ON THE QOS FEASIBILITY REGION IN THE VECTOR BROADCAST CHANNEL

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ABSTRACT

We investigate the geometry of the feasible Quality of Service (QoS) region in the vector broadcast channel when the available transmit power is unbounded. It turns out that a complete description of the feasible QoS region attains its simplest form in the minimum mean square error (MMSE) domain although most of the literature handles feasibility in the SINR domain. As our main contribution, we show that the closure of the feasible MMSE region is a polytope corresponding to a hyper-cube that maybe is cropped by an additional half-space constraint. Interestingly, this half-space constraint is the only relevant one which separates feasibility from infeasibility and it reflects a lower bound on the sum MMSE. Under the assumption of regular channels, this lower bound does not depend on the channel realization but solely depends on the number of users and antennas deployed at the base station. Testing feasibility of given QoS targets is easily performed by first converting the QoS targets into upper bounds on the MMSEs and afterwards verifying that the sum of target MMSEs is larger than the difference between antennas at the base station and the number of users. The derived results can be used to decide whether a new user with given QoS requirements can be admitted to the system, and if not, how the requirements have to be adapted such that they become feasible.

***Index Terms*—** QoS region, feasibility check

1. INTRODUCTION

Minimizing the transmit power that is needed to satisfy given QoS requirements is a frequently arising optimization problem, see for example [1–5]. While such QoS constraints are usually given as lower bounds on the signal-to-interference-and-noise ratio (SINR), any other metric like MMSE and data rate under Gaussian signaling which follows from a one-to-one mapping of the SINR may be chosen. If at most as many single-antenna users are served as the base station has antennas and if the channel matrix is regular, arbitrary QoS requirements can be satisfied due to the existence of the zero-forcing solution, see [4]. However, when more users shall be served than degrees of freedom are available, feasibility cannot be guaranteed for any set of QoS requirements. In order to detect feasibility which is a prerequisite for the power minimization algorithm, an additional balancing algorithm has so far been used where the ratios of any two SINR targets are fixed and the individual absolute SINRs are afterwards maximized. In contrast to the power minimization problem, balancing is always feasible and can be used to determine whether a given set of target SINRs is feasible or not, see [6–11] for the more general SIR balancing without power limitation.

The disadvantage of this kind of detecting feasibility is that the balancing algorithm has to be executed once per set of SINR constraints. A modification of the QoS targets leads to the necessity of running the feasibility check again, which is computationally demanding. Moreover, the SIR balancing algorithm depends on the channel matrix, whereas we show that the feasibility region is independent of the specific channel realization as long as it satisfies the regular channel constraint. Finally, only little knowledge about the underlying geometric structure of the feasible QoS region is obtained by means of the balancing algorithm. For example, in [11, 12], the feasibility region is defined as the set of SI(N)R tuples for which the spectral radius of a scaled interference coupling matrix is not larger than one, from which almost no structural properties can be deduced. In [13], feasibility of given QoS requirements is investigated for a single-antenna CDMA system. Therein, the existence of codes satisfying the sum MSE inequality is derived. We generalize this and show that for arbitrary full rank codes, the inequality is sufficient.

2. SYSTEM AND CHANNEL MODEL

Given perfect channel state information, the vector broadcast channel (BC) and the dual vector multiple access channel (MAC) share the same MSE region under a sum power constraint, e.g. [14, 15]. Thus, we may describe the maximum feasible MMSE region of the BC in its dual MAC and feasibility can be detected there as well. The main advantage of the dual MAC is its simple description by the powers of the transmitting users, since the optimum receive beamformers can be computed independently and are known to maximize the receive SINR and can be chosen to minimize the MSE. In the dual MAC, the K user system is described by only K nonnegative real-valued scalars representing the virtual uplink powers, whereas K complex-valued N-dimensional vectors are needed in the BC for the transmit beamformers. Here, N denotes the number of antennas at the base station, and we define the set of user indices $\mathcal{K} := \{1, \ldots, K\}$. The transmit powers of the K users in the dual MAC are denoted by $p_1, \ldots, p_K$ and all data symbols are assumed to have unit variance. A frequency flat channel is assumed for the transmission from the users to the base station and the matrix $\boldsymbol{H} = [\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K] \in \mathbb{C}^{N \times K}$ contains the channel vectors of all K users as its columns. At the base station (receiver in the MAC), zero-mean Gaussian noise $\boldsymbol{\eta} \in \mathbb{C}^N$ with covariance matrix $\mathrm{E}[\boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{H}}] = \sigma^2 \mathbf{I}_N$ is added.

