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***Abstract*— This paper addresses the joint optimization of linear transmitters and receivers in a multi-user *multiple-input multiple-output* (MIMO) *broadcast channel* (BC) system under the assumption of perfect *channel state information* (CSI) at both transmitter and receivers. Our first design criterion is the sum *mean square error* (MSE), which we seek to minimize. In a second approach, we need to satisfy *Quality of Service* (QoS) requirements by means of as few transmit power as possible or try to fulfill these requests as good as possible given a fixed sum transmit power. Since the downlink problem is difficult to handle, we formulate an equivalent uplink problem by exploiting the duality between both scenarios. We present efficient iterative solutions, which deliver the optimum transmit and receive matrices. The performance of our algorithms is studied theoretically and via simulations, we compare the design criteria with existing precoding algorithms.**

I. Introduction

In a multi-user downlink scenario, each of the decentralized receivers may be equipped with a single antenna (multi-user MISO system) or multiple antennas (multi-user MIMO system). The decentralization implies the necessity of preequalization at the transmitter to combat the inter-user interference. Linear pre-processing was considered in [1] based on the *minimum* MSE and *zeroforcing* (ZF) criterion, where the receivers were restricted to apply *the same* scalar weight. Jointly optimizing the linear transmit and receive matrices based on *Quality of Service* (QoS) constraints or sum-MSE minimization was considered in [2], [3], [4] for the MISO case and was extended to the MIMO case in [5], [6], [7], [8]. However, as most of these algorithms are based on the *alternating optimization* of receivers and transmitter, they suffer from a very slow convergence speed at high SNR values if not initialized in a clever fashion, or may even not converge to the optimum.

Our main contributions are on the one hand efficient iterative solutions, that jointly optimize the linear precoder at the centralized transmitter and the linear decoders at the decentralized receivers for the *multiple-input multiple-output* (MIMO) *broadcast channel* (BC) based on different design criteria. On the other hand, a novel simplified uplink-downlink MSE duality is presented facilitating the conversion of an uplink receiver/transmitters pair into a downlink receivers/transmitter pair.

As the downlink optimization is in general very complicated and non-convex, we make use of the duality between uplink and downlink and solve the equivalent uplink problem instead, which features the nice property to have a better mathematical structure with less coupling of the variables. In [9], [10], and [3], the duality in a linear system was proven in terms of information capacity, SINR, and *mean square error* (MSE), respectively. Contrary to [7], where the MSE duality is shown making use of the SINR duality in [10], we directly deduce the MSE duality for the MIMO case without detouring. Our MSE duality is defined user-wise making the uplink-downlink transformation very simple. This transformation is necessary, as the equivalent uplink model is solved instead of the downlink problem. Having found the optimum matrices in the equivalent *multiple access channel* (MAC), they have to be transformed into solutions of the BC channel.

We consider two design criteria: 1) Minimization of the sum-MSE (overall system efficiency) and 2) Assuring given *Quality of Service* user request (QoS based design). For the first problem, iterative solutions already exist, cf. [7]. However, we present a new algorithm with drastically reduced complexity. Based on a user-wise formulation, we also solve the QoS problem and deliver an algorithmic solution.

Our paper is organized as follows. In Section III, we prove the duality between uplink and downlink in terms of MSE for a multi-user MIMO system, then we derive the optimal uplink receiver in Section IV. In Section V, we deal with the iterative solution of the weighted sum-MSE minimization problem using a gradient projection approach [11], and provide a special solution for the MISO sum-MSE minimization in Section VI. Next, the QoS problem is discussed in Section VII. Finally, simulation results are presented in Section VIII.

II. System Model And Notation

We consider a K-users MIMO BC with an M antennas transmitter and the *k*th receiver has N_k antennas. Fig. 1 shows a general linear MIMO BC system, where $\boldsymbol{H}_k^{\mathrm{H}} \in \mathbb{C}^{N_k \times M}$, $k = 1, \ldots, K$, is the channel matrix of user k. The vector $\boldsymbol{s}_k \in \mathbb{C}^{B_k}$ comprises the B_k uncorrelated unit variance symbols of user k which are assumed to be uncorrelated with other users' symbols. The vectors $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_K$ refer to zero-mean white noise with variance σ_n^2 for each component. The centralized linear precoders are denoted by $\boldsymbol{P}_k^{\mathrm{DL}} \in \mathbb{C}^{M \times B_k}$, and $\boldsymbol{G}_k^{\mathrm{DL}} \in \mathbb{C}^{B_k \times N_k}$ are the decentralized receive filters. With these definitions, the estimate $\hat{\boldsymbol{s}}_k$ for the symbol $\boldsymbol{s}_k$ of user k reads as

$$\hat{\boldsymbol{s}}_k = \boldsymbol{G}_k^{\mathrm{DL}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{P}_k^{\mathrm{DL}} \boldsymbol{s}_k + \sum_{j \neq k} \boldsymbol{G}_k^{\mathrm{DL}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{P}_j^{\mathrm{DL}} \boldsymbol{s}_j + \boldsymbol{G}_k^{\mathrm{DL}} \boldsymbol{\eta}_k. \quad (1)$$

With uncorrelated unit variance entries of the signal $\boldsymbol{s}_k$, we compute the downlink MSE $\varepsilon_k^{\mathrm{DL}} = \mathrm{E}[\|\hat{\boldsymbol{s}}_k - \boldsymbol{s}_k\|_2^2]$ of user k

$$\varepsilon_k^{\mathrm{DL}} = \mathrm{tr}\Big(\mathbf{I}_{B_k} - \boldsymbol{G}_k^{\mathrm{DL}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{P}_k^{\mathrm{DL}} - \boldsymbol{P}_k^{\mathrm{DL,H}} \boldsymbol{H}_k \boldsymbol{G}_k^{\mathrm{DL,H}} + \sum_{j=1}^{K} \boldsymbol{G}_k^{\mathrm{DL}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{P}_j^{\mathrm{DL}} \boldsymbol{P}_j^{\mathrm{DL,H}} \boldsymbol{H}_k \boldsymbol{G}_k^{\mathrm{DL,H}} + \sigma_n^2 \boldsymbol{G}_k^{\mathrm{DL}} \boldsymbol{G}_k^{\mathrm{DL,H}}\Big). \quad (2)$$

From (2), we see that the MSE $\varepsilon_k^{\mathrm{DL}}$ of user k depends on all precoding matrices $\boldsymbol{P}_j^{\mathrm{DL}}$ which are strongly coupled by the sum power constraint. Thus, we formulate an equivalent uplink problem with a better mathematical structure, as the precoders are decoupled and only a joint power constraint has to be considered.

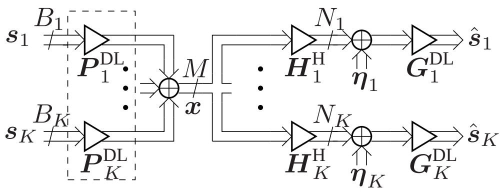

Fig. 1. Linear downlink model.

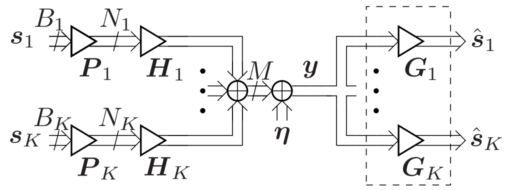

Fig. 2. Equivalent linear uplink model.

III. Linear Downlink/Uplink Duality

The dual uplink channel is shown in Fig. 2 and consists of a K-users MIMO uplink channel (MIMO MAC), where each of the dual uplink channels is the conjugate transpose of the corresponding BC channel. $\boldsymbol{P}_k$ denotes the transmit matrix and $\boldsymbol{G}_k$ the centralized receive matrix of user k.[1] The estimate $\hat{\boldsymbol{s}}_k$ in the uplink can be expressed as

$$\hat{\boldsymbol{s}}_k = \boldsymbol{G}_k\boldsymbol{H}_k\boldsymbol{P}_k\boldsymbol{s}_k + \sum\nolimits_{j\neq k}\boldsymbol{G}_k\boldsymbol{H}_j\boldsymbol{P}_j\boldsymbol{s}_j + \boldsymbol{G}_k\boldsymbol{\eta}, \tag{3}$$

and the uplink MSE $\varepsilon_k^{\mathrm{UL}} = \mathrm{E}[\|\hat{\boldsymbol{s}}_k - \boldsymbol{s}_k\|_2^2]$ of user k can easily be found to be

$$\begin{aligned}\varepsilon_k^{\mathrm{UL}} = \operatorname{tr}\Big(&\mathbf{I}_{B_k} - \boldsymbol{G}_k\boldsymbol{H}_k\boldsymbol{P}_k - \boldsymbol{P}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{G}_k^{\mathrm{H}} \\ &+ \sum\nolimits_{j=1}^{K}\boldsymbol{G}_k\boldsymbol{H}_j\boldsymbol{P}_j\boldsymbol{P}_j^{\mathrm{H}}\boldsymbol{H}_j^{\mathrm{H}}\boldsymbol{G}_k^{\mathrm{H}} + \sigma_n^2\boldsymbol{G}_k\boldsymbol{G}_k^{\mathrm{H}}\Big).\end{aligned} \tag{4}$$

In what follows, we establish the uplink/downlink duality for linear multi-user MIMO systems.

