]. The Goldstone bosons then correspond to linear combinations of the KK modes from the former 5D scalar and the non-physical fluctuations around the Higgs VEV. This method is convenient within the so-called *perturbative approach* of EWSB, which is widely used in the literature [103, 183, 186–188]. It means that one first ignores all effects from EWSB and then treats the Higgs coupling as a perturbation after the KK expansion has been performed.² As a consequence of the perturbative approach, the bulk equations of motion as well as the boundary conditions follow from the free 5D action and are not affected by the Higgs VEV. The approach is particularly convenient as the effects from EWSB can be treated as small perturbations to the mass matrices arising from the EDIM setup. We will also use this approach in the following, and choose to work in the gauge $V_5 = 0$ together with the constraint $\partial_\mu V^\mu = 0$.

Gauge Symmetry Breaking on the UV brane

As already mentioned in Section 5.1, the variation of the full 5D action separates into two pieces (5.13). From the first part the bulk equation of motions can be deduced while the second one requires boundary conditions which are consistent with the action principle. The natural BCs for pure gauge theories on an interval read

$$\partial_5 W_{L,\mu}^a|_{0,L} = 0 \quad (\text{Neumann BC}) \quad \text{and} \quad W_{L,5}^a|_{0,L} = 0 \quad (\text{Dirichlet BC}). \quad (5.20)$$

However, the introduction of boundary scalar fields on both branes which develop an infinitely large VEV allow for the opposite choice of BCs (see [184, 193] for further details)

$$W_{L,\mu}^a|_{0,L} = 0 \quad (\text{Dirichlet BC}) \quad \text{and} \quad \partial_5 W_{L,5}^a|_{0,L} = 0 \quad (\text{Neumann BC}). \quad (5.21)$$

²The complementary approach, solving the equations of motion already in the presence of EWSB, has been followed e.g. in [184, 189–192].

Analogously a mixture of the above BCs for the different boundaries may be obtained if the decoupling scalar is added on only one boundary of the extra dimension. In the following we use the abbreviations $+$ ($-$) for a Neumann (Dirichlet) BC and assign them to the bracket (UV IR).

With the general KK decomposition ansatz (5.19), we explicitly solved the EOM in the Appendix B, and found that the gauge KK profiles are given by

$$f_{\text{gauge}}^{(0)}(y) = 1 \quad \text{and} \\ f_{\text{gauge}}^{(n)}(y) = \frac{e^{ky}}{N_n} \left[J_1\left(\frac{m_n}{k}e^{ky}\right) + b_1(m_n)Y_1\left(\frac{m_n}{k}e^{ky}\right) \right], \quad (n = 1, 2, \dots). \quad (5.22)$$

A zero mode profile $f_{\text{gauge}}^{(0)}(y)$ exists only for $(++)$ BCs, $J_1(x)$ ($Y_1(x)$) denote the Bessel functions of first (second) kind, and the normalisation factor N_n (B.52) follows from the normalisation condition. While the zero mode profile is flat, the specific form of $f_{\text{gauge}}^{(n)}(y)$ in (5.22) implies that the excited gauge KK profiles are localised near the IR brane. The explicit expressions for $b_1(m_n)$ and the mass m_n of the n -th KK mode, defined by $(\partial_\mu\partial^\mu + m_n^2)\phi^{(n)} = 0$, depend on the specific choice of BCs on the branes.

For fields with $(++)$ boundary conditions, the profiles have to fulfil

$$\partial_y f_{\text{gauge}}^{(n)}(y) \Big|_{y=0,L} = 0 \quad (5.23)$$

and one obtains the relation [103]

$$b_1(m_n) = -\frac{J_1(m_n/k) + m_n/k J_1'(m_n/k)}{Y_1(m_n/k) + m_n/k Y_1'(m_n/k)} = b_1(m_n e^{kL}). \quad (5.24)$$

This can only be solved numerically with the following solution

$$m_1^{\text{gauge}}(++) \simeq 2.45f \equiv M_{++}, \quad (5.25)$$

where we have introduced the effective new physics scale $f = ke^{-kL} \sim \mathcal{O}(1 \text{ TeV})$.

Similarly for $(-+)$ fields, which have to fulfil

$$f_{\text{gauge}}^{(n)}(y) \Big|_{y=0} = \partial_y f_{\text{gauge}}^{(n)}(y) \Big|_{y=L} = 0, \quad (5.26)$$

one finds

$$b_1(m_n) = -\frac{J_1(m_n/k)}{Y_1(m_n/k)} = -\frac{J_1(m_n e^{kL}/k) + m_n e^{kL}/k J_1'(m_n e^{kL}/k)}{Y_1(m_n e^{kL}/k) + m_n e^{kL}/k Y_1'(m_n e^{kL}/k)}, \quad (5.27)$$

with the numerical result

$$m_1^{\text{gauge}}(-+) \simeq 2.40f \equiv M_{-+}. \quad (5.28)$$

Thus, the $\sim 2\%$ suppression of the numerical solution in the latter case is a direct consequence of the different BCs on the UV brane [186]. Note that the KK masses for the gauge bosons

neither depend on the gauge group nor on the size of the gauge coupling, but are universal for all gauge bosons with the same BCs. Whereas fields with $(++)$ BCs have zero modes in addition to their massive KK modes, fields with mixed BCs only contain massive KK modes. In order to avoid non-observed light gauge bosons apart from the SM content, we require the following set of BCs

$$\begin{aligned} W_{L,\mu}^a(++), \quad B_\mu(++), \\ W_{R,\mu}^b(-+), \quad Z_{X,\mu}(-+), \end{aligned} \quad (5.29)$$

where $a = 1, 2, 3$, and $b = 1, 2$. Remember that the BCs for the 4D gauge field automatically imply opposite BCs for its fifth component (5.20)–(5.21). The fact that the RSc gauge content does not include any 4D gauge fields with Dirichlet BC on the IR brane (5.29) then confirms our choice of the $V_5 = 0$ gauge.

The above given BCs (5.29) can be realised by adding a scalar on the UV brane which transforms as a doublet under $SU(2)_R$ and carries a non-trivial $U(1)_X$ charge $Q_X = 1/2$. In developing an infinite VEV, the scalar decouples from the theory. The BCs induce the symmetry breakdown

$$SU(2)_L \times SU(2)_R \times P_{LR} \times U(1)_X \xrightarrow{\text{UV brane}} SU(2)_L \times U(1)_Y \quad (5.30)$$

on the UV brane, where the quantum numbers are related by

$$\frac{Y}{2} = T_R^3 + Q_X. \quad (5.31)$$

The new linear combinations of the fields are given by

$$Z_{X,\mu} = \cos \phi W_{R,\mu}^3 - \sin \phi X_\mu, \quad B_\mu = \sin \phi W_{R,\mu}^3 + \cos \phi X_\mu, \quad (5.32)$$

where

$$\cos \phi = \frac{g}{\sqrt{g^2 + g_X^2}}, \quad \sin \phi = \frac{g_X}{\sqrt{g^2 + g_X^2}}. \quad (5.33)$$

At this stage, the zero modes of the gauge bosons $W_{L,\mu}^a$ and B_μ are massless, but in the course of EWSB they will receive small mass contributions of $\mathcal{O}(v^2)$. After the diagonalisation of the mass matrices, the zero modes get small admixtures of higher KK modes and lead to the gauge mass eigenstates which can be identified with the SM gauge bosons W_μ^\pm , Z_μ and A_μ .

Anticipating the effects of EWSB, it will be useful to follow [186] and define the fields

$$W_{L,\mu}^\pm = \frac{W_{L,\mu}^1 \mp iW_{L,\mu}^2}{\sqrt{2}}, \quad W_{R,\mu}^\pm = \frac{W_{R,\mu}^1 \mp iW_{R,\mu}^2}{\sqrt{2}}, \quad (5.34)$$

as well as the electrically neutral linear combinations

$$Z_\mu = \cos \psi W_{L,\mu}^3 - \sin \psi B_\mu, \quad A_\mu = \sin \psi W_{L,\mu}^3 + \cos \psi B_\mu, \quad (5.35)$$

where again $\sin \psi$ is given in terms of the gauge couplings (see (5.33) for the definition of ϕ)

$$\cos \psi = \frac{1}{\sqrt{1 + \sin^2 \phi}}, \quad \sin \psi = \frac{\sin \phi}{\sqrt{1 + \sin^2 \phi}} = \sin \phi \cos \psi. \quad (5.36)$$

Due to the above mentioned mixing between the gauge boson zero and KK modes, $\sin \psi$ differs from $\sin \theta_W$ (3.30) by corrections of $\mathcal{O}(v^2/f^2)$.

5.3 Higgs Sector and Electroweak Symmetry Breaking

In the previous section, we discussed the breaking of the EW bulk gauge symmetry to the SM gauge group through an appropriate choice of boundary conditions of the gauge bosons on the UV brane. To mimic the standard EWSB, $SU(2)_L \times U(1)_Y \rightarrow U(1)_Q$, we introduce a Higgs field which transforms as a singlet under the $U(1)_X$ bulk symmetry ($Q_X(H) = 0$) and as a bidoublet under $SU(2)_L \times SU(2)_R$. In contrast to the global transformation behaviour of the Higgs bidoublet in (3.35), we choose $U_R^T(x)$ instead of $U_R^\dagger(x)$ as transformation matrix,

$$H' = U_L(x) H U_R^T(x) = e^{i\alpha_L^a(x)T_L^a} H e^{i\alpha_R^b(x)(T_R^b)^T} = e^{i\alpha_L^a(x)T_L^a} H e^{i\alpha_R^b(x)(T_R^b)^*}, \quad (5.37)$$

such that the various components of the Higgs bidoublet have the following assignments of $SU(2)_{L,R}$ isospin quantum numbers (T_L^3, T_R^3)

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \sim \begin{pmatrix} (+\frac{1}{2}, +\frac{1}{2}) & (+\frac{1}{2}, -\frac{1}{2}) \\ (-\frac{1}{2}, +\frac{1}{2}) & (-\frac{1}{2}, -\frac{1}{2}) \end{pmatrix}. \quad (5.38)$$

Since the conjugated Higgs $\tilde{H} = \sigma^2 H^* \sigma^2$ has the same transformation behaviour as H one can impose the self-duality condition $H \stackrel{!}{=} \tilde{H}$. This requirement implies the two independent conditions

$$H_{11} = H_{22}^* \quad \text{and} \quad H_{12} = -H_{21}^*, \quad (5.39)$$

such that the degrees of freedom are reduced from eight to four real parameters and H can be represented by

$$H = \begin{pmatrix} H_{11} & H_{12} \\ -H_{12}^* & H_{11}^* \end{pmatrix} := H_{a\alpha}. \quad (5.40)$$

By construction, the Higgs Lagrangian

$$\mathcal{L}_{\text{Higgs}} = \sqrt{G} \left((D_M H)^\dagger_{a\alpha} (D^M H)_{a\alpha} - V(H) \right), \quad (5.41)$$

with the covariant derivative

$$(D_M H)_{a\alpha} = \partial_M H_{a\alpha} + ig(\tau^c)_{ab} W_{L,M}^c H_{b\alpha} + ig(\tau^\gamma)_{\alpha\beta} W_{R,M}^\gamma H_{a\beta} \quad (5.42)$$

is gauge invariant under local $SU(2)_L \times SU(2)_R$ transformations. Thus, if the Higgs VEV induces the breaking

$$SU(2)_L \times SU(2)_R \times P_{LR} \rightarrow SU(2)_V \times P_{LR}, \quad (5.43)$$

an unbroken custodial $SU(2)_V$ symmetry is preserved, which protects the ρ -parameter from radiative corrections as discussed in Section 3.5. However, here the custodial symmetry needs to be gauged to protect the Higgs sector since a global symmetry in the CFT corresponds to a gauge symmetry in the 5D theory [194].

As the scalar KK modes turn out to be much heavier than the gauge and fermionic resonances [103], we neglect them in what follows and truncate the KK expansion already after the zero mode according to

$$H(x, y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} H^{(n)}(x) f_H^{(n)}(y) = \frac{1}{\sqrt{L}} H^{(0)}(x) f_H^{(0)}(y) + \dots \equiv \frac{1}{\sqrt{L}} H(x) h(y) + \dots, \quad (5.44)$$

where the Higgs potential $V(H)$ generates a non-vanishing VEV only for the Higgs zero mode. As we do not specify the Higgs potential, we cannot solve the bulk equations of motion explicitly, but merely assume the zero mode profile

$$h(y) = \sqrt{2kL(\beta - 1)} e^{kL} e^{\beta k(y-L)}. \quad (5.45)$$

This form corresponds to the general solution for a zero mode in the limit $\beta \gg 1$ for a Higgs field localised near the IR brane (B.66), and fulfils the normalisation condition (B.51)

$$\frac{1}{L} \int_0^L dy e^{-2ky} h(y)^2 = 1. \quad (5.46)$$

The VEV of the Higgs zero mode respects a residual $SU(2)_V$ symmetry (see Section 3.5), if and only if it is invariant under the (infinitesimal) transformation of (5.37) with $\alpha_R^b(x) = \alpha_L^a(x) = \alpha^a(x)$ and $T_R^b = T_L^a$

$$\langle H(x) \rangle' = \langle H(x) \rangle + i\alpha^a(x) T_L^a \langle H(x) \rangle + \langle H(x) \rangle i\alpha^a(x) T_L^{a*} \stackrel{!}{=} \langle H(x) \rangle. \quad (5.47)$$

For the self-dual Higgs field (5.40) this condition is fulfilled for

$$\langle H(x) \rangle = \begin{pmatrix} 0 & -v/2 \\ v/2 & 0 \end{pmatrix}, \quad (5.48)$$

where v denotes the 4D VEV with the value $v = 246$ GeV. Parameterising the Higgs bidoublet in a linear manner

$$H(x) = \begin{pmatrix} \pi^+/\sqrt{2} & -(h^0 - i\pi^0)/2 \\ (h^0 + i\pi^0)/2 & \pi^-/\sqrt{2} \end{pmatrix}, \quad (5.49)$$

$h^0(x)$ represents the neutral real scalar Higgs field that develops the non-vanishing VEV v , and π^+, π^-, π^0 are the Goldstone fluctuations according to the three broken generators. In (5.49), the upper index indicates the electric charges of the fields and the factors of two are chosen such that all scalar fields are canonically normalised.

Combining the two symmetry breaking steps, we see that the low-energy effective theory is described by the spontaneous breaking

$$SU(2)_L \times U(1)_Y \rightarrow U(1)_Q, \quad (5.50)$$

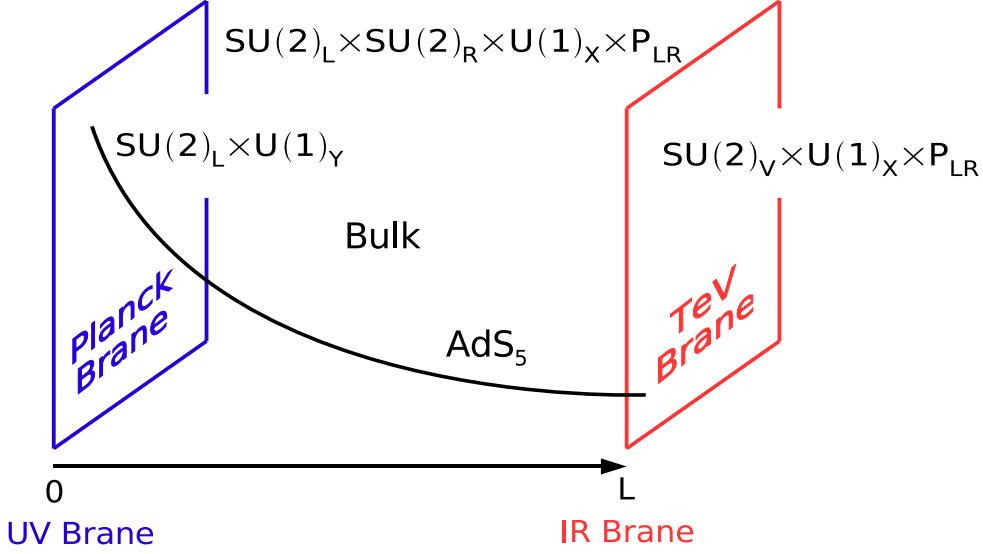


Figure 5.1: *EW symmetry breaking pattern of the RSc model.*

as required by phenomenology. Moreover, the hypercharge operator $Y/2 = T_R^3 + Q_X$ is modified to

$$Q = T_L^3 + \frac{Y}{2} \stackrel{(5.31)}{=} T_L^3 + T_R^3 + Q_X. \quad (5.51)$$

The EW symmetry breaking pattern of the model is displayed in Figure 5.1.

5.4 Gauge Boson Masses

In this chapter we determine the contribution to the gauge boson masses originating from the Higgs mechanism, where we concentrate only on the zeroth and first KK gauge boson modes. Including the KK masses from the extra-dimensional setup, we collect the mass contributions for the neutral and charged gauge bosons and place them in 3×3 mass matrices. The explicit diagonalisation of the latter will be subject of the following section.

QED and QCD Gauge Boson Masses

Due to the unbroken gauge invariance of QED and QCD, the gluon and photon fields including their KK modes do not couple to the Higgs boson at leading order in perturbation theory. Their masses are solely given by the extra-dimensional setup

$$\begin{aligned} M_{A^{(0)}} &= 0, & M_{A^{(1)}} &= m_1^{\text{gauge}(++)} = M_{++}, \\ M_{G^{(0)}} &= 0, & M_{G^{(1)}} &= m_1^{\text{gauge}(++)} = M_{++}, \end{aligned} \quad (5.52)$$

and neither mixings with each other nor with the neutral EW gauge bosons Z and Z_X arise.

EW Gauge Boson Masses

The EW gauge bosons ($W_L^{(0,1)\pm}$, $W_R^{(1)\pm}$, $Z^{(0,1)}$, $Z_X^{(1)}$) get $\mathcal{O}(v^2)$ masses from their couplings to the Higgs boson, which are contained in the Higgs kinetic term

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} \supset \frac{g^2}{L} e^{-2ky} & \left[(\tau^c W_{L,\mu}^c)_{ab} \langle H(x) \rangle_{b\alpha} + \langle H(x) \rangle_{a\beta} (\tau^\gamma W_{R,\mu}^\gamma)_{\beta\alpha}^T \right]^\dagger \\ & \left[(\tau^c W_L^{c,\mu})_{ab} \langle H(x) \rangle_{b\alpha} + \langle H(x) \rangle_{a\beta} (\tau^\gamma W_R^{\gamma,\mu})_{\beta\alpha}^T \right] h(y)^2 dy, \end{aligned} \quad (5.53)$$

where we used the explicit expressions for the RS metric. Inserting the Higgs VEV (5.48) and switching into the basis of $W_{L,R}^\pm$,

$$W_{L,R}^1 = \frac{W_{L,R}^+ + W_{L,R}^-}{\sqrt{2}}, \quad W_{L,R}^2 = i \frac{W_{L,R}^+ - W_{L,R}^-}{\sqrt{2}}, \quad (5.54)$$

we obtain the mass contributions to the charged EW gauge bosons

$$\mathcal{L}_{\text{Higgs}} \supset \frac{g^2 v^2}{4L} e^{-2ky} h(y)^2 (W_L^+ W_L^- + W_R^+ W_R^- - W_L^+ W_R^- - W_R^+ W_L^-), \quad (5.55)$$

and the neutral EW gauge bosons

$$\mathcal{L}_{\text{Higgs}} \supset \frac{g^2 v^2}{8L} e^{-2ky} h(y)^2 (W_L^3 W_L^3 + W_R^3 W_R^3 - 2 W_R^3 W_L^3). \quad (5.56)$$

Charged EW Gauge Boson Masses

Expanding also the gauge boson fields in (5.55) up to their first KK modes, the Lagrangian can be reformulated by

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} \supset \frac{g^2 v^2}{4L^2} e^{-2ky} h(y)^2 & \left[W_L^{(0)+} W_L^{(0)-} \right. \\ & + g(y) (W_L^{(0)+} W_L^{(1)-} + W_L^{(1)+} W_L^{(0)-}) - \tilde{g}(y) (W_L^{(0)+} W_R^{(1)-} + W_R^{(1)+} W_L^{(0)-}) \\ & \left. + g(y)^2 W_L^{(1)+} W_L^{(1)-} + \tilde{g}^2(y) W_R^{(1)+} W_R^{(1)-} - g(y) \tilde{g}(y) (W_L^{(1)+} W_R^{(1)-} + W_R^{(1)+} W_L^{(1)-}) \right], \end{aligned} \quad (5.57)$$

where we have introduced the short-hand notation

$$g(y) = f_{\text{gauge}}^{(1)}(y, (++) \quad (5.58)$$

for the bulk shape function of $Z^{(1)}$, $W_L^{(1)\pm}$, and

$$\tilde{g}(y) = f_{\text{gauge}}^{(1)}(y, (-+)) \quad (5.59)$$

for the bulk shape function of $Z_X^{(1)}$, $W_R^{(1)\pm}$. In order to obtain the masses in the effective 4D theory, we perform the integral over the extra dimension. The first term in (5.57) with the

two left-handed flat zero mode profiles simplifies due to the normalisation condition of the Higgs zero mode profile (5.46). For the other terms, we define the overlap integrals containing one KK mode

$$\mathcal{I}_1^+ = \frac{1}{L} \int_0^L dy e^{-2ky} g(y) h(y)^2, \quad \mathcal{I}_1^- = \frac{1}{L} \int_0^L dy e^{-2ky} \tilde{g}(y) h(y)^2, \quad (5.60)$$

corresponding to integrals arising from the second line in (5.57), or two KK modes

$$\begin{aligned} \mathcal{I}_2^{++} &= \frac{1}{L} \int_0^L dy e^{-2ky} g(y)^2 h(y)^2, & \mathcal{I}_2^{--} &= \frac{1}{L} \int_0^L dy e^{-2ky} \tilde{g}(y)^2 h(y)^2, \\ \mathcal{I}_2^{-+} &= \frac{1}{L} \int_0^L dy e^{-2ky} g(y) \tilde{g}(y) h(y)^2, \end{aligned} \quad (5.61)$$

from the third line. Including the heavy KK masses for the gauge fields according to (5.25) and (5.28) in addition to the Higgs-induced mass terms, one finds that the complete mass matrix for the charged gauge bosons contained in

$$\mathcal{L}_{\text{mass}}^{\text{charged}} = \begin{pmatrix} W_L^{(0)+} & W_L^{(1)+} & W_R^{(1)+} \end{pmatrix} \mathcal{M}_{\text{charged}}^2 \begin{pmatrix} W_L^{(0)-} \\ W_L^{(1)-} \\ W_R^{(1)-} \end{pmatrix}, \quad (5.62)$$

reads explicitly

$$\mathcal{M}_{\text{charged}}^2 = \begin{pmatrix} \frac{g^2 v^2}{4L} & \frac{g^2 v^2}{4L} \mathcal{I}_1^+ & -\frac{g^2 v^2}{4L} \mathcal{I}_1^- \\ \frac{g^2 v^2}{4L} \mathcal{I}_1^+ & M_{++}^2 + \frac{g^2 v^2}{4L} \mathcal{I}_2^{++} & -\frac{g^2 v^2}{4L} \mathcal{I}_2^{-+} \\ -\frac{g^2 v^2}{4L} \mathcal{I}_1^- & -\frac{g^2 v^2}{4L} \mathcal{I}_2^{-+} & M_{-+}^2 + \frac{g^2 v^2}{4L} \mathcal{I}_2^{--} \end{pmatrix}. \quad (5.63)$$

As the mass dimension of g^2/L is zero, the entries of the squared 4D mass matrix indeed have the right dimension. The off-diagonal elements in (5.63) induce mixings between the various modes which will be determined in the next section.

Neutral EW Gauge Boson Masses

The same procedure leads to the Higgs contribution for the neutral EW gauge boson masses. Expressing $W_{L,R}^3$ in terms of the physical fields Z , Z_X , A with the help of

$$\begin{aligned} W_L^3 &= \cos \psi Z + \sin \psi A, \\ W_R^3 &= Z_X \cos \phi - \frac{\sin^2 \psi}{\cos \psi} Z + \sin \psi A, \end{aligned} \quad (5.64)$$

the terms in (5.56) lead to the following mass terms of Z and Z_X

$$\mathcal{L}_{\text{Higgs}} \supset \frac{g^2 v^2}{8L} e^{-2ky} h(y)^2 \left(-2 \frac{\cos \phi}{\cos \psi} Z Z_X + \frac{1}{\cos^2 \psi} Z^2 + \cos^2 \phi Z_X^2 \right). \quad (5.65)$$

Applying the KK expansion and integrating over the extra dimension, the masses of the neutral electroweak gauge bosons, including the heavy KK masses, reads

$$\mathcal{L}_{\text{mass}}^{\text{neutral}} = \frac{1}{2} \begin{pmatrix} Z^{(0)} & Z^{(1)} & Z_X^{(1)} \end{pmatrix} \mathcal{M}_{\text{neutral}}^2 \begin{pmatrix} Z^{(0)} \\ Z^{(1)} \\ Z_X^{(1)} \end{pmatrix}, \quad (5.66)$$

where

$$\mathcal{M}_{\text{neutral}}^2 = \begin{pmatrix} \frac{g^2 v^2}{4L \cos^2 \psi} & \frac{g^2 v^2 \mathcal{I}_1^+}{4L \cos^2 \psi} & -\frac{g^2 v^2 \cos \phi \mathcal{I}_1^-}{4L \cos \psi} \\ \frac{g^2 v^2 \mathcal{I}_1^+}{4L \cos^2 \psi} & M_{++}^2 + \frac{g^2 v^2 \mathcal{I}_2^{++}}{4L \cos^2 \psi} & -\frac{g^2 v^2 \cos \phi \mathcal{I}_2^{-+}}{4L \cos \psi} \\ -\frac{g^2 v^2 \cos \phi \mathcal{I}_1^+}{4L \cos \psi} & -\frac{g^2 v^2 \cos \phi \mathcal{I}_2^{-+}}{4L \cos \psi} & M_{-+}^2 + \frac{g^2 v^2 \cos^2 \phi \mathcal{I}_2^{--}}{4L} \end{pmatrix}. \quad (5.67)$$

Again, mixing of the modes with same electric charge are induced by EWSB.

5.5 Analytic Diagonalisation of the EW Gauge Boson Mass Matrices

In order to find the physical mass eigenstates, the mass matrices $\mathcal{M}_{\text{charged}}^2$ and $\mathcal{M}_{\text{neutral}}^2$ have to be diagonalised. Being real and symmetric, this can be achieved by a rotation with the orthogonal transformation matrices \mathcal{G}_W and \mathcal{G}_Z . The gauge eigenstates are then related to the mass eigenstates (W^\pm , W_H^\pm , W'^\pm) and (Z , Z_H , Z') according to

$$\begin{pmatrix} W_L^{(0)\pm} \\ W_L^{(1)\pm} \\ W_R^{(1)\pm} \end{pmatrix} = \mathcal{G}_W^T \begin{pmatrix} W^\pm \\ W_H^\pm \\ W'^\pm \end{pmatrix}, \quad \begin{pmatrix} Z^{(0)} \\ Z^{(1)} \\ Z_X^{(1)} \end{pmatrix} = \mathcal{G}_Z^T \begin{pmatrix} Z \\ Z_H \\ Z' \end{pmatrix}. \quad (5.68)$$

The hierarchy between the $\mathcal{O}(v^2)$ mass contributions from EWSB and the heavy KK masses $M_{++}^2 \sim M_{-+}^2 \sim M^2 \sim f^2$ from the extra-dimensional setup, allows for a perturbative diagonalisation with respect to the expansion parameter

$$\epsilon = \frac{g^2 v^2}{4LM^2} \sim \mathcal{O}\left(\frac{v^2}{f^2}\right). \quad (5.69)$$

As the two EW gauge boson mass matrices (5.63) and (5.67) have the same hierarchical structure, we can conduct the diagonalisation procedure for both cases simultaneously by considering the symmetric matrix

$$A = M^2 \begin{pmatrix} A_{11} \epsilon & A_{12} \epsilon & A_{13} \epsilon \\ A_{12} \epsilon & 1 + A_{22} \epsilon & A_{23} \epsilon \\ A_{13} \epsilon & A_{23} \epsilon & 1 + A_{33} \epsilon \end{pmatrix}. \quad (5.70)$$

The coefficients A_{ij} ($i, j = 1, 2, 3$) are arbitrary, but of $\mathcal{O}(1)$ in order not to spoil the hierarchy. In the Appendix B.6, we calculate the corresponding eigenvalues up to $\mathcal{O}(\epsilon^2)$ and the eigenvectors up to $\mathcal{O}(\epsilon)$, with the help of two different approaches. In the ‘‘direct’’ one, in which

we have in principle the exact formulae for determining the eigenvalues and eigenvectors at hand and we use the ϵ expansion to avoid the increasing complexity order by order. The second method is based on the algorithm of Rayleigh–Schrödinger, where, due to the nature of a perturbation theory, the expansion is implemented from the early beginning. The three real eigenvalues of the matrix A in terms of the general elements A_{ij} in both approaches are found to be

$$\lambda_1 = A_{11}M^2\epsilon - (A_{12}^2 + A_{13}^2)M^2\epsilon^2 \quad (5.71)$$

$$\begin{aligned} \lambda_{2,3} = M^2 + \frac{M^2}{2}(A_{22} + A_{33} \pm B)\epsilon \\ + \frac{M^2}{2B}(\pm 4A_{12}A_{13}A_{23} + A_{12}^2(B \pm F) + A_{13}^2(B \mp F))\epsilon^2. \end{aligned} \quad (5.72)$$

The auxiliary quantities B and F used in (5.72) stand for

$$F = A_{22} - A_{33}, \quad B = \sqrt{4A_{23}^2 + F^2}, \quad \text{with } B^2 > F^2, \quad (5.73)$$

from which we receive the relation

$$A_{23} = \text{sgn}[A_{23}] \frac{1}{2} \sqrt{B^2 - F^2}. \quad (5.74)$$

Obviously, the eigenvalues $\lambda_{2,3}$ are degenerate at zeroth order in perturbation theory. However, if $B \neq 0$, the degeneracy is lifted at $\mathcal{O}(\epsilon)$. The corresponding normalised eigenvectors to $\mathcal{O}(\epsilon)$ accuracy are summarised by

$$\begin{aligned} v_{\lambda_1, \text{norm}}^T &= \left(1, \quad -A_{12}\epsilon, \quad -A_{13}\epsilon \right), \\ v_{\lambda_2, \text{norm}}^T &= \frac{\left((2A_{12}A_{23} + (B-F)A_{13})\epsilon, \quad 2A_{23} - \frac{(B-F)X}{B^2}\epsilon, \quad (B-F) + \frac{2A_{23}X}{B^2}\epsilon \right)}{\sqrt{2B(B-F)}}, \\ v_{\lambda_3, \text{norm}}^T &= \frac{\left((-2A_{13}A_{23} + (B-F)A_{12})\epsilon, \quad (B-F) + \frac{2A_{23}X}{B^2}\epsilon, \quad -2A_{23} + \frac{(B-F)X}{B^2}\epsilon \right)}{\sqrt{2B(B-F)}}, \end{aligned} \quad (5.75)$$

where we have introduced the short-hand notation

$$X = FA_{12}A_{13} + A_{23}(A_{13}^2 - A_{12}^2). \quad (5.76)$$

To zeroth order in perturbation theory, the eigenvectors $v_{\lambda_2, \text{norm}}$ and $v_{\lambda_3, \text{norm}}$ span the 2-dimensional degenerate subspace. Since its columns represent an orthogonal rotation matrix, we define the corresponding rotation angle ξ by

$$\sin \xi := \frac{2|A_{23}|}{\sqrt{2B(B-F)}} \stackrel{(5.74)}{=} \sqrt{\frac{1}{2} + \frac{F}{2B}}, \quad \cos \xi := \frac{(B-F)}{\sqrt{2B(B-F)}} = \sqrt{\frac{1}{2} - \frac{F}{2B}}. \quad (5.77)$$

Introducing in addition the definitions

$$\sin \chi := -\text{sgn}[A_{23}] \sin \xi + \frac{X}{B^2} \cos \xi \epsilon, \quad \cos \chi := \cos \xi + \frac{X}{B^2} \text{sgn}[A_{23}] \sin \xi \epsilon, \quad (5.78)$$

and utilising $\text{sgn}[A_{23}] \sin \xi = \sin[\pm\xi]$ for $A_{23} \gtrless 0$, (5.75) can be brought into the compact form

$$\begin{aligned} v_{\lambda_1, \text{norm}}^T &= \left(1, -A_{12}\epsilon, -A_{13}\epsilon \right), \\ v_{\lambda_2, \text{norm}}^T &= \left((A_{12} \sin[\pm\xi] + A_{13} \cos \xi) \epsilon, -\sin \chi, \cos \chi \right), \\ v_{\lambda_3, \text{norm}}^T &= \left((-A_{13} \sin[\pm\xi] + A_{12} \cos \xi) \epsilon, \cos \chi, \sin \chi \right). \end{aligned} \quad (5.79)$$

Explicit Expressions for Charged EW Gauge Bosons

Being equipped with the formulae of the previous subsection, it is straightforward to give the explicit expressions for the charged and neutral gauge boson mass eigenvalues as well as the corresponding mass eigenstates. Introducing the parametrisation we used in [120],

$$M_{++}^2 = M^2 + av^2, \quad M_{-+}^2 = M^2 - av^2, \quad (5.80)$$

one can identify the elements of the charged gauge boson mass matrix with

$$\begin{aligned} A_{11} &= 1, & A_{22} &= \frac{4La}{g^2} + \mathcal{I}_2^{++}, & A_{33} &= -\frac{4La}{g^2} + \mathcal{I}_2^{--}, \\ A_{12} &= \mathcal{I}_1^+, & A_{13} &= -\mathcal{I}_1^-, & A_{23} &= -\mathcal{I}_2^{+-}, \end{aligned} \quad (5.81)$$

where $\text{sgn}[A_{23}] = -1$ and numerically the parameter $a \sim \mathcal{O}(1)$ for $f \sim \mathcal{O}(1 \text{ TeV})$.