3. GEOMETRY OF THE FEASIBLE MSE REGION

Consider a two user scenario where the two channel vectors $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are colinear (e.g., if the base station has only $N = 1$ antenna). It can easily be shown that the upper boundary of the maximum feasible SINR region is given by [9]

$$\gamma_1 \gamma_2 = 1, \tag{1}$$

where γ_k denotes the SINR of user k. This boundary is only asymptotically achieved when $p_1/\sigma^2 \to \infty$ and $p_2/\sigma^2 \to \infty$. A generalization to more than two users is not straightforward in the SINR domain and does not feature such a simple relation as in (1). For example, when a third user is added to the single-antenna system, the feasible SINR region reads as

$$2\gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3 < 1, \tag{2}$$

see (21) at the end of this section. In the following, we will derive the exact description of the feasible region in the MSE domain for an arbitrary number of users. Throughout this paper, we assume that the channel vectors of any user subset $\mathcal{I} \subseteq \mathcal{K}$ with cardinality $1 \leq |\mathcal{I}| \leq K$ satisfy the rank relation

$$\mathrm{rank}(\boldsymbol{H}_{\mathcal{I}}) = \min\{|\mathcal{I}|, N\} \;\; \forall \mathcal{I} \subseteq \mathcal{K}. \tag{3}$$

The case of singular channels not fulfilling (3) is discussed in [16]. In (3), the matrix $\boldsymbol{H}_{\mathcal{I}}$ consists of the channel vectors of all users belonging to $\mathcal{I}$. Given these channel properties, we can prove the following theorem:

Theorem 3.1: *The closure of the feasible MMSE region in the vector broadcast channel with regular channels satisfying (3) is a polytope $\mathcal{P}$ whose bounding half-spaces are the individual box constraints $0 \leq \varepsilon_k \leq 1 \; \forall k \in \mathcal{K}$ and the sum MMSE constraint $\sum_{k\in\mathcal{K}} \varepsilon_k \geq K - N$. By means of a positive power allocation with finite sum power, any point belonging to the interior of the polytope can be achieved. For MMSEs equal to one no power is allocated to the respective user.*

From above theorem, we directly conclude the next corollary, which has already been proven in [4].

Corollary 3.2. *In the K-user vector broadcast channel obeying (3) with an $N \geq K$ antenna base station, arbitrary QoS requirements satisfying $0 < \varepsilon_k \leq 1 \; \forall k$ are feasible with finite sum power.*

For the proof of Theorem 3.1, we define $\boldsymbol{P} = \mathrm{diag}\{p_k\}_{k=1}^K$ as the diagonal matrix containing the powers of all users in the set $\mathcal{K}$. Then, the MMSE receive filter for user k reads as

$$\boldsymbol{g}_k^{\mathrm{T}} = \sqrt{p}_k \boldsymbol{h}_k^{\mathrm{H}} \left(\boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^{\mathrm{H}} + \sigma^2 \mathbf{I}_N\right)^{-1} \tag{4}$$

and achieves the minimum mean square error

$$\begin{aligned} \varepsilon_k &= 1 - p_k \boldsymbol{h}_k^{\mathrm{H}} \left(\boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^{\mathrm{H}} + \sigma^2 \mathbf{I}_N\right)^{-1} \boldsymbol{h}_k \\ &= \left[\left(\mathbf{I}_K + \sigma^{-2} \boldsymbol{P}^{\frac{1}{2}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{H} \boldsymbol{P}^{\frac{1}{2}}\right)^{-1}\right]_{k,k}. \end{aligned} \tag{5}$$

