Achievable Rates in the SIMO-Uplink/MISO-Downlink of an FDD System with Imperfect CSI

Mario Castañeda, Anup Chathoth and Josef A. Nossek

Lehrstuhl für Netzwerktheorie und Signalverarbeitung
Technische Universität München, 80290 München, Germany
{maca,anko,jono}@nws.ei.tum.de

***Abstract*—In practical systems, the available CSI for coherent transmission can not always be assumed to be perfect. For instance, in the MISO-downlink of a single-user FDD system with multiple antennas at the BS and a single antenna at the user, the available transmit CSI is actually estimated, quantized, outdated and affected by feedback errors. On the other hand, in the reverse link of this FDD system, i.e. the SIMO-uplink, the available receive CSI is only estimated. The two links of this FDD system represent a two-way system, in which we encounter a tradeoff between the MISO-downlink and SIMO-uplink rate due to the feedback of the transmit CSI in the uplink. Taking into account this tradeoff, we analyze the achievable rates in both links of the described FDD system considering imperfect CSI and limited feedback.**

I. Introduction

Deploying multiple antennas increases considerably the capacity of wireless communication links but under the assumption of perfect *channel state information* (CSI) at the transmitter and receiver [1]. In practice, however, the available CSI at the receiver and transmitter is *imperfect*. Consider the downlink and uplink of a single-user *frequency division duplex* (FDD) system in an isolated cell with a *base station* (BS) equipped with M multiple antennas and a user with a single antenna. In both links, we assume coherent transmission based on imperfect CSI. In the *single-input multiple-output* (SIMO) uplink of this system, the available *receive* CSI for *maximum ratio combining* (MRC) at the BS is an estimate of the current channel, obtained with T_{UL} training symbols in the uplink. In the *multiple-input single-output* (MISO) downlink of this system, the *transmit* CSI for *maximum rate transmission* (MRT) becomes available at the BS in a three-step process. First, the downlink channel is *estimated* at the user employing T_{DL} training symbols in the downlink. Afterwards, the channel estimate is *quantized* with B bits using random vector quantization and finally, the quantized channel estimate is fed back to the BS using $\frac{B}{2}$ QPSK symbols. Considering an error-prone feedback link, i.e. the uplink, the feedback could be received *erroneously*. Additionally, the obtained CSI could be *outdated* due to the feedback delay. Hence, the transmit CSI available for the downlink transmission is estimated, quantized, outdated and affected by feedback errors.

In one-way systems, the tradeoff between training and data payload has been investigated in [2]. Nevertheless, in an FDD system further resources are allocated in the uplink for the feedback of the transmit CSI to the BS and hence, we have an additional tradeoff between the downlink and uplink rates due to the feedback. There has been, however, only limited work that considers this tradeoff in two-way systems. The asymptotic tradeoff between training, feedback and data payload in one-way MISO systems has been discussed in [3], where the uplink is basically solely employed for feedback. The tradeoff between the rates in both links of a two-way MIMO system with limited feedback and beamforming has been considered in [4] with an uneven power allocation between the training, feedback and data payload. In this work, we do not allow such an uneven power allocation because of practical issues. The tradeoff between training, feedback and data payload in a multi-user FDD downlink with zero-forcing has been presented in [5] assuming imperfect CSI with feedback errors and feedback delay, by employing a common training phase and a dedicated training phase.

Due to the dependency between the two links in an FDD system, there is no single figure of merit as in a one-way system. In this work, therefore, we are interested in the achievable pairs of uplink and downlink rates under coherent beamforming based on imperfect CSI in a single-user FDD system with limited feedback. In order to reduce the feedback error probability, we also consider the case of coded feedback. The paper is structured as follows. Section II discusses the system model, while the probability of feedback errors is derived in Section III. The achievable downlink rates as a function of the number of feedback bits B is discussed in Section IV, while the achievable pairs of uplink and downlink rates is treated in Section V. We conclude the paper in Section VI.

