Quantifying Diversity and Correlation in Rayleigh Fading MIMO Communication Systems

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Abstract— **The performance of MIMO systems and the usefulness of dedicated signal processing and channel coding techniques is strongly affected by presence of fading correlation. For instance, space-time block coding is favored by low fading correlation, while beam-forming techniques are advantageous at high amounts of correlation. In this paper we define an easy to compute quantitative description of fading correlation and diversity present in a MIMO channel. This definition offers the possibility to build equivalence classes of channels which offer the same amount of diversity or correlation. It turns out, that channels from such an equivalence class perform essentially equivalently with respect to channel capacity or throughput.**

I. Introduction

MIMO communication systems recently have drawn considerable attention in the area of wireless communications as they promise huge capacity increase [1] in a fading wireless environment. It is known, that fading correlation has significant impact on the performance of MIMO systems. Depending on how much the transmitter is aware of the fading correlation, the effect can be capacity decreasing [2], [3] or even capacity increasing [9].

Fading correlation is directly connected to the diversity gain of a MIMO system. It is also strongly related to MIMO antenna gain [10] and multiplexing gain. The close relation of fading correlation with these three elementary gains provided by a MIMO system, suggests to search for a quantitative description of fading correlation. Such an attempt has already been made in [8] for the SIMO case, which provides a generalized definition of receive diversity order. This definition computes the ratio of variance of SNR after maximum ratio combining of all received signals and the variance of SNR of a single received signal. In this paper we define a measure of diversity and correlation which

- does not assume maximum ratio combining
- is usable for the MIMO case
- is able to separate receive and transmit diversity
- is also applicable in some non-Gaussian fading cases.

Interestingly, for the case of receive diversity, this results in the same definition as the one given in [8] for the SIMO case. With our definition we can

- quantify the amount of correlation and diversity present in the channel. This allows for instance to decide which of two MIMO channels has stronger correlation or provides more diversity. In [6] and [7] the principle of majorization [4] of eigenvalue profiles is used to order MIMO channels by their amount of correlation. It is interesting to note, that the ordering obtained by the principle of majorization is always the same as the ordering obtained by the correlation measure. The reverse is however not true, since the principle of majorization sometimes does not yield a result.
- build equivalence classes of channels which offer the same amount of diversity or correlation. It turns out, that channels having the same correlation measure perform essentially equivalently with respect to channel capacity or throughput. This result may have impact on simulation aspects, as the simulation results obtained for one channel can essentially be used for the whole equivalence class.

II. System Model

In the following we will assume a frequency flat fading MIMO channel, with N transmit and M receive antennas, described by its channel matrix $\boldsymbol{H} \in \mathcal{C}^{M \times N}$ which is composed of complex, circularly symmetric random variables, which exhibit certain correlations. In the absence of receiver noise, the received signal $\boldsymbol{y} \in \mathcal{C}^{M}$ can be written as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{s}, \tag{1}$$

where $\boldsymbol{s} \in \mathcal{C}^{N}$ contains the data from N independent data streams. For the derivation of the diversity and correlation measures, we assume uncorrelated and unity power data signals, i.e. $\mathrm{E}\left[\boldsymbol{s}\boldsymbol{s}^{\mathrm{H}}\right] = \boldsymbol{I}_N$, where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix and $\mathrm{E}\left[.\right]$ is the expectation operation, while $(.)^{\mathrm{H}}$ represents complex conjugate transpose. Let us stack all columns of the channel matrix $\boldsymbol{H}$ into one $K = M \cdot N$ dimensional channel vector

$$\boldsymbol{h} = \mathbf{vec}[\boldsymbol{H}] \in \mathcal{C}^{K \times 1}, \tag{2}$$

where we have used $\mathbf{vec}[.]$ as the column stacking operation. We can describe the correlations between the random entries of $\boldsymbol{H}$ by the correlation matrix

$$\boldsymbol{R} = \mathrm{E}[\boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}]. \tag{3}$$

The channel vector $\boldsymbol{h}$ can then be written as

$$\boldsymbol{h} = \mathbf{vec}[\boldsymbol{H}] = \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{g}. \tag{4}$$

The vector $g \in \mathcal{C}^{K\times 1}$ is populated by i.i.d. complex random variables $g_k = \zeta_k + j\eta_k$, which meet for $k = 1, \ldots, K$ the following condition:

$$\begin{aligned} \zeta_k \text{ and } \eta_k: &\quad \text{i.i.d. real random variables} \\ \mathrm{E}[g_k] &= 0 \\ \mathrm{E}\left[|g_k|^2\right] &= 1 \\ \mathrm{E}\left[|g_k|^4\right] &= 2. \end{aligned} \tag{5}$$

This condition covers the important special case, where the (g_k) are complex, circularly symmetric, zero-mean, *Gaussian* random variables, which leads to correlated *Rayleigh* fading. Even though other distributions exist which fulfill (5), we will concentrate on Rayleigh fading in the following.

