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Orthogonal Polynomials: Interaction between Orthogonality in L^2-spaces and Orthogonality in Reproducing Kernel Spaces

Frank Hofmaier

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Vorsitzender:		Univ.-Prof. Dr. Gregor Kemper
Prüfer der Dissertation:	1.	apl. Prof. Dr. Domenico Castrigiano
	2.	Univ.-Prof. Dr. Rupert Lasser
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About this Work

Given a sequence $(P_n)_{n\geq 0}$ of polynomials in one complex variable where $P_0 \equiv 1$ and $\deg(P_n) = n$ for all n, we look for a Borel mesaure μ supported on some subset of $\mathbb{C}$ such that $\int \overline{P_n}\, P_m \,\mathrm{d}\mu = \delta_{n,m}$ which we then will refer to as an *orthonormalizing measure* (om) for $(P_n)_n$. If there exists an om μ, we are also interested in the questions whether $(P_n)_n$ is an *orthonormal basis* (onb) in L^2_μ and whether μ is uniquely determined.

The classical approach to this problem uses the multiplication operator D defined by $(Dp)(z) := z\,p(z)$ for $p \in \mathbb{C}[z]$. One constructs an abstract Hilbert space $\mathcal{H}$ containing $\mathbb{C}[z]$ as a dense linear subspace such that the given polynomials $(P_n)_n$ form an orthonormal basis and D becomes a densely defined (not necessarily bounded) linear operator in $\mathcal{H}$. If there exists an om then we can isometrically embed $\mathcal{H}$ into L^2_μ and multiplication by z in L^2_μ is a normal operator extending D. In particular, there exists an om if and only if D is *subnormal*, i.e. has a normal extension which, however, might only exist in a larger space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace. Unfortunately, in general, it is comparatively difficult to determine whether an operator is subnormal or not.

In addition to the abstract space $\mathcal{H}$, we will construct a *Reproducing Kernel Hilbert Space* (RKHS) consisting of complex-valued functions defined on $E := \big\{z \in \mathbb{C} : \big(P_n(z)\big)_n \in \ell^2\big\}$ with kernel

$$K : E \times E \to \mathbb{C}\,, \qquad K(z,w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w)\,;$$

denote that space by $\mathcal{H}(K)$. We point out that E might be finite or even empty and if E is infinite, it still can happen that $(P_n)_n$ is not an orthogonal system in $\mathcal{H}(K)$. Yet, in many cases there exists an om μ and $\mathcal{H}(K)$ can be isometrically embedded into L^2_μ. This is of particular interest because the elements of L^2_μ, in general, only are classes of functions while the members of an RKHS are functions defined pointwisely on E. Making use of this fact, we will be able to deduce particular qualities of om; for example, if the operator D is essentially normal then there exists a unique om μ, $(P_n)_n$ is an onb in L^2_μ, $E = \big\{z \in \mathbb{C} : \mu(\{z\}) > 0\big\}$, and $(P_n)_n$ is an onb in the kernel space $\mathcal{H}(K)$ if and only if μ is discrete. It can also be shown that if there exists an om μ such that $\mu(E) = 1$ then $(P_n)_n$ is an onb in the associated RKHS. Another remarkable fact – to name just a few – is that, if the interior of E is non-empty then there exists an open set G that is dense in E and where all the members of $\mathcal{H}(K)$ are holomorphic. If, in addition, there exists an om μ such that $(P_n)_n$ is an onb in L^2_μ then G must be a μ-nullset and $\Lambda_\mu := \big\{z \in \mathbb{C} : \mu(\{z\}) > 0\big\}$ cannot have a limit point in G. It remains an open question whether it is possible that the largest such set G is a *proper* subset of the interior of E.

As convergence in the $\mathcal{H}(K)$-norm implies pointwise convergence on E, we will also be able to make statements about the limits of pointwisely – on a subset of $\mathbb{C}$ – convergent series of polynomials, concerning the domain where that limit is holomorphic. In connection with orthogonal polynomials on the unit circle $\partial\mathbb{D}$ we can, given any compact uncountable Lebesgue[1]-nullset $K \subset \partial\mathbb{D}$, construct an om μ such that $\mu(K) > 0$ and a sequence of polynomials $(q_n)_n$ which is convergent pointwisely in the open unit disk $\mathbb{D}$ and Lebesgue-almost everywhere in $\partial\mathbb{D}$ with limit 0 while μ-almost everywhere on K the limit is 1.

Orthogonal polynomials, especially in the real case, have been studied intensely for the past 70 years. The textbooks by Szegő [Sze] and Chihara [Chi] may serve as standard references. A short historical note on orthogonal polynomials can be found in [As, pp. 12-13]; for an overview concerning applications to random matrices, discrete Schrödinger operators, and birth-and-death processes, see [As, Ch. 6], for instance.
Polynomials orthogonal with respect to measures supported on the unit circle are very well understood, too. A two-volume monograph by Simon [Si] presents a vast treatise on this topic.
Polynomial sequences for which one tries to find orthonormalizing measures may arise from numerical problems such as interpolation or quadrature, for example. As mentioned before, existence of orthonormalizing measures is equivalent to subnormality of the multiplication operator D. For the theory of bounded subnormal operators, we refer to [Con1]; unbonded subnormal operators have been observed in [StSz1], [StSz2], and [StSz3]; see also [CaKl2] and [Kl2].

Our work is organized as follows. The first chapter is a short introduction stating the main problem and presenting the ideas how to handle it.

In chapter 2 we construct reproducing kernel spaces $\mathcal{H}(K)$ associated to a given sequence $(P_n)_n$. We will gather some statements about the elements of $\mathcal{H}(K)$ concerning (amongst other things) analyticity and we also realize some RKHS as a subspace of an L^2-space.

Chapter 3 is the heart of this work presenting all the theoretical aspects on orthonormalizing measures, the multiplication operator D, its normal extensions, and all the remarkable connections between L^2_μ and $\mathcal{H}(K)$ mentioned above.

Finally, in chapter 4 we study some special cases; in particular, orthogonal polynomials on the real line and on the unit circle – both of which have been studied intensely in the past. However, we see how they fit into the theory as special cases of the more general problem. Afterwards, we conclude the work with a variety of examples.

[1] Here we refer to one-dimensional Lebesgue measure on the unit circle.

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Contents

1 Introduction

Throughout this work, for any complex Hilbert space $\mathcal{H}$, we will denote its inner product and norm by $\langle \cdot , \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively. Note that the inner product is linear in the *second* argument. We may omit the index if it is clearly understood.
Furthermore, for a measure μ on the σ-algebra $\mathfrak{B}(\mathbb{C})$ of Borel subsets of $\mathbb{C}$, let L^2_μ be the Hilbert space of classes of (complex valued) functions square integrable with respect to μ, with inner product $\langle f , g \rangle = \int \overline{f} \cdot g \; \mathrm{d}\mu$.

Finally, let $\mathbb{N}$ denote the *positive* integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

1.1 Orthonormalizing Measures

1.1.1 Definition. Let $(P_n)_{n\geq 0}$ be a sequence in $\mathbb{C}[z]$ such that $P_0 \equiv 1$ and $\deg(P_n) = n$ for all n. A measure μ on $\mathfrak{B}(\mathbb{C})$ is called an **orthonormalizing measure** (om) for $(P_n)_n$, if all $p \in \mathbb{C}[z]$ are square integrable with respect to μ and

$$\int \overline{P_n}\, P_m \; \mathrm{d}\mu = \delta_{n,m} \; ,$$

i.e., if $(P_n)_n$ is an orthonormal system in the Hilbert space L^2_μ. We regard $\mathbb{C}[z]$ as a linear subspace of L^2_μ via the canonical embedding.

Note that, at this point, we do not require that $(P_n)_n$ be an orthonormal *basis* (in Hilbert space sense); we will write P^2_μ for the closure of $\mathbb{C}[z]$ in L^2_μ.

Furthermore, as $1 = \int |P_0|^2 \; \mathrm{d}\mu = \int 1 \; \mathrm{d}\mu = \mu(\mathbb{C})$, an om always is a probability measure. Note also that, in general, $\mathrm{supp}(\mu) \neq \mathbb{C}$. In particular, μ with $\mathrm{supp}(\mu) \subset \mathbb{R}$ will play a special role and have been widely studied. The textbooks by Szegő [Sze], Chihara [Chi], and van Assche [As] may be stated as references here; in section 4.1 we will discuss some general results about polynomials orthogonal w.r.t. a measure supported on the real line.

Polynomials with om supported on (possibly subsets of) the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ have also been examined in depth. Here the recently published two-volume book by Barry Simon [Si] definitely has to be mentioned.

On the other hand, if μ is a probability measure on $\mathfrak{B}(\mathbb{C})$ such that every $p \in \mathbb{C}[z]$ is square integrable with respect to μ, then μ obviously is an om for such a sequence of polynomials: One only has to apply the Gram-Schmidt algorithm in L^2_μ to $1, z, z^2, \ldots$, say.

For more detailed information on measure theoretical issues, we refer to [Bau], [El], or also [LL, Ch. 1 and 2].

The Newton Polynomials,

$$P_n(z) := (-1)^n \binom{z-1}{n} = \frac{(-1)^n}{n!} (z-1)(z-2)\cdots(z-n) ,$$

may serve as a first example. They form an orthonormal set in the space L^2_μ where μ is given by

$$\mathrm{d}\mu(x,y) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \frac{|\Gamma(\frac{n+1}{2}+\mathrm{i}y)|^2}{\Gamma(n+1)} \, \mathrm{d}y \qquad (z = x + \mathrm{i}y) ,$$

see 4.5.7 for details.

In section 4.5 we will give many more examples of polynomials and their orthonormalizing measures as well as sequences of polynomials which do not have an om.

We now turn to our central question: Which sequences of polynomials admit an om?