II. System Model

We assume the coherence time of the uplink and downlink channels to be T symbols, which is the duration of a time slot. The uplink (UL) of the FDD *two-way* system is a SIMO channel where the M receive antennas at the BS are used to perform maximum ratio combining and hence, the effective *single-input single-output* SISO channel of the D_{UL} data symbols at time slot n is

$$\mathbf{y}_{\text{UL}}[n] = \sqrt{P_{\text{UL}}}\mathbf{w}_{\text{UL}}^{\text{H}}[n]\mathbf{h}_{\text{UL}}[n]\mathbf{s}_{\text{UL}}[n] + \mathbf{u}[n], \tag{1}$$

where $\mathbf{y}_{\text{UL}}[n] \in \mathbb{C}^{D_{\text{UL}}}$ are the received signals at the BS after the beamforming, $\mathbf{s}_{\text{UL}}[n] \in \mathbb{C}^{D_{\text{UL}}}$ are the unit-variance transmit signals, $\mathbf{w}_{\text{UL}}[n] \in \mathbb{C}^{M}$ is the received beamforming vector with unit norm, $\mathbf{h}_{\text{UL}}[n] \in \mathbb{C}^M$ is the uplink SIMO channel, $\mathbf{u}[n] \in \mathbb{C}^{D_{\text{UL}}}$ is the *additive white Gaussian noise* (AWGN) with zero mean and variance σ_u^2 and P_{UL} is the transmit power at the user. The elements of the uplink channel $\mathbf{h}_{\text{UL}}[n]$ are i.i.d. complex Gaussian random variables with zero mean and unit variance. The uplink channel $\mathbf{h}_{\text{UL}}[n]$ is constant for the duration of a time slot, i.e., T symbols. The first T_{UL} symbols in the time slot are used to obtain an *minimum mean square error* (MMSE) estimate $\hat{\mathbf{h}}_{\text{UL}}[n]$ of the uplink channel, while the last $\frac{B}{2}$ symbols are reserved for the feedback of the quantized version of the downlink channel estimate $\hat{\mathbf{h}}_{\text{DL}}[n]$ to the BS, and the remaining $D_{\text{UL}} = T - T_{\text{UL}} - \frac{B}{2}$ symbols are used to transmit data from the user to the BS. The available receive CSI in the UL at the BS is $\hat{\mathbf{h}}_{\text{UL}}[n]$ and the uplink channel $\mathbf{h}_{\text{UL}}[n]$ can be written as $\mathbf{h}_{\text{UL}}[n] = \hat{\mathbf{h}}_{\text{UL}}[n] + \mathbf{e}_{\text{UL}}[n]$, where $\mathbf{e}_{\text{UL}}[n]$ is the error vector whose elements are i.i.d. zero-mean complex Gaussian random variables with variance $\sigma^2_{e_{\text{UL}}} = \frac{1}{1+P_{\text{UL}}T_{\text{UL}}/\sigma_u^2}$ [2]. The elements of $\hat{\mathbf{h}}_{\text{UL}}[n]$ are i.i.d. zero-mean complex Gaussian random variables with variance $1 - \sigma^2_{e_{\text{UL}}}$. With the available receive CSI, the beamforming vector for MRC is $\mathbf{w}_{\text{UL}}[n] = \frac{\hat{\mathbf{h}}_{\text{UL}}[n]}{\|\hat{\mathbf{h}}_{\text{UL}}[n]\|_2}$, so we rewrite (1) as

$$\mathbf{y}_{\text{UL}}[n] = \sqrt{P_{\text{UL}}}\;\|\hat{\mathbf{h}}_{\text{UL}}[n]\|_2 \mathbf{s}_{\text{UL}}[n] + \mathbf{z}'[n], \tag{2}$$

where $\mathbf{z}'[n] = \sqrt{P_{\text{UL}}}\;\mathbf{w}^{\text{H}}_{\text{UL}}[n]\mathbf{e}_{\text{UL}}[n]\mathbf{s}_{\text{UL}}[n] + \mathbf{u}[n]$ is the effective noise with variance $\sigma^2_{z'} = \sigma^2_{e_{\text{UL}}} P_{\text{UL}} + \sigma_u^2$.

The downlink (DL) of the FDD *two-way* system is a MISO channel where the M transmit antennas at the BS are used to perform maximum ratio transmission. Hence, the effective SISO channel of of the D_{DL} data symbols at time slot n in the downlink is given by

$$\mathbf{y}_{\text{DL}}[n] = \sqrt{P_{\text{DL}}}\;\mathbf{w}^{\text{H}}_{\text{DL}}[n]\mathbf{h}_{\text{DL}}[n]\mathbf{s}_{\text{DL}}[n] + \mathbf{v}[n], \tag{3}$$

where $\mathbf{y}_{\text{DL}}[n] \in \mathbb{C}^{D_{\text{DL}}}$ are the received signals, $\mathbf{s}_{\text{DL}}[n] \in \mathbb{C}^{D_{\text{DL}}}$ are the transmit symbols with unit variance, $\mathbf{w}_{\text{DL}}[n] \in \mathbb{C}^M$ is the beamforming vector with unit norm, $\mathbf{h}_{\text{DL}}[n] \in \mathbb{C}^M$ is the downlink MISO channel and $\mathbf{v}[n] \in \mathbb{C}^{D_{\text{DL}}}$ is the AWGN with zero mean and variance σ_v^2 at time slot n. Additionally, P_{DL} is the transmit power available at the BS. We assume Rayleigh fading, i.e. the elements of the channel vector $\mathbf{h}_{\text{DL}}[n]$ are i.i.d. complex Gaussian random variables with zero mean and unit variance.