Instead of putting the entries into the general formulae, we will give for simplicity only the expressions in the limit we took in [120]. According to the approximation to calculate $\mathcal{O}(v^2/f^2)$ corrections to the couplings of W^\pm and Z but to include only $\mathcal{O}(1)$ couplings involving heavy gauge boson mass eigenstates, we set the coefficients δ^{ij} in the ansatz

$$\mathcal{I}_2^{--} = \mathcal{I}_2, \quad \mathcal{I}_2^{-+} = \mathcal{I}_2 \left(1 + \delta^{-+} \frac{v^2}{f^2} \right), \quad \mathcal{I}_2^{++} = \mathcal{I}_2 \left(1 + \delta^{++} \frac{v^2}{f^2} \right), \quad (5.82)$$

to zero and thus a universal \mathcal{I}_2 will show up in the expressions. To this approximation, we receive the auxiliary quantities for the charged gauge boson masses

$$F \sim \frac{8La}{g^2}, \quad B \sim \frac{2}{g^2} \sqrt{g^4 \mathcal{I}_2^2 + 16L^2 a^2}, \quad (5.83)$$

which enter the expressions for the mass eigenvalues

$$\begin{aligned} M_W^2 &= \frac{g^2 v^2}{4L} - \frac{g^4 v^4}{16L^2 M^2} ((\mathcal{I}_1^+)^2 + (\mathcal{I}_1^-)^2), \\ M_{W', W_H}^2 &= M^2 + \frac{v^2}{4L} \left(g^2 \mathcal{I}_2 \pm \sqrt{g^4 \mathcal{I}_2^2 + 16L^2 a^2} \right). \end{aligned} \quad (5.84)$$

The corresponding eigenvectors span the orthogonal transformation matrix

$$G_W^T = \begin{pmatrix} 1 & \epsilon (\mathcal{I}_1^+ \cos \xi - \mathcal{I}_1^- \sin \xi) & -\epsilon (\mathcal{I}_1^+ \sin \xi + \mathcal{I}_1^- \cos \xi) \\ -\epsilon \mathcal{I}_1^+ & \cos \xi & -\sin \xi \\ \epsilon \mathcal{I}_1^- & \sin \xi & \cos \xi \end{pmatrix}, \quad (5.85)$$

where

$$\sin \xi \sim \sqrt{\frac{1}{2} + \frac{2La}{\sqrt{g^4 \mathcal{I}_2^2 + 16L^2 a^2}}}, \quad \cos \xi \sim \sqrt{\frac{1}{2} - \frac{2La}{\sqrt{g^4 \mathcal{I}_2^2 + 16L^2 a^2}}}. \quad (5.86)$$

According to equation (5.68), G_W^T relates the mass eigenstates (W^\pm , W_H^\pm , W'^\pm) to the gauge eigenstates ($W_L^{(0)\pm}$, $W_L^{(1)\pm}$, $W_R^{(1)\pm}$).

Explicit Expressions for Neutral EW Gauge Bosons

For the neutral EW gauge bosons, we identify the following elements corresponding to the neutral gauge boson mass matrix in (5.67)

$$\begin{aligned} A_{11} &= \frac{1}{\cos^2 \psi}, & A_{22} &= \frac{4La}{g^2} + \frac{\mathcal{I}_2^{++}}{\cos^2 \psi}, & A_{33} &= -\frac{4La}{g^2} + \mathcal{I}_2^{--} \cos^2 \phi, \\ A_{12} &= \frac{\mathcal{I}_1^+}{\cos^2 \psi}, & A_{13} &= -\frac{\mathcal{I}_1^- \cos \phi}{\cos \psi}, & A_{23} &= -\frac{\mathcal{I}_2^{+-} \cos \phi}{\cos \psi}. \end{aligned} \quad (5.87)$$

Using the above mentioned approximation and the relation (5.36) of the angle ϕ , the auxiliary quantities can be expressed as ψ -dependent functions with a universal \mathcal{I}_2 integral:

$$F \sim \frac{8La}{g^2} + \frac{2\mathcal{I}_2 \sin^2 \psi}{\cos^2 \psi}, \quad B \sim \frac{2}{g^2 \cos \psi} \sqrt{(g^4 \mathcal{I}_2^2 + 16L^2 a^2) \cos^2 \psi + 8Lag^2 \mathcal{I}_2 \sin^2 \psi}. \quad (5.88)$$

Again we omit the $\mathcal{O}(\epsilon^2)$ corrections to the mass eigenvalues of the neutral EW gauge bosons, which are then given by

$$\begin{aligned} M_Z^2 &= \frac{g^2 v^2}{4L \cos^2 \psi} - \frac{g^4 v^4}{16L^2 M^2 \cos^2 \psi} \left(\frac{(\mathcal{I}_1^+)^2}{\cos^2 \psi} + (\mathcal{I}_1^-)^2 \cos^2 \phi \right), \\ M_{Z', Z_H}^2 &= M^2 + \frac{v^2}{4L} \left(g^2 \mathcal{I}_2 \pm \frac{\sqrt{(g^4 \mathcal{I}_2^2 + 16L^2 a^2) \cos^2 \psi + 8Lag^2 \mathcal{I}_2 \sin^2 \psi}}{\cos \psi} \right). \end{aligned} \quad (5.89)$$

The eigenvectors $v_{\lambda_1, \text{norm}}$, $v_{\lambda_3, \text{norm}}$, $v_{\lambda_2, \text{norm}}$ correspond to the columns of G_Z^T

$$G_Z^T = \begin{pmatrix} 1 & \left(-\frac{\mathcal{I}_1^- \cos \phi}{\cos \psi} \sin \xi + \frac{\mathcal{I}_1^+}{\cos^2 \psi} \cos \xi \right) \epsilon & -\left(\frac{\mathcal{I}_1^+}{\cos^2 \psi} \sin \xi + \frac{\mathcal{I}_1^- \cos \phi}{\cos \psi} \cos \xi \right) \epsilon \\ -\frac{\mathcal{I}_1^+}{\cos^2 \psi} \epsilon & \cos \xi & -\sin \xi \\ \frac{\mathcal{I}_1^- \cos \phi}{\cos \psi} \epsilon & \sin \xi & \cos \xi \end{pmatrix}, \quad (5.90)$$

with the explicit expressions for $\sin \xi$ and $\cos \xi$

$$\begin{aligned} \sin \xi &\sim \left(\frac{1}{2} + \frac{4La \cos^2 \psi + g^2 \mathcal{I}_2 \sin^2 \psi}{2 \cos \psi \sqrt{(g^4 \mathcal{I}_2^2 + 16L^2 a^2) \cos^2 \psi + 8Lag^2 \mathcal{I}_2 \sin^2 \psi}} \right)^{1/2}, \\ \cos \xi &\sim \left(\frac{1}{2} - \frac{4La \cos^2 \psi + g^2 \mathcal{I}_2 \sin^2 \psi}{2 \cos \psi \sqrt{(g^4 \mathcal{I}_2^2 + 16L^2 a^2) \cos^2 \psi + 8Lag^2 \mathcal{I}_2 \sin^2 \psi}} \right)^{1/2}. \end{aligned} \quad (5.91)$$

As can be guessed from the assignment of the entries in (5.81) and (5.87), the results for the neutral gauge bosons reduce to the ones for the charged gauge bosons in the limit $\psi = \phi \rightarrow 0$.

5.6 Fermions in Warped Extra Dimensions

This section is devoted to fermions living in warped extra dimensions. After a short summary of basic concepts, e.g. the construction of the fermionic action, we focus on the specific particle content of the RSc model. For simplicity, we restrict ourselves to the quark sector.

As is well known, fermions are described by spinor fields belonging to the spin-1/2 representation of the Lorentz group. In extending the Lorentz group to the special orthogonal group $SO(1, n - 1)$ for theories with $n - 4$ extra dimensions, a problem arises for odd-dimensional space-times. In those, the generalised chirality operator is one of the Lorentz generators itself and does not commute with the other Lorentz generators any more. Thus, the spinor representation is in an irreducible representation of the Lorentz group and an analogue decomposition into two chiral Weyl spinors does not exist. However, chirality can be reintroduced through compactification, i.e. in our setup by imposing certain BCs within the interval approach. Since the BCs allow either for a right- or left-handed fermion zero mode, one has to double the fermion spectrum in order to obtain a chiral 4D effective theory that contains the SM fermion content.

5.6.1 Construction of the Warped Fermionic Action

The fermionic action has to be invariant under local frame rotations (Lorentz transformations $SO(1, n - 1)$) as well as under general coordinate transformations (diffeomorphism group $GL(n, \mathbb{R})$). Due to the equivalence principle we can find at every point x_0 a set of coordinates $\xi_{x_0}^A$, which are locally inertial at x_0 . Then the metric in any non-inertial system is given by

$$G_{MN}(x, y) = \eta_{AB} e_M^A(x, y) e_N^B(x, y), \quad (5.92)$$

with the *vielbein* $e_M^A(x, y) = \partial_M \xi_{x_0}^A(x, y)$. The vielbein relates the tangent frame, where the metric and the Dirac matrices are constants in space-time and which is the appropriate framework for the spinor formalism, with the coordinate space, in which the metric and Dirac matrices explicitly depend on the space-time coordinates. The vielbein in five dimensions can be represented for the RS metric (5.1) by

$$e_M^A(x, y) = \begin{cases} 1 & \text{for } A = M = 5, \\ e^{-ky} & \text{for } A = M = \mu. \\ 0 & \text{otherwise.} \end{cases} \quad (5.93)$$

For the inverse vielbein defined through

$$E_A^M(x, y) = \eta_{AB} G^{MN}(x, y) e_N^B(x, y), \quad (5.94)$$

an analogous expression exists

$$E_A^M(x, y) = \begin{cases} 1 & \text{for } A = M = 5, \\ e^{ky} & \text{for } A = M = \mu, \\ 0 & \text{otherwise.} \end{cases} \quad (5.95)$$

The relation between the coordinate-dependent gamma matrices $\Gamma^M(x, y)$ in curved space and the space-time independent gamma matrices γ^A of the tangent space is given by

$$\Gamma^M(x, y) = E_A^M(x, y) \gamma^A. \quad (5.96)$$

With the above definitions, the usual Clifford algebra in flat space

$$\{\gamma^A, \gamma^B\} = 2\eta^{AB} \quad (5.97)$$

can easily be translated into the curved space, where the gamma matrices fulfil

$$\{\Gamma^M(x, y), \Gamma^N(x, y)\} = 2G^{MN}(x, y). \quad (5.98)$$

Taking into account that the Clifford algebra (5.97) implies $(\gamma_{5D}^5)^2 = -1$ in contrast to the usual 4D definition of $\gamma_{4D}^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ with $(\gamma_{4D}^5)^2 = +1$, we include a factor of i into the definition, such that the 5D gamma matrices are related to the 4D gamma matrices as follows

$$\gamma_{5D}^A = \{\gamma^\mu, -i\gamma_{4D}^5\} = \{\gamma^\mu, \gamma^0\gamma^1\gamma^2\gamma^3\}. \quad (5.99)$$

In order to construct a covariant derivative that ensures the invariance of the Lagrangian under Lorentz and general coordinate transformation, the so-called *spin connection* ω_M has to be added to the gauge-covariant derivative D_M according to

$$\nabla_M = D_M + \omega_M. \quad (5.100)$$

With the help of the Christoffel symbols

$$\Gamma_{MP}^N = \frac{1}{2}G^{NR}(\partial_P G_{MR} + \partial_M G_{PR} - \partial_R G_{MP}), \quad (5.101)$$

the explicit expression for the spin connection can be written as

$$\omega_M = e_N^A (\partial_M E_B^N + \Gamma_{MP}^N E_B^P) \frac{\sigma_A^B}{2}, \quad (5.102)$$

where we introduced also the definition $\sigma_{AB} := \frac{1}{4}[\gamma_A, \gamma_B]$. For the RS metric (5.1) a straightforward calculation, which can be found in the Appendix B.1, yields

$$\omega_M = \begin{cases} \frac{i}{2}k e^{-ky} \gamma_\mu \gamma_{4D}^5 & \text{for } M = \mu, \\ 0 & \text{for } M = 5. \end{cases} \quad (5.103)$$

Starting with the requirement that the action should be real, or equivalently, that the corresponding Hamiltonian should be hermitian, one would make the following ansatz for the fermionic 4D Lagrangian coupled to the non-abelian gauge fields V_μ^a

$$S_{4D} = \int d^4x \frac{i}{2} (\bar{\Psi} \gamma^\mu (\partial_\mu - igV_\mu^a T^a) \Psi + \text{h.c.}). \quad (5.104)$$

In 4D one usually performs an integration by parts in the hermitian conjugate part. Neglecting the arising boundary terms due to the assumptions of vanishing field configurations in the limit $x \rightarrow \pm\infty$, the Lagrangian coincides with the one given in (2.13).

Remember that this approximation is not justified if an extra dimension of finite size is involved. Thus, we proceed as proposed in [195] and take the symmetric and hermitian Hamiltonian as a convenient starting point for the construction of the 5D fermionic Lagrangian

$$S = \int d^5x \sqrt{G} \left(\frac{1}{2} \bar{\Psi} (i \Gamma^M \nabla_M - ck) \Psi + \text{h.c.} \right), \quad (5.105)$$

with c denoting the 5D bulk Dirac mass of the fermion field in units of k . We will focus on this mass parameter c in the next subsection as it determines the localisation of the fermion zero mode profile along the extra dimension. As shown in the Appendix B.1, the action (5.105) can be brought into the form

$$S = \int d^5x \sqrt{G} \left(E_A^M \frac{i}{2} \bar{\Psi} \gamma^A (D_M - \overleftarrow{D}_M^\dagger) \Psi + E_A^M \frac{i}{2} \bar{\Psi} \{ \gamma^A, \omega_M \} \Psi - ck \bar{\Psi} \Psi \right), \quad (5.106)$$

whereupon the spin connection term drops out for the specific case of the RS metric. By convention, the derivatives in (5.106) act only on spinor fields, but not on metric factors like $E_A^M(x, y)$ or $\gamma^A(x, y)$.

5.6.2 KK Decomposition and Localisation of Fermionic Modes

In the first main part of this thesis, we already encountered the possibility to explain the wide range of quark masses through a dynamical spurion potential, which gives rise to a flavour symmetry breaking cascade via generating non-zero VEVs of the various spurion components within the Yukawa matrices. Also extra-dimensional models provide an explanation for the vast differences in quark masses, especially the hierarchy between the heavy third generation and the lighter first and second generation. The key point is the localisation freedom of the fermionic zero mode profiles $f_{L,R}^{(0)}(y)$ which arise in the KK decomposition

$$\Psi_{L,R}(x, y) = \frac{e^{2ky}}{\sqrt{L}} \sum_{n=0}^{\infty} \psi_{L,R}^{(n)}(x) f_{L,R}^{(n)}(y). \quad (5.107)$$

Under the assumption that the 5D Yukawa matrices are anarchic with complex $\mathcal{O}(1)$ entries, the hierarchy in the SM flavour parameters is directly related to the different localisation of the zero mode profiles [102, 103]. The reason is, that the effective 4D Yukawa couplings emerge

from overlap integrals of the Higgs shape function with the fermionic zero mode profiles along the extra dimension. In Subsection 5.7.2 we will explicitly demonstrate this statement via deriving the effective 4D Yukawa coupling matrices for the specific fermion content of the RSc model (see equation (5.125)). The larger the overlap of the fermion profiles with the IR brane localised Higgs, the larger the generated coupling and mass after EWSB will be. The method of solving the flavour hierarchy problem solely through geometry in EDIM models is known as *split fermion mechanism* [104, 105].

Localisation of Fermion Zero Mode Profiles — Solution to the Flavour Puzzle

In order to make the localisation feature more transparent, we absorb the factor of e^{ky} , occurring in the orthonormality condition for the fermion zero mode profiles,

$$\frac{1}{L} \int_0^L dy e^{ky} f_{L,R}^{(0)}(y) f_{L,R}^{(0)}(y) = 1, \quad (5.108)$$

into the shape functions along the extra dimension. Thus, with respect to the flat metric, their profiles change into

$$f_{L,R}^{(0)}(y, c) = \sqrt{\frac{(1 \mp 2c)kL}{e^{(1 \mp 2c)kL} - 1}} e^{\mp cky} \longrightarrow \hat{f}_{L,R}^{(0)}(y, c) = \sqrt{\frac{(1 \mp 2c)kL}{e^{(1 \mp 2c)kL} - 1}} e^{(\frac{1}{2} \mp c)ky}. \quad (5.109)$$

The specific form of $\hat{f}_{L,R}^{(0)}(y, c)$ suggests to differentiate between the two cases with $c > 1/2$ and $c < 1/2$, respectively.

- For $c > 1/2$ the normalisation factor in (5.109) is $\mathcal{O}(1)$ and $\hat{f}_L^{(0)}(y, c)$ is peaked around $y = 0$, i.e. the UV brane. As the overlap with the Higgs boson profile on or near the IR brane is small, so are the masses of the fermionic zero modes $\psi_{L,R}^{(0)}(x)$ which correspond to the SM fermions up to small admixtures with higher KK modes of $\mathcal{O}(v^2/f^2)$. This is the appropriate scenario for the lighter first two generations of quarks.
- For $c < 1/2$ the second term in the denominator of (5.109) can be neglected and the shape function

$$\hat{f}_{L,R}^{(0)}(y, c) \simeq \sqrt{(1 \mp 2c)kL} e^{(\frac{1}{2} \mp c)k(y-L)} \quad (5.110)$$

is strongly peaked towards $y = L$, i.e. the IR brane. The overlap with the Higgs profile is enormous and after EWSB a heavy mass is produced. Thus, this setup is favoured for the description of the third quark generation and especially for the heavy top quark.

The localisation of the fermionic zero mode profiles corresponding to the above two cases, as well as that of the flat fermion profile with $c = 1/2$, is visualised in Figure 5.2. Note that the chosen BCs (see Appendix B.4) determine whether there exists a left-handed or a right-handed zero mode for a specific 5D fermion representation. Correspondingly, the mass parameters $c_{L,R}$ can generally differ from each other. This freedom can help to satisfy certain

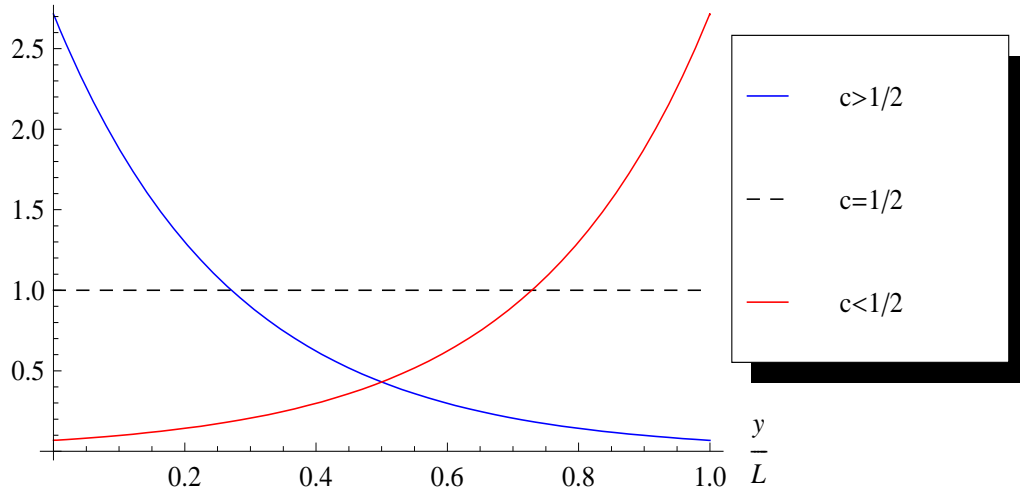


Figure 5.2: Localisation of fermion zero mode profiles for $c = 0.6, 0.5, 0.4$.

features in EW precision studies [112–114, 117] as well as in flavour physics [107, 188], while keeping the fermion masses of their natural size. This is in particular relevant for the third quark generation.

Localisation of Fermion KK Modes

The shape functions of the fermionic KK modes are given in (B.49) with $s = 1$,

$$f_{L,R}^{(n)}(y, c, \text{BC}) = \frac{e^{\frac{ky}{2}}}{N_n} \left(J_\nu \left(\frac{m_n}{k} e^{ky} \right) + b_\nu(m_n) Y_\nu \left(\frac{m_n}{k} e^{ky} \right) \right), \quad (5.111)$$

where $\nu = |c \pm 1/2|$ for left- (right-)handed modes, and explicit expressions for N_n and $b_\nu(m_n)$ can be found in [103]. The form of (5.111) implies that all KK modes are localised near the IR brane and there is no localisation freedom as it was the case for the zero mode profiles.

In summary [103, 196], the bulk mass parameter c is universal for all KK modes of a given fermion field, including the zero mode if one exists. The value of c controls the localisation of the zero mode along the extra dimension, which in turn can lead to a mass hierarchy of the SM quarks after EWSB. Despite the fact that the split fermion mechanism supplies a solution to the flavour puzzle, it gives rise to a flavour problem (see discussion in Subsection 5.8.6).

5.6.3 Fermion Content of the RSc Model

The specific fermion content of the RSc model has been motivated by the introduction of a custodial protection symmetry of the T parameter and the measured value of the $Zb_L\bar{b}_L$ coupling which is in nearly perfect agreement with the SM prediction [108, 110, 112–114, 116, 117, 197]. To this end, the left-handed bottom quark with $T_L^3 = -1/2$ has to satisfy the condition $T_R^3 = T_L^3$. This can be achieved by placing it in the lower right corner of a bidoublet

$(\mathbf{2}, \mathbf{2})_{2/3}$ under the $SU(2)_L \times SU(2)_R$ symmetry. Being in the same doublet of $SU(2)_L$, the quantum number assignment of the left-handed top quark, $(1/2, -1/2)$, follows immediately. In order not to break explicitly the global $U(3)^3$ bulk flavour symmetry of the quarks for vanishing Yukawa matrices and bulk mass matrices, we also embed the residual two left-handed quark doublets into bidoublets. To reproduce the proper hypercharge $Y/2 = T_R^3 + Q_X$ (5.31) of the quarks, one then needs to set $Q_X = 2/3$. In consequence, the right-handed quarks must have the same $U(1)_X$ quantum number in order to allow for non-vanishing Yukawa couplings that will produce SM masses of $\mathcal{O}(v^2)$ after EWSB. Using again the above relation, the SM hypercharges of the up-type quarks U_R^i (down-type quarks D_R^i) can be created by choosing $T_R^3 = 0$ ($T_R^3 = -1$). Hence, we need three $O(4)$ multiplets per generation ($i = 1, 2, 3$) to reproduce the SM quark content. According to the relation $Q = T_L^3 + T_R^3 + Q_X$ (5.51), we indicate the electric charge Q as a subscript of each field

$$\begin{aligned}
(\mathbf{2}, \mathbf{2})_{2/3} : (\xi_{1L}^i)_{a\alpha} &:= \xi_{1L}^i = \begin{pmatrix} \chi_L^{u_i}(-+)_{5/3} & q_L^{u_i}(++)_{2/3} \\ \chi_L^{d_i}(-+)_{2/3} & q_L^{d_i}(++)_{-1/3} \end{pmatrix}, \\
(\mathbf{1}, \mathbf{1})_{2/3} : \xi_{2R}^i &= U_R^i(++), \\
(\mathbf{3}, \mathbf{1})_{2/3} \oplus (\mathbf{1}, \mathbf{3})_{2/3} : \xi_{3R}^i &= \tilde{T}_{3R}^i \oplus \tilde{T}_{4R}^i = \begin{pmatrix} \psi_R^i(-+)_{5/3} \\ U_R^i(-+)_{2/3} \\ D_R^i(-+)_{-1/3} \end{pmatrix} \oplus \begin{pmatrix} \psi_R^{\prime i}(-+)_{5/3} \\ U_R^{\prime i}(-+)_{2/3} \\ D_R^{\prime i}(++)_{-1/3} \end{pmatrix}. \quad (5.112)
\end{aligned}$$

Obviously, the triplets with total isospin ($T_L^3 + T_R^3 = +1, 0, -1$) are given in the basis ($\tau_{\pm} = (\tau_1 \pm i\tau_2)/\sqrt{2}, \tau_3$). With respect to the usual Pauli matrix basis, which we will use later on, the components read

$$(T_{3R}^i)^a := \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_R^i + D_R^i) \\ \frac{i}{\sqrt{2}}(\psi_R^i - D_R^i) \\ U_R^i \end{pmatrix}, \quad (T_{4R}^i)^\alpha := \begin{pmatrix} \frac{1}{\sqrt{2}}(\psi_R^{\prime i} + D_R^i) \\ \frac{i}{\sqrt{2}}(\psi_R^{\prime i} - D_R^i) \\ U_R^{\prime i} \end{pmatrix}, \quad (5.113)$$

such that

$$\begin{aligned}
(T_{3R}^i)_{ab} &= (T_{3R}^i)^c (\tau^c)_{ab} := T_{3R}^i = \begin{pmatrix} U_R^i/2 & \psi_R^i/\sqrt{2} \\ D_R^i/\sqrt{2} & -U_R^i/2 \end{pmatrix}, \\
(T_{4R}^i)_{\alpha\beta} &= (T_{4R}^i)^\gamma (\tau^\gamma)_{\alpha\beta} := T_{4R}^i = \begin{pmatrix} U_R^{\prime i}/2 & \psi_R^{\prime i}/\sqrt{2} \\ D_R^{\prime i}/\sqrt{2} & -U_R^{\prime i}/2 \end{pmatrix}, \quad (5.114)
\end{aligned}$$

where we introduced a matrix notation as also done in (5.112). Reversing the BCs in (5.112), one obtains the corresponding states of opposite chirality. Remember that this doubling of the fermion spectrum is needed for the construction of a chiral 4D theory. The fields with $(++)$ BCs have massless zero modes and correspond to the SM quarks after EWSB. The remaining fields only have massive KK modes and thus we have to deal with the following additional

heavy fermionic states in this model

$$\begin{aligned}
Q = 5/3 : & \quad \chi^{u_i(n)}, \psi^{t_i(n)}, \psi^{m_i(n)}, \\
Q = 2/3 : & \quad q^{u_i(n)}, U^{i(n)}, U^{t_i(n)}, U^{m_i(n)}, \chi^{d_i(n)}, \\
Q = -1/3 : & \quad q^{d_i(n)}, D^{i(n)}, D^{t_i(n)},
\end{aligned} \tag{5.115}$$

where $n = 1, 2, \dots$. Remember that the bulk mass parameter is equal for all components of a given fermion representation ξ_m^i , where $m = 1, 2, 3$ indicates the three different $O(4)$ multiplets and i stands for the flavour index. In general, c_m are arbitrary hermitian 3×3 matrices in flavour space, but in the following we choose to work in the ‘‘special basis’’ in which they are real and diagonal.³ To this end, we parameterise them by three real parameters $c_m^i \equiv \text{Diag}(c_m^1, c_m^2, c_m^3)$ for each multiplet m .