1.1.2 Main Problem. Let $(P_n)_{n\geq 0}$ be a sequence of polynomials in $\mathbb{C}[z]$ satisfying $P_0 \equiv 1$ and $\deg(P_n) = n$ for all n.

- Is there an om for $(P_n)_n$?
- If there exists an om, is it unique ?
- If μ is an om for $(P_n)_n$, is P^2_μ a *proper* subspace of L^2_μ ?

Without further assumptions on the given polynomials, these questions can not be answered. It is our aim to state necessary and/or sufficient criteria for the existence or uniqueness of orthonormalizing measures as well as properties of these measures which can be deduced from certain qualities of the P_n.

The following is closely related to 1.1.2.

1.1.3 The Complex Moment Problem. Let $(s_{ij})_{i,j\in\mathbb{N}_0}$ be a doubly indexed sequence of complex numbers. Is there a measure μ on $\mathfrak{B}(\mathbb{C})$ such that $\int \overline{z}^i \, z^j \, \mathrm{d}\mu = s_{ij}$ for all i, j ? If such a measure exists, $(s_{ij})_{i,j\in\mathbb{N}_0}$ is called a **complex moment sequence**.

This generalizes the **Hamburger moment problem** where, given a sequence $(m_n)_{n\geq 0}$ of real numbers, one asks for a measure μ supported on a subset of $\mathbb{R}$ such that $\int x^n \, \mathrm{d}\mu = m_n$ for all n.

For further reference on the complex moment problem we mention two papers by Kilpi [Ki1] and [Ki2] as well as [StSz2]. Treatises on the Hamburger moment problem can be found in [Ak] and [ShTa]; see also [BCh] or [BD], for example.

1.1.4 Proposition. *Let $(P_n)_n$ be a sequence of polynomials as in* 1.1.1*. Then there are unique coefficients $a_{ij} \in \mathbb{C}$ such that $z^n = \sum_{k=0}^{n} a_{kn} P_k$ for all $n \in \mathbb{N}_0$. Furthermore, set*

$$s_{ij} := \sum_{k=0}^{i} \overline{a_{ki}}\, a_{kj}\ .$$

Then $(s_{ij})_{i,j}$ is a complex moment sequence associated to μ if and only if μ is an om for $(P_n)_n$.

The proof is a simple calculation and can be found in [Kl2, 1.2.2], for instance.

1.2 Hessenberg Operators

In the following, $(P_n)_{n\geq 0}$ will always be a sequence of polynomials as in 1.1.1. These polynomials form a vector space basis of $\mathbb{C}[z]$ and there is a uniquely defined inner product on $\mathbb{C}[z]$ such that $\langle\, P_n\,, P_m\,\rangle = \delta_{n,m}$.
Let now $\mathcal{H}$ denote the abstract completion of $\mathbb{C}[z]$ with respect to the norm induced by this inner product. Then, by construction, $(P_n)_{n\geq 0}$ is an orthonormal basis (onb) of the Hilbert space $\mathcal{H}$ and if an om μ exists then $\mathcal{H}$ is isometrically isomorphic to P^2_μ via $P_n \mapsto P_n$.

The operator D defined below has become a standard tool, see e.g. [Sz4] or [Kl2], to deal with the question on exsistence of orthonormalizing measures for a given sequence of polynomials.

1.2.1 Definition. Let $D : \mathbb{C}[z] \to \mathbb{C}[z]$ be the multiplication operator defined by $(Dp)(z) := z\,p(z)$.

We point out, that D is a densely defined linear operator in $\mathcal{H}$ with invariant domain $\mathrm{dom}(D) = \mathbb{C}[z]$. Moreover, D is cyclic, i.e. the linear span of $\{D^n P_0 : n \in \mathbb{N}_0\}$ (which is equal to $\mathbb{C}[z]$, of course), is dense in $\mathcal{H}$.
As, for all $n \in \mathbb{N}_0$, DP_n is a polynomial of degree $n+1$, there exist $d_{0n}, \dots, d_{n+1,n} \in \mathbb{C}$ such that

$$DP_n = \sum_{i=0}^{n+1} d_{in}\, P_i \tag{1.1}$$

and, with respect to the onb $(P_n)_n$ of $\mathcal{H}$, D is represented by the matrix

$$(d_{ij})_{i,j\geq 0} = \begin{pmatrix} d_{00} & d_{01} & d_{02} & \cdots \\ d_{10} & d_{11} & d_{12} & \cdots \\ 0 & d_{21} & d_{22} & \cdots \\ 0 & 0 & d_{32} & \cdots \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $d_{n+1,n} \neq 0$ for all n.

Note that $d_{ij} = 0$ if $i > j+1$. A matrix with this property is called (upper) **Hessenberg matrix**. Accordingly, we say that D is a **Hessenberg operator**.
Note also that, in terms of this matrix, the elements of $\mathbb{C}[z]$ correspond to the vectors having only finitely many non-zero entries. However, the formal multiplication with this matrix might have a larger domain. In 3.2.7 we will examine the matrix representation of D more closely.

1.2.2 Example (Unilateral Shift). Let $P_n(z) := z^n$ for $n \in \mathbb{N}_0$. Then,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{P_n(\mathrm{e}^{\mathrm{i}t})}\, P_m(\mathrm{e}^{\mathrm{i}t})\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(m-n)t}\ \mathrm{d}t = \delta_{n,m}\,.$$

Hence, $\mu := \frac{1}{2\pi}\lambda$, where λ is the one-dimensional Lebesgue measure on the unit circle, is an om for $(P_n)_n$. The Polynomials $(P_n)_n$, however, do not form an orthonormal basis of L^2_μ. An onb is given, for example, by $(z^n)_{n\in\mathbb{Z}}$. This shows that P^2_μ is a proper subspace of L^2_μ.
Here we have $DP_n = P_{n+1}$ for all n, i.e. $d_{n+1,n} = 1$ and $d_{ij} = 0$ if $i \neq j+1$. Thus D is the unilateral shift on $\mathbb{C}[z]$ with respect to the basis $(P_n)_n$. As D is continuous, it has a continuous extension on P^2_μ and, as we shall see, a uniquely determined normal extension on the entire space L^2_μ. Clearly, this extension is the multiplication by z on L^2_μ. We will return to this example in 1.3.8.

Note that, in general, D may be unbounded.

1.3 Subnormal Operators

Concerning subnormality we start with some definitions. For further reference on linear operators in Hilbert space see e.g. [AG], [EE], and [W], or textbooks on functional analysis such as [Ru2].

1.3.1 Definition. Let T be a (not necessarily bounded) densely defined linear operator in a Hilbert space $\mathcal{H}$ and T^* its adjoint.

We say that the operator T is **hyponormal** if $\operatorname{dom}(T) \subset \operatorname{dom}(T^*)$ and $\|T^*x\| \leq \|Tx\|$ for all $x \in \operatorname{dom}(T)$.
A hyponormal operator T is **formally normal** if $\|T^*x\| = \|Tx\|$ for all $x \in \operatorname{dom}(T)$.
An operator T is **normal** if it is formally normal and $\operatorname{dom}(T) = \operatorname{dom}(T^*)$.
Furthermore, T is **subnormal** if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a linear subspace and a normal operator N in $\mathcal{K}$ such that $\operatorname{dom}(T) \subset \operatorname{dom}(N)$ and $Tx = Nx$ for all $x \in \operatorname{dom}(T)$. The operator N is then called a **normal extension** of T.
Finally, T is **essentially normal** if T is closable and its closure $\overline{T}$ is normal.

1.3.2 Proposition. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\varphi : \Omega \to \mathbb{C}$ measurable. Set $B := \{f \in L^2_\mu : \varphi \cdot f \in L^2_\mu\}$ and define $M_\varphi : B \to L^2_\mu$, $Mf := \varphi \cdot f$.*
Then M_φ is a normal operator in L^2_μ with adjoint $M^*_\varphi = M_{\overline{\varphi}}$ (the bar denoting complex conjugation).

Proof: Following [W, 4.1 Beispiel 1], we show that M_φ is densely defined to ensure that M^*_φ exists. For $n \in \mathbb{N}$ set $X_n := \{x \in \Omega : |\varphi(x)| \leq n\}$. Then, for all n, $X_n \in \mathcal{A}$, $X_n \subset X_{n+1}$, and $\bigcup_{n=1}^\infty X_n = \Omega$. Now take $f \in L^2_\mu$ and define $f_n := \mathbf{1}_{X_n} f$ for $n \in \mathbb{N}$, where $\mathbf{1}_X$ denotes the indicator function of the set X. Then $f_n \in \operatorname{dom}(M_\varphi) = B$ for all n and $f_n \to f$ in L^2_μ as $n \to \infty$.
As f was arbitrarily chosen, we see that $\operatorname{dom}(M_\varphi)$ is dense in L^2_μ.

Obviously, $\operatorname{dom}(M_\varphi) = \operatorname{dom}(M_{\overline{\varphi}})$ and $\|M_\varphi f\| = \|M_{\overline{\varphi}} f\|$ for all $f \in \operatorname{dom}(M_\varphi)$, as well as $\int \overline{f}\, \varphi g \,\mathrm{d}\mu = \int \overline{\overline{\varphi} f}\, g \,\mathrm{d}\mu$ for all $f, g \in \operatorname{dom}(M_\varphi)$, hence M_φ is normal and $M^*_\varphi = M_{\overline{\varphi}}$. $\square$

If μ is a probability measure on $\mathfrak{B}(\mathbb{C})$ and $\mathbb{C}[z] \subset L^2_\mu$, we write M_μ for the operator M_φ in L^2_μ where $\varphi(z) := z$.
Note that $\operatorname{dom}(M_\mu)$ for such μ always contains $\mathbb{C}[z]$. Moreover, $\mathbb{C}[z]$ is an invariant subspace of M_μ, i.e. $M_\mu p \in \mathbb{C}[z]$ for all $p \in \mathbb{C}[z]$.