Theorem 1: Given the dimensions $B_1,\ldots,B_K$ of the symbol vectors $\boldsymbol{s}_k$, the MIMO BC channel and the dual MIMO MAC channel achieve the same MSE region, by using all possible sets of linear precoders and receivers under a fixed sum power constraint.

Proof: To prove the MSE duality between uplink and downlink, we show that for any set of precoders $\boldsymbol{P}_k$ and receivers $\boldsymbol{G}_k$ describing the uplink system and achieving certain user-wise MSE values, there exists at least one set of linear precoders and receivers for the dual BC channel that achieves the same MSE values under the same sum power and vice versa (see the following two subsections). ■

A. Uplink to Downlink Transformation

Given an uplink system, an equivalent downlink model is obtained by switching the roles of transmitters and receivers and scaling them with K strictly positive constants α_k and α_k^{-1}, respectively, i.e.,

$$\boldsymbol{P}_k^{\mathrm{DL}} = \alpha_k\boldsymbol{G}_k^{\mathrm{H}} \text{ and } \boldsymbol{G}_k^{\mathrm{DL}} = \alpha_k^{-1}\boldsymbol{P}_k^{\mathrm{H}}, \quad \forall k. \tag{5}$$

Setting $\varepsilon_k^{\mathrm{UL}} = \varepsilon_k^{\mathrm{DL}}$ yields a linear system of equations for α_k^2:

$$\boldsymbol{T}_{\mathrm{L}}\begin{bmatrix}\alpha_1^2\\ \vdots\\ \alpha_K^2\end{bmatrix} = \sigma_n^2\begin{bmatrix}\operatorname{tr}\left(\boldsymbol{P}_1\boldsymbol{P}_1^{\mathrm{H}}\right)\\ \vdots\\ \operatorname{tr}\left(\boldsymbol{P}_K\boldsymbol{P}_K^{\mathrm{H}}\right)\end{bmatrix}, \tag{6}$$

where

$$\boldsymbol{T}_{\mathrm{L},k,j} = \begin{cases}\sum\limits_{i\neq k}\operatorname{tr}\big(\boldsymbol{G}_k\boldsymbol{H}_i\boldsymbol{P}_i\boldsymbol{P}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}}\boldsymbol{G}_k^{\mathrm{H}}\big) + \sigma_n^2\operatorname{tr}\left(\boldsymbol{G}_k\boldsymbol{G}_k^{\mathrm{H}}\right) & k=j,\\ -\operatorname{tr}\left(\boldsymbol{G}_j\boldsymbol{H}_k\boldsymbol{P}_k\boldsymbol{P}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{G}_j^{\mathrm{H}}\right) & k\neq j.\end{cases}$$

Wee see that $\boldsymbol{T}_{\mathrm{L}}$ is a strictly (column) diagonally dominant real-valued matrix[2], so it is non-singular ($|\boldsymbol{T}_{\mathrm{L}}| > 0$); moreover, it has strictly positive diagonal entries and negative off-diagonal entries, thus all entries of the inverse matrix $\boldsymbol{T}_{\mathrm{L}}^{-1}$ are non-negative (the diagonal entries are strictly positive).[3] Summing up the rows in (6), we get:

$$\sum\nolimits_{k=1}^{K}\operatorname{tr}\left(\alpha_k^2\boldsymbol{G}_k\boldsymbol{G}_k^{\mathrm{H}}\right) = \sum\nolimits_{k=1}^{K}\operatorname{tr}\left(\boldsymbol{P}_k\boldsymbol{P}_k^{\mathrm{H}}\right). \tag{7}$$

We see that there is always a strictly positive[4] solution vector $\left[\alpha_1^2,\ldots,\alpha_K^2\right]^{\mathrm{T}}$, such that the uplink system can be transformed into an equivalent downlink system with the same individual MSEs using the same sum power (due to Eqn. 7).

B. Downlink to Uplink Transformation

Conversely, it can be shown by the same reasoning that every downlink system can be transformed into an equivalent uplink system with the same individual MSEs and with the same sum-power P_{tr}. The uplink transceivers are

$$\boldsymbol{P}_k = \bar{\alpha}_k\boldsymbol{G}_k^{\mathrm{DL,H}}, \quad \boldsymbol{G}_k = \frac{1}{\bar{\alpha}_k}\boldsymbol{P}_k^{\mathrm{DL,H}}. \tag{8}$$

$\bar{\alpha}_1^2$ to $\bar{\alpha}_K^2$ satisfy the following system of equations:

$$\bar{\boldsymbol{T}}_{\mathrm{L}}\begin{bmatrix}\bar{\alpha}_1^2\\ \vdots\\ \bar{\alpha}_K^2\end{bmatrix} = \sigma_n^2\begin{bmatrix}\operatorname{tr}\left(\boldsymbol{P}_1^{\mathrm{DL}}\boldsymbol{P}_1^{\mathrm{DL,H}}\right)\\ \vdots\\ \operatorname{tr}\left(\boldsymbol{P}_K^{\mathrm{DL}}\boldsymbol{P}_K^{\mathrm{DL,H}}\right)\end{bmatrix}, \tag{9}$$

where the strictly (column) diagonally dominant real-valued matrix $\bar{\boldsymbol{T}}_{\mathrm{L}}$ has a similar structure as $\boldsymbol{T}_{\mathrm{L}}$ in (6). Thus, a strictly positive solution to (9) exists and the downlink can be transformed into an uplink with the same sum power and the same individual user-wise MSEs.

IV. Optimal Receiver

To design the system, we first derive the optimum receive matrices, assuming all transmitters $\boldsymbol{P}_k$ to be fixed. Then, we deal with the difficult part, i.e., the derivation of the optimum transmit matrices.

For given precoders $\boldsymbol{P}_k$ and a cost function which is increasing in every MSE ε_k, the optimum linear receiver $\boldsymbol{G}_k$ detecting the symbol vector $\boldsymbol{s}_k$ of user k turns out to be the MMSE receiver minimizing the MSE ε_k individually:

$$\boldsymbol{G}_k = \boldsymbol{P}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{T}^{-1} = \boldsymbol{P}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}(\boldsymbol{H}_k\boldsymbol{P}_k\boldsymbol{P}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}} + \boldsymbol{R}_k)^{-1}, \tag{10}$$

with the substitutions

$$\begin{aligned}\boldsymbol{T} &= \sum\nolimits_i\boldsymbol{H}_i\boldsymbol{P}_i\boldsymbol{P}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}} + \sigma_n^2\mathbf{I},\\ \boldsymbol{R}_k &= \sum\nolimits_{i\neq k}\boldsymbol{H}_i\boldsymbol{P}_i\boldsymbol{P}_i^{\mathrm{H}}\boldsymbol{H}_i^{\mathrm{H}} + \sigma_n^2\mathbf{I}.\end{aligned} \tag{11}$$

[1] The uplink suffix 'UL' is omitted here to simplify the notation.

[2] $\boldsymbol{T}_{\mathrm{L},k,k} > \sum_{j\neq k}|\boldsymbol{T}_{\mathrm{L},j,k}| \quad \forall k.$

[3] The proof follows from considering the explicit formula of the adjoint matrix.

[4] We assume that all $\boldsymbol{P}_k \neq \mathbf{0}$, since we have to consider only the active users. For the other users, the duality is evident. Therefore, $\alpha_k > 0 \quad \forall k$.

The MSE of user k then reduces to

$$\varepsilon_k = \mathrm{tr}(\boldsymbol{E}_k) = \mathrm{tr}(\mathbf{I} - \boldsymbol{P}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{T}^{-1} \boldsymbol{H}_k \boldsymbol{P}_k). \tag{12}$$

Here, $\boldsymbol{E}_k$ denotes the error covariance matrix of user k.
Since the cost functions considered in the following are increasing in every MSE ε_k, the MMSE receivers are always optimum, and we can plug (10) into our optimizations to end up with problems depending only on the precoders $\boldsymbol{P}_k$.

