On Proportional Fairness in Nonconvex Wireless Systems

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***Abstract*—We investigate the concept of proportional fairness in the context of nonconvex rate regions. For convex rate regions, Kelly's definition of proportional fairness is equivalent to maximizing a sum of logarithms. For nonconvex rate regions, this equivalence does not always hold. In particular, a proportionally fair point in the sense of Kelly's definition may not exist. We propose the concept of local proportional fairness (LPF) and characterize its properties. Under mild conditions, an LPF point always exists, and all LPF points are Pareto optimal. Moreover, we show that in a multiantenna downlink with linear precoding, computing an LPF point represents a practically solvable problem.**

I. Introduction

We consider a wireless system in which $K > 1$ users compete for limited resources. A fundamental problem in such a system is to find a criterion on which resource allocation is to be based. In his seminal paper [1], Kelly proposed the concept of *proportional fairness* (PF) as a criterion for resource allocation. Let $\mathcal{R} \subset \mathbb{R}_+^K$ denote the set of feasible rates. Throughout this paper, it is assumed that $\mathcal{R}$ is compact and nontrivial. We say that $\mathcal{R}$ is nontrivial if there exists $\boldsymbol{r} \in \mathcal{R}$ such that $r_k > 0, \forall k$. Kelly defines PF as follows [1]: A vector of rates $\boldsymbol{r}^* \in \mathcal{R} \cap \mathbb{R}_{++}^K$ is proportionally fair if for any other feasible vector $\boldsymbol{r}$, the aggregate of proportional changes is zero or negative, i.e.,

$$\sum_{k=1}^{K} \frac{r_k - r_k^*}{r_k^*} \leq 0, \forall \boldsymbol{r} \in \mathcal{R}. \tag{1}$$

Kelly also showed the relationship between proportional fairness and maximizing a sum of logarithms,

$$\max_{\boldsymbol{r} \in \mathcal{R}} \sum_{k=1}^{K} \ln r_k. \tag{2}$$

Let $\mathcal{R}^*$ denote the set of global maximizers of (2). If $\mathcal{R}$ is convex, compact, and nontrivial, then $\mathcal{R}^*$ has a single element, and this rate vector is proportionally fair [1]. From this result, it follows that if $\mathcal{R}$ is convex, compact, and nontrivial, a proportionally fair vector always exists, and the proportionally fair rate allocation is unique.

The performance of wireless systems can be improved significantly by allowing for interference between users. This particularly holds true in multiantenna systems, where optimum transmission strategies often do not result in a complete interference suppression. On the other hand, interference can result in nonconvex rate regions, see, e.g., [2]. While any rate region can be convexified by time-sharing, a time-sharing mode can significantly complicate the problem of finding an optimum resource allocation [3]. This paper investigates the concept of proportional fairness in the context of compact, but not necessarily convex rate regions.

If $\mathcal{R}$ is not convex, the existence of a feasible rate vector that fulfills the definition of proportional fairness (1) cannot be guaranteed – in other words, there exist nontrivial compact sets $\mathcal{R}$ such that no vector $\boldsymbol{r} \in \mathcal{R}$ is proportionally fair. We provide an example of such a set in Section II. Based on this observation, it is desirable to extend the concept of proportional fairness in a way such that a proportionally fair rate vector (in the extended sense) always exists.

For compact and nontrivial $\mathcal{R}$, the set $\mathcal{R}^*$ is always well-defined and nonempty . Accordingly, one approach to extend the concept of proportional fairness to nonconvex $\mathcal{R}$ is to declare all points in $\mathcal{R}^*$ as proportionally fair. This approach was taken in [4], [5], [6], [7], [8]. As pointed out before, however, none of the maximizers in $\mathcal{R}^*$ may fulfill (1). Moreover, Kelly's definition is based on the notion of aggregates of proportional changes – that the proportionally fair point maximizes a sum of logarithms (if $\mathcal{R}$ is convex) is a property of PF, but not its definition. Based on this observation, we propose the concept of *local* proportional fairness: Let $\hat{\mathcal{R}}(\boldsymbol{r}^*)$ denote a local approximation of $\mathcal{R}$ at $\boldsymbol{r}^*$ based on the tangent cone of $\mathcal{R}$ at $\boldsymbol{r}^*$. We say that a vector of rates $\boldsymbol{r}^*$ is *locally proportional fair* (LPF) if $\boldsymbol{r}^* \in \mathcal{R} \cap \mathbb{R}_{++}^K$ and if for any other vector $\boldsymbol{r} \in \hat{\mathcal{R}}(\boldsymbol{r}^*)$, the aggregate of proportional changes is zero or negative.