Let now $(P_n)_{n \geq 0}$ be a sequence of polynomials as in 1.1.1. If there exists an om μ, we regard D as an operator in L^2_μ. Then M_μ is a normal extension of D. In other words, we have just proved the following.

1.3.3 Corollary. *If there exists an om μ for $(P_n)_{n \geq 0}$, then D is subnormal.*

To prove that the contrary is also true we will make use of the spectral theorem for normal operators, see A.1.3 in the appendix. The correspondence between the existence of orthonormalizing measures on the one hand and subnormality of the multiplication operator D on the other hand is also the main topic of [Kl2], [CaKl1], and [CaKl2]. The theory of subnormal operators and their normal extensions has been vastly treated in a series of three papers by Stochel and Szafraniec [StSz1], [StSz2], [StSz3] and a famous textbook by Conway [Con1] presents the theory of bounded subnormal operators. A proof of the next theorem can be found e.g. in [Kl2, 1.3.3]. For bounded operators, see also [Con1, II.§5]. We will present a proof in A.2.1 in the appendix.

1.3.4 Theorem. *For $(P_n)_{n\geq 0}$ as in* 1.1.1 *define $\mathcal{H}$ and D as before. Assume that there exists a normal extension N of D in a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and denote by $\sigma(N)$ the spectrum of N and by E the spectral measure of N.*

Then the measure μ on $\mathfrak{B}(\mathbb{C})$ given by $\mu(\Delta) := \langle P_0 , E(\Delta)P_0 \rangle$ is an om and $\operatorname{supp}(\mu) \subset \sigma(N)$. *Furthermore, L^2_μ can be embedded isometrically into $\mathcal{K}$ such that $D \subset M_\mu \subset N$ and M_μ is a normal extension of D (in L^2_μ) with* $\operatorname{supp}(\mu) = \sigma(M_\mu)$.

After introducing two more definitions we can proceed to the main statement of this section.

1.3.5 Definition. Let T be an operator in a Hilbert space $\mathcal{K}$ and $\mathcal{H}$ a closed subspace of $\mathcal{K}$. Denote by P the projection from $\mathcal{K}$ onto $\mathcal{H}$. We say that $\mathcal{H}$ **reduces** T if $PT \subset TP$, i.e. for all $x \in \operatorname{dom}(T)$, we have $Px \in \operatorname{dom}(T)$ and $PTx = TPx$.

1.3.6 Definition. Let T be a subnormal operator in a Hilbert space $\mathcal{H}$ and N a normal extension of T in a Hilbert space $\mathcal{K}$. The operator N is called a **minimal normal extension** of T, if the only closed subspace of $\mathcal{K}$ containing $\mathcal{H}$ and reducing N is the space $\mathcal{K}$ itself.

Remark. Stochel and Szafraniec [StSz3] call this **minimal normal extension of spectral type**. In [StSz3, Proposition 1] they show that N is a minimal normal extension of spectral type of T if and only if $\{E(\Delta)x : x \in \mathcal{H}, \Delta \in \mathfrak{B}(\mathbb{C})\}$ is dense in $\mathcal{K}$ where E denotes the spectral measure of N. This justifies their name for this property. As, for our purposes, we do not need another kind of minimality, we will keep with the shorter.

Now we can state the main result concerning the existence of orthonormalizing measures in terms of the theory of subnormal operators: There is a bijective correspondence between the orthonormalizing measures for $(P_n)_n$ and – up to unitary equivalence – the minimal normal extensions of the Hessenberg operator D. More precisely, the following holds.

1.3.7 Theorem. *If, in the situation of* 1.3.4, *N is a minimal normal extension of D then there exists an isomorphism $\beta : L^2_\mu \to \mathcal{K}$ such that $\beta(p) = p$ for all $p \in \mathbb{C}[z]$ and $\beta^{-1}N\beta = M_\mu$.*

If, on the other hand, μ is an om for $(P_n)_n$ then M_μ is a minimal normal extension of D. In particular, if D is subnormal then there always exists a minimal normal extension.

Furthermore, if D is essentially normal then its only minimal normal extension is $\overline{D}$, there exists a unique om μ, and $(P_n)_n$ is an orthonormal basis of L^2_μ.

For a proof, see A.2.4 in the appendix.

We point out that there need not be a unique minimal extension. It may happen that there exist “different” normal extensions $N_\mathcal{K}$ and $N_\mathcal{L}$ of D in spaces $\mathcal{K}$ and $\mathcal{L}$, respectively, such that there is no isomorphism $\beta : \mathcal{K} \to \mathcal{L}$ which is the identity on $\mathcal{H}$ satisfying $\beta^{-1} N_{\mathcal{K}} \beta = N_{\mathcal{L}}$.

Unfortunately, in general, one does not know if a linear operator is subnormal or not while, as mentioned before, in the case of a symmetric operator, we can give an answer. For the theory of self-adjoint extensions of symmetric operators, see [AG, Kap. VIII], [Ru2, 13.20], or [W, Kap. 8], for example. Suppose now that D is symmetric. Then the matrix representation of D is tri-diagonal with non-vanishing conjugate entries in the upper and lower diagonal and reals in the main diagonal; such operators are commonly called **Jacobi operators**. If such D is subnormal then its normal extensions must be self-adjoint, hence their spectral measures and, therefore, also the corresponding orthonormalizing measures are supported in $\mathbb{R}$.

It is a well known fact, that D has a self-adjoint extension in $\mathcal{H}$ if and only if the deficiency indices $\dim(\operatorname{ran}(D \pm \mathrm{i}\cdot\mathrm{id})^{\perp})$ are equal. In 3.1.3 we will be able to compute these deficiency indices using the given polynomials $(P_n)_n$.

The theory of real orthogonal ploynomials, like the theory of self-adjoint extensions, has been widely examined. As standard references see the textbooks by Szegő [Sze] and Chihara [Chi]. We will deal with this situation in section 4.1. Many properties of real orthogonal polynomials, however, do not even have an equivalent in our more general problem. In particular, formally normal or hyponormal operators will not be an adequate counterpart to symmetric operators in the real case. As a matter of fact, every subnormal operator is hyponormal, see [J, I. Example 4] or [OSch][2], while there exist hyponormal and even formally normal operators that are not subnormal. Examples to the latter can be found in a short note by Schmüdgen [Sch] and in [Cod]. We now proceed to a subnormal operator which is not formally normal.

1.3.8 Unilateral Shift (continued). The best-known example of a subnormal operator which does not have a normal extension in the same space, is the unilateral shift in $\ell^2(\mathbb{N}_0)$. Let $\{e_0, e_1, \ldots\}$ be the standard basis of $\ell^2(\mathbb{N}_0)$ and define $S : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$ for $i \in \mathbb{N}_0$. It is not difficult to see that its adjoint is given by $S^* e_i = e_{i-1}$ for $i \geq 1$, $S^* e_0 = 0$, and hence S is not normal and not formally normal, either.

However, S is subnormal. To see that, we understand $\ell^2(\mathbb{N}_0)$ as a linear subspace of $\ell^2(\mathbb{Z})$ via the canonical embedding. Denote the standard basis of $\ell^2(\mathbb{Z})$ by $\{e_i : i \in \mathbb{Z}\}$ and define $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $Te_i := e_{i+1}$ for $i \in \mathbb{Z}$.

One can easily see that T^* is given by $T^* e_i = e_{i-1}$ for $i \in \mathbb{Z}$ and that $\operatorname{dom}(T) = \operatorname{dom}(T^*) = \ell^2(\mathbb{Z})$ with $\|Tx\| = \|T^*x\|$ for all $x \in \ell^2(\mathbb{Z})$. Thus T is normal and extends S.

Note that T^* is not an extension of S^*.

Let us now return to 1.2.2, where $P_n(z) = z^n$ for $n \in \mathbb{N}_0$. If we identify P_μ^2 with $\ell^2(\mathbb{N}_0)$ via $P_n \mapsto e_n$, then S and also T extend the Hessenberg operator D. Hence D is subnormal. Moreover, we can identify L_μ^2 and $\ell^2(\mathbb{Z})$ via $z^n \mapsto e_n$, $n \in \mathbb{Z}$, and see that $T = M_\mu$ is a minimal normal extension of D.

[2]Ôta and Schmüdgen [OSch] use the terms *hyponormal* for what we call *subnormal* and *formally hyponormal* for what we call *hyponormal*.

1.4 Reproducing Kernel Hilbert Spaces (RKHS)

Assume that μ is an om for a sequence of polynomials $(P_n)_{n\geq 0}$ as in 1.1.1 and that there exists $z \in \mathbb{C}$ such that $\sum_{n\geq 0} |P_n(z)|^2 < \infty$.

Then $k_z := \sum_{n\geq 0} \overline{P_n(z)}\, P_n$ is well-defined and $\langle\, k_z \,, P_m \,\rangle_{L^2_\mu} = P_m(z)$ for all m. By linearity,

$$\langle\, k_z \,, p \,\rangle_{L^2_\mu} = p(z) \qquad \text{for all } p \in \mathbb{C}[z]\,.$$

Recall that P^2_μ denotes the closure of $\mathbb{C}[z]$ in L^2_μ. For any $f \in P^2_\mu$, there exists a sequence $(q_n)_n$ in $\mathbb{C}[z]$ such that $q_n \to f$ with respect to the L^2_μ-norm and for every such sequence, $q_n(z) = \langle\, k_z \,, q_n \,\rangle_{L^2_\mu} \to \langle\, k_z \,, f \,\rangle$ as $n \to \infty$. Therefore, to $f \in P^2_\mu$ we can assign a function pointwisely defined on the set $E := \left\{ z \in \mathbb{C} : \sum_{n\geq 0} |P_n(z)|^2 < \infty \right\}$ in a canonical way.

This approach leads to a Reproducing Kernel Hilbert Space; we start with some definitions.