To account for the feedback delay, we consider *temporally correlated block fading* similar to [5], i.e. $\mathbf{h}_{\text{DL}}[n]$ is assumed to be constant for the coherence time of T symbols and is correlated with the downlink channel in the previous time slot $\mathbf{h}_{\text{DL}}[n-1]$ according to a first order Markov model:

$$\mathbf{h}_{\text{DL}}[n] = \sqrt{\alpha}\,\mathbf{h}_{\text{DL}}[n-1] + \sqrt{1-\alpha}\,\mathbf{g}_{\text{DL}}[n-1], \tag{4}$$

where the elements of $\mathbf{g}_{\text{DL}}[n-1] \in \mathbb{C}^M$ are i.i.d. zero-mean unit-variance complex Gaussian random variables and are uncorrelated with $\mathbf{h}_{\text{DL}}[n-1]$ and $\sqrt{\alpha}$ is the correlation coefficent, with $\alpha \in [0,1]$. We assume that α is unknown.

The first T_{DL} symbols of each time slot in the downlink are used to obtain a MMSE estimate of the channel and the remaining $D_{\text{DL}} = T - T_{\text{DL}}$ symbols are used to transmit information from the BS to the user. We assume that the user does not make use of previous estimates. Let us denote the downlink channel estimate obtained with the T_{DL} training symbols at time slot n as $\hat{\mathbf{h}}_{\text{DL}}[n]$. Based on this estimate, the downlink channel can be written as $\mathbf{h}_{\text{DL}}[n] = \hat{\mathbf{h}}_{\text{DL}}[n] + \mathbf{e}_{\text{DL}}[n]$, where $\mathbf{e}_{\text{DL}}[n]$ is the error vector whose elements are i.i.d. zero-mean complex Gaussian random variables with variance $\sigma^2_{e_{\text{DL}}}$ given as [2]

$$\sigma^2_{e_{\text{DL}}} = \begin{cases} \dfrac{1+\rho_{\text{DL}}(M - T_{\text{DL}})}{1+\rho_{\text{DL}}M} & \text{for} \quad T_{\text{DL}} < M \\ \dfrac{1}{1+\rho_{\text{DL}}T_{\text{DL}}} & \text{for} \quad T_{\text{DL}} \geq M \end{cases}, \tag{5}$$

where $\rho_{\text{DL}} = \frac{P_{\text{DL}}}{M\sigma_v^2}$. Additionally, the elements of the MMSE estimate $\hat{\mathbf{h}}_{\text{DL}}$ are i.i.d. zero-mean Gaussian random variables with variance $(1 - \sigma^2_{e_{\text{DL}}})$.

The downlink channel estimate $\hat{\mathbf{h}}_{\text{DL}}$ is first available only at the receiver, i.e. at the user, and for the BS to have access to the CSI of the downlink channel, the downlink channel estimate $\hat{\mathbf{h}}_{\text{DL}}$ must be quantized and then relayed back to the BS with the limited feedback of B bits. For this, we employ the *random vector quantization* (RVQ) scheme [6], where we assume the existence, at the transmitter (BS) and at the receiver (user), of a codebook with 2^B random beamforming vectors $\mathbf{t}_j$, $j = 1, \ldots, 2^B$, i.i.d isotropically distributed over the M-dimensional unit sphere. For instance, the channel estimate $\hat{\mathbf{h}}_{\text{DL}}[n-1]$ is quantized by selecting the beamforming vector $\mathbf{w}'_{\text{DL}}[n]$ (to be used at time slot n) that best matches $\hat{\mathbf{h}}_{\text{DL}}[n-1]$

$$\mathbf{w}'_{\text{DL}}[n] = \underset{\mathbf{t}_j}{\text{argmax}}\;\; |\mathbf{t}_j^{\text{H}}\hat{\mathbf{h}}_{\text{DL}}[n-1]|^2. \tag{6}$$