We show the following results:

1) If $\mathcal{R}$ is compact and nontrivial, there always exists an LPF point.
2) If $\mathcal{R}$ is compact, nontrivial, and normal, all LPF points are Pareto optimal.
3) If $\mathcal{R}$ is convex, LPF and PF are equivalent.

In addition, we demonstrate that in a multiantenna downlink with linear precoding, an LPF point can be found by standard local methods for unconstrained optimization. This result shows that in terms of problem complexity, there is a fundamental difference between LPF and maximizing a sum of logarithms. For nonconvex $\mathcal{R}$, maximizing a sum of logarithms over $\mathcal{R}$ is a nonconvex problem. Thus, in general, computing a point in $\mathcal{R}^*$ is practically feasible only if the number of users is small [9].

A. Notation and Preliminaries

Let $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of nonnegative and positive real numbers, respectively. Given a vector $\boldsymbol{r} \in \mathbb{R}^K$ and a set $\mathcal{R} \subseteq \mathbb{R}^K$, the set $\boldsymbol{r} + \mathcal{R}$ is defined as follows:

$$\boldsymbol{r} + \mathcal{R} = \{\boldsymbol{r} + \boldsymbol{r}', \boldsymbol{r}' \in \mathcal{R}\}.$$

Order relations $\geq, >$, and $\gg$ between two vectors $\boldsymbol{r}, \boldsymbol{r}' \in \mathbb{R}^K$ are defined as follows:

$$\begin{aligned}
\boldsymbol{r}' \geq \boldsymbol{r} &\Leftrightarrow \boldsymbol{r}' \in \boldsymbol{r} + \mathbb{R}_+^K, \\
\boldsymbol{r}' > \boldsymbol{r} &\Leftrightarrow \boldsymbol{r}' \in \boldsymbol{r} + (\mathbb{R}_+^K \setminus \{\boldsymbol{0}\}), \\
\boldsymbol{r}' \gg \boldsymbol{r} &\Leftrightarrow \boldsymbol{r}' \in \boldsymbol{r} + \mathbb{R}_{++}^K.
\end{aligned}$$

Given a set $\mathcal{R} \subset \mathbb{R}^K$, the Pareto boundary of $\mathcal{R}$ is given by

$$\mathcal{P} = \left\{\boldsymbol{r} \in \mathcal{R} : \mathcal{R} \cap \boldsymbol{r} + \mathbb{R}_+^K = \{\boldsymbol{r}\}\right\}.$$

A function $f : \mathcal{D} \subseteq \mathbb{R}^K \to \mathbb{R}$ is strictly increasing if

$$\boldsymbol{r}, \boldsymbol{r}' \in \mathcal{D}, \boldsymbol{r}' > \boldsymbol{r} \Rightarrow f(\boldsymbol{r}') > f(\boldsymbol{r}).$$

A set $\mathcal{R} \subseteq \mathbb{R}_+^K$ is normal if

$$\boldsymbol{r}' \in \mathcal{R}, \boldsymbol{r} \in \mathbb{R}_+^K, \boldsymbol{r}' \geq \boldsymbol{r} \Rightarrow \boldsymbol{r} \in \mathcal{R}.$$

Note that this property is also known as comprehensive [8] or coordinate-convex [7].

The closure of a set $\mathcal{S}$ is denoted by $\mathrm{cl}(\mathcal{S})$, and ∇f denotes the gradient of a function f. Finally, $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{1}$ the vector of all ones.

II. An Example

In this section, we show with an example that there exist compact and nontrivial rate regions for which no proportionally fair rate vector exists.