From the first line of (5), we can observe that applying MMSE receive filters according to (4) leads to the box constraints

$$0 \leq \varepsilon_k \leq 1 \;\; \forall k \in \mathcal{K} \tag{6}$$

which contribute to the polytope. A necessary condition for a particular user i to achieve the lower bound $\varepsilon_i = 0$ asymptotically with equivalence is that his power goes to infinity, i.e., $p_i \to \infty$. Given an antenna configuration with $N < K$, not all MMSEs can be chosen arbitrarily small *simultaneously*. With the individual MMSEs from (5), we can express the sum MMSE for arbitrary N via

$$\sum_{k\in\mathcal{K}} \varepsilon_k = K - N + \mathrm{tr}\left[\left(\mathbf{I}_N + \sigma^{-2} \boldsymbol{H}\boldsymbol{P}\boldsymbol{H}^{\mathrm{H}}\right)^{-1}\right] \tag{7}$$

resulting from the first line of (5). As the inverse matrix in (7) is positive definite, $N - \mathrm{rank}(\boldsymbol{H})$ is a lower bound on its trace. Thus, any nonnegative power allocation satisfies

$$\sum_{k\in\mathcal{K}} \varepsilon_k \geq K - \mathrm{rank}(\boldsymbol{H}) \tag{8}$$

with strict inequality for finite sum power, and with equality if all[1] powers $p_1, \ldots, p_K$ raise to infinity. Under the regular channels assumption (3), the rank of the channel matrix is given by $\mathrm{rank}(\boldsymbol{H}) = \min\{N, K\}$, and the antenna configuration $N \geq K$ does not entail any limitations on the sum MSE since the resulting lower bound $\sum_{k\in\mathcal{K}} \varepsilon_k \geq 0$ from (8) is already included in the box constraints (6). The feasible MMSE region is then completely described by (6). Therefore, the case $N \geq K$ with regular channels allows for arbitrary QoS requirements, which nevertheless have to meet (6). In particular, any nonnegative SINR tuple is feasible since the maximum SINR γ_k and the minimum MSE ε_k are related via $\gamma_k = \frac{1}{\varepsilon_k} - 1$ and ε_k can be made arbitrarily close to zero. However, when $K > N$, the polytope in (6) is cropped by the lower bound on the sum MSE in (8), which in conjunction with $\mathrm{rank}(\boldsymbol{H}) = \min\{N, K\} = N$ simplifies to

$$\sum_{k\in\mathcal{K}} \varepsilon_k > K - N \;\; \text{for } \|\boldsymbol{p}\|_1 < \infty. \tag{9}$$

This obviously limits the set of feasible QoS requirements. Note that the lower bound $K - N$ in (9) can be achieved with equality when $p_k \to \infty \, \forall k \in \mathcal{K}$.

So far, we have proven that any positive power allocation with finite sum power achieves an MMSE tuple inside the polytope, see (9) and (6) with strict inequality for finite sum power. To complete the proof for Theorem 3.1, we also have to show the converse, namely that there exists a power allocation $\boldsymbol{p}$ for any desired MMSE tuple belonging to $\mathrm{int}(\mathcal{P})$. Because only then, the mapping from the powers to the MMSEs in (5) is surjective in the interior of the polytope $\mathcal{P}$ defined in Theorem 3.1. The proof goes as follows: First, we rewrite the mapping from the powers $\boldsymbol{p}$ to the MMSE tuple $\varepsilon_1, \ldots, \varepsilon_K$ as a fixed point equation at the optimum power allocation. Since the arising function in the fixed point equation is increasing and concave, there is at most one fixed point according to [17, 18]. If the fixed point does exist, the mapping from powers to MMSEs is then not only surjective, but also injective and thus bijective. For any target MMSE tuple for which a fixed point exists, there is a power allocation $\boldsymbol{p}$ obtaining it. Fortunately, Kennan also defines some sufficient conditions for the *existence* of a fixed point in [17]. We show that these conditions are met only if the target MMSE tuple is taken from the interior of the polytope $\mathcal{P}$, which will then complete the proof of Theorem 3.1.