1.4.1 Definition. Let E be an arbitrary non-empty set and $\mathcal{F}(E)$ be the vector space of all functions $f : E \to \mathbb{C}$. Furthermore, let $\mathcal{H}(K)$ be a linear subspace of $\mathcal{F}(E)$ which is a Hilbert space with the following property: For every $z \in E$ there exists $K_z \in \mathcal{H}(K)$ such that $f(z) = \langle\, K_z \,, f \,\rangle$ for all $f \in \mathcal{H}(K)$.
Then $\mathcal{H}(K)$ is called a **Reproducing Kernel Hilbert Space** (RKHS) with **domain** E and $K : E \times E \to \mathbb{C}$, $K(z,w) := K_z(w) = \langle\, K_w \,, K_z \,\rangle$ is the **reproducing kernel** or just **kernel** of $\mathcal{H}(K)$. The functions K_z are also sometimes referred to as **kernel functions**.

Note that L^2-spaces cannot match this definiton when their elements are not functions but equivalence classes of functions.

Note also that K_z is unique: Take $h \in \mathcal{H}(K)$ such that $f(z) = \langle\, h \,, f \,\rangle$ for all $f \in \mathcal{H}(K)$, then $\langle\, K_z - h \,, f \,\rangle = 0$ for all $f \in \mathcal{H}(K)$ and hence $K_z - h = 0$.

Obviously, $K(z,z) = \langle\, K_z \,, K_z \,\rangle = \|K_z\|^2 \geq 0$ and $K(z,w) = \overline{K(w,z)}$ for all $z, w \in E$. Without loss of generality we can assume $K(z,z) > 0$ for all $z \in E$ because if $K(z_0, z_0) = 0$ for some $z_0 \in E$ then $K_{z_0} = 0$ in $\mathcal{H}(K)$ and $0 = \langle\, K_{z_0} \,, f \,\rangle = f(z)$ for all $z \in E$. In this case we can simply restrict to $E \setminus \{z_0\}$, see A.3.6 in the appendix for a more detailed discussion.

Moreover, $\langle\, K_z \,, f \,\rangle = 0$ for all $z \in E$ is equivalent to $f = 0$; hence $\{K_z : z \in E\}$ is total in $\mathcal{H}(K)$.

1.4.2 Example. For any non-empty set E, the space $\ell^2(E)$ is an RKHS with kernel $K(z,w) = \delta_{z,w}$ because, for $\alpha = (\alpha_\lambda)_{\lambda\in E} \in \ell^2(E)$ and $z \in E$, we have

$$\alpha_z = \sum_{\lambda\in E} \delta_{z,\lambda}\, \alpha_\lambda = \langle\, K_z \,, \alpha \,\rangle\,.$$

In particular, as $\dim \ell^2(E) = |E|$, an RKHS need not be separable.

A fundamental work concerning RKHS is the survey by Aronszajn [Ar]. An introduction to this topic can also be found in [Me] as well as in [Do, Chapter X]. We will state the most important properties of RKHS here, too, including some very straightforward proofs.

1.4.3 Lemma. *A Hilbert space $\mathcal{H} \subset \mathcal{F}(E)$ is an RKHS if and only if point evaluation $f \mapsto f(z)$ is a continuous linear functional for every $z \in E$.*

Proof: Obviously, if $\mathcal{H}$ is an RKHS with kernel K and domain E then, for fixed $z \in E$, the mapping $f \mapsto f(z) = \langle K_z , f \rangle$ is continuous and linear.

Conversely, let now $f \mapsto f(z)$ be continuous. According to the Riesz representation theorem, there exists $K_z \in \mathcal{H}$ such that $\langle K_z , f \rangle = f(z)$ for all $f \in \mathcal{H}$. $\square$

Another important property of RKHS is that convergence in the norm always implies pointwise convergence.

1.4.4 Lemma. *Let $(f_n)_n$ be a sequence in $\mathcal{H}(K)$ which is weakly convergent to f, say, i.e. $\lim_{n\to\infty} \langle h , f_n \rangle = \langle h , f \rangle$ for every $h \in \mathcal{H}(K)$. Then $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in E$.*
Moreover, if $f_n \to f$ (in the norm) as $n \to \infty$ then $(f_n)_n$ is uniformly convergent in every subset of E where $K(z,z)$ is bounded.

Proof: The first assertion follows immediately from $g(z) = \langle K_z , g \rangle$ for all $g \in \mathcal{H}(K)$.
Let now $\|f_n - f\| \to 0$ as $n \to \infty$ and $c > 0$, $A \subset E$ such that $K(z,z) \le c$ for all $z \in A$.
Cauchy-Schwarz yields

$$|f_n(z) - f(z)| = |\langle K_z , f_n - f \rangle| \le \|K_z\| \; \|f_n - f\| \le c^{\frac{1}{2}} \; \|f_n - f\|$$

for all $z \in A$. Hence $f_n \to f$ uniformly on A. $\square$

In the setting of the above definition, for arbitrary finite subsets $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$, we have

$$0 \le \left\langle \sum_{i=1}^{n} \overline{c_i}\, K_{z_i} , \sum_{i=1}^{n} \overline{c_i}\, K_{z_i} \right\rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} c_j\, \overline{c_k}\, \langle K_{z_j} , K_{z_k} \rangle = \sum_{k=1}^{n} \sum_{j=1}^{n} \overline{c_k}\, c_j\, K(z_k, z_j)\,. \tag{1.2}$$

The converse is also true:

1.4.5 Proposition. *Let E be an arbitrary set and $K : E \times E \to \mathbb{C}$ a map such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i}\, c_j\, K(z_i, z_j) \ge 0$$

for all finite sets $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$.
Then K is the kernel function of an RKHS. More precisely, there is a unique linear subspace V of $\mathcal{F}(E)$ such that $\mathcal{H}(K) = V$.

We will provide a proof in the appendix, see A.3.1.

Remark. In other words, the premises of this theorem read *for any* $\{z_1, \ldots, z_n\} \subset E$ *the matrix* $(k_{ij})_{1\le i,j\le n}$ *with* $k_{ij} := K(z_i, z_j)$ *is positive semi-definite.* Therefore, K is sometimes, as in [Do] or [Me, Kap. V], referred to as **positive matrix**.

1.4.6 Lemma. *Let $\mathcal{H}(K)$ be an RKHS with domain E. If $(h_\iota)_{\iota\in I}$ is an onb in $\mathcal{H}(K)$ then $K(z,w) = \sum\limits_{\iota\in I} \overline{h_\iota(z)} h_\iota(w)$ for all $z, w \in E$.*

Proof: Using the Parseval equation, we see

$$K(z,w) = \langle K_w \,, K_z \rangle = \sum_{\iota\in I} \langle K_w \,, h_\iota \rangle \, \langle h_\iota \,, K_z \rangle = \sum_{\iota\in I} h_\iota(w)\, \overline{h_\iota(z)}\,. \qquad \square$$

For $(P_n)_{n\ge 0}$ as in 1.1.1 we aim to construct an RKHS such that $(P_n)_n$ is an onb. As we have just seen, its kernel must be of the form $K(z,w) = \sum\limits_{n\ge 0} \overline{P_n(z)} P_n(w)$ for $z, w \in E$ with suitable $E \subset \mathbb{C}$.
We will deal with this topic in the next chapter.

As for now, we conclude this chapter with a necessary and sufficient criterion for a function to be a member of a given RKHS. For a proof see A.3.4 in the appendix and also [Sz1] or [Sz2].

1.4.7 RKHS-Test. *Let $\mathcal{H}(K)$ be an RKHS with domain E and f a complex-valued function on E. Then $f \in \mathcal{H}(K)$ if and only if there exists $r > 0$ such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i} c_j \, K(z_i, z_j) \ge r \left| \sum_{i=1}^{n} c_i \, f(z_i) \right|^2$$

for all finite sets $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Note that if this holds for some $r_0 > 0$ then it holds for all $0 < r < r_0$.

2 Orthogonality in RKHS

Given a sequence of polynomials $(P_n)_{n\geq 0}$ as in 1.1.1, we will in this chapter state necessary and sufficient conditions for the existence of an RKHS such that $(P_n)_n$ is an orthonormal basis (onb) in this space. For $E := \left\{z \in \mathbb{C} : \sum\limits_{n\geq 0} |P_n(z)|^2 < \infty\right\}$ define

$$K : E \times E \to \mathbb{C}\,, \qquad K(z,w) := \sum_{n\geq 0} \overline{P_n(z)} P_n(w)\,.$$

As we shall see, K is the kernel of some RKHS but $(P_n)_n$ need not be an onb of this space.

2.1 Series Expansions of Kernel Functions

We now construct an RKHS from a (pointwisely on a set E) convergent series of functions. The following theorem contains [Sz3, Proposition 3.1].

2.1.1 Theorem. *Let $(h_\iota)_{\iota\in I}$, I an index set, be a family of complex-valued functions defined on a set E such that $\sum\limits_{\iota\in I} |h_\iota(z)|^2 < \infty$ for all $z \in E$. Then*

$$K : E \times E \to \mathbb{C}\,, \qquad K(z,w) := \sum_{\iota\in I} \overline{h_\iota(z)} h_\iota(w)$$

is the kernel of a reproducing kernel Hilbert space $\mathcal{H}(K)$. For any $\alpha = (\alpha_\iota)_\iota \in \ell^2(I)$,

$$\beta(\alpha) := \sum_{\iota\in I} \alpha_\iota\, h_\iota$$

is summable in $\mathcal{H}(K)$ and defines a continuous linear map $\beta : \ell^2(I) \to \mathcal{H}(K)$ with $\|\beta\| \leq 1$. Moreover, $\{h_\iota : \iota \in I\}$ is total in $\mathcal{H}(K)$.