Define a function $f : \mathbb{R}_{++}^K \to \mathbb{R}$ as follows:

$$f(\boldsymbol{r}) = \sum_{k=1}^{K} \ln r_k. \tag{3}$$

Note that f is strictly increasing, and

$$\nabla f(\boldsymbol{r}) \in \mathbb{R}_{++}^K, \forall \boldsymbol{r} \in \mathbb{R}_{++}^K. \tag{4}$$

Moreover,

$$\nabla f(\boldsymbol{r}^*)^{\mathrm{T}}(\boldsymbol{r} - \boldsymbol{r}^*) = \sum_{k=1}^{K} \frac{r_k - r_k^*}{r_k^*}. \tag{5}$$

Define the following level sets:

$$\underline{\mathcal{S}}_f(\gamma) = \left\{\boldsymbol{r} \in \mathbb{R}_{++}^K : f(\boldsymbol{r}) \leq \gamma\right\}, \tag{6}$$

$$\overline{\mathcal{S}}_f(\gamma) = \left\{\boldsymbol{r} \in \mathbb{R}_{++}^K : f(\boldsymbol{r}) \geq \gamma\right\}. \tag{7}$$

Due to the fact that f is strictly concave and smooth on $\mathbb{R}_{++}^K$,

$$\nabla f(\boldsymbol{r}^*)^{\mathrm{T}}(\boldsymbol{r} - \boldsymbol{r}^*) > 0, \forall \boldsymbol{r} \in \overline{\mathcal{S}}_f(f(\boldsymbol{r}^*)) \setminus \{\boldsymbol{r}^*\}. \tag{8}$$

Now let $K = 2$ and define a rate region $\mathcal{R}$ as follows:

$$\mathcal{R} = \left\{\boldsymbol{r} \in \mathbb{R}_+^2 : r_k \leq \exp(2)\right\} \cap \mathrm{cl}(\underline{\mathcal{S}}_f(3)). \tag{9}$$

The set $\mathcal{R}$ is compact and nontrivial. Only points on the Pareto boundary of $\mathcal{R}$ are candidates for PF points, as for all other points in $\mathcal{R}$ the rate of at least one user can be increased without decreasing the rate of any other user, leading to a positive aggregate proportional change. The Pareto boundary of $\mathcal{R}$ is given by

$$\mathcal{P} = \mathcal{R} \cap \overline{\mathcal{S}}_f(3) = \{(\exp(t+1), \exp(2-t)) : t \in [0,1]\}. \tag{10}$$

Accordingly, any point $\boldsymbol{r}^* \in \mathcal{R}$ with $f(\boldsymbol{r}^*) = 3$ is a candidate for PF. But according to (5) and (8), for any such point there exists another point $\boldsymbol{r} \in \mathcal{P}$ such that

$$\nabla f(\boldsymbol{r}^*)^{\mathrm{T}}(\boldsymbol{r} - \boldsymbol{r}^*) = \sum_{k=1}^{2} \frac{r_k - r_k^*}{r_k^*} > 0. \tag{11}$$

Consequently, no point in $\mathcal{R}$ fulfills (1) – in other words, there exists no proportionally fair rate vector. In contrast, note that any point in $\mathcal{P}$ is a global maximizer of (2), i.e., $\mathcal{R}^* = \mathcal{P}$.

Notably, in [8] it was shown that the classical Nash Bargaining Solution (NBS) [10], which was originally defined for convex utility sets only, can be extended to so-called strictly log-convex utility sets. By slightly modifying the above example, however, it can be shown that there exist strictly log-convex sets that do not contain a proportionally fair point. In particular, for such sets the NBS point that results if the disagreement point is the zero vector is not proportionally fair. This result constitutes another example for the difference between maximizing a sum of logarithms (or, equivalently, the Nash product with disagreement point $\boldsymbol{0}$) and proportional fairness on the family of nonconvex compact sets.