Given MMSE targets $\varepsilon_k^{\mathrm{target}}$ with $k \in \mathcal{K}$, we assume that the power allocation $\check{\boldsymbol{p}}$ achieves these targets. Rewriting the first line of (5) with the help of the matrix inversion lemma leads to the well-known result

$$\varepsilon_k = \frac{1}{1 + p_k \boldsymbol{h}_k^{\mathrm{H}} \left(\sigma^2 \mathbf{I} + \sum_{\ell\neq k} \boldsymbol{h}_\ell \boldsymbol{h}_\ell^{\mathrm{H}} p_\ell\right)^{-1} \boldsymbol{h}_k}.$$

Equating $\varepsilon_k = \varepsilon_k^{\mathrm{target}}$ at the hypothetical fixed point $\check{\boldsymbol{p}}$, we obtain

$$\check{p}_k = f_k(\check{\boldsymbol{p}}; \boldsymbol{\varepsilon}^{\mathrm{target}}) := \frac{\frac{1}{\varepsilon_k^{\mathrm{target}}} - 1}{\boldsymbol{h}_k^{\mathrm{H}} \left(\sigma^2 \mathbf{I} + \sum_{\ell\neq k} \boldsymbol{h}_\ell \boldsymbol{h}_\ell^{\mathrm{H}} \check{p}_\ell\right)^{-1} \boldsymbol{h}_k}, \tag{10}$$

where $f_k(\boldsymbol{p}; \boldsymbol{\varepsilon}^{\mathrm{target}})$ is well known from SINR balancing and satisfies the interference function properties defined in [9]. In particular, $f_k(\boldsymbol{p}; \cdot)$ is positive, (quasi-)increasing, and the function $f_k(\boldsymbol{p}; \cdot) - p_k$ is strictly radially quasiconcave for all k (see [17] for the definition). Hence, there is at most one fixed point according to [17, Corollary

[1] For $K > N$, it suffices that at least N powers go to infinity

1]. For the existence of a unique fixed point, Theorem 3 in [17] requires

$$\boldsymbol{f}(\mathbf{0};\boldsymbol{\varepsilon}^{\text{target}}) \geq \mathbf{0}, \tag{11}$$

$$\exists \boldsymbol{a} > \mathbf{0} \text{ with } \boldsymbol{f}(\boldsymbol{a};\boldsymbol{\varepsilon}^{\text{target}}) > \boldsymbol{a}, \tag{12}$$

$$\exists \boldsymbol{b} > \boldsymbol{a} \text{ with } \boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) < \boldsymbol{b}. \tag{13}$$

The first requirement (11) immediately follows from (10):

$$f_k(\mathbf{0};\boldsymbol{\varepsilon}^{\text{target}}) = \sigma^2 \frac{\frac{1}{\varepsilon_k^{\text{target}}} - 1}{\|\boldsymbol{h}_k\|_2^2} \geq 0 \text{ for } 0 \leq \varepsilon_k^{\text{target}} \leq 1.$$

If there is a user whose target MMSE is equal to one, simply no power has to be allocated to that user and he can be discarded for the power computation of the remaining users. Hence, we assume $\varepsilon_k^{\text{target}} < 1\ \forall k \in \mathcal{K}$ in the following. For the second requirement (12), we choose $\boldsymbol{a} = a\mathbf{1}$ as the scaled all-ones vector. Using (10), we find

$$f_k(\boldsymbol{p};\boldsymbol{\varepsilon}^{\text{target}}) \geq \sigma^2 \frac{\frac{1}{\varepsilon_k^{\text{target}}} - 1}{\|\boldsymbol{h}_k\|_2^2} \text{ for } \boldsymbol{p} \geq \mathbf{0},$$

from which we find the upper bound $\bar{a}$ for a.

$$\bar{a} = \sigma^2 \min_{k\in\mathcal{K}} \frac{\frac{1}{\varepsilon_k^{\text{target}}} - 1}{\|\boldsymbol{h}_k\|_2^2} \tag{14}$$

Note that $\bar{a} > 0$ for $\varepsilon_k^{\text{target}} < 1$. Thus, choosing $a < \bar{a}$ and setting $\boldsymbol{a} = a\mathbf{1}$ satisfies $\boldsymbol{f}(\boldsymbol{a};\boldsymbol{\varepsilon}^{\text{target}}) > \boldsymbol{a}$. For the third requirement (13), we need to find a power vector $\boldsymbol{b}$ for which $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) < \boldsymbol{b}$. The cases $K \leq N$ and $K > N$ will be treated separately. When $K \leq N$, the function $f_k(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}})$ can be upper bounded by lower bounding its denominator. Defining the set $\mathcal{I}_k := \mathcal{K} \setminus \{k\}$, we introduce