Fermion Lagrangian of the RSc Model

Adapting the covariant derivatives of the RSc model, contained in the general 5D action (5.105), to the gauge transformation behaviour of the various quark fields, we are able to formulate the fermionic Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{fermion}} = \frac{1}{2} \sqrt{G} \sum_{i=1}^3 & \left[(\bar{\xi}_1^i)_{\alpha\alpha} i\Gamma^M (D_M^1)_{ab,\alpha\beta} (\xi_1^i)_{b\beta} + (\bar{\xi}_1^i)_{\alpha\alpha} (i\Gamma^M \omega_M - c_1^i k) (\xi_1^i)_{\alpha\alpha} \right. \\
& + \bar{\xi}_2^i (i\Gamma^M D_M^2 + i\Gamma^M \omega_M - c_2^i k) \xi_2^i \\
& + (\bar{T}_3^i)_a i\Gamma^M (D_M^3)_{ab} (T_3^i)_b + (\bar{T}_3^i)_a (i\Gamma^M \omega_M - c_3^i k) (T_3^i)_a \\
& \left. + (\bar{T}_4^i)_\alpha i\Gamma^M (D_M^4)_{\alpha\beta} (T_4^i)_\beta + (\bar{T}_4^i)_\alpha (i\Gamma^M \omega_M - c_3^i k) (T_4^i)_\alpha \right] + \text{h.c.}, \tag{5.116}
\end{aligned}$$

where summation over repeated indices is understood. Explicitly, the covariant derivatives D_M^i are given by

$$\begin{aligned}
(D_M^1)_{ab,\alpha\beta} &= \left(\partial_M + \frac{i}{2} g_s \lambda^A G_M^A + i g_X Q_X X_M \right) \delta_{ab} \delta_{\alpha\beta} \\
&\quad + i g (\tau^c)_{ab} W_{L,M}^c \delta_{\alpha\beta} + i g (\tau^\gamma)_{\alpha\beta} W_{R,M}^\gamma \delta_{ab}, \\
D_M^2 &= \partial_M + \frac{i}{2} g_s \lambda^A G_M^A + i g_X Q_X X_M, \\
(D_M^3)_{ab} &= \left(\partial_M + \frac{i}{2} g_s \lambda^A G_M^A + i g_X Q_X X_M \right) \delta_{ab} + g \varepsilon^{abc} W_{L,M}^c, \\
(D_M^4)_{\alpha\beta} &= \left(\partial_M + \frac{i}{2} g_s \lambda^A G_M^A + i g_X Q_X X_M \right) \delta_{\alpha\beta} + g \varepsilon^{\alpha\beta\gamma} W_{R,M}^\gamma, \tag{5.117}
\end{aligned}$$

with the generators $-i\varepsilon^{abc}$ and $-i\varepsilon^{\alpha\beta\gamma}$ of the adjoint triplet representations of $SU(2)_L$ and $SU(2)_R$, respectively. Recall that despite having the same matrix structure, the $SU(2)_L$ and $SU(2)_R$ generators act on different internal spaces. Writing out the ‘‘+h.c.’’ term in (5.116)

³This can always be achieved by appropriate field redefinitions of the ξ^i multiplets.

explicitly, the two terms including the spin connection ω_M cancel each other in a general RS setup. Under the hermitian conjugation we understand to transpose the whole matrix, e.g. the bidoublet in (5.112), first, and then apply the usual hermitian conjugation ($\bar{\Psi} = \Psi^\dagger \gamma_0$) separately to each entry that represents a four-dimensional Dirac vector.

5.7 Flavour Structure

5.7.1 Constructing Gauge-Invariant Yukawa Couplings

This section is dedicated to the construction of the 5D Yukawa couplings of the fermions to the Higgs boson and the determination of the quark mass matrices after EWSB. Corresponding to the specific bulk symmetry group, the Yukawa coupling terms have to preserve the full $O(4) \sim SU(2)_L \times SU(2)_R \times P_{LR}$ symmetry. Utilising the transformation behaviour of the Higgs bidoublet and the fermion multiplets

$$H' = U_L H U_R^T, \quad (\xi_1^i)' = U_L (\xi_1^i) U_R^T, \quad (T_3^i)' = U_L (T_3^i) U_L^\dagger, \quad (T_4^i)' = U_R (T_4^i) U_R^\dagger, \quad (5.118)$$

the most general Yukawa Lagrangian is given by

$$\mathcal{L}_{\text{Yuk}} = -\sqrt{2}\sqrt{G} \sum_{i,j=1}^3 (-\lambda_{ij}^u \text{Tr} [\bar{\xi}_{1L}^i \cdot H] \xi_{2R}^j + \sqrt{2} \lambda_{ij}^d (\text{Tr} [\bar{\xi}_{1L}^i \cdot T_{3R}^j \cdot H] + \text{Tr} [\bar{\xi}_{1L}^i \cdot H \cdot T_{4R}^j])) + \text{h.c.} \quad (5.119)$$

While the first coupling proportional to λ_{ij}^u contributes, after EWSB, only to the mass matrix of the $+2/3$ charge quarks, the second term proportional to λ_{ij}^d contributes to all $+5/3$, $+2/3$ and $-1/3$ mass matrices. This is a direct consequence of T_3^j and T_4^j being placed in the adjoint representations of $SU(2)_L$ and $SU(2)_R$, respectively. The two factors of $\sqrt{2}$ in equation (5.119) are chosen such that the zero mode fermions, which can be identified after EWSB — up to $\mathcal{O}(v^2/f^2)$ mixings — with the SM fermions, have Yukawa couplings with the same scaling factor $v/\sqrt{2}$ as in the SM. To see this explicitly, we first insert the KK decomposition of the Higgs field and project out the neutral Higgs component $h^0(x)$ which develops a 4D effective VEV $\langle h^0(x) \rangle = v$ after EWSB,

$$\begin{aligned} \mathcal{L}_{\text{Yuk}}^{4D} \supset & \frac{1}{\sqrt{L}} \int_0^L dy \sum_{i,j=1}^3 \sqrt{G} \left(-\frac{1}{\sqrt{2}} \bar{\chi}_L^{u_i} \lambda_{ij}^d \psi_R^{j'} + \frac{1}{\sqrt{2}} \bar{\chi}_L^{u_i} \lambda_{ij}^d \psi_R^{j''} - \frac{1}{\sqrt{2}} \bar{q}_L^{d_i} \lambda_{ij}^d D_R^j + \frac{1}{\sqrt{2}} \bar{q}_L^{d_i} \lambda_{ij}^d D_R^{j'} \right. \\ & + \frac{1}{2} \bar{\chi}_L^{d_i} \lambda_{ij}^d U_R^{j'} + \frac{1}{2} \bar{q}_L^{u_i} \lambda_{ij}^d U_R^{j'} - \frac{1}{2} \bar{\chi}_L^{d_i} \lambda_{ij}^d U_R^{j''} - \frac{1}{2} \bar{q}_L^{u_i} \lambda_{ij}^d U_R^{j''} \\ & \left. + \frac{1}{\sqrt{2}} \bar{\chi}_L^{d_i} \lambda_{ij}^u U_R^j - \frac{1}{\sqrt{2}} \bar{q}_L^{u_i} \lambda_{ij}^u U_R^j + \text{h.c.} \right) h^0(x) h(y). \quad (5.120) \end{aligned}$$

While the first $\sqrt{2}$ -factor in (5.119) determines the scaling factor of the up-quark mass terms, the second one is responsible for the factor of the down-quark mass terms. Note that there is also the correct overall minus sign for the SM fermion mass terms (to compare with (3.51)).

5.7.2 4D Effective Yukawa Coupling Matrices

In order to obtain the 4D effective Yukawa couplings, we have to KK expand also the fermion modes in (5.120) to be able to apply the integration over the extra dimension. Again we restrict ourselves to the zero modes and first-excited KK modes. We assign a superscript (0) to the zero modes in order to distinguish them from the excited KK modes, for which we will make the index $n = 1$ implicit. According to their electric charges $Q = +5/3, +2/3$ and $-1/3$, we will group the fermion modes into the following vectors.

The $+5/3$ charge vectors have only excited KK states

$$\begin{aligned}\Psi_L^{5/3} &= (\chi_L^{u_i}(-+), \psi_L^i(+), \psi_L^i(-))^T, \\ \Psi_R^{5/3} &= (\chi_R^{u_i}(+-), \psi_R^i(-), \psi_R^i(+))^T,\end{aligned}\quad (5.121)$$

where the flavour index $i = 1, 2, 3$ runs over the three quark generations. Thus we deal with 9-dimensional vectors.

The 18-dimensional vectors, corresponding to the charge $+2/3$ mass matrix, contain zero modes in their first components

$$\begin{aligned}\Psi_L^{2/3} &= (q_L^{u_i(0)}(++), q_L^{u_i}(++), U_L^i(+), U_L^i(-), \chi_L^{d_i}(-), U_L^i(--))^T, \\ \Psi_R^{2/3} &= (U_R^{i(0)}(++), q_R^{u_i}(--), U_R^i(-), U_R^i(+), \chi_R^{d_i}(+), U_R^i(++))^T.\end{aligned}\quad (5.122)$$

Equivalently, this is the case for the 12-dimensional charge $-1/3$ vectors

$$\begin{aligned}\Psi_L^{-1/3} &= (q_L^{d_i(0)}(++), q_L^{d_i}(++), D_L^i(+), D_L^i(-))^T, \\ \Psi_R^{-1/3} &= (D_R^{i(0)}(++), q_R^{d_i}(--), D_R^i(-), D_R^i(++))^T.\end{aligned}\quad (5.123)$$

The shape functions, corresponding to the r -th and s -th component of Ψ_L^Q and Ψ_R^Q in (5.121)–(5.123), are denoted by $f_{L,r}^Q(y, c_m^i)$ and $f_{R,s}^Q(y, c_m^i)$, respectively.

Inserting the KK decomposition of the fermions (5.107) into the Yukawa Lagrangian (5.120), the factor e^{4ky}/L combines with $\sqrt{G/L}$ to an overall prefactor $1/L^{3/2}$. Absorbing this factor together with the prefactors of 2 and $\sqrt{2}$ into the definition of the 4D effective Yukawa matrices, here schematically shown for $[Y_{ij}^{(5/3)}]_{rs}$

$$\mathcal{L}_{\text{mass}}^{5/3} = v \sum_{i,j=1}^3 \underbrace{\left[\frac{1}{\sqrt{2}L^{3/2}} \int_0^L dy \lambda_{ij}^d f_{L,r}^{5/3}(y) f_{R,s}^{5/3}(y) h(y) \right]}_{[Y_{ij}^{(5/3)}]_{rs}} \bar{\Psi}_{L,r}^{5/3}(x) \Psi_{R,s}^{5/3}(x) + \text{h.c.}, \quad (5.124)$$

one obtains the whole set of relations between the 4D effective Yukawa matrices and the

original 5D Yukawa matrices

$$\begin{aligned}
[Y_{ij}^{(5/3)}]_{rs} &= \frac{1}{\sqrt{2}L^{3/2}} \int_0^L dy \lambda_{ij}^d f_{L,r}^{5/3}(y) f_{R,s}^{5/3}(y) h(y), \\
[Y_{ij}^{(2/3)}]_{rs} &= \frac{1}{2L^{3/2}} \int_0^L dy \lambda_{ij}^d f_{L,r}^{2/3}(y) f_{R,s}^{2/3}(y) h(y), \\
[\tilde{Y}_{ij}^{(2/3)}]_{rs} &= \frac{1}{\sqrt{2}L^{3/2}} \int_0^L dy \lambda_{ij}^u f_{L,r}^{2/3}(y) f_{R,s}^{2/3}(y) h(y), \\
[Y_{ij}^{(-1/3)}]_{rs} &= \frac{1}{\sqrt{2}L^{3/2}} \int_0^L dy \lambda_{ij}^d f_{L,r}^{-1/3}(y) f_{R,s}^{-1/3}(y) h(y).
\end{aligned} \tag{5.125}$$

The goal is now to construct and diagonalise the quark mass matrices. Thereby, an analytic diagonalisation is impossible due to the large dimension of the mass matrices. Hence we will only sketch the notation for the rotation matrices, whose actual values have to be determined numerically.

5.7.3 Quark Mass Matrices

In addition to the mass contributions resulting from the 4D effective Yukawa couplings after EWSB, fermionic KK masses from the extra-dimensional setup contribute to the fermion mass matrices. They can be obtained from solving the bulk equations of motion. In what follows we will use the 3×3 KK fermion mass matrices $M_m^{\text{KK}}(\text{BC-L})$, where $m = 1, 2, 3$ again labels the different multiplet representations in (5.112), and (BC-L) are the BCs for the left-handed modes. Actually, these matrices depend on the bulk mass parameter and on the BCs. In terms of the mode vectors (5.121)–(5.123), the mass matrices are contained in

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= -\bar{\Psi}_L^{5/3} \mathcal{M}^{5/3} \Psi_R^{5/3} + \text{h.c.} \\
&\quad -\bar{\Psi}_L^{2/3} \mathcal{M}^{2/3} \Psi_R^{2/3} + \text{h.c.} \\
&\quad -\bar{\Psi}_L^{-1/3} \mathcal{M}^{-1/3} \Psi_R^{-1/3} + \text{h.c.} .
\end{aligned} \tag{5.126}$$

The quark mass matrix for the $+5/3$ charge fermions

$$\mathcal{M}^{5/3} = \begin{pmatrix} M_1^{\text{KK}}(-+) & v[Y_{ij}^{(5/3)}]_{12} & -v[Y_{ij}^{(5/3)}]_{13} \\ v[Y_{ij}^{(5/3)}]_{21}^\dagger & M_3^{\text{KK}}(+-) & 0 \\ -v[Y_{ij}^{(5/3)}]_{31}^\dagger & 0 & M_3^{\text{KK}}(+-) \end{pmatrix} \tag{5.127}$$

is diagonalised by a biunitary transformation, which defines the rotation into the mass eigenstate basis

$$\mathcal{L}_{\text{mass}}^{5/3} = -\underbrace{\bar{\Psi}_L^{5/3} \mathcal{X}_L}_{\bar{\Psi}_{L,\text{mass}}^{5/3}} \underbrace{\mathcal{X}_L^\dagger \mathcal{M}^{5/3} \mathcal{X}_R}_{\mathcal{M}_{\text{diag}}^{5/3}} \underbrace{\mathcal{X}_R^\dagger \Psi_R^{5/3}}_{\Psi_{R,\text{mass}}^{5/3}} + \text{h.c.} . \tag{5.128}$$

Obviously, the mass eigenstates of charge $5/3$ are all heavy.

The mass matrix of the charge $+2/3$ fermions reads

$$\mathcal{M}^{2/3} = \begin{pmatrix} v[\tilde{Y}_{ij}^{(2/3)}]_{00} & 0 & -v[Y_{ij}^{(2/3)}]_{02} & v[Y_{ij}^{(2/3)}]_{03} & 0 & v[\tilde{Y}_{ij}^{(2/3)}]_{05} \\ v[\tilde{Y}_{ij}^{(2/3)}]_{10} & M_1^{\text{KK}}(++) & -v[Y_{ij}^{(2/3)}]_{12} & v[Y_{ij}^{(2/3)}]_{13} & 0 & v[\tilde{Y}_{ij}^{(2/3)}]_{15} \\ 0 & -v[Y_{ij}^{(2/3)}]_{21}^\dagger & M_3^{\text{KK}}(+-) & 0 & -v[Y_{ij}^{(2/3)}]_{24}^\dagger & 0 \\ 0 & v[Y_{ij}^{(2/3)}]_{31}^\dagger & 0 & M_3^{\text{KK}}(+-) & v[Y_{ij}^{(2/3)}]_{34}^\dagger & 0 \\ -v[\tilde{Y}_{ij}^{(2/3)}]_{40} & 0 & -v[Y_{ij}^{(2/3)}]_{42} & v[Y_{ij}^{(2/3)}]_{43} & M_1^{\text{KK}}(-+) & -v[\tilde{Y}_{ij}^{(2/3)}]_{45} \\ 0 & v[\tilde{Y}_{ij}^{(2/3)}]_{51}^\dagger & 0 & 0 & -v[\tilde{Y}_{ij}^{(2/3)}]_{54}^\dagger & M_2^{\text{KK}}(--) \end{pmatrix}, \quad (5.129)$$

where the index 0 reminds us that a zero mode fermion is involved. Again, the diagonalisation, involving the unitary 18×18 matrices $\mathcal{U}_{L,R}$

$$\mathcal{M}_{\text{diag}}^{2/3} = \mathcal{U}_L^\dagger \mathcal{M}^{2/3} \mathcal{U}_R, \quad (5.130)$$

has to be performed numerically and thus we do not give the explicit expressions. Nevertheless one can imagine that the off-diagonal entries in the first column and row mix the light zero modes with the heavy KK modes. However, due to the hierarchy between the KK masses and the relatively small masses from EWSB effects, this mixing will be suppressed by $\mathcal{O}(v^2/f^2)$.

The same argument holds for the mass matrix of the fermions with charge $-1/3$

$$\mathcal{M}^{-1/3} = \begin{pmatrix} v[Y_{ij}^{(-1/3)}]_{00} & 0 & -v[Y_{ij}^{(-1/3)}]_{02} & v[Y_{ij}^{(-1/3)}]_{03} \\ v[Y_{ij}^{(-1/3)}]_{10} & M_1^{\text{KK}}(++) & -v[Y_{ij}^{(-1/3)}]_{12} & v[Y_{ij}^{(-1/3)}]_{13} \\ 0 & -v[Y_{ij}^{(-1/3)}]_{21}^\dagger & M_3^{\text{KK}}(+-) & 0 \\ 0 & v[Y_{ij}^{(-1/3)}]_{31}^\dagger & 0 & M_3^{\text{KK}}(--) \end{pmatrix}, \quad (5.131)$$

which is diagonalised by 12×12 unitary matrices according to

$$\mathcal{M}_{\text{diag}}^{-1/3} = \mathcal{D}_L^\dagger \mathcal{M}^{-1/3} \mathcal{D}_R, \quad (5.132)$$

with the corresponding rotation of the $\Psi_{L,R}^{-1/3}$ vector

$$\Psi_{L,R,\text{mass}}^{-1/3} = \mathcal{D}_{L,R}^\dagger \Psi_{L,R}^{-1/3}. \quad (5.133)$$

As the quark mass matrices cannot be diagonalised analytically, we present the following expressions with the fermions still given in their flavour eigenstate basis.

5.8 Flavour Violation within the RSc Model

In this section we elucidate the origin of *flavour violation* (FV) in the RSc model and consider some implications coming from phenomenology. As already mentioned previously, the model contains two different sources of flavour violation — the 5D Yukawa matrices and the 5D fermion bulk mass matrices.⁴

Counting of Parameters

Besides the 18 real parameters (R) and 18 complex phases (P) of the two 5D Yukawa matrices $\lambda_{ij}^{u,d}$, the three hermitian bulk mass matrices $c_{1,2,3}^i$ of each quark representation add 18R and 9P to the theory. In the limit of vanishing $\lambda_{ij}^{u,d}$ and $c_{1,2,3}^i$, the maximal quark flavour symmetry of the 5D theory $U(3)^3$ is identical to the SM one. As the non-vanishing matrices still leave an unbroken $U(1)_B$ baryon number symmetry, $3 \times 9 - 1 = 26$ parameters can be eliminated corresponding to the generators of the broken flavour symmetry $U(3)^3/U(1)_B$. Subtracting the 9R and 17P, we are left with 27R and 10P. Thus, compared to the 9R and 1P in the SM, the theory contains 18 additional real parameters and 9 new complex phases, which can evidently be identified with the parameters of the three bulk mass matrices. As we will illustrate in the following, these new parameters represent an additional source of flavour violation with respect to models incorporating the MFV assumption. In particular, corresponding to the above counting of parameters, new CP-violating phases are present. Moreover, new flavour changing effective 4-quark operators arise at tree level that are either absent or strongly suppressed within the SM. Fortunately, the built-in *RS-GIM mechanism* [106] can suppress the FCNC interactions and prevent this NP model from leading to disastrous phenomenological predictions.

5.8.1 Fermion-Gauge Boson Interactions

The 5D interactions between fermions and gauge bosons stem from the covariant derivatives (5.117), which are contained in the kinetic terms of the fermion Lagrangian (5.116). Similar to the derivation of the 4D effective Yukawa couplings, we start with the fundamental 5D interactions and perform the KK decomposition up to the first KK modes. The effective 4D couplings result from the overlap integrals of the gauge boson profiles with the fermion shape functions

$$g_{nkl}^{4D}(i, m) = \frac{g}{L^{3/2}} \int_0^L dy e^{ky} f^{(n)}(y, c_m^i) f^{(k)}(y, c_m^i) f_{\text{gauge}}^{(l)}(y), \quad (5.134)$$

where $(n, k, l = 0, 1, \dots)$ denote the different KK levels. Note that the covariant derivatives only couple fermions within the same gauge multiplet, so that their bulk mass parameters are necessarily equal.

⁴We do not include kinetic mixing terms on the branes which would represent another source of FV.

Couplings to Gauge Boson Zero Modes

Following the perturbative approach of EWSB, we first neglect the effects coming from EWSB and consider a flat gauge boson zero mode profile $f_{\text{gauge}}^{(0)}(y) = 1$. In this case, the overlap integral in (5.134) simplifies due to the orthonormality condition of the fermion fields (5.108). As only couplings between equal fermion KK levels $n = k$ are allowed, (5.134) reduces to the simple tree-level matching condition of the 5D coupling constant with the flavour universal 4D effective gauge coupling

$$g_{kk0}^{4D} = \frac{g}{\sqrt{L}}. \quad (5.135)$$

Couplings to Higher Gauge Boson Modes

Since the integration over the extra dimension cannot be carried out explicitly, the couplings to excited KK gauge boson modes do not simplify significantly and the right-hand side of (5.134) retains its form. Note that the bulk mass parameter is the same for all fermions of one gauge multiplet $(\xi_1^i, \xi_2^i, \xi_3^i)$, but depends on the flavour index $i = 1, 2, 3$. This property was already used in Subsection 5.6.2 to locate the fermion zero modes profiles of different flavours at different positions along the extra dimension and thus provides an explanation for the hierarchies of the quark masses and mixings. Here, in an analogue manner they are responsible for the fact that the effective 4D couplings of fermionic zero modes to KK gauge bosons are non-universal in flavour space. Since the KK gauge profiles are localised towards the IR brane, their overlap intervals with the light quarks are similar in magnitude and much smaller than the overlap integral with the heavy third generation.

According to the perturbative approach, we then treat the Higgs VEV as a small perturbation that induces mixings among the various modes. Rotating the gauge bosons as well as the fermions into their physical mass eigenstates, causes the following two different effects of flavour violation:

- As can be seen from the rotation matrices (5.85) and (5.90), the SM gauge bosons correspond to the gauge boson zero modes up to a small admixture of higher-excited KK modes with identical electric charge. According to the above discussion, the SM weak gauge bosons have non-universal couplings to the fermion zero modes. As the diagonal but non-universal coupling matrix does not commute with the rotation matrices, which transform the fermions to their mass eigenstate basis, flavour violation arises already at tree level (FV1). However, the strength of these flavour-violating contributions are controlled by the RS–GIM mechanism: While the non-universality of the first and second generation is small due to the small splitting along the extra dimension, the third generation is protected from large flavour transitions due to very small mixing angles with the first two generations.

- The previous source of FV has its origin in the gauge boson mixing and is also present in the limit of vanishing heavy fermion KK modes, which are given in (5.115). There is a second FV source (FV2) which is based on the mixings of the SM-like zero modes with the heavy KK quarks of same electric charge that are induced by the diagonalisation of the quark mass matrices. The impact of these KK fermions on the SM fermion couplings in the RSc model has been considered in [198] within an effective theory framework, where the heavy fermions have been integrated out at tree level by using their EOMs. In general, the contribution from KK fermions tends to be numerically smaller than the one from gauge boson mixing [190, 198].

5.8.2 Neutral Currents

To illustrate the above general statements, we consider the neutral current involving the SM neutral gauge boson Z with the quarks being still in their flavour eigenstate basis

$$\begin{aligned}
J_\mu(Z) = & \overline{\Psi}_L^{5/3} \gamma_\mu A_L^{5/3}(Z) \Psi_L^{5/3} + \overline{\Psi}_R^{5/3} \gamma_\mu A_R^{5/3}(Z) \Psi_R^{5/3} \\
& + \overline{\Psi}_L^{2/3} \gamma_\mu A_L^{2/3}(Z) \Psi_L^{2/3} + \overline{\Psi}_R^{2/3} \gamma_\mu A_R^{2/3}(Z) \Psi_R^{2/3} \\
& + \overline{\Psi}_L^{-1/3} \gamma_\mu A_L^{-1/3}(Z) \Psi_L^{-1/3} + \overline{\Psi}_R^{-1/3} \gamma_\mu A_R^{-1/3}(Z) \Psi_R^{-1/3}.
\end{aligned} \tag{5.136}$$

The matrices $A_{L,R}^Q(Z)$ with $Q = 2/3, -1/3, 5/3$ have the dimensions 18×18 , 12×12 and 9×9 , respectively. They are flavour-diagonal, i.e. all 3×3 submatrices $[A_{L,R}^Q(Z)]_{rs}$, consisting of the elements $[A_{L,R}^Q]_{ij}(Z)_{rs}$, $i, j = 1, 2, 3$, are diagonal in the flavour space ($[A_{L,R}^Q]_{ij}(Z)_{rs} = 0$ for $i \neq j$). We denote the position of the submatrices corresponding to the entries of the mode vectors with r, s . For instance $r, s = 0, \dots, 5$ in the case of $[A_{L,R}^{2/3}(Z)]_{rs}$, which is in full analogy to the subscripts of the Yukawa matrices used in the mass matrix $\mathcal{M}^{2/3}$.

The small admixtures of higher KK modes, forming the SM Z boson,

$$Z = Z^{(0)} - \epsilon \frac{\mathcal{I}_1^+}{\cos^2 \psi} Z^{(1)} + \epsilon \frac{\mathcal{I}_1^- \cos \phi}{\cos \psi} Z_X^{(1)}, \tag{5.137}$$

make the diagonal 3×3 coupling submatrices to quark zero modes flavour non-universal at order $\mathcal{O}(\epsilon)$. In the Appendix B.8, we give a detailed derivation of the Feynman rule for the vertex $\overline{q}_L^{u_i(0)} q_L^{u_i(0)} Z$, which corresponds to the coupling submatrix $i[A_L^{2/3}(Z)]_{00}$. Together with the other quark zero mode couplings to the Z boson, we display the results in Table 5.1.

The non-universality effects of $\mathcal{O}(\epsilon)$ are caused by the flavour dependence of the overlap integrals \mathcal{R}_{00}^i and \mathcal{P}_{00}^i , which involve a higher KK gauge mode as follows. While we define

$$\mathcal{R}_{nk}^i(\text{BC})_{L,R} = \frac{1}{L} \int_0^L dy e^{ky} f_{L,R}^{(n)}(y, c_m^i, \text{BC}) f_{L,R}^{(k)}(y, c_m^i, \text{BC}) g(y) \tag{5.138}$$

for the overlap integrals including the KK mode $Z^{(1)}$ or $W_L^{(1)\pm}$, we use the short-hand notation for the ones involving the $Z_X^{(1)}$ KK mode

$$\mathcal{P}_{nk}^i(\text{BC})_{L,R} = \frac{1}{L} \int_0^L dy e^{ky} f_{L,R}^{(n)}(y, c_m^i, \text{BC}) f_{L,R}^{(k)}(y, c_m^i, \text{BC}) \tilde{g}(y). \tag{5.139}$$

Zero mode couplings to the Z boson		
$Q = 2/3$ quarks		
$\bar{q}_L^{u_i(0)} q_L^{u_i(0)} Z$	$-i\gamma^\mu \left[g_Z(q^{u_i}) - \epsilon g_Z(q^{u_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{00}^i(++)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_1 \mathcal{P}_{00}^i(++)_L \right]$	$(0,0)$
$\bar{U}_R^{i(0)} U_R^{i(0)} Z$	$-i\gamma^\mu \left[g_Z(U^i) - \epsilon g_Z(U^i) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{00}^i(++)_R + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_3 \mathcal{P}_{00}^i(++)_R \right]$	$(0,0)$
$Q = -1/3$ quarks		
$\bar{q}_L^{d_i(0)} q_L^{d_i(0)} Z$	$-i\gamma^\mu \left[g_Z(q^{d_i}) - \epsilon g_Z(q^{d_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{00}^i(++)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_1 \mathcal{P}_{00}^i(++)_L \right]$	$(0,0)$
$\bar{D}_R^{i(0)} D_R^{i(0)} Z$	$-i\gamma^\mu \left[g_Z(D^i) - \epsilon g_Z(D^i) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{00}^i(++)_R + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_5 \mathcal{P}_{00}^i(++)_R \right]$	$(0,0)$

Table 5.1: *Couplings involving zero modes and the Z boson. These zero modes correspond to the SM quark fields when the rotation to fermion mass eigenstates is performed.*

The shape functions $\tilde{g}(y) \neq g(y)$ were defined in (5.58) and (5.59), and depend weakly on their respective BCs. After the rotation into the quark mass eigenstate basis, the small non-universality in the couplings induce FCNC transitions already at tree level (FV1).

Due to the presence of heavy KK fermions, the coupling matrices $A_{L,R}^{2/3}(Z)$ and $A_{L,R}^{-1/3}(Z)$ contain off-diagonal submatrices in their first row and column ($[A_{L,R}^Q(Z)]_{rs}$, $Q = \frac{2}{3}, -\frac{1}{3}$, $r \neq s$, $r = 0$ or $s = 0$) corresponding to couplings between the SM-like zero modes with their KK partners (e.g. $\bar{q}_L^{u_i(0)} q_L^{u_i} Z$ in Table 5.2). In addition there are 3×3 building blocks on the diagonal axis of $A_{L,R}^Q(Z)$, devoted to the submatrices $[A_{L,R}^Q(Z)]_{rs}$, $r = s \neq 0$, which induce couplings of the heavy KK fermions to the Z boson (e.g. $\bar{U}_L^i U_L^i Z$ in Table 5.3). While the coupling matrices $A_{L,R}^Q(Z)$ are diagonal in the limit of vanishing gauge boson mixing ($Z \rightarrow Z^{(0)}$), they are nevertheless not proportional to a 18×18 , 12×12 and 9×9 unit matrix. Rotating the fermions to their mass eigenstates, non-diagonal mixings in flavour space occur and FCNC transitions are mediated (FV2).