Note that in (6), we have written the beamforming vector $\mathbf{w}'_{\text{DL}}$ with a prime compared to $\mathbf{w}_{\text{DL}}$ shown in (3), since in case of feedback errors, the applied beamforming vector $\mathbf{w}_{\text{DL}}[n]$ will not be the same as the beamforming vector $\mathbf{w}'_{\text{DL}}[n]$ desired by the user. After the quantization, the user feeds back the B bits representing the index of the beamforming vector $\mathbf{w}'_{\text{DL}}[n]$ in the codebook. Let us rewrite (3) as

$$\begin{aligned} \mathbf{y}_{\text{DL}}[n] &= \sqrt{P_{\text{DL}}}\;\mathbf{w}^{\text{H}}_{\text{DL}}[n]\left(\hat{\mathbf{h}}_{\text{DL}}[n] + \mathbf{e}_{\text{DL}}[n]\right)\mathbf{s}_{\text{DL}}[n] + \mathbf{v}[n] \\ &= \sqrt{P_{\text{DL}}}\;\mathbf{w}^{\text{H}}_{\text{DL}}[n]\hat{\mathbf{h}}_{\text{DL}}[n]\mathbf{s}_{\text{DL}}[n] + \mathbf{z}[n]. \end{aligned} \tag{7}$$

where $\mathbf{z}[n] = \sqrt{P_{\text{DL}}}\;\;\mathbf{w}^{\text{H}}_{\text{DL}}[n]\mathbf{e}_{\text{DL}}[n]\mathbf{s}_{\text{DL}}[n] + \mathbf{v}[n]$ is the effective noise. The variance of the elements of $\mathbf{z}[n]$ is $\sigma_z^2 = \sigma^2_{e_{\text{DL}}} P_{\text{DL}} + \sigma_v^2$.

III. Feedback Error Probability

Let us now consider the feedback of the uncoded B bits. For instance at time slot $n-1$, the user feeds back the B uncoded bits representing the index of the beamforming vector $\mathbf{w}'_{\text{DL}}[n]$ computed in (6) using $\frac{B}{2}$ QPSK symbols. We employ QPSK for the feedback due to robustness, since higher modulation schemes suffer from higher symbol error probabilites. We do not assume that the feedback bits are coded with the data since the data is usually coded over many time slots and hence, the BS would have to wait more time before being able to use the feedback. Also, we assume the B feedback bits are also sent without any error detection, such that if at least one feedback symbol is in error, there is a *total* loss of the feedback, since no optimized labelling scheme of the feedback bits is employed as in [5].

After a feedback error event, the index received by the BS corresponds to a different beamforming vector than the one intended by the user. Therefore, the beamforming vector applied by the BS, i.e., $\mathbf{w}_{\text{DL}}[n]$ (c.f. (3)), would be different than the beamforming vector $\mathbf{w}'_{\text{DL}}[n]$, whose index was fed back by the user (c.f. (6)). The beamforming vector applied by the BS $\mathbf{w}_{\text{DL}}[n]$ would be completely *uncorrelated* with the actual downlink channel $\mathbf{h}_{\text{DL}}[n]$ such that the performance would be the same as if there were no feedback, i.e. as with random beamforming.

Let us compute the feedback error probability by assuming that feedback, like the data, is detected in the uplink using *maximum ratio combining* (MRC) with the M receiving antennas at the BS. Recall that the receive CSI for MRC is the uplink channel estimate $\hat{\mathbf{h}}_{\text{UL}}$ with error variance $\sigma^2_{e_{\text{UL}}}$. With MRC the SNR in the uplink at time slot n for the feedback (and also for the uplink data) would be given by

$$\gamma_{\text{F}}[n] \;=\; \frac{P_{\text{UL}}\|\hat{\mathbf{h}}_{\text{UL}}[n]\|_2^2}{\sigma_u^2 + P_{\text{UL}}\sigma^2_{e_{\text{UL}}}}. \qquad (8)$$

Using [7, (6.23)], the symbol error probability p_{s} for uncoded QPSK with SNR $\gamma_{\text{F}}[n]$ can be approximated as $p_{\text{s}}(\gamma_{\text{F}}[n]) \approx 2\text{Q}\left(\sqrt{\gamma_{\text{F}}[n]}\right)$, where $Q(\bullet)$ is the Q-function. Since one symbol error leads to a total feedback loss, the average feedback error probability p_ϵ is given as

$$\begin{aligned} p_\epsilon &= \text{E}[1-(1-p_{\text{s}}(\gamma_{\text{F}}))^{\frac{B}{2}}] \\ p_\epsilon &\approx 1-(1-\text{E}[p_{\text{s}}(\gamma_{\text{F}})])^{\frac{B}{2}}, \end{aligned} \qquad (9)$$