III. Local Proportional Fairness

As shown in the previous section, for general $\mathcal{R}$, there may not exist a rate vector that is proportionally fair. The idea underlying local proportional fairness is to approximate $\mathcal{R}$ at a candidate point $\boldsymbol{r}^*$ by a set $\hat{\mathcal{R}}(\boldsymbol{r}^*)$. The approximation of $\mathcal{R}$ at $\boldsymbol{r}^*$ is defined as follows:

$$\hat{\mathcal{R}}(\boldsymbol{r}^*) = \{\boldsymbol{r}^* + \boldsymbol{v}, \boldsymbol{v} \in \mathcal{T}(\boldsymbol{r}^*)\} \cap \mathbb{R}_+^K, \tag{12}$$

where $\mathcal{T}(\boldsymbol{r}^*)$ denotes the tangent cone of $\mathcal{R}$ at $\boldsymbol{r}^*$ [11]. The tangent cone $\mathcal{T}(\boldsymbol{r}^*)$ provides information about the local structure of $\mathcal{R}$ near $\boldsymbol{r}^*$.

Our definition of *local proportional fairness* is as follows: A point $\boldsymbol{r}^* \in \mathcal{R} \cap \mathbb{R}_{++}^K$ is locally proportional fair in $\mathcal{R}$ if for any other vector $\boldsymbol{r} \in \hat{\mathcal{R}}(\boldsymbol{r}^*)$, the aggregate of proportional changes is nonpositive:

$$\sum_{k=1}^{K} \frac{r_k - r_k^*}{r_k^*} \leq 0, \forall \boldsymbol{r} \in \hat{\mathcal{R}}(\boldsymbol{r}^*). \tag{13}$$

Obviously, $\boldsymbol{r}^*$ is LPF in $\mathcal{R}$ if it is proportionally fair in $\hat{\mathcal{R}}(\boldsymbol{r}^*)$.

Again, LPF is closely related to the maximization of a sum of logarithms: If f is chosen as in (3), then (13) is equivalent to

$$\nabla f(\boldsymbol{r}^*)^{\mathrm{T}} \boldsymbol{v} \leq 0, \forall \boldsymbol{v} \in \mathcal{T}(\boldsymbol{r}^*). \tag{14}$$

Eq. (14) represents a necessary condition for $\boldsymbol{r}^*$ being a local maximizer of f over $\mathcal{R} \cap \mathbb{R}^K_{++}$ [11]. It is important to note, however, that Eq. (14) is only a necessary, not a sufficient condition – as a consequence, any local maximizer of f over $\mathcal{R} \cap \mathbb{R}^K_{++}$ is LPF, but an LPF point is not necessarily a local maximizer. Let $\mathcal{R}^{\mathrm{LPF}}$ denote the set of LPF points in $\mathcal{R}$. From the perspective of maximizing a sum of logarithms, the set $\mathcal{R}^{\mathrm{LPF}}$ can be partitioned into three sets: the set of points that satisfy (14), but are not local maximizers, the set of local maximizers that are not global maximizers, and the set of global maximizers. If a sum of logarithms is to be maximized, there is an obvious hierarchy among these sets. In contrast, from the perspective of aggregate proportional changes over an approximated rate region, the three sets are equivalent.

According to (14), every local maximizer of f over $\mathcal{R} \cap \mathbb{R}^K_{++}$ is LPF. As $\mathcal{R}^*$ contains the global maximizers,

$$\mathcal{R}^* \subseteq \mathcal{R}^{\mathrm{LPF}}. \tag{15}$$

Moreover, $\mathcal{R}^*$ is nonempty for compact and nontrivial $\mathcal{R}$, thus $\mathcal{R}^{\mathrm{LPF}}$ is nonempty. This proves the existence of an LPF point: *If $\mathcal{R}$ is compact and nontrivial, there always exists a feasible rate vector that is LPF.*