The explicit expression for the $A_L^{2/3}(Z)$ matrix can be read off from the Feynman rules given in the Tables 5.1–5.3. Thereby, the brackets in the right column denote the placements rs of the submatrices within the 18×18 matrix. The ones for the other coupling matrices follow from the Feynman rules we have given in the appendix of [61].

The current given in the gauge and quark mass eigenstate basis

$$\begin{aligned}
J_\mu(Z) = & \bar{\Psi}_{L,\text{mass}}^{5/3} \gamma_\mu B_L^{5/3}(Z) \Psi_{L,\text{mass}}^{5/3} + \bar{\Psi}_{R,\text{mass}}^{5/3} \gamma_\mu B_R^{5/3}(Z) \Psi_{R,\text{mass}}^{5/3} \\
& + \bar{\Psi}_{L,\text{mass}}^{2/3} \gamma_\mu B_L^{2/3}(Z) \Psi_{L,\text{mass}}^{2/3} + \bar{\Psi}_{R,\text{mass}}^{2/3} \gamma_\mu B_R^{2/3}(Z) \Psi_{R,\text{mass}}^{2/3} \\
& + \bar{\Psi}_{L,\text{mass}}^{-1/3} \gamma_\mu B_L^{-1/3}(Z) \Psi_{L,\text{mass}}^{-1/3} + \bar{\Psi}_{R,\text{mass}}^{-1/3} \gamma_\mu B_R^{-1/3}(Z) \Psi_{R,\text{mass}}^{-1/3}, \quad (5.140)
\end{aligned}$$

Off-diagonal couplings to the Z boson		
$Q = 2/3$ quarks		
$\bar{q}_L^{u_i(0)} q_L^{u_i} Z$	$-i\gamma^\mu \left[-\epsilon g_Z(q^{u_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{01}^i(++)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_1 \mathcal{P}_{01}^i(++)_L \right]$	(0,1)
$\bar{q}_L^{u_i} q_L^{u_i(0)} Z$	$-i\gamma^\mu \left[-\epsilon g_Z(q^{u_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{10}^i(++)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_1 \mathcal{P}_{10}^i(++)_L \right]$	(1,0)

Table 5.2: Couplings involving the Z boson and a single left-handed zero mode of electric charge $Q = 2/3$.

Heavy fermion couplings to the Z boson		
$Q = 2/3$ quarks		
$\bar{q}_L^{u_i} q_L^{u_i} Z$	$-i\gamma^\mu \left[g_Z(q^{u_i}) - \epsilon g_Z(q^{u_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{11}^i(++)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_1 \mathcal{P}_{11}^i(++)_L \right]$	(1,1)
$\bar{U}_L^i U_L^i Z$	$-i\gamma^\mu \left[g_Z(U^i) - \epsilon g_Z(U^i) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{11}^i(+-)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_3 \mathcal{P}_{11}^i(+-)_L \right]$	(2,2)
$\bar{U}_L^{ii} U_L^{ii} Z$	$-i\gamma^\mu \left[g_Z(U^{ii}) - \epsilon g_Z(U^{ii}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{11}^i(+)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_3 \mathcal{P}_{11}^i(+)_L \right]$	(3,3)
$\bar{\chi}_L^{d_i} \chi_L^{d_i} Z$	$-i\gamma^\mu \left[g_Z(\chi^{d_i}) - \epsilon g_Z(\chi^{d_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{11}^i(--)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_2 \mathcal{P}_{11}^i(--)_L \right]$	(4,4)
$\bar{U}_L^i U_L^i Z$	$-i\gamma^\mu \left[g_Z(U^i) - \epsilon g_Z(U^i) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{11}^i(--)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \kappa_3 \mathcal{P}_{11}^i(--)_L \right]$	(5,5)

Table 5.3: Couplings involving the Z boson and the heavy left-handed fermions of electric charge $Q = 2/3$.

defines the new coupling matrices

$$\begin{aligned} B_{L,R}^{5/3}(Z) &= \mathcal{X}_{L,R}^\dagger A_{L,R}^{5/3}(Z) \mathcal{X}_{L,R}, & B_{L,R}^{2/3}(Z) &= \mathcal{U}_{L,R}^\dagger A_{L,R}^{2/3}(Z) \mathcal{U}_{L,R}, \\ B_{L,R}^{-1/3}(Z) &= \mathcal{D}_{L,R}^\dagger A_{L,R}^{-1/3}(Z) \mathcal{D}_{L,R}. \end{aligned} \quad (5.141)$$

Similar expressions for the currents $J_\mu(Z_H)$, $J_\mu(Z')$ follow for the gauge boson mass eigenstates Z_H and Z' . Note that flavour-violating couplings of Z and Z' to left-handed down quarks are protected by the custodial symmetry of the model (see [121, 122, 198] for further details).

5.8.3 Charged Currents

After rotating the gauge eigenstates $W_L^{(0)\pm}$, $W_L^{(1)\pm}$, $W_R^{(1)\pm}$ into their mass eigenstates W^\pm , W_H^\pm , W'^\pm , the charged current for the SM gauge bosons W^\pm has the following structure

$$\begin{aligned} J_\mu(W^\pm) &= \overline{\Psi}_L^{2/3} \gamma_\mu G_L(W^+) \Psi_L^{-1/3} + \overline{\Psi}_R^{2/3} \gamma_\mu G_R(W^+) \Psi_R^{-1/3} \\ &\quad + \overline{\Psi}_L^{5/3} \gamma_\mu \tilde{G}_L(W^+) \Psi_L^{2/3} + \overline{\Psi}_R^{5/3} \gamma_\mu \tilde{G}_R(W^+) \Psi_R^{2/3} + \text{h.c.} \end{aligned} \quad (5.142)$$

Corresponding to the size of the vectors $\Psi_{L,R}^{2/3}$, $\Psi_{L,R}^{-1/3}$, $\Psi_{L,R}^{5/3}$, the matrix $G_{L,R}$ ($\tilde{G}_{L,R}$) is a 18×12 (9×18) matrix in the model under consideration. Explicit expressions can again be obtained from the Feynman rules in the appendix of [61]. The line of argument is the same as in the neutral case: There are two effects of flavour violation — one from KK fermion mixing and the other from gauge boson mixing. Schematically, the charged current in the fermion mass eigenstate basis can be noted as

$$\begin{aligned} J_\mu(W^\pm) &= \overline{\Psi}_{L,\text{mass}}^{2/3} \gamma_\mu H_L(W^+) \Psi_{L,\text{mass}}^{-1/3} + \overline{\Psi}_{R,\text{mass}}^{2/3} \gamma_\mu H_R(W^+) \Psi_{R,\text{mass}}^{-1/3} \\ &\quad + \overline{\Psi}_{L,\text{mass}}^{5/3} \gamma_\mu \tilde{H}_L(W^+) \Psi_{L,\text{mass}}^{2/3} + \overline{\Psi}_{R,\text{mass}}^{5/3} \gamma_\mu \tilde{H}_R(W^+) \Psi_{R,\text{mass}}^{2/3} + \text{h.c.}, \end{aligned} \quad (5.143)$$

where the currents $J_\mu(W_H^\pm)$ and $J_\mu(W'^\pm)$ can be derived analogously, and where

$$H_{L,R}(W^+) = \mathcal{U}_{L,R}^\dagger G_{L,R}(W^+) \mathcal{D}_{L,R} \quad \text{and} \quad \tilde{H}_{L,R}(W^+) = \mathcal{X}_{L,R}^\dagger \tilde{G}_{L,R}(W^+) \mathcal{U}_{L,R}. \quad (5.144)$$

In analogy to the SM, we define the CKM matrix to be the 3×3 submatrix placed in the upper left corner of the final mixing matrix $[H_L(W^+)]_{00}$. The matrix $G_L(W^+)$ deviates from the unit matrix due to the two different flavour-violating effects and thus does not commute with the rotation matrices which implies a non-unitary CKM matrix. However, the non-unitarity effects are small, as both contributions are of $\mathcal{O}(\epsilon)$. With respect to the SM and all other MFV models, where the CKM matrix is the only relic of the rotation matrices, the latter explicitly appear in this model in the charged and neutral currents (5.140)–(5.141) and (5.143)–(5.144).

As flavour violation in charged currents in the SM arises also at tree level, the additional contributions seem, at first glance, not to be so restricting as the constraints coming from the FCNCs. Nevertheless, the W boson mediates right-handed weak interactions such that there are new operators compared to the SM contribution. In addition the new heavy gauge bosons W'^\pm and W_H^\pm may give sizeable contributions as well.

5.8.4 Photonic and Gluonic Currents

Due to the unbroken gauge invariance of QCD and QED, the various photonic and gluonic modes do not mix with each other. Thus, according to our general remarks in Subsection 5.8.1, only the flavour non-universal couplings to massive KK modes induce FCNC processes. Restricting ourselves to the first KK mode, the coupling matrices of the massive photonic current

$$J_\mu(A^{(1)}) = \overline{\Psi}_{L,R}^Q \gamma_\mu A_{L,R}^Q(A^{(1)}) \Psi_{L,R}^Q, \quad (5.145)$$

and the one occurring in the massive gluonic current

$$J_\mu^a(G^{(1)}) = \overline{\Psi}_{L,R}^Q \gamma_\mu T^a A_{L,R}^Q(G^{(1)}) \Psi_{L,R}^Q, \quad (5.146)$$

are related by

$$\frac{A_{L,R}^Q(G^{(1)})}{g_s} = \frac{A_{L,R}^Q(A^{(1)})}{Qe}. \quad (5.147)$$

As can be seen from the generic coupling overlap integral in (5.134), from which the coupling matrices are constructed, $A_{L,R}^Q(G^{(1)})$ and $A_{L,R}^Q(A^{(1)})$ depend on the flavour index i and on the fermion chirality L, R . Explicit expressions for the $A_{L,R}^Q(G^{(1)})$ and $A_{L,R}^Q(A^{(1)})$ matrices can be obtained from the Feynman rules given in the appendix of [61]. After rotation into the fermion mass eigenbasis, the currents are given by

$$J_\mu(A^{(1)}) = \overline{\Psi}_{L,R,\text{mass}}^Q \gamma_\mu B_{L,R}^Q(A^{(1)}) \Psi_{L,R,\text{mass}}^Q, \quad (5.148)$$

and

$$J_\mu^a(G^{(1)}) = \overline{\Psi}_{L,R,\text{mass}}^Q \gamma_\mu T^a B_{L,R}^Q(G^{(1)}) \Psi_{L,R,\text{mass}}^Q, \quad (5.149)$$

where the proportionality of $A_{L,R}^Q(A^{(1)})$, $A_{L,R}^Q(G^{(1)})$ in (5.147) remains valid for the matrices $B_{L,R}^Q(A^{(1)})$ and $B_{L,R}^Q(G^{(1)})$. The $B_{L,R}^Q$ are non-diagonal in flavour space and mediate tree-level FCNC processes, which we will focus on in the next subsections.

5.8.5 Tree-Level Contribution of KK Gluons to $\Delta F = 2$ Transitions

In this subsection, we will discuss the main new features of tree-level KK gluon contributions to the particle-antiparticle mixings $K^0 - \overline{K}^0$ and $B_{s,d}^0 - \overline{B}_{s,d}^0$ [199]. In the SM the off-diagonal element M_{12}^p in the neutral K ($p = K$) and $B_{s,d}$ ($p = s, d$) meson mass matrices has its origin from one-loop box diagrams. The contributions stem from the single operator

$$(\overline{s}d)_{V-A}(\overline{s}d)_{V-A} = [\overline{s}\gamma_\mu(1 - \gamma_5)d] [\overline{s}\gamma^\mu(1 - \gamma_5)d], \quad (5.150)$$

in case of $K^0 - \overline{K}^0$ mixing, and

$$(\overline{b}q)_{V-A}(\overline{b}q)_{V-A} = [\overline{b}\gamma_\mu(1 - \gamma_5)q] [\overline{b}\gamma^\mu(1 - \gamma_5)q], \quad \text{with } q = s, d, \quad (5.151)$$

correspondingly for the $B_{s,d}^0 - \bar{B}_{s,d}^0$ mixing. Detailed formulae for $(M_{12}^p)_{\text{SM}}$ may be found in [121]. The relevant piece of the Lagrangian which contains the coupling of the lightest KK gluons $G_\mu^{(1)a}$ ($a = 1, \dots, 8$) to the down-type quarks (D^i , $i = 1, 2, 3$) can be read off from the gluonic current (5.149)

$$\mathcal{L}^{\text{QCD}} = - \sum_i \bar{D}^i \gamma^\mu T^a (\varepsilon_L(i) P_L + \varepsilon_R(i) P_R) D^i G_\mu^{(1)a}. \quad (5.152)$$

With $\varepsilon_{L,R}(i)$ we denote the diagonal elements of the 3×3 matrices

$$\hat{\varepsilon}_{L,R} = \text{Diag} (\varepsilon_{L,R}(1), \varepsilon_{L,R}(2), \varepsilon_{L,R}(3)), \quad (5.153)$$

which correspond to the flavour submatrices in the upper left corner of the left- and right-handed coupling matrices $[A_{L,R}^{-1/3}(G^{(1)})]_{00}$. Identifying $g(y)$ with the shape function of the first KK gluon modes $G_\mu^{(1)a}$, the explicit expressions for $\varepsilon_{L,R}(i)$ follow from the overlap integral (5.134)

$$\varepsilon_{L,R}(i) = \frac{g_s}{L^{3/2}} \int_0^L dy e^{ky} \left(f_{L,R}^{(0)}(y, c_D^i) \right)^2 g(y) \stackrel{(5.135)}{=} \frac{g_s^{\text{4D}}}{L} \int_0^L dy e^{ky} \left(f_{L,R}^{(0)}(y, c_D^i) \right)^2 g(y), \quad (5.154)$$

where we used the tree-level matching relation for the strong coupling constant in the absence of brane kinetic terms. Furthermore $f_{L,R}^{(0)}(y, c_D^i)$ has to be taken from (5.109) and $g(y)$ from (5.22). Since the bulk mass parameters and in consequence also the shape functions $f_{L,R}^{(0)}(y, c_D^i)$ are chirality dependent, $\varepsilon_L(i)$ is not equal to $\varepsilon_R(i)$ and parity is broken by QCD-like interactions in this model. Remember that the couplings $\varepsilon_{L,R}(i)$ are in addition flavour non-universal and a rotation into the fermion mass eigenstates through the rotation matrices $\mathcal{D}_{L,R}$ produces the non-diagonal matrices

$$\Delta_{L,R} = \mathcal{D}_{L,R}^\dagger \hat{\varepsilon}_{L,R} \mathcal{D}_{L,R}. \quad (5.155)$$

Their non-diagonal complex elements $\Delta_{L,R}^{ij}$, $i \neq j$, introduce new flavour and CP-violating interactions with respect to the SM. Rewriting the Lagrangian (5.152) into the fermion mass eigenstate basis yields

$$\mathcal{L}^{\text{QCD}} \equiv - \left[\mathcal{L}_L^{\text{QCD}} + \mathcal{L}_R^{\text{QCD}} \right], \quad (5.156)$$

where

$$\begin{aligned} \mathcal{L}_L^{\text{QCD}} &= \left(\Delta_L^{sd} (\bar{s}_L \gamma^\mu T^a d_L) + \Delta_L^{bd} (\bar{b}_L \gamma^\mu T^a d_L) + \Delta_L^{bs} (\bar{b}_L \gamma^\mu T^a s_L) \right) G_\mu^{(1)a}, \\ \mathcal{L}_R^{\text{QCD}} &= \left(\Delta_R^{sd} (\bar{s}_R \gamma^\mu T^a d_R) + \Delta_R^{bd} (\bar{b}_R \gamma^\mu T^a d_R) + \Delta_R^{bs} (\bar{b}_R \gamma^\mu T^a s_R) \right) G_\mu^{(1)a}. \end{aligned} \quad (5.157)$$

The KK gluon contributions to $K^0 - \bar{K}^0$ mixing in Figure 5.3 arise at second-order in the S-matrix expansion, keeping the following relevant terms in the interaction Lagrangian

$$\mathcal{L}_{\text{int}}(x) = - \left(\Delta_L^{sd} (\bar{s}_L(x) \gamma^\mu T^a d_L(x)) G_\mu^{(1)a} + \Delta_R^{sd} (\bar{s}_R(x) \gamma^\mu T^a d_R(x)) G_\mu^{(1)a} \right). \quad (5.158)$$

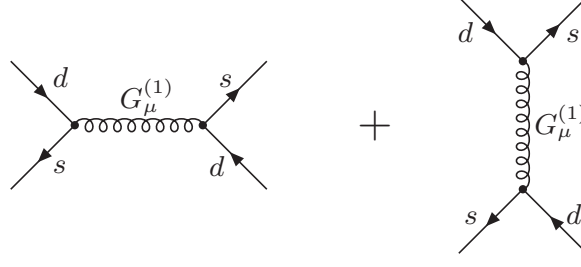


Figure 5.3: Tree-level contribution of KK gluons to $K^0 - \bar{K}^0$ mixing.

Due to the heavy KK mass of the gluon, we take the approximation $p_{\text{gluon}}^2 \ll M_{\text{KK}}^{(1)}$ such that the gluon propagator shrinks to a pointlike four-quark coupling, known as effective vertex. The corresponding effective Hamiltonian for this $\Delta S = 2$ transition is then given by

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{1}{2 \left(M_{\text{KK}}^{(1)}\right)^2} \left(\left(\Delta_L^{sd}\right)^2 (\bar{s}_L \gamma_\mu T^a d_L) (\bar{s}_L \gamma^\mu T^a d_L) + \left(\Delta_R^{sd}\right)^2 (\bar{s}_R \gamma_\mu T^a d_R) (\bar{s}_R \gamma^\mu T^a d_R) + 2\Delta_L^{sd} \Delta_R^{sd} (\bar{s}_L \gamma_\mu T^a d_L) (\bar{s}_R \gamma^\mu T^a d_R) \right), \quad (5.159)$$

with analogous expressions for the $B_{d,s}^0 - \bar{B}_{d,s}^0$ effective Hamiltonians. In Appendix B.7 we carry out the transformation of the operator basis in (5.159) into the one used in [200]:

$$\begin{aligned} \mathcal{Q}_1^{VLL} &= (\bar{s} \gamma_\mu P_L d) (\bar{s} \gamma^\mu P_L d), & \mathcal{Q}_1^{VRR} &= (\bar{s} \gamma_\mu P_R d) (\bar{s} \gamma^\mu P_R d), \\ \mathcal{Q}_1^{LR} &= (\bar{s} \gamma_\mu P_L d) (\bar{s} \gamma^\mu P_R d), & \mathcal{Q}_2^{LR} &= (\bar{s} P_L d) (\bar{s} P_R d). \end{aligned} \quad (5.160)$$

We also calculate the Wilson coefficients at the scale $\mu \sim \mathcal{O}(M_{\text{KK}})$ for the effective Hamiltonian, depending on the new basis

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{1}{4 \left(M_{\text{KK}}^{(1)}\right)^2} (C_1^{VLL} \mathcal{Q}_1^{VLL} + C_1^{VRR} \mathcal{Q}_1^{VRR} + C_1^{LR} \mathcal{Q}_1^{LR} + C_2^{LR} \mathcal{Q}_2^{LR}), \quad (5.161)$$

which are found to be

$$\begin{aligned} C_1^{VLL}(M_{\text{KK}}) &= \frac{2}{3} \left(\Delta_L^{sd}\right)^2, & C_1^{VRR}(M_{\text{KK}}) &= \frac{2}{3} \left(\Delta_R^{sd}\right)^2 \\ C_1^{LR}(M_{\text{KK}}) &= -\frac{2}{3} \Delta_L^{sd} \Delta_R^{sd}, & C_2^{LR}(M_{\text{KK}}) &= -4 \Delta_L^{sd} \Delta_R^{sd}. \end{aligned} \quad (5.162)$$

The result (5.162) is valid for three colour degrees of freedom and confirms the results of [201]. Note that we suppressed all colour indices in the above expressions, however, they are given in full detail in Appendix B.7. In the new basis one can directly take the formulae for the anomalous dimension matrices from [200], which state the renormalisation group evolution of the Hamiltonian valid at the KK scale down to low energies. The above example illustrates that in the RSc model the new operators \mathcal{Q}_1^{VRR} , \mathcal{Q}_1^{LR} and \mathcal{Q}_2^{LR} are involved in FCNC transitions — with respect to the single operator \mathcal{Q}_1^{VLL} present in the SM.

5.8.6 Tree-Level Contribution of the KK Photon to $\Delta F = 2$ Transitions

The KK photon contribution follows from the formulae of the KK gluon contributions through the replacements $G^{(1)} \rightarrow A^{(1)}$, $g_s^{4D} \rightarrow e^{4D}$ and $T^a \rightarrow 1$. Thus, the sum of the left- and right-handed components of \mathcal{L}^{QED}

$$\mathcal{L}^{\text{QED}} \equiv - \left[\mathcal{L}_L^{\text{QED}} + \mathcal{L}_R^{\text{QED}} \right], \quad (5.163)$$

are given by

$$\begin{aligned} \mathcal{L}_L^{\text{QED}} &= \left(\Delta_L^{sd}(A^{(1)}) (\bar{s}_L \gamma^\mu d_L) + \Delta_L^{bd}(A^{(1)}) (\bar{b}_L \gamma^\mu d_L) + \Delta_L^{bs}(A^{(1)}) (\bar{b}_L \gamma^\mu s_L) \right) A_\mu^{(1)}, \\ \mathcal{L}_R^{\text{QED}} &= \left(\Delta_R^{sd}(A^{(1)}) (\bar{s}_R \gamma^\mu d_R) + \Delta_R^{bd}(A^{(1)}) (\bar{b}_R \gamma^\mu d_R) + \Delta_R^{bs}(A^{(1)}) (\bar{b}_R \gamma^\mu s_R) \right) A_\mu^{(1)}. \end{aligned} \quad (5.164)$$

Since this result is already given in the basis of (5.160), the corrections to the Wilson coefficients in (5.162) from the KK photon follow immediately

$$\begin{aligned} (\Delta C_1^{VLL}(M_{\text{KK}}))^{\text{QED}} &= 2 \left(\Delta_L^{sd}(A^{(1)}) \right)^2, \\ (\Delta C_1^{VRR}(M_{\text{KK}}))^{\text{QED}} &= 2 \left(\Delta_R^{sd}(A^{(1)}) \right)^2, \\ (\Delta C_1^{LR}(M_{\text{KK}}))^{\text{QED}} &= 4 \left(\Delta_L^{sd}(A^{(1)}) \right) \left(\Delta_R^{sd}(A^{(1)}) \right), \\ (\Delta C_2^{LR}(M_{\text{KK}}))^{\text{QED}} &= 0. \end{aligned} \quad (5.165)$$

Implications from Numerical Studies

As has been found in various numerical studies, the fully anarchic approach in which the hierarchy of fermion masses and weak mixings is assumed to originate solely from geometry is challenging. Particularly, in case of an IR-brane localised Higgs, the excessive contribution of the chirality flip operators Q_2^{LR} , Q_1^{LR} to the CP-violating parameter ε_K in the K meson system implies a lower bound on the lightest KK gluon mass around 20 TeV [121, 201]. However, as was also demonstrated in [121], if a modest hierarchy and some fine-tuning is reintroduced into the fundamental 5D Yukawa matrices, regions in parameter space exist in which the constraints coming from ε_K , the off-diagonal mixing amplitudes ΔM_{12}^K , ΔM_{12}^d , ΔM_{12}^s and the mixing-induced CP asymmetry $S_{\psi K_S}$ can be fulfilled simultaneously for KK masses as low as $M_{\text{KK}} \simeq (2 - 3)$ TeV. Moreover, as was also stated there, the KK gluons give the dominant contribution to ε_K and ΔM_{12}^K while the EW tree-level contributions of Z_H , Z' can compete with the KK gluons in the case of $B_{d,s}$ physics observables. The contribution of the KK photon can be safely neglected in both cases. Also the one from the Z boson, which is controlled by the custodial symmetry, is found to be negligible. Within the same framework also rare K and B meson decays were studied [122] as well as contributions to ε_K from Higgs FCNCs [202, 203]. Recent reviews can also be found in [204] and [205].

Instead of deviating from the fully anarchic approach, the KK scale can be lowered down to $\mathcal{O}(5)$ TeV if the Higgs is in the bulk and one-loop matching of the strong gauge coupling is

included [206]. Important ingredient in this analysis is the ability to raise the overall size of the 5D down-type Yukawa coupling that suppresses the contributions to ε_K but simultaneously let the contribution to $B \rightarrow X_s \gamma$ grow. Incorporating also the bound on ϵ'/ϵ , the flavour problem becomes even more serious and the KK scale is again pushed up to $\mathcal{O}(8)$ TeV [207], beyond the LHC reach [186, 208].

To prevent the theory from large FCNCs, [209] introduces bulk and brane flavour symmetries. Flavour violation occurs via kinetic mixing terms of the right-handed fields on the UV brane. It is shown that this approach incorporates the GIM mechanism [106] and the SM CKM picture can be reproduced. However, the natural explanation of the flavour puzzle has to be abandoned in this setup.

Keeping this feature, another alternative for suppressing dangerous down-type FCNCs in WED models was proposed in [210] and [211], where the MFV paradigm is transferred to the 5D bulk. In the so-called “5D MFV” model, the 5D anarchic Yukawa matrices are assumed to be the only sources which break the 5D bulk FS $U(3)^3$. In consequence, the 5D vector-like 3×3 mass matrices are expressed in terms of $Y_{U,D}$ according to

$$\begin{aligned} c_Q &= \alpha_Q \cdot \mathbb{1} + r_u \beta_Q Y_U Y_U^\dagger + r_d \gamma_Q Y_D Y_D^\dagger \\ c_U &= \alpha_U \cdot \mathbb{1} + \gamma_u Y_U^\dagger Y_U \\ c_D &= \alpha_D \cdot \mathbb{1} + \beta_d Y_D^\dagger Y_D. \end{aligned} \tag{5.166}$$

In the limit of vanishing r_u , the down sector is completely aligned and one can choose a basis in which Y_D and the bulk mass matrices c_Q and c_D are simultaneously diagonal. Thus the 4D down-quark mass matrix is diagonal in the basis where the couplings to the bulk gauge fields are diagonal (but non-universal) and hence there are no tree-level FCNCs involving down-type quarks. This special limit can be realised for example with the requirement that there is a bulk FS $SU(3)_{Q_L} \times SU(3)_{D_R}$ which is broken only by the VEV of a bifundamental down-type scalar field $\langle y_d \rangle \sim Y_D$. It was pointed out in [211] that the alignment assumption allows for a KK mass scale as low as 2 – 3 TeV.

Chapter 6

Summary and Outlook

In this work we presented two possible answers to the question: “Why are the quark masses and mixing angles in the SM so much different?” This quark flavour puzzle is connected to the “obscure” part of the SM — the so far unobserved Higgs boson, which is supposed to be an elementary field of scalar nature.

We gave a brief summary of the basic features of gauge theories, including the formalism of chiral gauge anomalies, and continued having a closer look at the SM where we focused on the Higgs sector. Being a doublet under $SU(2)_L$, the Higgs also allows for gauge-invariant fermion mass terms via chiral Yukawa couplings. The only relic of the two 3×3 complex SM Yukawa matrices after the diagonalisation are six real quark masses and four CKM parameters that are physically observable.

In the dMFV model we tried to “revive” the static SM Yukawa coupling matrices, in promoting them to dynamical scalar spurion fields. While retaining the SM fermion content, we dealt with new flavour gauge bosons since we augmented the SM gauge group with the local flavour symmetry group $SU(3)_{Q_L} \times SU(3)_{U_R} \times SU(3)_{D_R}$. The Yukawa matrices transform as bifundamental objects to restore the FS in the Yukawa interactions. Hence there is less arbitrariness than in the SM where they have been introduced to ensure the most generic renormalisable and gauge-invariant Yukawa Lagrangian. To account for the observed hierarchy in the quark masses and mixing angles, the FS was broken in a sequential fashion by the VEVs of the two Yukawa matrix spurion fields [157]. Being a singlet under the flavour gauge group, the SM Higgs boson did not participate in this breaking and its VEV did not contribute to the masses of the new scalars and flavour gauge bosons either. Corresponding to the different breaking scales the masses of the new heavy gauge bosons and Higgs modes are hierarchically ordered, which becomes relevant when integrating out the new heavy degrees of freedom to obtain an effective theory.

Before doing so, we had to find an appropriate parameterisation of the Yukawa spurions in which the physically relevant scalar fluctuations around the physical masses and mixing parameters are disentangled from the Goldstone modes [90]. The Goldstone bosons, corre-

sponding to the broken symmetry generators of the local part of the FS, disappear from the particle spectrum in the unitary gauge. Requiring that there are no linear mixing terms of scalar fields with gauge boson fields in the spurion kinetic terms when working in the unitary gauge, we were able to give the parameterisation for the two- and three-family case. We pointed out that the physical scalar fluctuations around the CKM angles, that directly lead to FCNCs in the effective low-energy theory, appear in both Yukawa matrices regardless to the choice of basis of the Yukawa VEVs. In the three-family case kinetic mixings between the spurion fluctuations around the three CKM angles occurred which we diagonalised in order to assign the non-trivial flavour structure solely to the Yukawa sector.

As already mentioned in the introduction, our setup necessarily has to be understood in the context of an effective field theory framework for two reasons. First, the Yukawa interactions are described by dimension-5 operators due to the canonical dimension of the Yukawa matrices. Second, a quantum theory involving anomalies of the chiral FS itself or mixed anomalies with the SM gauge group can be consistently formulated only if the theory originates from a more fundamental anomaly-free theory [89]. We restricted ourselves to construct the effective Lagrangian for a subgroup of the local symmetry, which arises as an intermediate step in the breaking sequence. By adding higher-dimensional operators involving the Goldstone fields, we showed that they can cancel the anomalous fermion contributions and thus can formally restore the gauge invariance. While the mixed anomalies are removed by choosing the counterterms appropriately, the effect of the anomalies can be absorbed into the masses of the heavy gauge bosons of the broken flavour symmetry.