where the last step follows first by obtaining an upper bound through Jensen's inequality (by noting that $1-(1-p_{\text{s}}(\gamma_{\text{F}}))^{\frac{B}{2}}$ is a concave function over $p_{\text{s}} \in [0,1]$) and then by noting that this upper bound is tight for small values of p_{s}. The symbol error probability $\text{E}[p_{\text{s}}(\gamma_{\text{F}})]$ averaged over the uplink channel realizations for QPSK with MRC can be computed by making use of the result from [7, (7.20)]. $\text{E}[p_{\text{s}}(\gamma_{\text{F}})]$ with QPSK symbols and MRC with M antennas can be shown is given as

$$\text{E}[p_{\text{s}}(\gamma_{\text{F}})] = 2\left(\frac{1-\kappa}{2}\right)^M \sum_{m=0}^{M-1} \binom{M-1+m}{m} \left(\frac{1+\kappa}{2}\right)^m, \qquad (10)$$

where

$$\kappa \;=\; \sqrt{\frac{1-\sigma^2_{e_{\text{UL}}}}{1+2\frac{\sigma_u^2}{P_{\text{UL}}}+\sigma^2_{e_{\text{UL}}}}}.$$

The upper bound on the feedback error probability follows by plugging (10) in (9). It is clear that feedback error probability p_ϵ depends on the uplink resource allocation, i.e. T_{UL} and B, which are fixed, and on the average uplink SNR, which is a long term parameter. Hence, the feedback error probability p_ϵ remains constant over many transmission blocks. Note that as T_{UL} increases, the uplink SNR $\gamma_{\text{F}}[n]$ given in (8) increases and consequently the feedback error probability p_ϵ decreases.

With a probability $1-p_\epsilon$ we have correct feedback, i.e. $\mathbf{w}_{\text{DL}}[n] = \mathbf{w}'_{\text{DL}}[n]$, and with a probability p_ϵ we have erroneous feedback, i.e. $\mathbf{w}_{\text{DL}}[n] \neq \mathbf{w}'_{\text{DL}}[n]$. After an error event, the user's assumption about the beamforming vector is wrong. Although the BS and the user are unaware of the feedback errors, we can assume that the BS and the user know the feedback error probability p_ϵ.

IV. Achievable Rates in the Downlink

The capacity with imperfect CSI is unknown in general and in such a case, one can instead recur to bounds on the capacity in the downlink with imperfect CSI. A lower bound of the capacity under beamforming (c.f. (7)) with imperfect CSI which is estimated, quantized, outdated and affected by feedback errors is given by [8, Theorem 3.3]:

$$\begin{aligned} C_{\text{DL,lb}} \;=\; & \frac{D_{\text{DL}}}{T}(1-p_\epsilon)\left(1-\frac{1}{\sqrt{2\pi}}\frac{\sigma_\eta}{\mu_\eta}\right)\times \\ & \log_2\left(1+\frac{(\alpha(M\text{E}[\nu]-1)+1)(1-\sigma_e^2)^3}{\sigma_e^2+\frac{\sigma_v^2}{P}}\right) \\ & + \frac{D_{\text{DL}}}{T}\, p_\epsilon \log_2(\text{e})\, \text{e}^{\frac{\sigma_v^2}{P_{\text{DL}}}} \text{E}_1\left(\frac{\sigma_v^2}{P_{\text{DL}}}\right) \\ & -\frac{1}{T}\left(p_\epsilon \log_2(\text{e})\, \text{e}^{\frac{\sigma_v^2}{P_{\text{DL}}}} \sum_{k=1}^{D_{\text{DL}}} \text{E}_k\left(\frac{\sigma_v^2}{P_{\text{DL}}}\right) + h_{\text{b}}(p_\epsilon)\right), \end{aligned} \qquad (11)$$

where E_k denotes the generalized exponential integral $\text{E}_k(z) = \int_1^\infty \frac{\text{e}^{-zt}}{t^k}\text{d}t$, $h_{\text{b}}(p_\epsilon)$ is the binary entropy function with probability p_ϵ and with $\text{E}[\nu]$ and $\frac{\sigma_\eta}{\mu_\eta}$ given by (20) and (24) in [8], respectively.