Let $\mathcal{P}$ denote the Pareto boundary of $\mathcal{R}$. From the strict monotonicity of f defined in (3), it follows that $\mathcal{R}^* \subseteq \mathcal{P}$ [12]. Under the assumption that $\mathcal{R}$ is normal, the same holds for the set of LPF points:

$$\mathcal{R}^{\mathrm{LPF}} \subseteq \mathcal{P}.$$

In other words, *if $\mathcal{R}$ is compact, nontrivial, and normal, all LPF points are Pareto optimal*. The proof is by contradiction: Let $\boldsymbol{r}^*$ in $\mathcal{R}^{\mathrm{LPF}}$. Moreover, assume that $\boldsymbol{r}^*$ is not Pareto optimal. Accordingly, there exists $\boldsymbol{r}' \in \mathcal{R}$ such that $\boldsymbol{r}' > \boldsymbol{r}^*$. Thus, $\boldsymbol{v} = \boldsymbol{r}' - \boldsymbol{r}^* > \boldsymbol{0}$. Due to the fact that $\mathcal{R}$ is normal,

$$\boldsymbol{r}^* + \alpha \boldsymbol{v} \in \mathcal{R}, \forall \alpha \in [0, 1].$$

Consequently, $\boldsymbol{v} \in \mathcal{T}(\boldsymbol{r}^*)$. But $\nabla f(\boldsymbol{r}^*) \in \mathbb{R}^K_{++}$ and $\boldsymbol{v} \in \mathbb{R}^K_{+}$, thus

$$\nabla f(\boldsymbol{r}^*)^{\mathrm{T}} \boldsymbol{v} > 0,$$

which contradicts the assumption that $\boldsymbol{r}^*$ in $\mathcal{R}^{\mathrm{LPF}}$.

If $\mathcal{R}$ is convex, (14) is equivalent to [11]

$$\nabla f(\boldsymbol{r}^*)^{\mathrm{T}} (\boldsymbol{r} - \boldsymbol{r}^*) \leq 0, \forall \boldsymbol{r} \in \mathcal{R}. \tag{16}$$

By choosing f as in (3), (16) is identical to (1). As a result, *for convex $\mathcal{R}$, LPF and PF are equivalent*.

IV. LPF in a MU-MISO Downlink

In this section, an achievable rate region in a K-user multiple-input, single-output (MISO) downlink with linear precoding is considered. The transmitter has N antennas, while each of the K receivers has a single antenna. Interference is treated as noise. The transmitter is subject to a power constraint. Using the duality relation between uplink and downlink [13], a parameterization of the rate region $\mathcal{R}$ can be given as follows:

$$\mathcal{R} = \left\{ \boldsymbol{r}(\boldsymbol{p}) : \boldsymbol{p} \in \mathbb{R}^K_{+}, \|\boldsymbol{p}\|_1 \leq P \right\}, \tag{17}$$

with the uplink rates

$$r_k(\boldsymbol{p}) = \log_2 \det \left(\boldsymbol{I} + \left(\sigma^2 \boldsymbol{I} + \sum_{q \neq k} \boldsymbol{h}_q \boldsymbol{h}_q^{\mathrm{H}} p_q \right)^{-1} \boldsymbol{h}_k \boldsymbol{h}_k^{\mathrm{H}} p_k \right). \tag{18}$$

We assume that $\mathcal{R}$ is nontrivial.

Due to the nonconvexity of the function $f(\boldsymbol{r}(\boldsymbol{p}))$, determining an element of $\mathcal{R}^*$ represents a hard problem. As illustrated in [2], the rate region $\mathcal{R}$ may be nonconvex. Accordingly, the elements of $\mathcal{R}^*$ correspond to global optima of a nonconvex problem. As shown in [3], an element of $\mathcal{R}^*$ can be found by using methods from global optimization, but such a solution is practically feasible for small K only [9]. In contrast, computing an LPF point is simple, as shown in the following.

The rate region $\mathcal{R}$ is normal. Thus, all LPF points lie on the Pareto boundary of $\mathcal{R}$. The Pareto boundary corresponds to the rate vectors where full transmission power is used [13]. Define a parameterization $\boldsymbol{\phi}$ as follows:

$$\boldsymbol{\phi}(\boldsymbol{\mu}) = \boldsymbol{r}(\boldsymbol{p}^0 + \boldsymbol{Q}\boldsymbol{\mu}),$$

where $\boldsymbol{p}^0 = \frac{P}{K}\boldsymbol{1}$ and the columns of $\boldsymbol{Q}$ constitute a basis of the nullspace of $\boldsymbol{1}^{\mathrm{T}}$, i.e., $\boldsymbol{1}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{0}$. A parameterization of the Pareto boundary is then given by

$$\mathcal{P} = \left\{ \boldsymbol{\phi}(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathbb{R}^{K-1}, \boldsymbol{p}^0 + \boldsymbol{Q}\boldsymbol{\mu} \geq \boldsymbol{0} \right\}. \tag{19}$$