Integrating out the heavy gauge bosons and scalar fields at tree level by means of the equations of motion, we derived the 4-quark operators that share — according to our ansatz — the basic features of MFV. While the heavy $SU(2)_{D_R}$ gauge bosons involve only flavour-diagonal currents with non-trivial colour structure, the $U(1)_X$ gauge boson has non-universal flavour couplings to left-handed down quarks, leading to FCNCs after rotation into their mass eigenbases. The scalar fluctuations around the CKM angles directly lead to FCNCs in the effective low-energy theory, which may be checked experimentally. The lightest scalar particle — and therefore the last to be integrated out in the sequence — corresponds to the fluctuation around the Cabibbo angle. Its mass has to be sufficiently large in order to guarantee that the induced flavour-changing transitions are in line with the experimental constraints from precision measurements in the K and B meson system and thus will set a lower bound on the FS breaking scale Λ . The smallness of the first-generation Yukawa couplings implies that the global chiral $U(1)_{u_R} \times U(1)_{d_R}$ symmetry is broken at the very last step of the breaking sequence. We motivated to leave the chiral $U(1)$ factors as global symmetries in order to allow for a modified Peccei–Quinn mechanism in which the Goldstone modes dynamically lead to a vanishing effective θ -parameter in QCD and thus resolve the strong CP problem. The couplings of physical axion fields in such a scenario are strongly suppressed by the UV scale of the effective theory. The phenomenological implications of this scenario for flavour

observables accessible by future experiments at the LHC or at Super- B factories has still to be worked out. Moreover, this picture could also be adopted in the lepton sector, where a non-linear representation of the lepton flavour symmetry group was already presented in [212].

The second main part of this thesis is devoted to the RSc model with fermions allowed to propagate in the 5D bulk and a Higgs boson localised on or near the IR brane [99]. Originally motivated by a resolution of the gauge hierarchy problem, RS models can simultaneously address the flavour puzzle.

We presented the gauge boson content of the RSc model with its enlarged electroweak bulk gauge group $SU(2)_L \times SU(2)_R \times U(1)_X$ and discussed its breakdown by an appropriate choice of BCs on the UV brane as well as by the usual Higgs mechanism. The hierarchy between the gauge boson mass contributions coming from the extra-dimensional setup and from the Higgs mechanism offered us the possibility of an analytic perturbative diagonalisation of the mass matrices and is reflected in the expansion parameter $\epsilon = g^2 v^2 / (4LM^2) \sim \mathcal{O}(v^2/f^2)$. For this purpose, we introduced the general formalism of the Rayleigh–Schrödinger perturbation theory for the degenerate case. We consistently determined the gauge boson mass eigenvalues up to $\mathcal{O}(\epsilon^2)$ and the mass eigenstates up to $\mathcal{O}(\epsilon)$, which poses a higher accuracy than we displayed in [120].

Having summarised the general aspects about fermions living in EDIMs like the localisation freedom of their zero mode profiles, we introduced a specific fermion content which ensures — together with the enlarged gauge group — a custodial protection of the T parameter and the $Zb_L\bar{b}_L$ coupling. Restricting ourselves to the zero and first-excited KK modes, we constructed the 4D effective Yukawa matrices that arise from the most general gauge-invariant 5D Yukawa Lagrangian and thereby observed that the hierarchical structure can emerge from an overlap integral over non-uniform localised zero mode quark profiles along the extra dimension. Since the quark mass matrices after EWSB have to be diagonalised numerically, we derived the 4D effective Feynman rules involving gauge boson mass eigenstates while the fermions are still given in their gauge eigenstates [120].

Due to the new flavour-violating source in form of the 5D bulk masses, the RSc model has a rich flavour structure far beyond models with MFV. While the distortion of the quark zero profiles allows for a solution to the flavour puzzle, it also implies flavour-dependent couplings to the KK gauge bosons which induce tree-level FCNCs after the rotation to the fermion mass eigenbasis. A further contribution, even if negligible with regard to the gauge boson mixing, comes from the mixing of the SM quarks with the heavy KK fermions. After having studied the structure of the neutral and charged weak currents, we introduced the quark couplings to the massive KK gluon and KK photon. This is followed by a discussion of tree-level KK gluon and KK photon contributions to the particle-antiparticle mixings in the K and B meson sector, where we demonstrated that new operators are present in the effective Hamiltonian — in particular the dangerous flavour-changing LR-4-fermion operators.

We commented that RSc models which satisfy all existing $\Delta F = 2$ and electroweak pre-

cision constraints with KK masses $M_{\text{KK}} \simeq 2 - 3 \text{ TeV}$ reachable at the LHC exist if a modest hierarchy in the 5D Yukawas [121,122] is reintroduced or some sort of alignment in the down-type sector [210,211] is assumed. As also mentioned in [211], it would be interesting to consider a 5D MFV model with fully gauged $SU(3)^3$ bulk FS group without alignment and to carefully work out its dynamics. Having in mind the 4D dMFV model, one may argue that it is then quite “natural” to allow for a hierarchy in the 5D Yukawa matrices resulting from the breakdown of the 5D bulk symmetry group.

Anyway, the very different approaches in solving the flavour puzzle have in common that the hierarchy in the 4D Yukawa matrices is generated dynamically. While in the RSc model the latter is generated by an overlap integral over “dynamical” fermion wave functions, the hierarchy in the dMFV is generated by dynamical scalar fields that obtain non-vanishing VEVs through an appropriately chosen spurion potential.

Which road to take in building models beyond the SM will hopefully be shown by the upcoming experiments at the LHC and/or in the interplay of a future linear collider [213]. The extensive LHC program [214,215] covers EW, flavour and QCD precision measurements which allow a deeper understanding of the SM and of possible deviations indicating new physics. In particular a more accurate determination of the W boson and the top quark mass can be performed, as well as a precise measurement of the strong coupling constant, parton distribution functions, a detailed study of rare decays, CKM elements and CP violation — especially in the B meson sector.

A highlight would be the direct detection of supersymmetric particles which provide the proper framework for grand unified theories and a deeper insight of space-time symmetries. Furthermore, the LHC allows for the approval of additional spatial dimensions by the exploration of heavy KK modes or of theories containing heavy gauge bosons like Z' models. However, of utmost importance for the LHC is to proof the (non-)existence of the last missing particle of the SM, the famous Higgs boson, and with it to test the very mechanism of spontaneous electroweak symmetry breaking.

Appendix A

Addendum to Chapter 4

In this appendix we demonstrate that the coefficient of the second term in the anomaly contribution follows from the first one by the usage of the Wess–Zumino consistency condition (see Section 2.5). The first term can be obtained via a direct calculation of the triangle diagrams. To simplify the notation, we begin with a summary of the formalism of differential forms, as given in [131].

A.1 Differential Forms

A general p -form is constructed from an antisymmetric tensor with p indices via

$$\Phi_p = \Phi_{\mu_1 \dots \mu_p}(x) \left(\frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p} \right), \quad (\text{A.1})$$

where the *wedge product* is defined as

$$dx^\mu \wedge dx^\nu := dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu = -dx^\nu \wedge dx^\mu. \quad (\text{A.2})$$

In the following the wedge-product symbol is omitted and dx^μ is treated as an anticommuting Grassmann object. If the rank p exceeds the dimension m of the manifold M the wedge product vanishes. The commutation law for a p -form with a q -form is given by

$$\alpha_p \beta_q = (-1)^{pq} \beta_q \alpha_p, \quad (\text{A.3})$$

as each $dx^{\mu_{q_i}}$ ($q_i = 1, \dots, q$) has to be commuted p times with $dx^{\mu_{p_i}}$ ($p_i = 1, \dots, p$), giving in total q factors of $(-1)^p$. In particular, odd forms with odd ranks always anticommute.

An exterior derivative $d = \frac{\partial}{\partial x^\nu} dx^\nu$ acting on a p -form like

$$d\Phi_p = \partial_\nu \Phi_{\mu_1 \dots \mu_p}(x) \left(\frac{1}{p!} dx^\nu dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_p} \right), \quad (\text{A.4})$$

transforms it into a $(p + 1)$ -form. The fact that the derivatives are symmetric while the wedge product is antisymmetric leads to the important property that the exterior derivative

is nilpotent

$$d^2\Phi_p = 0. \quad (\text{A.5})$$

The exterior derivative further obeys the *antiderivation rule*

$$d(\alpha_p\beta_q) = (d\alpha_p)\beta_q + (-)^p\alpha_p(d\beta_q). \quad (\text{A.6})$$

The *dual p-form* is defined by the so-called *Hodge * operation* which transforms p -forms into $(m-p)$ -forms,

$$*\Phi_p = \Phi_{\mu_1\dots\mu_p} \frac{1}{p!} *(dx^{\mu_1}\dots dx^{\mu_p}) = \Phi_{\mu_1\dots\mu_p} \frac{1}{p!} \left(\epsilon^{\mu_1\dots\mu_p \mu_{p+1}\dots\mu_m} \frac{1}{(m-p)!} dx^{\mu_{p+1}}\dots dx^{\mu_m} \right). \quad (\text{A.7})$$

In Minkowski space the following relations hold

$$\frac{1}{m!} \epsilon_{\mu_1\dots\mu_m} dx^{\mu_1}\dots dx^{\mu_m} = dx^0\dots dx^{m-1}, \quad (\text{A.8})$$

$$dx^{\mu_1}\dots dx^{\mu_m} = -\epsilon^{\mu_1\dots\mu_m} dx^0\dots dx^{m-1}, \quad (\text{A.9})$$

where the second follows from the first one with the identity

$$\epsilon_{\mu_1\dots\mu_m} \epsilon^{\mu_1\dots\mu_m} = -m!. \quad (\text{A.10})$$

Non-Abelian Gauge Anomaly within Differential Form

We now give a few examples for the usage of differential forms. In Yang-Mills theory, the gauge field can be described by the 1-form

$$\tilde{A} = \tilde{A}_\mu dx^\mu \quad \text{with} \quad \tilde{A}_\mu = A_\mu^a \tilde{T}^a = -i A_\mu^a T^a. \quad (\text{A.11})$$

The field strength tensor

$$F_{\mu\nu} = F_{\mu\nu}^a \tilde{T}^a = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + [\tilde{A}_\mu, \tilde{A}_\nu], \quad (\text{A.12})$$

can be rewritten as a 2-form:

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \frac{1}{2} (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + [\tilde{A}_\mu, \tilde{A}_\nu]) dx^\mu dx^\nu = d\tilde{A} + \frac{1}{2} [\tilde{A}, \tilde{A}] = d\tilde{A} + \tilde{A}^2. \quad (\text{A.13})$$

With the introduction of the covariant derivative

$$D = D_\mu dx^\mu = (D_\mu^a \tilde{T}^a) dx^\mu = (\partial_\mu + [\tilde{A}_\mu, \]) dx^\mu = d + [\tilde{A}, \], \quad (\text{A.14})$$

the field strength tensor can be reformulated by the compact form $F = D\tilde{A}$.

Finally, we want to rewrite the non-abelian anomaly for L - and R -fields

$$G^a[\tilde{A}_\mu^{L,R}(x)] = D_\mu^{ab}[\tilde{A}_\mu^{L,R}] j_{L,R}^{\mu b} = \mp \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} [\tilde{T}^a \partial_\mu (\tilde{A}_\nu^{L,R} \partial_\rho \tilde{A}_\sigma^{L,R} + \frac{1}{2} \tilde{A}_\nu^{L,R} \tilde{A}_\rho^{L,R} \tilde{A}_\sigma^{L,R})]. \quad (\text{A.15})$$

Therefore we use the definition of the non-abelian L - and R -currents as a 1-form

$$j_{L,R} = j_{L,R}^a \tilde{T}^a = j_{L,R,\mu}^a \tilde{T}^a dx^\mu, \quad (\text{A.16})$$

to define the corresponding dual currents (3-forms)

$$*j_{L,R} = \frac{1}{1!(m-1)!} j_{L,R}^\mu \varepsilon_{\mu\nu\rho\sigma} dx^\nu dx^\rho dx^\sigma. \quad (\text{A.17})$$

The anomaly arises from the exterior derivative acting on the dual L - and R -currents:

$$\begin{aligned} (D * j_{L,R})^a &= \frac{1}{(m-1)!} D_\alpha^{ab} j_{L,R}^{\mu b} \varepsilon_{\mu\nu\rho\sigma} dx^\alpha dx^\nu dx^\rho dx^\sigma \\ &\stackrel{(\text{A.9})}{=} \frac{1}{(m-1)!} D_\alpha^{ab} j_{L,R}^{\mu b} \varepsilon_{\mu\nu\rho\sigma} (-\varepsilon^{\alpha\nu\rho\sigma}) dx^0 \dots dx^3 \\ &= \frac{1}{(m-1)!} D_\alpha^{ab} j_{L,R}^{\mu b} ((m-1)! \delta_\mu^\alpha) dx^0 \dots dx^3 \\ &= D_\alpha^{ab} j_{L,R}^{\alpha b} dx^0 dx^1 dx^2 dx^3 \\ &= \mp \frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{Tr} [\tilde{T}^a \partial_\mu (\tilde{A}_\nu^{L,R} \partial_\rho \tilde{A}_\sigma^{L,R} + \frac{1}{2} \tilde{A}_\nu^{L,R} \tilde{A}_\rho^{L,R} \tilde{A}_\sigma^{L,R})] dx^0 dx^1 dx^2 dx^3 \\ &= \pm \frac{1}{24\pi^2} \text{Tr} [\tilde{T}^a \partial_\mu (\tilde{A}_\nu^{L,R} \partial_\rho \tilde{A}_\sigma^{L,R} + \frac{1}{2} \tilde{A}_\nu^{L,R} \tilde{A}_\rho^{L,R} \tilde{A}_\sigma^{L,R})] dx^\mu dx^\nu dx^\rho dx^\sigma \\ &= \pm \frac{1}{24\pi^2} \text{Tr} [\tilde{T}^a d(\tilde{A}^{L,R} d\tilde{A}^{L,R} + \frac{1}{2} \tilde{A}^{L,R} \tilde{A}^{L,R} \tilde{A}^{L,R})]. \end{aligned} \quad (\text{A.18})$$

Hence, the non-abelian gauge anomaly within differential forms reads

$$G^a[\tilde{A}^{L,R}] = (D * j_{L,R})^a = \pm \frac{1}{24\pi^2} \text{Tr} [\tilde{T}^a d(\tilde{A}^{L,R} d\tilde{A}^{L,R} + \frac{1}{2} (\tilde{A}^{L,R})^3)], \quad (\text{A.19})$$

and correspondingly

$$G(v, \tilde{A}^{L,R}) = \int dx v^a(x) G^a[\tilde{A}^{L,R}](x) = \pm \frac{1}{24\pi^2} \int dx \text{Tr} [v d(\tilde{A}^{L,R} d\tilde{A}^{L,R} + \frac{1}{2} (\tilde{A}^{L,R})^3)]. \quad (\text{A.20})$$

A.2 Graded Algebra

We have seen in Section 2.3 that the BRST operator δ_v increases the ghost number by one unit and in the previous Section A.1 that the exterior differential d raises the form degree by one unit. Combining the form and ghost degree through a sum of both degrees to a total degree,

$$\text{Deg}_{\text{total}}(X) = \text{Deg}_{\text{form}}(X) + \text{Deg}_{\text{ghost}}(X), \quad (\text{A.21})$$

defines the *graded algebra* [131]. Both operators act as antiderivations ($\delta_v^2 = d^2 = 0$) on this algebra and they are required to satisfy

$$\delta_v d + d \delta_v = 0. \quad (\text{A.22})$$

For the graded algebra the commutator is defined by

$$[P, Q] = PQ - (-1)^{\text{Deg}_{\text{total}}(P) \times \text{Deg}_{\text{total}}(Q)}QP, \quad (\text{A.23})$$

for example,

$$[\tilde{A}, v] = \tilde{A}v + v\tilde{A}, \quad [v, v] = 2v^2. \quad (\text{A.24})$$

Using differential forms, the first two BRST transformations in (2.20) can be rewritten as

$$\delta_v \tilde{A} = -Dv = -dv - [\tilde{A}, v] \stackrel{(\text{A.24})}{=} -dv - \tilde{A}v - v\tilde{A}, \quad \delta_v v = -v^2, \quad (\text{A.25})$$

which is constructed in such a way that the nilpotency of the BRST operator is fulfilled, e.g.

$$\begin{aligned} \delta_v^2 \tilde{A} &= \delta_v(-Dv) = \delta_v(-dv - [\tilde{A}, v]) = +d(\delta_v v) - \delta_v(\tilde{A}v + v\tilde{A}) \\ &= -d(v^2) - (-Dv)v + \tilde{A}(-v^2) + v^2\tilde{A} + v(-Dv) \\ &= -(dv)v + v(dv) + (dv)v + \tilde{A}v^2 + v\tilde{A}v - \tilde{A}v^2 + v^2\tilde{A} - v(dv) - v\tilde{A}v - v^2\tilde{A} = 0. \end{aligned} \quad (\text{A.26})$$

A.3 Usage of the Wess–Zumino Consistency Condition

The aim of this section is to show that the WZCC $\delta_v G(v, \tilde{A}) = 0$ introduced in (2.35) uniquely determines the constants c_1, \dots, c_4 of the most general form of the left-handed anomaly

$$G(v, \tilde{A}) = \frac{1}{24\pi^2} \int_M \text{Tr} [v d\tilde{A}d\tilde{A} + c_1 v(d\tilde{A})\tilde{A}^2 + c_2 v\tilde{A}(d\tilde{A})\tilde{A} + c_3 v\tilde{A}^2(d\tilde{A}) + c_4 v\tilde{A}^4]. \quad (\text{A.27})$$

In the following we let the generalised gauge operator act on the different terms in the above sum, where all expressions have to be understood as arguments of the trace:

- $\delta_v(vd\tilde{A}d\tilde{A}) = v\tilde{A}dv d\tilde{A} - vd\tilde{A}d\tilde{A}v - vdv\tilde{A}d\tilde{A} + v(d\tilde{A})\tilde{A}dv - vd\tilde{A}dv\tilde{A},$
- $\delta_v(v(d\tilde{A})\tilde{A}^2) = vd\tilde{A}(dv)\tilde{A} - v(d\tilde{A})\tilde{A}dv + v\tilde{A}(dv)\tilde{A}^2 - v(dv)\tilde{A}^3 - v(d\tilde{A})\tilde{A}^2v,$
- $\delta_v(v\tilde{A}(d\tilde{A})\tilde{A}) = vdv(d\tilde{A})\tilde{A} - v\tilde{A}(d\tilde{A})dv - v\tilde{A}^2(dv)\tilde{A} + v\tilde{A}(dv)\tilde{A}^2 - v\tilde{A}(d\tilde{A})\tilde{A}v,$
- $\delta_v(v\tilde{A}^2(d\tilde{A})) = v(dv)\tilde{A}(d\tilde{A}) - v\tilde{A}(dv)(d\tilde{A}) - v\tilde{A}^2(d\tilde{A})v + v\tilde{A}^3dv - v\tilde{A}^2(dv)\tilde{A},$
- $\delta_v(v\tilde{A}^4) = v^2\tilde{A}^4 - v\tilde{A}(dv)\tilde{A}^2 + v\tilde{A}^2(dv)\tilde{A} - v\tilde{A}^3dv. \quad (\text{A.28})$

Since there are no other terms proportional to \tilde{A}^4 that could probably cancel the term arising in the last line of (A.28), we conclude that the coefficient c_4 has to be zero. Actually we do not need to require that the various terms all add up to zero, but only that they can be written as a derivative of some local function, so that its integral over the manifold vanishes. Having started with the single term containing no derivative, we continue with a proper rearrangement of the contributions which contain exactly one derivative, namely

$$\bullet (c_1 + c_2)(dv)\tilde{A}^2v\tilde{A} - (c_2 + c_3)v\tilde{A}^2(dv)\tilde{A} \quad (\text{A.29})$$

$$\bullet c_3(dv)v\tilde{A}^3 - c_1v(dv)\tilde{A}^3 + c_1v^2(d\tilde{A})\tilde{A}^2 + c_2v^2\tilde{A}(d\tilde{A})\tilde{A} + c_3v^2\tilde{A}^2(d\tilde{A}). \quad (\text{A.30})$$

As the ghost fields do not simply commute with the gauge fields, we differentiate between the terms with ghosts alternating with the gauge fields in (A.29) from the ones in which they are separated from each other as in (A.30). The terms of the first line could probably form a total derivative $d(v\tilde{A}^2v\tilde{A})$, but, as a term involving a derivative acting on a gauge field is missing, the coefficients have to be chosen such that all terms vanish. Thus, we impose the condition $c_1 = -c_2 = c_3 = c$, which in turn implies that the remaining terms in (A.30) add up to the total derivative

$$c d(v^2\tilde{A}^3). \quad (\text{A.31})$$

For the terms containing two derivatives we proceed in the same manner. First, we collect all terms in which the gauge fields are sandwiched between two ghost fields

$$(1-c)v\tilde{A}(dv)(d\tilde{A}) + (1-c)\tilde{A}v(d\tilde{A})(dv). \quad (\text{A.32})$$

Realising that the “missing” third term, needed for a completion of the total derivative, would vanish under the trace anyway

$$v(d\tilde{A})v(d\tilde{A}) = -(d\tilde{A})v(d\tilde{A})v = -v(d\tilde{A})v(d\tilde{A}) = 0, \quad (\text{A.33})$$

we can summarise the above two terms according to

$$(1-c)d(v\tilde{A}vd\tilde{A}). \quad (\text{A.34})$$

The remaining terms in (A.28), where the gauge fields and ghosts are not mixed up, contain at least one derivative acting on the gauge field

$$v^2(d\tilde{A})(d\tilde{A}) + (c-1)v(dv)\tilde{A}(d\tilde{A}) + (1-c)(dv)v(d\tilde{A})\tilde{A} - cv(dv)(d\tilde{A})\tilde{A} + c(dv)v\tilde{A}(d\tilde{A}). \quad (\text{A.35})$$

As in addition a term with two dv 's is missing, one may guess that these terms arise from a total derivative of the following structure

$$d(v^2\tilde{A}d\tilde{A} + v^2(d\tilde{A})\tilde{A}) = (dv)v\tilde{A}d\tilde{A} - v(dv)\tilde{A}d\tilde{A} + 2v^2d\tilde{A}d\tilde{A} + (dv)v(d\tilde{A})\tilde{A} - v(dv)(d\tilde{A})\tilde{A}. \quad (\text{A.36})$$

Indeed, if one imposes $c \stackrel{!}{=} 1/2$, the terms in (A.35) can be brought into the form

$$\frac{1}{2}d(v^2\tilde{A}(d\tilde{A}) + v^2(d\tilde{A})\tilde{A}). \quad (\text{A.37})$$

Thus, we have shown that the WZCC uniquely fixes the coefficients to

$$c_1 = -c_2 = c_3 = c = \frac{1}{2}. \quad (\text{A.38})$$

Using this condition, the anomaly can be reformulated as

$$\begin{aligned} G(v, \tilde{A}) &= \frac{1}{24\pi^2} \int_M \text{Tr} [v d\tilde{A}d\tilde{A} + \frac{1}{2}v(d\tilde{A})\tilde{A}^2 - \frac{1}{2}v\tilde{A}(d\tilde{A})\tilde{A} + \frac{1}{2}v\tilde{A}^2(d\tilde{A})] \\ &= \frac{1}{24\pi^2} \int_M \text{Tr} [v d(\tilde{A}d\tilde{A} + \frac{1}{2}\tilde{A}^3)], \end{aligned} \quad (\text{A.39})$$

which confirms the result given in (2.32) and in (A.20).

A.4 Diagonalisation of the Spurion Kinetic Terms

As mentioned in Subsection 4.4.2, one has the freedom to diagonalise the spurion kinetic terms contained in the kinetic term of the up-type Yukawa matrix $\text{Tr}[(\partial_\mu Y_U)(\partial^\mu Y_U^\dagger)]$ separately from that in the down-type one, and vice versa. Actually, we will proceed with the diagonalisation and normalisation of $\text{Tr}[(\partial_\mu Y_D)(\partial^\mu Y_D^\dagger)]$, because it accidentally does not contain any mixings. Thus we can leave out the first step and normalise the terms in

$$\text{Tr}[(\partial_\mu Y_D)(\partial^\mu Y_D^\dagger)] = \frac{1}{4}y_s^2(\partial_\mu \tilde{\eta}_{12}(x))^2 + \frac{1}{4}y_b^2(\partial_\mu \tilde{\eta}_{13}(x))^2 + \frac{1}{4}\frac{(y_b^2 - y_s^2)^2}{y_b^2 + y_s^2}(\partial_\mu \tilde{\eta}_{23}(x))^2, \quad (\text{A.40})$$

by using the following redefinitions

$$\tilde{\eta}_{12}(x) \rightarrow \frac{2}{y_s} \check{\eta}_{12}(x), \quad \tilde{\eta}_{13}(x) \rightarrow \frac{2}{y_b} \check{\eta}_{13}(x), \quad \tilde{\eta}_{23}(x) \rightarrow \frac{2\sqrt{y_b^2 + y_s^2}}{y_b^2 - y_s^2} \check{\eta}_{23}(x). \quad (\text{A.41})$$

The next step is to diagonalise the kinetic mixing terms induced by $\text{Tr}[(\partial_\mu Y_U)(\partial^\mu Y_U^\dagger)]$. To this end we introduce the matrix notation

$$\left(\partial_\mu \check{\eta}_{12}(x), \quad \partial_\mu \check{\eta}_{13}(x), \quad \partial_\mu \check{\eta}_{23}(x) \right) A \begin{pmatrix} \partial^\mu \check{\eta}_{12}(x) \\ \partial^\mu \check{\eta}_{13}(x) \\ \partial^\mu \check{\eta}_{23}(x) \end{pmatrix}. \quad (\text{A.42})$$

Observing that the off-diagonal elements of the real matrix A are small compared to the diagonal entries, we are going to use this hierarchy to diagonalise the matrix perturbatively. As motivated in the description of the Rayleigh–Schrödinger method in the Appendix B.5, we divide A into an “unperturbed” matrix A_0 and a “perturbed” matrix A_1

$$A = A_0 + \epsilon A_1 = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} + \epsilon \begin{pmatrix} 0 & A_{12} & A_{13} \\ A_{12} & 0 & A_{23} \\ A_{13} & A_{23} & 0 \end{pmatrix}, \quad (\text{A.43})$$

where we made the smallness of the off-diagonal CKM matrix elements (as given in the linear representation in (4.63)) more transparent by redefining $\theta_{ij} \rightarrow \epsilon \theta_{ij}$, $i, j = 1, 2, 3$.

The explicit values of the diagonal elements of A_0 are given by

$$A_{11} = \frac{y_s^2}{y_c^2}, \quad A_{22} = \frac{y_b^2}{y_t^2}, \quad A_{33} = \frac{(y_b^2 - y_s^2)^2 (y_c^2 + y_t^2)}{(y_b^2 + y_s^2)(y_c^2 - y_t^2)^2}. \quad (\text{A.44})$$

Obviously, an appropriate orthonormal basis of zeroth eigenvectors of the unperturbed matrix A_0 is represented by

$$|1^0\rangle^T = \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}, \quad |2^0\rangle^T = \begin{pmatrix} 0, & 1, & 0 \end{pmatrix}, \quad |3^0\rangle^T = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix}, \quad (\text{A.45})$$

with corresponding zeroth-order eigenvalues $E_1^0 = A_{11}$, $E_2^0 = A_{22}$ and $E_3^0 = A_{33}$. As A_{33} equals A_{22} in the limit $y_s \ll y_b$ and $y_c \ll y_t$, we will not use this approximation already at

this stage to avoid having to deal with the formulae for the degenerate case (see also Appendix B.5). There are no first-order corrections to these eigenvalues due to the fact that A_1 has no diagonal entries, which implies that $\langle k^0|A_1|k^0\rangle = 0$ for the above basis with $k = 1, 2, 3$ (see also formula (B.96)).

