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Time Domain Model Reduction By Moment Matching

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ABSTRACT

This dissertation delivers new results in the model reduction of large-scale linear time-invariant dynamical systems. In particular, it suggests solutions for the well-known problem of finding a suitable interpolation point in order reduction by moment matching. As a first step, a new time-domain model order reduction method based on matching some of the first Laguerre coefficients of the impulse response is presented. Then, the equivalence between the classical moment matching and the Laguerre-based reduction approaches both in time- and frequency-domain is shown. In addition, this equivalence is generalized to include a larger family of coefficients known as *generalized Markov parameters*. This allows a first time-domain interpretation of the moment matching approach which has been until now developed and applied only in the frequency domain. Moreover, using this equivalence, the open problem of choosing an *optimal* expansion point in the rational Krylov subspace reduction methods (moment matching about $s_0 \neq 0$) is reformulated to the problem of finding the *optimal* parameter α in the Laguerre-based reduction methods. Based on the Laguerre representation of the system, two methods for the choice of the Laguerre parameter and, consequently, the single expansion point in rational interpolation order reduction are presented. Accordingly, different model reduction algorithms are suggested. The importance of these approaches lies in the fact that they try to approximate the impulse response of the original system, have a simple structure, are numerically efficient, and are suitable for the reduction of large-scale systems.

To my family and my wife.

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GLOSSARY

Notations

$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{R}^n$	The set of all vectors of dimension n with real entries
$\mathbb{R}^{n\times m}$	The set of all $n \times m$ matrices with real entries
$\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}$	State space matrices of the original MIMO state space system
$\mathbf{E}, \mathbf{A}, \mathbf{b}, \mathbf{c}^T$	State space matrices of the original SISO state space system
$\mathbf{x}$	The vector of state variables of the original state space system
n	Order of the original state space model
$\mathbf{E}_r, \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r$	State space matrices of the reduced MIMO state space system
$\mathbf{E}_r, \mathbf{A}_r, \mathbf{b}_r, \mathbf{c}_r^T$	State space matrices of the reduced SISO state space system
$\mathbf{x}_r$	The vector of state variables of the reduced state space system
q	Order of the reduced state space model
u	The input function
$\mathbf{u}$	The vector of input functions
y	The output function
$\mathbf{y}$	The vector of output functions
m_i	i-th moment of a SISO system
$\mathbf{m}_i$	i-th moment of a MIMO system
M_i	i-th Markov parameter of a SISO system
$\mathbf{M}_i$	i-th Markov parameter of a MIMO system

$\mathcal{K}_Q(.,.)$	The Krylov subspace
$\mathbf{V}, \mathbf{W}$	Projection matrices for the reduction in state space
$\mathcal{L}_2(R_+)$	The Hilbert space.
$l_i(t)$	the i-th Laguerre polynomial
$\phi_i^\alpha(t)$	the i-th Laguerre function
$\mathcal{H}_2$	The Hardy space

Abbreviations

LTI	Linear Time Invariant
TBR	Truncated Balanced Realization
MEMS	Micro-Electro-Mechanical System
SISO	Single Input Single Output
MIMO	Multi Input Multi Output
FEM	Finite Element Method
SVD	Singular Value Decomposition
MOR	Model Order Reduction
LHS	Left Hand Side
RHS	Right Hand Side
I/O	Input-Output
HSV	Hankel Singular Value

Chapter 1

INTRODUCTION

The problem of model order reduction (MOR) of linear and nonlinear dynamical systems has been widely studied in the last two decades and still considered nowadays as a hot topic. From one side, this is due to the increasing capability of methods and computers to accurately model real-world systems, for instance, in VLSI circuits design [70], civil engineering, aerospace engineering [83], earthquake engineering [61], mechanical and Micro-Electro-Mechanical Systems (MEMS) [8, 72], in molecular biology and environmental engineering [5]. And from the other side, it is due to the unavoidable requirement of being able to handle those generated models for the purpose of e.g. simulation (also real-time), optimization, and control.

Given a large-scale linear or nonlinear system with a predefined output, the model reduction problem can be simply formulated as finding a *simpler* system that *approximates* the behavior of the original one. As linear systems are generally described by a mathematical model that involves mainly first and/or second order differential equations, simpler, in this context means having fewer states. In other words, the lower the order of the differential equations, the simpler the model is and the easier to work with. The approximation concept is based generally on the minimization of some predefined errors between the original and approximated outputs. In the literature, different error systems have been applied and accordingly reduction methods minimizing these differences between the systems have been presented. For instance, H_2 and H_∞ error norms have been

considered in [43] and [49] respectively. Also, approximation errors based on the Hankel operator are widely used and are the key idea of the famous Balancing and Truncation (TBR) method [35]. Krylov-based order reduction can be also considered to zero the error between the original and reduced transfer functions at predefined frequencies.

Nowadays, the definition of the model reduction problem can be extended to include several properties of the system to be preserved by the reduction method, in addition to minimizing the approximation error. For instance, it may be required to preserve the stability or passivity of the original system, the matrix special structure or sparsity, or to reduce the computational cost of the method so that it could be applied to very high-order system and keep the reduction step numerically justified.

1.1 Main approaches of order reduction

After around twenty years of research in the field of MOR of linear systems, the developed methods and algorithms can be classified under two main approaches: the Singular Value Decomposition (SVD)-based approximation and the Krylov subspace methods.

1.1.1 Balancing and truncation

The most famous and commonly used method in SVD-based approaches is the Lyapunov balancing and truncation known also as *truncated balanced realization (TBR)* [5, 35, 63]. The basic idea of this approach is to try to delete the states that do not contribute 'considerably' to the system input-output behavior. In order to study the contribution of each state, the input-output response is divided into input-to-state and state-to-output responses. The *controllability gramian*, which is defined as the integral of the product of the input-to-state response and its complex conjugate,

$$\mathbf{P} = \int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau,$$

is used as an indicator of the amount of input energy needed to reach a certain state. Whereas the *observability gramian*, defined as the integral of the state-to-output response and its complex conjugate,

$$\mathbf{Q} = \int_0^\infty e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}\tau}d\tau,$$

is used as an indicator of the effect of any given initial state on the output.

It can be shown that these gramians can be calculated by solving the following two Lyapunov equations involving matrices in the size of the original model:

$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{BB}^T = \mathbf{0},$$
$$\mathbf{A}^T\mathbf{Q} + \mathbf{QA} + \mathbf{C}^T\mathbf{C} = \mathbf{0}.$$

However, states that are difficult to control, i. e. requiring a large amount of input energy, may not be difficult to observe and vice-versa. Thus, there is a need for a certain representation of the system which makes the states difficult to control also difficult to observe and vice-versa. This can be achieved by the so-called *balancing transformation*. Balancing was first introduced by Mullis and Roberts [64] in the design of digital filters and was later brought to the field of control and order reduction by Moore in the beginning of the 80's [63]. By balancing, a state transformation $\mathbf{T}$ is found which, after being applied to the system matrices, makes the controllability and observability gramians diagonal, equal, and having the *Hankel singular values* (HSV) on their diameter. By definition, the Hankel singular values of an observable, controllable, and stable system are the positive square roots of the eigenvalues of the product of the gramians and are input-output invariants.

Now, for a balanced stable system, the contribution of each state to the input-output behavior can be measured by the relative magnitude of its Hankel singular value. Hence, after balancing the original system and computing the Hankel singular values, a clear idea about the contribution of each state to the system I/O behavior can be extracted. The truncation of the least contributing states results in the reduced order model.

One of the most important advantages of balancing and truncation is preserving the stability of the original model, i. e. if the original high-order model is stable, then the obtained reduced model will be also stable. Also, a global apriori error bound between the original and reduced systems exists and depends only on the neglected Hankel singular values [35]. In [14, 58, 74, 75], different reduction approaches that modify the original TBR were introduced. The method of balancing is then changed with the aim of achieving better approximations or preserving system properties like passivity.

While TBR is theoretically attractive and yields to excellent approximate models in practice, its use for model reduction of large scale systems (a few ten thousands) is costly by growing computational complexity in solving two Lyapunov equations. Many works tried to extend the application of TBR to higher order systems by parallel computation [13, 14], and approximate solution of Lyapunov or Sylvester equations, leading to low rank gramians, and resulting in what is known nowadays as approximate balancing and truncation [40, 45, 68, 69, 81].

1.1.2 Krylov subspace methods

The second main approach in MOR of linear systems is *Krylov-based order reduction* which is a projection-based method that was initially developed to solve large-scale linear mathematical problems, and at a later stage applied to the control engineering field and subsequently to model order reduction [29, 32, 33, 37, 48, 86].

In this family of methods, the aim is to match some of the first coefficients of the Taylor series expansion of the transfer functions of the original and reduced models. When the series expansion is developed about a finite point, we speak about moment matching and Padé interpolation however when it is about $s = \infty$, we speak about Markov parameter matching and partial realization. This is in fact the reason behind calling this approach *moment matching* or *order reduction by moment matching*.

Krylov subspaces are needed in this approach to calculate the projection matrices that

are shown to form bases of these subspaces when aiming at moment and/or Markov parameter matching. To this end, numerically efficient and iterative methods known as the Lanczos [57], the Arnoldi [6] and the two-sided Arnoldi [24, 77] algorithms are employed. Once calculated, these projection matrices are applied to the system matrices of the original model resulting in a reduced-order model matching some of the moments and/or the Markov parameters.

When comparing TBR to Krylov subspaces methods, it can be clearly stated that TBR is superior to Krylov subspaces when stability of the reduced system and error bound are in question. However, Krylov subspaces methods are superior to TBR when computational effort and memory requirements are in question. These methods involve mainly matrix-vector multiplications and the reduced order model is obtained directly by a projection to a lower dimensional subspace calculated through a completely iterative approach. This makes them suitable for the reduction of very large-scale systems (hundreds of thousands), far above the orders achieved by balancing and truncation.

1.2 Thesis contributions

In model order reduction of large-scale systems, Krylov subspace methods are considered one of the most successful and widely used approaches. However, it is still early to speak about an automatic Krylov-based order reduction where the user just defines how accurate the reduced-order model should be and keep the rest for the algorithm, as in the case of TBR. This is mainly due to the absence of a global error bound and to the fact that several parameters of this approach have still to be heuristically determined by the user based on the results obtained. These parameters are mainly the order of the reduced model and the number and location of the expansion points s_0 about which the moments are to be matched. The latter problem constitutes one of the main focuses of this work. Here, the case of a single expansion point is considered as it is the most common case in practice and the numerically cheapest way to calculate the reduced model. In

addition, the interpretation of these methods is restricted to be a local approximation of the frequency response of the original system, where the frequency range of interest is determined by the expansion point(s). Hence, moment matching methods can not guarantee a good approximation of the time responses of the original system. This is due to the fact that it is generally hard in practice to predict the time-domain approximation from the frequency-domain one.

The contributions of this dissertation can be summarized as follows:

- A new time-domain Laguerre-based order reduction method based on matching the Laguerre coefficients of the impulse response of the original and reduced systems is introduced. This method is shown to be computationally efficient and applicable to large-scale systems by employing suitable Krylov subspaces.

- The complete equivalence between all Laguerre-based methods and moment matching about a single interpolation is proven both in the time and frequency-domain.

- A new time-domain interpretation for moment matching via the Laguerre representation of the system is presented.

- New time-domain algorithms for the choice of the expansion point s_0 in Krylov-based MOR are developed.

These results offer now the possibility to have a look at moment matching from a time-domain perspective and make a step towards a completely automatic Krylov-based order reduction.

1.3 Dissertation overview

This introductory chapter is concluded with summaries of the different chapters of this dissertation.

In the next chapter, a brief overview of the relevant definitions of linear algebra and system theory is presented including a. o. Krylov subspace, projection, and stability of dynamical systems. In addition, the Laguerre polynomials and functions together with some of their important properties are introduced.

The third chapter provides a complete overview of Krylov-based model order reduction. The framework of this method together with all the theoretical and numerical aspects are presented. A special section is dedicated to describe the main properties of this approach from the mathematical and engineering point of views. The chapter is concluded with a detailed study of the main open problems of this method.

In chapter 4, as an alternative to moment matching, a reduction method based on matching the coefficients of the Laguerre series expansion of the transfer function is presented. Using the fact that a Laguerre series expansion can be connected to a Taylor series by a certain bilinear transformation, this method uses the numerically robust and efficient algorithms, implemented for moment matching, to achieve Laguerre coefficients matching.

A new time-domain model order reduction method based on the Laguerre function expansion of the impulse response is presented in chapter 5. The Laguerre coefficients of the impulse response of the reduced-order model, which is calculated using a projection whose matrices form bases of appropriate Krylov subspaces, match, up to a given order, those of the original system. It is also shown that this matching of coefficients applies to the response on any other input signal $u(t) \in \mathcal{L}_2(R_+)$.

In Chapter 6, the equivalence between the classical moment matching and the Laguerre-based reduction approach in frequency-domain is shown. This equivalence is then generalized to include a family of coefficients known as *generalized Markov parameters*. A similar equivalence between the time-domain Laguerre-based approach and moment matching about a single expansion point is also presented. Moreover, the connection between MOR in continuous and discrete-time is investigated and the equivalence between matching the Markov parameters in the z-domain and the moment matching about a single point is

shown.

In chapter 7, several solutions for the well-known problem of finding a suitable interpolation point in order reduction via moment matching by Krylov subspaces are presented. By using the equivalence property of moment matching and Laguerre-based order reduction in time-domain, the problem is reformulated as finding the best choice for the free parameter α in the Laguerre basis. Minimizing appropriate cost functions using the system's Laguerre representation is the key point toward finding this optimal interpolation point.

The dissertation is concluded with a summary of the results and possible future works.

Chapter 2

PRELIMINARIES

In this chapter, definitions and properties from linear algebra, linear system theory, and Laguerre polynomials, relevant to this dissertation, are presented. Krylov subspaces, projections, stability, the Tustin transformation, controllability and observability, together with the definition of Laguerre polynomials and their various properties are briefly described.

2.1 Linear algebra

One of the main components of Krylov-based order reduction are Krylov subspaces, which are quite famous in linear algebra and almost unknown in the field of automatic control. As the reduced system in all the MOR methods of this work is obtained by a projection to a lower dimensional Krylov subspace, it is of great importance to review the definition of this subspace and of a projector and its application to solve linear systems.

2.1.1 The Krylov subspace

A Krylov subspace is spanned by a certain sequence of vectors generated by a matrix and a vector as follows: Given a matrix $\widetilde{\mathbf{A}}$ and a starting vector $\widetilde{\mathbf{b}}$, the q-th Krylov subspace $\mathcal{K}_q(\widetilde{\mathbf{A}}, \widetilde{\mathbf{b}})$ is spanned by the following sequence of q column vectors called basic vectors [5, 7, 33]:

$$\mathcal{K}_q(\widetilde{\mathbf{A}}, \widetilde{\mathbf{b}}) = span\{\widetilde{\mathbf{b}}, \widetilde{\mathbf{A}}\widetilde{\mathbf{b}}, \cdots, \widetilde{\mathbf{A}}^{q-1}\widetilde{\mathbf{b}}\}, \tag{2.1}$$

where $\widetilde{\mathbf{A}} \in \mathbb{R}^{n\times n}$, and $\widetilde{\mathbf{b}} \in \mathbb{R}^n$ is called starting vector. The first linearly independent basic vectors constitute what is known as basis of a Krylov subspace.

Now when there exist more than one starting vector i.e. $\widetilde{\mathbf{b}}$ is not a vector but a matrix $\widetilde{\mathbf{B}}$, equation (2.1) is generalized to the *block Krylov subspace* as follows:

$$\mathcal{K}_q(\widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}) = span\{\widetilde{\mathbf{B}}, \widetilde{\mathbf{A}}\widetilde{\mathbf{B}}, \cdots, \widetilde{\mathbf{A}}^{q-1}\widetilde{\mathbf{B}}\}, \tag{2.2}$$

where $\widetilde{\mathbf{A}} \in \mathbb{R}^{n\times n}$, and $\widetilde{\mathbf{B}} \in \mathbb{R}^{n\times m}$ are the starting vectors. The block Krylov subspace with m starting vectors can be considered as a union of m Krylov subspaces defined for each starting vector [76].

2.1.2 Orthogonal projection

A matrix $\mathbf{P} \in R^{n\times n}$ is called a projector onto a subspace $\mathcal{S} \subseteq \mathbb{R}^n$ if it satisfies:

$$range(\mathbf{P}) = \mathcal{S}, \text{ and } \mathbf{P}^2 = \mathbf{P}. \tag{2.3}$$

From this definition, it can be shown that if $\mathbf{x} \in \mathbb{R}^n$, then it can be rewritten as

$$\mathbf{x} = \mathbf{P}\mathbf{x} + (\mathbf{I} - \mathbf{P})\mathbf{x}.$$

This allows to decompose the subspace $\mathbb{R}^n$ into two subspaces as,

$$\mathbb{R}^n = span(S_1 + S_2), \tag{2.4}$$

where $range(\mathbf{P}) = S_1$ and $null(\mathbf{P}) = S_2$. In other words, a projector $\mathbf{P}$ separates $\mathbb{R}^n$ into two subspaces and projects onto S_1 along or parallel to S_2.

Based on this subspace decomposition, two bases are required to obtain a matrix representation of a projector: a basis $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q]$ for the subspace S_1 and $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_q]$ for the subspace S_2. If these bases are biorthogonal, i.e. $\mathbf{W}^T\mathbf{V} = \mathbf{I}$, then it follows that the matrix projector is,

$$\mathbf{P} = \mathbf{V}\mathbf{W}^T, \tag{2.5}$$

otherwise,

$$\mathbf{P} = \mathbf{V}(\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T. \tag{2.6}$$

Definition 2.1. *An orthogonal projector is one that projects onto a subspace S_1 along a space S_2 where S_1 and S_2 are orthogonal. Algebraically, a projector is orthogonal when $\mathbf{P}^T = \mathbf{P}$.*

It should be noted that orthogonal projectors do not have necessarily biorthogonal bases.

2.1.3 Petrov Galerkin projection

Assume we want to try to express the solution (or state) $\mathbf{x}(t) \in \mathbb{R}^n$ of the system (2.9) with only q variables. The solution can be thus written as $\mathbf{x}(t) = \mathbf{V}\mathbf{x}_r(t)$, where $\mathbf{x}_r(t) \in \mathbb{R}^n$ and $\mathbf{V} \in \mathbb{R}^{n\times q}$. If this approximated solution is put into the original state-space model (2.9), we get,

$$\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{B}\mathbf{u}(t) + \epsilon(t), \tag{2.7}$$

where $\epsilon(t)$ is the residual. For $\mathbf{x}(t) = \mathbf{V}\mathbf{x}_r(t)$ to be an exact solution, $\epsilon(t)$ should be equal to zero for all t.

This system is generally over determined as it has n equations but only q unknowns in $\mathbf{x}_r(t)$. To find a unique solution, it is generally imposed that the projection (given by $\mathbf{W}^T$) of the residual $\epsilon(t)$ onto the subspace spanned by $\mathbf{V}$ is zero, leading to $\mathbf{W}^T\epsilon(t) = 0$.

This results in the equation,

$$\mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{B}\mathbf{u}(t), \tag{2.8}$$

which is exactly defined as the *Petrov Galerkin projection* of (2.9) with $\mathbf{P} = \mathbf{V}\mathbf{W}^T$ under the assumption that $\mathbf{V}$ and $\mathbf{W}$ are biorthogonal, and $\mathbf{P} = \mathbf{V}(\mathbf{W}^T\mathbf{V})^{-1}\mathbf{W}^T$ otherwise. This approach constitutes in fact the fundamental of all projection methods for model order reduction, including the ones involved in this work. Notice that if $\mathbf{W}^T = \mathbf{V}^T$, this projection is called a *Galerkin projection.*

2.2 Linear dynamical systems

This section presents the necessary background on LTI dynamical systems based on [5, 47] and the references therein. In addition, an overview of the properties induced by the Tustin and bilinear transformations when applied to a state-space model is described.

Consider the continuous-time state-space dynamical system

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{2.9}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ are the constant system's matrices, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, $\mathbf{x}(t) \in \mathbb{R}^n$ are respectively the input, output and states vectors of the system. For SISO systems, i. e. $p = m = 1$, the matrices $\mathbf{B}$ and $\mathbf{C}$ become the vectors $\mathbf{b}$ and $\mathbf{c}^T$, the vectors $\mathbf{u}$ and $\mathbf{y}$ become the scalars u and y.

After integration and assuming zero initial conditions, the state equation of system (2.9) becomes

$$\mathbf{E}\mathbf{x}(t) = \mathbf{A}\int_0^t \mathbf{x}(\tau)d\tau + \mathbf{b}\int_0^t u(\tau)d\tau, \tag{2.10}$$

and its impulse response can be shown to be

$$\mathbf{h}(t) = \mathbf{C}\mathbf{x}(t) = \mathbf{C}e^{(\mathbf{E}^{-1}\mathbf{A})t}\mathbf{E}^{-1}\mathbf{B}. \tag{2.11}$$

The transfer function of the system (2.9) in the Laplace domain is then:

$$\mathbf{H}(s) = \mathcal{L}(\mathbf{h}(t)) = \int_0^\infty \mathbf{h}(t)e^{-st}dt = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}. \tag{2.12}$$

Similarly, by considering the discrete-time state-space dynamical system,

$$\begin{cases} \mathbf{E}_d\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k), \\ \mathbf{y}(k) = \mathbf{C}_d\mathbf{x}(k). \end{cases} \tag{2.13}$$

its impulse response assuming zero initial conditions can be shown to be,

$$\mathbf{h}(k) = \mathbf{C}_d\mathbf{E}_d\mathbf{A}_d^{k-1}\mathbf{E}_d^{-1}\mathbf{B}_d, \tag{2.14}$$

and its transfer function in the z-domain,

$$\mathbf{H}(z) = \mathcal{Z}(\mathbf{h}(k)) = \sum_{n=0}^{\infty} \mathbf{h}(n) z^{-n} = \mathbf{C}_d (z\mathbf{E}_d - \mathbf{A}_d)^{-1} \mathbf{B}_d. \tag{2.15}$$

2.2.1 The Tustin transformation

The Tustin transformation is one possibility to transform the continuous-time system (2.9) in the s-domain to a time-discrete system (2.13) in the z-domain, given a sampling period T.

Based on the fact that $z = e^{sT}$, the Tustin transformation is chosen to be the Padé approximation of the exponential function having a first order polynomial in the numerator and denominator. This is in fact the highest first order function possible to approximate an exponential function:

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} \approx \frac{1 - sT/2}{1 + sT/2}. \tag{2.16}$$

its inverse is then:

$$\begin{aligned} s &= \frac{1}{T} ln(z) = \frac{2}{T} \left[\frac{z-1}{z+1} + \frac{1}{3} \left(\frac{z-1}{z+1} \right)^3 + \frac{1}{5} \left(\frac{z-1}{z+1} \right)^5 + \frac{1}{7} \left(\frac{z-1}{z+1} \right)^7 + \cdots \right] \\ &\approx \frac{2}{T} \frac{z-1}{z+1}. \end{aligned}$$

Hence, this bilinear transformation maps the open left half of the complex plane onto the inside of the unit circle, and the imaginary axis onto the unit circle. The state-space matrices in the z-domain (discrete-time system) can be derived from those in the s-domain (assuming $\mathbf{E} = \mathbf{I}$, otherwise multiply the state equation by $\mathbf{E}^{-1}$) as follows [5]:

$$\mathbf{A}_d = (w_0 \mathbf{I} + \mathbf{A})(w_0 \mathbf{I} - \mathbf{A})^{-1} \tag{2.17}$$

$$\mathbf{B}_d = \sqrt{2w_o}(w_0 \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \tag{2.18}$$

$$\mathbf{C}_d = \sqrt{2w_o} \mathbf{C} (w_0 \mathbf{I} - \mathbf{A})^{-1} \tag{2.19}$$

$$\mathbf{D}_d = \mathbf{D} - \mathbf{C}(w_0 \mathbf{I} - \mathbf{A})^{-1} \mathbf{B} \tag{2.20}$$

where $w_0 = 2/T$ is not an eigenvalue of $\mathbf{A}$.

2.2.2 The Bilinear/Möbius transformation

A more general transformation than the Tustin transformation is the bilinear transformation, also known as Möbius transformation:

$$s = \frac{au + b}{cu + d} \Longleftrightarrow u = \frac{b - ds}{cs - a} \text{ with } ad - bc \neq 0 \tag{2.21}$$

In fact, all the common methods (e.g. prewarped Tustin, forward or backward rectangular) that transform a continuous system into a discrete one, and vice versa, are special cases of this transformation (with $u = z$).

The continuous open right half plane defined by $Re(s) > 0$ is mapped by this bilinear transformation to a region in the u-plane, interior or exterior of the circle C_u, having its center (x_c, y_c) and radius r according to the following equation [22]:

$$x_c = -\frac{ad + bc}{2ac}, \quad y_c = 0, \quad r^2 = \left(\frac{ad + bc}{2ac}\right)^2 - \frac{db}{ac}.$$

Note that the circle C_u is the unit circle if $-b/a = d/c = \pm 1$.

Lemma 2.1. *The state-space matrices in the u-domain can be derived from those in the s-domain as follows [22]:*

$$\mathbf{A}_u = (d\mathbf{A} - b\mathbf{I})(a\mathbf{I} - c\mathbf{A})^{-1} \tag{2.22}$$

$$\mathbf{B}_u = (ad - bc)(a\mathbf{I} - c\mathbf{A})^{-1}\mathbf{B} \tag{2.23}$$

$$\mathbf{C}_u = \mathbf{C}(a\mathbf{I} - c\mathbf{A})^{-1} \tag{2.24}$$

$$\mathbf{D}_u = \mathbf{D} + c\mathbf{C}(a\mathbf{I} - c\mathbf{A})^{-1}\mathbf{B} \tag{2.25}$$

This transformation will play a major role in Chapter 6 where the results of the equivalence between the Laguerre-based order reduction and moment matching are presented. Also, it will be employed to establish the connection between moment and Markov parameter matching in the continuous and discrete-time domains.

2.2.3 Stability and passivity

Definition 2.2. *The continuous dynamical system* (2.9) *(assuming* $\mathbf{E} = \mathbf{I}$*, otherwise multiply the state equation by* $\mathbf{E}^{-1}$*) is:*

- *asymptotically stable, if and only if all the eigenvalues of* $\mathbf{A}$ *have negative real parts.*
- *stable, if and only if all eigenvalues of* $\mathbf{A}$ *have nonpositive real parts, and, in addition, all pure imaginary eigenvalues have multiplicity one.*
- *Bounded Input Bounded Output (BIBO) stable, if its impulse response is absolutely integrable, i. e.* $\int_0^\infty |h(t)|dt < \infty$

Accordingly, the discrete system (2.13), is asymptotically stable if all the eigenvalues of $\mathbf{A}$ are inside the unit circle, and stable, when they have a norm at most one and those on the unit circle have multiplicity one. For more details and a more general definition of stability, see e. g. [5, 80].

Roughly speaking, passive systems are systems that do not generate energy. In other words, the energy dissipated in the system is never greater than the energy supplied to it. In fact, it can be proven that any linear dynamical system is passive if and only if its transfer matrix is positive real [31].