III. Diversity Measure

A. Motivation

Let us think about the sum γ of K i.i.d. random variables γ_k

$$\gamma = \sum_{k=1}^{K} \gamma_k, \quad \text{with} \quad \gamma_k = |\alpha_k|^2$$

and $\alpha_k \in \mathcal{N}_\mathcal{C}(0,1)$ being i.i.d. zero mean, unity variance Gaussian distributed random variables. This describes a K-th order diversity in uncorrelated and equal average power Rayleigh fading. The probability density function of γ becomes a Nakagami distribution of the order K. Let us compute the ratio of the squared expected value of γ and its variance:

$$\frac{(\mathrm{E}[\gamma])^2}{\mathrm{var}[\gamma]} = K,$$

where var[.] denotes the variance of a random variable. This tells us, how much the random variable γ varies relative to its average value. The larger this number is, the smaller is the relative variance in γ, i.e., the more diversity is available. Furthermore, the numerical value of this ratio is equal to the diversity order, i.e., K in this case. This motivates the following generic definition of a measure of diversity for MIMO systems.

B. Definition of Diversity Measure

Definition 1: The **Diversity Measure** $\Psi(\boldsymbol{R})$ of a Rayleigh fading MIMO system described by the channel matrix $\boldsymbol{H}$ with correlation matrix $\boldsymbol{R} = \mathrm{E}\left[\mathbf{vec}[\boldsymbol{H}]\mathbf{vec}[\boldsymbol{H}]^{\mathrm{H}}\right]$ is given by

$$\Psi(\boldsymbol{R}) = \left(\frac{\mathrm{tr}\,\boldsymbol{R}}{||\boldsymbol{R}||_{\mathrm{F}}}\right)^2. \tag{6}$$

Here the symbol tr is used for the trace operator, while $||.||_{\mathrm{F}}$ declares the Frobenius norm.

C. Properties of the Diversity Measure

The connection between the definition of diversity measure given in section III-B and the motivation provided in section III-A is made by the following

Theorem 1: The **Diversity Measure** $\Psi(\boldsymbol{R})$ of a correlated fading MIMO system from Definition 1 has the property

$$\Psi(\boldsymbol{R}) = \frac{(\mathrm{E}[\gamma])^2}{\mathrm{var}[\gamma]}, \quad \text{where} \tag{7}$$

$$\gamma := \mathrm{E}\left[||y||_2^2 \mid \boldsymbol{H}\right] = ||\boldsymbol{H}||_{\mathrm{F}}^2 \tag{8}$$

is the receive signal power for a given MIMO channel matrix $\boldsymbol{H}$ from (4) which meets condition (5). □ The proof can be found in [11]. The diversity measure has several further properties, including

- $1 \leq \Psi(\boldsymbol{R}) \leq K$, where $\boldsymbol{R} \in \mathcal{C}^{K\times K}$.
- If the first L eigenvalues of the correlation matrix $\boldsymbol{R}$ are positive and identical and the remaining eigenvalues vanish, i.e. $\lambda = \lambda_1 = \ldots = \lambda_L > \lambda_{L+1} = \ldots = 0$, the diversity measure becomes $\Psi = L$.

D. Independent Receive and Transmit Correlation

If the matrix $\boldsymbol{R}$ can be decomposed into the tensor product of two matrices $\boldsymbol{R}_{\mathrm{Tx}} \in \mathcal{C}^{N\times N}$ and $\boldsymbol{R}_{\mathrm{Rx}} \in \mathcal{C}^{M\times M}$

$$\boldsymbol{R} = \frac{1}{\mathrm{tr}\,\boldsymbol{R}_{\mathrm{Tx}}} \boldsymbol{R}_{\mathrm{Tx}}^{\mathrm{T}} \otimes \boldsymbol{R}_{\mathrm{Rx}}, \tag{9}$$

the channel matrix $\boldsymbol{H}$ from (4) becomes

$$\boldsymbol{H} = \frac{1}{\sqrt{\mathrm{tr}\,\boldsymbol{R}_{\mathrm{Tx}}}} \boldsymbol{R}_{\mathrm{Rx}}^{\frac{1}{2}} \boldsymbol{G} \boldsymbol{R}_{\mathrm{Tx}}^{\frac{1}{2}}, \tag{10}$$

where $\mathbf{vec}[G] = g$. Here $\boldsymbol{R}_{\mathrm{Rx}} = \mathrm{E}[\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}}]$ is the *receive* correlation matrix, while $\boldsymbol{R}_{\mathrm{Tx}} = \mathrm{E}[\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}]$ is the *transmit* correlation matrix. We can write the diversity measure as