$$\boldsymbol{\Pi}_k = \mathbf{I}_N - \boldsymbol{H}_{\mathcal{I}_k}\left(\boldsymbol{H}_{\mathcal{I}_k}^{\text{H}}\boldsymbol{H}_{\mathcal{I}_k}\right)^{-1}\boldsymbol{H}_{\mathcal{I}_k}^{\text{H}}$$

as the projector into the null-space of all channel vectors except the k-th one that features $\text{rank}(\boldsymbol{\Pi}_k) = N - K + 1 \geq 1$ due to $K \leq N$. Herewith, we lower bound the denominator of $f_k(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}})$ via

$$\boldsymbol{h}_k^{\text{H}}\big(\sigma^2\mathbf{I} + \sum_{\ell\neq k}\boldsymbol{h}_\ell\boldsymbol{h}_\ell^{\text{H}}b_\ell\big)^{-1}\boldsymbol{h}_k \geq \sigma^{-2}\boldsymbol{h}_k^{\text{H}}\boldsymbol{\Pi}_k\boldsymbol{h}_k, \tag{15}$$

which is valid for all $\boldsymbol{b} \geq \mathbf{0}$. Equality in (15) only holds for $b_k \to \infty\ \forall k \in \mathcal{I}_k$. Note that $\boldsymbol{h}_k^{\text{H}}\boldsymbol{\Pi}_k\boldsymbol{h}_k = 1/[(\boldsymbol{H}^{\text{H}}\boldsymbol{H})^{-1}]_{k,k}$ is larger than zero due to (3). Hence, an upper bound for $f_k(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}})$ is given by

$$f_k(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) \leq \sigma^2\Big(\frac{1}{\varepsilon_k^{\text{target}}} - 1\Big)\big[(\boldsymbol{H}^{\text{H}}\boldsymbol{H})^{-1}\big]_{k,k}, \tag{16}$$

and when $K \leq N$ and if $\boldsymbol{b}$ is chosen such that

$$b_k > \sigma^2\Big(\frac{1}{\varepsilon_k^{\text{target}}} - 1\Big)\big[(\boldsymbol{H}^{\text{H}}\boldsymbol{H})^{-1}\big]_{k,k}\ \forall k \in \mathcal{K}, \tag{17}$$

the third requirement (13) is satisfied. Since $\boldsymbol{a} = a\mathbf{1}$ with $0 < a < \bar{a}$ and $\bar{a}$ defined in (14), choosing $\boldsymbol{b}$ according to (17) satisfies $\boldsymbol{b} > \boldsymbol{a}$, which is also required in (13). Since the three conditions (11)–(13) are fulfilled, Theorem 3.1 is proven for $N \geq K$.

To show the existence of a power vector $\boldsymbol{b}$ in (13) when $K > N$ is slightly more complicated. First, we set the power allocation to $\boldsymbol{b} = \alpha\boldsymbol{b}_0$, where $\boldsymbol{b}_0$ is taken from the interior of the unit simplex

$$\mathcal{S} := \big\{\boldsymbol{x}\big|\sum_{k\in\mathcal{K}} x_k = 1 \wedge x_k \geq 0\ \forall k\big\}$$

of dimension $K-1$ and α will later go to infinity. Since $\boldsymbol{b}_0 \in \text{int}(\mathcal{S})$, the strict inequality $\boldsymbol{b}_0 > \mathbf{0}$ holds. For $\alpha \to \infty$, we may omit the scaled identity $\sigma^2\mathbf{I}$ in the denominator of $f_k(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}})$ in (10), as the matrix $\sum_{\ell\neq k}\boldsymbol{h}_k\boldsymbol{h}_k^{\text{H}}b_{0,\ell}\alpha$ has rank N for $K > N$ and all its eigenvalues grow beyond all limits when $\alpha \to \infty$ and $\boldsymbol{b}_0 > \mathbf{0}$. Then, the fixed point equation (10) can be rewritten as

$$b_{0,k} = f_k^{\infty}(\boldsymbol{b}_0;\boldsymbol{\varepsilon}^{\text{target}}) := \frac{\frac{1}{\varepsilon_k^{\text{target}}} - 1}{\boldsymbol{h}_k^{\text{H}}\big(\sum_{\ell\neq k}\boldsymbol{h}_\ell\boldsymbol{h}_\ell^{\text{H}}b_{0,\ell}\big)^{-1}\boldsymbol{h}_k}, \tag{18}$$