However, the explicit expressions for the off-diagonal elements of A_1 ,

$$\begin{aligned} A_{12} &= -\theta_{23} y_b y_s \frac{y_c^2 - y_t^2}{y_c^2 y_t^2} \sim \theta_{23} \frac{y_b y_s}{y_c^2}, \\ A_{13} &= \theta_{13} \frac{y_s y_t^2 (-y_b^2 + y_s^2) (-3y_c^2 + y_t^2)}{y_c^2 \sqrt{y_b^2 + y_s^2} (y_c^2 - y_t^2)^2} \sim -\theta_{13} \frac{y_b y_s}{y_c^2}, \\ A_{23} &= \theta_{12} \frac{y_b y_c^2 (y_b^2 - y_s^2) (y_c^2 - 3y_t^2)}{y_t^2 \sqrt{y_b^2 + y_s^2} (y_c^2 - y_t^2)^2} \sim -3\theta_{12} \frac{y_b^2 y_c^2}{y_t^4}, \end{aligned} \quad (\text{A.46})$$

enter the formula given in (B.100) for the calculation of the first-order corrections to the eigenvectors. In the case of non-degeneracy the formula simplifies to

$$|k^1\rangle = \sum_{k' \neq k, k'=1}^3 \frac{\langle k'^0|A_1|k^0\rangle}{(E_k^0 - E_{k'}^0)} |k'^0\rangle, \quad (\text{A.47})$$

with the explicit first-order corrections to the eigenvectors given by

$$\begin{aligned} |1^1\rangle^T &= \left(0, \frac{y_b y_s (y_c^2 - y_t^2) \theta_{23}}{y_b^2 y_c^2 - y_s^2 y_t^2}, \frac{y_s y_t^2 (y_b^2 - y_s^2) \sqrt{y_b^2 + y_s^2} (-3y_c^2 + y_t^2) \theta_{13}}{(y_b^2 y_c^2 - y_s^2 y_t^2) (y_s^2 (-3y_c^2 + y_t^2) + y_b^2 (y_c^2 + y_t^2))} \right), \\ |2^1\rangle^T &= \left(-\frac{y_b y_s (y_c^2 - y_t^2) \theta_{23}}{y_b^2 y_c^2 - y_s^2 y_t^2}, 0, \frac{y_b y_c^2 (y_b^2 - y_s^2) \sqrt{y_b^2 + y_s^2} (y_c^2 - 3y_t^2) \theta_{12}}{(y_b^2 y_c^2 - y_s^2 y_t^2) (y_b^2 (y_c^2 - 3y_t^2) + y_s^2 (y_c^2 + y_t^2))} \right), \\ |3^1\rangle^T &= \left(-\frac{y_s y_t^2 (y_b^2 - y_s^2) \sqrt{y_b^2 + y_s^2} (-3y_c^2 + y_t^2) \theta_{13}}{(y_b^2 y_c^2 - y_s^2 y_t^2) (y_s^2 (-3y_c^2 + y_t^2) + y_b^2 (y_c^2 + y_t^2))}, -\frac{y_b y_c^2 (y_b^2 - y_s^2) \sqrt{y_b^2 + y_s^2} (y_c^2 - 3y_t^2) \theta_{12}}{(y_b^2 y_c^2 - y_s^2 y_t^2) (y_b^2 (y_c^2 - 3y_t^2) + y_s^2 (y_c^2 + y_t^2))}, 0 \right). \end{aligned} \quad (\text{A.48})$$

Together with the zeroth eigenvectors they form the orthogonal transformation matrix

$$G = \begin{pmatrix} |1^0\rangle^T + \epsilon |1^1\rangle^T \\ |2^0\rangle^T + \epsilon |2^1\rangle^T \\ |3^0\rangle^T + \epsilon |3^1\rangle^T \end{pmatrix}, \quad (\text{A.49})$$

which diagonalises A up to $\mathcal{O}(\epsilon^2)$ corrections

$$G(A_0 + \epsilon A_1)G^T = \text{Diag}(A) = \text{Diag}(A_{11}, A_{22}, A_{33}) + \mathcal{O}(\epsilon^2). \quad (\text{A.50})$$

Simultaneously, the spurion fields are transformed into the new basis

$$\begin{pmatrix} \hat{\eta}_{12} & \hat{\eta}_{13} & \hat{\eta}_{23} \end{pmatrix}^T = G \begin{pmatrix} \check{\eta}_{12} & \check{\eta}_{13} & \check{\eta}_{23} \end{pmatrix}^T, \quad (\text{A.51})$$

in which then also the spurion kinetic terms arising from the up-type Yukawas are diagonal,

$$\text{Tr}[(\partial_\mu Y_U)(\partial^\mu Y_U^\dagger)] = \frac{y_s^2}{y_c^2} (\partial_\mu \hat{\eta}_{12}(x))^2 + \frac{y_b^2}{y_t^2} (\partial_\mu \hat{\eta}_{13}(x))^2 + \frac{(y_b^2 - y_s^2)^2 (y_c^2 + y_t^2)}{(y_b^2 + y_s^2) (y_c^2 - y_t^2)^2} (\partial_\mu \hat{\eta}_{23}(x))^2. \quad (\text{A.52})$$

Remember that the latter unitary transformation does not influence the form of the normalised contributions from the kinetic term of the down-type Yukawa spurion matrix

$$\text{Tr}[(\partial_\mu Y_D)(\partial^\mu Y_D^\dagger)] = (\partial_\mu \hat{\eta}_{12}(x))^2 + (\partial_\mu \hat{\eta}_{13}(x))^2 + (\partial_\mu \hat{\eta}_{23}(x))^2. \quad (\text{A.53})$$

Without loss of generality, we choose to rescale the fields according to

$$\hat{\eta}_{12}(x) \rightarrow -\frac{1}{y_s} \tilde{\eta}_{12}(x), \quad \hat{\eta}_{13}(x) \rightarrow -\frac{1}{y_b} \tilde{\eta}_{13}(x), \quad \hat{\eta}_{23}(x) \rightarrow -\frac{\sqrt{y_b^2 + y_s^2}}{(y_b^2 - y_s^2)} \tilde{\eta}_{23}(x), \quad (\text{A.54})$$

which leads to the symmetric form of the spurion kinetic terms presented in (4.68).

Appendix B

Addendum to Chapter 5

B.1 Fermionic Action and RS Spin Connection Term

In this section we show that the action given in (5.105)

$$S = \int d^5x \sqrt{G} \left(\frac{1}{2} \bar{\Psi} (i \Gamma^M \nabla_M - ck) \Psi + \text{h.c.} \right), \quad (\text{B.1})$$

with $\nabla_M = D_M + \omega_M$, can be brought into the shape

$$S = \int d^5x \sqrt{G} \left(E_A^M \frac{i}{2} \bar{\Psi} \gamma^A (D_M - \overleftarrow{D}_M^\dagger) \Psi + E_A^M \frac{i}{2} \bar{\Psi} \{ \gamma^A, \omega_M \} \Psi - ck \bar{\Psi} \Psi \right). \quad (\text{B.2})$$

$\overleftarrow{D}_M^\dagger = \overleftarrow{\partial}_M + ig V_M^a T^a$ denotes the hermitian conjugate covariant derivative for the specific case of the non-abelian gauge fields V_M^a , whereas the partial derivative acts again solely on the spinor fields. With the definition $\gamma_{5D}^5 = -i\gamma_{4D}^5$ (see (5.99)), the hermicity conditions, which are usually imposed on the 4D gamma matrices,

$$(\gamma^\alpha)^\dagger = -\gamma^\alpha, \quad \alpha = 1, 2, 3, \quad (\gamma^5)^\dagger = \gamma^5, \quad \gamma^0 (\gamma^\alpha)^\dagger \gamma^0 = \gamma^\alpha, \quad (\text{B.3})$$

can be combined into

$$(\gamma^A)^\dagger = -\gamma^A, \quad \gamma^0 \gamma^{A\dagger} \gamma^0 = \gamma^A, \quad A = 1, 2, 3, 5, \quad (\text{B.4})$$

where $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^0)^2 = \mathbb{1}$. These relations are needed for the calculation of the various hermitian conjugate terms in (B.1), e.g.

$$\left(\frac{i}{2} E_A^M \bar{\Psi} \gamma^A \partial_M \Psi \right)^\dagger = -\frac{i}{2} E_A^M (\Psi^\dagger \overleftarrow{\partial}_M) \gamma^0 \gamma^0 \gamma^{A\dagger} \gamma^0 \Psi \stackrel{(\text{B.4})}{=} -\frac{i}{2} (\bar{\Psi} \overleftarrow{\partial}_M) \Gamma^M \Psi, \quad (\text{B.5})$$

and of the term which contains the coupling to the non-abelian gauge fields

$$\left(\frac{i}{2} E_A^M \bar{\Psi} \gamma^A (-ig V_M^a T^a) \Psi \right)^\dagger \stackrel{(\text{B.4})}{=} -\frac{i}{2} \bar{\Psi} (ig V_M^a T^a) \Gamma^M \Psi. \quad (\text{B.6})$$

For the calculation of the term involving the spin connection

$$\left(\frac{i}{2}E_A^M\bar{\Psi}\gamma^A\omega_M\Psi\right)^\dagger = -\frac{i}{2}E_A^M\Psi^\dagger\gamma^0\gamma^0\omega_M^\dagger\gamma^0\gamma^0\gamma^A\gamma^0\Psi = -\frac{i}{2}E_A^M\bar{\Psi}\gamma^0\omega_M^\dagger\gamma^0\gamma^A\Psi, \quad (\text{B.7})$$

we need to know $\gamma^0\omega_M^\dagger\gamma^0$. Concentrating on the Dirac algebra and neglecting all metric factors of the spin connection, it follows independently of the chosen metric:

$$\gamma^0\omega_M^\dagger\gamma^0 \stackrel{(5.102)}{\sim} \gamma^0[\gamma^A, \gamma^B]^\dagger\gamma^0 \stackrel{(B.4)}{=} \gamma^0(\gamma^B\gamma^A - \gamma^A\gamma^B)\gamma^0 = -[\gamma^A, \gamma^B] \sim -\omega_M. \quad (\text{B.8})$$

Thus, (B.7) can be rewritten according to

$$\left(\frac{i}{2}E_A^M\bar{\Psi}\gamma^A\omega_M\Psi\right)^\dagger = \frac{i}{2}\bar{\Psi}\omega_M\Gamma^M\Psi. \quad (\text{B.9})$$

Inserting $(-1/2\bar{\Psi}ck\Psi)^\dagger = -1/2\bar{\Psi}ck\Psi$ into (B.1) and using the above results, yields

$$S = \int d^5x\sqrt{G}\left(\frac{i}{2}\bar{\Psi}\Gamma^M(D_M - \overleftarrow{D}_M^\dagger)\Psi + \frac{i}{2}\bar{\Psi}\{\Gamma^M, \omega_M\}\Psi - ck\bar{\Psi}\Psi\right), \quad (\text{B.10})$$

which is equivalent to (B.2) (with $E_A^M\gamma^A = \Gamma^M$).

The next aim of this section is to confirm the result of the RS spin connection given in (5.103),

$$\omega_M = e_N^A(\partial_M E_B^N + \Gamma_{MP}^N E_B^P)\frac{\sigma_A^B}{2} \stackrel{\text{RS}}{=} \begin{cases} \frac{i}{2}ke^{-ky}\gamma_\mu\gamma_{4D}^5 & \text{for } M = \mu, \\ 0 & \text{for } M = 5. \end{cases} \quad (\text{B.11})$$

To this end, we have to evaluate all metric factors for the specific case of the RS metric. According to formula (5.95) the derivative $\partial_5 E_B^N$ contained in the fifth component of the spin connection

$$\omega_5 = e_N^A(\partial_5 E_B^N + \Gamma_{5P}^N E_B^P)\frac{\sigma_A^B}{2}, \quad (\text{B.12})$$

is non-zero only for the case of $B = N \neq 5$. Moreover, (5.93) implies that $A = N$ for $e_N^A \neq 0$ and thus the whole first term of (B.12) vanishes because $\sigma_N^N = 0$.

Concerning the second term of (B.12), the Christoffel symbol simplifies to

$$\Gamma_{5P}^N = \frac{1}{2}G^{NR}(\partial_P G_{5R} + \partial_5 G_{PR} - \partial_R G_{5P}) = \frac{1}{2}G^{NR}\partial_5 G_{PR}, \quad (\text{B.13})$$

as $G_{5R} = G_{5P} = 1$ for $R = 5, P = 5$, and $G_{5R} = G_{5P} = 0$ otherwise. We finally obtain

$$\begin{aligned} \omega_5 &= \frac{1}{2}e_N^A G^{NR}(\partial_5 G_{PR})E_B^P\frac{\sigma_A^B}{2} = e_N^A G^{N\nu}(\partial_5 e^{-2ky}\eta_{\mu\nu})E_B^\mu\frac{\sigma_A^B}{4} \\ &= e_\gamma^A e^{2ky}\eta^{\gamma\nu}(\partial_5 e^{-2ky}\eta_{\mu\nu})e^{ky}\frac{\sigma_A^\mu}{4} = (\partial_5 e^{-2ky})\frac{\sigma_\nu^\nu}{4} = 0. \end{aligned} \quad (\text{B.14})$$

Recognising that the first term of ω_μ in (B.11) gives no contribution

$$e_N^A(\partial_\mu E_B^N)\frac{\sigma_A^B}{2} = 0, \quad (\text{B.15})$$

we are left with the calculation of the following term

$$\omega_\mu = e_N^A \Gamma_{\mu P}^N E_B^P \frac{\sigma_A^B}{2} = e_N^A \frac{1}{2} G^{NR} (\partial_P G_{\mu R} + \partial_\mu G_{PR} - \partial_R G_{\mu P}) E_B^P \frac{\sigma_A^B}{2}. \quad (\text{B.16})$$

As the metric tensors are x -independent, only derivatives involving the fifth component are relevant for the calculation, which then yields

$$\begin{aligned} \omega_\mu &= \frac{1}{2} e_N^A G^{NR} (\partial_5 G_{\mu R}) E_B^5 \frac{\sigma_A^B}{2} - \frac{1}{2} e_N^A G^{N5} (\partial_5 G_{\mu P}) E_B^P \frac{\sigma_A^B}{2} \\ &= -k e_N^A G^{N\nu} (e^{-2ky} \eta_{\mu\nu}) \frac{\sigma_A^5}{2} - k (e^{-2ky} \eta_{\mu\nu}) E_B^\nu \frac{\sigma_5^B}{2} \\ &= -k e_\gamma^A \eta^{\gamma\nu} \eta_{\mu\nu} \frac{\sigma_A^5}{2} - k (e^{-2ky} \eta_{\mu\nu}) e^{ky} \frac{\sigma_5^\nu}{2} \\ &= -k e^{-ky} \frac{\sigma_\mu^5}{2} + k e^{-ky} \frac{\sigma_\mu^5}{2} = -\frac{k}{2} e^{-ky} \gamma_\mu \gamma^5 = \frac{i}{2} k e^{-ky} \gamma_\mu \gamma_{4D}^5. \end{aligned} \quad (\text{B.17})$$

Finally we demonstrate that the spin connection term cancels in the fermionic Lagrangian (B.10) for the special case of a RS metric

$$E_A^M \{ \gamma^A, \omega_M \} = e^{ky} \left\{ \gamma^\mu, \frac{i}{2} k e^{-ky} \gamma_\mu \gamma_{4D}^5 \right\} = \frac{i}{2} k (4 \gamma_{4D}^5 + \gamma_\mu \gamma_{4D}^5 \gamma^\mu) \stackrel{\gamma_\mu \gamma^\mu = 4}{=} 0. \quad (\text{B.18})$$

B.2 Bulk Equations of Motion

Bulk Equations of Motion — Fermions

Setting all gauge interaction terms in (B.10) to zero, the fermionic action is given by

$$S_{\text{fermion}} = \int d^5 x \sqrt{G} \left(\frac{i}{2} (\overline{\Psi} \Gamma^M (\partial_M \Psi)) - \frac{i}{2} (\overline{\Psi} \overleftarrow{\partial}_M) \Gamma^M \Psi - ck \overline{\Psi} \Psi \right). \quad (\text{B.19})$$

Performing an integration by parts over the 4D space in the second term, then leads to

$$S_{\text{fermion}} = \int d^5 x e^{-4ky} \left(i e^{ky} \overline{\Psi} \gamma^\mu (\partial_\mu \Psi) + \frac{1}{2} \overline{\Psi} \gamma_{4D}^5 \partial_5 \Psi - \frac{1}{2} \overline{\Psi} \overleftarrow{\partial}_5 \gamma_{4D}^5 \Psi - ck \overline{\Psi} \Psi \right), \quad (\text{B.20})$$

where we used the relation (5.99) to transfer the 5D gamma matrices into the 4D ones. With $\gamma_{4D}^5 = -\gamma_5^{4D}$ and the introduction of the projection operators (2.26), the action can be reexpressed through

$$\begin{aligned} S_{\text{fermion}} &= \int d^5 x e^{-4ky} \left(i e^{ky} \overline{\Psi}_L \gamma^\mu (\partial_\mu \Psi_L) + i e^{ky} \overline{\Psi}_R \gamma^\mu (\partial_\mu \Psi_R) - \frac{1}{2} (\overline{\Psi}_L \partial_5 \Psi_R - \overline{\Psi}_R \partial_5 \Psi_L) \right. \\ &\quad \left. + \frac{1}{2} (\overline{\Psi}_L \overleftarrow{\partial}_5 \Psi_R - \overline{\Psi}_R \overleftarrow{\partial}_5 \Psi_L) - ck (\overline{\Psi}_L \Psi_R + \overline{\Psi}_R \Psi_L) \right). \end{aligned} \quad (\text{B.21})$$

In Section 5.1 we derived the 5D analogue of the 4D equations of motion (following from the first term of equation (5.13)). Applying it to the right-handed fermion fields, the EOM is given by

$$\left[\frac{\partial \mathcal{L}}{\partial \overline{\Psi}_R} - \partial_M \left(\frac{\partial \mathcal{L}}{\partial (\partial_M \overline{\Psi}_R)} \right) \right] \delta \overline{\Psi}_R \stackrel{!}{=} 0. \quad (\text{B.22})$$

Inserting the various components

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_R} &= e^{-4ky} \left(ie^{ky} \gamma^\mu \partial_\mu \Psi_R - \frac{1}{2} \partial_5 \Psi_L - ck \Psi_L \right), \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi}_R)} = 0, \\ \partial_5 \frac{\partial \mathcal{L}}{\partial (\partial_5 \bar{\Psi}_R)} &= \partial_5 \left(e^{-4ky} \frac{1}{2} \Psi_L \right) = -2ke^{-4ky} \Psi_L + e^{-4ky} \frac{1}{2} \partial_5 \Psi_L, \end{aligned} \quad (\text{B.23})$$

into the EOM, we receive the condition

$$e^{-3ky} i \gamma^\mu \partial_\mu \Psi_R - e^{-4ky} \partial_5 \Psi_L + 2ke^{-4ky} \Psi_L - cke^{-4ky} \Psi_L \stackrel{!}{=} 0. \quad (\text{B.24})$$

Multiplying the equation with e^{2ky} and then using the identity

$$\partial_5 (e^{-2ky} \Psi_L) = -2ke^{-2ky} \Psi_L + e^{-2ky} \partial_5 \Psi_L, \quad (\text{B.25})$$

the above equation (B.24) simplifies to

$$ie^{ky} \gamma^\mu \partial_\mu \hat{\Psi}_R - \partial_5 \hat{\Psi}_L - ck \hat{\Psi}_L = 0, \quad (\text{B.26})$$

where we have rescaled the fermions according to $\hat{\Psi}_{L,R} = e^{-2ky} \Psi_{L,R}$.

In full analogy, the EOM of the left-handed fermion fields can be obtained:

$$ie^{ky} \gamma^\mu \partial_\mu \hat{\Psi}_L + \partial_5 \hat{\Psi}_R - ck \hat{\Psi}_R = 0. \quad (\text{B.27})$$

In order to decouple the two first-order differential equations, we first multiply (B.26) with $(ie^{ky} \gamma^\nu \partial_\nu)$ and then insert (B.27):

$$(-e^{2ky} \partial^\mu \partial_\mu + e^{ky} \partial_5 (e^{-ky} \partial_5) - c(c-1)k^2) \hat{\Psi}_R = 0. \quad (\text{B.28})$$

Analogously, we derive the second-order differential equation for the left-handed fields

$$(-e^{2ky} \partial^\mu \partial_\mu + e^{ky} \partial_5 (e^{-ky} \partial_5) - c(c+1)k^2) \hat{\Psi}_L = 0. \quad (\text{B.29})$$

Obviously, both results can be combined into the final result

$$\left[e^{2ky} \eta^{\mu\nu} \partial_\mu \partial_\nu - e^{ky} \partial_5 (e^{-ky} \partial_5) + (c(c \pm 1)k^2) \right] \hat{\Psi}_{L,R} = 0. \quad (\text{B.30})$$

Bulk Equations of Motion — Scalars

Based on the scalar action of the form

$$S_{\text{scalar}} = \int d^5x \sqrt{G} \left[(D_M \phi)^\dagger (D^M \phi) - m_\phi^2 \phi^\dagger \phi \right], \quad (\text{B.31})$$

the following terms

$$\frac{\partial \mathcal{L}}{\partial (\partial_M \phi)} = \sqrt{G} (G^{MN} \partial_N \phi + G^{PM} \partial_P \phi) = \sqrt{G} G^{MN} \partial_N \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi} = -2\sqrt{G} m_\phi^2 \phi, \quad (\text{B.32})$$

contribute to the equations of motion

$$\frac{1}{\sqrt{G}} \partial_M (\sqrt{G} G^{MN} \partial_N \phi) + m_\phi^2 \phi = 0. \quad (\text{B.33})$$

Inserting the RS metric factors, finally yields

$$(e^{2ky} \eta^{\mu\nu} \partial_\mu \partial_\nu - e^{4ky} \partial_5 (e^{-4ky} \partial_5) + m_\phi^2) \phi = 0. \quad (\text{B.34})$$

Bulk Equations of Motion — Gauge Bosons

For the gauge fields the free action is given by

$$S_{\text{gauge}} = - \int d^5x \sqrt{G} \frac{1}{4} F_{MN} F^{MN} = - \int d^5x \sqrt{G} \frac{1}{4} F_{MN} F_{LS} G^{LM} G^{SN}, \quad (\text{B.35})$$

where the interaction term in the field strength tensor is neglected so that

$$F_{MN} = \partial_M V_N - \partial_N V_M. \quad (\text{B.36})$$

Varying the action S_{gauge} with respect to V_R , we obtain the EOM for the gauge fields:

$$\partial_P \frac{\partial \mathcal{L}}{\partial (\partial_P V_R)} = -\partial_P \left(\sqrt{G} F_{LS} G^{LP} G^{SR} \right) = 0. \quad (\text{B.37})$$

In the special case of a RS metric, the EOM can be reformulated as

$$\begin{aligned} & \partial_P \left(e^{-4ky} (E_A^L E_B^P \eta^{AB}) (E_C^S E_D^R \eta^{CD}) F_{LS} \right) \\ &= \partial_P \left(e^{-4ky} (\delta_{L\mu}^{2ky} \delta_{P\nu} \eta^{\mu\nu} + \delta_{L5} \delta_{P5} \eta^{55}) (e^{2ky} \delta_{S\gamma} \delta_{R\kappa} \eta^{\gamma\kappa} + \delta_{S5} \delta_{R5} \eta^{55}) \right) F_{LS} \\ &= \partial_\nu (\eta^{\mu\nu} \delta_{R\kappa} \eta^{\gamma\kappa}) \partial_\mu V_\gamma - \partial_5 (e^{-2ky} \delta_{R\kappa} \eta^{\gamma\kappa}) \partial_5 V_\gamma, \end{aligned} \quad (\text{B.38})$$

where we used the gauge-fixing condition $V_5 = 0$ and $\partial_\mu V^\mu = 0$ in the last step. Finally, we multiply the last equation by e^{2ky} and obtain

$$\left[e^{2ky} \eta^{\mu\nu} \partial_\mu \partial_\nu - e^{2ky} \partial_5 (e^{-2ky} \partial_5) \right] V^\gamma = 0. \quad (\text{B.39})$$

B.3 KK Decomposition — Bulk Profiles of Wave Functions

The EOM of all fields can be combined into a single second-order differential equation

$$\left[e^{2ky} \eta^{\mu\nu} \partial_\mu \partial_\nu - e^{s ky} \partial_5 (e^{-s ky} \partial_5) + M_\Phi^2 \right] \Phi(x^\mu, y) = 0, \quad (\text{B.40})$$

where

$$\Phi = \{ e^{-2ky} \Psi_{L,R}, \phi, V^\gamma \}, \quad s = \{1, 4, 2\}, \quad M_\Phi^2 = \{c(c \pm 1)k^2, m_\phi^2, 0\}. \quad (\text{B.41})$$

The different sign of the first term in (B.40) with respect to the result given in equation (11) of [103] is due to the different sign convention of the metric tensor. Substituting $z = e^{ky}$ in the general EOM (B.40) gives

$$\left[-z^2 \eta^{\mu\nu} \partial_\mu \partial_\nu + (1-s) z k^2 \partial_z + z^2 k^2 \partial_z^2 - M_\Phi^2 \right] \Phi = 0. \quad (\text{B.42})$$

After expanding Φ into its KK modes $\phi^{(n)}$ and profiles $f^{(n)}(y)$

$$\Phi(x, y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} \phi^{(n)}(x) f^{(n)}(y) \quad (\text{B.43})$$

and introducing the masses m_n ($n = 1, \dots$) of the KK excitation $\phi^{(n)}$, corresponding to $(\partial_\mu \partial^\mu + m_n^2) \phi^{(n)} = 0$, the above differential equation reads

$$\left[z^2 m_n^2 + (1-s)k^2 z \partial_z + k^2 z^2 \partial_z^2 - M_\Phi^2 \right] f^{(n)}(y) = 0. \quad (\text{B.44})$$

Dividing by k^2 and using the short-hand notation $\partial_z f^{(n)}(y) = f'^{(n)}$, $\partial_z^2 f^{(n)}(y) = f''^{(n)}$, the equation can be rewritten according to

$$z^2 f''^{(n)} + (1-s)z f'^{(n)} + \frac{1}{k^2} \left(z^2 m_n^2 - M_\Phi^2 \right) f^{(n)} = 0. \quad (\text{B.45})$$

The general differential equation of type $x^2 y'' + axy' + (bx^m + c)y = 0$ with $m \neq 0$ and $b \neq 0$ is solved by [216]

$$y = x^{\frac{1-a}{2}} \left[C_1 J_\nu \left(\frac{2}{m} \sqrt{bx^{\frac{m}{2}}} \right) + C_2 Y_\nu \left(\frac{2}{m} \sqrt{bx^{\frac{m}{2}}} \right) \right], \quad (\text{B.46})$$

where $J_1(x)$ ($Y_1(x)$) denote the Bessel functions of first (second) kind and

$$\nu := \frac{1}{m} \sqrt{(1-a)^2 - 4c}. \quad (\text{B.47})$$

Together with the identifications

$$x = z; \quad y = f^{(n)}; \quad a = (1-s); \quad b = \frac{m_n^2}{k^2}; \quad m = 2; \quad c = -\frac{M_\Phi^2}{k^2}, \quad (\text{B.48})$$

we thus get the result

$$f^{(n)}(y) = e^{\frac{sky}{2}} \left[C_1 J_\nu \left(\frac{m_n}{k} e^{ky} \right) + C_2 Y_\nu \left(\frac{m_n}{k} e^{ky} \right) \right] = \frac{e^{\frac{sky}{2}}}{N_n} \left[J_\nu \left(\frac{m_n}{k} e^{ky} \right) + b_\nu(m_n) Y_\nu \left(\frac{m_n}{k} e^{ky} \right) \right], \quad (\text{B.49})$$

with

$$\nu = \sqrt{\left(\frac{s}{2} \right)^2 + \frac{M_\Phi^2}{k^2}}. \quad (\text{B.50})$$

Requiring $f^{(n)}(y)$ in (B.49) to fulfil the orthonormality condition,

$$\frac{1}{L} \int_0^L dy e^{(2-s)ky} f^{(n)}(y) f^{(m)}(y) = \delta_{nm}, \quad (\text{B.51})$$

determines the normalisation constant N_n

$$N_n^2 = \frac{1}{L} \int_0^L dy e^{2ky} \left[J_\nu \left(\frac{m_n}{k} e^{ky} \right) + b_\nu(m_n) Y_\nu \left(\frac{m_n}{k} e^{ky} \right) \right]^2. \quad (\text{B.52})$$

The coefficient $b_\nu(m_n)$ as well as the KK masses m_n are specified by the chosen boundary conditions and have to be solved numerically.