Proof: For $z \in E$ set $h(z) := (h_\iota(z))_{\iota\in I}$. Now take $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$. Then

$$\sum_{i=1}^n \sum_{j=1}^n \overline{c_i} c_j\, K(z_i, z_j) = \sum_{i=1}^n \sum_{j=1}^n \overline{c_i} c_j \langle\, h(z_i)\,, h(z_j)\,\rangle_{\ell^2(I)} = \Big\| \sum_{i=1}^n c_i\, h(z_i) \Big\|^2_{\ell^2(I)} \geq 0 \qquad (2.1)$$

and, according to 1.4.5, $\mathcal{H}(K)$ exists.

For $\alpha = (\alpha_\iota)_\iota \in \ell^2(I)$, define $f : E \to \mathbb{C}$, $f(z) := \sum_{\iota\in I} \alpha_\iota h_\iota(z) = \langle\, \overline{\alpha}\,, h(z)\,\rangle_{\ell^2(I)}$.

In order to show $f \in \mathcal{H}(K)$, we use Cauchy-Schwarz and (2.1) to obtain

$$\begin{aligned}\Big|\sum_{i=1}^n c_i\, f(z_i)\Big|^2 &= \Big|\Big\langle\, \overline{\alpha}\,, \sum_{i=1}^n c_i\, h(z_i)\,\Big\rangle_{\ell^2(I)}\Big|^2 \\ &\le \|\alpha\|^2_{\ell^2(I)} \cdot \Big\|\sum_{i=1}^n c_i\, h(z_i)\Big\|^2_{\ell^2(I)} = \|\alpha\|^2_{\ell^2(I)} \sum_{i=1}^n \sum_{j=1}^n \overline{c_i} c_j\, K(z_i, z_j)\,.\end{aligned}$$

Thus the RKHS-test 1.4.7 implies $f \in \mathcal{H}(K)$.

We next show that $\sum_{\iota\in I} \alpha_\iota h_\iota$ is summable in $\mathcal{H}(K)$ with sum f.

Let $\mathcal{H}_0 := \operatorname{lin}\{K_z : z \in E\}$ and $u = \sum_{i=1}^n b_i K_{z_i}$, see also (A.3) in the appendix. Then

$$\big|\langle\, u\,, f\,\rangle_{\mathcal{H}(K)}\big|^2 = \Big|\sum_{i=1}^n \overline{b_i}\; f(z_i)\Big|^2 \le \|\alpha\|^2_{\ell^2(I)} \sum_{i=1}^n \sum_{j=1}^n \overline{b_i} b_j\, K(z_i, z_j) = \|\alpha\|^2_{\ell^2(I)} \cdot \|u\|^2_{\mathcal{H}(K)}\;.$$

According to 1.4.1, $\mathcal{H}_0$ is dense in $\mathcal{H}(K)$; this yields $|\langle\, g\,, f\,\rangle_{\mathcal{H}(K)}|^2 \le \|\alpha\|^2_{\ell^2(I)} \cdot \|g\|^2_{\mathcal{H}(K)}$ for all $g \in \mathcal{H}(K)$.

In particular, $\|f\|^4_{\mathcal{H}(K)} \le \|\alpha\|^2_{\ell^2(I)} \cdot \|f\|^2_{\mathcal{H}(K)}$. Thus $\|f\|_{\mathcal{H}(K)} \le \|\alpha\|_{\ell^2(I)}$.

Therefore, the linear map

$$\beta : \ell^2(I) \to \mathcal{H}(K)\,, \qquad \beta(\alpha)(z) := \sum_{\iota\in I} \alpha_\iota\, h_\iota(z)$$

(pointwise limit for $z \in E$) is well-defined and continuous with $\|\beta\| \le 1$.

Let now $\varepsilon > 0$. There exists a finite set $J_\varepsilon \subset I$ such that, for every finite $J \subset I$ with $J_\varepsilon \subset J$, the sequence $\widetilde{\alpha} \in \ell^2(I)$ given by $\widetilde{\alpha}_\iota = \alpha_\iota$ for $\iota \in J$, and $\widetilde{\alpha}_\iota = 0$ otherwise, satisfies $\|\alpha - \widetilde{\alpha}\|_{\ell^2(I)} < \varepsilon$. Using continuity of β, we get $\|f - \sum_{\iota\in J} \alpha_\iota\, h_\iota\|_{\mathcal{H}(K)} < \|\beta\| \cdot \varepsilon \le \varepsilon$. Thus $f = \sum_{\iota\in I} \alpha_\iota\, h_\iota$ in $\mathcal{H}(K)$.

In particular, $\sum_{\iota\in I} \overline{h_\iota(z)}\, h_\iota = K_z$ for $z \in E$ and

$$f(z) = \langle\, K_z\,, f\,\rangle_{\mathcal{H}(K)} = \Big\langle \sum_{\iota\in I} \overline{h_\iota(z)}\, h_\iota\,, f\,\Big\rangle_{\mathcal{H}(K)} = \sum_{\iota\in I} h_\iota(z) \langle\, h_\iota\,, f\,\rangle_{\mathcal{H}(K)} \tag{2.2}$$

for all $f \in \mathcal{H}(K)$.

For the last assertion take $f \in \{h_\iota : \iota \in I\}^\perp$. Then, by (2.2), $f(z) = 0$ for all $z \in E$, i.e. $f = 0$, which proves that $\{h_\iota : \iota \in I\}$ is total in $\mathcal{H}(K)$. $\square$

Note that, as β need not be one-to-one, in general $\alpha_\iota \ne \langle\, h_\iota\,, f\,\rangle_{\mathcal{H}(K)}$.

In 1.4.6 we have seen that the kernel always has a series expansion with respect to an onb. We will now turn to the question whether $(h_\iota)_{\iota\in I}$ is an onb of $\mathcal{H}(K)$.

2.1.2 Lemma. *In the situation of* 2.1.1, $(h_\iota)_{\iota\in I}$ *is an onb of* $\mathcal{H}(K)$ *if and only if* β *is an isomorphism.*

Proof: Let $(e_\kappa)_{\kappa\in I}$ be the standard onb of $\ell^2(I)$, i.e. $[e_\kappa]_\iota = \delta_{\iota\kappa}$. Then $\beta(e_\kappa) = h_\kappa$ for all $\kappa \in I$ which shows that $(h_\kappa)_{\kappa\in I}$ is an onb if and only if β is an isomorphism. □

Recall the definition $\mathcal{H}_0 := \operatorname{lin}\{K_z : z \in E\}$, see also (A.3). Set $W_0 := \operatorname{lin}\{\overline{h(z)} : z \in E\}$, where $h(z) = (h_\iota(z))_{\iota\in I}$ and the bar denotes complex conjugation. Then W_0 is a linear subspace of $\ell^2(I)$ and $\beta\big(\overline{h(z)}\big) = \sum_{\iota\in I} \overline{h_\iota(z)}\, h_\iota = K_z$ for all $z \in E$. Thus $\beta(W_0) = \mathcal{H}_0$.

Furthermore, let W be the closure of W_0 in $\ell^2(I)$. Then $\ell^2(I) = W \oplus W^\perp$.

2.1.3 Lemma. *Using the above notation,* $\beta : \ell^2(I) \to \mathcal{H}(K)$ *is a partial isometry with* $\mathcal{N}(\beta) = W^\perp$.

Proof: For $\{z_1, \dots, z_n\} \subset E$ and $c_1, \dots, c_n \in \mathbb{C}$ we have

$$\Big\|\beta\Big(\sum_{i=1}^n c_i\overline{h(z_i)}\Big)\Big\|^2_{\mathcal{H}(K)} = \Big\|\sum_{i=1}^n c_i\, \beta\big(\overline{h(z_i)}\big)\Big\|^2_{\mathcal{H}(K)}$$
$$= \Big\|\sum_{i=1}^n c_i\, K_{z_i}\Big\|^2_{\mathcal{H}(K)} \overset{(1.2)}{=} \sum_{i=1}^n \sum_{j=1}^n c_i\, \overline{c_j}\, K(z_i, z_j) \overset{(2.1)}{=} \Big\|\sum_{i=1}^n \overline{c_i}\, h(z_i)\Big\|^2_{\ell^2(I)} .$$

Hence $\|\beta(\alpha)\|_{\mathcal{H}(K)} = \|\alpha\|_{\ell^2(I)}$ for all $\alpha \in W_0$.

As β is continuous, it isometrically maps W onto the closure of $\mathcal{H}_0$ in $\mathcal{H}(K)$, which is $\mathcal{H}(K)$ itself.

Finally, if $\alpha \in W^\perp$ then

$$0 = \langle\, \overline{h(z)}\,, \alpha\, \rangle_{\ell^2(I)} = \sum_{\iota\in I} \alpha_\iota\, h_\iota(z) = \beta(\alpha)(z) \qquad \text{for all } z \in E\,.$$

Thus $\beta(\alpha) = 0$. □

Combining 2.1.2 and 2.1.3, we have the following necessary and sufficient criterion for $(h_\iota)_{\iota\in I}$ being an onb of $\mathcal{H}(K)$.

2.1.4 Corollary. *In the situation of* 2.1.1, $(h_\iota)_{\iota\in I}$ *is an onb of* $\mathcal{H}(K)$ *if and only if* $V := \big\{\big(\overline{h_\iota(z)}\big)_{\iota\in I} : z \in E\big\}$ *is total in* $\ell^2(I)$.

Proof: According to 2.1.2, $(h_\iota)_{\iota\in I}$ is an onb of $\mathcal{H}(K)$ if and only if β is an isomorphism. The partial isometry β is an isomorphism if and only if its null space is trivial, i.e. $W^\perp = \{0\}$. The latter is equivalent to the totality of V in $\ell^2(I)$ because W is the closure of the linear span of V. □

Recall that, for every $\kappa \in I$, we have $\beta(e_\kappa) = h_\kappa$, where $\{e_\iota : \iota \in I\}$ denotes the standard unit vectors of $\ell^2(I)$. If $(h_\iota)_\iota$ is an onb in $\mathcal{H}(K)$ then β is an isomorphism $\ell^2(I) \to \mathcal{H}(K)$. If, on the other hand, $\langle h_\kappa , h_\iota \rangle_{\mathcal{H}(K)} \neq 0$ for some $\iota \neq \kappa \in I$, then at least one of the vectors e_ι, e_κ must not belong to W. Analogously, if $\|h_\iota\|_{\mathcal{H}(K)} \neq 1$ for some ι then β is not an isomorphism and $W \neq \ell^2(I)$.