V. Weighted Sum-MSE Optimization

Our goal is to design the precoder matrices for the downlink that minimize a *weighted* sum of the MSEs, namely $\sum_k w_k \varepsilon_k$ with positive scalars w_k under a sum power constraint. The introduction of weighting scalars enables us to consider different design criteria such as an MMSE design, QoS based design, or equal user MSEs (fairness). Using the duality, this is equivalent to solving the same optimization problem in the dual uplink:

$$\min_{\{\boldsymbol{P}_1,\ldots,\boldsymbol{P}_K\}} \sum_k w_k \varepsilon_k \quad \text{s.t.:} \sum_k \mathrm{tr}(\boldsymbol{P}_k \boldsymbol{P}_k^{\mathrm{H}}) \leq P_{\mathrm{tr}}. \tag{13}$$

Using (12), the KKT conditions of the weighted sum-MSE minimization read as:

$$\mu \boldsymbol{P}_k \stackrel{!}{=} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{T}^{-1} (w_k \boldsymbol{T} - \boldsymbol{S}) \boldsymbol{T}^{-1} \boldsymbol{H}_k \boldsymbol{P}_k, \tag{14}$$

where

$$\boldsymbol{S} = \sum\nolimits_i w_i \boldsymbol{H}_i \boldsymbol{P}_i \boldsymbol{P}_i^{\mathrm{H}} \boldsymbol{H}_i^{\mathrm{H}}, \tag{15}$$

and the Lagrangian multiplier $\mu \in \mathbb{R}$ with $\mu \geq 0$. If we multiply (14) for each k with $\boldsymbol{P}_k^{\mathrm{H}}$ from the left and take its trace, we observe that the weighting w_k satisfies nearly the same linear system of equations as the scalars α_k of the uplink/downlink transformation (cf. Eq. 5) except for a constant μ/σ_n^2 (cf. Eq. 6).[5] We get the following relation:

$$\frac{\alpha_1^2}{w_1} \stackrel{!}{=} \frac{\alpha_2^2}{w_2} \stackrel{!}{=} \cdots \stackrel{!}{=} \frac{\alpha_K^2}{w_K}. \tag{16}$$

Thus, all α_k are equal for the special case of sum-MSE minimization ($w_k = \text{const}\ \forall k$). Hence, they can be computed directly with minimum complexity from the transmit power constraint, which is the big advantage of our duality compared to the one given in [3] and [7].

In order to solve constrained optimization problems, the standard unconstrained gradient algorithm can be modified to take into account the constraints. The modified gradient algorithm is called the *projected* gradient algorithm and its iteration is defined as follows [11]:

$$\boldsymbol{P}^{(\ell+1)} = [\boldsymbol{P}^{(\ell)} - \eta \boldsymbol{M}^{-1} \nabla^* f(\boldsymbol{P}^{(\ell)})]_{\perp}, \tag{17}$$

where ∇ corresponds to the matrix valued nabla operator (*Jacobian* matrix), $[.]_{\perp}$ denotes the projection operator onto the hypersphere with radius $\sqrt{P_{\mathrm{tr}}}$, η is the step size, and $\boldsymbol{M}$ represents a preconditioning matrix, which is chosen to be $\boldsymbol{M}^{-1} = \sqrt{\frac{P_{\mathrm{tr}}}{\|\nabla f\|_{\mathrm{F}}^2}}\mathbf{I}$. In this way, the speed of the algorithm becomes almost independent from the SNR.[6]

Algorithm 1 shows the pseudo-code of the iterative gradient projection solution. The iteration is divided into two parts: In part one (steps 4 to 6), the standard gradient is computed, whereas in part two (step 7), the projection onto the constraint set is done. Steps 12 to 16 perform the uplink/downlink conversion. The convergence of this algorithm is proved by means of a descent argument [11]:

Algorithm 1 MIMO Weighted Sum-MSE Algorithm

1: **Initialize:** $\boldsymbol{P}_k^{(0)}(1:B_k, 1:B_k) \leftarrow \sqrt{\frac{P_{\mathrm{tr}}}{\sum B_k}}\mathbf{I}_{B_k} \forall k$
   $d \leftarrow 2$, $l \leftarrow 0$
2: **repeat**
3: $\quad \ell \leftarrow \ell + 1$
4: $\quad \boldsymbol{T}^{(\ell)} \leftarrow \sigma_n^2 \mathbf{I} + \sum_k \boldsymbol{H}_k \boldsymbol{P}_k^{(\ell-1)} \boldsymbol{P}_k^{(\ell-1),\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}}$
5: $\quad \boldsymbol{S}^{(\ell)} \leftarrow \sum_k w_k \boldsymbol{H}_k \boldsymbol{P}_k^{(\ell-1)} \boldsymbol{P}_k^{(\ell-1),\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}}$
6: $\quad$ Gradient update for $\boldsymbol{P}_k^{(\ell-1)}$
   $\quad \forall k : \delta \boldsymbol{P}_k^{(\ell)} \leftarrow \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{T}^{(\ell)-1} (w_k \boldsymbol{T}^{(\ell)} - \boldsymbol{S}^{(\ell)}) \boldsymbol{T}^{(\ell)-1} \boldsymbol{H}_k \boldsymbol{P}_k^{(\ell-1)}$
   $\quad \forall k : \delta \boldsymbol{P}_k^{(\ell)} \leftarrow \sqrt{\frac{P_{\mathrm{tr}}}{\sum_i \left\|\delta \boldsymbol{P}_i^{(\ell)}\right\|_{\mathrm{F}}^2}} \delta \boldsymbol{P}_k^{(\ell)}$ (scaled gradient)
   $\quad \forall k : \boldsymbol{P}_k^{(\ell)} \leftarrow \frac{1}{d} \delta \boldsymbol{P}_k^{(\ell)} + \boldsymbol{P}_k^{(\ell-1)}$
7: $\quad \forall k : \boldsymbol{P}_k^{(\ell)} \leftarrow \sqrt{\frac{P_{\mathrm{tr}}}{\sum_i \left\|\boldsymbol{P}_i^{(\ell)}\right\|_{\mathrm{F}}^2}} \boldsymbol{P}_k^{(\ell)}$ (projection)
8: $\quad$ **if** $\sum_k w_k \mathrm{tr}(\boldsymbol{E}_k^{(\ell)}) > \sum_k w_k \mathrm{tr}(\boldsymbol{E}_k^{(\ell-1)})$ **then**
9: $\quad\quad d \leftarrow d + 1, \quad \ell \leftarrow \ell - 1$
10: $\quad$ **end if**
11: **until** desired accuracy for $\boldsymbol{E}_k$ is achieved
12: Uplink/downlink conversion:
   $\alpha_0 \leftarrow \sqrt{\frac{P_{\mathrm{tr}}}{\sum_i \mathrm{tr}(w_i \boldsymbol{P}_i^{\mathrm{H}} \boldsymbol{H}_i^{\mathrm{H}} \boldsymbol{T}^{-2} \boldsymbol{H}_i \boldsymbol{P}_i)}}$
13: **for** $k = 1, \ldots, K$ **do**
14: $\quad \alpha_k \leftarrow \alpha_0 \sqrt{w_k}$
15: $\quad \boldsymbol{G}_k^{\mathrm{DL}} \leftarrow \frac{1}{\alpha_k} \boldsymbol{P}_k^{\mathrm{H}}, \quad \boldsymbol{P}_k^{\mathrm{DL}} \leftarrow \alpha_k \boldsymbol{T}^{-1} \boldsymbol{H}_k \boldsymbol{P}_k$
16: **end for**

Theorem 2: Suppose f is bounded below and Lipschitzian with the Lipschitz constant L, and $0 < \eta < 2/L$. The sequence generated by the gradient projection algorithm then converges. Furthermore, the limit point of this sequence satisfies the first order KKT optimality condition. In particular, if f is convex then the algorithm converges to the global minimum.

Proof: See [11]. ∎

The parameter η ensures the convergence of the algorithm. It is determined by successive reduction, i.e., $\eta = 1/d$ where d is initialized with 2 and is incremented as soon as the objective tends to increase.
This approach features excellent convergence properties (see Fig. 3) compared to alternating optimization of receivers and transmitters. Moreover, this algorithm presents interesting properties from an implementation point of view, since it requires only one inverse in each iteration (computational complexity is $\mathcal{O}(M^3)$).

For the case of sum-MSE minimization ($w_k = 1\ \forall k$), the cost function simplifies to

$$\sum\nolimits_k \varepsilon_k = \sum\nolimits_k B_k - M + \sigma_n^2 \mathrm{tr}(\boldsymbol{T}^{-1}), \tag{18}$$

which is jointly convex with respect to all $\boldsymbol{Q}_k = \boldsymbol{P}_k \boldsymbol{P}_k^{\mathrm{H}}$. Hence, Algorithm 1 converges to the global optimum in this case due to Theorem 2. The KKT conditions then read as:

$$\mu \boldsymbol{P}_k \stackrel{!}{=} \sigma_n^2 \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{T}^{-2} \boldsymbol{H}_k \boldsymbol{P}_k\ \forall k. \tag{19}$$

In the following section, we derive an efficient algorithm that solves the KKT conditions in (19) for the MISO case with very low complexity.