Definition 2.3. *A square* $(m = p)$ *transfer matrix* $\mathbf{H}(s) : \mathbb{C} \mapsto (\mathbb{C}^{m\times m} \cup \infty)$ *is positive real if*

1. $\mathbf{H}(s)$ *has no pole in the right half complex plane, and*
2. $\mathbf{H}(s^*) = (\mathbf{H}(s))^*$ *for all* $s \in \mathbb{C}$*, and*
3. $Re\left(\mathbf{w}^H\mathbf{H}(s)\mathbf{w}\right) \geq 0$ *for all* $s \in \mathbb{C}$ *with* $Re(s) > 0$ *and* $\mathbf{w} \in \mathbb{C}^m$*.*

2.2.4 Controllability and observability

Definition 2.4. *A dynamical system is controllable if there exists a finite energy input* $\tilde{\mathbf{u}}(t)$ *that takes its state* $\mathbf{x}(t)$ *from any initial value* $\mathbf{x}_0$ *to zero in a finite time.*

It can be shown that the LTI dynamical system (2.9) is controllable if and only if its controllability matrix $\mathcal{C}(\mathbf{A}, \mathbf{B})$, as defined below, has full rank,

$$rank(\mathcal{C}(\mathbf{A}, \mathbf{B})) = rank\left(\left[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \cdots, \mathbf{A}^{n-1}\mathbf{B}\right]\right) = n.$$

Definition 2.5. *A dynamical system is observable if there exists a finite time* $\tilde{t}$ *such that from the given input* $\mathbf{u}(t)$ *and the output* $\mathbf{y}(t)$ *over the interval* $[0, \tilde{t}]$*, the initial value* $\mathbf{x}_0$ *can be uniquely determined.*

It can be shown that the LTI dynamical system (2.9) is observable if and only if its observability matrix $\mathcal{O}(\mathbf{A}, \mathbf{C})$, as defined below, has full rank,

$$rank(\mathcal{O}(\mathbf{A}, \mathbf{C})) = rank\left(\left[\mathbf{C}^T, \mathbf{A}^T\mathbf{C}^T, \left(\mathbf{A}^\mathbf{T}\right)^2\mathbf{C}^T, \cdots, \left(\mathbf{A}^\mathbf{T}\right)^{n-1}\mathbf{C}^T\right]\right) = n.$$

2.3 Laguerre polynomials and functions

Named after the famous French mathematician Edmond Laguerre, Laguerre polynomials became in the last years an efficient mathematical tool for engineers working in the field of identification, order reduction and filters design. Their orthogonality and their various properties made them a widely used family of polynomials in the engineering community.

Restricting the interest into the fields of model reduction and system approximation, the Laguerre polynomials and functions are presented in time, frequency, and z-domain, together with some of their basic properties. Other relevant properties will be presented in details prior to their use in the coming chapters.

2.3.1 Definition

The i-th *Laguerre polynomial* is defined as follows [84]:

$$l_i(t) = \frac{e^t}{i!}\frac{d^i}{dt^i}(e^{-t}t^i), \tag{2.26}$$

and the scaled *Laguerre functions* are defined as [84]:

$$\phi_i^\alpha(t) = \sqrt{2\alpha}e^{-\alpha t}l_i(2\alpha t), \tag{2.27}$$

where α is a positive scaling parameter called time-scale factor. It is clear from the previous definition that the Laguerre functions are exponentially decreasing functions over the time where the free parameter α determines their time-spreading. This is one of the properties that makes them very suitable to approximate the impulse response of LTI systems, unlike the Laguerre polynomials that are unbounded increasing functions (see figures 2.1 and 2.2).

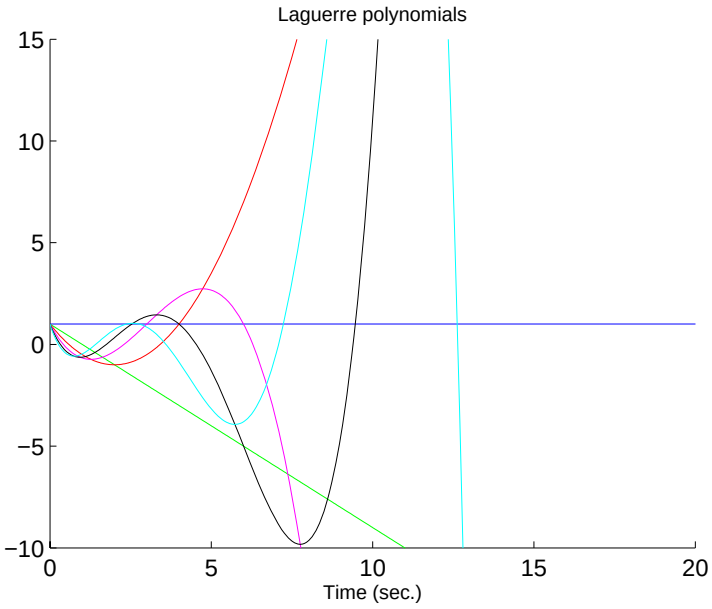

Figure 2.1: The first six Laguerre polynomials.

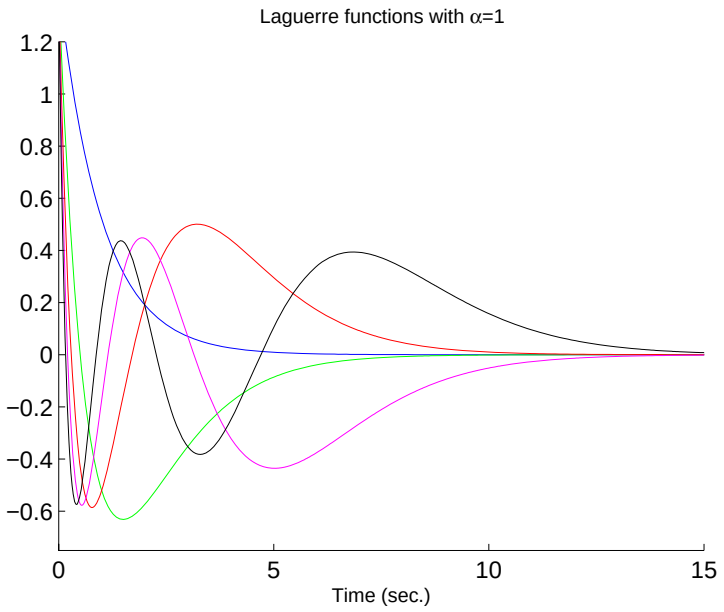

Figure 2.2: The first five Laguerre functions with the parameter $\alpha = 1$.

In frequency domain, the sequence of Laplace transforms of the scaled Laguerre functions (2.27) can be shown to be [84]:

$$\Phi_i^\alpha(s) = \mathcal{L}\left(\phi_i^\alpha(t)\right) = \frac{\sqrt{2\alpha}}{s+\alpha}\left(\frac{s-\alpha}{s+\alpha}\right)^i, \qquad n = 0, 1, \ \cdots . \tag{2.28}$$

and in the z-domain,

$$\Phi_i^\alpha(z) = \mathcal{Z}\left(\phi_i^\alpha(t)\right) = \frac{\sqrt{1-\alpha^2}}{z-\alpha}\left(\frac{1-\alpha z}{z-\alpha}\right)^{k-1}, \qquad k = 1, 2, \ \cdots . \tag{2.29}$$

with $|\alpha| < 1$.

2.3.2 Orthonormality

The Laguerre functions defined in time and frequency domains (2.26), (2.27), and (2.29) are orthonormal over $[0, \infty]$, $[-\infty, \infty]$ and on the unit circle, respectively ,

$$\int_0^\infty \phi_i^\alpha(t)\phi_j^\alpha(t)dt = \frac{1}{2\pi}\int_{-j\infty}^{j\infty} \Phi_n^\alpha(s)\Phi_m^\alpha(-s)ds = \frac{1}{2\pi j}\oint_C \Phi_m^\alpha(z)\Phi_n^\alpha(z^{-1})z^{-1}dz = \delta_{mn} \tag{2.30}$$

where δ_{mn} is the standard Dirac function that is equal to 1 when $m = n$ and 0 in all other cases [84, 91].

2.3.3 Time and frequency-domain orthonormal basis

The scaled Laguerre functions of (2.27) form in time-domain a uniformly bounded orthonormal basis for the Hilbert space $\mathcal{L}_2(R_+)$ of all real-valued and square integrable functions $f(t)$ on R_+ with the following inner product [91]:

$$< f, g >= \int_0^\infty f(t)g(t)dt. \tag{2.31}$$

Hence, any function $f \in \mathcal{L}_2(R_+)$ can be expanded as

$$f(t) = \sum_{n=0}^{\infty} f_n \phi_n^\alpha(t), \tag{2.32}$$

where f_n are the Laguerre coefficients of $f(t)$ which due to the orthogonality property can be calculated as $f_n =< f, \phi_n^\alpha >$.

Similarly, in the frequency domain, considering the Hardy space $\mathcal{H}_2$ consisting of all complex valued functions that are analytic and square integrable in the open right-half plane, the corresponding inner product is defined as follows:

$$< F, G >= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)G(-j\omega)d\omega. \tag{2.33}$$

Now, as the Hardy space is the Laplace transform of the Hilbert space defined previously, the Laplace transform of an orthonormal basis of $\mathcal{L}_2(R_+)$ is an orthonormal basis of $\mathcal{H}_2$.

Hence, any function $F \in \mathcal{H}_2$ can be expanded as

$$F(s) = \sum_{n=0}^{\infty} F_n \Phi_n^\alpha(s), \tag{2.34}$$

where F_n are the Laguerre coefficients of $F(s)$ defined as $F_n =< F, \Phi_n^\alpha >$.

Similarly, the above derivation can be done with the z-transform of the Laguerre function (2.29), however with the following inner product:

$$< f, g >= \frac{1}{2\pi j} \oint_C \Phi_m^\alpha(z)\Phi_n^\alpha(z^{-1})z^{-1}dz, \tag{2.35}$$

where C is a contour around the origin in the z-domain, generally taken as $|z| = 1$.

2.3.4 The Laguerre differential equation

The Laguerre polynomials (2.26) can be shown to be the canonical solutions of the second-order linear differential equation [84]:

$$x\ddot{y} + (1 - x)\dot{(y)} + ny = 0 \text{ with } n \neq 0,$$

which results in the famous Laguerre differential equation:

$$t\ddot{l}_i(t) + (1 - t)\dot{l}_i(t) + il_i(t) = 0 \text{ with } n \neq 0,$$

This equation will play a major role when calculating the optimal parameter α of the Laguerre function in Chapter 7.

Chapter 3

KRYLOV-BASED ORDER REDUCTION

In reduced order modeling of large scale systems, the methods based on matching the moments and/or Markov parameters of the original and reduced models using the Krylov subspaces are among the best choices [5, 7, 29, 32, 33, 37, 48, 76, 86]. These methods find the reduced order model in a relatively short time with a good numerical accuracy via a projection whose columns form bases of particular Krylov subspaces. The main advantage of this approach is that it requires a low computational effort and small memory storage especially when compared to other reduction approaches. However, generally speaking, the stability of the reduced order model can not be guaranteed and neither an *a priori* nor an *a posteriori* error bound measuring the accuracy of the approximation exists.

In this chapter, Krylov-based model order reduction is introduced together with the definitions of moments and Markov parameters. The projection framework and the theorems for matching the moments about one or several expansion points and/or the Markov parameters are reviewed. In addition, the main properties of this approach together with some of its major open problems are discussed in detail.

3.1 Moments and Markov parameters

The transfer matrix (2.12) of the system (2.9) can be rewritten as:

$$\mathbf{H}(s) = -\mathbf{C}(\mathbf{I} - s\mathbf{A}^{-1}\mathbf{E})^{-1}\mathbf{A}^{-1}\mathbf{B}, \tag{3.1}$$

and by making use of the Neumann expansion,

$$(\mathbf{I} - s\mathbf{A}^{-1}\mathbf{E})^{-1} = \sum_{i=0}^{\infty}(\mathbf{A}^{-1}\mathbf{E}s)^i \tag{3.2}$$

reformulated to the Taylor series,

$$\mathbf{H}(s) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{C}\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{B}s - \cdots - \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B}s^i - \cdots . \tag{3.3}$$

The moments about zero, are consequently defined as being the negative coefficients of the Taylor series expansion about zero of the system's transfer function and are calculated as [5, 37, 76]:

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B} \qquad i = 0, 1, \cdots . \tag{3.4}$$

They can also be shown using (3.3) to be the value and subsequent derivatives of the transfer function $\mathbf{H}(s)$ at the point $s = 0$:

$$\mathbf{m}_i^0 = -\frac{1}{i}\frac{d^i\mathbf{H}(s)}{ds^i}\bigg|_{s=0} . \tag{3.5}$$

By replacing s in the transfer matrix with the shifted variable $(s - s_0)$,

$$\mathbf{H}(s) = \mathbf{C}[(s - s_0)\mathbf{E} - (\mathbf{A} - s_0\mathbf{E})]^{-1}\mathbf{B}, \tag{3.6}$$

and repeating the derivation above, the moments about s_0 can be computed by replacing $\mathbf{A}$ by $(\mathbf{A} - s_0\mathbf{E})$ in (3.4) (assuming that $(\mathbf{A} - s_0\mathbf{E})$ is nonsingular),

$$\mathbf{m}_i^{s_0} = \mathbf{C}((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E})^i(\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{B} \qquad i = 0, 1, \cdots . \tag{3.7}$$

The Markov parameters are defined as being the coefficient of the Taylor series of the transfer matrix when s_0 in (3.6) tends to infinity ($s_0 \to \infty$) [5]. Using the infinite geometric series, the transfer matrix $\mathbf{H}(s)$ can be reformulated as,

$$\begin{aligned}
\mathbf{H}(s) &= \frac{1}{s}\mathbf{C}\left(\mathbf{I} - \frac{1}{s}\mathbf{E}^{-1}\mathbf{A}\right)^{-1}\mathbf{E}^{-1}\mathbf{B} \\
&= \frac{1}{s}\mathbf{C}\left[\mathbf{I} + \left(\mathbf{E}^{-1}\mathbf{A}\right)s^{-1} + \left(\mathbf{E}^{-1}\mathbf{A}\right)^2 s^{-2} + \cdots\right]\mathbf{E}^{-1}\mathbf{B} \\
&= \mathbf{C}\mathbf{E}^{-1}\mathbf{B}s^{-1} + \mathbf{C}\left(\mathbf{E}^{-1}\mathbf{A}\right)\mathbf{E}^{-1}\mathbf{B}s^{-2} + \mathbf{C}\left(\mathbf{E}^{-1}\mathbf{A}\right)\mathbf{E}^{-1}\mathbf{B}s^{-3} + \cdots ,
\end{aligned}$$

which allows calculating the Markov parameters in closed form as:

$$\mathbf{M}_i = \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^i\mathbf{E}^{-1}\mathbf{B} \qquad i = 0, 1, \cdots . \tag{3.8}$$

In addition, it can be shown that the i-th Markov parameter is in fact equal to the i-th derivative of the impulse response of the system (2.9) at time zero [47],

$$\mathbf{M}_i = \left.\frac{d^i\mathbf{h}(t)}{dt^i}\right|_{t=0}. \tag{3.9}$$

Hence, the first Markov parameter $\mathbf{M}_0$ is the system's impulse response value at $t = 0$.

3.2 Moment and Markov parameters matching: the SISO case

The aim of order reduction by moment matching is to find a reduced order model of order $q \ll n$, whose moments match some of those of the original one [5, 33, 86]. When some of the moments about $s = 0$ are matched, the reduced-order model is known as Padé approximant. For the case of $s = s_0$, we speak about a shifted Padé approximant.

If the following projection:

$$\mathbf{x} = \mathbf{V}\mathbf{x}_r + \epsilon, \quad \mathbf{V} \in \mathbb{R}^{n\times q},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x}_r \in \mathbb{R}^q, \tag{3.10}$$

with $q < n$, is applied to the system (2.9) for the SISO case, and after multiplying the state equation by the transpose of a matrix $\mathbf{W} \in \mathbb{R}^{n\times q}$, the following model with reduced order q is found:

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{V}\mathbf{x}_r(t). \end{cases} \tag{3.11}$$

Consequently, the system matrices of the reduced order model in state space are:

$$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{b}_r = \mathbf{W}^T\mathbf{b}, \quad \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}. \tag{3.12}$$

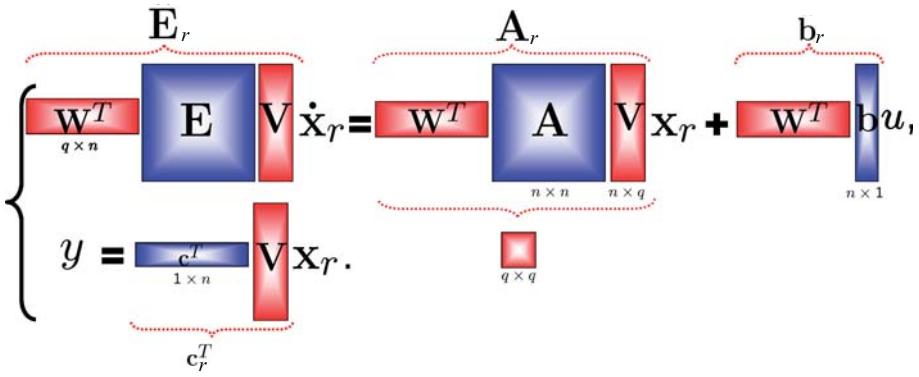

Figure 3.1: The reduced order model by projection.

Figure 3.1 illustrates the concept of order reduction of dynamical system by projection. In order to match the moments of the transfer function of the original and reduced order models about the point $s_0 = 0$, without explicitly calculating them, the projection matrices $\mathbf{V}$ and $\mathbf{W}$ should form a basis of a suitably chosen Krylov subspace as stated in the following two theorems:

Theorem 3.1. *[5, 33, 37] If the columns of the matrix* $\mathbf{V}$ *used in (3.11), form a basis for the Krylov subspace* $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ *and the matrix* $\mathbf{W} \in \mathbb{R}^{n \times q}$ *is chosen such the matrix* $\mathbf{A}_r$ *is nonsingular, then the first* q *moments about zero of the original and reduced order systems match.*

The subspace $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ is called *input Krylov subspace* and the order reduction method implementing theorem 3.1 is called *one-sided Krylov subspace method.*

Dual to the input Krylov subspace is the *output Krylov subspace* $\mathcal{K}_q(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$. Using the duality property, and this Krylov subspace, matching of the first q moments can be achieved by choosing the matrix $\mathbf{W}$ as a basis of the output Krylov subspace and the matrix $\mathbf{V}$ arbitrarily. The method is also considered as a one-sided Krylov subspace method.

One typical and common choice of the projection matrices in one-sided Krylov subspaces

methods is $\mathbf{W} = \mathbf{V}$. This choice may be advantageous for some specific large scale models when aiming at preserving the stability or passivity of the reduced order [32, 65].

The previous theorems have in fact one degree of freedom which is the free choice of one of the two required projection matrices, $\mathbf{V}$ and $\mathbf{W}$. If the results of those two theorems are combined, the number of matched moments can be doubled to $2q$.

Theorem 3.2. *[5, 33, 37] If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$ *used in (3.11), form bases for the Krylov subspaces* $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ *and* $\mathcal{K}_q(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$ *respectively, then the first* $2q$ *moments about zero of the original and reduced order systems match. It is assumed that* $\mathbf{A}$ *and* $\mathbf{A}_r$ *are invertible.*

The order reduction method implementing theorem 3.2 is called *two-sided Krylov subspace method* as it involves both the input and output Krylov subspaces.

In all the previously presented theorems, the aim was matching the moments about $s_0 = 0$. However, these theorems can be easily extended to match the moments about $s_0 \neq 0$ by simply substituting the matrix $\mathbf{A}$ by the matrix $(\mathbf{A} - s_0\mathbf{E})$ in all the involved Krylov subspaces. Consequently, the input and output Krylov subspaces become respectively,

$$\mathcal{K}_q((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b}), \tag{3.13}$$

$$\mathcal{K}_q((\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{c}). \tag{3.14}$$

In order to approximate the high frequencies of the original model ($s \to \infty$), a reduced order model that matches some of the Markov parameters is required. This model, known as *partial realization*, captures mainly the very fast dynamics of the system as these parameters corresponds to the impulse response and its derivatives at $t = 0$ (see (3.9)). Up to the author's knowledge, methods that extend the partial realization of large-scale systems to different time segments are not available.

A more general approach that tries to capture the slow and fast dynamics of the original system consists of simultaneously matching some of the first moments and Markov parameters. This is in fact the most general formulation of Krylov-based order reduction.

Theorem 3.3. *If the columns of the matrix* $\mathbf{V}$ *used in (3.11), form a basis for the Krylov subspace* $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^l\mathbf{A}^{-1}\mathbf{b})$ *where* $l \in \mathbb{Z}$ *and* $0 \leq l \leq q$ *and the matrix* $\mathbf{W} \in \mathbb{R}^{n\times q}$ *is chosen such the matrices* $\mathbf{A}_r$ *and* $\mathbf{E}_r$ *are nonsingular, then the first* $q-l$ *moments and the first* l *Markov parameters of the original and reduced order systems match.*

Also in this method, the number of matched moments and Markov parameters can be doubled by considering the suitable input and output Krylov subspaces when calculating the projection matrices $\mathbf{V}$ and $\mathbf{W}$.

Theorem 3.4. *If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$ *used in (3.11), form bases for the Krylov subspaces* $\mathcal{K}_q(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^{l_1}\mathbf{A}^{-1}\mathbf{b})$ *and* $\mathcal{K}_q(\mathbf{A}^{-T}\mathbf{E}^T, (\mathbf{E}^{-T}\mathbf{A}^T)^{l_2}\mathbf{A}^{-T}\mathbf{c})$ *respectively, where* $l_1, l_2 \in \mathbb{Z}$ *and* $0 \leq l_1, l_2 \leq q$*, then the first* $2q - l_1 - l_2$ *moments and the first* $l_1 + l_2$ *Markov parameters of the original and reduced order systems match.*

3.3 Moment and Markov parameters matching: the MIMO case

By using the block Krylov subspaces that were introduced in (2.2), all the previous theorems defining the projection matrices $\mathbf{V}$ and $\mathbf{W}$ for moment and/or Markov parameters matching can be generalized to be applied to MIMO systems. The resulting generalized reduced order system of (3.11) with m inputs and p outputs is of the form:

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{V}\mathbf{x}_r(t). \end{cases} \tag{3.15}$$

For instance, the input and output block Krylov subspaces that generalize theorem 3.2 for MIMO systems are: $\mathcal{K}_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathcal{K}_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$. Hence, similar to what has been presented in section (3.2), $\frac{q_1}{m}$ moments will be matched in a one-sided method, and $\frac{q_1}{m} + \frac{q_2}{p}$ in a two-sided one. Note that, for the projection matrices $\mathbf{V}$ and $\mathbf{W}$ to have the appropriate dimensions, the order of the reduced system should be a multiple of the number of inputs and outputs.

3.4 Rational interpolation

When a reduced-order model is calculated so that it interpolates the frequency response and its derivatives at multiple expansion points, it is denoted as multipoint Padé approximant or rational interpolant [3].

Assuming the moments are to be matched about $s_1, \cdots s_k$, k different Krylov subspaces are to be considered and a projection matrix whose columns form a basis of the union of these subspaces is to be found. In other words, it is aimed at matching q_i moments about each of the points s_i with $q_1 + \cdots + q_k = q$.

Theorem 3.5. *If the matrix* $\mathbf{V}$ *used in (3.11) is chosen such that,*

$$\bigcup_{i=1}^{k} \mathcal{K}_{q_i}\left((\mathbf{A}-s_i\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-s_i\mathbf{E})^{-1}\mathbf{b}\right) \subseteq colspan\mathbf{V} \tag{3.16}$$

with an arbitrary full rank matrix $\mathbf{W}$*, then the first* q_i *moments about* s_i *of the original and reduced models match, for* $i = 1, \cdots, k$*. It is assumed that the matrix pencil* $(\mathbf{A}-s_i\mathbf{E})$ *and the reduced order pencil* $(\mathbf{A}_r - s_i\mathbf{E}_r)$ *are both nonsingular.*

Similarly to the previous theorems, these results can be generalized for the two-sided method and for the MIMO case. In a two-sided method, it will be then possible to match the double number of moments about each of the points s_i while keeping the order of the reduced system equal to q.

3.5 Main properties of Krylov-subspace methods

This approach guarantees moment and/or Markov parameters matching without the need of explicitly calculating these coefficients. This is an important advantage for large-scale systems for which this calculation is numerically ill-conditioned and thus considered unfeasible.

Also, the resulting reduced-order model by this family of methods is in fact a Hermite interpolant of the transfer function as it matches some of the coefficients of its Taylor

series expansion about a certain point. Now, when considering two-sided methods, thus matching the maximum number of $2q$ coefficients, it can be shown that the reduced order model is a Padé approximant. In fact, $2q$ is the maximum number because there are q free coefficients available respectively in the numerator and denominator of the transfer function of a reduced order system of order q.

From the numerical point of view, Krylov-subspace methods are considered to require a relatively low computational effort and memory storage which favors them for the reduction of large systems. For matching any given number of moments about a single expansion point, only one LU factorization of the pencil $(\mathbf{A} - s_i\mathbf{E})$ is required. The remaining operations are then only matrix-vector multiplications. Depending on the order of the reduced system, each of the columns of the projection matrices $\mathbf{V}$ and $\mathbf{W}$ is iteratively calculated from the previous one using the numerically stable Arnoldi [6] or Lanczos [57] algorithms or one of their modified versions. Finally, the reduced model is directly calculated through a projection to the lower dimensional subspace. Consequently, in addition to the computational cost of the LU factorization, this approach needs only $\mathcal{O}(q.n^2)$ flops when dealing with dense systems and roughly $\mathcal{O}(q.n)$ for sparse ones, and a memory requirement of only $\mathcal{O}(q.n)$.

When examining the reduced order model delivered by this approach, it is remarked that, through the projection framework, the physical interpretability of the original states gets lost. However, an approximation of the original state vector can be always calculated by a back projection, i. e. $\mathbf{x} \approx \mathbf{V}\mathbf{x}_r$.

In addition, stability, which is one of the most important properties for the analysis of dynamical systems, can not be generally preserved. In fact, there is no well established and accepted method that guarantees the stability of the reduced system (in one- or two-sided methods) under the only assumption that the original system is stable. For further application of the reduced system in simulation, control or optimization, there is a need to measure its accuracy and thus the quality of the approximation achieved by

the Krylov-based model reduction. Generally speaking, this is not possible due to the absence of a general error bound that describes the difference between the original and reduced models.

Due to these facts, it is still early to speak about an automatic Krylov-based order reduction where the user has just to define how accurate the reduced-order model should be and keep the rest to the algorithm, as in the case of TBR. What is currently happening in practice can be described by an iterative process, where the user has to specify a certain set of parameters (the reduced order, the frequency of interest ...) to which the different algorithms and the projection matrices are adjusted, examine the resulting reduced-order models, and then re-tune the parameters until satisfactory results are achieved.

3.6 Numerical algorithms

After a detailed presentation of the Krylov-based MOR framework, the question that is still to be answered is how to numerically calculate the matrices $\mathbf{V}$ and $\mathbf{W}$.

Based on the definition of the Krylov subspace (2.1) and for instance theorem 3.1, a trivial approach would be to use a *"for" loop* that calculates the matrix $\mathbf{A}^{-1}\mathbf{E}$ and its powers in the input Krylov subspace. Even though this would be theoretically possible, practically the obtained matrix $\mathbf{V}$, that is supposed to be full rank, is not, even for moderate values of n and q, as its columns tend to become linearly dependent. Hence, the use of the Arnoldi and Lanczos algorithms or one of their improved versions is unavoidable for the proper calculation of the projection matrices, and thus to guarantee obtaining a reduced order model of order q.

3.6.1 The Arnoldi algorithm

The most popular algorithm in one-sided methods for the calculation of the projection matrices is the Arnoldi algorithm [6, 31], which finds an orthonormal basis for a given

Krylov subspace. Specifically, it finds a set of normalized vectors that are orthogonal to each others, i.e. satisfying the following equality:

$$\mathbf{V}^T\mathbf{V} = \mathbf{I}, \tag{3.17}$$

and form a basis for the given Krylov subspace. In each step, one more vector orthogonal to all the previous ones is constructed and normalized. However, when q is relatively large, this algorithm may produce a set of basic vectors that are not linearly independent anymore. In this case, what is called *deflation* should be introduced to delete linearly dependent vectors (up to a certain accuracy) [23]. In addition to finding an orthonormal basis $\mathbf{V}$ for the given Krylov subspace, this algorithm results in a specific structure of the reduced order system matrices $\mathbf{B}_r$ and $\mathbf{A}_r$ [76]. For instance, for a SISO system with a single starting vector, the reduced matrix $\mathbf{A}_r$ is an upper Hessenberg matrix where all elements under the diameter, except for the sub-diagonal entries, are zero and the vector $\mathbf{b}_r$ is a multiple of the first unit vector.