B.4 Zero Mode Profiles

Fermionic Zero Mode Profile

The most obvious solution to enforce the vanishing of the boundary terms in (5.13), arising from the variation of the action (B.21)

$$\delta S_{\text{bound}} = \frac{1}{2} \int d^4x \left[e^{-4ky} (\Psi_L \delta \bar{\Psi}_R - \bar{\Psi}_R \delta \Psi_L + \Psi_R \delta \bar{\Psi}_L - \bar{\Psi}_L \delta \Psi_R) \right]_0^L, \quad (\text{B.53})$$

is to set one of the two spinors to zero on the endpoints, for example the right-handed one:

$$\Psi_R|_{0,L} = 0. \quad (\text{B.54})$$

With this condition also $\delta \Psi_R|_{0,L} = 0$ is valid and the full boundary variation term vanishes. However, as the bulk EOM (B.26) has to be satisfied on the endpoints of the interval as well, Ψ_L does not remain arbitrary [195]. With $\Psi_R|_{0,L} = 0$, the condition from the bulk EOM simplifies to:

$$(\partial_5 + ck) \widehat{\Psi}_L|_{0,L} = 0. \quad (\text{B.55})$$

Analogously, if we choose the left-handed spinor to vanish at the boundaries

$$\Psi_L|_{0,L} = 0, \quad (\text{B.56})$$

the condition for the right-handed field reads

$$(\partial_5 - ck) \widehat{\Psi}_R|_{0,L} = 0. \quad (\text{B.57})$$

Separating the variables with the help of the KK decomposition

$$\widehat{\Psi}_{L,R}(x, y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} \psi_{L,R}^{(n)}(x) f^{(n)}(y) = \frac{1}{\sqrt{L}} \psi_{L,R}^{(0)}(x) f^{(0)}(y) + \dots = \widehat{\Psi}_{L,R}^0(x, y) + \dots \quad (\text{B.58})$$

and using $i\eta^{\mu\nu} \gamma_\mu \partial_\nu \psi_{L,R}^{(n)}(x) = m_n \psi_{L,R}^{(n)}(x)$ with $m_0 = 0$ for the zero mode $\psi_{L,R}^{(0)}(x)$, the bulk EOMs (B.26) and (B.27) also decouple:

$$(\partial_5 \pm ck) \widehat{\Psi}_{L,R}^0(x, y) = \frac{1}{\sqrt{L}} \psi_{L,R}^{(0)}(x) (\partial_5 \pm ck) f^{(0)}(y) = 0. \quad (\text{B.59})$$

Moreover, there will always exist a zero mode since the boundary conditions (B.55) and (B.57) are trivially the same as the EOMs (B.59). The general solution is given by

$$f_{L,R}^{(0)}(y) = \frac{e^{\mp cky}}{N_{L,R}^0}, \quad (\text{B.60})$$

with the normalisation constant following from the orthonormal condition

$$(N_{L,R}^0)^2 = \frac{1}{L} \int_0^L e^{(1 \mp 2c)ky} dy = \frac{1}{(1 \mp 2c)kL} \left(e^{(1 \mp 2c)kL} - 1 \right). \quad (\text{B.61})$$

Thus the left- and right-handed fermion zero mode profiles are given by

$$f_{L,R}^{(0)}(y) = \sqrt{\frac{(1 \mp 2c)kL}{e^{(1 \mp 2c)kL} - 1}} e^{\mp cky}. \quad (\text{B.62})$$

Scalar Zero Mode Profile:

Assuming a bulk scalar mass $m_\phi^2 = ak^2$, the bulk EOM of the scalar field (B.40) with $s = 4$ is specified by

$$(\partial_5^2 - 4k\partial_5 - ak^2)\phi = 0. \quad (\text{B.63})$$

Using the KK decomposition $\phi(x, y) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} \phi^{(n)}(x) f_{\text{scalar}}^{(n)}(y)$, this equation is solved by the zero mode profile ($m_0 = 0$)

$$f_{\text{scalar}}^{(0)}(y) = C_1 e^{(2-\sqrt{4+a})ky} + C_2 e^{(2+\sqrt{4+a})ky}. \quad (\text{B.64})$$

A non-vanishing solution ($C_1 \neq 0$, $C_2 \neq 0$) only exists if a boundary mass term $m_\phi = 2\beta k(\delta(y) - \delta(y-L))$ with $\beta = 2 \pm \sqrt{4+a}$ is introduced [196], which implies the modified Neumann condition

$$\left(\partial_5 \phi^{(0)} - \beta k \phi^{(0)}\right)\Big|_{0,L} = 0. \quad (\text{B.65})$$

For $\beta = 2 - \sqrt{4+a}$ the coefficient C_2 vanishes while for $\beta = 2 + \sqrt{4+a}$ this is the case for C_1 . Consequently, the two solutions can be summarised by

$$f_{\text{scalar}}^{(0)}(y) = \sqrt{\frac{2kL(\beta-1)}{e^{2kL(\beta-1)} - 1}} e^{\beta ky}. \quad (\text{B.66})$$

Gauge Boson Zero Mode Profile

After KK decomposition, the EOM of the gauge boson zero mode (B.39) can be written as

$$(\partial_5^2 - 2k\partial_5) f_{\text{gauge}}^{(0)}(y) = 0, \quad (\text{B.67})$$

which has the general solution

$$f_{\text{gauge}}^{(0)}(y) = C_1 \frac{e^{2ky}}{2k} + C_2. \quad (\text{B.68})$$

It is easy to see that zero modes only exist for Neumann boundary conditions on both boundaries. In this case it turns out that $C_1 = 0$ and the correctly normalised zero mode for the gauge boson is flat:

$$f_{\text{gauge}}^{(0)}(y) = 1. \quad (\text{B.69})$$

B.5 Derivation of the Rayleigh–Schrödinger Formulae

The formulae for the *non-degenerate* Rayleigh–Schrödinger perturbation theory can be found in many textbooks about quantum mechanics, e.g. [217, 218]. In deriving the formulae for the Rayleigh–Schrödinger perturbation theory in the *degenerate* case, we follow closely [219] wherein the general case of a hermitian operator is considered. For our specific problem, we

restrict ourselves to a symmetric $D \times D$ matrix A , which is linear in ϵ . Thus, it can be decomposed into the sum of an unperturbed matrix A_0 and a small perturbation matrix A_1 :

$$A = A_0 + \epsilon A_1. \quad (\text{B.70})$$

Expanding the exact eigenvectors $|n_l\rangle$ and exact eigenvalues E_{n_l} of the eigenvalue problem

$$(A_0 + \epsilon A_1)|n_l\rangle = E_{n_l}|n_l\rangle \quad (\text{B.71})$$

in powers of the expansion parameter ϵ , we obtain

$$|n_l\rangle = |n_l^0\rangle + \epsilon |n_l^1\rangle + \dots, \quad E_{n_l} = E_{n_l}^0 + \Delta_{n_l} = E_{n_l}^0 + \epsilon \Delta_{n_l}^1 + \dots. \quad (\text{B.72})$$

While the index n denotes different eigenvalues, l counts the number of eigenvectors sharing the same eigenvalue in the case of degeneracy. Inserting the ansatz (B.72) into (B.71), and comparing the coefficients of m -th order in ϵ , one gets the result

$$(E_{n_l}^0 - A_0)|n_l^m\rangle = (A_1 - \Delta_{n_l}^1)|n_l^{m-1}\rangle - \sum_{i=2}^m \Delta_{n_l}^i |n_l^{m-i}\rangle. \quad (\text{B.73})$$

We suppose that the unperturbed eigenvalue problem

$$A_0|n_l^0\rangle = E_{n_l}^0|n_l^0\rangle \quad (\text{B.74})$$

has only one degenerate subspace \mathcal{L}_0 of dimension x . Then the index n takes the values $n = 1, \dots, D - x + 1$, where the x -fold eigenvalue belongs to the fixed index $n = p$ according to

$$E_{n_l}^0 \ni E_{p_l}^0 = E_p^0, \quad l = 1, \dots, x. \quad (\text{B.75})$$

The corresponding eigenvectors $|p_l^0\rangle$ of E_p^0 , which fulfil the equation

$$A_0|p_l^0\rangle = E_p^0|p_l^0\rangle, \quad l = 1, \dots, x, \quad (\text{B.76})$$

span the x -dimensional subspace \mathcal{L}_0 . We define a projector onto this subspace by

$$P_0 = \sum_{r=1}^x |p_r^0\rangle \langle p_r^0|. \quad (\text{B.77})$$

The complementary non-degenerate subspace \mathcal{L}'_0 incorporates the $D - x$ different eigenvalues

$$E_{n_l}^0 \ni E_k^0, \quad k \neq p, \quad k = 1, \dots, D - x. \quad (\text{B.78})$$

Due to the non-degeneracy we will omit the additional index l , so that the eigenvectors are given by the relation

$$A_0|k^0\rangle = E_k^0|k^0\rangle, \quad (\text{B.79})$$

and the analogue projector reads

$$P'_0 = \sum_{k'=1}^{D-x} |k'^0\rangle \langle k'^0|. \quad (\text{B.80})$$

The projectors P_0, P'_0 fulfil the usual relations as given in (2.27) for the chiral projectors $P_{L,R}$.

Normalisation of the Eigenvectors

The eigenvectors of a symmetric matrix are orthogonal for different eigenvalues. Supposing that the degeneracy of \mathcal{L}_0 is lifted at some order in perturbation theory, the following equation holds

$$\langle n_{l'} | n_l \rangle = 0, \quad \text{for } n \neq n' \quad \text{and/or} \quad l \neq l'. \quad (\text{B.81})$$

Expanding the eigenvectors into powers of ϵ and taking the limit $\epsilon \rightarrow 0$, we require that the orthogonality has to remain

$$\langle n_{l'}^0 | n_l^0 \rangle = 0, \quad \text{for } n \neq n' \quad \text{and/or} \quad l \neq l'. \quad (\text{B.82})$$

For convenience, we normalise the unperturbed eigenvectors $|n_l^0\rangle$ according to

$$\langle n_{l'}^0 | n_l^0 \rangle = \delta_{n'l} \delta_{l'l}. \quad (\text{B.83})$$

Furthermore, we impose the condition $\langle n_l^0 | n_l \rangle \stackrel{!}{=} 1$

$$\langle n_l^0 | n_l \rangle \stackrel{(\text{B.72})}{=} \langle n_l^0 | (|n_l^0\rangle + \epsilon |n_l^1\rangle + \epsilon^2 |n_l^2\rangle + \dots) = 1 + \epsilon \langle n_l^0 | n_l^1 \rangle + \epsilon^2 \langle n_l^0 | n_l^2 \rangle + \dots, \quad (\text{B.84})$$

from which we conclude that

$$\langle n_l^0 | n_l^i \rangle = \delta_{0i}. \quad (\text{B.85})$$

This means that the higher-order corrections are orthogonal to the associated unperturbed eigenvector. The normalisation conditions (B.83) and (B.85), involving the eigenvectors of the non-degenerate subspace, can be rewritten as

$$\langle k'^0 | k^0 \rangle = \delta_{k'k}, \quad \langle k^0 | k^i \rangle = \delta_{0i}. \quad (\text{B.86})$$

The corresponding relations for the eigenvectors spanning the degenerate subspace read

$$\langle p_r^0 | p_l^0 \rangle = \delta_{rl}, \quad \langle p_l^0 | p_l^i \rangle = \delta_{0i}. \quad (\text{B.87})$$

$\Delta_{k'}^1$ and Contributions to $|k^1\rangle$ and $|p_l^1\rangle$ within the Non-Degenerate Subspace

Applying the projector P'_0 onto (B.73), we receive the following eigenvalue equation to the first-order in perturbation theory ($m = 1$)

$$P'_0(E_{n_l}^0 - E_{k'}^0) |n_l^1\rangle = P'_0(A_1 - \Delta_{n_l}^1) |n_l^0\rangle. \quad (\text{B.88})$$

For the case of $E_{n_l}^0 = E_{k'}^0$ ($n_l = k'$), the first-order corrections $\Delta_{k'}^1$ to the eigenvalues $E_{k'}^0$ can then be calculated as

$$\Delta_{k'}^1 = \langle k'^0 | A_1 | k'^0 \rangle. \quad (\text{B.89})$$

For the inequality $E_{n_l}^0 \neq E_{k'}^0$, there are two different possibilities for the equation

$$\sum_{k'=1}^{D-x} (E_{n_l}^0 - E_{k'}^0) |k'^0\rangle \langle k'^0 | n_l^1 \rangle = \sum_{k'=1}^{D-x} |k'^0\rangle \langle k'^0 | (A_1 - \Delta_{n_l}^1) | n_l^0 \rangle \quad (\text{B.90})$$

to be true. Either $E_{n_l}^0 = E_k^0 \neq E_{k'}^0$ is an eigenvalue of the non-degenerate subspace or $E_{n_l}^0 = E_{p_l}^0 \neq E_{k'}^0$ is an eigenvalue of the degenerate subspace.

For the first case $E_{n_l}^0 = E_k^0 \neq E_{k'}^0$ it follows the contribution to the first-order correction of the eigenvector k^0 within the non-degenerate subspace

$$\sum_{k' \neq k, k'=1}^{D-x} |k'^0\rangle \langle k'^0 | k^1 \rangle \stackrel{\text{(B.86)}}{=} \sum_{k'=1}^{D-x} |k'^0\rangle \langle k'^0 | k^1 \rangle = \sum_{k' \neq k, k'=1}^{D-x} |k'^0\rangle \frac{\langle k'^0 | A_1 | k^0 \rangle}{(E_k^0 - E_{k'}^0)}. \quad (\text{B.91})$$

For $E_{n_l}^0 = E_{p_l}^0$ one obtains the first-order correction to the eigenvector p_l^0 with respect to the non-degenerate subspace

$$\sum_{k'=1}^{D-x} |k'^0\rangle \langle k'^0 | p_l^1 \rangle = \sum_{k'=1}^{D-x} |k'^0\rangle \frac{\langle k'^0 | A_1 | p_l^0 \rangle}{(E_{p_l}^0 - E_{k'}^0)}. \quad (\text{B.92})$$

$\Delta_{p_r}^1$ and Contributions to $|k^1\rangle$ and $|p_l^1\rangle$ within the Degenerate Subspace

Applying in an analogous manner the projector P_0 onto (B.73), the corresponding equation for $m = 1$ reads

$$P_0(E_{n_l}^0 - E_{p_r}^0)|n_l^1\rangle = P_0(A_1 - \Delta_{n_l}^1)|n_l^0\rangle. \quad (\text{B.93})$$

Assuming $E_{n_l}^0 = E_{p_l}^0 (= E_{p_r}^0)$ and using the normalisation condition (B.87), one can conclude that

$$\sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | (E_{p_l}^0 - E_{p_r}^0) | p_l^1 \rangle = 0 = \sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | A_1 | p_l^0 \rangle - \sum_{r=1}^x |p_r^0\rangle \Delta_{p_l}^1 \delta_{rl}. \quad (\text{B.94})$$

Comparing the above coefficients, the following equation has to hold

$$\Delta_{p_l}^1 \delta_{rl} = \langle p_r^0 | A_1 | p_l^0 \rangle. \quad (\text{B.95})$$

The Kronecker-Delta on the left-hand side of (B.95) indicates that one has to find a basis of eigenvectors spanning the degenerate subspace, in which the elements of A_1 are diagonal. Within this new basis, whereby we make the change of the notation implicit, the first-order corrections are given by the diagonal elements of A_1 :

$$\Delta_{p_r}^1 = \langle p_r^0 | A_1 | p_r^0 \rangle. \quad (\text{B.96})$$

For $E_{n_l}^0 = E_k^0 (\neq E_{p_r}^0)$ equation (B.93) reads

$$\sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | (E_k^0 - E_{p_r}^0) | k^1 \rangle = \sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | (A_1 - \Delta_k^1) | k^0 \rangle = \sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | A_1 | k^0 \rangle, \quad (\text{B.97})$$

which yields the first-order contributions to $|k^1\rangle$ from the degenerate subspace

$$\sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | k^1 \rangle = \sum_{r=1}^x |p_r^0\rangle \frac{\langle p_r^0 | A_1 | k^0 \rangle}{E_k^0 - E_{p_r}^0}. \quad (\text{B.98})$$

As the unperturbed eigenvectors $|k'^0\rangle, |p_r^0\rangle$ span a complete orthonormal system of the D -dimensional space, we can interpret $|\tilde{k}^1\rangle$ as the basis-transformed vector

$$|\tilde{k}^1\rangle = \sum_{k=1}^{D-x} |k'^0\rangle \langle k'^0|k^1\rangle + \sum_{r=1}^x |p_r^0\rangle \langle p_r^0|k^1\rangle, \quad (\text{B.99})$$

where we will again make the change of notation implicit and neglect the tilde in the following. With the help of the latter formula we can summarise the two contributions to $|k^1\rangle$ obtained in (B.91) and (B.98):

$$|k^1\rangle = \sum_{k' \neq k, k'=1}^{D-x} \frac{\langle k'^0|A_1|k^0\rangle}{(E_k^0 - E_{k'}^0)} |k'^0\rangle + \sum_{r=1}^x \frac{\langle p_r^0|A_1|k^0\rangle}{(E_k^0 - E_{p_r}^0)} |p_r^0\rangle. \quad (\text{B.100})$$

In (B.94) we have seen that the operator $(E_{p_l}^0 - A_0)$ acting on the degenerate subspace $\langle p_r^0|$ is singular. Thus, we cannot invert it in order to get the contributions to $|p_l^1\rangle$ from the eigenvectors which span the degenerate subspace $|p_r^0\rangle, r \neq l$. However, one can define a ‘‘pseudo’’-projector according to

$$\tilde{P}'_0 = \frac{P'_0}{E_{p_r}^0 - E_{k'}^0} = \sum_{k'} \frac{|k'^0\rangle \langle k'^0|}{E_{p_r}^0 - E_{k'}^0} \quad (\text{B.101})$$

and apply it onto (B.73) for $m = 1$:

$$P'_0 \frac{E_{n_l}^0 - E_{k'}^0}{E_{p_r}^0 - E_{k'}^0} |n_l^1\rangle = \tilde{P}'_0 (A_1 - \Delta_{n_l}^1) |n_l^0\rangle = \tilde{P}'_0 A_1 |n_l^0\rangle. \quad (\text{B.102})$$

For $E_{n_l}^0 = E_{p_l}^0 (= E_p^0 = E_{p_r}^0)$ and using $P'_0 = 1 - \sum_{r=1}^x |p_r^0\rangle \langle p_r^0|$, it follows

$$|p_l^1\rangle = \sum_{r=1}^x |p_r^0\rangle \underbrace{\langle p_r^0|p_l^1\rangle}_{\text{unknown coefficients}} + \tilde{P}'_0 A_1 |p_l^0\rangle. \quad (\text{B.103})$$

These unknown coefficients will appear in the calculation of the second-order corrections to the eigenvalues as we will see in the next subsection.

$\Delta_{k'}^2$ and $\Delta_{p_r}^2$

The second-order corrections to the eigenvalues of the non-degenerate subspace follow from (B.73) with $m = 2$,

$$(E_{n_l}^0 - A_0) |n_l^2\rangle = (A_1 - \Delta_{n_l}^1) |n_l^1\rangle - \Delta_{n_l}^2 |n_l^0\rangle, \quad (\text{B.104})$$

by applying the projector P'_0 on both sides for the case $E_{n_l}^0 = E_{k'}^0$:

$$\Delta_{k'}^2 = \langle k'^0|A_1|k'^1\rangle. \quad (\text{B.105})$$

Inserting the contribution to $|k^1\rangle$, which was derived for the eigenvalues of the non-degenerate subspace $E_{n_l}^0 = E_k^0$, yields for $k = k'$

$$\Delta_{k'}^2 = \sum_{r=1}^x \frac{\langle k'^0 | A_1 | p_r^0 \rangle \langle p_r^0 | A_1 | k'^0 \rangle}{E_{k'}^0 - E_{p_r}^0} = \frac{\langle k'^0 | A_1 P_0 A_1 | k'^0 \rangle}{E_{k'}^0 - E_{p_r}^0}. \quad (\text{B.106})$$

Analogously, applying the projector P_0 on (B.104) leads to

$$P_0(E_{n_l}^0 - E_{p_r}^0) |n_l^2\rangle = P_0(A_1 - \Delta_{n_l}^1) |n_l^1\rangle - P_0 \Delta_{n_l}^2 |n_l^0\rangle. \quad (\text{B.107})$$

For the degenerate eigenvalues $E_{n_l}^0 = E_{p_l}^0$ it follows, that

$$\Delta_{p_l}^1 \sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | p_l^1 \rangle + \Delta_{p_l}^2 \sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | p_l^0 \rangle = \sum_{r=1}^x |p_r^0\rangle \langle p_r^0 | A_1 | p_l^1 \rangle. \quad (\text{B.108})$$

Using the normalisation conditions, the coefficients of $|p_r^0\rangle$ for each r have to fulfil

$$\begin{aligned} \Delta_{p_l}^1 \langle p_r^0 | p_l^1 \rangle + \Delta_{p_l}^2 \delta_{rl} &= \langle p_r^0 | A_1 | p_l^1 \rangle \stackrel{(\text{B.103})}{=} \sum_{r'=1}^x \langle p_r^0 | A_1 | p_{r'}^0 \rangle \langle p_{r'}^0 | p_l^1 \rangle + \langle p_r^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle \\ &\stackrel{(\text{B.95})}{=} \sum_{r'=1}^x \Delta_{p_{r'}}^1 \delta_{r'r} \langle p_{r'}^0 | p_l^1 \rangle + \langle p_r^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle \\ &= \Delta_{p_r}^1 \langle p_r^0 | p_l^1 \rangle + \langle p_r^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle. \end{aligned} \quad (\text{B.109})$$

Concerning (B.109), we distinguish between the following three cases

$$l = r : \Delta_{p_l}^2 = \langle p_l^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle = \frac{\langle p_l^0 | A_1 P'_0 A_1 | p_l^0 \rangle}{E_{p_l}^0 - E_{k'}^0}, \quad (\text{B.110})$$

$$l \neq r, \Delta_{p_l}^1 = \Delta_{p_r}^1 : \Delta_{p_l}^2 \delta_{rl} = \langle p_r^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle, \quad (\text{B.111})$$

$$l \neq r, \Delta_{p_l}^1 \neq \Delta_{p_r}^1 : \langle p_r^0 | p_l^1 \rangle = \frac{\langle p_r^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle}{\Delta_{p_l}^1 - \Delta_{p_r}^1}. \quad (\text{B.112})$$

(B.110) gives us the second-order corrections to the eigenvalues. In (B.111) the degeneracy is not lifted at first-order perturbation theory and $A_1 \tilde{P}'_0 A_1$ has to be diagonalised in the degenerate subspace according to the argument given above for the first-order correction of the eigenvectors. The unknown coefficients drop out of the last equation (B.112), corresponding to the case in which the degeneracy has removed. Thus we finally receive the first-order corrections to the eigenvectors of the degenerate subspace:

$$|p_l^1\rangle = \sum_{r \neq l, r=1}^x |p_r^0\rangle \frac{\langle p_r^0 | A_1 \tilde{P}'_0 A_1 | p_l^0 \rangle}{\Delta_{p_l}^1 - \Delta_{p_r}^1} + \sum_{k'} |k'^0\rangle \frac{\langle k'^0 | A_1 | p_l^0 \rangle}{E_{p_r}^0 - E_{k'}^0}. \quad (\text{B.113})$$

We will give an explicit example of the usage of the above formulae in the next section.

B.6 Analytic Diagonalisation of the Hierarchical Matrix

The aim of this section is to diagonalise the symmetric matrix given in (5.70) by taking advantage of its hierarchical structure. We discuss two different perturbative approaches and demonstrate that both supply the same result.

Analytic Diagonalisation I: “Direct” Calculation

The characteristic polynomial $P(\lambda)$, following from the characteristic equation

$$P(\lambda) = \text{Det}(A - \lambda \mathbf{1}) = 0, \quad (\text{B.114})$$

can be solved analytically up to the fourth power, e.g. with the solutions proposed by Cardano [220]. In particular, for a 3×3 matrix A , the characteristic equation is given by the cubic form

$$P(\lambda) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0. \quad (\text{B.115})$$

As we are only interested in the solution with real eigenvalues, we follow the procedure proposed in [221] and give a short summary of the relevant formulae therein.

The solutions of (B.115), which correspond to the eigenvalues of A , can be calculated by

$$\lambda_i = \frac{\sqrt{p}}{3}x_i - \frac{1}{3}c_2, \quad i = 1, 2, 3 \quad \text{with} \quad p = c_2^2 - 3c_1, \quad (\text{B.116})$$

where the explicit expressions x_i depend on the sign of the parameter $q = -\frac{27}{2}c_0 - c_2^3 + \frac{9}{2}c_2c_1$. In our case, with the entries (5.81) and (5.87), q is negative ($q = -1 + \mathcal{O}(\epsilon)$) and

$$x_1 = -2 \cos \phi, \quad x_2 = \cos \phi - \sqrt{3} \sin \phi, \quad x_3 = \cos \phi + \sqrt{3} \sin \phi, \quad (\text{B.117})$$

with

$$\phi = \frac{1}{3} \arctan \left[\frac{1}{q} \sqrt{27 \left(\frac{1}{4} c_1^2 (p - c_1) + c_0 \left(q + \frac{27}{4} c_0 \right) \right)} \right]. \quad (\text{B.118})$$

Inserting the entries of A and expanding up to $\mathcal{O}(\epsilon^2)$, the eigenvalues are found to be

$$\begin{aligned} \lambda_1 &= A_{11}M^2\epsilon - (A_{12}^2 + A_{13}^2)M^2\epsilon^2, \\ \lambda_{2,3} &= M^2 + \frac{M^2}{2}(A_{22} + A_{33} \pm B)\epsilon \\ &\quad + \frac{M^2}{2B}(\pm 4A_{12}A_{13}A_{23} + A_{12}^2(B \pm F) + A_{13}^2(B \mp F))\epsilon^2. \end{aligned} \quad (\text{B.119})$$

The abbreviations introduced in (B.119) stand for

$$F = A_{22} - A_{33}, \quad B = \sqrt{4A_{23}^2 + F^2}, \quad \text{with} \quad B^2 > F^2, \quad (\text{B.120})$$

which implies

$$A_{23} = \text{sgn}[A_{23}] \frac{1}{2} \sqrt{B^2 - F^2}. \quad (\text{B.121})$$

In the 3-dimensional space, the associated eigenvectors v_{λ_i} can be calculated efficiently by using cross products [221]. With A^j denoting the j -th column of A and e_j the j -th unit vector, the real eigenvectors result from the formula

$$v_{\lambda_i} = (A^1 - \lambda_i e_1) \times (A^2 - \lambda_i e_2). \quad (\text{B.122})$$

In order to build up the orthogonal transformation matrix $G_A^T = (v_{\lambda_1, \text{norm}}, v_{\lambda_2, \text{norm}}, v_{\lambda_3, \text{norm}})$, the eigenvectors in (B.122) still have to be normalised ($v_{\lambda_i, \text{norm}} = N_{\lambda_i} v_{\lambda_i}$). For the concrete matrix A , all normalisation factors N_{λ_i} are proportional to $1/\epsilon$ and thus, knowing the eigenvectors $v_{\lambda_i, \text{norm}}$ up to $\mathcal{O}(\epsilon)$, requires to calculate the eigenvalues up to $\mathcal{O}(\epsilon^2)$.

Finally, the normalised eigenvectors can be summarised by

$$\begin{aligned} v_{\lambda_1, \text{norm}}^T &= \left(-1, \quad A_{12} \epsilon, \quad A_{13} \epsilon \right), \\ v_{\lambda_2, \text{norm}}^T &= \frac{\left((2A_{12}A_{23} + (B-F)A_{13}) \epsilon, \quad 2A_{23} - \frac{(B-F)X}{B^2} \epsilon, \quad (B-F) + \frac{2A_{23}X}{B^2} \epsilon \right)}{\sqrt{2B(B-F)}}, \\ v_{\lambda_3, \text{norm}}^T &= \frac{\left((-2A_{13}A_{23} + (B-F)A_{12}) \epsilon, \quad (B-F) + \frac{2A_{23}X}{B^2} \epsilon, \quad -2A_{23} + \frac{(B-F)X}{B^2} \epsilon \right)}{\sqrt{2B(B-F)}}, \end{aligned} \quad (\text{B.123})$$

where we have used the short-hand notation

$$X = FA_{12}A_{13} + A_{23}(A_{13}^2 - A_{12}^2). \quad (\text{B.124})$$

Analytic Diagonalisation II: Rayleigh–Schrödinger Perturbation Theory

Due to the implemented hierarchy between the various entries, we may also use the algorithm of Rayleigh–Schrödinger, introduced in Section B.5, to calculate the eigenvalues and eigenvectors. As discussed there, the first step is to define the basis of the unperturbed eigenvectors corresponding to the unperturbed eigenvalue problem

$$A_0 |n_i^0\rangle = E_{n_i}^0 |n_i^0\rangle. \quad (\text{B.125})$$

In our example A_0 is already diagonal and the eigenvalues can be read off. One can identify a two-fold ($x = 2$) degeneracy at zeroth order in perturbation theory, i.e. the two different eigenvalues $E_1^0 = 0$ and $E_{2_1}^0 = E_{2_2}^0 = E_2^0 = M^2$ belong to three different eigenvectors $|1^0\rangle$, $|2_1^0\rangle$ and $|2_2^0\rangle$. While the eigenvector, corresponding to the projector onto the non-degenerate subspace (see (B.80) with $k' = 1$)

$$P'_0 = |1^0\rangle\langle 1^0|, \quad (\text{B.126})$$

is given by $|1^0\rangle = (1, 0, 0)^T$, the basis of eigenvectors, which define the projector onto the two-dimensional degenerate subspace (see (B.77) with $p = 2$ and $r = 1, 2$)

$$P_0 = |2_1^0\rangle\langle 2_1^0| + |2_2^0\rangle\langle 2_2^0|, \quad (\text{B.127})$$

has to be found by diagonalising the 2-dimensional submatrix of A_1 (see also [219] for details). Normalising the set of unperturbed eigenvectors

$$\begin{aligned} |1^0\rangle^T &= \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}, \\ |2_1^0\rangle^T &= \frac{\begin{pmatrix} 0, & 2A_{23}, & (B-F) \end{pmatrix}}{\sqrt{2B(B-F)}}, \\ |2_2^0\rangle^T &= \frac{\begin{pmatrix} 0, & (B-F), & -2A_{23} \end{pmatrix}}{\sqrt{2B(B-F)}}, \end{aligned} \quad (\text{B.128})$$

according to

$$\langle n_l^0 | n_l^0 \rangle = \delta_{nn'} \delta_{ll'}, \quad (\text{B.129})$$

implies that all higher-order corrections $|n_l^i\rangle$ with $i = 1, 2, \dots$ are orthogonal to the unperturbed eigenvectors ($\langle n_l^0 | n_l^i \rangle = 0$). As the eigenvectors $|2_1^0\rangle, |2_2^0\rangle$ form the columns of an orthogonal rotation matrix in the 2-dimensional subspace, it is convenient to define the corresponding rotation angle ξ

$$\sin \xi := \frac{2|A_{23}|}{\sqrt{2B(B-F)}} \stackrel{(\text{B.121})}{=} \sqrt{\frac{1}{2} + \frac{F}{2B}}, \quad \cos \xi := \frac{(B-F)}{\sqrt{2B(B-F)}} = \sqrt{\frac{1}{2} - \frac{F}{2B}}, \quad (\text{B.130})$$

where ξ lies in the first quadrant as $|B| > |F|$. Using the basis of zeroth-order eigenvectors (B.128), the $\mathcal{O}(\epsilon)$ contributions to the eigenvalues (B.89), (B.96) are given by

$$\begin{aligned} \Delta_1^1 &= \langle 1^0 | A_1 | 1^0 \rangle = M^2 A_{11}, \\ \Delta_{2_1,2_2}^1 &= \langle 2_{1,2}^0 | A_1 | 2_{1,2}^0 \rangle = \frac{1}{2} M^2 (A_{22} + A_{33} \pm B). \end{aligned} \quad (\text{B.131})$$

Obviously, if $B \neq 0$ the degeneracy of the eigenvalues is removed at first-order perturbation theory. As mentioned in the previous subsection as well as in the Section B.5, one has to know the eigenvalues up to second order to be able to calculate the first-order corrections to the unperturbed eigenvectors consistently. Including the second-order corrections

$$\begin{aligned} \Delta_1^2 &\stackrel{(\text{B.106})}{=} -M^2 (A_{12}^2 + A_{13}^2), \\ \Delta_{2_1,2_2}^2 &\stackrel{(\text{B.110})}{=} \frac{M^2}{2B} (\pm 4A_{23}A_{12}A_{13} + (B \pm F)A_{12}^2 + (B \mp F)A_{13}^2), \end{aligned} \quad (\text{B.132})$$

the eigenvalues up to $\mathcal{O}(\epsilon^2)$ are equal to the formulae given in (B.119). Using the expression for A_{23} in terms of B and F (B.121), the second-order contributions $\Delta_{2_1,2_2}^2$ to the eigenvalues can then be rewritten as

$$\Delta_{2_1,2_2}^2 = \frac{M^2}{2B} (\sqrt{B \pm F} A_{12} \operatorname{sgn}[A_{23}] \pm \sqrt{B \mp F} A_{13})^2. \quad (\text{B.133})$$

The first-order correction to the eigenvector $|1^0\rangle$ can be calculated via the formula

$$|1^1\rangle^T \stackrel{(\text{B.100})}{=} \sum_{r=1}^2 \frac{\langle 2_r^0 | A_1 | 1^0 \rangle}{(E_1^0 - E_{2_r}^0)} |2_r^0\rangle^T = \begin{pmatrix} 0, & -A_{12}, & -A_{13} \end{pmatrix}. \quad (\text{B.134})$$

Note that there is no further contribution within the non-degenerate subspace itself, as it is only 1-dimensional. However, there are two different contributions to the first-order corrections of the unperturbed eigenvectors belonging to the degenerate subspace. One comes from the complementary space spanned by $|1^0\rangle$ according to

$$\begin{aligned} |2_1^1\rangle^T &\supset \frac{\langle 1^0 | A_1 | 2_1^0 \rangle}{E_{2_1}^0 - E_1^0} |1^0\rangle^T = \frac{(2A_{12}A_{23} + A_{13}(B-F))}{\sqrt{2B(B-F)}} \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}, \\ |2_2^1\rangle^T &\supset \frac{\langle 1^0 | A_1 | 2_2^0 \rangle}{E_{2_2}^0 - E_1^0} |1^0\rangle^T = \frac{(-2A_{13}A_{23} + A_{12}(B-F))}{\sqrt{2B(B-F)}} \begin{pmatrix} 1, & 0, & 0 \end{pmatrix}, \end{aligned} \quad (\text{B.135})$$

which is naturally orthogonal with respect to the unperturbed eigenvector. The second contribution stems from corrections within the degenerate subspace

$$\begin{aligned} |2_1^1\rangle^T &\supset |2_2^0\rangle \frac{\langle 2_2^0 | A_1 | 1^0 \rangle \langle 1^0 | A_1 | 2_1^0 \rangle}{(\Delta_{2_1}^1 - \Delta_{2_2}^1)(E_{2_2}^0 - E_1^0)} = \frac{\begin{pmatrix} 0, & -(B-F)X, & 2A_{23}X \end{pmatrix}}{B^2 \sqrt{2B(B-F)}}, \\ |2_2^1\rangle^T &\supset |2_1^0\rangle \frac{\langle 2_1^0 | A_1 | 1^0 \rangle \langle 1^0 | A_1 | 2_2^0 \rangle}{(\Delta_{2_2}^1 - \Delta_{2_1}^1)(E_{2_1}^0 - E_1^0)} = \frac{\begin{pmatrix} 0, & 2A_{23}X, & (B-F)X \end{pmatrix}}{B^2 \sqrt{2B(B-F)}}. \end{aligned} \quad (\text{B.136})$$

Note that the above contributions are indeed orthogonal to the corresponding unperturbed eigenvectors $|2_{1,2}^0\rangle$, as they should due to the normalisation condition (B.129).