which means that $\boldsymbol{b}_0 - \boldsymbol{f}^{\infty}(\boldsymbol{b}_0;\boldsymbol{\varepsilon}^{\text{target}}) = \mathbf{0}$. Obviously, the MMSE tuple obtained with the power allocation $\boldsymbol{b} = \alpha\boldsymbol{b}_0$ and $\alpha \to \infty$ satisfies $\sum_{k\in\mathcal{K}}\varepsilon_k = K - N$, see (8). Since (18) and (8) evolve from (5), the target MMSE tuple $\boldsymbol{\varepsilon}^{\text{target}}$ also has to satisfy $\|\boldsymbol{\varepsilon}^{\text{target}}\|_1 = K - N$ for $\boldsymbol{f}^{\infty}(\boldsymbol{b}_0;\boldsymbol{\varepsilon}^{\text{target}}) = \boldsymbol{b}_0$. So far we have shown that a power allocation $\boldsymbol{b} = \alpha\boldsymbol{b}_0$ with $\boldsymbol{b}_0 \in \text{int}(\mathcal{S})$ and $\alpha \to \infty$ achieves an MMSE tuple which satisfies $\boldsymbol{\varepsilon} \in \text{int}(\mathcal{B})$, where

$$\mathcal{B} := \Big\{\boldsymbol{\varepsilon}\Big|\sum_{k\in\mathcal{K}}\varepsilon_k = K - N \wedge 0 \leq \varepsilon_k \leq 1\ \forall k \in \mathcal{K}\Big\}$$

denotes the plane that separates feasibility from infeasibility. Now, we show the converse, i.e., that there always exists a unique $\boldsymbol{b}_0 \in \text{int}(\mathcal{S})$ in the power allocation $\boldsymbol{b} = \alpha\boldsymbol{b}_0$ with $\alpha \to \infty$ for *any* MMSE tuple taken from $\boldsymbol{\varepsilon} \in \text{int}(\mathcal{B})$. When $\alpha \to \infty$, the SINR metric reduces to the SIR without noise component. From the various SIR-balancing papers, see for example [7, 9, 11], we know that *any* ratio between individual SIRs can be balanced since balancing is always feasible. The main objective of the SIR balancing is to find the maximum common scalar r of all SIRs:

$$\underset{r,\boldsymbol{b}_0}{\text{maximize}}\ r \quad \text{s.t.: } \text{SIR}_k = r \cdot \text{SIR}_{0,k}\ \forall k \in \mathcal{K} \tag{19}$$

with the SIR definition

$$\text{SIR}_k := b_{0,k}\boldsymbol{h}_k^{\text{H}}\Big(\sum_{\ell\neq k}\boldsymbol{h}_\ell\boldsymbol{h}_\ell^{\text{H}}b_{0,\ell}\Big)^{-1}\boldsymbol{h}_k.$$

Assuming $\text{SIR}_{0,k} > 0\ \forall k$ and using $\gamma_k = \frac{1}{\varepsilon_k} - 1$, the SIR ratio constraints can be converted into MMSE constraints:

$$\text{SIR}_k = r \cdot \text{SIR}_{0,k} \Leftrightarrow \varepsilon_k = \frac{1}{r(\frac{1}{\varepsilon_{0,k}} - 1) + 1}, \tag{20}$$