For a detailed implementation of the Arnoldi algorithm with deflation using modified Gram-Schmidt orthogonalization [76] refer to Appendix A.

3.6.2 The Lanczos algorithm

The most popular algorithm in two-sided methods for the calculation of the projection matrices $\mathbf{V}$ and $\mathbf{W}$ simultaneously is the Lanczos algorithm [29, 57], which generates two sequences of basis vectors spanning the input and output Krylov subspaces. The generated basis vectors are also orthogonal to each other, i.e.,

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}. \tag{3.18}$$

The first and standard version of this algorithm is limited to the Krylov subspace with one starting vector and consequently can not be used for the reduction of MIMO system. In [2], this disadvantage was overcome and a Lanczos algorithm that works with multiple starting vectors was presented. The standard version suffers from the loss of bi-orthogonality when the number of iterations increases, thus the algorithm use is limited

to reasonable order reduced model. This problem was solved in [15] by introducing what is defined as reorthogonalization and deflation.

In addition to finding two bases orthogonal to each other for the given Krylov subspaces, this algorithm results also in a specific structure of the reduced order system matrices $\mathbf{B}_r$ and $\mathbf{A}_r$ [76]. For instance, for a SISO system with a single starting vector and considering reorthogonalization and deflation, the resulting reduced matrix $\mathbf{A}_r$ is an upper Hessenberg matrix where all elements under the diameter, except for the sub-diagonal entries, are zero and the vector $\mathbf{b}_r$ is a multiple of the first unit vector. However, when using the standard Lanczos algorithm the matrix $\mathbf{A}_r$ will be tridiagonal.

For an implementation of the Lanczos algorithm with full orthogonalization and deflation [76] refer to Appendix A.

3.6.3 The two-sided Arnoldi algorithm

This algorithm is based on the original Arnoldi algorithm presented in section (3.6.1) but with the performance of the Lanczos algorithm and without numerical problems. It is used in two-sided Krylov subspace methods and results in the same reduced order model obtained by the Lanczos algorithm. However, it is easier to implement and is numerically more stable.

Note that in the Lanczos algorithm the bases are orthogonal to each other, i.e. $\mathbf{W}^T\mathbf{V} = \mathbf{I}$, however for the two-sided Arnoldi algorithm, each basis is itself orthonormal, i.e. $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ and $\mathbf{W}^T\mathbf{W} = \mathbf{I}$.

For a detailed implementation of this algorithm when applied for order reduction [76] refer to Appendix A.

3.7 Open problems in Krylov-based MOR

After around a decade of research in the field of Krylov-based MOR, numerous theoretical and numerical problems of this approach have been tackled and solved, but many are still open or were only solved for particular systems or under specific conditions. In this section, some of the most relevant open problems are exposed, together with an overview of the important work that has been achieved towards solving them.

3.7.1 Passivity/Stability preservation

It is well known that stability is one of the most important requirements for any dynamical system for its analysis, proper operation and control. Consequently, for the reduced order model to be valid and adopted instead of the original one, and even before discussing the accuracy and effectiveness of its approximation, it should be stable. This however can not be generally guaranteed by Krylov subspace methods. The simplest and most *intuitive* solution to this problem would be to vary the order q of the reduced system until obtaining a stable reduced model. However, what if all the low order models are unstable? or the stable q-th order model is not an acceptable approximation of the original system?

After a survey of the literature treating the stability preservation problem of Krylov subspace methods, it is easily observed that until now no general method can guarantee the stability of the reduced system under the only assumption that the original system is stable. However, several works tackled this problem and several methods that can be grouped under three main approaches have been suggested. The first group are passivity-based methods, see e. g. [33, 50, 65], which offered a very high accuracy and guaranteed the passivity, and thus stability, of the reduced model assuming a passive original system. The second group are post-processing methods which remove the unstable poles of the reduced system by using restarted algorithms like the implicitly and explicitly restarted Arnoldi and Lanczos algorithms [9, 36, 45]. The third group are interpolation-based methods which preserve the passivity of the original system through the interpolation of

the *spectral zeros* [4, 82].

The drawbacks of the passivity-based methods are mainly their limitation to a narrow set of systems and their restriction to the one-sided Krylov methods which have generally a lower accuracy than the two-sided methods. For the case of the restarted algorithms, they are mostly implemented for the SISO case, and have a relatively high numerical effort compared to the classical case. In addition, they can not preserve the moment matching after deleting the unstable poles and can not always guarantee to find a stable reduced model with a finite number of restarts. The interpolation methods have proven to be theoretically very promising however they are clearly numerically more expensive than the classical Krylov-based MOR. In addition, several questions related to the choice of the interpolation points which should not only preserve the passivity but also result in a good system approximation, and to the choice of some others parameters in this method, have still to be answered. An interesting work in this direction can be found in [71], where several algorithms for the calculation of the dominant poles of large-scale systems have been presented.

3.7.2 A time-domain interpretation

In order to investigate the Krylov-based MOR from a time-domain perspective, the definitions of moments about a general expansion point s_0 and Markov parameters in time-domain and their possible significance are discussed.

Let us first calculate the moments about $s_0 = 0$ in the time-domain. Consider the definition of the Laplace transform of the impulse response,

$$H(s) = \int_0^\infty h(t)e^{-st}dt.$$

After a Taylor series expansion of the exponential term,

$$\begin{aligned} H(s) &= \int_0^\infty h(t)\left[1 - st + \frac{s^2t^2}{2!} - \frac{s^3t^3}{3!} + \cdots\right]dt \\ &= \int_0^\infty h(t)dt - s\int_0^\infty th(t)dt + s^2\int_0^\infty \frac{t^2}{2!}h(t)dt + \cdots . \end{aligned}$$

Thus, the ith moment about $s_0 = 0$ in time-domain is:

$$m_i = \frac{(-1)^i}{i!} \int_o^\infty t^i h(t) dt.$$

This definition can be found in e.g. [5], and is frequently used in physics and statistics.

Now, using the fact that the moments of $H(s)$ about s_0 are equal to the moments of $H(s + s_0)$ about $s_0 = 0$,

$$\begin{aligned} H(s+s_0) &= \int_0^\infty h(t) e^{-s-s_0 t} dt = \int_0^\infty h(t) e^{-s_0 t} \left[1 - st + \frac{s^2 t^2}{2!} - \frac{s^3 t^3}{3!} + \cdots \right] dt \\ &= \int_0^\infty h(t) e^{-s_0 t} dt - s \int_0^\infty t e^{-s_0 t} h(t) dt + s^2 \int_0^\infty \frac{t^2}{2!} e^{-s_0 t} h(t) dt + \cdots, \end{aligned}$$

which leads to the time-domain definition of the moments about s_0,

$$m_i^{s_0} = \frac{(-1)^i}{i!} \int_0^\infty t^i e^{-s_0 t} h(t) dt.$$

As the functions $t^i e^{-s_0 t}$ and t^i are not orthogonal, the time-domain moments are not the coefficients of the series expansion of the impulse response $h(t)$ having $t^i e^{-s_0 t}$ or t^i as basis functions. This can be easily proven based on the generalized Fourier series theory, or by contradiction, assuming first that $h(t) = \sum_{i=0}^\infty m_i t^i$, and then showing that:

$$\int_0^\infty t^j h(t) dt = \int_0^\infty t^j \left(\sum_{i=0}^\infty m_i t^i \right) dt \neq m_j.$$

Hence, when calculating a reduced-order model matching the moments in the frequency-domain, these time-domain moments are also matched however without offering any useful theoretical or physical time-domain interpretation.

On the other hand, it is is well known that the i-th Markov parameter is in fact the i-th derivative of the value of the impulse response at time zero [47] and that the first Markov parameter M_0 is the impulse response at $t = 0$ (see (3.9)). However, neither a method nor an interpretation for an extended partial realization at some other time instant are relevant. In [92], a theoretical definition of the *modified Markov parameters* as being the coefficients of the series $(s - s_0)^{-i}$ has been presented. However, no system theoretical interpretation or relevance to model reduction were discussed.

In frequency-domain, order reduction by moment matching is known to be a local approximation of the frequency response of the original system. The frequency range of interest is determined by the interpolation point about which the moments in the frequency-domain are matched. Consequently, this family of methods can not directly guarantee a good approximation of the impulse response, as it is quite hard in most practical cases, to predict the accuracy of the time-domain response of the reduced-order model from its frequency-domain one.

Accordingly, it can be generally concluded that no clear and useful time-domain interpretation of the Krylov-based model reduction approach exists.

This open problem has been a focus point in this work, and a new time-domain method based on matching the Laguerre coefficients of the impulse response is presented in Chapter 5. In addition, a time-interpretation for MOR by moment matching based on the Laguerre representation of the system is presented in Chapter 6.

3.7.3 A general error bound

Another open problem in Krylov-based order reduction is the absence of a general error bound describing the difference between the original and reduced models for the complete spectrum, while being at the same time numerically cheap to be calculated. The existing methods presented e.g. in [10, 37] are only valid under some special conditions or offer an approximation (or local approximation) of the reduction error. In [11], several heuristic error indicators were presented, however without any theoretical proof. The first one is based on the observation that the error between the original and reduced systems is approximately equal to the one between two successive reduced systems. The second error indicator is based on the observation that the HSV of the reduced system approximate those of the original one. This error indicator is in fact an approximation of the global error bound of TBR by using the HSV of the reduced system instead of those of the original one.

To conclude, it should be noted that in order to be able to derive an exact error bound, the knowledge of both the reduced-order model and the error system are required. This indirectly implies the need to involve the original system in any error bounds calculation which becomes numerically unfeasible when dealing with large-scale systems.

3.7.4 Choice of the reduced order

Up to the author's knowledge, there are very few works dealing with the choice of the order of the reduced system in Krylov-based order reduction based on a stopping criterion connecting the original and reduced systems. This is in fact due to the absence of a general error bound that, if existing, would allow to adaptively determines the reduced order required to satisfy a prescribed error tolerance. The majority of the available methods connect the choice of the reduced order (or a stopping criterion) to the calculation of the projection matrices. For instance, the reduced order can be determined when in the Arnoldi or Lanczos algorithms no more new linearly independent columns, up to a certain accuracy (deflation principle), can be found, e.g. [77]. Another approach employs the error indicators mentioned in the previous section and increases the reduced order until a certain approximated error tolerance is satisfied.

However, in practice, the user typically specifies a reduced order to which the algorithm and the projection matrices are then adjusted, examines the results, and then reduces or increases the order until satisfactory results and stability are achieved.

3.7.5 Choice of the expansion point(s)

One of the most important parameter in Krylov subspace method is the expansion point about which the moments are matched. The value and number of the expansion point(s) steer the quality and the "location" of the approximation of the frequency response and the numerical effort needed to calculate the reduced order model.

By definition, s_0 is the point around which the Taylor series of the transfer function of the

system is expanded. Accordingly, matching the moments and/or the Markov parameters about this point can be directly interpreted from a system theory point of view and employed to describe the similarity between the original and reduced models based on the following facts:

1. With $s_0 = 0$, the reduced and original model have the same DC gain and steady state accuracy is achieved.

2. Small values of s_0 result in a reduced model with a good approximation of the slow dynamics.

3. Large values of s_0 and/or matching the Markov parameters result in a reduced model approximating the high frequencies and thus the transient response.

4. When matching the moments about different frequency points s_0, a better approximation on a wider frequency band or on a specific frequency band of interest can be achieved.

Even though these facts give an idea on the choice of the expansion point, no exact value of s_0 can be derived based on them.

Consequently, the choice of the interpolation point(s) in moment matching became and is still an active field of research. In [37], the basic principles for this choice have been established and the different possibilities involving real and/or imaginary, single and multiple expansion points, have been considered and discussed, leading to the following conclusions:

1. A purely imaginary expansion point leads to very good local approximation and to a very slow convergence at all frequencies away from s_0.

2. A real interpolation point offer a good approximation in a large neighborhood around s_0, except around some lightly damped eigenvalues on the imaginary axis.

3. The combination of real, imaginary, and complex expansion points is generally preferred over a single interpolation point, however the choice and number of these points is not straightforward.

In [82], the problem of passivity preserving order reduction has been addressed and a rational Krylov algorithm with q interpolation points selected as spectral zeros of the original transfer function has been presented (see section 3.7.1). In [39], an iteratively corrected rational Krylov algorithm (IRKA) for H_2 model reduction of SISO systems has been suggested. The interpolation-based idea consists of matching only the first moment about q expansion points chosen at the mirror image of the poles of the reduced system. These results have been generalized to the MIMO and $H_{2\alpha}$ cases. For more details, refer to [56] and the references therein. The determination of appropriate starting values and the convergence of this iterative algorithm are still considered as open questions. In addition, the complexity and numerical costs of these approaches still play significant roles especially when compared to a reduction with a single expansion point.

In summary, the choice of s_0 is still often an ad-hoc process, where the user has to try several expansion points (or several ones simultaneously) and pick up the one(s) resulting in a satisfactory approximation for his system. The choice $s_0 = 0$ is widely spread and comes generally at the first place as it often delivers very good result in a large neighborhood of the low-frequency part of the spectrum, including the steady state.

This open problem constitutes one focus of this thesis and several algorithms for the choice of a single expansion point in Krylov-based MOR, ensuring a good approximation of the impulse response, will be presented in Chapter 7.

3.7.6 Systems with high number of terminals

It is well-known that Krylov subspace methods are nowadays well suited for the reduction of large-scale SISO systems or MIMO systems with very few inputs and outputs. However, these methods can not be anymore applicable when the original system has a

high number of terminals which in some cases may get as high as the number of state variables. This is typical in many engineering fields, for instance, in the field of VLSI design and simulation.

The problem lies in fact in the calculation of the projection matrices $\mathbf{V}$ and $\mathbf{W}$ and thus in the resulting reduced order. Assuming a system having order n with m inputs and p outputs, the input and output block Krylov subspaces will have then m and p starting vectors, respectively. Hence, for every additional moment to be matched, m columns need to be added to the $\mathbf{V}$ matrix, and p to $\mathbf{W}$. Thus, the order of the reduced model will be increased by m, if $m = p$, and to the next common multiple of m and p, if $m \neq p$. As an example, consider a system having an order of 5,000 with 500 inputs and outputs. Just by matching the first 10 moments, the reduced order model will have the order of the original one!

The recently proposed methods SVDMOR [28] and RecMOR [30] which are considered as first works in this field, are based on approximating the input and output matrices using the dominant singular vectors of the transfer function at steady state. A main drawback of these approaches is that they are limited to systems where the responses at different I/O ports are highly correlated, which is not generally the case, and that the computed reduced order model do not preserve any moment matching properties. In addition, the frequency dependence of these I/O responses is not considered as the SVD of the transfer matrix with $s = 0$ is employed. A solution for the latter problem was presented in [67] where the frequency dependence has been considered and moment matching in the sense of tangential interpolation has been proven. Another interesting method based on the superposition principle in linear systems can be found in [12]. The reduction is performed separately using each column of the input matrix and the output response of the original multi-input system is approximated by the summation of the output responses of each of the single-input reduced systems.

3.8 Illustrative example

In the last decade, numerous works related to the Krylov subspace methods have been published and the efficiency and suitability of these methods have been demonstrated by various numerical examples. Many of these examples can be found in the Benchmark collection of [18]. In order not to repeat the known results from the literature, the numerical simulations in this section will be targeting the open problem of Krylov-based MOR that is treated in this work, namely, the choice of the expansion point s_0. To

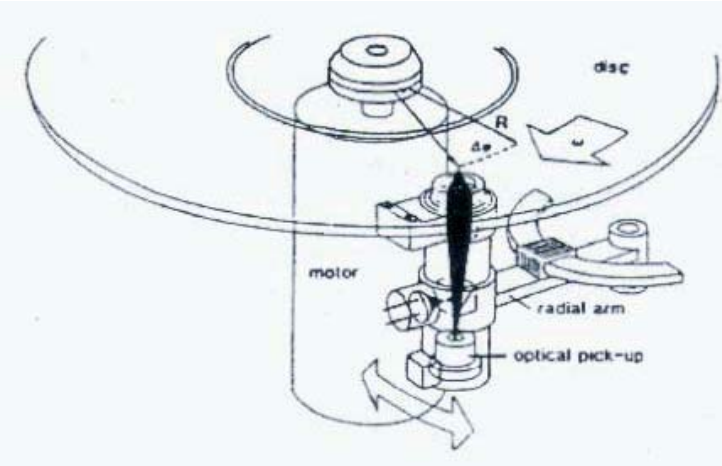

Figure 3.2: The schematic of the CD player

this aim, the widely spread Compact Disk (CD) player model [1] is considered. The most important part in this system is the optical unit consisting of different lenses, laser diode, photodetectors and micro actuators (see figure 3.2). The main task here is to control the arm holding the optical unit to read the required track on the disk and to adjust the position of the focusing lens to adjust the depth of the laser beam penetrating the disc. In order to achieve this task, the system should be modeled by finite element method (FEM) leading to a differential equation of order 120. From figure 3.3, the effect of the choice of the expansion point s_0 on the impulse response approximation is studied. The order of the reduced model has been set to a constant, here $q = 10$, and

[1] available online at http://www.win.tue.nl/niconet/NIC2/benchmodred.html.

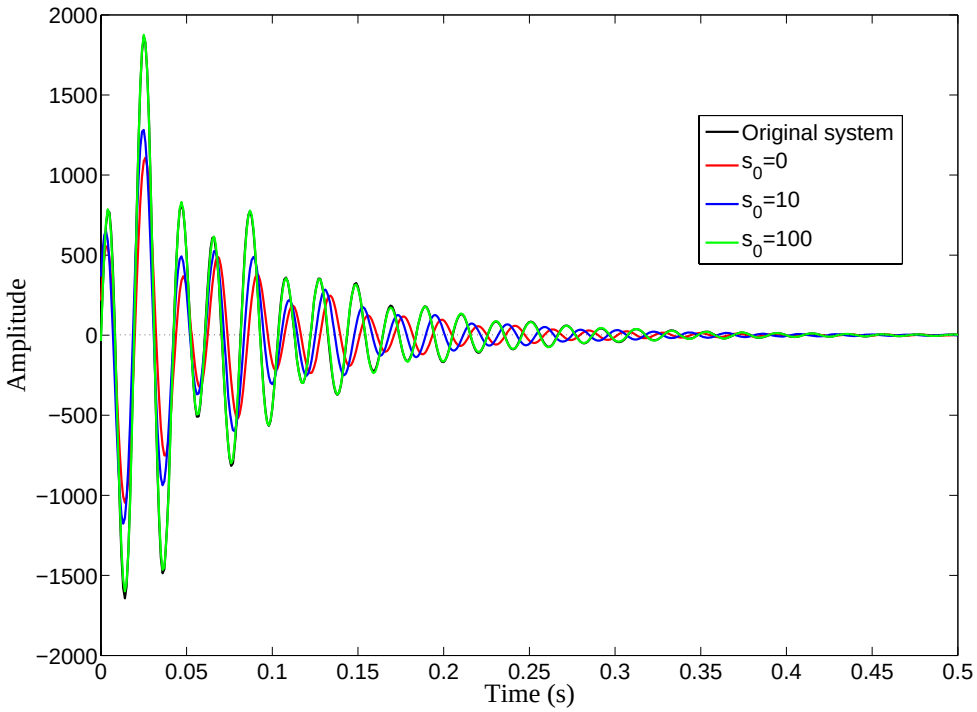

Figure 3.3: The impulse responses of the original system and the reduced order models of order 10.

the expansion point has been varied. Clearly, the best choice for this system and this specific reduced order is $s_0 = 100$ which was obtained here by trial and error. It should be noted that when the expansion point is chosen to be very large, the approximation is then concentrated at the beginning of the impulse response, i.e. transient behavior. This is to be expected as matching the moments about a high value of s_0 results in a system that approximates the system's behavior as t tends to zero. When trying to examine the effect of the choice of the expansion point in the frequency-domain, figure 3.4 shows that the reduced order model corresponding to $s_0 = 100$ is the only model matching the resonant peak of the Bode diagram, which partly explains the excellent approximation of the impulse response in figure 3.3. However, generally speaking, none of the reduced systems offer an acceptable overall approximation of the original frequency response.

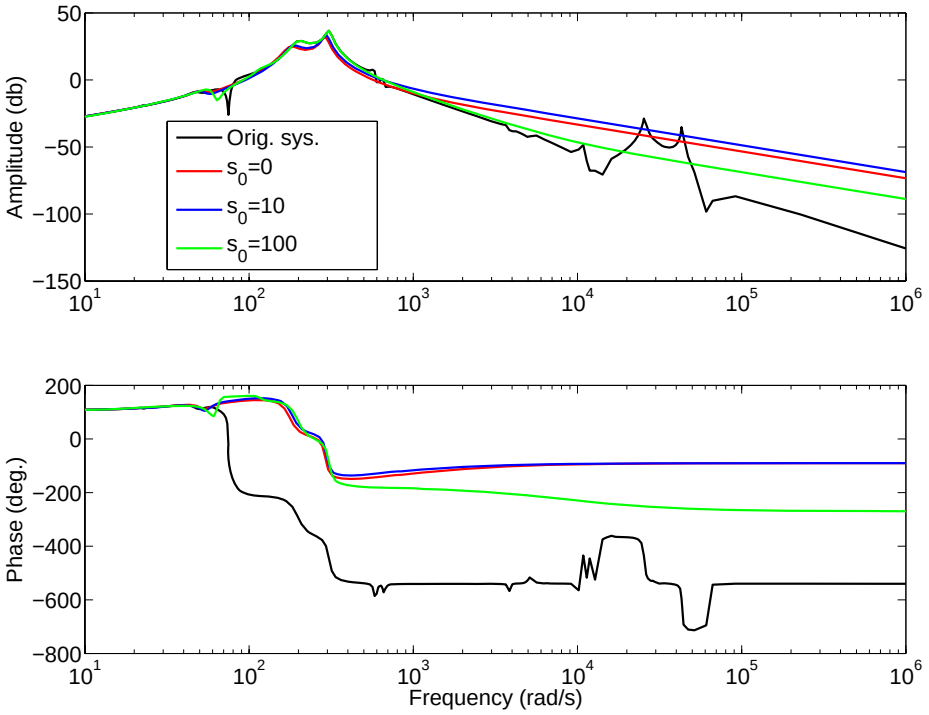

Figure 3.4: The Bode diagram of the original system and the reduced order models of order 10.

This shows from one side that it is more difficult to approximate the original system in the frequency domain (more degrees of freedom are needed) and from the other side that the connection between a well-approximated time and frequency response is not as clear as one may think.

These facts, which are not restricted to the current example, confirms what was already presented in section 3.7 and proves that there is a need for a method that calculates an 'optimal' expansion point guaranteeing a good approximation in the time-domain.

Chapter 4

LAGUERRE-BASED MODEL REDUCTION IN FREQUENCY DOMAIN

The interesting properties of orthogonal functions have made them a powerful tool for modeling, identification, simplification, signal processing and controller design. The most commonly used orthogonal functions by Engineers and Mathematicians are Laguerre, Chebyshev, Legendre, and Kautz. Among all these sets of orthogonal functions, Laguerre polynomials are widely employed because of their interesting properties: the definition on the wide range $[0, \infty]$, forming a basis for the Hilbert space, the recursive calculation of each polynomial, and the special form of their Laplace transform which consists of the product of a low-pass filter and some first order all-pass filters.

Since the early sixties, many researchers have been trying to approximate finite and infinite-dimensional systems based on truncating the Laguerre series expansion and have investigated the proper choice of the approximation order and the optimal value of the time-scale factor; see for instance [38, 62, 89] and the references therein. For system identification, employing the Laguerre functions has been proposed by several authors; e.g. [20, 87]. Some other applications may be found in the signal processing community for filters design [91], and in control engineering for the design of decentralized adaptive controllers [1]. The Laguerre series expansion has been also integrated into the Balancing and Truncation reduction method in order to avoid the expensive computation of the gramians [93].

To improve the results of moment matching, to preserve stability, or to provide an error bound, some researchers have tried to replace the Taylor series by other series expansions. For instance in [46] and [17], a transformed Legendre polynomial and a Chebyshev expansion have been respectively used for order reduction.

In [52], the Laguerre series was first used for the purpose of order reduction of state-space models based on matching some of the first Laguerre coefficients instead of the moments of the transfer function; see also [42]. An advantage of the Laguerre functions is that the corresponding series expansion can be transformed to a Taylor series by applying a certain bilinear transformation. This property simplifies the reduction procedure and makes it possible to use the numerically robust and efficient algorithms implemented for moment matching. In [26], this method has been generalized for the reduction of second-order systems.

The Laguerre-based order reduction has been further investigated and generalized by several other authors, [53, 55]. A reduction method based on truncating the Laguerre series is presented in [90]. Although this method preserves stability and provides an error bound, it locates all poles of the reduced system at a single real point making it difficult to approximate systems with dominant complex conjugate poles.

In this chapter, the basic Laguerre-based order reduction approach for state-space systems which was first presented in [52] is reviewed.

4.1 Moments and Laguerre coefficients

For the ease of presentation, the SISO case is considered here, however all the results in this chapter can be easily generalized to the MIMO case.

Consider the dynamical system of the form:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{x}(t). \end{cases} \tag{4.1}$$

It is assumed that the the transfer function $H(s)$ of any state space model (2.12) belongs to the Hardy space $\mathcal{H}_2$. Consequently, it can be expanded into the orthonormal basis formed by the Laplace transforms of the scaled Laguerre functions $\phi_n^\alpha(t)$ as:

$$H(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} = \sum_{i=0}^{\infty} F_i \Phi_i^\alpha(s), \tag{4.2}$$

which is equivalent to:

$$H(s) = \mathbf{c}^T(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b} = \frac{\sqrt{2\alpha}}{s+\alpha} \sum_{i=0}^{\infty} F_i \left(\frac{s-\alpha}{s+\alpha}\right)^i, \tag{4.3}$$

with the F_i being the Laguerre coefficients. By considering the following bilinear transformation,

$$s = \alpha \frac{1+v}{1-v}, \tag{4.4}$$

the Laguerre expansion is mapped from the s-domain into the v-domain, leading to the following equality:

$$H(s)\,\big|_{s(v)} = H(v) = (1-v)\underbrace{\mathbf{c}^T[(\alpha\mathbf{E} - \mathbf{A}) + v(\alpha\mathbf{E} + \mathbf{A})]^{-1}\mathbf{b}}_{H_v(v)} = (1-v)\frac{1}{\sqrt{2\alpha}} \sum_{i=0}^{\infty} F_i v^i. \tag{4.5}$$

Neglecting the factor $\frac{1}{\sqrt{2\alpha}}$, the Laguerre coefficients F_i are now the moments of the system $H_v(v)$ in the v-domain having the following system matrices:

$$\mathbf{E}_v = \alpha\mathbf{E} + \mathbf{A}, \tag{4.6}$$

$$\mathbf{A}_v = -\alpha\mathbf{E} + \mathbf{A}. \tag{4.7}$$

Since $H_v(v)$ is the only high order and dynamic part of the system $H(v)$, it can be inferred that a qth-order Padé approximation of the modified transfer function $H_v(v)$ in the v-domain is equivalent to a qth-order Laguerre approximation of $H(s)$ in the s-domain [52].

4.2 The order reduction approach

Based on the mapping of the Laguerre expansion of the transfer function from the s-domain into a Taylor series in the v-domain, the Laguerre coefficients F_i of the original

system in the s-domain are now the moments of an intermediate model that we call *modified state space model* in the v-domain.

The next step after calculating this intermediate model, is to reduce it using suitable Krylov subspace methods, i.e. moment matching. Thus, the obtained reduced order model, still in the v-domain, matches the moments of the modified original one, which are in fact the Laguerre coefficients of the original state-space model. Finally, this modified reduced order model is transformed back to the s-domain to result in the required reduced-order model.