An upper bound on the capacity in the downlink with imperfect CSI can be derived by assuming that the user knows the exact channel vector $\mathbf{h}_{\text{DL}}[n]$ and the actual beamforming vector applied by the BS $\mathbf{w}_{\text{DL}}[n]$, and is given by [8, Theorem 3.4]: $C_{\text{DL,ub}} =$

$$\begin{aligned} & \frac{D_{\text{DL}}}{T}(1-p_\epsilon)\log_2\left(1+\frac{P_{\text{DL}}}{\sigma_v^2}\alpha(1-\sigma^2_{e_{\text{DL}}})^3\,(M\text{E}[\nu]-1)+(1-\sigma^2_{e_{\text{DL}}})\right) \\ & + \frac{D_{\text{DL}}}{T}\, p_\epsilon \log_2(\text{e})\text{e}^{\frac{\sigma_v^2}{P_{\text{DL}}}} \text{E}_1\left(\frac{\sigma_v^2}{P_{\text{DL}}}\right), \end{aligned} \qquad (12)$$

where the first term corresponds to the capacity with correct feedback and the second term corresponds to the capacity achieved after there is a feedback error.

With these two bounds, we can now present some simulation results to show the effect of the feedback errors on the downlink capacity. Let us for now assume that the uplink is solely employed for the feedback, i.e. by setting the uplink training length T_{UL} as large as possible: $T_{\text{UL}} = T - \frac{B}{2}$. This leads to the smallest possible feedback error probability and our described FDD two-way systems becomes actually a one-way system. Fig. 1 depicts the achievable downlink capacity as a function of the number of feedback bits B for $M = 6$, $T = 100$, $\frac{P}{\sigma_v^2} = 8$ dB, $\alpha = 1$ (no outdating) and $\frac{P_{\text{UL}}}{\sigma_u^2} = 0$ dB. The solid curve with the square markers represent the capacity with perfect CSI and the dashed curve represents also the capacity with perfect CSI but multiplied with a factor $\frac{T-T_{\text{DL}}}{T} < 1$, i.e. including part of the rate loss due to training. These two curves are unachievable in a real system with finite resources, but they are just shown here as a reference. The next set of curves depicted are the upper and lower bounds with imperfect CSI but assuming correct feedback, i.e. the terms with the factor $(1-p_\epsilon)$ in (11) and (12), respectively. For each B, the optimum training length is determined by finding numerically the T_{DL} that maximizes each of the bounds for the given setting. It can be observed that both bounds increase with B and for $B \geq 32$ the capacity increase is just marginal.

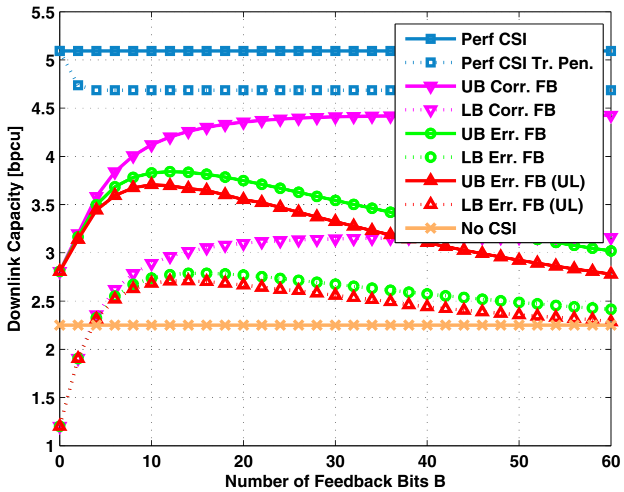

Fig. 1. Optimum Number of Feedback Bits: Downlink Capacity vs. Feedback.

The set of curves with the circular markers are the upper bound and the lower bound considering feedback errors, i.e. (12) and (11). As before, for each B, the optimum downlink training length is determined by finding the T_{DL} that maximizes each of the bounds. Since both bounds follow a similar behavior and have a maximum at almost the same value, we can expect that the optimum number of feedback bits for the true capacity is also between $B = 10$ and $B = 12$, feedback bits. The capacity increases with B at first, since the reduction in the quantization error overcompensates the increase of the feedback error probability p_ϵ. After the optimum $B = 10/B = 12$, the quantization error could be further reduced by increasing B but this cannot compensate the increase of p_ϵ. As B keeps on increasing, the downlink capacity is dominated by the capacity achieved with only erroneous feedback which is the same as without feedback or equivalently to random beamforming, i.e. with no CSI. Hence, as $B \to \infty$ the capacity converges to the capacity with no CSI as shown in the figure. Hence, we have a tradeoff between the quantization error and the feedback error probability.