where $\varepsilon_{0,k} = 1/(1 + \text{SIR}_{0,k})$. Exploiting the fact that in case of infinite power allocation for every user, the sum MMSE is given by $K - N$ [see (8)], we find the radius r in the SIR domain. According to (20), it is easy to see that $\sum_{k\in\mathcal{K}}\varepsilon_k$ is decreasing in r and therefore, $\sum_{k\in\mathcal{K}}\varepsilon_k = K - N$ has a unique solution for r. In particular, $r = 1$ if all $\text{SIR}_{0,\cdot}$ are chosen such that $\sum_{k\in\mathcal{K}} 1/(1 + \text{SIR}_{0,k}) = K - N$. This proves that for any target MMSE tuple $\boldsymbol{\varepsilon}^{\text{target}}$ taken from $\text{int}(\mathcal{B})$ and thus satisfying $\|\boldsymbol{\varepsilon}^{\text{target}}\|_1 = K - N$, there exists an asymptotic power allocation $\boldsymbol{b} = \alpha\boldsymbol{b}_0$ with $\alpha \to \infty$ and $\boldsymbol{b}_0 \in \text{int}(\mathcal{S})$ for which $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) = \boldsymbol{b}$. A slight relaxation of the target MMSE tuple $\boldsymbol{\varepsilon}^{\text{target}}$ with $\|\boldsymbol{\varepsilon}^{\text{target}}\|_1 = K - N$ to $\boldsymbol{\varepsilon}'^{,\text{target}} = \beta\boldsymbol{\varepsilon}^{\text{target}} > \boldsymbol{\varepsilon}^{\text{target}}$ with $\beta > 1$ yields $\|\boldsymbol{\varepsilon}'^{,\text{target}}\|_1 > K - N$. Clearly, the scaling β must be small enough such that $\boldsymbol{\varepsilon}'^{,\text{target}}$ fulfills (6). Since $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon})$ in (10) is strictly *decreasing* in $\boldsymbol{\varepsilon}$, we have $\boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) > \boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}'^{,\text{target}})$ and therefore,

$$\boldsymbol{b} = \boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) \text{ if } \|\boldsymbol{\varepsilon}^{\text{target}}\|_1 = K - N,$$

$$\boldsymbol{b} > \boldsymbol{f}(\boldsymbol{b};\boldsymbol{\varepsilon}^{\text{target}}) \text{ if } \|\boldsymbol{\varepsilon}^{\text{target}}\|_1 > K - N.$$

Hence, the third requirement (13) for the existence of a unique fixed point is satisfied for any $\varepsilon^{\text{target}}$ whose sum MMSE is larger than $K-N$. Since $\alpha \to \infty$ and $\boldsymbol{b}_0 > \boldsymbol{0}$, we have $\boldsymbol{b} = \alpha \boldsymbol{b}_0 > \boldsymbol{a}$. This completes the proof for Theorem 3.1.

Based on these observations, we can infer that the smallest *common* MMSE $\bar{\varepsilon} := \varepsilon_1 = \ldots = \varepsilon_K$ reads as

$$\bar{\varepsilon} = \begin{cases} \frac{K-N}{K} = 1 - \frac{N}{K} & \text{for } K \geq N \\ 0 & \text{for } K \leq N \end{cases}$$

and leads via $\gamma_k = \frac{1}{\varepsilon_k} - 1$ to the largest common SIR

$$\bar{\gamma} = \frac{1}{\bar{\varepsilon}} - 1 = \begin{cases} \frac{N}{K-N} & \text{for } K \geq N \\ \infty & \text{for } K \leq N \end{cases}$$

which has already been observed in [4].

Examples: In the single antenna case, the SINRs γ_1 and γ_2 of the two users have to satisfy $\gamma_1\gamma_2 < 1$ in case of finite sum power as shown in [9]. Using Theorem 3.1, we are now able to extend the feasible SINR region to the case of $K = 3$ users and $N = 1$ antenna at the base station. The bijective mapping $\gamma_k = \frac{1}{\varepsilon_k} - 1$ converts the condition $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 > K - N = 2$ to

$$2\gamma_1\gamma_2\gamma_3 + \gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3 < 1. \tag{21}$$

Adding a second antenna to the base station, the MMSEs have to satisfy $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 > K - N = 1$. In the SINR domain, this requires

$$\gamma_1\gamma_2\gamma_3 - (\gamma_1 + \gamma_2 + \gamma_3) < 2$$

to hold for finite sum power.

4. CONCLUSION

The geometry of the feasible QoS region in the vector broadcast channel has been derived when the available transmit power is not limited. We have shown that the maximum feasible MMSE region is a hyper-cube that is cropped when more users shall be served than antennas are available at the base station. This geometric structure is independent of the channel realization as long as all column subsets of the channel matrix define a nonsingular matrix. Based on this new observation, feasibility of given QoS requirements can be detected with practically no computational complexity. Instead of complex balancing algorithms, only the sum of the target MMSEs has to be compared to the difference between the number of antennas at the base station and the number of served users to find out, whether the QoS constraints are feasible or not.

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