Summarising all contributions within the Rayleigh–Schrödinger perturbation theory, the eigenvectors up to $\mathcal{O}(\epsilon)$ are represented by

$$\begin{aligned} |1\rangle^T &= \begin{pmatrix} 1, & -A_{12}\epsilon, & -A_{13}\epsilon \end{pmatrix}, \\ |2_1\rangle^T &= \frac{\begin{pmatrix} (2A_{12}A_{23} + A_{13}(B-F))\epsilon, & 2A_{23} - \frac{(B-F)X}{B^2}\epsilon, & (B-F) + \frac{2A_{23}X}{B^2}\epsilon \end{pmatrix}}{\sqrt{2B(B-F)}}, \\ |2_2\rangle^T &= \frac{\begin{pmatrix} (-2A_{13}A_{23} + A_{12}(B-F))\epsilon, & (B-F) + \frac{2A_{23}X}{B^2}\epsilon, & -2A_{23} + \frac{(B-F)X}{B^2}\epsilon \end{pmatrix}}{\sqrt{2B(B-F)}}, \end{aligned} \quad (\text{B.137})$$

which coincide with (B.123). Introducing the definitions

$$\sin \chi := -\text{sgn}[A_{23}] \sin \xi + \frac{X}{B^2} \cos \xi \epsilon, \quad \cos \chi := \cos \xi + \frac{X}{B^2} \text{sgn}[A_{23}] \sin \xi \epsilon, \quad (\text{B.138})$$

and $\text{sgn}[A_{23}] \sin \xi = \sin[\pm\xi]$ for $A_{23} \gtrless 0$, (B.137) can be brought into the compact form

$$\begin{aligned} |1\rangle^T &= \begin{pmatrix} 1, & -A_{12}\epsilon, & -A_{13}\epsilon \end{pmatrix}, \\ |2_1\rangle^T &= \begin{pmatrix} (A_{12} \sin[\pm\xi] + A_{13} \cos \xi) \epsilon, & -\sin \chi, & \cos \chi \end{pmatrix}, \\ |2_2\rangle^T &= \begin{pmatrix} (-A_{13} \sin[\pm\xi] + A_{12} \cos \xi) \epsilon, & \cos \chi, & \sin \chi \end{pmatrix}. \end{aligned} \quad (\text{B.139})$$

We showed that both methods provide the same results up to the given order in the ϵ expansion. However, as the second one uses the expansion in the small parameter already from the beginning, the calculation is much more transparent.

B.7 Basis Transformation of the Effective Hamiltonian $\mathcal{H}_{\text{eff}}^{\Delta S=2}$

In this section we bring the effective Hamiltonian $\mathcal{H}_{\text{eff}}^{\Delta S=2}$ (5.159) into the basis where the operators are diagonal in colour space. Writing out the colour indices explicitly, $\mathcal{H}_{\text{eff}}^{\Delta S=2}$ reads

$$\begin{aligned} \mathcal{H}_{\text{eff}}^{\Delta S=2} = & \frac{1}{2 \left(M_{\text{KK}}^{(1)} \right)^2} \left[\left(\Delta_L^{sd} \right)^2 \left(\bar{s}_{L,\alpha} \gamma_\mu T_{\alpha\beta}^a d_{L,\beta} \right) \left(\bar{s}_{L,\rho} \gamma^\mu T_{\rho\sigma}^a d_{L,\sigma} \right) \right. \\ & + \left(\Delta_R^{sd} \right)^2 \left(\bar{s}_{R,\alpha} \gamma_\mu T_{\alpha\beta}^a d_{R,\beta} \right) \left(\bar{s}_{R,\rho} \gamma^\mu T_{\rho\sigma}^a d_{R,\sigma} \right) \\ & \left. + 2 \Delta_L^{sd} \Delta_R^{sd} \left(\bar{s}_{L,\alpha} \gamma_\mu T_{\alpha\beta}^a d_{L,\beta} \right) \left(\bar{s}_{R,\rho} \gamma^\mu T_{\rho\sigma}^a d_{R,\sigma} \right) \right]. \quad (\text{B.140}) \end{aligned}$$

Using the Fierz identities for the Gell-Mann matrices, which are given in terms of the $SU(3)_c$ generators $T_{\alpha\beta}^a = \lambda_{\alpha\beta}^a/2$

$$T_{\alpha\beta}^a T_{\rho\sigma}^a = \frac{1}{2} \left(\delta_{\alpha\sigma} \delta_{\beta\rho} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\rho\sigma} \right), \quad (\text{B.141})$$

as well as the Fierz identities for rearranging products of fermion bilinears [222]

$$\begin{aligned} & \bullet \left(\bar{s}_\alpha \gamma_\mu P_L d_\beta \right) \left(\bar{s}_\beta \gamma^\mu P_L d_\alpha \right) = \left(\bar{s}_\alpha \gamma_\mu P_L d_\alpha \right) \left(\bar{s}_\beta \gamma^\mu P_L d_\beta \right) = \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_L d \right), \\ & \bullet \left(\bar{s}_\alpha \gamma_\mu P_R d_\beta \right) \left(\bar{s}_\beta \gamma^\mu P_R d_\alpha \right) = \left(\bar{s}_\alpha \gamma_\mu P_R d_\alpha \right) \left(\bar{s}_\beta \gamma^\mu P_R d_\beta \right) = \left(\bar{s}_\mu \gamma_\mu P_R d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right), \\ & \bullet \left(\bar{s}_\alpha \gamma_\mu P_L d_\beta \right) \left(\bar{s}_\beta \gamma^\mu P_R d_\alpha \right) = -2 \left(\bar{s}_\alpha P_L d_\alpha \right) \left(\bar{s}_\beta P_R d_\beta \right) = -2 \left(\bar{s} P_L d \right) \left(\bar{s} P_R d \right), \end{aligned} \quad (\text{B.142})$$

the various terms in (B.140) can be reformulated:

$$\begin{aligned} & \bullet \left(\bar{s}_\alpha \gamma_\mu T_{\alpha\beta}^a P_L d_\beta \right) \left(\bar{s}_\rho \gamma^\mu T_{\rho\sigma}^a P_L d_\sigma \right) = \frac{N-1}{2N} \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_L d \right), \\ & \bullet \left(\bar{s}_\alpha \gamma_\mu T_{\alpha\beta}^a P_R d_\beta \right) \left(\bar{s}_\rho \gamma^\mu T_{\rho\sigma}^a P_R d_\sigma \right) = \frac{N-1}{2N} \left(\bar{s}_\mu \gamma_\mu P_R d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right), \\ & \bullet \left(\bar{s}_\alpha \gamma_\mu T_{\alpha\beta}^a P_L d_\beta \right) \left(\bar{s}_\rho \gamma^\mu T_{\rho\sigma}^a P_R d_\sigma \right) = - \left(\bar{s}_\mu \gamma_\mu P_R d \right) \left(\bar{s}_\mu \gamma^\mu P_L d \right) - \frac{1}{2N} \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right). \end{aligned} \quad (\text{B.143})$$

Inserting (B.143) into (B.140), the effective Hamiltonian in the new basis is given by

$$\begin{aligned} \mathcal{H}_{\text{eff}}^{\Delta S=2} = & \frac{1}{2 \left(M_{\text{KK}}^{(1)} \right)^2} \left[\left(\Delta_L^{sd} \right)^2 \frac{N-1}{2N} \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_L d \right) \right. \\ & + \left(\Delta_R^{sd} \right)^2 \frac{N-1}{2N} \left(\bar{s}_\mu \gamma_\mu P_R d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right) \\ & \left. + 2 \Delta_L^{sd} \Delta_R^{sd} \left(- \left(\bar{s} P_L d \right) \left(\bar{s} P_R d \right) - \frac{1}{2N} \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right) \right) \right]. \quad (\text{B.144}) \end{aligned}$$

Using the abbreviations already introduced in (5.160),

$$\begin{aligned} \mathcal{Q}_1^{VLL} &= \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_L d \right), & \mathcal{Q}_1^{VRR} &= \left(\bar{s}_\mu \gamma_\mu P_R d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right), \\ \mathcal{Q}_1^{LR} &= \left(\bar{s}_\mu \gamma_\mu P_L d \right) \left(\bar{s}_\mu \gamma^\mu P_R d \right), & \mathcal{Q}_2^{LR} &= \left(\bar{s} P_L d \right) \left(\bar{s} P_R d \right), \end{aligned} \quad (\text{B.145})$$

the effective Hamiltonian (B.144) is reexpressed by

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{1}{4 \left(M_{\text{KK}}^{(1)}\right)^2} \left[(\Delta_L^{sd})^2 \frac{N-1}{N} \mathcal{Q}_1^{VLL} + (\Delta_R^{sd})^2 \frac{N-1}{N} \mathcal{Q}_1^{VRR} - 4 \Delta_L^{sd} \Delta_R^{sd} \mathcal{Q}_2^{LR} - \frac{2}{N} \Delta_L^{sd} \Delta_R^{sd} \mathcal{Q}_1^{LR} \right]. \quad (\text{B.146})$$

Thus, the Wilson coefficients of

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{1}{4 \left(M_{\text{KK}}^{(1)}\right)^2} \left[C_1^{VLL} \mathcal{Q}_1^{VLL} + C_1^{VRR} \mathcal{Q}_1^{VRR} + C_1^{LR} \mathcal{Q}_1^{LR} + C_2^{LR} \mathcal{Q}_2^{LR} \right] \quad (\text{B.147})$$

with respect to the new basis are specified by

$$\begin{aligned} C_1^{VLL}(M_{\text{KK}}) &= \frac{N-1}{N} \left(\Delta_L^{sd}\right)^2 \stackrel{(N=3)}{=} \frac{2}{3} \left(\Delta_L^{sd}\right)^2 \\ C_1^{VRR}(M_{\text{KK}}) &= \frac{N-1}{N} \left(\Delta_R^{sd}\right)^2 \stackrel{(N=3)}{=} \frac{2}{3} \left(\Delta_R^{sd}\right)^2 \\ C_1^{LR}(M_{\text{KK}}) &= \frac{2}{N} \Delta_L^{sd} \Delta_R^{sd} \stackrel{(N=3)}{=} -\frac{2}{3} \Delta_L^{sd} \Delta_R^{sd} \\ C_2^{LR}(M_{\text{KK}}) &= -4 \Delta_L^{sd} \Delta_R^{sd}, \end{aligned} \quad (\text{B.148})$$

where we also give the results for three colour degrees of freedom ($N = 3$).

B.8 Effective 4D Feynman Rules: Two Examples

For demonstration we will calculate the 4D Feynman rules for the couplings of the fermion zero modes to the Z boson as well as the triple-gauge vertices involving the Z boson. After expanding the S-matrix, decomposing the field operators into Fourier series, carrying out all possible Wick contractions, the Feynman amplitude for the vertex is equal to the prefactor of the interaction Lagrangian of the involved particles up to a factor of i . The KK decomposition allows to obtain a 4D effective theory from the 5-dimensional full theory.

Zero Mode Fermion Couplings to the Z Boson

We begin with the calculation of the 3-dimensional diagonal coupling submatrix $[A_L^{2/3}(Z)]_{00}$ (upper left corner of $A_L^{2/3}(Z)$), which determines the coupling of the SM Z gauge boson to the SM up-type quarks corresponding to the vertex $\bar{q}_L^{u_i(0)} q_L^{u_i(0)} Z$. Being aware that the Z boson appears in X , W_L^3 , W_R^3 and that $q_L^{u_i}$ is embedded in the bidoublet ξ_{1L}^i , the relevant terms in $\mathcal{L}_{\text{fermion}}$ stemming from the covariant derivative D_M^1 look like

$$\mathcal{L}_{\text{int}}^{4\text{D}} \supset - \int_0^L dy \sqrt{G} e^{ky} \left(\bar{q}_L^{u_i} \gamma^\mu (g_X Q_X X_\mu) q_L^{u_i} + \bar{q}_L^{u_i} \gamma^\mu (g T_L^3 W_{L,\mu}^3) q_L^{u_i} + \bar{q}_L^{u_i} \gamma^\mu (g T_R^3 W_{R,\mu}^3) q_L^{u_i} \right). \quad (\text{B.149})$$

With the help of the formulae in (5.64) which give $W_{L,R}^3$ in terms of the physical fields, and the analogue relation for X

$$X = \cos \phi \cos \psi A - \cos \phi \sin \psi Z - \sin \phi Z_X, \quad (\text{B.150})$$

we can rewrite the equation (B.149), where we omit the terms including couplings to the photon:

$$\begin{aligned} \mathcal{L}_{\text{int}}^{4\text{D}} \supset & \int_0^L dy e^{-3ky} \bar{q}_L^{u_i} \gamma_\mu (g_X Q_X (\cos \phi \sin \psi Z^\mu + \sin \phi Z_X^\mu)) q_L^{u_i} \\ & - e^{-3ky} \bar{q}_L^{u_i} \gamma_\mu (g T_L^3 \cos \psi Z^\mu) q_L^{u_i} - e^{-3ky} \bar{q}_L^{u_i} \gamma_\mu \left(g T_R^3 \cos \phi Z_X^\mu - \frac{\sin^2 \psi}{\cos \psi} Z^\mu \right) q_L^{u_i}. \end{aligned} \quad (\text{B.151})$$

As discussed in Subsection 5.8.1, the integration over the extra dimension after the KK decomposition can be carried out explicitly for the couplings to gauge boson zero mode profiles. Thus, the couplings to the $Z^{(0)}$ mode in (B.151) simplify to the flavour-universal 4D effective couplings:

$$\left(\frac{g_X}{\sqrt{L}} Q_X \cos \phi \sin \psi - \frac{g}{\sqrt{L}} T_L^3 \cos \psi + \frac{g}{\sqrt{L} \cos \psi} T_R^3 \sin^2 \psi \right) \bar{q}_L^{u_i(0)} \gamma^\mu Z_\mu^{(0)} q_L^{u_i(0)}. \quad (\text{B.152})$$

Reexpressing Q_X by means of equation (5.51), the contributions can be summarised as

$$-\frac{g}{\sqrt{L} \cos \psi} (T_L^3 - \sin^2 \psi Q) \bar{q}_L^{u_i(0)} \gamma^\mu Z_\mu^{u_i(0)}, \quad (\text{B.153})$$

where we also used the fact that $Z^{(0)}$ corresponds to the SM Z boson up to small admixtures with Z_H and Z' after EWSB.

The couplings to the first KK mode $Z^{(1)}$ can be derived analogously by remembering that the appropriate overlap integral \mathcal{R}_{nk}^i , which involves the shape function $g(y)$ of (5.58), is defined in (5.138):

$$-\frac{g}{\sqrt{L} \cos \psi} \mathcal{R}_{00}^i(++)_L (T_L^3 - \sin^2 \psi Q) \bar{q}_L^{u_i(0)} \gamma^\mu Z_\mu^{(1)} q_L^{u_i(0)}. \quad (\text{B.154})$$

After EWSB, the KK mode $Z^{(1)}$ is rotated into its mass eigenstate and given as a linear combination of physical fields. Concentrating on the contribution $Z^{(1)} \ni -\epsilon \mathcal{I}_1^+ / \cos^2 \psi Z$, (B.154) contains the following couplings to the SM Z boson

$$\epsilon \frac{g}{\sqrt{L} \cos \psi} \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_{00}^i(++)_L (T_L^3 - \sin^2 \psi Q) \bar{q}_L^{u_i(0)} \gamma^\mu Z_\mu q_L^{u_i(0)}. \quad (\text{B.155})$$

Finally, the two contributions involving $Z_X^{(1)}$, which contains the SM Z boson according to $Z_X^{(1)} \ni \mathcal{I}_1^- \frac{\cos \phi}{\cos \psi} \epsilon Z$ after EWSB, lead to

$$-\epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- \mathcal{P}_{00}^i(++)_L (g T_R^3 \cos \phi - g_X Q_X \sin \phi) \bar{q}_L^{u_i(0)} \gamma^\mu Z_\mu q_L^{u_i(0)}, \quad (\text{B.156})$$

where the corresponding overlap integral \mathcal{P}_{nk}^i was introduced in (5.139) with the shape function $\tilde{g}(y)$ of $Z_X^{(1)}$ given in (5.59). Utilising $g_X = g \frac{\sin \phi}{\cos \phi}$, the equation can be rewritten by

$$-\epsilon \frac{\cos \phi}{\cos \psi} \frac{g}{\cos \phi} (T_R^3 - (T_R^3 + Q_X) \sin^2 \phi) \mathcal{I}_1^- \mathcal{P}_1^i(++)_L \bar{q}_L^{u_i(0)} \gamma^\mu Z_\mu q_L^{u_i(0)}. \quad (\text{B.157})$$

In summary, the couplings of the left-handed up-type zero mode quarks to the SM Z boson (corresponding to $i[A_L^{2/3}(Z)]_{00}$) read

$$-i\gamma^\alpha \left[g_Z(q^{u_i}) - \epsilon g_Z(q^{u_i}) \frac{1}{\cos^2 \psi} \mathcal{I}_1^+ \mathcal{R}_1^i(++)_L + \epsilon \frac{\cos \phi}{\cos \psi} \mathcal{I}_1^- g_{Z_X}(q^{u_i}) \mathcal{P}_1^i(++)_L \right], \quad (\text{B.158})$$

where the coupling constants are given by

$$\begin{aligned} g_Z(q^{u_i}) &= \frac{g}{\sqrt{L} \cos \psi} (T_L^3(q^{u_i}) - \sin^2 \psi Q(q^{u_i})), \\ g_{Z_X}(q^{u_i}) &= \frac{g}{\sqrt{L} \cos \phi} (T_R^3(q^{u_i}) - (T_R^3(q^{u_i}) + Q_X(q^{u_i})) \sin^2 \phi), \end{aligned} \quad (\text{B.159})$$

and the quantum numbers of q^{u_i} can be taken from (5.112). Adapting the formula (B.158) to the quantum numbers and representation index m of the other quarks, we reproduce the expressions as given in Table 5.1.

Effective 4D Feynman Rules for Triple-Gauge Vertices

In non-abelian gauge theories the presence of the gauge boson self-interaction term $gf^{abc}V_\mu^b V_\nu^c$ generates vertices with three gauge bosons — called triple-gauge vertices. For instance, the relevant terms for the $SU(2)_L$ gauge bosons are contained in the gauge kinetic term

$$\mathcal{L}_{\text{gauge}}^{4D} \ni - \int_0^L dy \frac{1}{4} L_{\mu\nu}^a L^{a,\mu\nu} \supset - \int_0^L dy \frac{1}{2} (\partial_\mu W_{L,\nu}^a - \partial_\nu W_{L,\mu}^a) (-g\epsilon^{abc} W_L^{b,\mu} W_L^{c,\nu}), \quad (\text{B.160})$$

where $\epsilon^{123} = 1$. Using the formulae (5.54) and (5.64), the Lagrangian (B.160) can be rewritten in terms of $W_{L,R}^\pm$, Z and A :

$$\begin{aligned} ig \int_0^L dy \left[& \left(-(\partial_\mu W_{L,\nu}^+ - \partial_\nu W_{L,\mu}^+) W_L^{-,\mu} + (\partial_\mu W_{L,\nu}^- - \partial_\nu W_{L,\mu}^-) W_L^{+,\mu} \right) (\cos \psi Z^\nu + \sin \psi A^\nu) \right. \\ & \left. + \cos \psi (\partial_\mu Z_\nu - \partial_\nu Z_\mu) W_L^{+,\nu} W_L^{-,\mu} + \sin \psi (\partial_\mu A_\nu - \partial_\nu A_\mu) W_L^{+,\nu} W_L^{-,\mu} \right]. \end{aligned} \quad (\text{B.161})$$

Inserting the relations $\partial_\mu W_{L,\nu}^+ = -ik_\mu W_{L,\nu}^+$, $\partial_\mu W_{L,\nu}^- = -ip_\mu W_{L,\nu}^-$ and $\partial_\mu Z_\nu = -iq_\mu Z_\nu$, where k, p, q denote the incoming momenta of W_L^+ , W_L^- , Z , the contribution involving the Z boson is given by

$$\bullet g \cos \psi \int_0^L dy [\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu] W_L^{+,\mu} W_L^{-,\nu} Z^\rho, \quad (\text{B.162})$$

and the analogue coupling to the photon by

$$\bullet g \sin \psi \int_0^L dy [\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu] W_L^{+,\mu} W_L^{-,\nu} A^\rho. \quad (\text{B.163})$$

Hence, the Dirac structure (DS) of all effective 4D Feynman diagrams, obtained after KK decomposition and integration over the fifth dimension, will be the same so that we introduce the following abbreviation

$$[\text{DS}]_{\mu\nu\rho} = [\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu]. \quad (\text{B.164})$$

For illustration, we only deduce the triple vertex involving the SM Z boson. Furthermore, we will neglect all $\mathcal{O}(\epsilon)$ contributions. In this limit, the SM Z boson equals the zero mode $Z^{(0)}$ and the 5D overlap integral reduces to the orthonormality condition of the gauge boson profiles ((B.51) for $s=2$)

$$\frac{g \cos \psi}{\sqrt{L}} [\text{DS}]_{\mu\nu\rho} \left(W_L^{(0)+,\mu} W_L^{(0)-,\nu} Z^\rho + W_L^{(1)+,\mu} W_L^{(1)-,\nu} Z^\rho \right). \quad (\text{B.165})$$

According to our chosen approximation, the following relations hold

$$W_L^{(0)\pm} \simeq W^\pm, \quad W_L^{(1)\pm} \simeq \cos \chi W_H^\pm - \sin \chi W'^\pm. \quad (\text{B.166})$$

Inserting them into (B.165), yields

$$\begin{aligned} \frac{g \cos \psi}{\sqrt{L}} [\text{DS}]_{\mu\nu\rho} \left(W^{+,\mu} W^{-,\nu} Z^\rho + \cos^2 \chi W_H^{+,\mu} W_H^{-,\nu} Z^\rho + \sin^2 \chi W'^{+,\mu} W'^{-,\nu} Z^\rho \right. \\ \left. - \cos \chi \sin \chi W_H^{+,\mu} W'^{-,\nu} Z^\rho - \cos \chi \sin \chi W'^{+,\mu} W_H^{-,\nu} Z^\rho \right). \quad (\text{B.167}) \end{aligned}$$

In addition, the $SU(2)_R$ gauge bosons have the following couplings to the Z boson:

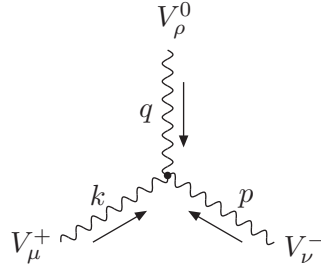
$$\begin{aligned} ig \int_0^L dy \left[\left(-(\partial_\mu W_{R,\nu}^+ - \partial_\nu W_{R,\mu}^+) W_R^{-,\mu} + (\partial_\mu W_{R,\nu}^- - \partial_\nu W_{R,\mu}^-) W_R^{+,\mu} \right) (-\sin \psi \sin \phi Z^\rho) \right. \\ \left. - \sin \psi \sin \phi (\partial_\mu Z_\nu - \partial_\nu Z_\mu) W_R^{+,\nu} W_R^{-,\mu} \right] \\ = -g \sin \psi \sin \phi \int_0^L dy [\text{DS}]_{\mu\nu\rho} W_R^{+,\mu} W_R^{-,\nu} Z^\rho. \quad (\text{B.168}) \end{aligned}$$

With $W_R^{(1)\pm} \simeq \sin \chi W_H^\pm + \cos \chi W'^\pm$, the above contribution, after performing the integration over the extra dimension, is given by

$$-\frac{g \sin \psi \sin \phi}{\sqrt{L}} [\text{DS}]_{\mu\nu\rho} \left(\sin^2 \chi W_H^{+,\mu} W_H^{-,\nu} Z^\rho + \cos^2 \chi W'^{+,\mu} W'^{-,\nu} Z^\rho \right) \quad (\text{B.169})$$

$$+ \sin \chi \cos \chi W_H^{+,\mu} W'^{-,\nu} Z^\rho + \sin \chi \cos \chi W'^{+,\mu} W_H^{-,\nu} Z^\rho. \quad (\text{B.170})$$

Summarising all terms contributing to a given vertex, we collect the respective couplings involving the Z boson in Table B.1 and illustrate the Feynman diagrams for the various triple-gauge vertices with Dirac structure $[\text{DS}]_{\mu\nu\rho}$ by



, where $V_\mu^+ = W_\mu^+, W_{H,\mu}^+, W_\mu'^+$, $V_\nu^- = W_\nu^-, W_{H,\nu}^-, W_\nu'^-$, $V_\rho^0 = Z_\rho$ and k, p, q are their incoming momenta.

Couplings to the Z boson	
W^+W^-Z	$i\frac{g}{\sqrt{L}}\cos\psi + \mathcal{O}(\epsilon^2)$
$W_H^+W^-Z$	$\mathcal{O}(\epsilon)$
W'^+W^-Z	$\mathcal{O}(\epsilon)$
$W_H^+W_H^-Z$	$i\frac{g}{\sqrt{L}}(\cos\psi\cos^2\chi - \sin\phi\sin\psi\sin^2\chi)$
$W'^+W_H^-Z$	$-i\frac{g}{\sqrt{L}}\sin\chi\cos\chi(\cos\psi + \sin\phi\sin\psi)$

Table B.1: Triple-gauge boson couplings to the Z boson. The coupling of $W^+W_H^-Z$ is equal to $W_H^+W^-Z$, the same is valid for $W^+W'^-Z$ and W'^+W^-Z as well as $W_H^+W'^-Z$ and $W'^+W_H^-Z$.

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Danksagung

- An erster Stelle gilt mein Dank Thorsten Feldmann für die intensive Zusammenarbeit und hervorragende Betreuung als Doktorvater. Darüber hinaus bin ich dankbar einen Menschen kennengelernt zu haben, der immer ein offenes Ohr und aufmunternde Worte für andere hat. Es würde mich sehr freuen, wenn wir uns auch in Zukunft noch über Physik, Musik und alles mögliche unterhalten.
- Als nächstes möchte ich mich bei Andrzej J. Buras bedanken, der mir die Möglichkeit gab, an seinem Lehrstuhl diese Doktorarbeit anfertigen zu können.
- Vielen Dank auch an André H. I. Hoang für die Zweitkorrektur dieser Arbeit.
- Tausend Dank an Björn Duling, mit dem ich nicht nur etliche Bahnen im Olybad zurücklegte, sondern der mir während der Promotion immer mit Rat und Tat zur Seite stand. Vielen Dank für Deine wertvollen Kommentare bei der Durchsicht dieser Arbeit. Ich hoffe, dass wir uns auch in Zukunft nicht aus den Augen verlieren werden.
- Vielen Dank an Selma Uhlig für die schönen gemeinsamen Stunden im Italienisch- und Bauchtanzkurs. Ich hoffe auch hier auf Fortsetzung.
- Meinen herzlichsten Dank an Felix Schwab für das “Verbessern” dieser Arbeit.
- Vielen Dank an die Kollegen, mit denen ich zusammen arbeiten durfte:
Thorsten Feldmann und Thomas Mannel im Rahmen des dMFV-Projekts, Monika Blanke, Andrzej J. Buras, Björn Duling und Katrin Gemmler in punkto Extradimensionen und Wolfgang Altmannshofer, Andrzej J. Buras, Diego Guadagnoli, David Straub im SUSY-GUT-Projekt.
- Danke auch an Stefan Recksiegel und Elke Krüger — ohne Eure Hilfe wäre oft gar nichts mehr weiter gegangen!
- Vielen Dank an Björn, Christoph, David, Katrin, Michael, Thorsten, Tillmann, Wolfgang und Yasutaka für viele unterhaltsame Gespräche innerhalb der Kaffeerrunde.
- Weiterhin gilt mein Dank meinen treuen Freunden die ich in den letzten Jahren vernachlässigt habe und die mir in den seltenen Treffen immer wieder in kürzester Zeit sehr nahe standen.
- Vor allem möchte ich meinen Eltern für ihre liebevolle moralische und finanzielle Unterstützung während der Zeit des gesamten Studiums und der Doktorarbeit danken. Ohne ihren Beistand wäre das Studium gar nicht möglich gewesen.
Vielen Dank liebe Mama, dass Du zu jeder Tages- und Nachtzeit seit meiner Geburt für mich da bist und in jeglicher Hinsicht an mich glaubst.
Vielen Dank auch an meinen Bruder Christian für seine Ehrlichkeit und Hilfsbereitschaft.

- Zum Schluss möchte ich mich bei meinem Freund Anton Poschenrieder bedanken, der mir in der Phase des Aufschreibens der Doktorarbeit immer wieder Mut zusprach. Vielen Dank für das x-fache Korrekturlesen dieser Arbeit und für das stetige Aufbauen meines Selbstvertrauens. Zudem vielen Dank für die schöne gemeinsame Zeit die wir bisher zusammen verbrachten — insbesondere die inspirierenden Wochen in Paris.