V. Achievable Rates in the DL and UL

The previous result is under the assumption that the uplink is solely employed for feedback. Now let us consider the transmission of data also in the uplink, i.e. like the described two-way FDD system in Section II. In this case, the downlink capacity would be further degraded as can be seen in the bounds labelled as "Err. Fb (UL)" in Fig. 1. This is a consequence of the reduction of the uplink SNR due to the decrease of the uplink training. Hence, due to the two-way nature of the system, we have to consider the achievable rates in the the downlink and uplink of a single-user FDD system with limited feedback. In a two-way system there is no single figure of merit as in a one-way system. Therefore, we propose to solve the following biobjective maximization problem, which represents a lower bound on the achievable boundary of the uplink and downlink rates with imperfect CSI:

$$\begin{aligned} &\max \quad (\, C_{\text{DL,lb}} \,,\, C_{\text{UL,lb}} \,) \\ &\text{s.t.} \quad 0 \leq B < 2T,\ 0 \leq T_{\text{DL}} \leq T,\ \ 0 \leq T_{\text{UL}} \leq T - B/2, \end{aligned} \tag{13}$$

where $C_{\text{DL,lb}}$ and $C_{\text{UL,lb}}$ is a lower bound on the downlink capacity and uplink capacity with imperfect CSI, respectively. We employ lower bounds, since the the capacity with imperfect CSI in general is unknown. $C_{\text{DL,lb}}$ has been already presented in (11). A lower bound on the uplink capacity $C_{\text{UL,lb}}$ with beamforming based on imperfect CSI (c.f (2)) is given by $C_{\text{UL,lb}} =$

$$\frac{T - T_{\text{UL}} - \frac{B}{2}}{T} \log_2(\mathrm{e}) \mathrm{e}^{\frac{\sigma_u^2 + P_{\text{UL}}\sigma_{e\text{UL}}^2}{P_{\text{UL}}(1-\sigma_{e\text{UL}}^2)}} \sum_{k=1}^{M} \mathrm{E}_k \left(\frac{\sigma_u^2 + P_{\text{UL}}\sigma_{e\text{UL}}^2}{P_{\text{UL}}(1-\sigma_{e\text{UL}}^2)} \right). \tag{14}$$

The solution to (13) is a Pareto boundary. We have confirmed the convexity of the boundary with simulations and hence, the boundary can be obtained by finding the downlink and uplink rates which maximize the weighted sum of the downlink and uplink capacity lower bounds:

$$\max \quad \mu\, C_{\text{DL,lb}} + (1-\mu)\, C_{\text{UL,lb}}, \tag{15}$$

for $\mu \in [0, 1]$. The solution can be obtained similarly to the solution to the optimization given in [9, Eq. (14)], where μ is practically $\mu = 0.5$.

An example of the resulting boundary of the achievable rates in the downlink and uplink is shown in Fig. 2 for different coherence lengths T, $M = 10$, $\alpha = 0$ and SNR $= 0$ dB per transmit antenna in the downlink and uplink. We point out that each point on the boundary is achieved with different downlink training, uplink training and number of feedback bits. Although they are unachievable in practice, we also include the uplink and downlink capacities with perfect CSI as a reference. For small T, the maximum downlink rate is achieved when the uplink rate $C_{\text{UL,lb}} \to 0$. The maximum uplink rate can be achieved with a $C_{\text{DL,lb}} > 0$ by setting $B = 0$. With no feedback, the performance in the downlink is non-zero since it is the same as with random beamforming. As T increases, the achievable rates in both the uplink and downlink increase since the relative overhead due to the training and feedback decreases and so can the estimation error and quantization error. However, note that for any T the gap between the uplink capacity with imperfect CSI and perfect CSI is smaller than the gap in the downlink, since the CSI in the uplink is only estimated while the CSI in the downlink is estimated, quantized and affected by feedback errors.

The circle shown on each curve represent the set of downlink and uplink rates which maximize the downlink-uplink sum capacity lower bound with $\mu = 0.5$. Note that for large T, this solution represents a good figure of merit for the two-way FDD system.

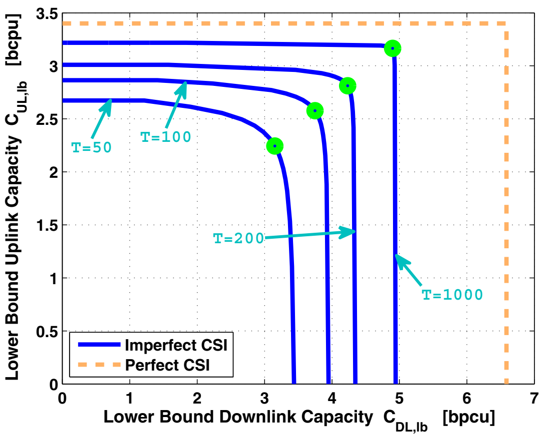

Fig. 2. Achievable rates with uncoded feedback, $M = 10$.