The Krylov subspace reduction methods introduced in the previous chapter together with their corresponding projection framework are employed to calculate the reduced order system according to the following theorems:

Theorem 4.1. *If the columns of the matrix* $\mathbf{V}$ *used in (3.11), form a basis for the Krylov subspace* $\mathcal{K}_q(\mathbf{A}_v^{-1}\mathbf{E}_v, \mathbf{A}_v^{-1}\mathbf{b})$ *and the matrix* $\mathbf{W} \in \mathbb{R}^{n\times q}$ *is chosen such the matrix* $\mathbf{A}_v$ *and* $\mathbf{W}^T\mathbf{A}_v\mathbf{V}$ *are nonsingular, then the first* q *Laguerre coefficients of the original and reduced order systems match.*

Proof: Consider the modified original model in the v-domain formed by the matrices $\mathbf{E}_v$, $\mathbf{A}_v$, and having the transfer function function defined in the LHS of (4.5). If the projection matrices $\mathbf{V}$ and $\mathbf{W}$ calculated according to this theorem are applied to this model, a reduced model called *modified reduced model* is obtained. This model matches according to theorem 3.1, q moments of the original modified model, which are in fact the Laguerre coefficients of the original model. The modified reduced system is still in the v-domain and have the following form:

$$\begin{cases} \mathbf{W}^T\mathbf{E}_v\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}_v\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{V}\mathbf{x}_r(t), \end{cases} \tag{4.8}$$

The transfer function of this system in the v-domain is:

$$\mathbf{c}^T\mathbf{V}(v\mathbf{W}^T\mathbf{E}_v\mathbf{V} - \mathbf{W}^T\mathbf{A}_v\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = \frac{1}{\sqrt{2\alpha}}\sum_{i=0}^{\infty} F_{ri}v^i \tag{4.9}$$

Applying the back transformation from the v-domain to the s-domain as defined in (4.4), and after substituting the matrices in the v-domain by their values in (4.6), (4.7), we get:

$$\frac{(s+\alpha)}{2\alpha}\underbrace{\mathbf{c}^T\mathbf{V}(s\mathbf{W}^T\mathbf{E}\mathbf{V}-\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B}}_{H_r(s)}=\frac{1}{\sqrt{2\alpha}}\sum_{i=0}^{\infty}F_{ri}\left(\frac{s-\alpha}{s+\alpha}\right), \tag{4.10}$$

where the first q Laguerre coefficients F_{r0} to F_{rq-1} are the same as the first q of the original model F_0 to F_{q-1}.

Consequently, from (4.10), $H_r(s)$ can be written in the s-domain as:

$$H_r(s)=\frac{\sqrt{2\alpha}}{s+\alpha}\sum_{i=0}^{\infty}F_{ri}\left(\frac{s-\alpha}{s+\alpha}\right), \tag{4.11}$$

which is in fact the required Laguerre approximation of the reduced system transfer function. ■

Remark 4.1. This transfer function corresponds to the reduced system in the s-domain obtained by directly applying the projection matrices $\mathbf{V}$ and $\mathbf{W}$ to the original model (see eq. (4.10)) and thus no need to calculate the modified reduced model and apply a back transformation.

In other words, the modified system matrices $\mathbf{E}_u$ and $\mathbf{A}_u$ are only involved in the calculation of $\mathbf{V}$ and $\mathbf{W}$, and the projection is directly applied to the original model in the s-domain to calculate the reduced system, ensuring the Laguerre coefficients matching.

A typical and common choice in theorem 4.1 is $\mathbf{W}=\mathbf{V}$, however in order to double the number of matched Laguerre coefficients, the calculation of the matrices $\mathbf{V}$ and $\mathbf{W}$ should be done according to the following theorem:

Theorem 4.2. *If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$ *used in (3.11), form bases for the Krylov subspaces* $\mathcal{K}_q(\mathbf{A}_v^{-1}\mathbf{E}_v,\mathbf{A}_v^{-1}\mathbf{b})$ *and* $\mathcal{K}_q(\mathbf{A}_v^{-T}\mathbf{E}_v^T,\mathbf{A}_v^{-T}\mathbf{c})$ *respectively, then the first* $2q$ *Laguerre coefficients of the original and reduced order systems match. It is assumed that* $\mathbf{A}_v$ *and* $\mathbf{W}^T\mathbf{A}_v\mathbf{V}$ *are invertible.*

Proof: Similar proof of theorem 4.1. However the number of matched Laguerre coefficients is doubled to $2q$ by using theorem 3.2. ■

The numerical calculation of the projection matrices is achieved by applying the known Lanczos or Arnoldi algorithms or one of their improved versions [5, 6, 24, 57, 76], as the formulation of both theorems is very similar to that of moment matching.

The discussion related to the possible choices of the free parameter α will be skipped here as it constitutes one of the main subject of this thesis and will be detailed in the following chapters.

Chapter 5

A NEW TIME-DOMAIN REDUCTION SCHEME USING LAGUERRE FUNCTIONS

In chapter 3, it was shown that Krylov subspace methods are restricted to being a local approximation of the frequency response, and thus, can not generally guarantee a good overall approximation of the impulse response.

Hence, it is more natural to do order reduction directly in the time-domain through the approximation of the system's impulse response while benefiting from the numerical and computational advantages of the Krylov subspace methods. The first work in this direction appeared to be [41], where some of the first derivatives of the time response of the nonlinear system and those of its corresponding reduced-order model are matched. However, up the author's knowledge, no work dealing with linear system exists, except the works based on matching the Markov parameters which is limited to matching the impulse response and its derivatives at $t = 0$.

Lately, based on the success of several methods approximating the impulse response using orthogonal polynomials for low-order systems, several works tried to improve and further develop these methods to make them suitable for the reduction of large-scale systems. The key point consists of projecting the time response of the original system onto an efficiently calculated lower order dimensional subspace spanned by an orthogonal basis.

Consequently, some of the first coefficients of the infinite orthogonal series expansion of the impulse response of the reduced order model match those of the original one. For instance, in [88], the Chebyshev expansion has been used in time-domain for passive model order reduction of interconnect networks. In [19], a time-domain approach involving the Laguerre polynomials and some Krylov subspaces for the approximation of the impulse response has been presented. A disadvantage of this method is that it employed the Laguerre polynomials which are known to form an unbounded basis for the Hilbert space $\mathcal{L}_2(R_+)$, and thus are inadequate for the approximation of the impulse response of stable systems.

In this chapter, the results of [19] are generalized to a new purely time-domain Krylov-based model reduction involving the Laguerre functions - instead of the Laguerre polynomials - which are exponentially decreasing functions and form a bounded orthonormal basis for $\mathcal{L}_2(R_+)$. In this new approach, the Laguerre coefficients of the impulse response of the reduced-order model, which is calculated using a projection whose matrices form basis of appropriate Krylov subspaces, match, up to a given order, those of the original system. In the next chapter, it will be shown that the obtained reduced-order model in time-domain, is equivalent to the one obtained by the classical moment matching around a single expansion point in frequency-domain. Accordingly, a new time-domain interpretation for the rational interpolation problem is deduced.

5.1 Approximation of the state vector

As a first step towards the approximation of the impulse response, the state vector $\mathbf{x}(t)$ is approximated using the Laguerre functions as

$$\mathbf{x}(t) \approx \mathbf{x}_q(t) = \sum_{i=0}^{q-1} \mathbf{l}_i \phi_i^{\alpha}(t). \tag{5.1}$$

Based on [59, 84], the integral of $\mathbf{x}(t)$ can be expressed as

$$\int_0^t \mathbf{x}(\tau)d\tau = \frac{1}{\alpha}\sum_{i=0}^{\infty}\left(\mathbf{l}_i + 4\sum_{j=0}^{i-1}(-1)^{i+j}\mathbf{l}_j\right)\phi_i^{\alpha}(t), \tag{5.2}$$

and thus the integral equation of the state equation

$$\mathbf{E}\mathbf{x}(t) = \mathbf{A}\int_0^t \mathbf{x}(\tau)d\tau + \mathbf{b}\int_0^t u(\tau)d\tau, \tag{5.3}$$

can be rewritten, after replacing $\mathbf{x}(t)$ by $\mathbf{x}_q(t)$ and assuming $u(t) = \delta(t)$, as:

$$\mathbf{E}\sum_{i=0}^{q-1}\mathbf{l}_i\phi_i^{\alpha}(t) - \frac{\mathbf{A}}{\alpha}\sum_{i=0}^{q-1}\mathbf{l}_i\phi_i^{\alpha}(t) - \frac{4\mathbf{A}}{\alpha}\sum_{i=0}^{q-1}\sum_{j=0}^{i-1}(-1)^{i+j}\mathbf{l}_j\phi_i^{\alpha}(t) = \mathbf{b}.$$

This equation can be represented in matrix form by

$$\mathbf{\Phi}\begin{bmatrix}\mathbf{l}_{q-1}\\ \vdots \\ \mathbf{l}_1 \\ \mathbf{l}_0\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0 \\ \mathbf{b}\end{bmatrix}, \tag{5.4}$$

where

$$\mathbf{\Phi}=\begin{bmatrix}(\alpha\mathbf{E}-\mathbf{A})^{-1} & -4\mathbf{A} & 4\mathbf{A} & -4\mathbf{A} & 4\mathbf{A} \\ & \cdots & \cdots & & \\ & & (\alpha\mathbf{E}-\mathbf{A})^{-1} & -4\mathbf{A} & 4\mathbf{A} \\ & & & (\alpha\mathbf{E}-\mathbf{A})^{-1} & -4\mathbf{A} \\ & & & & \alpha\,(\alpha\mathbf{E}-\mathbf{A})^{-1}\end{bmatrix},$$

with q in this case being an even integer.

As the first q entries of the vector on the RHS of (5.4) are zero, the Laguerre coefficients $\mathbf{l}_i$ of $\mathbf{x}_q(t)$ can be then expressed using the following recursive formulae:

$$\mathbf{l}_0 = \alpha\,(\alpha\mathbf{E}-\mathbf{A})^{-1}\,\mathbf{b}, \tag{5.5}$$

$$\mathbf{l}_i = (\alpha\mathbf{E}-\mathbf{A})^{-1}\sum_{j=0}^{i-1}(-1)^{i+j}4\mathbf{A}\mathbf{l}_j, \quad 1 \le i \le q-1. \tag{5.6}$$

Based on the approximation (5.1), $\mathbf{x}_q(t)$ can be reformulated as:

$$\mathbf{x}_q(t) = \mathbf{L}_q \begin{bmatrix} \phi_0^\alpha(t) \\ \phi_1^\alpha(t) \\ \vdots \\ \phi_{q-1}^\alpha(t) \end{bmatrix}, \tag{5.7}$$

with

$$\mathbf{L}_q = \begin{bmatrix} \mathbf{l}_0 & \mathbf{l}_1 & \cdots & \mathbf{l}_{q-1} \end{bmatrix}. \tag{5.8}$$

Thus, $\mathbf{x}_q(t)$ lies in the subspace spanned by the columns of $\mathbf{L}_q$ for all t.

In this section, for the ease of presentation, it was considered that the state vector $\mathbf{x}_q(t)$ consists of a truncated Laguerre series expansion. However in the proposed method it is assumed that an infinite series, which coefficients are matched, is involved.

5.2 The reduction approach

The key idea of the reduction method presented in this section consists of projecting the state vector $\mathbf{x}(t)$ of the original system (2.9) onto the q-th subspace spanned by the first q Laguerre functions $\phi_i^\alpha(t)$. This results in a reduced order model whose impulse response's Laguerre coefficients match some of the first coefficients of the original response $y(t)$. For the simplicity of exposition, the SISO case is only considered, however all the results presented in this section can be easily generalized to the MIMO case.

Similar to the classical moment matching, the reduced order system is obtained by applying the known projection $\mathbf{x}(t) = \mathbf{V}\hat{\mathbf{x}}(t)$, $\mathbf{V} \in \mathbb{R}^{q\times n}, q < n$ to the system (2.9) and multiplying the state equation by the transpose of the matrix $\mathbf{W}$, e.g. with $\mathbf{W} = \mathbf{V}$,

$$\begin{cases} \overbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}^{\hat{\mathbf{E}}}\dot{\hat{\mathbf{x}}}(t) = \overbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}^{\hat{\mathbf{A}}}\hat{\mathbf{x}}(t) + \overbrace{\mathbf{W}^T\mathbf{b}}^{\hat{\mathbf{b}}}u(t), \\ y(t) = \underbrace{\mathbf{c}^T\mathbf{V}}_{\hat{\mathbf{c}}^T}\hat{\mathbf{x}}(t), \end{cases} \tag{5.9}$$

where q is the order of the reduced system.

Lemma 5.1. *If the columns of* $\mathbf{V}$ *used in (5.9), form an orthonormal basis for the subspace spanned by the columns of* $\mathbf{L}_q$*, then the first* q *Laguerre coefficients of the Laguerre series expansions of the original and reduced state vectors satisfy*

$$\mathbf{l}_i = \mathbf{V}\hat{\mathbf{l}}_i, \qquad 0 \le i \le q-1. \tag{5.10}$$

with $\mathbf{x}(t) = \sum_{i=0}^{\infty} \mathbf{l}_i \phi_i^{\alpha}(t)$ *and* $\hat{\mathbf{x}}(t) = \sum_{i=0}^{\infty} \hat{\mathbf{l}}_i \phi_i^{\alpha}(t)$.

Proof: After integration, the state equation of system (5.9) becomes

$$\mathbf{W}^T\mathbf{E}\mathbf{V}\hat{\mathbf{x}}(t) - \mathbf{W}^T\mathbf{A}\mathbf{V}\int_0^t \hat{\mathbf{x}}(\tau)d\tau = \mathbf{W}^T\mathbf{b}, \tag{5.11}$$

with $\hat{\mathbf{x}}(t)$ as its solution. As the coefficients $\mathbf{l}_i \in colspan(\mathbf{V})$, they can be written as a linear combination of the columns of $\mathbf{V}$,

$$\mathbf{l}_i = \mathbf{V}\mathbf{z}_i, \qquad 0 \le i \le q-1.$$

By substituting the above equation in (5.4) and multiplying both sides by $\mathbf{W}^T$, we get

$$\begin{bmatrix} \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} \\ & \cdots & \cdots & & \\ & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} \\ & & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} \\ & & & & \alpha\left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{z}_{q-1} \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{z}_1 \\ \mathbf{z}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \hat{\mathbf{b}} \end{bmatrix} \tag{5.12}$$

Now, from (5.11) and (5.4), the equations for $\hat{\mathbf{l}}_i$ are found to be

$$\begin{bmatrix} \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} \\ & \cdots & \cdots & & \\ & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} & 4\hat{\mathbf{A}} \\ & & & \left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} & -4\hat{\mathbf{A}} \\ & & & & \alpha\left(\alpha\hat{\mathbf{E}}-\hat{\mathbf{A}}\right)^{-1} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{l}}_{q-1} \\ \vdots \\ \vdots \\ \vdots \\ \hat{\mathbf{l}}_1 \\ \hat{\mathbf{l}}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \hat{\mathbf{b}} \end{bmatrix} \tag{5.13}$$

The proof is completed by comparing (5.13) and (5.12). ∎

Remark 5.1. Due to the fact that the Laguerre functions $\phi_i^\alpha(t) = \sqrt{2\alpha}e^{-\alpha t}l_i(2\alpha t)$ are not polynomials, a proof of lemma 5.1 by simply comparing the coefficients in $\mathbf{x}(t) = \mathbf{V}\hat{\mathbf{x}}(t)$, i.e. $\sum_{i=0}^{\infty}\mathbf{l}_i\phi_i^\alpha(t) = \mathbf{V}\sum_{i=0}^{\infty}\hat{\mathbf{l}}_i\phi_i^\alpha(t)$, is not possible.

Based on the previous lemma, the main theorem describing the choice of the projection matrix in order to match the Laguerre coefficients of the impulse response is then:

Theorem 5.1. *If the columns of* $\mathbf{V}$ *used in (5.9), form an orthonormal basis for the subspace spanned by the columns of* $\mathbf{L}_q$*, and* $\mathbf{W}$ *is chosen such that* $(\mathbf{A} - \alpha\mathbf{E})$ *and* $\mathbf{W}^T(\mathbf{A} - \alpha\mathbf{E})\mathbf{V}$ *are nonsingular, then the first* q *Laguerre coefficients of the Laguerre series expansions of the impulse response of the original and reduced systems match, i.e.,*

$$f_i = \hat{f}_i, \qquad 0 \le i \le q-1, \tag{5.14}$$

with $h(t) = \sum_{i=0}^{\infty} f_i\phi_i^\alpha(t)$ *and* $\hat{h}(t) = \sum_{i=0}^{\infty}\hat{f}_i\phi_i^\alpha(t)$.

Proof: From (2.9), we have $f_i = \mathbf{c}^T\mathbf{l}_i$, and similarly from (5.9), $\hat{f}_i = \hat{\mathbf{c}}^T\hat{\mathbf{l}}_i$. Based on the fact that $\mathbf{W}^T\mathbf{V} = \mathbf{I}$ and lemma 5.1, it can be easily shown that $\hat{\mathbf{l}}_i = \mathbf{W}^T\mathbf{l}_i$. Hence, using Lemma 5.1, $\mathbf{V}\mathbf{W}^T\mathbf{l}_i = \mathbf{V}\mathbf{W}^T\mathbf{V}\hat{\mathbf{l}}_i = \mathbf{V}\hat{\mathbf{l}}_i = \mathbf{l}_i$. Finally,

$$\hat{f}_i = \hat{\mathbf{c}}^T\hat{\mathbf{l}}_i = \mathbf{c}^T\mathbf{V}\mathbf{W}^T\mathbf{l}_i = \mathbf{c}^T\mathbf{l}_i = f_i.$$

■

Even though the previous theorem offers a theoretical proof and tool to find a reduced order model that matches the Laguerre coefficients, it suffers from the famous ill-conditioning problem that occurs when explicitly calculating the Laguerre coefficients. Fortunately, it can be shown that the subspace spanned by the columns of $\mathbf{L}_q$ is a Krylov subspace involving the system matrices. This guarantees matching the Laguerre coefficients without explicitly calculating them, similar to the moment matching case.

Lemma 5.2. *The subspace spanned by the columns of* $\mathbf{L}_q$ *is equal to the Krylov subspace:*

$$\mathcal{K}_q\left((\mathbf{A} - \alpha\mathbf{E})^{-1}\mathbf{A}, (\mathbf{A} - \alpha\mathbf{E})^{-1}\mathbf{b}\right). \tag{5.15}$$

Proof: Let $\mathbf{l}_i$ and $\mathbf{k}_i$ be the basic blocks of the matrix $\mathbf{L}_q$ and the Krylov subspace $\mathcal{K}_q$, respectively. It is shown that the basic blocks of the two subspaces span the same space by proving that the ith basic block of each subspace can be written as a linear combination of the first i blocks of the other.

As α is a constant, and based on (5.5), it is clear that the starting vectors are the same, i.e., $\mathbf{l}_0 = -\alpha\mathbf{k}_0$. Recall that multiplying any basic block by a minus sign or a constant does not affect the spanned subspace. For the next two basic blocks, we have,

$$\begin{aligned}
\mathbf{l}_1 &= (\mathbf{A}-\alpha\mathbf{E})^{-1}(4\mathbf{A}\mathbf{l}_0) = -4\alpha(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b} = -4\alpha\mathbf{k}_1.\\
\mathbf{l}_2 &= (\mathbf{A}-\alpha\mathbf{E})^{-1}[4\mathbf{A}(-\mathbf{l}_0+\mathbf{l}_1)]\\
&= -\mathbf{l}_1 + 4\alpha(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\\
&= 4\alpha\mathbf{k}_1 + 4\alpha\mathbf{k}_2.
\end{aligned}$$

Now consider that $\mathbf{l}_n = \sum_{j=1}^{n}\beta_j\mathbf{k}_j$ for $n = 2,\cdots,q-1$, where β_n is a constant. For an even[1] $i = q$, we have, based on (5.6),

$$\begin{aligned}
\mathbf{l}_q &= (\mathbf{A}-\alpha\mathbf{E})^{-1}[4\mathbf{A}(-\mathbf{l}_0+\mathbf{l}_1+\cdots-\mathbf{l}_{q-1})]\\
&= -4\mathbf{l}_1 + 4\mathbf{l}_2 + \cdots - 4(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}\mathbf{l}_{q-1} = \sum_{j=1}^{q-2}\beta'_j\mathbf{k}_j - 4(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}\mathbf{l}_{k-1}\\
&= \sum_{j=1}^{q-2}\beta'_j\mathbf{k}_j - (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}\sum_{j=1}^{q-1}\beta'_j\mathbf{k}_j = \sum_{j=1}^{q-2}\beta'_j\mathbf{k}_j - \sum_{j=2}^{q}\beta'_j\mathbf{k}_j.
\end{aligned}$$

This part of the proof is completed by induction leading to

$$colspan(\mathbf{L}_k) \subset \mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}, (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right). \tag{5.16}$$

Now, the columns of $\mathcal{K}_q$ are shown to be a linear combination of those of the matrix $\mathbf{L}_k$.

It is clear that the two first vectors are the same up to a constant, i.e., $\mathbf{k}_0 = -\frac{1}{\alpha}\mathbf{l}_0$ and $\mathbf{k}_1 = -\frac{1}{4\alpha}\mathbf{l}_1$.

[1]for an odd q the same proof is valid, however the subsequent derivation will have the opposite alternating sign of the coefficients because of the $(-1)^{i+j}$ in (5.6).

The equation (5.6) can be reformulated as,

$$\mathbf{l}_i = (\alpha\mathbf{E} - \mathbf{A})^{-1} \sum_{j=0}^{i-1} (-1)^{i+j} 4\mathbf{A}\mathbf{l}_j = \mathbf{l}_{i-1} + 4\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}\mathbf{l}_{i-1}, \tag{5.17}$$

which for $i = 2$ becomes

$$\begin{aligned} \mathbf{l}_2 &= -\mathbf{l}_1 + 4\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}\mathbf{l}_1 \\ &= -\mathbf{l}_1 - 4^2\alpha\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}\mathbf{k}_1 \\ &= -\mathbf{l}_1 - 4^2\alpha\mathbf{k}_2 \end{aligned} \tag{5.18}$$

leading to

$$\mathbf{k}_2 = -\frac{1}{4^2\alpha}(\mathbf{l}_1 + \mathbf{l}_2). \tag{5.19}$$

Similarly, by using (5.19) and (5.18),

$$\begin{aligned} \mathbf{l}_3 &= -\mathbf{l}_2 + 4\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}\mathbf{l}_2 \\ &= -\mathbf{l}_2 + 4\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}[-4^2\alpha\mathbf{k}_2 - \mathbf{l}_1] \\ &= -\mathbf{l}_2 - 4^3\alpha\mathbf{k}_3 - \mathbf{l}_1 - \mathbf{l}_2 \end{aligned}$$

leading to

$$\mathbf{k}_3 = -\frac{1}{4^3\alpha}(\mathbf{l}_1 + 2\mathbf{l}_2 + \mathbf{l}_3). \tag{5.20}$$

Now assume that $\mathbf{k}_n = -\frac{1}{4^n\alpha}\sum_{j=0}^{n-1}\binom{n-1}{j}\mathbf{l}_{j+1}$ for $n = 2, \cdots, q-1$. For $n+1$, we have,

$$\begin{aligned} \mathbf{l}_{n+1} &= -\mathbf{l}_n + 4\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}\mathbf{l}_n \\ &= -\mathbf{l}_n + 4\alpha\,(\alpha\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{A}\left[-4^n\alpha\mathbf{k}_n - \binom{n-1}{0}\mathbf{l}_1 - \binom{n-1}{1}\mathbf{l}_2 - \cdots - \binom{n-1}{n-2}\mathbf{l}_{n-1}\right] \\ &= -\mathbf{l}_n - 4^{n+1}\mathbf{k}_{n+1} - \binom{n-1}{0}(\mathbf{l}_1 + \mathbf{l}_2) - \binom{n-1}{1}(\mathbf{l}_3 + \mathbf{l}_2) - \cdots - \binom{n-1}{n-2}(\mathbf{l}_n + \mathbf{l}_{n-1}), \end{aligned}$$

which, by regrouping the different $\mathbf{l}_i$ terms, leads to

$$\mathbf{k}_{n+1} = -\frac{1}{4^{n+1}\alpha}\sum_{j=0}^{n}\binom{n}{j}\mathbf{l}_{j+1}.$$

This part of the proof is completed by induction leading to

$$\mathcal{K}_q\left((\mathbf{A} - \alpha\mathbf{E})^{-1}\mathbf{A}, (\mathbf{A} - \alpha\mathbf{E})^{-1}\mathbf{b}\right) \subset colspan(\mathbf{L}_k). \tag{5.21}$$

Finally,

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A},(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right)=colspan(\mathbf{L}_k). \tag{5.22}$$

■

The importance of this theorem lies in the fact that it allows a numerically stable and computationally efficient way of calculating the projection matrix $\mathbf{V}$ using the well-known Arnoldi and Lanczos algorithms or one of their improved versions [5, 76]. In addition, it reflects the direct dependance of the projection matrix $\mathbf{V}$, and thus the reduced system, on the parameter α. By varying α, different basis functions $\phi_i^{\alpha}(t)$ are generated and consequently the error-spreading of the approximation of the impulse response along the temporal axis can be controlled. The possible choices of the parameter α will be detailed in Chapter 7.

By using the duality principle, it can be stated that if the columns of the matrix $\mathbf{W}$ used in (5.9), form a basis for the Krylov subspace $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{A}^T,(\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right)$, and $\mathbf{V}$ is chosen such that $(\mathbf{A}-\alpha\mathbf{E})$ and $\mathbf{W}^T(\mathbf{A}-\alpha\mathbf{E})\mathbf{V}$ are nonsingular, then the first q Laguerre coefficients of the Laguerre series expansions of the impulse response of the original and reduced systems match.

Now, if the choices of the matrix $\mathbf{V}$ and $\mathbf{W}$ are combined, the number of matched Laguerre coefficients can be doubled to $2q$.

Theorem 5.2. *If the columns of the matrices* $\mathbf{V}$ *and* $\mathbf{W}$*, form bases for the Krylov subspaces* $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A},(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right)$ *and* $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{A}^T,(\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right)$ *respectively, then the first* $2q$ *Laguerre coefficients of the impulse response of the original and reduced order systems match. It is assumed that* $(\mathbf{A}-\alpha\mathbf{E})$ *and* $\mathbf{W}^T(\mathbf{A}-\alpha\mathbf{E})\mathbf{V}$ *are nonsingular.*

Proof: According to theorem 5.1, the first q Laguerre coefficients match. The vector $(\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}$ is in the output Krylov subspace and can be written as a linear combination

of the columns of matrix $\mathbf{W}$,

$$\begin{aligned} \exists \mathbf{r}_0 \in \mathbb{R}^q \quad &: \quad (\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c} = \mathbf{W}\mathbf{r}_0, \\ \text{Thus,}\ \ \mathbf{c}^T\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}_r &= \mathbf{c}^T\left((\alpha\mathbf{E}-\mathbf{A})^{-1}(\alpha\mathbf{E}-\mathbf{A})\right)\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}_r \\ &= \mathbf{r}_0^T\mathbf{W}^T(\alpha\mathbf{E}-\mathbf{A})\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}_r \\ &= \mathbf{r}_0^T\mathbf{W}^T\mathbf{A}\mathbf{V} = \mathbf{c}^T(\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}\mathbf{V}. \end{aligned} \tag{5.23}$$

In addition, based on (5.5) and lemma 5.1,

$$\begin{aligned} \mathbf{V}\hat{\mathbf{l}}_{q-1} &= \mathbf{l}_{q-1} \\ &= (\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}\left[(-1)^{(q-1)}4\alpha\mathbf{I}+\cdots-4(\alpha\mathbf{E}-\mathbf{A})^{-(q-2)}\mathbf{A}^{q-2}\right](\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}. \end{aligned} \tag{5.24}$$

The qth Laguerre coefficient of the impulse response of the reduced system is

$$\begin{aligned} \mathbf{c}_r^T\hat{\mathbf{l}}_q &= \mathbf{c}^T\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}\left[(-1)^q4\alpha\mathbf{I}+\cdots-4(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-(q-1)}\mathbf{A}^{q-1}\right](\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b} \\ &= \mathbf{c}^T\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}_r\left[(-1)^q4\alpha\mathbf{I}+\cdots-4(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-(q-2)}\mathbf{A}^{q-2}\right](\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r \\ &+ \ \ 4\mathbf{c}^T\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}_r(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-(q-1)}\mathbf{A}_r^{q-1}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r. \end{aligned}$$

Now, by using (5.23) and (5.24),

$$\mathbf{c}_r^T\hat{\mathbf{l}}_q = \mathbf{c}^T\mathbf{l}_{q-1} + \underbrace{4\mathbf{c}^T\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{A}_r}_{\mathbf{c}^T(\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}\mathbf{V}}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-(q-1)}\mathbf{A}_r^{q-1}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r. \tag{5.25}$$

Due to the fact that the $(q-1)th$ Laguerre coefficient is matched and the subspace spanned by the columns of $\mathbf{L}_q$ is equivalent to $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A},(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right)$, we have

$$\mathbf{V}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-(q-1)}\mathbf{A}_r^{q-1}(\alpha\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r = (\alpha\mathbf{E}-\mathbf{A})^{-(q-1)}\mathbf{A}^{q-1}(\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}. \tag{5.26}$$

Now, by replacing the previous equation in (5.27),

$$\mathbf{c}_r^T\hat{\mathbf{l}}_q = \mathbf{c}^T\mathbf{l}_{q-1} + \mathbf{c}^T(\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{A}(\alpha\mathbf{E}-\mathbf{A})^{-(q-1)}\mathbf{A}^{q-1}(\alpha\mathbf{E}-\mathbf{A})^{-1}\mathbf{b} = \mathbf{c}^T\mathbf{l}_q \tag{5.27}$$

which completes the proof.