$$\Psi(\boldsymbol{R}) = \Psi(\boldsymbol{R}_{\mathrm{Tx}}) \cdot \Psi(\boldsymbol{R}_{\mathrm{Rx}}), \tag{11}$$

which decomposes into the product of a *transmit* diversity measure $\Psi(\boldsymbol{R}_{\mathrm{Tx}})$ and a *receive* diversity measure $\Psi(\boldsymbol{R}_{\mathrm{Rx}})$. In case only the *transmit* correlation matrix is *different* from a scaled identity matrix, the channel defined in (10) is said to be *semi-correlated* [10].

IV. Correlation Measure

The measure for diversity in Rayleigh fading MIMO systems, that has been defined in Chapter III is based on correlation matrices, and therefore directly related to the correlation properties of a MIMO channel. In this section, based on the defined diversity measure, we provide a definition of correlation measure.

A. Definition of MIMO Correlation Measure

We state the generic definition of our measure of correlation.

Definition 2: The **Correlation Measure** $\Phi(\boldsymbol{R})$ of a $L \times L$ correlation matrix $\boldsymbol{R}$ is given by

$$\Phi(\boldsymbol{R}) = \sqrt{\frac{1 - L/\Psi(\boldsymbol{R})}{1 - L}}. \tag{12}$$

This generic definition of the correlation measure $\Phi(\boldsymbol{R})$ can be applied to measure receive, transmit or the total correlation depending on the choice of $\boldsymbol{R}$. The motivation stems from the following observation. Let us have a look at a specific correlation matrix $\boldsymbol{R} \in \mathcal{C}^{L \times L}$:

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho^* & 1 & \rho & \cdots & \rho \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho^* & \rho^* & \rho^* & \cdots & 1 \end{bmatrix}.$$

Since in this case, the correlation properties are essentially captured by the single parameter ρ, we want our proposed correlation measure to yield the same result. Since the diversity measure in this situation is

$$\Psi(\boldsymbol{R}) = \frac{L}{1 + (L-1) \cdot |\rho|^2},$$

by substituting into (12), we obtain $\Phi(\boldsymbol{R}) = |\rho|$, as desired.

B. Special cases

There are two special cases, we want to bring to the reader's attention. They are characterized by their eigenvalue profile.

1) **Rank one situation**: the correlation matrix has only one non-zero eigenvalue. This represents the strongest possible correlation. The correlation measure from (12) yields

$$\Phi = 1,$$

2) **Uncorrelated and equal power situation**: all eigenvalues of the $L \times L$ correlation matrix are identical and positive. The correlation measure this time yields

$$\Phi = 0.$$

Note, that we always have $0 \leq \Phi \leq 1$. Therefore, by virtue of Definition 2, we have a way to quantify the amount of correlation, from $\Phi = 0$ representing the uncorrelated case, up to $\Phi = 1$ for maximum correlation.

C. Basic properties

The correlation measure has some interesting properties.

- For a correlation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,K} \\ \rho_{1,2}^* & 1 & \rho_{2,3} & \cdots & \rho_{2,K} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \rho_{1,K}^* & \rho_{1,K-1}^* & \rho_{1,K-2}^* & \cdots & 1 \end{bmatrix}$$

the correlation measure computes to

$$\Phi = \sqrt{\frac{1}{K(K-1)} \sum_{n=1}^{K} \sum_{\substack{m=1 \\ m \neq n}}^{K} |\rho_{n,m}|^2}.$$

Hence, the correlation measure is the root mean square of the magnitudes of all correlation coefficients.

- The larger the correlation measure, the stronger the correlation and the less diversity is available.
- For large numbers of both receive and transmit antennas there is the asymptotic property $\lim_{N,M \to \infty} \Psi \cdot \Phi^2 = 1$.

V. Applications

There are several applications of the diversity and correlation measures defined in Chapters III and IV, respectively. They include the following:

- *Establishment of equivalence classes*:
  By grouping together different correlation matrices which have the same diversity or correlation measure, a so called equivalence class is obtained. It turns out that MIMO channels from one such equivalence class offer similar performance in terms of ergodic capacity and throughput. One element out of the equivalence class can then be used as a representative for the whole class. This has impact for physical layer simulation of mobile communication systems, as only a small number of representative channel types have to be simulated.
- *Build an order relation of MIMO channels*:
  The diversity and correlation measures define an order relation according to which correlation matrices can be sorted by their amount of correlation.
- *Statistical analysis of correlation matrices*:
  Sometimes the correlation matrices are modeled as random variables themselves. The diversity and correlation measures can be used to analyze the statistical properties of diversity and correlation associated with these random correlation matrices. This is especially useful for analysis of data obtained through field measurement.
- *Classification of channel types*:
  The amount of diversity or correlation can be used to classify a MIMO channel. For instance one could define three classes, which collect channels of low, medium and high correlation, respectively. Since different transmit and receive signal processing may be used for different amount of correlation, the classification may decide upon the proper signal processing or its parameters. This may include selection of number of transmitted data streams, associated modulation schemes and distribution of transmit power, as well as selection between transmit processing algorithms which are built on diversity (like space-time block coding) or beam-forming oriented schemes, which profit from higher correlation. Since the diversity and the correlation measures can be computed with both