[5]Remember that $\boldsymbol{G}_k = \boldsymbol{P}_k^{\mathrm{H}} \boldsymbol{H}_k^{\mathrm{H}} \boldsymbol{T}^{-1}$

[6]The function is nearly flat for a high SNR. Thus, the Jacobian has a small Frobenius norm, which makes this scaling important.

Algorithm 2 MISO Sum-MSE Minimization

1: **Initialize:** $p_k^{(0)} \leftarrow 0\ \forall k, \ell \leftarrow 1$
2: **repeat**
3: $\quad \boldsymbol{T}^{(\ell)} \leftarrow \sigma_n^2\mathbf{I} + \sum_k p_k^{(\ell),2}\boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}}$
4: $\quad$ Water-filling: Sort users decreasing in $\|\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k\|_2$. $k_0 \leftarrow K+1$
5: $\quad$ **repeat**
6: $\quad\quad k_0 \leftarrow k_0 - 1$
7: $\quad\quad \mu^{-1/2} \leftarrow \frac{P_{\mathrm{tr}}+\sum_{k=1}^{k_0}(\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k)^{-1}}{\sigma_n\sum_{k=1}^{k_0}\left\|\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k\right\|_2(\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k)^{-1}}$
8: $\quad$ **until** $(\mu^{-1/2}\sigma_n\|\boldsymbol{R}_{k_0}^{(\ell),-1}\boldsymbol{h}_{k_0}\|_2 - 1) \geq 0$
9: $\quad \bar{p}_k^{(\ell+1),2} \leftarrow \frac{1}{\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k}(\mu^{-1/2}\sigma_n\|\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k\|_2 - 1)_+$
10: $\quad$ **if** $\ell < 5$ **then**
11: $\quad\quad p_k^{(\ell+1),2} \leftarrow \bar{p}_k^{(\ell+1),2}, \forall k$
12: $\quad$ **else**
13: $\quad\quad p_k^{(\ell+1),2} \leftarrow \frac{1}{K}\bar{p}_k^{(\ell+1),2} + \frac{K-1}{K}p_k^{(\ell),2}, \forall k$
14: $\quad$ **end if**
15: $\quad \ell \leftarrow \ell + 1$
16: **until** $|\sum_k \varepsilon_k^{(\ell+1)} - \sum_k \varepsilon_k^{(\ell)}| < \epsilon$
17: $\alpha \leftarrow \sqrt{\frac{P_{\mathrm{tr}}}{\sum p_k^2\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{T}^{-2}\boldsymbol{h}_k}}$
18: $g_k^{\mathrm{DL}} \leftarrow \frac{p_k}{\alpha}$, $\boldsymbol{p}_k^{\mathrm{DL}} \leftarrow \alpha\boldsymbol{T}^{-1}\boldsymbol{h}_k p_k$

VI. Sum-MSE Minimization for the MISO System

In the equivalent SIMO uplink, the precoders are simple scalars p_k. The resulting system of nonlinear equations can be solved iteratively using the Jacobi method by computing p_k via (19) for $k = 1, \ldots, K$ in a parallel fashion (assuming the other precoders to be fixed). Then, (19) reduces to

$$\frac{\mu}{\sigma_n^2} = \boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{T}^{-2}\boldsymbol{h}_k, \tag{20}$$

and applying the inversion lemma to $\boldsymbol{T} = \boldsymbol{R}_k + \boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}}p_k^2$ yields

$$\boldsymbol{T}^{-1} = \boldsymbol{R}_k^{-1} - \frac{\boldsymbol{R}_k^{-1}\boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{-1}}{p_k^{-2} + \boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{-1}\boldsymbol{h}_k}. \tag{21}$$

The right hand side expression from (20) can then be computed to be

$$\begin{aligned}\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{T}^{-2}\boldsymbol{h}_k = \|\boldsymbol{T}^{-1}\boldsymbol{h}_k\|_2^2 &= \left\|\frac{\boldsymbol{R}_k^{-1}\boldsymbol{h}_k}{1+p_k^2\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{-1}\boldsymbol{h}_k}\right\|_2^2 \\ &= \frac{1}{\left(1+p_k^2\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{-1}\boldsymbol{h}_k\right)^2}\left\|\boldsymbol{R}_k^{-1}\boldsymbol{h}_k\right\|_2^2.\end{aligned} \tag{22}$$

The Jacobi iteration for step $\ell+1$ follows from plugging (22) into (20) and solving it for p_k^2:

$$\bar{p}_k^{(\ell+1),2} = \frac{1}{\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k}\left(\frac{\sigma_n}{\sqrt{\mu}}\left\|\boldsymbol{R}_k^{(\ell),-1}\boldsymbol{h}_k\right\|_2 - 1\right)_+, \tag{23}$$

where μ is chosen to satisfy the transmit power constraint and $(x)_+$ denotes $\max(x, 0)$. Note that $\bar{p}_k^{(\ell+1),2}$ minimizes $\beta(\boldsymbol{p}) = \sum_{k=1}^{K}\mathrm{tr}[(\sigma_n^2\mathbf{I} + p_k^2\boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}} + \sum_{i\neq k}p_i^{(\ell),2}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathrm{H}})^{-1}]$ under the constraint $\|\boldsymbol{p}\|_2^2 \leq P_{\mathrm{tr}}$. Thus, $\beta(\bar{\boldsymbol{p}}^{(\ell+1)}) \leq \beta([p_1^{(\ell)}, \ldots, p_K^{(\ell)}]^{\mathrm{T}})$, where $\bar{\boldsymbol{p}}^{(\ell+1)} = [\bar{p}_1^{(\ell+1)}, \ldots, \bar{p}_K^{(\ell+1)}]^{\mathrm{T}}$.

Above iteration doesn't necessarily converge. However, the following scaled version converges to the global minimizer:

$$p_k^{(\ell+1),2} \leftarrow \frac{1}{K}\bar{p}_k^{(\ell+1),2} + \frac{K-1}{K}p_k^{(\ell),2}. \tag{24}$$

Again, K is the number of users.

Proposition 1: The sequence $\sum_k \varepsilon_k^{(\ell)}$ generated by the scaled Jacobi iteration in (24) is monotone decreasing in ℓ and converges to the global minimum, independently of the initialization.

Proof: The sum-MSE is given by (18), thus we have:

$$\begin{aligned}\mathrm{tr}(\boldsymbol{T}^{(\ell+1),-1}) &= \mathrm{tr}\left[\left(\sigma_n^2\mathbf{I} + \sum_i p_i^{(\ell+1),2}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathrm{H}}\right)^{-1}\right] \\ &= \mathrm{tr}\left[\left(\sigma_n^2\mathbf{I} + \sum_i\left(\frac{K-1}{K}p_i^{(\ell),2} + \frac{1}{K}\bar{p}_i^{(\ell+1),2}\right)\boldsymbol{h}_i\boldsymbol{h}_i^{\mathrm{H}}\right)^{-1}\right] \\ &= \mathrm{tr}\left[\left(\frac{1}{K}\sum_k\left(\sigma_n^2\mathbf{I} + \bar{p}_k^{(\ell+1),2}\boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}} + \sum_{i\neq k}p_i^{(\ell),2}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathrm{H}}\right)\right)^{-1}\right] \\ &\leq \frac{1}{K}\sum_k\mathrm{tr}\left[\left(\sigma_n^2\mathbf{I} + \bar{p}_k^{(\ell+1),2}\boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}} + \sum_{i\neq k}p_i^{(\ell),2}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathrm{H}}\right)^{-1}\right] \\ &\leq \frac{1}{K}\sum_k\mathrm{tr}\left[\left(\sigma_n^2\mathbf{I} + \sum_i p_i^{(\ell),2}\boldsymbol{h}_i\boldsymbol{h}_i^{\mathrm{H}}\right)^{-1}\right] = \mathrm{tr}(\boldsymbol{T}^{(\ell),-1}),\end{aligned} \tag{25}$$

where the first inequality follows from the convexity of $\mathrm{tr}(\boldsymbol{A}^{-1})$ and the second one from the Jacobi iteration in (23). Therefore every iteration step given by (24) decreases the MSE. Due to the non-negativity of the cost function, the sequence generated by the algorithm converges to a limit point from the Heine Borel covering theorem. Moreover, since the limit point of this sequence surely satisfies the KKT conditions and since the problem is convex, the algorithm converges to the global optimum. ■