This proof is repeated with the next vector of $\mathbf{W}$ until showing that $\mathbf{c}_r^T\hat{\mathbf{l}}_{2q-1} = \mathbf{c}^T\mathbf{l}_{2q-1}$ and thus that $2q$ Laguerre coefficients match. ■

5.3 Matching the Laguerre coefficients of the system's response to different inputs

Consider that the dynamical system (2.9) has been formulated in terms of the Laguerre functions $\phi_i^\alpha(t)$, i.e.,

$$\mathbf{u}(t) = \sum_{i=0}^{\infty} u_i \phi_i^\alpha(t), \tag{5.28}$$

$$\mathbf{h}(t) = \sum_{i=0}^{\infty} f_i \phi_i^\alpha(t), \tag{5.29}$$

$$\mathbf{y}(t) = \sum_{i=0}^{\infty} y_i \phi_i^\alpha(t), \tag{5.30}$$

with $u(t)$, $h(t)$ and $y(t)$ being respectively the system's input, impulse response and output. Assume further that this system has been reduced using the method of the previous section. Consequently, the Laguerre coefficients of the impulse response of the reduced-order model match, up to a given order, those of the original system, i.e. $\hat{f}_i = f_i,$ for $i = 0, \cdots, q-1$.

It is well known that in order to calculate the system's response to inputs different than the impulse function, the input in question has to be convoluted with the system's impulse response,

$$y(t) = h(t) * u(t) = \sum_{m=0}^{\infty} f_m \phi_m^\alpha(t) * \sum_{n=0}^{\infty} u_n \phi_n^\alpha(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_m u_n \phi_m^\alpha(t) * \phi_n^\alpha(t) \tag{5.31}$$

Now inspired from the work in [21], the following convolution property of the Laguerre function can be derived:

Lemma 5.3. *For any two Laguerre functions $\phi_m^\alpha(t)$ and $\phi_n^\alpha(t)$,*

$$\phi_m^\alpha(t) * \phi_n^\alpha(t) = \frac{1}{\sqrt{2\alpha}} \left(\phi_{m+n}^\alpha(t) - \phi_{m+n+1}^\alpha(t) \right). \tag{5.32}$$

Proof:

$$\begin{aligned}\mathcal{L}\left[\phi_m^\alpha(t) * \phi_n^\alpha(t)\right] &= \phi_m^\alpha(s)\phi_n^\alpha(s) = 2\alpha\frac{(s-\alpha)^m}{(s+\alpha)^{m+1}}\frac{(s-\alpha)^n}{(s+\alpha)^{n+1}} = 2\alpha\frac{(s-\alpha)^{m+n}}{(s+\alpha)^{m+n+2}} \\ &= \left(\frac{(s-\alpha)^{m+n}}{(s+\alpha)^{m+n+1}} - \frac{(s-\alpha)^{m+n+1}}{(s+\alpha)^{m+n+2}}\right) = \frac{1}{\sqrt{2\alpha}}\left(\phi_{m+n}^\alpha(s) - \phi_{m+n+1}^\alpha(s)\right) \\ &= \frac{1}{\sqrt{2\alpha}}\left(\mathcal{L}\left[\phi_{m+n}^\alpha(t)\right] - \mathcal{L}\left[\phi_{m+n+1}^\alpha(t)\right]\right).\end{aligned}$$

■

Hence, it can be shown that the reduced-order model, obtained by the approach introduced in this chapter, does not only match the coefficients of the impulse response but also those of the system's response to different inputs. This result is formulated and proven in the following theorem:

Theorem 5.3. *The reduced order model calculated by the time-domain approach matching the laguerre coefficients of the impulse response, matches also the coefficients of the system's response to any input* $u(t) \in \mathcal{L}_2(R_+)$.

Proof: The response of the reduced system to an input $u(t)$ in the form (5.28) is

$$\hat{y}(t) = \sum_{p=0}^{\infty}\hat{y}_p\phi_p^\alpha(t) = \hat{h}(t) * u(t) = \sum_{m=0}^{\infty}\hat{f}_m\phi_m^\alpha(t) * \sum_{n=0}^{\infty}u_n\phi_n^\alpha(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\hat{f}_m u_n\phi_m^\alpha(t) * \phi_n^\alpha(t).$$

Using lemma 5.3, the previous equation becomes

$$\hat{y}(t) = \sum_{p=0}^{\infty}\hat{y}_p\phi_p^\alpha(t) = \frac{1}{\sqrt{2\alpha}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\hat{f}_m u_n\left(\phi_{m+n}^\alpha(t) - \phi_{m+n+1}^\alpha(t)\right).$$

However, $\phi_p^\alpha(t)$ can only appear on the r.h.s. of the equation above when either $m+n=p$ or $m+n+1=p$. Thus,

$$\hat{y}_p\phi_p^\alpha(t) = \sum_{s=0}^{s=p}\hat{f}_s\phi_{p-s}^\alpha(t) - \sum_{s=0}^{s=p-1}u_s\phi_{p-1-s}^\alpha(t). \tag{5.33}$$

As it is assumed that the reduced and original systems have the same input, and that $\hat{f}_i = f_i$, for $i = 0, \cdots, q$, it can be concluded based on (5.33) that $\hat{y}_p = y_p$ for $p = 0, \cdots, q$.

■

5.4 Illustrative example

In the next chapter, the equivalence between the time-domain Laguerre-based and moment matching about a single expansion will be shown. Hence, at this stage, it is not really meaningful to show simulation results due to the fact that the reduced-order models obtained by both methods are exactly the same. However, it is interesting to illustrate the effect of varying the parameter α on the approximation of the impulse response. It should be noted that an infinite Laguerre series expansion is considered, however only the first five Laguerre functions are plotted, which generally speaking, is enough to get an idea of the time-spread of those functions. The original system considered here is the CD player introduced in Chapter 3 and its reduced-order model of order 10.

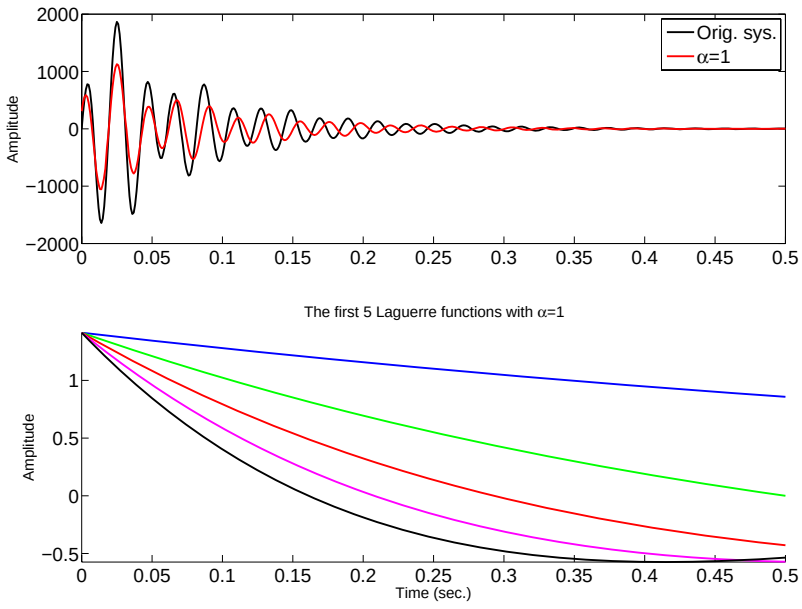

Figure 5.1: The impulse responses of the original and reduced systems ($q = 10$) together with the Laguerre functions for $s_0 = \alpha = 1$.

In figure 5.1, it can be seen that the decay of the first Laguerre functions within the considered time-window of interest $0 - 0.5$ seconds is very slow, which results in a bad approximation of the system's impulse response. In figure 5.2, the value of α is

increased which resulted in a faster decreasing Laguerre functions within the time-window of interest. This lead to a better approximation then the previous case. With $\alpha = 200$, fast decaying functions suitable for the dynamics of this system were generated resulting in a very good approximation of the impulse response as shown in figure 5.3. In fact,

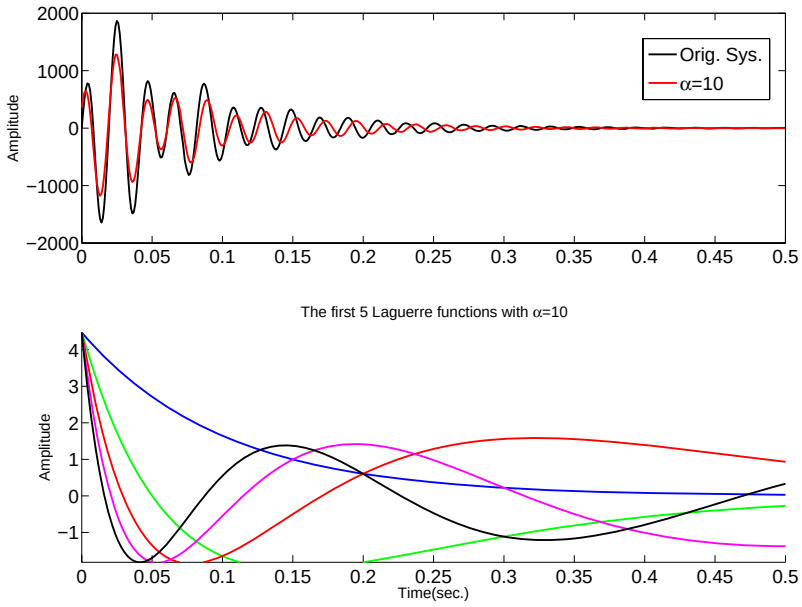

Figure 5.2: The impulse responses of the original and reduced systems ($q = 10$) together with the Laguerre functions for $s_0 = \alpha = 10$.

by varying α a time horizon of interest is implicitly introduced. Hence, the quality and location of the approximation can be controlled, where lower values of α spread out the region of approximation, and higher ones makes it tighter. Consequently, depending on the system's dynamics, a suitable value has to be found based on some knowledge of the original impulse response.

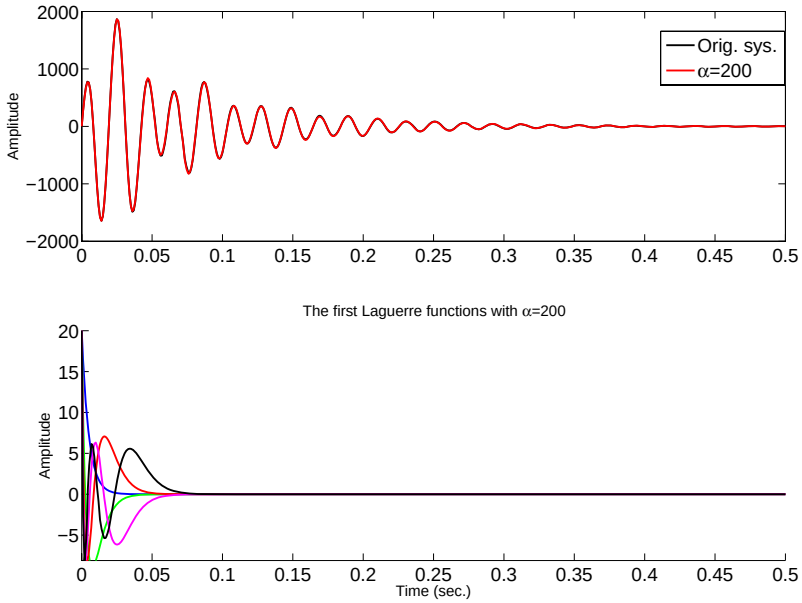

Figure 5.3: The impulse responses of the original and reduced systems ($q = 10$) together with the Laguerre functions for $s_0 = \alpha = 200$.

Chapter 6

THE EQUIVALENCE

As the Laguerre-based order reduction, both in frequency and time-domain, has been receiving more and more attention and started to be considered as an independent reduction approach which is developing parallel to that of moment matching, it is of great importance to investigate the connection between these two approaches and under which conditions they are equivalent.

Towards this aim, the connection between the Krylov subspaces involved in each of the approaches will be first analyzed. Then, the effect of these subspaces on the transfer function of each of the obtained reduced models is described.

This analysis will lead to three main results that can be summarized as follows:

1. In Section 6.3, it will be shown that all the Laguerre-based reduction methods in frequency-domain, e.g. [42, 52], are equivalent to the classical moment matching about a single expansion point [27].

2. In section 6.4, this equivalence is then generalized to the case of the so-called *generalized Markov parameters*. It is shown that by matching the moment about s_0, this family of coefficients are matched including the Laguerre coefficients in frequency-domain as a special case.

3. In Section 6.5, the equivalence between the time-domain Laguerre-based reduction approach of Chapter 5 [25], and moment matching about a single expansion point

is shown. This results in a time-domain interpretation of the classical moment matching approach based on the fact that the reduced order models obtained by both approaches are exactly the same.

6.1 Property of Krylov subspaces

In the following, a lemma and a theorem, that exploit some properties of the Krylov subspaces, are introduced. These properties will play a key role in this chapter when showing the equivalence between the Laguerre-based and moment matching reduction approaches, both in time-and frequency-domain.

Lemma 6.1. *The Krylov subspaces* $\mathcal{K}_q(\mathbf{M},\mathbf{v})$ *and* $\mathcal{K}_q(\mathbf{N},\mathbf{v})$ *with* $\mathbf{M}+c\mathbf{N}=\gamma\mathbf{I}$ *where* $0\neq c,\gamma\in\mathbb{R}$*, are identical.*

Proof: From definition (2.1) we have:

$$\mathcal{K}_q(\mathbf{M},\mathbf{v}) = Range(\mathbf{S}^1),\ \mathbf{S}^1 = \left[\begin{array}{cccc}\mathbf{v} & \mathbf{M}\mathbf{v} & \mathbf{M}^2\mathbf{v} & \cdots & \mathbf{M}^{q-1}\mathbf{v}\end{array}\right],$$
$$\mathcal{K}_q(\mathbf{N},\mathbf{v}) = Range(\mathbf{S}^2),\ \mathbf{S}^2 = \left[\begin{array}{cccc}\mathbf{v} & \mathbf{N}\mathbf{v} & \mathbf{N}^2\mathbf{v} & \cdots & \mathbf{N}^{q-1}\mathbf{v}\end{array}\right].$$

As $\mathbf{N}=\frac{1}{c}(\gamma\mathbf{I}-\mathbf{M})$, $\mathbf{S}^2$ can be rewritten as $\left[\begin{array}{cccc}\mathbf{s}_0^2 & \mathbf{s}_1^2 & \cdots & \mathbf{s}_{q-1}^2\end{array}\right]$ with

$$\mathbf{s}_j^2 = \left(\frac{\gamma\mathbf{I}-\mathbf{M}}{c}\right)^j\mathbf{v} = \sum_{i=0}^{j} k_{ij}\mathbf{M}^i\mathbf{v} = \sum_{i=0}^{j} k_{ij}\mathbf{s}_i^1, \tag{6.1}$$

where $k_{ij}\in\mathbb{R}$ and $\mathbf{s}_i^1$ is the ith column of $\mathbf{S}^1$. From (6.1), it can be seen that the columns of $\mathbf{S}^2$ are a linear combination of those of $\mathbf{S}^1$, and thus $\mathcal{K}_q(\mathbf{N},\mathbf{v})\subset\mathcal{K}_q(\mathbf{M},\mathbf{v})$. Similarly, because $\mathbf{M}=\gamma\mathbf{I}-c\mathbf{N}$, $\mathcal{K}_q(\mathbf{M},\mathbf{v})\subset\mathcal{K}_q(\mathbf{N},\mathbf{v})$, which leads to $\mathcal{K}_q(\mathbf{N},\mathbf{v})=\mathcal{K}_q(\mathbf{M},\mathbf{v})$. ■

This lemma expresses in fact the *shift invariance property* of Krylov subspaces however reformulated for the purpose of connecting the subspaces involved in the Laguerre-based reduction and moment matching.

Theorem 6.1. *The Krylov subspaces* $\mathcal{K}_q\left((\mathbf{P}-\alpha\mathbf{Q})^{-1}\mathbf{Q},\mathbf{v}\right)$ *and* $\mathcal{K}_q\left(\gamma(\mathbf{P}-\alpha\mathbf{Q})^{-1}(\mathbf{P}-\beta\mathbf{Q}),\mathbf{v}\right)$ *with* $\alpha\neq\beta$ *and* $\gamma\neq 0$*, are identical.*

Proof: Set $\mathbf{N} = (\mathbf{P} - \alpha\mathbf{Q})^{-1}\mathbf{Q}$ and $\mathbf{M} = \gamma(\mathbf{P} - \alpha\mathbf{Q})^{-1}(\mathbf{P} - \beta\mathbf{Q})$. $\mathbf{M}$ can be written as:

$$\begin{aligned}\mathbf{M} &= \gamma(\mathbf{P} - \alpha\mathbf{Q})^{-1}(\mathbf{P} - \beta\mathbf{Q} + \alpha\mathbf{Q} - \alpha\mathbf{Q}) \\ &= \gamma\mathbf{I} + \gamma(\alpha - \beta)[(\mathbf{P} - \alpha\mathbf{Q})^{-1}\mathbf{Q}] = \gamma\mathbf{I} - c\mathbf{N}.\end{aligned}$$

By applying Lemma 6.1, the proof is completed. ∎

Remark 6.1. This theorem and Lemma 6.1 can be easily generalized to the multiple starting vectors case. We note that some results related to the Theorem 6.1 can be found in [60] and [37] but used in a different context.

6.2 Invariance property

Another main component for proving the equivalence between the two reduction approaches is the invariance property of the transfer function of the reduced order model to the change of basis of the Krylov subspaces.

Theorem 6.2. *The transfer function of the reduced-order model depends only on the choice of the Krylov subspaces and not on the bases of these subspaces.*

Proof: Consider two reduced order models by using pairs of bases $\mathbf{V}_1, \mathbf{W}_1$ and $\mathbf{V}_2, \mathbf{W}_2$. The corresponding reduced order models are

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\dot{\mathbf{x}}_{r1}(t) = \mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{x}_{r1}(t) + \mathbf{W}_1^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{x}_{r1}(t), \end{cases} \tag{6.2}$$

$$\begin{cases} \mathbf{W}_2^T\mathbf{E}\mathbf{V}_2\dot{\mathbf{x}}_{r2}(t) = \mathbf{W}_2^T\mathbf{A}\mathbf{V}_2\mathbf{x}_{r2}(t) + \mathbf{W}_2^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}\mathbf{V}_2\mathbf{x}_{r2}(t). \end{cases} \tag{6.3}$$

The columns of the matrices $\mathbf{V}_2$ and $\mathbf{W}_2$ are, respectively, in the input and output Krylov subspaces. Hence, they can be written as a linear combination of the other bases which are the columns of matrices $\mathbf{V}_1$ and $\mathbf{W}_1$, and there exist square matrices $\mathbf{Q}_v$ and $\mathbf{Q}_w$ of compatible dimensions such that,

$$\mathbf{V}_2 = \mathbf{V}_1\mathbf{Q}_v, \ \mathbf{W}_2 = \mathbf{W}_1\mathbf{Q}_w. \tag{6.4}$$

Since $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{W}_1$ and $\mathbf{W}_2$ have full rank, the matrices $\mathbf{Q}_v$ and $\mathbf{Q}_w$ are invertible. By substituting equation (6.4) into the system (6.3) and multiplying both sides of the state equation by the invertible matrix $\mathbf{Q}_w^{-T}$ we get,

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\mathbf{Q}_v\dot{\mathbf{x}}_{r2}(t) = \mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}(t) + \mathbf{W}_1^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}(t). \end{cases}$$

Applying the state transformation $\mathbf{z} = \mathbf{Q}_v\mathbf{x}_{r2}$ to this system, converts it into (6.2). Thus, the reduced order models (6.2) and (6.3) have the same transfer functions. ■

6.3 The equivalence in frequency-domain

After reducing the original system by the Laguerre-based method of chapter 4, and matching for instance $2q$ Laguerre coefficients, the original and reduced model transfer functions $H(s)$ and $H_r(s)$ can be written based on (4.3) as:

$$H(s) = \frac{\sqrt{2\alpha}}{s+\alpha}\left[F_0 + F_1\left(\frac{s-\alpha}{s+\alpha}\right) + \ldots + F_{2q-1}\left(\frac{s-\alpha}{s+\alpha}\right)^{2q-1} + F_{2q}\left(\frac{s-\alpha}{s+\alpha}\right)^{2q} + \ldots\right]$$

$$H_r(s) = \frac{\sqrt{2\alpha}}{s+\alpha}\left[F_{r0} + F_{r1}\left(\frac{s-\alpha}{s+\alpha}\right) + \ldots + F_{r2q-1}\left(\frac{s-\alpha}{s+\alpha}\right)^{2q-1} + F_{r2q}\left(\frac{s-\alpha}{s+\alpha}\right)^{2q} + \ldots\right]$$

with $F_i = F_{ri}$ for $i = 0, \ldots, 2q-1$. By calculating the first two derivatives of the reduced system's transfer function at $s = s_0 = \alpha$, we get:

$$m_{r0}^{\alpha} = H_r(s)|_{s=s_0=\alpha} = \frac{F_{r0}}{\sqrt{2\alpha}} = H(s)|_{s=s_0=\alpha} = m_0^{\alpha} \tag{6.5}$$

$$\begin{aligned} m_{r1}^{\alpha} = \left.\frac{dH_r(s)}{ds}\right|_{s=s_0=\alpha} &= \sqrt{2\alpha}\left(\frac{-F_{r0}}{(s+\alpha)^2} + \frac{F_{r1}(3\alpha - s)}{(s+\alpha)^3}\right)\Bigg|_{s=s_0=\alpha} \\ &= \frac{F_{r1} - F_{r0}}{2\alpha\sqrt{2\alpha}} = \left.\frac{dH(s)}{ds}\right|_{s=s_0=\alpha} = m_1^{\alpha}. \end{aligned} \tag{6.6}$$

with m_i^{α} and m_{ri}^{α} being the ith moment about α of the original and reduced systems, respectively. The jth derivative of $H(s)$ and $H_r(s)$ at $s = \alpha$, includes only the first j Laguerre coefficients, because the remaining terms vanish as they have an $(s-\alpha)$ factor in the numerator that becomes zero at $s = \alpha$. Moreover, it can be shown that all the derivatives are a linear combination of the Laguerre coefficients and the constant α.

Theorem 6.3. *When matching some of the first (up to $2q$) Laguerre coefficients of the transfer functions of the original and reduced models, the same number of the first moments about $s = s_0 = \alpha$ will also match.*

Proof: The moments of a transfer function are defined as its derivatives with respect to s about a certain point s_0. As in (6.5) and (6.6), it can be shown by straightforward calculations that some of the first (up to $2q$) derivatives of $H(s)$ and $H_r(s)$ match due to the matching of some of the first (up to $2q$) Laguerre coefficients F_i. This implies that the same number of moments is also matched. ∎

Even though the above theorem establishes already the connection between both reduction approaches and their corresponding coefficients, it does not imply that the reduced systems found by the two methods are the same.

Now, considering a projection-based reduction approach, we know that a reduced system obtained by a two-sided reduction method is unique. Using theorem 6.3, it can be stated that, in this case, the reduced systems calculated by moment matching and Laguerre coefficients matching are the same. However, nothing can be said about the reduced systems obtained by a one-sided method.

In order to consider this problem in a more general way, the Krylov subspaces involved in every method are investigated. It is well known that to match some of the first moments about $s_0 = \alpha$, the projection matrices $\mathbf{V}$ and $\mathbf{W}$ should be chosen as bases of the Krylov subspaces,

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right), \tag{6.7}$$

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right), \tag{6.8}$$

and to match the Laguerre coefficients, they should constitute bases of the subspaces

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}(\mathbf{A}+\alpha\mathbf{E}), (\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right), \tag{6.9}$$

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}(\mathbf{A}+\alpha\mathbf{E})^{T}, (\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right). \tag{6.10}$$

Based on these subspaces, the following theorem generalizes the results of theorem 6.3 by showing the equivalence of the two approaches for the one and two-sided methods.

Theorem 6.4. *The transfer functions of the reduced systems obtained by a one-sided (*$\mathbf{W} = \mathbf{V}$*) or a two-sided Krylov subspace method by matching either the moments about* s_0 *or the Laguerre coefficients, are equal if* $s_0 = \alpha$*.*

Proof: By applying Theorem 6.1 to the subspaces pair (6.7) and (6.9) and to (6.8) and (6.10) with $s_0 = \alpha$, $\gamma = 1$ and $\beta = -\alpha$, it can be concluded that the subspaces involved in the Laguerre-based order reduction and moment matching about $s_0 = \alpha$ are the same. Theorem 6.2 completes the proof. ■

Remark 6.2. Based on Theorem 6.4, if order reduction is carried out to match some of the first moments about s_0, the same number of the first Laguerre coefficients with the parameter $\alpha = s_0$ automatically match. Similarly, if order reduction is carried out to match some of the first Laguerre coefficients with the parameter α, the same number of moments about $s_0 = \alpha$ automatically match.

It should be noted that the fact of matching the Laguerre coefficients implies matching the moments about an appropriate point (and vice versa) is totally independent of the method used to find the reduced system. However, it is worth mentioning that the equivalence result of Theorem 6.4 covers the different Laguerre reduction methods presented in [19, 42, 52].

Remark 6.3. Theorem 6.4 states that the reduced systems obtained by the two different methods have the same input-output behavior, however, they may generally possess different realizations. Since the two methods employ equal subspaces with a common starting vector, the resulting reduced models will have exactly the same realization when using the same numerical algorithm (e.g. Lanczos or Arnoldi) for the calculation of the projection matrices.

This equivalence gives one more credit to moment matching as it implies that by only matching the moments about a certain frequency point, the Laguerre coefficients are

implicitly matched. Also, as it establishes the missing link between these two methods, a possibility is offered to exploit the interesting properties of the Laguerre functions, like their orthogonality (which does not have a counterpart in the Padé-Krylov approach) for the improvement of moment matching-based reduction methods (and possibly vice versa). For instance, the results of this section reformulate the open problem of choosing an optimal expansion point in the rational Krylov subspace reduction methods, to the problem of finding the optimal parameter α in the Laguerre-based reduction methods [53, 89].