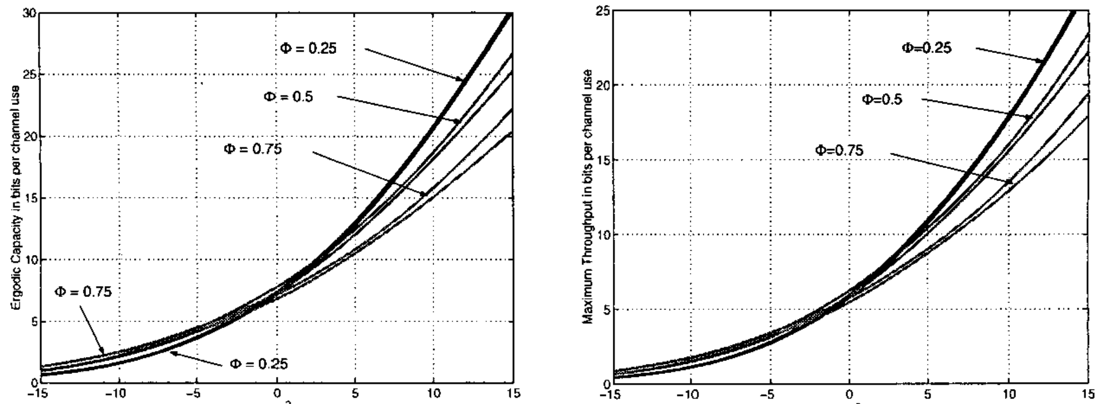

Fig. 1. Left: Ergodic channel capacity vs. transmit power for transmit correlation matrices from the sets $\mathcal{S}_{0.25}$, $\mathcal{S}_{0.5}$ and $\mathcal{S}_{0.75}$. Right: Same for throughput. For each of the sets two lines are shown, which represent the range of ± 1 standard deviation around the average. MIMO-Eigenbeamforming is applied at the transmitter. The number of receive and transmit antennas equals 8 each.

very low and constant complexity, a decision based on this criterion is attractive for real-time applications.

For sake of brevity we will discuss only the first application in a little bit more detail. The correlation (or diversity) measure allows the definition of equivalence classes $\mathcal{S}_\Phi$ of correlation matrices, which all have the same correlation measure Φ, that is

$$\mathcal{S}_\Phi = \{\boldsymbol{R} \mid \Phi(\boldsymbol{R}) = \Phi\}. \quad (13)$$

These equivalence classes have one important feature: the performance of MIMO systems with respect to channel capacity and throughput is essentially equivalent for all correlation matrices out of the equivalence class. In order to demonstrate this property in a random channel we provide a sample numerical result in Figure 1. The ergodic capacity and the throughput are shown as a function of transmit power for correlation matrices from the equivalence classes. We have a semi-correlated channel from (10), where the transmitter is aware of the transmit correlation matrix, which enables the application of Eigenbeamforming [9]. In order to obtain the results shown in Figure 1, we select randomly 250 transmit correlation matrices from each of the equivalence classes $\mathcal{S}_{0.25}$, $\mathcal{S}_{0.5}$ and $\mathcal{S}_{0.75}$ and compute throughput and ergodic capacity as functions of transmit power. Further, we compute the mean and standard deviation of the ergodic capacity and throughput over the 250 matrices for each equivalence class. As can be seen from Figure 1 transmit correlation matrices out of the same equivalence class lead to highly similar performance with respect to both ergodic capacity and throughput.

VI. Conclusion

In this paper a measure for diversity and correlation present in a wireless MIMO communication system is defined. With this definition it is not only possible to quantify the amount of correlation and diversity present in the channel, but also to classify channel types by their amount of diversity or correlation. Thereupon equivalence classes of channels can be defined, which offer the same amount of diversity or correlation. It is demonstrated, that channels with the same correlation measure, i.e. from the same equivalence class perform essentially equivalent with respect to both ergodic capacity and throughput. The correlation measure may also be used for other applications, like ranking the suitability of different signal processing and coding techniques, for instance space-time block-coding or beam-forming in correlated fading.

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