The iterative solution of the sum-MSE minimization is summarized in Algorithm 2. Lines 4 to 8 describe the *water-filling* procedure that determines the Lagrangian multiplier μ. To speed up the algorithm, the first four iterations are directly performed by the Jacobi iteration (23), which approaches the optimal solution rapidly but doesn't ensure convergence. Then, we switch to the scaled iteration from (24), which guarantees the convergence to the optimum. The divergence of the unscaled Jacobi iteration in (23) was observed for a high number of users ($K > 100$). Even in those cases, the first iterations directly performed with the Jacobi variant deliver an excellent starting point for the scaled iteration. Note that the matrices $\boldsymbol{R}_k^{-1}$ can all be computed from $\boldsymbol{T}^{-1}$ using the matrix inversion lemma, i.e.,

$$\boldsymbol{R}_k^{-1} = \boldsymbol{T}^{-1} + p_k^2\frac{\boldsymbol{T}^{-1}\boldsymbol{h}_k\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{T}^{-1}}{1 - p_k^2\boldsymbol{h}_k^{\mathrm{H}}\boldsymbol{T}^{-1}\boldsymbol{h}_k}. \tag{26}$$

Finally, the uplink to downlink conversion takes place in Lines 17 and 18. Simulations show that this combination converges very quickly (only 2 or 3 iterations are needed), independently from the number of users and antennas. Thus, it is more efficient and faster than the interior point approach suggested in [3] (see Fig. 4).

We can generalize this algorithm to the MIMO case by decomposing the filters $\boldsymbol{P}_k$ into $\boldsymbol{V}_k\boldsymbol{\Sigma}_k$, where $\boldsymbol{V}_k$ has unit norm columns and $\boldsymbol{\Sigma}_k$ is diagonal. A new update for $\boldsymbol{V}_k$ is found using the gradient projection method explained in the previous section and the power matrices $\boldsymbol{\Sigma}_k$ are updated similarly to the MISO iteration in (23). Due to the lack of space, this approach will be not discussed in details.

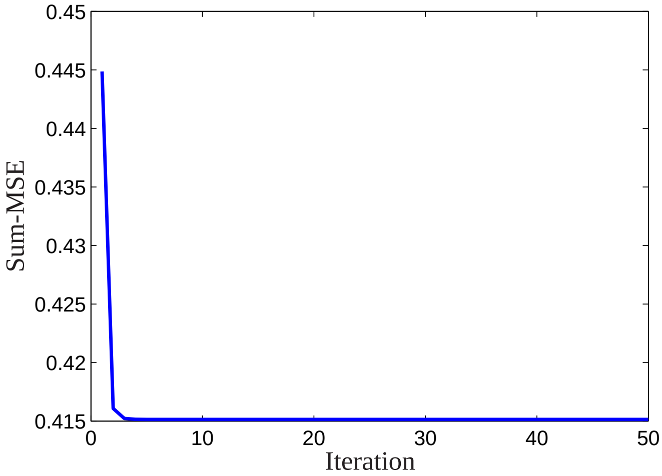

Fig. 3. Convergence of the MIMO weighted sum-MSE minimization algorithm; $M = 6$ transmit antennas, $K = 3$ users, $N_k = 2$ receive antennas per user, $B_k = 2$ streams per user, weights $w_k = 1\forall k$; SNR = 20dB.

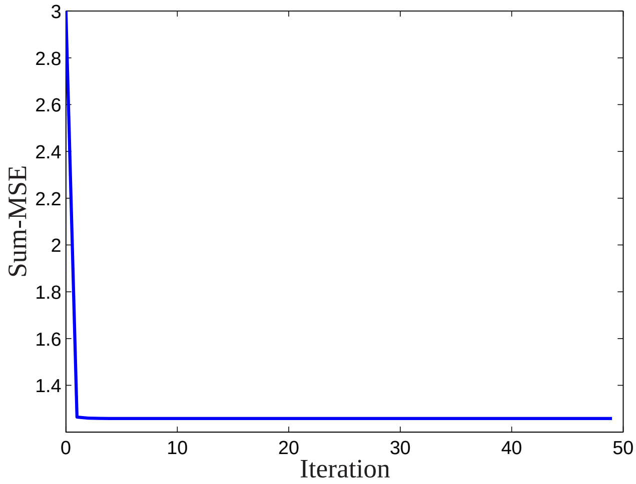

Fig. 4. Convergence of the MISO sum-MSE minimization algorithm; $M = 3$ transmit antennas, $K = 3$ users, SNR = 6dB.

VII. QoS Optimization

The multi-user MISO QoS/fairness problem has already been solved in the context of SINR optimization in [2]. The general MIMO case has been studied in [12] but only for fixed receivers. Two problems have been investigated: The first one consists of minimizing the total transmission power while satisfying a set of SINR constraints. The second one seeks to maximize the jointly achievable SINR margin under a total sum power constraint. We are interested in achieving different QoSs among the users rather than among subchannels separately. The user-wise QoS formulation is especially attractive if the subchannels of each individual user are jointly encoded and decoded. A possible metric for this problem is the mutual information I_k, which can be tightly related to the users' MSEs by means of the approximation

$$I_k = -\log_2|\boldsymbol{E}_k| \gtrapprox B_k \log_2(B_k/\varepsilon_k), \tag{27}$$

where we used the arithmetic-geometric sum inequality (for the eigenvalues of $\boldsymbol{E}_k$). A strict equality holds for $B_k = 1$. Thus, for the QoS/fairness problem, we use $\frac{B_k}{\varepsilon_k} - 1$ as the metric which is identical to the SINR in the single-stream case. We formulate the power minimization problem (28) and power assignment problem (29) as follows:

$$\min_{\{\boldsymbol{P}_1,\ldots,\boldsymbol{P}_K\}} \textstyle\sum_k \mathrm{tr}(\boldsymbol{P}_k\boldsymbol{P}_k^{\mathrm{H}}) \quad \text{s.t.: } \frac{B_k}{\varepsilon_k} - 1 \geq \gamma_k, \forall k \tag{28}$$

$$\begin{aligned}\max_{\{\boldsymbol{P}_1,\ldots,\boldsymbol{P}_K,\gamma_0\}} \gamma_0 \quad \text{s.t.: } & \textstyle\sum_k \mathrm{tr}(\boldsymbol{P}_k\boldsymbol{P}_k^{\mathrm{H}}) \leq P_{\mathrm{tr}}, \quad \text{and} \\ & \frac{B_k}{\varepsilon_k} - 1 \geq \gamma_0\gamma_k, \forall k,\end{aligned} \tag{29}$$

where $\gamma_k, k = 1, \ldots, K$ are given. Although these problems are non-convex, we note that the KKT conditions are sufficient. Consequently, they can be solved using nonlinear programming methods. First, we derive a fixed point iterative solution for the MISO case, which is numerically more efficient than the one in [2] and prove its convergence. Then, an algorithm for the general MIMO case is proposed. By means of our approach, it is possible to guarantee fairness not only among the users, but also among all subchannels.

A. QoS fixed point solution for MISO ($B_k = 1\ \forall k$)

1) Power minimization problem (28): It can be easily seen that the constraints in (28) are fulfilled with equality, otherwise we could decrease the power p_k related with inactive constraints and thus decrease the objective [2], [12]. Therefore, the optimal solution is given by

$$1/\varepsilon_k - 1 = \gamma_k, \forall k. \tag{30}$$

From Section IV we know that, the optimal receiver is the MMSE receiver (Wiener filter) and that the MMSE of each user has then the following form (see Eqn. 12 and Eqn. 21):

$$\varepsilon_k = \frac{1}{1 + p_k^2 \boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{R}_k^{-1} \boldsymbol{h}_k}. \tag{31}$$

With the QoS constraints in (30) and the MMSE in (31), we get a system of non-linear equations:

$$p_k^2 = \frac{\gamma_k}{\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{R}_k^{-1} \boldsymbol{h}_k} = f_k(\{p_i^2, i \neq k\}). \tag{32}$$

Obviously, this a fixed point problem. The functions $\boldsymbol{f} = \{f_k\}$ can easily be shown to satisfy the condition of [13], i.e., they are positive, monotone increasing, and concave (see the Jacobian of $\boldsymbol{f}$ in Eqn. 35). Thus, $\boldsymbol{f}$ has at most one fixed point, i.e., there is at most one solution. For the case $\mathrm{rank}(\boldsymbol{H}) = K$, where $\boldsymbol{H} = [\boldsymbol{h}_1, \cdots, \boldsymbol{h}_K]$, there is always a unique solution (see [12]).