From the numerical point of view, depending on the system to be reduced, one of the Laguerre-based or moment matching methods may be preferred, since they suggest the calculation of the projection matrices in two different ways based on different representations of the same Krylov subspace. However, when α tends to zero (or infinity) the Laguerre technique breaks down while the moment (or Markov parameter) matching approach remains theoretically valid and numerically applicable. In addition, for the case where $\mathbf{E} = \mathbf{I}$, the Laguerre-based formulation is numerically more expensive.

6.4 The generalized equivalence

The results of the previous section can be in fact taken a step further and generalized to the so-called *generalized Markov parameters*. These parameters, first introduced in [92], are obtained by generalizing the Taylor series to a series having a first order polynomial in the numerator and denominator, i.e.,

$$T(s) = \frac{\delta s + \eta}{\mu s + \beta}, \tag{6.11}$$

where δ, η, μ and β are real numbers satisfying $\delta\beta - \eta\mu \neq 0$. The importance of this formulation is that it includes many of the well-known series coefficients as a special case, as it will be shown subsequently.

Now, define the operator $u = T(s)$, and

$$\mathbf{G}(u) = \mathbf{H}(T^{-1}(u)) \tag{6.12}$$

where $T^{-1}(u)$ is the inverse transformation of $T(s)$.

Assuming that $\mathbf{G}(u)$ is proper, i.e.,

$$\lim_{u\to\infty} \|\mathbf{G}(u)\| < \infty \tag{6.13}$$

and based on Laurent's theorem, $\mathbf{G}(u)$ can be written as a series in the negative powers of u as,

$$\mathbf{G}(u) = \sum_{k=0}^{\infty} \mathbf{G}_k u^{-k}, \quad |u| > \rho \tag{6.14}$$

where $\mathbf{G}_k$ are real matrices called the generalized Markov parameters and ρ is the radius of a sufficiently large circle enclosing the origin and all the singular points of $\mathbf{G}(u)$ [92].

Many of the well-known coefficients used in order reduction constitute a subset of the previously defined $\mathbf{G}_k$. In fact, the choice of the parameters $(\delta, \eta, \mu, \beta)$ in the bilinear transformation $T(s)$ is responsible of determining the coefficients matched during the reduction procedure. For instance, with the set $(0, 1, 1, 0)$, the moments about zero, with $(1, 0, 0, 1)$ the Markov parameters, and with $(1, \alpha, 1, -\alpha)$ (see 4.4) the Laguerre coefficients in frequency-domain are matched [51, 92].

Theorem 6.5. *Matching the moments about β/μ, i.e. $(0, 1, \mu, \beta)$ is equivalent to matching the generalized Markov parameters $\mathbf{G}_k$.*

Proof: With $\delta, \mu \neq 0$, (otherwise we are restricted to moment or Markov parameters matching), the bilinear transformation (6.11) can be rewritten as,

$$T(s) = \gamma \frac{s + \alpha'}{s - \beta'}, \tag{6.15}$$

with $\gamma = -\delta/\mu$, $\alpha' = \eta/\delta$, and $\beta' = \beta/\mu$. Consequently, by similar analysis to section 6.3, the Krylov subspaces involved in the reduction approach are:

$$\mathcal{K}_q \left(\gamma(\mathbf{A} - \beta'\mathbf{E})^{-1}(\mathbf{A} + \alpha'\mathbf{E}), (\mathbf{A} - \beta'\mathbf{E})^{-1}\mathbf{B}\right), \tag{6.16}$$

$$\mathcal{K}_q \left(\gamma(\mathbf{A} - \beta'\mathbf{E})^{-T}(\mathbf{A} + \alpha'\mathbf{E})^{T}, (\mathbf{A} - \beta'\mathbf{E})^{-T}\mathbf{C}^{T}\right). \tag{6.17}$$

By means of Theorem 6.2, (6.16) and (6.17) are equivalent to the subspaces

$$\mathcal{K}_q\left((\mathbf{A}-\beta'\mathbf{E})^{-1}\mathbf{E},(\mathbf{A}-\beta'\mathbf{E})^{-1}\mathbf{B}\right), \tag{6.18}$$

$$\mathcal{K}_q\left((\mathbf{A}-\beta'\mathbf{E})^{-T}\mathbf{E}^T,(\mathbf{A}-\beta'\mathbf{E})^{-T}\mathbf{C}^T\right), \tag{6.19}$$

which are the subspaces needed to achieve moment matching about $s=\beta'$. ■

Even though the results of this section are theoretically very interesting, in practice, it is still unclear how to benefit from them. In other words, there are no methods that suggests a choice for the parameters α,β,γ, and δ in order for the corresponding reduced system to have certain properties or for the reduction method to preserve, for instance, stability.

6.5 Time-domain interpretation of moment matching

Based on the work in [27], where the equivalence between the classical moment matching and the Laguerre-based reduction approach in frequency-domain [52] has been shown, a similar equivalence between the time-domain Laguerre-based approach and moment matching about a single expansion point is presented in this section.

It was shown in Chapter 5 that in order to match the Laguerre coefficients of the impulse response, the projection matrices $\mathbf{V}$ and $\mathbf{W}$ should be chosen as bases of the Krylov subspaces,

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A},(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right), \tag{6.20}$$

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{A}^T,(\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right) \tag{6.21}$$

and to match the moments about s_0,

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E},(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{b}\right), \tag{6.22}$$

$$\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{E}^T,(\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{c}\right). \tag{6.23}$$

Theorem 6.6. *The Krylov subspaces pairs* $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E},\mathbf{v}\right)$ *and* $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A},\mathbf{v}\right)$*, and* $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{E}^T,\mathbf{v}\right)$ *and* $\mathcal{K}_q\left((\mathbf{A}-\alpha\mathbf{E})^{-T}\mathbf{A}^T,\mathbf{v}\right)$ *are identical.*

Proof: Set $\mathbf{N}=(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E}$ and $\mathbf{M}=(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{A}$. $\mathbf{M}$ can be rewritten as:

$$\begin{aligned}\mathbf{M} &= (\mathbf{A}-\alpha\mathbf{E})^{-1}(\mathbf{A}-\alpha\mathbf{E}+\alpha\mathbf{E})\\ &= \mathbf{I}+\alpha(\mathbf{A}-\alpha\mathbf{E})^{-1}\mathbf{E}\\ &= \mathbf{I}+\alpha\mathbf{N}.\end{aligned}$$

By applying Lemma 6.1 with $\gamma=1$ and $c=-\alpha$, the proof is completed. This proof can be slightly modified to prove the equivalence for the case of the second pair. ■

Based on what was presented, the main theorem stating the equivalence of the two approaches is the following:

Theorem 6.7. *Reducing a state space model in time-domain by matching the Laguerre coefficients of the impulse responses of the original and reduced models is equivalent to matching the moments of their transfer functions about* $s=\alpha$ *in the frequency-domain.*

Proof: Using theorem 6.6, it is shown that the subspaces involved in both approaches are equivalent, and using the fact that the transfer function of the reduced-order model depends only on the choice of the Krylov subspaces as stated in theorem 6.2, the proof is completed. ■

Based on Theorem 6.7, an important time-domain interpretation of the moment matching approach can be deduced. In fact, if order reduction is carried out in time-domain to match some of the first Laguerre coefficients with a certain parameter α as proposed in section 5, the same number of moments about $s_0=\alpha$ in the frequency-domain automatically match. Similarly, if order reduction is carried out in frequency-domain to match some of the first moments about s_0, the same number of the first Laguerre coefficients of the Laguerre series expansion of the impulse response with $\alpha=s_0$ automatically match.

The importance of this equivalence lies in the fact that it allows a first time interpretation of the moment matching approach which has been until now developed and applied only in the frequency domain. in the *time-domain*. In addition, by showing that the time scale factor α in the Laguerre functions corresponds to the expansion point in the frequency-domain, the open problem of choosing a suitable expansion point in the rational Krylov subspace reduction methods, is converted into the problem of finding an optimal time-scale α in the time-domain Laguerre-based methods.

Similarly to the frequency-domain case, Theorem 6.7 indirectly states that the reduced systems obtained by the two different methods have the same input-output behavior, however, they may generally possess different realizations. Now, since the two methods employ equal subspaces, the resulting reduced models will have exactly the same realization when using the same numerical algorithm (e.g. Lanczos or Arnoldi) for the calculation of the projection matrices.

6.6 The connection between MOR in discrete and continuous-time

Applying the Tustin transformation introduced in section 2.2.1, to discretize a given SISO[1] dynamical system (with $\mathbf{E} = \mathbf{I}$), results in a discrete-time system having the following matrices:

$$\mathbf{A}_d = (w_0\mathbf{I} + \mathbf{A})(w_0\mathbf{I} - \mathbf{A})^{-1} \tag{6.24}$$

$$\mathbf{b}_d = \sqrt{2w_o}(w_0\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \tag{6.25}$$

$$\mathbf{c}_d = \sqrt{2w_o}\mathbf{c}^T(w_0\mathbf{I} - \mathbf{A})^{-1} \tag{6.26}$$

$$\mathbf{D}_d = \mathbf{D} - \mathbf{c}^T(w_0\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \tag{6.27}$$

where $w_0 = 2/T$ is not an eigenvalue of $\mathbf{A}$.

[1]For the ease of presentation. The results are also valid for the MIMO case

Theorem 6.8. *The reduced system obtained by matching the Markov parameters in the z-domain is exactly equivalent to the one obtained by matching the moment about $s_0 = w_0$ in the s-domain.*

Proof: Based on what presented in chapter 3, the Krylov subspaces required to match the Markov parameters of the discrete-time system, are the following:

$$\mathcal{K}_q\left(\mathbf{A}_d, \mathbf{b}_d\right) \quad \text{and} \quad \mathcal{K}_q\left(\mathbf{A}_d^T, \mathbf{c}_d\right). \tag{6.28}$$

By replacing the matrices by their values in function of the continuous-time ones,

$$\begin{aligned}
\mathcal{K}_q\left(\mathbf{A}_d, \mathbf{b}_d\right) &= \mathcal{K}_q\left((w_0\mathbf{I}+\mathbf{A})(w_0\mathbf{I}-\mathbf{A})^{-1}, \sqrt{2w_o}\left(\mathbf{A}-w_0\mathbf{I}\right)^{-1}\mathbf{b}\right), \\
\mathcal{K}_q\left(\mathbf{A}_d^T, \mathbf{c}_d\right) &= \mathcal{K}_q\left((w_0\mathbf{I}-\mathbf{A})^{-T}(w_0\mathbf{I}+\mathbf{A})^{T}, \sqrt{2w_o}\left(\mathbf{A}-w_0\mathbf{I}\right)^{-T}\mathbf{c}\right).
\end{aligned}$$

Similar to the Laguerre series case, by applying lemma 6.1 and theorem 6.2, the proof is completed. ∎

It is well known that the Markov parameters in the z-domain are the values of the discrete-time impulse response. Hence, it can be stated based on the previous theorem that by matching q $(2q)$ moments about w_0, the discrete-time impulse responses of the original and reduced systems matches in the first q $(2q)$ steps. In addition, it can be remarked that the inverse of the sampling frequency T appears in the expansion point, which suggests that the latter should be chosen so that the state vectors of the discrete-time system contain sufficient information for describing the model.

These facts offer a possible answer to the question raised by Grimme in his Ph.D thesis [37] about the physical interpretation and good approximation results of real interpolation points. Since they are real, these points can not be connected to the accuracy of the approximation of the frequency response where only imaginary interpolation points of the form $s = jw$ can play a direct role.

6.7 Illustrative Example

To illustrate the equivalence between the moment matching and laguerre-based order reduction presented in this chapter, the following system is considered:

$$H(s) = \frac{(s+2)(s+4)}{(s+1)(s+2)^2(s+6)(s+10)}.$$

The corresponding state space matrices are:

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} -1 & 3 & 1.7321 & 0 & 0 \\ 0 & -2 & 1.7321 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -10 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}, \\
\mathbf{b}^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
\mathbf{c} &= \begin{bmatrix} 1.7321 & 1.7321 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

This system is reduced to order $q = 3$ using the two different methods. In order to match the moments about $s_0 = 0.5$, the projection matrix,

$$\mathbf{V}_m = \begin{bmatrix} -1.4888 & 2.2838 & -2.3204 \\ -0.4060 & 0.4260 & -0.2951 \\ -0.5861 & 0.3804 & -0.1799 \\ -1.4652 & 0.3650 & -0.0694 \\ -15.3846 & 2.3669 & -0.3641 \end{bmatrix} \times 10^{-2},$$

whose columns form a basis for the Krylov subspace $\mathcal{K}_3\left((\mathbf{A}-0.5\mathbf{I})^{-1}, (\mathbf{A}-0.5\mathbf{I})^{-1}\mathbf{b}\right)$, is used (with $\mathbf{W} = \mathbf{V}$), leading to the following reduced system H_m:

$$\begin{aligned}
\mathbf{A}_m &= \begin{bmatrix} -11.0319 & 1 & 0 \\ -37.4282 & -0.5 & 1 \\ -29.6068 & 0 & 0.5 \end{bmatrix}, \\
\mathbf{b}_m^T &= \begin{bmatrix} -11.5319 & -37.4282 & -29.6068 \end{bmatrix}, \\
\mathbf{c}_m &= \begin{bmatrix} -0.0387 & 0.0507 & -0.0471 \end{bmatrix}.
\end{aligned}$$

The first six moments of the original and reduced systems denoted as m and m_r are shown in figure 6.1. As expected, the first 3 moments are matching.

In order to match the Laguerre coefficients with $\alpha = 0.5$, the projection matrix,

$$\mathbf{V}_L = \begin{bmatrix} -1.4888 & 0.7949 & 0.7583 \\ -0.4060 & 0.0199 & 0.1509 \\ -0.5861 & -0.2057 & -0.0052 \\ -1.4652 & -1.1002 & -0.8047 \\ -15.3846 & -13.0178 & -11.0150 \end{bmatrix} \times 10^{-2},$$

whose columns form a basis for the Krylov subspace $\mathcal{K}_3\left((\mathbf{A}-0.5\mathbf{I})^{-1}(\mathbf{A}+0.5\mathbf{I}),\right.$ $\left.(\mathbf{A}-0.5\mathbf{I})^{-1}\mathbf{b}\right)$, is used (with $\mathbf{W}=\mathbf{V}$), leading to the reduced system H_L:

$$\begin{aligned} \mathbf{A}_L &= \begin{bmatrix} -3.21 & -2.71 & -2.71 \\ 21.79 & 22.29 & 22.79 \\ -29.61 & -29.61 & -29.11 \end{bmatrix}, \\ \mathbf{b}_L^T &= \begin{bmatrix} -3.71 & 21.79 & -29.61 \end{bmatrix}, \\ \mathbf{c}_L &= \begin{bmatrix} -0.03868 & 0.01206 & 0.0157 \end{bmatrix}. \end{aligned}$$

The first six Laguerre coefficients of the original and reduced systems denoted as F and F_r are shown in figure 6.1. As expected, the first 3 coefficients are matching.

Although the two systems H_m and H_L appears to be different, they are connected by the similarity transformation,

$$\mathbf{Q} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

and have the same transfer function,

$$H_m(s) = H_L(s) = \frac{-0.0584(s-4.484)(s+4.415)}{(s+5.921)(s+3.441)(s+0.67)}.$$

Moreover, the projection matrices satisfy the relationship $\mathbf{V}_L = \mathbf{V}_m\mathbf{Q}$ as stated in Lemma 6.1.

This confirms the results of Theorem 6.4 and therefore the reduced system H_m matches also the first 3 Laguerre coefficients of the original system and the reduced system H_L matches also the first 3 moments.

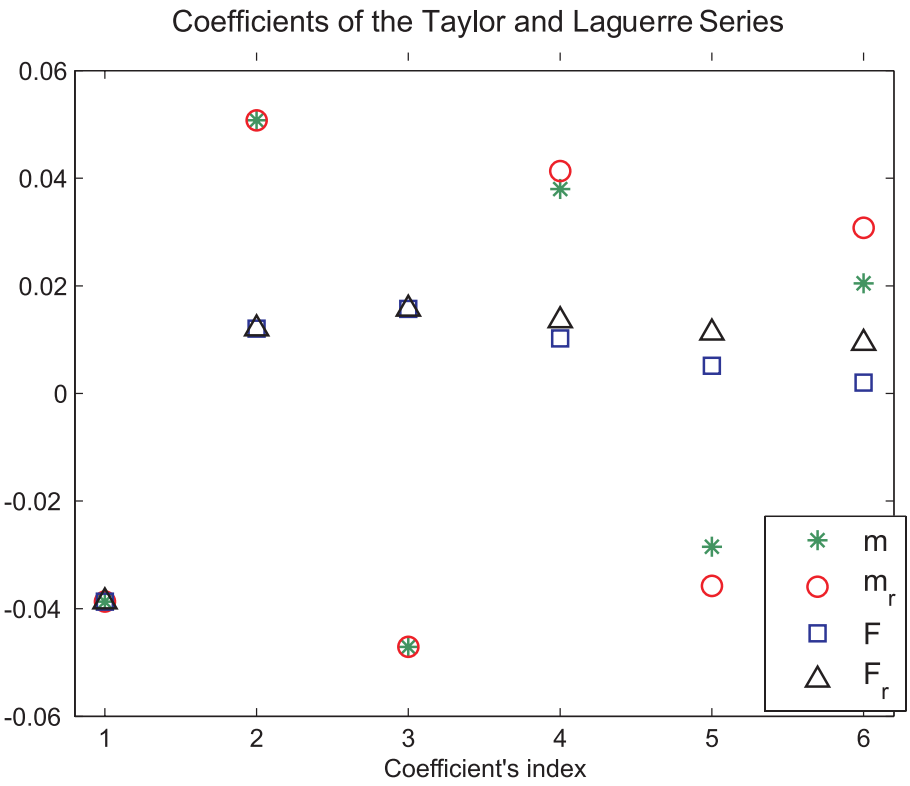

Figure 6.1: Moments and Laguerre coefficients of the original and reduced systems.

As illustrated in figure 6.1, the equivalence property does not imply that the Laguerre coefficients are equal to the moments but only states that matching one set of coefficients results in matching the other one.

Note that the current example has been chosen with a very low order to make the calculations more transparent and to easily illustrate the reduction steps and results.

Chapter 7

CHOICE OF THE INTERPOLATION POINT IN MOMENT MATCHING

Even though the reduced-order model by Krylov-based MOR methods is calculated, via a projection, in a relatively short time with a good numerical accuracy, the interpretation of these methods is restricted to being a local approximation of the frequency response of the original system. The frequency range of interest is determined by the so-called interpolation point about which the moments in the frequency-domain are matched. Hence, it can not be guaranteed that a good approximation of the time responses of the system can be achieved, as it is generally hard in practice to predict the time-domain approximation from the frequency one. It is then more natural to do order reduction directly in the time-domain, for instance, through the approximation of the impulse response.

Furthermore, the appropriate choice of the interpolation point in moment matching is not straightforward and is still an active field of research as its value and number can drastically affect the quality of the approximation and the numerical effort needed to calculate the reduced model. In the literature, different choices for the single or multiple interpolation points have been presented, targeting different aims. In [82], the problem of passivity preserving order reduction has been addressed and a rational Krylov algorithm with interpolation points selected as spectral zeros of the original transfer function has been presented. In [39], an iteratively corrected rational Krylov algorithm for H_2 model

reduction has been suggested. However, both methods are based on interpolating the transfer function at q or $2q$ points which is more expensive when compared to a single interpolation as considered in this work. In addition, these methods do not consider the time responses in any form. Up to date and to the author's knowledge, no method exists for the choice of a single interpolation point based on a time-domain perspective of moment matching.

Lately, several successful methods for approximating the impulse response using orthogonal polynomials have been proposed [19, 25, 52, 88, 90]. Among these approaches, the Laguerre-based reduction of [25] has shown to be very suitable for the reduction of large-scale systems as it can be reformulated (both in time and frequency domain) to benefit from the numerical and computational advantages of the Krylov subspace-based methods.

In order to optimize the approximation using the Laguerre basis functions, the choice of the Laguerre pole, also known as time-scale factor, is crucial. Numerous works treated this problem in system identification [87], approximation [34, 54, 62, 66, 85, 89], and signal processing [16].

Based on the results in the previous chapter, the open problem of choosing an optimal expansion point in the rational Krylov subspace reduction methods (moment matching about $s_0 \neq 0$) is now reformulated to the problem of finding an optimal parameter α in the Laguerre-based reduction methods. This makes it possible to be solved in the time-domain through the Laguerre representation of the original system.

In this chapter, it is first shown that the key parameter for the impulse response approximation of the original system can be calculated optimally in a closed-form by solving appropriate Lyapunov equations. Then, two time-domain methods for the choice of an optimal Laguerre parameter and consequently the single expansion point in rational interpolation order reduction are presented. Accordingly, different model reduction algorithms are suggested and their advantages and disadvantages are pointed out. The

importance of these approaches lies in the fact that they try to minimize the effect of the higher order terms in the infinite Laguerre series expansions of the impulse response, and that they simultaneously consider the moment matching approach, which is originally developed in frequency domain, from a time-domain perspective. In addition, the methods have a simple structure and are numerically efficient and thus suitable for the reduction of large-scale systems.

7.1 Property of the Laguerre Function

The key point to investigate the Laguerre parameter is the differential equation satisfied by the Laguerre functions. It is well-known that the Laguerre polynomial $l_i(t)$ satisfies the following differential equation [84],

$$t\ddot{l}_i(t) + (1-t)\dot{l}_i(t) + il_i(t) = 0.$$

Considering the Laguerre function and the variable $\tilde{t} = 2\alpha t$, the following relations hold,

$$\begin{aligned} l_i(\tilde{t}) &= \frac{1}{\sqrt{2\alpha}} e^{\alpha t} \phi_i^\alpha(t), \\ \frac{d}{d\tilde{t}} l_i(\tilde{t}) &= \frac{1}{2\alpha\sqrt{2\alpha}} e^{\alpha t} \left(\dot{\phi}_i^\alpha(t) + \alpha \phi_i^\alpha(t) \right), \\ \frac{d^2}{d\tilde{t}^2} l_i(\tilde{t}) &= \frac{1}{4\alpha^2\sqrt{2\alpha}} e^{\alpha t} \left(\ddot{\phi}_i^\alpha(t) + 2\alpha\dot{\phi}_i^\alpha(t) + \alpha^2 \phi_i^\alpha(t) \right). \end{aligned}$$

Combining these equations with the following equation,

$$2\alpha t\ddot{l}_i(2\alpha t) + (1-2\alpha t)\dot{l}_i(2\alpha t) + il_i(2\alpha t) = 0,$$

leads to the differential equation that is satisfied by the Laguerre function,

$$\begin{aligned} t\left(\ddot{\phi}_i^\alpha(t) + 2\alpha\dot{\phi}_i^\alpha(t) + \alpha^2\phi_i^\alpha(t) \right) + (1-2\alpha t)\left(\dot{\phi}_i^\alpha(t) + \alpha\phi_i^\alpha(t) \right) + 2\alpha i\phi_i^\alpha(t) = 0 \quad &\Rightarrow \\ t\ddot{\phi}_i^\alpha(t) + \dot{\phi}_i^\alpha(t) - \alpha^2 t\phi_i^\alpha(t) + \alpha\phi_i^\alpha(t) + 2\alpha i\phi_i^\alpha(t) = 0 \quad &\Rightarrow \\ -t\ddot{\phi}_i^\alpha(t) - \dot{\phi}_i^\alpha(t) + \alpha^2 t\phi_i^\alpha(t) = 2\alpha(i+\tfrac{1}{2})\phi_i^\alpha(t). \end{aligned} \tag{7.1}$$

The differential equation (7.1) which is found by a direct calculation in time-domain is the same as in [66] where it was derived in the s-domain using the Laplace transform of

the Laguerre function. This property will play a key role when solving for the optimal[1] α in the following sections.

7.2 An optimal Laguerre parameter

Following the results in [66] and assuming that $h(t)$ is the impulse response of a stable system, one can write

$$\begin{aligned}
&-\int_0^\infty t h(t)\ddot{h}(t)dt - \int_0^\infty h(t)\dot{h}(t)dt + \alpha^2 \int_0^\infty t h^2(t)dt \\
&= \int_0^\infty \left[-t\sum_{i=0}^{\infty} f_i \ddot{\phi}_i^\alpha(t) - \sum_{i=0}^{\infty} f_i \dot{\phi}_i^\alpha(t) + \alpha^2 t \sum_{i=0}^{\infty} f_i \phi_i^\alpha(t) \right] h(t)dt \\
&= \int_0^\infty \left[\sum_{i=0}^{\infty} f_i \left(-t\ddot{\phi}_i^\alpha(t) - \dot{\phi}_i^\alpha(t) + \alpha^2 t \phi_i^\alpha(t) \right) \right] h(t)dt.
\end{aligned}$$

By using the differential equation (7.1),

$$\begin{aligned}
\int_0^\infty \left[\sum_{i=0}^{\infty} f_i \left(-t\ddot{\phi}_i^\alpha(t) - \dot{\phi}_i^\alpha(t) + \alpha^2 t \phi_i^\alpha(t) \right) \right] h(t)dt &= \int_0^\infty \left[\sum_{i=0}^{\infty} f_i \left(2\alpha(i+\frac{1}{2}) \right) \phi_i^\alpha(t) \right] h(t)dt \\
= \sum_{i=0}^{\infty} \left[f_i \left(2\alpha(i+\frac{1}{2}) \right) \int_0^\infty \phi_i^\alpha(t) h(t)dt \right] &= \sum_{i=0}^{\infty} 2f_i^2 \alpha(i+\frac{1}{2}).
\end{aligned}$$

Consequently,

$$-\int_0^\infty t\, h(t)\ddot{h}(t)dt - \int_0^\infty h(t)\dot{h}(t)dt + \alpha^2 \int_0^\infty t\, h^2(t)dt = \sum_{i=0}^{\infty} 2f_i^2 \alpha(i+\frac{1}{2}).$$

Now define the cost function,

$$\min_\alpha J(\alpha) = \min_\alpha \sum_{i=0}^{\infty} i f_i^2. \tag{7.2}$$

By minimizing this cost function that imposes more weight on the coefficients with higher index, it can be shown that the convergence of the infinite sum of the Laguerre functions is accelerated. In other words, *minimizing $J(\alpha)$ leads to a basis where only few of the first*

[1] The definition of optimality in the context of this dissertation will be explained in the next section.

terms are almost enough to describe the impulse response in terms of Laguerre functions, as the higher indexed f_i are made small.

To calculate $J(\alpha)$, assume that $\lim_{t\to\infty} h(t) = 0$, $\lim_{t\to 0} h(t) < \infty$, $\lim_{t\to\infty} \dot{h}(t) = 0$ and $\sum_{i=0}^{\infty} f_i^2 = \|h(t)\|_2^2 = \int_0^\infty h^2(t)dt$ and,

$$-\int_0^\infty t\,h(t)\ddot{h}(t)dt - \int_0^\infty h(t)\dot{h}(t)dt + \alpha^2 \int_0^\infty t\,h^2(t)dt = 2\alpha J + \alpha \sum_{i=0}^{\infty} f_i^2. \tag{7.3}$$

Then,

$$-\int_0^\infty t\,h(t)\ddot{h}(t)dt = -\,t\,h(t)\dot{h}(t)\Big]_0^\infty + \int_0^\infty \dot{h}(t)(h(t) + t\dot{h}(t))dt \tag{7.4}$$

which leads to

$$\int_0^\infty t\,\dot{h}^2(t)dt + \alpha^2 \int_0^\infty t\,h^2(t)dt = 2\alpha J + \alpha\|h(t)\|_2^2.$$

Consequently,

$$J = \|h(t)\|_2^2 \frac{\alpha^2 M_1 + M_2}{2\alpha} - \frac{1}{2}\|h(t)\|_2^2, \tag{7.5}$$

$$\text{with} \quad M_1 = \frac{\int_0^\infty t\,h^2(t)dt}{\int_0^\infty h^2(t)dt}, \quad \text{and} \quad M_2 = \frac{\int_0^\infty t\,\dot{h}^2(t)dt}{\int_0^\infty h^2(t)dt}. \tag{7.6}$$

The optimal value of α, defined as the value minimizing the cost function $J(\alpha)$, can be found as follows

$$\frac{dJ}{d\alpha} = \|h(t)\|_2^2 \frac{2\alpha^2 M_1 - 2M_2}{4\alpha^2} = 0 \quad \Rightarrow$$

$$\alpha^* = \sqrt{\frac{M_2}{M_1}}\,,\; J^* = \|h(t)\|_2^2 \left(\sqrt{M_2 M_1} - \frac{1}{2}\right). \tag{7.7}$$

Based on their definition, the parameters M_1 and M_2 can be physically interpreted as being, respectively, the *decay rate* and *smoothness* of the system's impulse response.