Let $\boldsymbol{J}_\phi(\boldsymbol{\mu})$ denote the Jacobian of $\boldsymbol{\phi}$ at $\boldsymbol{\mu}$. The parameterization $\boldsymbol{\phi}$ is differentiable, thus the tangent space at $\boldsymbol{r}^* \in \mathcal{P} \cap \mathbb{R}^K_{++}$ is given by $\mathrm{span}\boldsymbol{J}_\phi(\boldsymbol{\phi}^{-1}(\boldsymbol{r}^*))$, and the tangent cone at $\boldsymbol{r}^*$ corresponds to the half-space

$$\mathcal{T}(\boldsymbol{r}^*) = \left\{ \boldsymbol{v} \in \mathbb{R}^K : \boldsymbol{n}^{\mathrm{T}} \boldsymbol{v} \leq 0 \right\}, \tag{20}$$

where $\boldsymbol{n} \in \mathbb{R}^K_{+}$ is a nonnegative solution of

$$\boldsymbol{J}_\phi(\boldsymbol{\phi}^{-1}(\boldsymbol{r}^*))^{\mathrm{T}} \boldsymbol{n} = \boldsymbol{0}.$$

Due to the fact that $\mathcal{R}$ is normal, such a solution exists.

From (14), (20) and $\nabla f(\boldsymbol{r}) \in \mathbb{R}^K_{++}$, it follows that $\boldsymbol{r}^* \in \mathcal{P} \cap \mathbb{R}^K_{++}$ is LPF if $\nabla f(\boldsymbol{r}^*) = \alpha \boldsymbol{n}$ for some $\alpha \in \mathbb{R}_{++}$, or, equivalently,

$$\boldsymbol{J}_\phi(\boldsymbol{\phi}^{-1}(\boldsymbol{r}^*))^{\mathrm{T}} \nabla f(\boldsymbol{r}^*) = \boldsymbol{0}. \tag{21}$$

Define a function $g : \mathbb{R}^{K-1} \to \mathbb{R}$,

$$g(\boldsymbol{\mu}) = f(\boldsymbol{\phi}(\boldsymbol{\mu})), \tag{22}$$

Note that

$$\nabla g(\boldsymbol{\mu}) = \boldsymbol{J}_{\phi}(\boldsymbol{\mu})^{\mathrm{T}} \nabla f(\boldsymbol{\phi}(\boldsymbol{\mu})).$$

With Eq. (21), it follows that any $\boldsymbol{\mu}^*$ satisfying

$$\nabla g(\boldsymbol{\mu}^*) = \mathbf{0} \tag{23}$$

yields an LPF point

$$\boldsymbol{r}^* = \boldsymbol{\phi}(\boldsymbol{\mu}^*).$$

Accordingly, any of the standard local methods for unconstrained optimization (gradient ascent, (quasi-)Newton) can be used to find an LPF point.

V. Conclusions

For convex rate regions, the concepts of proportional fairness and maximizing a sum of logarithms are equivalent. This equivalence does not hold for nonconvex rate regions. In particular, for nonconvex rate regions, there may not exist a proportionally fair rate vector. We proposed the concept of local proportional fairness (LPF), which, similar to the original definition of proportional fairness, is based on the aggregate of proportional changes. If the rate region is compact, nontrivial, and normal, an LPF solution always exists, and all LPF points are Pareto optimal. A significant advantage of LPF is that it represents a local property. In the MISO downlink with linear precoding, an LPF point can be found by standard local methods – as a result, finding an LPF point is a practically solvable problem in this setup. In contrast, finding a maximizer of a sum of logarithms over a nonconvex rate region is a global optimization problem that is practically solvable only if the number of users is small.

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