For $\iota \in I$ there exists a unique $b^{(\iota)} \in W$ such that $\beta(b^{(\iota)}) = h_\iota$ and, as $\beta(b^{(\iota)} - e_\iota) = h_\iota - h_\iota = 0$, we have $b^{(\iota)} - e_\iota \in W^\perp$.

Now, for arbitrary $f \in \mathcal{H}(K)$, let $a = (a_\iota)_\iota \in W$ such that $\beta(a) = f$. Then

$$\langle h_\iota , f \rangle_{\mathcal{H}(K)} = \langle b^{(\iota)} , a \rangle_{\ell^2(I)} = \langle b^{(\iota)} - e_\iota , a \rangle_{\ell^2(I)} + \langle e_\iota , a \rangle_{\ell^2(I)} .$$

Due to $b^{(\iota)} - e_\iota \in W^\perp$, the first summand vanishes, leaving $\langle h_\iota , f \rangle_{\mathcal{H}(K)} = \langle e_\iota , a \rangle_{\ell^2(I)} = a_\iota$ as well as $\beta\big([\langle h_\iota , f \rangle_{\mathcal{H}(K)}]_\iota\big) = f$. In particular, for $f = h_\kappa$, we get $\langle h_\iota , h_\kappa \rangle_{\mathcal{H}(K)} = b_\iota^{(\kappa)}$.

We have just proved the following lemma.

2.1.5 Lemma. *Let $f \in \mathcal{H}(K)$. Then $\big(\langle h_\iota , f \rangle_{\mathcal{H}(K)}\big)_\iota \in W$ and $f = \sum_{\iota \in I} \langle h_\iota , f \rangle_{\mathcal{H}(K)} h_\iota$ in $\mathcal{H}(K)$.*

Moreover, if $f = \sum_{\iota \in I} c_\iota h_\iota$ with $(c_\iota)_\iota \in W$ then $(c_\iota)_\iota = \big(\langle h_\iota , f \rangle_{\mathcal{H}(K)}\big)_\iota$.

Let B be the orthogonal projection in ℓ^2 onto W. Then $\beta = \beta \circ B$ and we have the following characterization of B.

2.1.6 Proposition. *For $x = (x_\iota)_\iota \in \ell^2(I)$, let $Bx := \left[\sum_{\iota \in I} \overline{b_\iota^{(\kappa)}}\, x_\iota \right]_\kappa$.*

Then $x \mapsto Bx$ is the orthogonal projection in $\ell^2(I)$ onto W.

Proof: We start with $x \in W^\perp$. As $b^{(\kappa)} \in W$, we have $\sum_{\iota \in I} \overline{b_\iota^{(\kappa)}}\, x_\iota = \langle b^{(\kappa)} , x \rangle_{\ell^2(I)} = 0$ for all $\kappa \in I$. Hence $Bx = 0$.

On the other hand, for $x \in W$, define $f := \beta(x) \in \mathcal{H}(K)$. Then $x_\iota = \langle h_\iota , f \rangle_{\mathcal{H}(K)}$ for $\iota \in I$ and

$$\sum_{\iota \in I} \overline{b_\iota^{(\kappa)}}\, x_\iota = \langle b^{(\kappa)} , x \rangle_{\ell^2(I)} = \langle h_\kappa , f \rangle_{\mathcal{H}(K)} = x_\kappa .$$

Thus $Bx = x$. □

Remark. Although $(h_\iota)_\iota$ need not be an onb in $\mathcal{H}(K)$, 2.1.5 shows that any $f \in \mathcal{H}(K)$ is represented by a Fourier-like series with respect to the sequence $(h_\iota)_\iota$. Such a sequence is commonly called a **Parseval frame**. For details on the theory of frames in Hilbert spaces we refer to [HL] or [Chr]. In view of this theory, our next theorem is a special case of [Chr, 5.4.7].

2.1.7 Theorem. *Let $\kappa \in I$. Then $\|h_\kappa\|_{\mathcal{H}(K)} \leq 1$ and*

$$\|h_\kappa\|_{\mathcal{H}(K)} = 1 \iff \langle h_\kappa , h_\iota \rangle_{\mathcal{H}(K)} = 0 \quad \textit{for all } \iota \neq \kappa \iff h_\kappa \notin \overline{\operatorname{lin}\{h_\iota : \iota \neq \kappa\}} .$$

Proof: As β is a partial isometry, $\|h_\kappa\|_{\mathcal{H}(K)} = \|\beta(e_\kappa)\|_{\mathcal{H}(K)} \le \|e_\kappa\|_{\ell^2(I)} = 1$.

Moreover, $\langle\, h_\kappa \,, h_\kappa \,\rangle_{\mathcal{H}(K)} = \|h_\kappa\|^2_{\mathcal{H}(K)} = \|b^{(\kappa)}\|^2_{\ell^2(I)} = \sum_{\iota\in I} \big|b_\iota^{(\kappa)}\big|^2 = \sum_{\iota\in I} \big|\langle\, h_\iota \,, h_\kappa \,\rangle_{\mathcal{H}(K)}\big|^2$.

Equivalently, $\langle\, h_\kappa \,, h_\kappa \,\rangle_{\mathcal{H}(K)}\big(1 - \langle\, h_\kappa \,, h_\kappa \,\rangle_{\mathcal{H}(K)}\big) = \sum_{\iota\neq\kappa} \big|\langle\, h_\iota \,, h_\kappa \,\rangle_{\mathcal{H}(K)}\big|^2$.

This shows $\langle\, h_\kappa \,, h_\kappa \,\rangle_{\mathcal{H}(K)} = 1 \iff \langle\, h_\iota \,, h_\kappa \,\rangle_{\mathcal{H}(K)} = 0$ for all $\iota \neq \kappa$.

Clearly, $\langle\, h_\iota \,, h_\kappa \,\rangle_{\mathcal{H}(K)} = 0$ for all $\iota \neq \kappa$ is equivalent to $h_\kappa \perp \overline{\mathrm{lin}\{h_\iota : \iota \neq \kappa\}}$, implying $h_\kappa \notin \overline{\mathrm{lin}\{h_\iota : \iota \neq \kappa\}}$.

For the converse, suppose $\langle\, h_\kappa \,, h_\kappa \,\rangle_{\mathcal{H}(K)} \neq 1$.

Then $\quad h_\kappa = \beta\big(b^{(\kappa)}\big) = \sum_{\iota\in I} b_\iota^{(\kappa)}\, h_\iota = \sum_{\iota\in I} \langle\, h_\iota \,, h_\kappa \,\rangle_{\mathcal{H}(K)}\, h_\iota \quad$ implies

$$h_\kappa = \frac{1}{1-\langle\, h_\kappa \,, h_\kappa \,\rangle_{\mathcal{H}(K)}} \sum_{\iota\neq\kappa} \langle\, h_\iota \,, h_\kappa \,\rangle_{\mathcal{H}(K)}\, h_\iota \,.$$

Hence $h_\kappa \in \overline{\mathrm{lin}\{h_\iota : \iota \neq \kappa\}}$. □

2.1.8 Notations. We now apply the previous results to a given sequence of polynomials. For $(P_n)_{n\ge 0}$ as in 1.1.1, set $E := \big\{z \in \mathbb{C} : \sum_{n\ge 0} |P_n(z)|^2 < \infty\big\}$,

$$K : E \times E \to \mathbb{C}\,, \qquad K(z,w) := \sum_{n\ge 0} \overline{P_n(z)}P_n(w)\,,$$

and denote the associated RKHS by $\mathcal{H}(K)$. Then, for all $f \in \mathcal{H}(K)$ and all $z \in E$,

$$f(z) = \langle\, K_z \,, f \,\rangle_{\mathcal{H}(K)} = \Big\langle \sum_{n\ge 0} \overline{P_n(z)}\, P_n \,, f \,\Big\rangle_{\mathcal{H}(K)} = \sum_{n\ge 0} \langle\, P_n \,, f \,\rangle_{\mathcal{H}(K)} P_n(z)\,. \tag{2.3}$$

In the following, when we write ℓ^2, we always mean $\ell^2(\mathbb{N}_0)$, i.e. the vectors in ℓ^2 start with the 0^{th} entry.

Define $P(E) := \big\{\big(\overline{P_n(z)}\,\big)_{n\ge 0} : z \in E\big\}$; in accordance to the previous results, let W_0 be the linear span of $P(E)$ and let W be the closure of W_0 in ℓ^2. Then, see 2.1.3, $\beta : \ell^2 \to \mathcal{H}(K)\,,\ \ (\alpha_n)_{n\ge 0} \mapsto \sum_{n\ge 0} \alpha_n\, P_n\,,$ is a partial isometry with initial space W.

Denote by α the canonical isomorphism from the abstract space $\mathcal{H}$ defined in section 1.2 onto ℓ^2. Then $\beta \circ \alpha$ maps $p \in \mathbb{C}[z] \subset \mathcal{H}$ to $p \in \mathcal{H}(K)$ and

$$\|p\|_{\mathcal{H}(K)} \le \|p\|_{\mathcal{H}} \qquad \text{for all } p \in \mathbb{C}[z]\,. \tag{2.4}$$

Furthermore, note that the map $z \mapsto \big(\,\overline{P_n(z)}\,\big)_n$ is one-to-one, as $\big(\,\overline{P_n(z)}\,\big)_n = \big(\,\overline{P_n(w)}\,\big)_n$ implies $p(z) = p(w)$ for all $p \in \mathbb{C}[z]$ and hence $z = w$.