7.3 Calculation of M_1 and M_2 for LTI systems

The main question arising in this context is the calculation of the optimal parameter α in practice. Although the problem looks complicated in the general case, it may be easily

solved for LTI systems. Consider the continuous-time dynamical system in state-space

$$\Sigma \; : \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{x}(t), \end{cases} \tag{7.8}$$

with $h(t) = \mathbf{c}^T e^{\mathbf{A}t}\mathbf{b}$ being its impulse response, assuming zero initial conditions. The square of the two-norm of this system is known to be,

$$\|h(t)\|_2^2 = \int_0^\infty h^2(t)dt = \mathbf{c}^T\mathbf{X}\mathbf{c}.$$

where $\mathbf{X}$ is the controllability gramian and satisfies the following Lyapunov equation,

$$\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^T + \mathbf{b}\mathbf{b}^T = \mathbf{0}, \text{ and } \mathbf{X} = \int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt. \tag{7.9}$$

Lemma 7.1. *For the system* (7.8)*, the optimal parameter that minimizes the cost function* (7.5) *can be calculated as follows:*

$$\alpha^* = \sqrt{\frac{\mathbf{c}^T\mathbf{A}\mathbf{Y}\mathbf{A}^T\mathbf{c}}{\mathbf{c}^T\mathbf{Y}\mathbf{c}}}, \tag{7.10}$$

where $\mathbf{Y}$ *is the solution of the Lyapunov equation,*

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T + \mathbf{X} = \mathbf{0}, \tag{7.11}$$

and $\mathbf{X}$ *is the controllability gramian.*

Proof: First the numerator of M_1 for a system of the form (7.8) is calculated,

$$\int_0^\infty t\, h^2(t)dt = \mathbf{c}^T \underbrace{\int_0^\infty t\, e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt}_{\mathbf{Y}}\, \mathbf{c}.$$

Assuming that the system is stable and considering equation (7.9), we have,

$$\begin{aligned} \mathbf{A}\mathbf{Y} &= \int_0^\infty t\, \mathbf{A}e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt \\ &= t\, e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}\Big]_0^\infty - \int_0^\infty t\, e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt - \int_0^\infty t\, e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt\mathbf{A}^T = \mathbf{0} - \mathbf{X} - \mathbf{Y}\mathbf{A}^T. \end{aligned}$$

Therefore,

$$\mathbf{AY} + \mathbf{YA}^T + \mathbf{X} = \mathbf{0}, \quad \text{and} \quad M_1 = \frac{\mathbf{c}^T\mathbf{Yc}}{\mathbf{c}^T\mathbf{Xc}}. \tag{7.12}$$

To calculate M_2, we have,

$$\begin{aligned}\int_0^\infty t\,\dot{h}^2(t)dt &= \mathbf{c}^T\mathbf{A}\int_0^\infty t\,e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T e^{\mathbf{A}^T t}dt\mathbf{A}^T\mathbf{c} \\ &= \mathbf{c}^T\mathbf{AYA}^T\mathbf{c} \Rightarrow M_2 = \frac{\mathbf{c}^T\mathbf{AYA}^T\mathbf{c}}{\mathbf{c}^T\mathbf{Xc}}.\end{aligned} \tag{7.13}$$

Applying equation (7.7) completes the proof. ■

Remark 7.1. If the original system is stable then $\mathbf{X}$ and $\mathbf{Y}$ are positive definite. Therefore, $M_1, M_2 > 0$ and α^* is real.

Remark 7.2. If the impulse response data is already available, for instance, after a system simulation, the calculation of M_1 and M_2 is then numerically possible and straightforward. All what is needed is to numerically evaluate the integrals involved in the definition of these parameters (7.6) and thus there will be no need for the lyapunov equations derived in Lemma 7.1.

Remark 7.3. The closed form calculation of the parameters M_1 and M_2 involves the controllability gramian known from the TBR reduction method. This is an interesting point to be further investigated, as it indirectly connects the moment matching method to the TBR method through the choice of the single interpolation point s_0.

According to Lemma 7.1, the cost function J can now be easily minimized and the optimal parameter α is found by (7.10). Such a calculation is straightforward as J depends only on the original system and the parameter α.

7.4 Rational Krylov with an optimal interpolation point

The previously presented approach for the choice of the parameter α can be employed to achieve an optimal approximation of the system's impulse response when representing it with a truncated Laguerre series while minimizing the number of coefficients. However, this series is not truncated when reducing the system using the Laguerre-based order reduction of Chapter 5. Hence, the idea of using α^* here as an optimal interpolation point in the Krylov-based order reduction. This is possible due to the equivalence results of the previous chapter where it was shown that this Laguerre parameter corresponds to the expansion point s_0 in moment matching.

Remark 7.4. Due to the fact that when matching the Laguerre coefficients of the impulse response, the coefficients of the system's response to all inputs $u(t) \in \mathcal{L}_2(R_+)$ are matched, this choice of α^* is then not only optimal for an impulse input but also for all these inputs $u(t) \in \mathcal{L}_2(R_+)$.

Accordingly, the following algorithm is suggested:

Algorithm 7.1. *Rational Krylov with an Optimal Point (RK-OP)*

1. *Solve the Lyapunov equations* (7.9), (7.11) *to calculate* $\mathbf{X}$ *and* $\mathbf{Y}$.

2. *Calculate* α^* *using* (7.10) *to minimize the function* (7.2).

3. *Find the reduced system* (3.11) *using* α^* *and with a given order* q *using* (3.13) *and* (3.14)*, i.e. classical moment matching about a single point* $s_0 = \alpha^*$.

Once the impulse response data is not available, the calculation of α^* will be costly as two Lyapunov equations in the size of the original system are to be solved. By finding an *approximate* solution of the Lyapunov equations involved in the RK-OP algorithm, the cost of calculation can be dramatically reduced.

In the following, $\mathbf{X}$ and $\mathbf{Y}$ are approximately calculated using a reduced system by means of the so-called Galerkin conditions [44, 73].

Consider the Lyapunov equation associated with the reduced system Σ_r in (3.11),

$$\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{X}_r + \mathbf{X}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \mathbf{W}^T\mathbf{b}\mathbf{b}^T\mathbf{W} = \mathbf{0}. \tag{7.14}$$

If we approximate the original controllability gramian as,

$$\mathbf{X} \approx \hat{\mathbf{X}} = \mathbf{V}\mathbf{X}_r\mathbf{V}^T, \tag{7.15}$$

and assuming $\mathbf{W} = \mathbf{V}$, the following Galerkin condition is satisfied,

$$\mathbf{W}^T\left(\mathbf{A}\hat{\mathbf{X}} + \hat{\mathbf{X}}\mathbf{A}^T + \mathbf{b}\mathbf{b}^T\right)\mathbf{W} = \mathbf{0}. \tag{7.16}$$

Using the approximate gramian to calculate $\mathbf{Y}$ results in

$$\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{Y}_r + \mathbf{Y}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \mathbf{X}_r = \mathbf{0}, \tag{7.17}$$

$$\mathbf{Y} \approx \hat{\mathbf{Y}} = \mathbf{V}\mathbf{Y}_r\mathbf{V}^T, \tag{7.18}$$

$$\mathbf{W}^T\left(\mathbf{A}\hat{\mathbf{Y}} + \hat{\mathbf{Y}}\mathbf{A}^T + \mathbf{X}_r\right)\mathbf{W} = \mathbf{0}. \tag{7.19}$$

Accordingly, the optimal parameter is then approximated as,

$$\alpha^* \approx \sqrt{\frac{\mathbf{c}^T\mathbf{A}\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{A}^T\mathbf{c}}{\mathbf{c}^T\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{c}}}. \tag{7.20}$$

Such an approximation depends on the reduced system itself and the optimization is hence not straightforward. To converge to an optimal solution as in the RK-OP algorithm, it is proposed to iterate between the optimal parameter and the reduced system starting from an initial parameter.

Algorithm 7.2. *Rational Krylov with an Iteratively Calculated Optimal Point (RK-ICOP)*

1. *Reduce the original system using an arbitrary value α_0 and set $i = 1$.*

2. *Solve the corresponding Lyapunov equations for the reduced system to calculate* $\mathbf{X}_r$ *and* $\mathbf{Y}_r$.

3. *Calculate the approximation of the optimal parameter* α_i^* *using* (7.20).

4. *Reduce the system using* α_i^* *with a given order* q.

5. *Increase* i *and go back to step 2.*

The algorithm may be terminated if $|\alpha_i - \alpha_{i-1}| \leq \epsilon$ for a given tolerance ϵ. The convergence of this algorithm will be discussed in the following section.

The idea of approximating the solution of the Lyapunov equations using Krylov subspaces is based on the first works in this direction presented in [44, 73]. In these works, a basis of the subspace $\mathcal{K}_q(\mathbf{A}, \mathbf{b})$ has been used to reduce the original system, then the Lyapunov equations were solved using the reduced system and their solutions projected back as in eq. 7.15 leading to the approximate solution. In the recent works [78] and [79], it was shown that the quality of the approximation can be dramatically improved when the Krylov subspaces involving the matrix $\mathbf{A}^{-1}$ and/or $\left(\mathbf{A} - \mathbf{s_0}\mathbf{I}^{-1}\right)$ are employed. These facts have been confirmed in this work, and it is suggested to employ the subspace $\mathcal{K}_q\left(\left(\mathbf{A} - \alpha\mathbf{I}^{-1}\right), \mathbf{b}\right)$. In addition, it was observed that for the IK-ICOP algorithm, the starting value of the parameter α does not affect the convergence.

Remark 7.5. A common method to approximate or identify complex systems is based on *truncating* the Laguerre series expansion. Assume that $h(t)$ is approximated by the sum of the first N terms as $\hat{h}(t) = \sum_{i=0}^{N-1} f_i \phi_i^{\alpha}(t)$. Then,

$$\|h(t) - \hat{h}(t)\|_2^2 = \sum_{i=N}^{\infty} f_i^2 \leq \frac{1}{N} \sum_{i=0}^{\infty} i f_i^2 = \frac{1}{N} J \Rightarrow$$
$$\frac{\|h(t) - \hat{h}(t)\|_2^2}{\|h(t)\|_2^2} \leq \frac{1}{N} \frac{\alpha^2 M_1 + M_2}{2\alpha} - \frac{1}{2N}. \tag{7.21}$$

This suggests to minimize J to find an optimal α that minimizes the upper bound of the relative error norm of the approximation. Such a reduced system usually does not

lead to satisfactory result mainly because all its poles are located in a single point $-\alpha$. Furthermore, although the cost function J appears as the upper bound of the error, it does not really reflect the magnitude of the error system. In most applications, the bound given above is far from the real two-norm of the error as the weighting i increases to infinity.

7.5 Rational krylov with an optimal error minimizing interpolation point

Consider the order reduction problem by matching the first q Laguerre coefficients. An alternative cost function to the one considered before would be to minimize the difference between the rest of the coefficients, i.e. the unmatched ones. This suggests the new cost function,

$$J_d = \sum_{i=q}^{\infty} i(f_i - f_{ri})^2 = \sum_{i=0}^{\infty} i f_i^2 + \sum_{i=0}^{\infty} i f_{ri}^2 - 2\sum_{i=0}^{\infty} i f_i f_{ri}. \tag{7.22}$$

$$\text{with } f_i = f_{ri} \text{ for } i = 1, \cdots, q-1. \tag{7.23}$$

The value of α suggested by minimizing the cost function J_d will lead to accurate reduced systems as the first coefficients are matched and the weighted sum of the unmatched ones is minimized.

In the following, the optimal parameter for this cost function is calculated. The first two terms in (7.22) are calculated using equation (7.5) and the result of section 7.2. For the last term, the method in section 7.1 is followed and the differential equation (7.1) is used,

$$\begin{aligned}
&-\int_0^{\infty} t\, h_r(t)\ddot{h}(t)dt - \int_0^{\infty} h_r(t)\dot{h}(t)dt + \alpha^2 \int_0^{\infty} t\, h(t)h_r(t)dt \\
&= \int_0^{\infty} \left[\sum_{i=0}^{\infty} f_i 2\alpha \left(i + \frac{1}{2}\right) \phi_i^{\alpha}(t)\right] h_r(t)dt \\
&= 2\alpha \sum_{i=0}^{\infty} i f_i f_{ri} + \alpha \sum_{i=0}^{\infty} f_i f_{ri} = 2\alpha \sum_{i=0}^{\infty} i f_i f_{ri} + \alpha \int_0^{\infty} h(t)h_r(t)dt.
\end{aligned}$$

Consequently, by simplifying the integral terms we have,

$$-\int_0^\infty t\, h_r(t)\ddot{h}(t)dt - \int_0^\infty h_r(t)\dot{h}(t)dt + \alpha^2 \int_0^\infty t\, h(t)h_r(t)dt = 2\alpha \sum_{i=0}^{\infty} i f_i f_{ri} + \int_0^\infty h(t)h_r(t)dt$$

$$\Rightarrow \sum_{i=0}^{\infty} i f_i f_{ri} = \frac{1}{2\alpha}\left(\int_0^\infty t\, \dot{h}_r(t)\dot{h}(t)dt + \alpha^2 \int_0^\infty t\, h(t)h_r(t)dt - \int_0^\infty h(t)h_r(t)dt\right)$$

Consider the original and its projected reduced system with $h_r(t) = \mathbf{c}^T\mathbf{V}e^{\mathbf{W}^T\mathbf{A}\mathbf{V}t}\mathbf{W}^T\mathbf{b}$, then,

$$\int_0^\infty h(t)h_r(t)dt = \mathbf{c}^T \int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T\mathbf{W}e^{\mathbf{V}^T\mathbf{A}^T\mathbf{W}t}dt\mathbf{V}^T\mathbf{c} = \mathbf{c}^T\tilde{\mathbf{X}}\mathbf{V}^T\mathbf{c}.$$

where $\tilde{\mathbf{X}}$ is the solution of the following Sylvester equation,

$$\mathbf{A}\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \mathbf{b}\mathbf{b}^T\mathbf{W} = \mathbf{0}. \tag{7.24}$$

Assuming that the original and reduced systems are stable and considering the solution of the Sylvester equation (7.24),

$$\int_0^\infty t\, h_r(t)h(t)dt = \mathbf{c}^T\tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{c}. \tag{7.25}$$

where the new variable $\tilde{\mathbf{Y}}$ satisfies,

$$\mathbf{A}\tilde{\mathbf{Y}} + \tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{A}^T\mathbf{W} + \tilde{\mathbf{X}} = \mathbf{0}, \text{ and } \tilde{\mathbf{Y}} = \int_0^\infty e^{\mathbf{A}t}\mathbf{b}\mathbf{b}^T\mathbf{W}e^{\mathbf{V}^T\mathbf{A}^T\mathbf{W}t}tdt$$

Finally, it is straightforward to show that,

$$\int_0^\infty t\, \dot{h}_r(t)\dot{h}(t)dt = \mathbf{c}^T\mathbf{A}\tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{A}^T\mathbf{W}\mathbf{V}^T\mathbf{c}. \tag{7.26}$$

For the ease of presentation, the preceding results are summarized in the following lemma.

Lemma 7.2. *Consider an LTI system that has been reduced by projection. The cost function J_d in equation* (7.22) *satisfies,*

$$J_d = \left(\frac{\alpha^2\tilde{M}_1 + \tilde{M}_2}{2\alpha} - \frac{\|h(t)\|_2}{2} - \frac{\|h_r(t)\|_2}{2} + \mathbf{c}^T\tilde{\mathbf{X}}\mathbf{V}^T\mathbf{c}\right),$$

where,

$$\tilde{M}_1 = \mathbf{c}^T \left(\mathbf{Y} + \mathbf{V}\mathbf{Y}_r\mathbf{V}^T - 2\tilde{\mathbf{Y}}\mathbf{V}^T \right) \mathbf{c}, \tag{7.27}$$

$$\tilde{M}_2 = \mathbf{c}^T \left(\mathbf{A}\mathbf{Y}\mathbf{A}^T + \mathbf{V}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{A}^T\mathbf{W}\mathbf{V}^T - 2\mathbf{A}\tilde{\mathbf{Y}}\mathbf{V}^T\mathbf{A}^T\mathbf{W}\mathbf{V}^T \right) \mathbf{c}. \tag{7.28}$$

What restricts the application of the cost function given in lemma 7.2 is the dependency of the RHS on the reduced system. However, if the reduced system is assumed to be given, the upper bound can be minimized leading to the *suboptimal* parameter $\alpha^* = \sqrt{\frac{\tilde{M}_2}{\tilde{M}_1}}$.

Algorithm 7.3. *Rational Krylov with an Optimal Error Minimizing Point (RK-OEMP)*

1. *Solve the Lyapunov equations for the original system to find* $\mathbf{X}$ *and* $\mathbf{Y}$*.*
2. *Reduce the original system using an arbitrary value* α_0 *and set* $i = 1$*.*
3. *Solve the corresponding Lyapunov and Sylvester equations for the reduced system to calculate* $\mathbf{X}_r$*,* $\mathbf{Y}_r$*,* $\tilde{\mathbf{X}}$ *and* $\tilde{\mathbf{Y}}$*.*
4. *Calculate the parameter* $\alpha_i = \sqrt{\frac{\tilde{M}_2}{\tilde{M}_1}}$ *where* $\tilde{M}_2$ *and* $\tilde{M}_1$ *are defined in theorem 7.2.*
5. *Reduce the system using* α_i *with a given order* q*.*
6. *Increase* i *and go back to step 3.*

It should be noted that the value of $\mathbf{X}$ and $\mathbf{Y}$ should be calculated only once and the best choice for the starting parameter is $\alpha_0 = \sqrt{\frac{M_2}{M_1}}$. To reduce the computational cost and avoid solving Lyapunov equations in the size of original system, similar to algorithm 7.2, the Galerkin method is employed leading to the next algorithm. Note that calculating $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ is not numerically expensive as they are $n \times q$ matrices with $q << n$.

Algorithm 7.4. *Rational Krylov with a Near Optimal Error Minimizing Point (RK-NOEMP)*

1. *Reduce the original system using an arbitrary value* α_0 *and set* $i = 1$.
2. *Solve the corresponding Lyapunov and Sylvester equations for the reduced system to calculate* $\mathbf{X}_r$, $\mathbf{Y}_r$, $\tilde{\mathbf{X}}$ *and* $\tilde{\mathbf{Y}}$ *and set* $\mathbf{Y} = \mathbf{V}\mathbf{Y}_r\mathbf{V}^T$.
3. *Calculate the parameter* $\alpha_i = \sqrt{\frac{\tilde{M}_2}{\tilde{M}_1}}$ *where* $\tilde{M}_2$ *and* $\tilde{M}_1$ *are defined in theorem 7.2.*
4. *Reduce the system using* α_i *with a given order* q.
5. *Increase* i *and go back to step 2.*

In order to analyze the convergence of the algorithms 7.2 and 7.4, consider the original system Σ that has been reduced to Σ_0 by matching the first q Laguerre coefficients associated with α_0. This system approximates some of the major dynamics of Σ from a Laguerre approximation point of view. In addition, based on the theory in, e.g. [44, 73] this reduced model can be already used to approximately solve the lyapunov equations (7.9), (7.11) that are involved in the calculation of the optimal parameter α^*.

In the next step, the parameter α_1 calculated from Σ_0 is employed to calculate the reduced system Σ_1. This is the first reduced system found based on the results of this section. Now, since α_2 is extracted from Σ_1, it is expected that the difference $\alpha_2 - \alpha^*$ tends to zero. Applying the proposed algorithms to several technical systems confirms this theoretical interpretation and shows that no significant changes occur in α_i, typically, after the *third* iteration. Also, it was remarked that as long as it results in a stable reduced model, the initial value α_0 can be arbitrarily chosen.

In fact, it is still an open problem to mathematically proof those observations and to quantize the convergence speed of each of the presented algorithms. In other words, it is required to prove that the information required from the impulse response (i.e. M_1 and

M_2) is preserved after an order reduction step using moment matching about α. The fact that the optimal parameter α in all the suggested algorithms is expressed as a Rayleigh quotient could offer a possibility to investigate those observations.

7.6 The discrete-time case

In this section, the optimization approach of this chapter is generalized to the case of discrete systems. Even though the general framework is very similar, the resulting optimal choice of the parameter α is different.

Consider the discrete system

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{b}_d u(k), \\ y(k) = \mathbf{c}_d^T\mathbf{x}(k). \end{cases} \tag{7.29}$$

If this system is stable, its impulse response can be expressed by using a discrete Laguerre expansion as:

$$h(n) = \sum_{i=0}^{\infty} f_i \phi_i^{\alpha}(n). \tag{7.30}$$

Similar to the continuous case and based on the same logic, the cost function

$$J_{dis}(\alpha) = \sum_{i=0}^{\infty} i f_i^2(\alpha), \tag{7.31}$$

is considered, and it is aimed at finding an optimal α which minimizes it.

Based on the work in [34], the cost function can be rewritten as,

$$J_{dis}(\alpha) = \frac{\alpha M_2 + (1-\alpha)^2 M_1 + \alpha - \alpha^2}{1-\alpha^2} \|h\|^2 \tag{7.32}$$

where

$$M_1 = \frac{1}{\|h\|^2}\sum_{n=0}^{\infty} nh^2(n), \tag{7.33}$$

$$M_2 = \frac{1}{\|h\|^2}\sum_{n=0}^{\infty} n(h(n+1)-h(n))^2, \tag{7.34}$$

$$\|h\|^2 = \sum_{n=0}^{\infty} h^2(n) = \sum_{n=0}^{\infty} f_n^2. \tag{7.35}$$

From (7.32), we have

$$\alpha^2(M_1 + J_{dis} - 1) + \alpha(M_2 - 2M_1 + 1) + (M_1 - J_{dis}) = 0 \tag{7.36}$$

Thus, for a real α, the following equation should be satisfied:

$$(M_2 - 2M_1 + 1)^2 - 4(M_1 + J_{dis} - 1)(M_1 - J) \geq 0,$$

leading to,

$$J_{dis_{min}} = \frac{1 + \sqrt{4M_1M_2 - M_2^2 - 2M_2}}{2} \tag{7.37}$$

$$\alpha^*_{dis} = \frac{2M_1 - 1 - M_2}{2M_1 - 1 + \sqrt{4M_1M_2 - M_2^2 - 2M_2}} \tag{7.38}$$

Based on the fact that $(4M_1 - M_2 - 2) = \sum_{n=0}^{\infty} n[h(n) + h(n+1)]^2$, which can not be negative, it can concluded that $(4M_1M_2 - M_2^2 - 2M_2) \geq 0$ is always true [34].

From the above equations it can be remarked that the problem formulation in the discrete-time domain results in a different optimal choice α^*_{dis}.

7.7 Calculation of M_1 and M_2 for LTI discrete systems

In this section, a closed form for the calculation of the optimal parameter α^*_{dis} and its main components M_1 and M_2 is derived.

Consider the discrete-time system (7.29) with its impulse response $h(n)$. Its two norm can be calculated using the Stein (discrete Lyapunov equation) equation,

$$\|h(n)\|^2 = \sum_{i=1}^{\infty} \mathbf{c}^T \mathbf{A}^{i-1} \mathbf{b}\mathbf{b}^T \mathbf{A}^{T(i-1)} \mathbf{c} = \mathbf{c}^T \left(\sum_{i=0}^{\infty} \mathbf{A}^i \mathbf{b}\mathbf{b}^T \mathbf{A}^{Ti} \right) \mathbf{c} = \mathbf{c}^T \mathbf{X} \mathbf{c}. \tag{7.39}$$

where $\mathbf{X}$ is the solution of the following equation,

$$\mathbf{A}\mathbf{X}\mathbf{A}^T - \mathbf{X} + \mathbf{b}\mathbf{b}^T = \mathbf{0}. \tag{7.40}$$

Following the above scheme,

$$\sum_{i=n}^{\infty} n h^2(n) = \mathbf{c}^T \left(\sum_{n=1}^{\infty} n \mathbf{A}^{n-1} \mathbf{b}\mathbf{b}^T \mathbf{A}^{T(n-1)} \right) \mathbf{c}.$$

Now, by defining $\mathbf{Y} = \sum_{i=1}^{\infty} n \mathbf{A}^{n-1} \mathbf{b}\mathbf{b}^T \mathbf{A}^{T(n-1)}$, it can be shown that

$$\mathbf{Y} = \sum_{i=0}^{\infty} (n+1) \mathbf{A}^n \mathbf{b}\mathbf{b}^T \mathbf{A}^{Tn} = \mathbf{A} \underbrace{\sum_{n=0}^{\infty} n \mathbf{A}^{n-1} \mathbf{b}\mathbf{b}^T \mathbf{A}^{T(n-1)}}_{\mathbf{Y}} \mathbf{A}^T + \underbrace{\sum_{n=0}^{\infty} \mathbf{A}^n \mathbf{b}\mathbf{b}^T \mathbf{A}^{Tn}}_{\mathbf{X}},$$

which can be summarized as,

$$\mathbf{A}\mathbf{Y}\mathbf{A}^T - \mathbf{Y} + \mathbf{X} = \mathbf{0}. \tag{7.41}$$

where $\mathbf{X}$ is the solution of the equation (7.40). Finally, this leads to the closed form expression of M_1,

$$M_1 = \frac{1}{\|h\|^2} \sum_{n=0}^{\infty} n h^2(n) = \frac{\mathbf{c}^T \mathbf{Y}^T \mathbf{c}}{\mathbf{c}^T \mathbf{X} \mathbf{c}}. \tag{7.42}$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^{\infty} n (h(n+1) - h(n))^2 &= \sum_{n=0}^{\infty} n \tilde{h}^2(n), \\ \tilde{h}(n) &= h(n+1) - h(n) = \mathbf{c}^T \mathbf{A}^n \mathbf{b} - \mathbf{c}^T \mathbf{A}^{n-1} \mathbf{b} \\ &= \mathbf{c}^T \mathbf{A}^{n-1} (\mathbf{A} - \mathbf{I}) \mathbf{b}. \end{aligned}$$

This is in fact the impulse response of the system,

$$\begin{cases} \mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n) + (\mathbf{A}-\mathbf{I})\mathbf{b}u(n), \\ y = \mathbf{c}^T\mathbf{x}(n). \end{cases}$$

Using the above system and similar to the calculation of M_1, it can be shown that

$$M_2 = \frac{1}{\|h\|^2}\sum_{n=0}^{\infty} n\tilde{h}^2(n) = \frac{\mathbf{c}^T\mathbf{W}^T\mathbf{c}}{\mathbf{c}^T\mathbf{X}\mathbf{c}}. \tag{7.43}$$

with

$$\mathbf{A}\mathbf{Z}\mathbf{A}^T - \mathbf{Z} + (\mathbf{A}-\mathbf{I})\mathbf{b}\mathbf{b}^T(\mathbf{A}^T-\mathbf{I}) = \mathbf{0}. \tag{7.44}$$

$$\mathbf{A}\mathbf{W}\mathbf{A}^T - \mathbf{W} + \mathbf{Z} = \mathbf{0}. \tag{7.45}$$

Accordingly, the algorithms presented for the continuous case can be modified and applied to the reduction of discrete systems. As an example, algorithm 7.1 modified and adapted to the discrete system case is presented.