A. Coded Feedback

In order to reduce the feedback error probability p_ϵ we propose to code the B feedback bits using a linear code of length K, i.e. with a code rate of $r = \frac{B}{K}$ and $K - B$ redundancy bits. The K bits are sent with $\frac{K}{2}$ QPSK symbols and therefore, the feedback now consists of $\frac{K}{2}$ QPSK symbols instead of $\frac{B}{2}$ QPSK symbols. To this end, we make use of the upper bounds on the maximum possible minimum distance $d_{\text{min}}(K, B)$ of a linear code with codeword length K and dimension B given in [10] for different B and K. In this case, the feedback error probability with coded feedback and hard decoding is reduced to

$$p_{\epsilon,\text{coded}} \approx 1 - \sum_{m=0}^{t} \begin{pmatrix} K \\ m \end{pmatrix} (\text{E}[p_\text{b}(\gamma_\text{F})])^m \, (1 - \text{E}[p_\text{b}(\gamma_\text{F})])^{K-m},$$

where $d_{\text{min}}(K, B)$ is obtained from [10], $t = \lfloor \frac{d_{\text{min}}(K,B)-1}{2} \rfloor$ is a lower bound on the number of feedback bit errors that can be corrected and $\text{E}[p_\text{b}(\gamma_\text{F}])$ is the average bit error probability with QPSK and MRC. The bit error probability $p_\text{b}(\gamma_\text{F})$ with a given uplink SNR γ_F is given by $p_\text{b}(\gamma_\text{F}) = \text{Q}\left(\sqrt{\gamma_\text{F}[n]}\right) \approx p_\text{s}(\gamma_\text{F})/2$. Therefore, $\text{E}[p_\text{b}(\gamma_\text{F}])$ can be derived similarly like $\text{E}[p_\text{s}(\gamma_\text{F}])$ derived in Section III. The reduction of the feedback error probability leads to an increase of the downlink capacity given in (12), since the capacity with correct feedback is larger than the capacity with erroneous feedback. However, this comes at the expense of a reduction of the uplink capacity because $\frac{K-B}{2}$ less symbols are available for the data, i.e. the pre-factor in (14) would be reduced to $\frac{T - T_\text{UL} - \frac{K}{2}}{T}$ with $K > B$. The tradeoff between the uplink and downlink with coded feedback is depicted in Fig. 3 for $T = 100$, $M = 4$, $\text{SNR} = 0$ dB per transmit antenna in the downlink and $\text{SNR} = -3$ dB in the uplink. As a reference we include the case without coding, i.e. $r = 1$. As the code rate r is decreased, the feedback error probability decreases since t increases. The downlink rate increases but with a slight decrease of the uplink rate as can be observed for the solution of the downlink-uplink sum capacity with $\mu = 0.5$, which is marked by the circle on each curve.

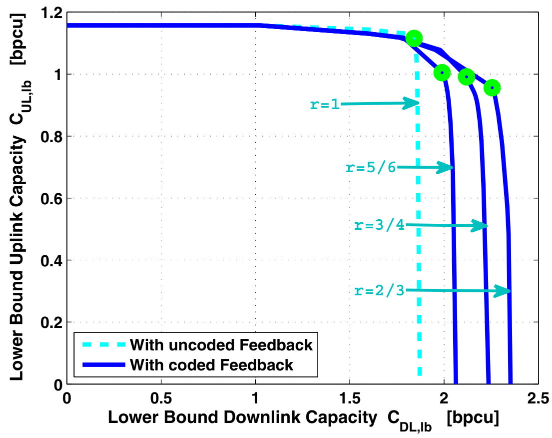

Fig. 3. Achievable rates with coded feedback, $M = 4$, $T = 100$.

VI. Conclusions

In this work we discuss the inherent tradeoff between the uplink rate and the downlink in a single-user two-way FDD system with limited feedback. It was observed that an interesting figure of merit for the two-way system is the sum of the downlink and uplink capacities since as T increases, with this solution one is able to achieve practically the largest possible rate on each link for each T. Hence, the downlink-uplink sum capacity can be employed as a figure of merit for optimizing the resource allocation in FDD systems. We have also shown the existence of an optimum number of feedback bits in the downlink, due to the feedback errors. Future work includes extending this framework to the multiuser case.

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