2.1.9 Corollary. *$(P_n)_n$ is an onb of $\mathcal{H}(K)$ if and only if $P(E)$ is total in ℓ^2, i.e. $W^\perp = \{0\}$.*

This was just the adaption of 2.1.4 to the situation above.

Obviously, if $|E| = \infty$ then the set $\{P_n : n \in \mathbb{N}_0\}$ is linearly independent in $\mathcal{H}(K)$ and, due to the last assertion in 2.1.1, $\mathbb{C}[z]$ is dense in $\mathcal{H}(K)$.

Therefore, we can apply Gram-Schmidt orthonormalization to $(P_n)_n$ in $\mathcal{H}(K)$ obtaining an onb $(Q_n)_{n\geq 0}$ of $\mathcal{H}(K)$ where Q_n is a polynomial of degree n and

$$K(z,w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w) = \sum_{n\geq 0} \overline{Q_n(z)}\, Q_n(w) \qquad \text{for all } z, w \in E\,.$$

Note that $Q_0(z) = \frac{1}{\|P_0\|_{\mathcal{H}(K)}}\, P_0(z) = \frac{1}{\|P_0\|_{\mathcal{H}(K)}}$; hence $(Q_n)_n$ need not match 1.1.1.

2.1.10 Series indexed by $\mathbb{N}_0$. Let $\alpha = (\alpha_n)_{n\geq 0} \in \ell^2$ and $f := \sum_{n\geq 0} \alpha_n P_n \in \mathcal{H}(K)$. Then

$$f(z) = \sum_{n\geq 0} \alpha_n\, P_n(z) = \sum_{n=0}^{\infty} \alpha_n\, P_n(z) \qquad \text{for } z \in E\,.$$

We will use the latter notation especially when we regard this as a series of functions defined on some subset of $\mathbb{C}$ without using the Hilbert space structure of $\mathcal{H}(K)$; it may also represent a convergent but not absolutely convergent series for some $\alpha \notin \ell^2(\mathbb{N}_0)$ or $z \notin E$, when the former does not make sense.

2.1.11 Example (Weighted Shift). For $n \in \mathbb{N}_0$, let $P_n(z) := b_n z^n$ where $b_n \in \mathbb{C} \setminus \{0\}$ and $b_0 = 1$. Then

$$E = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |b_n|^2\, |z|^{2n} < \infty \right\}$$

is an either open or closed disk centered at 0 (including the cases $E = \{0\}$ and $E = \mathbb{C}$). We now consider the case $E \neq \{0\}$. For $m \in \mathbb{N}_0$, (2.2) yields

$$b_m z^m = P_m(z) = \sum_{n=0}^{\infty} \langle\, P_n\,, P_m\,\rangle_{\mathcal{H}(K)}\, P_n(z) = \sum_{n=0}^{\infty} \langle\, P_n\,, P_m\,\rangle_{\mathcal{H}(K)}\, b_n z^n \qquad \text{for all } z \in E\,.$$

Now uniqueness of power series implies $\langle\, P_n\,, P_m\,\rangle_{\mathcal{H}(K)} = \delta_{nm}$. Thus $(P_n)_n$ is an orthonormal system in $\mathcal{H}(K)$. Moreover, according to 2.1.1, this sytem is total in $\mathcal{H}(K)$ and, consequently, an onb.

Here the multiplication operator D, defined as in 1.2.1, is a **weighted shift**: In its matrix representation, $d_{n+1,n} = \frac{b_n}{b_{n+1}}$ for all $n \in \mathbb{N}_0$ and $d_{ij} = 0$ if $i \neq j+1$.

We will return to this situation in 2.2.4 concerning analycity of the functions in $\mathcal{H}(K)$ and in section 4.4 regarding subnormality of weighted shifts.

Now we turn to the special case $b_n = 1$ for all $n \in \mathbb{N}_0$. Here $E = \{z \in \mathbb{C} : |z| < 1\}$ and $K(z,w) = \sum_{n=0}^{\infty} (\overline{z}\, w)^n = \frac{1}{1-\overline{z}\, w}$ for $z, w \in E$.

Recall 1.2.2 and 1.3.8 where we have seen that $(z^n)_{n\in\mathbb{Z}}$ is an onb in L^2_μ where μ is the normalized one-dimensional Lebesgue measure on the unit circle and we can identify $\mathcal{H}(K)$ with the closure of $\mathbb{C}[z]$ in L^2_μ.

2.1.12 Example. Let $P_0 :\equiv 1$ and $P_n(z) := \frac{1}{\sqrt{n!}} z^{n-1}(z-n)$ for $n \geq 1$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} |P_n(z)|^2 \;&=\; 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\, |z|^{2n-2}\; |z-n|^2 \;\;\leq\;\; 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\, |z|^{2n-2} \cdot 2\big(|z|^2 + n^2\big) \\ &=\; 1 + 2\sum_{n=1}^{\infty} \tfrac{1}{n!}\, |z|^{2n} + 2\sum_{n=1}^{\infty} \tfrac{n^2}{n!}\, |z|^{2n-2} \;<\; \infty \qquad \text{for all } z \in \mathbb{C}\,.\end{aligned}$$

Hence $E = \mathbb{C}$ and

$$\begin{aligned}K(z,w) &= \sum_{n=0}^{\infty} \overline{P_n(z)}\, P_n(w) = 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\, (\overline{z}\, w)^{n-1}(\overline{z} - n)(w-n) \\ &= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\, (\overline{z}\, w)^{n} - (\overline{z} + w) \sum_{n=1}^{\infty} \tfrac{1}{(n-1)!}\, (\overline{z}\, w)^{n-1} + \sum_{n=1}^{\infty} \tfrac{n}{(n-1)!}\, (\overline{z}\, w)^{n-1} \\ &= \exp(\overline{z}\, w) - (\overline{z} + w)\exp(\overline{z}\, w) + (\overline{z}\, w + 1)\exp(\overline{z}\, w) \\ &= \big(1 + (1-\overline{z})(1-w)\big)\; \exp(\overline{z}\, w) \qquad \text{for } z, w \in \mathbb{C}\,.\end{aligned}$$

We next show that $(P_n)_n$ is **not** an onb in $\mathcal{H}(K)$.

For $n \in \mathbb{N}_0$ set $x_n := \frac{1}{\sqrt{n!}}$. Then $x := (x_n)_{n \geq 0} \in \ell^2$ and

$$\begin{aligned}\sum_{n=0}^{\infty} x_n\, P_n(z) &= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\, z^{n-1}(z-n) \\ &= 1 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\, z^{n} - \sum_{n=1}^{\infty} \tfrac{1}{(n-1)!}\, z^{n-1} = \exp(z) - \exp(z) = 0 \qquad \text{for all } z \in \mathbb{C}\,.\end{aligned}$$

Thus $0 \neq x \in W^{\perp}$ and 2.1.9 implies that $(P_n)_n$ is not an onb in $\mathcal{H}(K)$. Moreover, in 4.5.2 we will be able to conclude that there exists no om for $(P_n)_n$.

These two examples show that, for $(P_n)_{n\geq 0}$ as in 1.1.1, both cases – being an onb in $\mathcal{H}(K)$ or not – can occur. If, in particular, E is finite then $\mathcal{H}(K)$ is finite-dimensional and $(P_n)_{n\geq 0}$ cannot be an onb in $\mathcal{H}(K)$. It may also happen that $E = \varnothing$.

2.2 Analytic Kernels

If the polynomials $(P_n)_n$ form an onb in $\mathcal{H}(K)$ then this space is isometrically isomorphic to the abstract space $\mathcal{H}$ defined in section 1.2 via $P_n \mapsto P_n$, and if there exists an om μ for $(P_n)_n$ then also to P^2_μ.

We observe that, on the one hand, point evaluation in $\mathcal{H}(K)$ is a continuous linear functional while, on the other hand, point evaluation in general is not even defined in L^2_μ. In this section we will state some sufficient conditions for the functions of $\mathcal{H}(K)$ being analytic – at least in some subset $G \subset E$ – in order to characterize $\mathcal{H}(K)$ as a proper (because in general not every element of L^2_μ has an analytic representative) subspace of

L^2_μ. In this situation, D is sometimes said to have an **analytic model**, see e.g. [StSz3]. At this point, however, we do not know any connections between the domain E of $\mathcal{H}(K)$ and the support of the om μ yet; they may be disjoint, which actually can happen, recall 1.2.2 and 1.3.8 where $P_n(z) = z^n$, $\operatorname{supp}(\mu)$ is the unit circle, and E is the open unit disk; see also section 4.2.

2.2.1 Notations. For $A \subset \mathbb{C}$ set $A^* := \{z : \overline{z} \in A\}$ and for $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, define $p^* \in \mathbb{C}[z]$ by $p^*(z) := \overline{a_0} + \overline{a_1}\, z + \cdots + \overline{a_n}\, z^n$.

Let now $(P_n)_{n\geq 0}$ be as in 1.1.1; we define $\mathcal{H}(K)$ as in 2.1.8 and regard $\mathbb{C}[z]$ as a dense (due to the last assertion in 2.1.1) linear subspace of $\mathcal{H}(K)$.

Furthermore, set $\kappa : E \to \mathbb{C}$, $\kappa(z) := K(z, z)$ and

$$H : E^* \times E \to \mathbb{C}\,, \qquad H(z, w) := K(\overline{z}, w) = \sum_{n\geq 0} P_n^*(z)\, P_n(w)\,.$$

Note that $\kappa(z) \geq 1$ for all $z \in E$ and that κ need not be a member of $\mathcal{H}(K)$.

2.2.2 Lemma. *Let G be an open non-empty subset of E such that all $f \in \mathcal{H}(K)$ are holomorphic in G. Then H is holomorphic in $G^* \times G$. In particular, κ is continuous in G.*

Proof: For fixed $z \in G^*$, the function $H(z, \cdot) = K_{\overline{z}} \in \mathcal{H}(K)$ is holomorphic in G.