Algorithm 7.5. *Rational Krylov with an Optimal Point (RK-OP)(Discrete case)*

1. *Solve the discrete Lyapunov equations* (7.40)*,* (7.41)*,* (7.44)*, and* (7.44) *to calculate* $\mathbf{X}$*,* $\mathbf{Y}$*,* $\mathbf{Z}$*, and* $\mathbf{W}$ *.*

2. *Calculate the parameters* M_1 *and* M_2 *using* (7.42) *and* (7.43)

3. *Calculate* α^*_{dis} *using* (7.38) *to minimize the function* (7.32)*.*

4. *Find the reduced system* (3.11) *using* α^*_{dis} *and with a given order* q *using* (3.13) *and* (3.14)*, i.e. classical moment matching about a single point* $s_0 = \alpha^*_{dis}$*.*

7.8 Illustrative examples

In order to illustrate the suitability and efficiency of each of the algorithms developed in this thesis, we apply them for the reduction of two different Benchmark models.

The results presented here are the one obtained using a one-sided Krylov method and will be compared with the results obtained from TBR and moment matching about $s_0 = 0$. This widely spread value of s_0 is generally used when no information about the original system is available as it often delivers very good result in a large neighborhood of the low-frequency part of the spectrum, including the steady state.

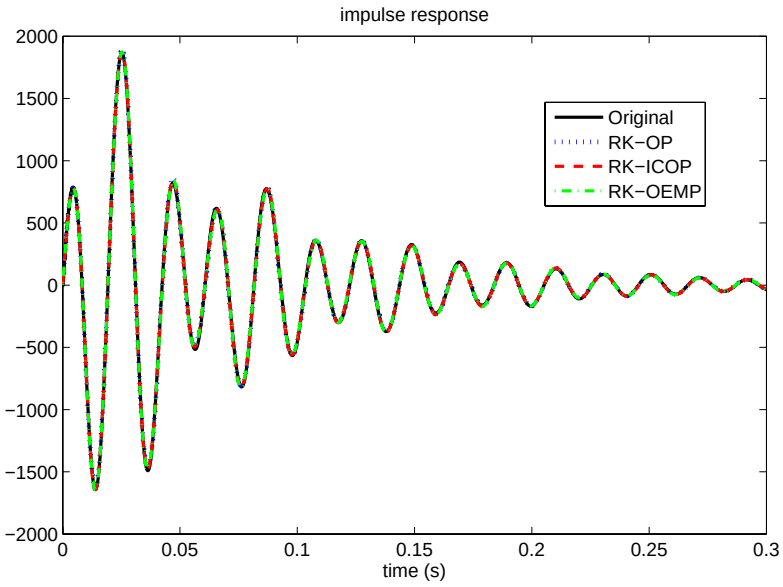

Figure 7.1: The impulse responses of the original and reduced systems.

7.8.1 The CD Player

The CD player model of order 120 which has been considered throughout this dissertation has been reduced to order 8 by applying the four new algorithms.

Minimizing the cost function (7.5) in RK-OP lead to $\alpha^* = 292.8794$. By running the algorithm 7.2, RK-ICOP, the parameter converged in three steps to $\alpha^* = 291.8036$ which has less than 0.4% error when compared to the solution using the full order model. The algorithm RK-OEMP lead to the optimized parameter $\alpha^* = 728.553$. For this system, the approximated version RK-NOEMP did not converge but oscillated around the value delivered by RK-OEMP. This in fact happened due to the approximation of the $\mathbf{Y}$ matrix

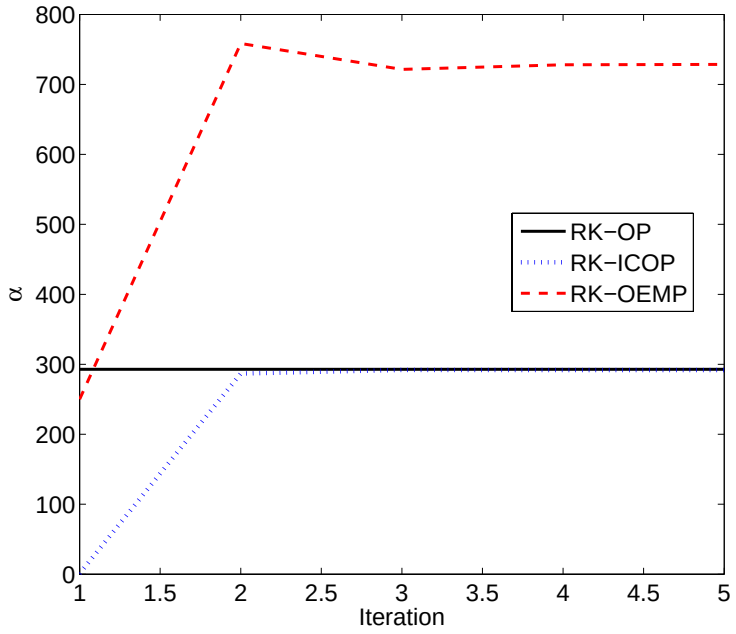

Figure 7.2: The expansion point for the CD player model.

by $\mathbf{V}\mathbf{Y}_r\mathbf{V}^T$ in both equations (7.27) and (7.28) when calculating $\tilde{M}_1$ and $\tilde{M}_2$. When the first term $\mathbf{c}^T\mathbf{Y}\mathbf{c}$ is well approximated by $\mathbf{c}^T\mathbf{V}\mathbf{Y}_r\mathbf{V}^T\mathbf{c}$, the relative approximation error is very small, however the value of $\tilde{M}_1$ will be significantly changed. This is due to the large absolute error when considering the sum of three terms of $\tilde{M}_1$ when no approximation is used, i. e. RK-OEMP. These facts are also valid for the calculation of $\tilde{M}_2$. Note that this sometimes results in a negative $\tilde{M}_1$ or $\tilde{M}_2$, leading to a breakdown of the algorithm. In addition, it has to be noted that this problem gets worse when the relative difference between f_i and f_{ri} gets small which explains the oscillations around the value obtained by the RK-OEMP algorithm. Figure 7.2 illustrates the parameter α in terms of iterations for all algorithms. A fast convergence to the desired value of parameter can be observed in the case of the RK-ICOP and RK-OEMP algorithms. The impulse response of the reduced systems is excellently approximated, as it can be seen in figure 7.1.

For this system, both methods, namely minimizing the two different cost functions (7.5)

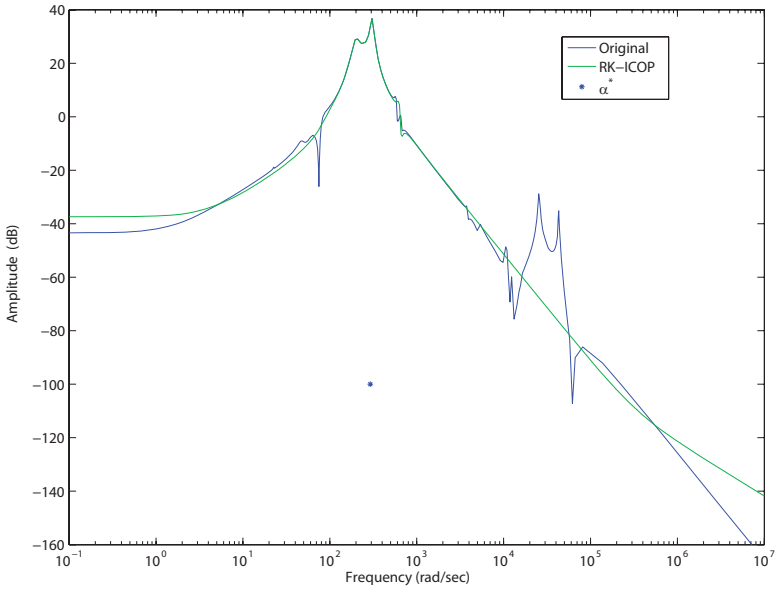

Figure 7.3: The amplitude of the Bode Diagram of the original CD player model and the reduced model by RK-ICOP.

and (7.22) lead to almost the same results in the time-domain even though they minimize different cost functions. When considering the approximation results in the frequency-domain, a very good matching of the peak region of the Bode diagram can be seen in figure 7.3. This in fact explains the excellent approximation of the impulse response. Moreover, it can be seen that the expansion point found by the algorithm RK-OP and its iterative version RK-ICOP tends to the resonant frequency of the Bode diagram, as shown in figure 7.3. Based on the fact that the Bode diagram of the original system is not or can not be calculated, this is a very important feature of the new algorithm. In other words, this algorithm automatically captures the peak region of the Bode diagram without the need of calculating or giving any hint about this diagram.

In Figure 7.4, the results of the suggested methods are compared with the results of the TBR method and the moment matching about $s_0 = 0$. It can be seen that the result of the new method are almost equal to those of the TBR and much better than those of

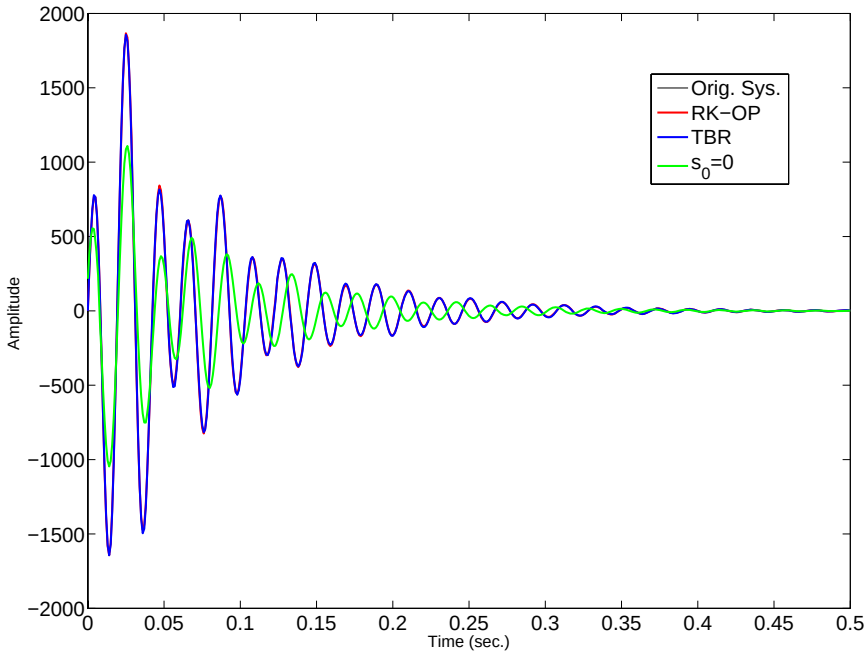

Figure 7.4: Comparison of the approximated impulse responses of the CD player.

moment matching about $s_0 = 0$. Note that the order of the considered reduced model is $q = 8$ for all the methods.

7.8.2 A random model

The random model from the collection of Benchmark examples [18] is considered. The system was randomly generated and has an order of 200. The real and complex conjugate poles of the system matrix $\mathbf{A}$ are all close to each others and no real pole dominance can be detected from the eigenvalues plot (see figure7.5). This results in an oscillating impulse response that takes around 10 minutes to approach zero. The system has been reduced to order 12. Minimizing the cost function (7.5) in RK-OP lead to $\alpha^* = 798.0345$. By running the algorithm 7.2, RK-ICOP, the parameter converged in two steps to $\alpha^* = 789.0346$ which has almost zero error when compared to the solution using the full order model.

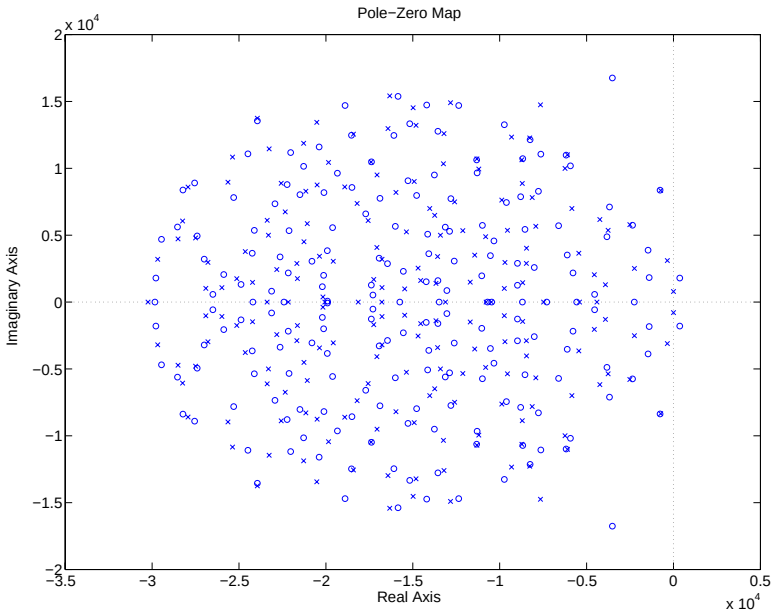

Figure 7.5: The pole-zero plot of the random model.

The two algorithms RK-OEMP and the approximated version RK-NOEMP lead, also after 2 iterations, to the optimized parameter $\alpha^* = 789.113$ and 798.245, respectively. The difference between the result using the full order model and the reduced one is less than 0.1%. The impulse response of the reduced systems is very well approximated until the settling time, as it can be seen in figure 7.6. Note that for this specific system both methods, i. e. RK-OP (RK-ICOP) and RK-OEMP (RK-NOEMP) delivered the same parameter $s_0 = \alpha^*$, however this is not the general rule.

Also here, the expansion point found by the algorithm RK-OP and its iterative version RK-ICOP tends to the resonant frequency of the Bode diagram, as shown in figure 7.8. This confirms one more time this special feature of the new suggested algorithm.

When compared with the TBR method, the result of the suggested methods showed to be almost the same. Note that no comparison with moment matching about $s_0 = 0$ was possible due to the instability of the obtained reduced model.

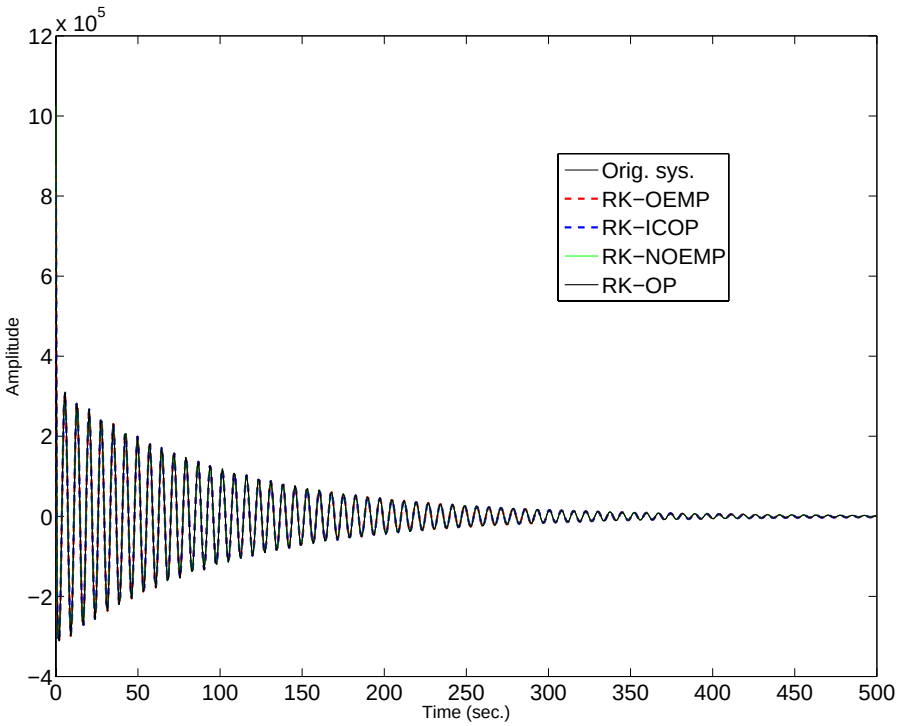

Figure 7.6: The impulse responses of the reduced systems of order 12.

Generally speaking, all examples investigated so far have confirmed that algorithm 7.2 converges quickly (typically, within max. 4 steps) towards the results of algorithm 7.1. The convergence of algorithm 7.4 can not be always guaranteed due to the numerical problems explained in the previous subsection. However, the obtained expansion point by both approaches results in an excellent approximation of the time-domain system responses.

In addition, numerous simulations have shown that the initial value of the parameter does not affect neither the convergence in general nor its speed, nor the accuracy of the result. However, all the reduced systems involved in the iterative process have to be stable for the proper functioning of the algorithms. In addition, it is needed that the user sets the reduced order to which the original model has to be reduced.

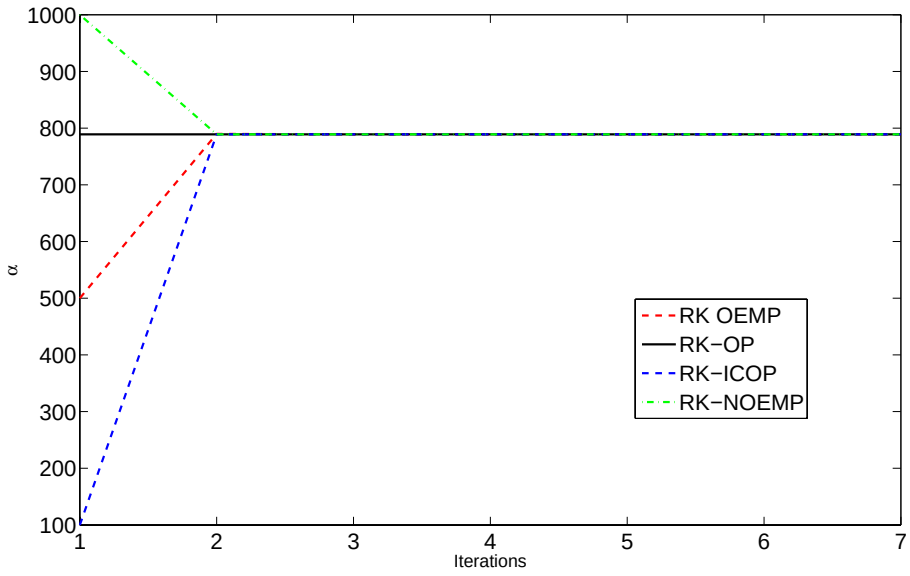

Figure 7.7: The expansion point for the random model.

Apart from that, it is observed that the approximation using the Galerkin conditions showed to be particularly useful and efficient when the order of the reduced system is high enough for a good approximation of the Lyapunov equations. Clearly, the choice of the reduced order is highly dependent on the complexity of the considered original system.

Finally, based on all the examples investigated, the superiority of the RK-OP algorithm and its iterative version RK-ICOP has been confirmed, even though the time and frequency-domain results of the four algorithms are very close to each other. This superiority is mainly due to the numerical stability of the iterative version and to the feature of capturing the peak of Bode diagram without the need of calculating this diagram.

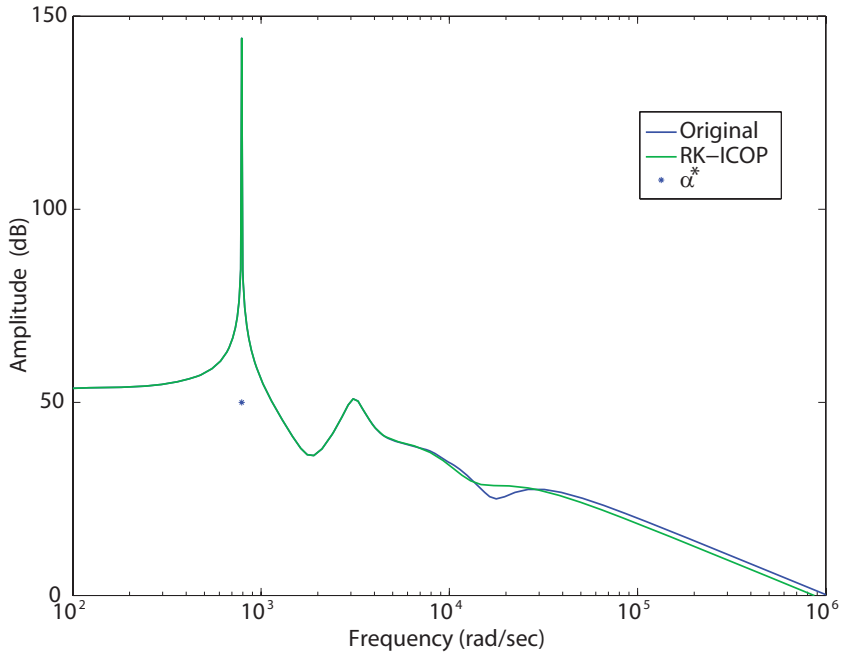

Figure 7.8: The amplitude of the Bode Diagram of the original random model and the reduced model by RK-ICOP.

Chapter 8

CONCLUSIONS AND FUTURE WORK

A new time-domain order reduction method based on the Laguerre function expansion of the impulse response has been presented. By showing that the subspace spanned by the Laguerre coefficients vectors is a Krylov subspace and thus can be computed very efficiently, the method can be applied for the reduction of large-scale systems. In addition, a time-domain interpretation of the classical moment matching approach has been developed. It is based on the fact that the reduced order model obtained by the new method and the one obtained by order reduction by rational interpolation around a single point, are exactly the same.

In fact, the equivalence between all the existing Laguerre-based order reduction method and moment matching around $s = \alpha$ was proved, both in time and frequency-domains. Moreover, it was shown that, by matching the moments, a family of coefficients called generalized Markov parameters (including the Laguerre coefficients as a special case) are also matched.

Then, two time-domain algorithms for the choice of a single expansion point in Krylov-based order reduction have been presented, both in the continuous and discrete-time cases. The algorithms are numerically cheap when approximating the solution of the required Lyapunov equations and deliver excellent approximation of the impulse response, once the order of the reduced system has been set by the user. Applying the proposed algorithms to several technical systems confirms their fast convergence and showed that negligible improvements in the accuracy of the reduced order model occur after the

fourth iteration step. In addition, the RK-OP and its iterative version RK-ICOP have the feature of calculating an expansion point that detects the peak of the Bode diagram without the need of calculating or giving any hint about this diagram.

The presented equivalence in this work establishes the missing link between the Laguerre-based order reduction and rational interpolation around a single point. Hence, a possibility is offered to exploit the interesting properties of the first method in order to solve many of the open problems of moment matching.

For instance, it would be now possible to further develop and adapt the numerous error bounds and indicators existing for the impulse response in the Laguerre representation to serve for moment matching. This will then allow to develop an algorithm for finding the minimum required reduced order to achieve a pre-defined approximation accuracy.

In addition it is of high interest to investigate the connection of the results of this dissertation to L_2 optimality or even to further develop them to achieve optimality in the sense of a well-known norm. Future works include also the generalization of the presented algorithms to the MIMO case and the investigation of the possibility to extend them to the case of rational interpolation with multiple points.

Apart from that, it would be interesting to further investigate the connection between the suggested approach and Balancing and Truncation as both methods share the use of the controllability gramian, and to make use of the automatic detection of the peak of the Bode diagram to develop, e. .g an aposteriori H_∞ error indicator.

Appendix A

NUMERICAL ALGORITHMS

In this appendix, the detailed implementation steps of the Arnoldi, Lanczos, and two-sided Arnoldi algorithms including deflation are presented as in [76].

A.1 Arnoldi Algorithm with deflation using modified Gram-Schmidt

1. Delete all linearly dependent starting vectors to find m_1 independent starting vectors for the given Krylov subspace then set

$$\mathbf{v}_1 = \frac{\mathbf{b}_1}{\|\mathbf{b}_1\|_2}$$

where $\mathbf{b}_1$ is the first starting vector after deleting the dependent starting vectors

2. For $j = 2, 3, \cdots$, do,

   (a) Calculation of the next vector: if $j \leq m_1$, the next vector is the j-th starting vector. Otherwise, the next vector is:

$$\mathbf{r}_j = \widetilde{\mathbf{A}}\mathbf{v}_{j-m_1}$$

   (b) Orthogonalization: Set $\hat{\mathbf{v}}_j = \mathbf{r}_j$ then for $i = 1$ to $j - 1$ do:

$$\begin{aligned} h_{i,j-1} &= \hat{\mathbf{v}}_j^T \mathbf{v}_i \\ \hat{\mathbf{v}}_j &= \hat{\mathbf{v}}_j - h_{i,j-1}\mathbf{v}_i \end{aligned}$$

(c) Normalization: if $\hat{\mathbf{v}}_j \neq 0$, the i-th column is

$$h_{i,j-1} = \|\hat{\mathbf{v}}_j\|_2 \ , \ \mathbf{v}_j = \frac{\hat{\mathbf{v}}_j}{h_{i,j-1}}$$

increase j and go to step (2a).

(d) Deflation: Reduce m_1 to $m_1 - 1$ and if m_1 is nonzero go to the next step and if m_1 is zero break the loop.

(e) Go to step (2a) without increasing j.

A.2 Lanczos Algorithm with deflation and full orthogonalization

1. Delete all linearly dependent starting vectors to find m_1 and p_1 independent starting vectors, $\breve{\mathbf{b}}_1, \cdots, \breve{\mathbf{b}}_{m_1}$ and $\hat{\mathbf{c}}_1, \cdots, \hat{\mathbf{c}}_{p_1}$ for the given input and output Krylov subspaces respectively then set

$$\mathbf{v}_1 = \frac{\breve{\mathbf{b}}_1}{\sqrt{|\hat{\mathbf{c}}_1^T \breve{\mathbf{b}}_1|}}, \quad \mathbf{w}_1 = \frac{\hat{\mathbf{c}}_1}{-\sqrt{|\hat{\mathbf{c}}_1^T \breve{\mathbf{b}}_1|}}$$

2. For $j = 2, 3, \cdots$, do,

   (a) Calculation of the next vector: for the input Krylov subspace, if $j \leq m_1$, then $\mathbf{r}_j = \breve{\mathbf{b}}_j$. Otherwise, the next vector is:

   $$\mathbf{r}_j = \breve{\mathbf{A}} \mathbf{v}_{j-m_1}$$

   for the output Krylov subspace, if $j \leq p_1$, then $\mathbf{r}_j = \hat{\mathbf{c}}_j$. Otherwise, the next vector is:

   $$\mathbf{l}_j = \hat{\mathbf{A}} \mathbf{w}_{j-p_1}$$

(b) Orthogonalization: Set $\hat{\mathbf{v}}_j = \mathbf{r}_j$ and $\hat{\mathbf{w}}_j = \mathbf{l}_j$ then for $i = 1$ to $j - 1$ do:

$$\hat{h}_{i,j-1} = \hat{\mathbf{w}}_j^T \mathbf{v}_i \quad , \quad \breve{h}_{i,j-1} = \hat{\mathbf{v}}_j^T \mathbf{w}_i$$
$$\hat{\mathbf{v}}_j = \hat{\mathbf{v}}_j - \breve{h}_{i,j-1} \mathbf{v}_i \quad , \quad \hat{\mathbf{w}}_j = \hat{\mathbf{w}}_j - \hat{h}_{i,j-1} \mathbf{w}_i$$

(c) Normalization: if $\hat{\mathbf{v}}_i^T \hat{\mathbf{w}}_i \neq 0$, then

$$\mathbf{v}_j = \frac{\hat{\mathbf{v}}_j}{\sqrt{|\hat{\mathbf{w}}_j^T \hat{\mathbf{v}}_j|}}, \quad \mathbf{w}_j = \frac{\hat{\mathbf{w}}_j}{-\sqrt{|\hat{\mathbf{w}}_j^T \hat{\mathbf{v}}_j|}}$$

increase j and go to step (2a).

(d) Deflation: If $\hat{\mathbf{v}}_j = 0$, reduce m_1 to $m_1 - 1$ and if m_1 and if m_1 is zero break the loop.

If $\hat{\mathbf{w}}_j = 0$, reduce p_1 to $p_1 - 1$ and if p_1 and is zero break the loop.

(e) Go to step (2a) without increasing j.

A.3 Two-sided Arnoldi algorithm

1. Choose the appropriate input and output Krylov subspaces for the original system, $\mathcal{K}_Q(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $\mathcal{K}_Q(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$.

2. Apply Arnoldi algorithm A.1 to the input Krylov subspace to find the matrix $\mathbf{V}$.

3. Apply Arnoldi algorithm A.1 to the output Krylov subspace to find the matrix $\mathbf{W}$.

4. Find the reduced order model as defined in (3.11).

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