Assuming that the problem is feasible, i.e., there is a unique solution $\bar{p}_1^2, \ldots, \bar{p}_K^2$, we desire to solve this system using a fixed point technique (also known as Picard-iteration method or direct iteration method). This method generates new updates $p_1^{(\ell+1),2}, \ldots, p_K^{(\ell+1),2}$ by evaluating the functions with their old arguments $p_1^{(\ell),2}, \ldots, p_K^{(\ell),2}$. In other words, the fixed point iteration for our problem is

$$p_k^{(\ell+1),2} = \frac{\gamma_k}{\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{R}_k^{(\ell),-1} \boldsymbol{h}_k} = \frac{\gamma_k p_k^{(\ell),2}}{\frac{1}{\varepsilon_k^{(\ell)}} - 1}. \tag{33}$$

For the starting point, we use $p_k^{(0),2} = f_k(0, \ldots, 0) = \frac{\gamma_k \sigma_n^2}{\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{h}_k}$. This fixed point method is summarized in Algorithm 3. The non-feasibility of the problem is detected by checking if the sum power exceeds a maximum power $P_{\max}$. We see that this algorithm needs only an inverse per iteration, all other computations have a lower order of complexity.

We now prove that this fixed iteration is locally contractive and thus converges. If the problem is feasible, i.e., there exist

Algorithm 3 MISO Power Minimization Algorithm

Initialize: $p_k^{(0)} \leftarrow \sqrt{\frac{\gamma_k \sigma_n^2}{\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{h}_k}} \quad \forall k, \ell \leftarrow 1$
repeat
$\quad \boldsymbol{P}^{(\ell)} \leftarrow \mathrm{diag}\{p_k^{(\ell)}\}_{k=1}^K$
$\quad \boldsymbol{E}^{(\ell)} \leftarrow \left(\mathbf{I} + \frac{1}{\sigma_n^2} \boldsymbol{P}^{(\ell)} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{H} \boldsymbol{P}^{(\ell)}\right)^{-1}$
$\quad p_k^{(\ell+1)} \leftarrow \sqrt{\frac{\gamma_k \boldsymbol{E}_{k,k}^{(\ell)}}{1 - \boldsymbol{E}_{k,k}^{(\ell)}}} p_k^{(\ell)}, \qquad \forall k$
$\quad \ell \leftarrow \ell + 1$
until $\left| p_k^{(\ell+1),2} - p_k^{(\ell),2} \right| < \epsilon, \forall k$ or $\sum_k p_k^{(l+1),2} > P_{\max}$
$\boldsymbol{G} \leftarrow \boldsymbol{P} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H} \boldsymbol{P}^2 \boldsymbol{H}^{\mathrm{H}} + \sigma_n^2 \mathbf{I} \right)^{-1}$
Compute the downlink transceivers using the uplink/downlink transformation (5)

a unique fixed point $\bar{p}_k^2 = f_k(\bar{p}_1^2, \ldots, \bar{p}_K^2)$, and the starting point is chosen sufficiently close to the fixed point, then the following condition is sufficient to guarantee the convergence:

$$\rho(\bar{\boldsymbol{J}}) = \sup\{|\lambda| : \lambda \text{ eigenvalue of } \bar{\boldsymbol{J}}\} < 1. \tag{34}$$

ρ is the spectral radius of the Jacobian matrix $\bar{\boldsymbol{J}}$ of $\boldsymbol{f}$ evaluated at the fixed point. To verify this, we calculate the Jacobian matrix:

$$\boldsymbol{J}_{k,i} = \frac{\partial f_k}{\partial (p_i^2)} = \begin{cases} \gamma_k \frac{\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{R}_k^{-1} \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{H}} \boldsymbol{R}_k^{-1} \boldsymbol{h}_k}{(\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{R}_k^{-1} \boldsymbol{h}_k)^2} > 0 & \text{if } k \neq i \\ 0 & \text{else,} \end{cases} \tag{35}$$

which means that every f_k is increasing in each argument $p_i, i \neq k$. Evaluating the Jacobian at the fixed point we obtain for the off-diagonal elements by means of (32)

$$\bar{\boldsymbol{J}}_{k,i} = \bar{p}_k^2 \frac{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k}{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k}. \tag{36}$$

To prove the convergence we make use of

Theorem 3 (Perron-Frobenius Theorem, 1912 [14]): If all elements $a_{i,j}$ of an irreducible matrix $\boldsymbol{A}$ are nonnegative, then $R = \min_{\boldsymbol{\theta}} M_{\boldsymbol{\theta}}$ is an eigenvalue of $\boldsymbol{A}$ and all the eigenvalues of $\boldsymbol{A}$ lie on the disk $|\lambda| \leq R$, where, if $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_K]^{\mathrm{T}}$ is a set of nonnegative numbers (which are not all zero simultaneously), $M_{\boldsymbol{\theta}} = \inf\left\{\mu : \mu\theta_k \geq \sum_{j=1}^K a_{k,j}\theta_j, 1 \leq k \leq K\right\}$, and $R = \min_{\boldsymbol{\theta}} M_{\boldsymbol{\theta}} = \rho(\boldsymbol{A})$ is the biggest eigenvalue and also the spectral radius of $\boldsymbol{A}$.

For $\boldsymbol{\theta}$ we use the vector $\bar{\boldsymbol{\theta}} = [\bar{p}_1^2, \ldots, \bar{p}_K^2]$.

$$\begin{aligned} \sum_{i=1}^K \bar{\boldsymbol{J}}_{k,i} \bar{p}_i^2 &= \frac{\bar{p}_k^2}{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k} \boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \Big(\sum_{i \neq k} \bar{p}_i^2 \boldsymbol{h}_i \boldsymbol{h}_i^{\mathrm{H}}\Big) \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k = \\ \frac{\bar{p}_k^2 \boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} (\mathbf{I} - \sigma_n^2 \bar{\boldsymbol{R}}_k^{-1}) \boldsymbol{h}_k}{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k} &= \left(1 - \sigma_n^2 \frac{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-2} \boldsymbol{h}_k}{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k}\right) \bar{p}_k^2, \end{aligned} \tag{37}$$

where the last step follows from the definition of $\boldsymbol{R}_k$ (see Eqn. 11). We obtain:

$$\begin{aligned} M_{\boldsymbol{\theta}} &= \inf\left\{\mu : \mu \bar{p}_k^2 \geq \left(1 - \sigma_n^2 \frac{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-2} \boldsymbol{h}_k}{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k}\right) \bar{p}_k^2, 1 \leq k \leq K\right\} \\ &= \sup\left\{1 - \sigma_n^2 \frac{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-2} \boldsymbol{h}_k}{\boldsymbol{h}_k^{\mathrm{H}} \bar{\boldsymbol{R}}_k^{-1} \boldsymbol{h}_k}, 1 \leq k \leq K\right\} < 1. \end{aligned} \tag{38}$$

Thus convergence is guarateed, as $\rho(\bar{\boldsymbol{J}}) = \min_{\boldsymbol{\theta}} M_{\boldsymbol{\theta}} < M_{\bar{\boldsymbol{\theta}}}$.

Algorithm 4 MISO Power Allocation Algorithm

Initialize: $p_k^{(0)} \leftarrow \sqrt{P_{\mathrm{tr}} \frac{\gamma_k}{\boldsymbol{h}_k^{\mathrm{H}} \boldsymbol{h}_k}} / \sqrt{\sum_i \frac{\gamma_i}{\boldsymbol{h}_i^{\mathrm{H}} \boldsymbol{h}_i}} \quad \forall k, \ \ell \leftarrow 1$
repeat
$\quad \boldsymbol{P}^{(\ell)} \leftarrow \mathrm{diag}\{p_k^{(\ell)}\}_{k=1}^K$
$\quad \boldsymbol{E}^{(\ell)} \leftarrow \left(\mathbf{I} + \frac{1}{\sigma_n^2} \boldsymbol{P}^{(\ell)} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{H} \boldsymbol{P}^{(\ell)}\right)^{-1}$
$\quad \gamma_0^{(\ell+1)} \leftarrow P_{\mathrm{tr}} / \Big(\sum_i \frac{\gamma_i \boldsymbol{E}_{i,i}^{(\ell)}}{1 - \boldsymbol{E}_{i,i}^{(\ell)}} p_i^{(\ell),2} \Big)$
$\quad p_k^{(\ell+1)} \leftarrow \sqrt{\frac{\gamma_k \gamma_0^{(\ell+1)} \boldsymbol{E}_{k,k}^{(\ell)}}{1 - \boldsymbol{E}_{k,k}^{(\ell)}}} p_k^{(\ell)} \quad \forall k$
$\quad \ell \leftarrow \ell + 1$
until $\left| p_k^{(\ell),2} - p_k^{(\ell-1),2} \right| < \epsilon, \forall k$
$\boldsymbol{G} \leftarrow \boldsymbol{P} \boldsymbol{H}^{\mathrm{H}} \left(\boldsymbol{H} \boldsymbol{P}^2 \boldsymbol{H}^{\mathrm{H}} + \sigma_n^2 \mathbf{I} \right)^{-1}$
Compute the downlink transceivers using the uplink/downlink transformation (5)

2) Power allocation problem (29): For the same reasons as in the previous subsection, the constraints in (29) are fulfilled with equality. Therefore, we can adjust the previous algorithm, so that γ_0 is determined iteratively to satisfy the sum power constraint, i.e.,

$$\gamma_0^{(\ell+1)} \leftarrow P_{\mathrm{tr}} / \Big(\sum_i \frac{\gamma_i \varepsilon_i^{(\ell)}}{1 - \varepsilon_i^{(\ell)}} p_i^{(\ell),2} \Big) \tag{39}$$

The powers p_k^2 are then updated in a similar way as in the power minimization problem. The resulting Algorithm 4 also converges, since the projection on the hypersphere with radius P_{tr} is a non-expansive operation.

B. QoS Iterative Solution for MIMO

As stated in the beginning of this section, we define the QoS problem user-wise, since it is easier to tackle then. Nevertheless in some applications it might be important to assure different SINR constraints among the streams. We show that under a user-wise QoS solution, some degrees of freedom always remain for each user in order to achieve restricted QoS constraints among its streams. In fact, user k can multiply its precoder $\boldsymbol{P}_k$ with a unitary matrix $\boldsymbol{U}_k$ from the right side without any influence on the other users. We see from (12) that this operation conserves the trace and the eigenvalues of the error covariance matrix $\boldsymbol{E}_k$, but its diagonal elements will be changed. In particular, this rotation matrix $\boldsymbol{U}_k$ can be chosen such that $\boldsymbol{E}_k$ has identical diagonal elements (*fairness*). An algorithm that computes such a matrix $\boldsymbol{U}_k$ is given in [15]. The same observation was done for the single user case in [16].

However, the distribution of each user's MSE on its streams cannot be done arbitrarily. The maximum *"unfairness"* that can be achieved is given by the eigenvalues of $\boldsymbol{E}_k$. This is a result of the Schur-Horn theorem [17] stating that a matrix with given eigenvalues $\boldsymbol{\lambda}$ and diagonal elements $\boldsymbol{d}$ can be constructed if and only if $\boldsymbol{\lambda}$ weakly majorizes $\boldsymbol{d}$. In other words (ascending order sorted λ_j and sorted d_j assumed now)

$$\sum_{j=i}^K \lambda_j \leq \sum_{j=i}^K d_j, \forall \quad 1 \leq i \leq K. \tag{40}$$

We see that the solutions of (28) and (29) satisfy only the first equation of (40) where $i = 1$. Therefore it is a necessary condition for the stream-wise QoS problems but not a sufficient one. For the fairness problem, both formulations (user- and stream-wise) are equivalent except that we need to do compute the rotation matrix $\boldsymbol{U}_k$ explained above. Allocating different MSEs for the individual streams lying between maximum 'unfairness' and complete 'fairness' can be achieved by the generalized Bendel-Mickey algorithm from [18].

Now, we derive an iterative solution for the power assignment problem (29) with a sum-power constraint. As stated earlier the first order conditions are sufficient, i.e., there is only a single local minimum which is the global optimum. The proof of this proposition is omitted due to the lack of space. Now, the KKT conditions for the problem (29) read as:

$$\begin{aligned}&\boldsymbol{P}_k \overset{!}{=} \boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{T}^{-1}(w_k\boldsymbol{T}-\boldsymbol{S})\boldsymbol{T}^{-1}\boldsymbol{H}_k\boldsymbol{P}_k,\\ &\varepsilon_k \leq \frac{B_k}{1+\gamma_0\gamma_k}, \forall k, \text{ and}\\ &\mathrm{tr}(\sum_k \boldsymbol{P}_k\boldsymbol{P}_k^{\mathrm{H}}) \leq P_{\mathrm{tr}},\end{aligned} \tag{41}$$

where $\boldsymbol{S}$ is defined in (15). We see the similarity to the weighted sum-MSE minimization problem in (14). But now, the scalars w_k are Lagrangian multipliers, that must be determined, and are not fixed parameters as in (14).

The proposed iterative solution in Algorithm 5 for solving this problem works as follow. The filters $\boldsymbol{P}_k$ are decomposed into $\boldsymbol{V}_k\boldsymbol{\Sigma}_k$, where $\boldsymbol{V}_k$ has unit norm columns and $\boldsymbol{\Sigma}_k$ is diagonal. For fixed $\boldsymbol{V}_k$, Line 4 updates the power matrices similarly to the MISO iteration in Algorithm 4. Next, assuming $\boldsymbol{V}_k\boldsymbol{\Sigma}_k$ to be fixed, the Lagrangian multipliers w_k are computed by solving the linear system of equation in (41); a new update for $\boldsymbol{V}_k$ is then found using the gradient projection method explained previously (Lines 5 to 13). Since both steps are contractive as we have seen in the previous sections, the algorithm converges to the optimum.

Algorithm 5 MIMO Power Allocation Algorithm

1: **Initialize:** $\sigma_{k,q}^{(0),2} \leftarrow \frac{P_{\mathrm{tr}}}{\sum B_k}$, $\boldsymbol{V}_k^{(0)}(1:B_k,1:B_k) \leftarrow \mathbf{I}_{B_k}$, Compute $\boldsymbol{E}_k^{(0)}$ with (12), $\forall k \forall q$, $d \leftarrow 2$, $\ell \leftarrow 0$
2: **repeat**
3: $\quad\ell \leftarrow \ell + 1$
4: $\quad\gamma_0^{(\ell)} \leftarrow P_{\mathrm{tr}}/\sum_{k,q}\frac{\gamma_k\boldsymbol{E}_k^{(\ell-1)}(q,q)}{1-\boldsymbol{E}_k^{(\ell-1)}(q,q)}\sigma_{k,q}^{(\ell-1),2}$,
$\quad\sigma_{k,q}^{(\ell)} \leftarrow \sqrt{\frac{\gamma_k\gamma_0^{(\ell)}\boldsymbol{E}_k^{(\ell-1)}(q,q)}{1-\boldsymbol{E}_k^{(\ell-1)}(q,q)}}\sigma_{k,q}^{(\ell-1)}$, $\forall k, q$ (power allocation)
$\quad\boldsymbol{\Sigma}_k^{(\ell)} = \mathrm{diag}\{\sigma_{k,q}^{(\ell)}\}_{q=1}^{B_k}$
5: $\quad\boldsymbol{T}^{(\ell)} \leftarrow \sigma_n^2\mathbf{I} + \sum_k \boldsymbol{H}_k\boldsymbol{V}_k^{(\ell-1)}\boldsymbol{\Sigma}_k^{(\ell),2}\boldsymbol{V}_k^{(\ell-1),\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}$
6: $\quad$Compute $w_k^{(\ell)}$ and $\boldsymbol{E}_k^{(\ell),\mathrm{temp}}$ with (41) and (12)
7: $\quad\boldsymbol{S}^{(\ell)} \leftarrow \sum_k w_k^{(\ell)}\boldsymbol{H}_k\boldsymbol{V}_k^{(\ell-1)}\boldsymbol{\Sigma}_k^{(\ell),2}\boldsymbol{V}_k^{(\ell-1),\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}$
8: $\quad$Gradient Update for $\boldsymbol{V}_k^{(\ell)}$ with fixed $\boldsymbol{\Sigma}_k^{(\ell)}$, $w_k^{(\ell)}$ ($\forall k$):
$\quad\delta\boldsymbol{V}_k^{(\ell)} \leftarrow \boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{T}^{(\ell),-1}(w_k^{(\ell)}\boldsymbol{T}^{(\ell)}-\boldsymbol{S}^{(\ell)})\boldsymbol{T}^{(\ell),-1}\boldsymbol{H}_k\boldsymbol{V}_k^{(\ell-1)}$
$\quad\delta\boldsymbol{v}_{k,q}^{(\ell)} \leftarrow \frac{\delta\boldsymbol{v}_{k,q}^{(\ell)}}{\left\|\delta\boldsymbol{v}_{k,q}^{(\ell)}\right\|_2}$, $\forall q$ (scaled gradient)
9: $\quad\boldsymbol{V}_k^{(\ell)} \leftarrow \frac{1}{d}\delta\boldsymbol{V}_k^{(\ell)} + \boldsymbol{V}_k^{(\ell-1)}$, $\boldsymbol{v}_{k,q}^{(\ell)} \leftarrow \frac{\boldsymbol{v}_{k,q}^{(\ell)}}{\left\|\boldsymbol{v}_{k,q}^{(\ell)}\right\|_2}$, $\forall q$
10: $\quad$Compute $\boldsymbol{E}_k^{(\ell)}$, $\quad\forall k$
11: $\quad$**if** $\sum_k w_k^{(\ell)}\mathrm{tr}(\boldsymbol{E}_k^{(\ell)}) > \sum_k w_k^{(\ell)}\mathrm{tr}(\boldsymbol{E}_k^{(\ell),\mathrm{temp}})$ **then**
12: $\quad\quad d \leftarrow d+1$; goto line 9
13: $\quad$**end if**
14: **until** desired accuracy for $\boldsymbol{E}_k$ is achieved
15: $\boldsymbol{G}_k^{\mathrm{DL}} \leftarrow \frac{1}{\alpha_k}\boldsymbol{\Sigma}_k\boldsymbol{V}_k^{\mathrm{H}}$, $\quad\boldsymbol{P}_k^{\mathrm{DL}} \leftarrow \alpha_k\boldsymbol{T}^{-1}\boldsymbol{H}_k\boldsymbol{V}_k\boldsymbol{\Sigma}_k$
where $\alpha_k \leftarrow \sqrt{\frac{w_kP_{\mathrm{tr}}}{\sum\mathrm{tr}(w_k\boldsymbol{\Sigma}_k\boldsymbol{V}_k^{\mathrm{H}}\boldsymbol{H}_k^{\mathrm{H}}\boldsymbol{T}^{-2}\boldsymbol{H}_k\boldsymbol{V}_k\boldsymbol{\Sigma}_k)}}$

VIII. Simulation Results

In our channel model, the entries of $\boldsymbol{H}_k$ are complex-valued realizations of independent zero-mean Gaussian random variables. For each channel realization, 100 16QAM modulated symbols are transmitted, and the bit error rates are averaged over 1000 channel realizations.

In Fig. 5, we compare the sum-MSE transceiver with the TxWF of [1], where all users apply the same scalar at the receivers. We choose a MISO system (one antenna per user), where $K = 3$ users are served by $M = 3$ transmit antennas. Furthermore, the user's channels do not have the same average power, i.e., $\mathrm{E}[\|\boldsymbol{h}_3\|_2^2] = 9\mathrm{E}[\|\boldsymbol{h}_2\|_2^2] = 100\mathrm{E}[\|\boldsymbol{h}_1\|_2^2]$, as for identical average powers only small gains can be achieved by extending the receivers toward different scalars. The stronger user (cross marker) dramatically benefits, whereas the weaker user (diamond marker) sees only few improvement. Thus the overall efficiency of the system slightly improves.

The comparison between the performance of the sum-MSE and the QoS transceivers is shown in Fig. 6 for a three user MIMO system (16QAM), where the *average* channel power of user 2 and 3 is four times stronger than that of user 1. The QoS transceiver features fairness among all subchannels ($\gamma_k = 1, \forall k \rightarrow$ minimizing the maximum $\varepsilon_{k,q}$), thus it assures equal BERs to all users (dashed lines). The sum-MSE transceiver has a better overall efficiency for low and middle SNR, but user 1 with the weak channel is disfavored compared to the others (solid lines). Nevertheless, the QoS transceiver, which guaranties *fairness* among all streams, outperforms the sum-MSE transceiver for high SNR, due to the fact that the performance of the system in this SNR range strongly depends on the stream with the worst SINR. The choice of the optimum design thus depends on the application and the SNR range. In Fig. 7, we compare our sum-MSE transceiver with existing MIMO approaches in the literature, in particular the block diagonal approach (also known as *zeroforcing* ZF) [19], [20]. Note that the algorithm in [19] requires that the number of transmit antennas must be greater than or equal to the total number of receive antennas in order to satisfy the null-space criterion. Therefore, we chose a system with $M = 6$ transmit antennas and $K = 3$ users with $N_k = 2$ receive antennas and $B_k = 2$ streams each. It is not surprising that the sum-MSE THP clearly outperforms the *zeroforcing* transceiver. The main advantage of the sum-MSE design is also its robustness with respect to rank deficient channels.

IX. Conclusion

We addressed the problem of jointly designing the linear transmitter and receivers for a multi-user MIMO system. Thanks to a general form of duality between downlink and uplink, we formulated all optimization problems in the equivalent uplink, and solved the KKT conditions iteratively. Several design criteria have been considered. Examples are the weighted sum-MSE minimization, the QoS, and the *fairness* optimization problems, for which we have provided algorithmic solutions. The choice of a certain criterion depends on the properties of the channel and the application. The presented algorithms have good convergence properties compared to alternating optimization of transmitter and receivers. Moreover,

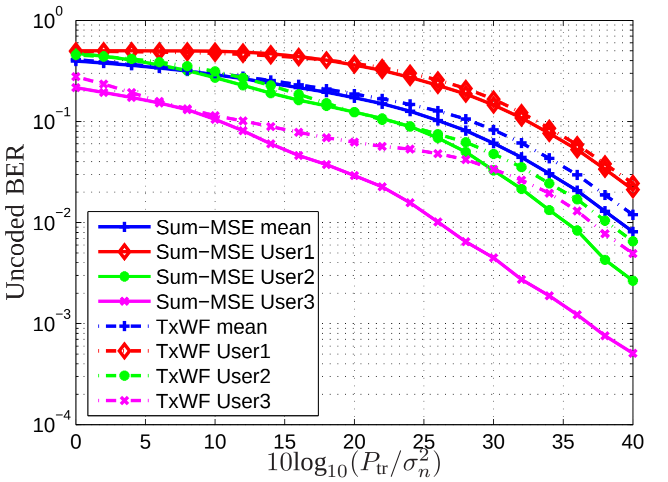

Fig. 5. TxWF and SumMSE design (MISO); $M = 3$ transmit antennas, $K = 3$ users; *average* channel power of user 3 is 9 times that of user 2 and 100 times that of user 1.

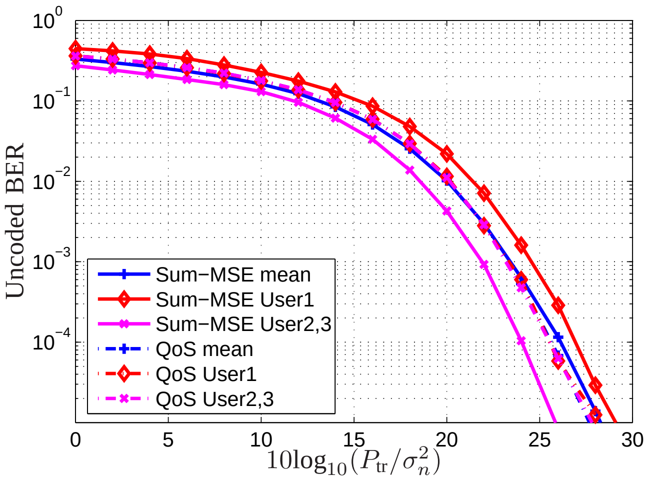

Fig. 6. QoS and sum-MSE design; $M = 8$ transmit antennas, $K = 3$ users, $N_k = 2$ receive antennas, $B_k = 2$ streams, $\forall k$; *average* channel power of user 2 and 3 is four times that of user 1.

they show interesting properties from an implementation point of view. Thanks to the joint optimization of the transmitters and receivers, our transceivers, which have no requirements on the dimensions of system (as opposed to the ZF approaches), outperform the existing *zeroforcing* solutions and offer excellent performance in different scenarios of the MIMO channel.

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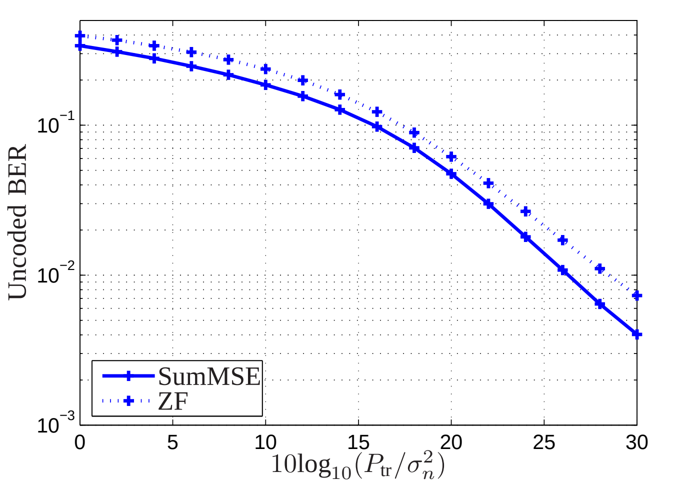

Fig. 7. Sum-MSE vs. ZF design; $M = 6$ transmit antennas, $K = 3$ users, $N_k = 2$ receive antennas, $B_k = 2$ streams, $\forall k$;

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