Now fix $w \in G$ and define $h_w : G^* \to \mathbb{C}$, $h_w(z) := H(z, w) = K(\overline{z}, w) = \overline{K(w, \overline{z})} = \overline{K_w(\overline{z})}$. Clearly, h_w is holomorphic in G^*, as K_w is holomorphic in G. Recall that functions of several complex variables are holomorphic if and only if they are holomorphic in each variable seperately, see e.g. [GR, I.A.2].

In particular, H is continuous in $G^* \times G$ which implies that $z \mapsto \kappa(z) = H(\overline{z}, z)$ is continuous in G. $\square$

Remark. Note that we did not have to require that $\mathcal{H}(K)$ contains polynomials here.

2.2.3 Lemma. *Let G be an open non-empty subset of E and suppose that for every compact $K \subset G$ there exists $c_K > 0$ such that $\kappa(z) \leq c_K$ for all $z \in K$.*

Then all $f \in \mathcal{H}(K)$ are holomorphic in G. In particular, K_z is holomorphic in G for every $z \in E$.

Moreover, H is holomorphic in $G^* \times G$ with

$$\partial_z H(z, w) = \sum_{n\geq 0} P_n^{*\prime}(z)\, P_n(w)\,, \qquad \partial_w H(z, w) = \sum_{n\geq 0} P_n^*(z)\, P_n'(w)\,.$$

Proof: As $\mathbb{C}[z]$ is dense in $\mathcal{H}(K)$, for arbitrary $f \in \mathcal{H}(K)$, there exists a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{C}[z]$ such that $\|f - q_n\|_{\mathcal{H}(K)} \to 0$ as $n \to \infty$. Then $q_n \to f$ uniformly on K (see 1.4.4). According to the Weierstrass convergence theorem (see e.g. [Re, 8.§4]), the limit of a pointwisely convergent sequence of holomorphic functions $G \to \mathbb{C}$ which is uniformly convergent on every compact $K \subset G$, is holomorphic in G and $q_n' \to f'$ uniformly on every compact subset of G.

Now fix $w \in E$. For $f = H(\,\cdot, w)$ and $q_n = \sum_{k=0}^{n} P_k(w)\, P_k^*$, we obtain $q_n' \to f'$ on G^*. Hence

$$\tfrac{\mathrm{d}}{\mathrm{d}z} H(z,w) = \sum_{n\geq 0} P_n^{*\prime}(z)\, P_n(w)\,.$$

Note that, for fixed $z \in E^*$, we get an analogous result concerning the derivative of H with respect to w on G.
Therefore, H is holomorphic in $G^* \times G$ with partial derivatives as asserted. $\square$

For $(z_0, w_0) \in G^* \times G$, there exist coefficients $a_{jk} \in \mathbb{C}$ and $r > 0$ such that

$$K(\overline{z}, w) = H(z,w) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}(z-z_0)^j(w-w_0)^k \qquad \text{for } |z-z_0| < r\,, |w-w_0| < r\,.$$

If, for example, $G = E$ is an open disk centered at z_0 then we can write

$$K(z,w) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \widetilde{a}_{jk}(\overline{z}-\overline{z_0})^j(w-z_0)^k \qquad \text{for all } z, w \in E$$

with some $\widetilde{a}_{jk} \in \mathbb{C}$. Here, K is often referred to as an **analytic kernel.**

2.2.4 Weighted Shift (continued). For $n \in \mathbb{N}_0$ let $P_n(z) := b_n z^n$, where $b_0 = 1$ and $b_n \in \mathbb{C} \setminus \{0\}$ for $n > 0$, see also 2.1.11. Assume $E \neq \{0\}$; then E is an open or closed disk centered at 0 or $E = \mathbb{C}$ and $(b_n z^n)_n \in \ell^2$ for all $z \in E$. Now, for $x = (x_n)_n \in \ell^2$ we set

$$f_x : E \to \mathbb{C}\,, \qquad f_x(z) := \sum_{n=0}^{\infty} x_n b_n z^n$$

and $\mathcal{F} := \{f_x : x \in \ell^2\}$. Then $\beta(x) := f_x$ defines linear map $\beta : \ell^2 \to \mathcal{F}$ which is one-to-one, as $f_x = 0$, i.e. $f_x(z) = 0$ for all $z \in E$, implies $x_n b_n = 0$ for all n and hence $x = 0$.

As β is onto by construction, β is a bijective map $\ell^2 \to \mathcal{F}$. We define an inner product on $\mathcal{F}$ by $\langle\, f_x\,, f_y\,\rangle_{\mathcal{F}} := \langle\, x\,, y\,\rangle_{\ell^2}$. Then $\mathcal{F} = \beta(\ell^2)$ and, in particular, $\{b_n z^n : n \in \mathbb{N}_0\}$ is an onb in $\mathcal{F}$.

For $\zeta \in E$, set $k(\zeta) := (\,\overline{b_n \zeta^n}\,)_n$. This yields

$$\langle\, f_{k(\zeta)}\,, f_y\,\rangle_{\mathcal{F}} = \langle\, k(\zeta)\,, y\,\rangle_{\ell^2} = \sum_{n=0}^{\infty} b_n \zeta^n \cdot y_n = f_y(\zeta) \qquad \text{for all } f_y \in \mathcal{F} \text{ and all } \zeta \in E\,.$$

Therefore, $\mathcal{F}$ is an RKHS with kernel

$$K(z,w) = \langle\, f_{k(w)}\,, f_{k(z)}\,\rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} b_n w^n\, \overline{b_n z^n} = \sum_{n=0}^{\infty} \overline{P_n(z)}\, P_n(w)$$

and thus coincides with $\mathcal{H}(K)$ when defined as in 2.1.8.

Furthermore, every $f \in \mathcal{H}(K) = \mathcal{F}$ is analytic.

2.2.5 Lemma. *Let $\mathcal{H}(K)$ be an RKHS with domain E such that all $f \in \mathcal{H}(K)$ are holomorphic in $G \subset E$.*
Then, for $z \in G$, there exists $L_z \in \mathcal{H}(K)$ such that $\langle L_z , f \rangle = f'(z)$ for all $f \in \mathcal{H}(K)$.

Proof: Fix $z \in G$ and choose a sequence $(h_n)_n$ in $\mathbb{C} \setminus \{0\}$ such that $z + h_n \in G$ for all n and $h_n \to 0$ as $n \to \infty$. Then, for all $f \in \mathcal{H}(K)$,

$$\left\langle \tfrac{1}{h_n}\big(K_{z+h_n} - K_z\big) , f \right\rangle = \tfrac{f(z+h_n)-f(z)}{h_n} \to f'(z) \qquad \text{as } n \to \infty .$$

A well-known corollary of the Banach-Steinhaus theorem, see e.g. [W, 4.24 d], implies that there exists $L_z \in \mathcal{H}(K)$ such that $\langle L_z , f \rangle = \lim\limits_{n\to\infty} \left\langle \tfrac{1}{h_n}\big(K_{z+h_n} - K_z\big) , f \right\rangle$ for all $f \in \mathcal{H}(K)$. This yields $\langle L_z , f \rangle = f'(z)$. □

Suppose now that $\mathcal{H}(K)$ is defined via polynomials as in 2.1.8 and $G \subset E$ exists such that all $f \in \mathcal{H}(K)$ are holomorphic in G. Let L_z be as in 2.2.5.

2.2.6 Theorem. *Let $z \in G$. Then $L_z = \sum\limits_{n\geq 0} \overline{P_n'(z)}\, P_n$. In particular, $\big(P_n'(z)\big)_n \in \ell^2$.*

Proof: According to 2.2.5, $\langle L_z , f \rangle = f'(z)$ for all $f \in \mathcal{H}(K)$.
Using 2.1.5, we get $(\langle P_n , L_z \rangle)_n \in \ell^2$ and $L_z = \sum\limits_{n\geq 0} \langle P_n , L_z \rangle P_n = \sum\limits_{n\geq 0} \overline{P_n'(z)}\, P_n$. □

Remark. Analogously, one can define higher-order derivatives $L_z^{(n)}$ satisfying $\langle L_z^{(n)} , f \rangle = f^{(n)}(z)$.

If the interior E° of E is non-empty, the question arises whether all $f \in \mathcal{H}(K)$ are holomorphic there. Recall that f can be approximated (in $\mathcal{H}(K)$ and therefore also pointwisely) by a sequence of polynomials. According to Osgood's theorem, see [Osg, Theorem I] and also [BM], the limit of a sequence of holomorphic functions which is pointwisely convergent in an open set U, is holomorphic in an open set that is dense in U. In the following, we will be able to chracterize a dense open subset of E° in our particular situation where all $f \in \mathcal{H}(K)$ are holomorphic; some intermediate results will be useful later on, too.

2.2.7 Lemma. *For $R > 0$, the set $A_R := \{z \in E : \kappa(z) \leq R\}$ is closed in $\mathbb{C}$.*

Proof: Let $(z_m)_{m\in\mathbb{N}}$ be a convergent sequence in $\mathbb{C}$ such that $z_m \in A_R$ for all $m \in \mathbb{N}$ and set $z_0 := \lim\limits_{m\to\infty} z_m$. Fix $N \in \mathbb{N}$; then

$$\sum_{n=0}^{N} |P_n(z_m)|^2 \leq \sum_{n=0}^{\infty} |P_n(z_m)|^2 = \kappa(z_m) \leq R \qquad \text{for all } m \in \mathbb{N}$$

and continuity implies $\sum\limits_{n=0}^{N} |P_n(z_0)|^2 \leq R$. Now $N \to \infty$ yields $\kappa(z_0) \leq R$.

Thus $z_0 \in A_R$ and A_R is a closed subset of $\mathbb{C}$. □

Note that $A_r \subset A_R$ whenever $0 < r < R$ and $E = \bigcup_{k\in\mathbb{N}} A_k$ is an F_σ-set.

Suppose that κ is bounded. In other words, there is $R > 0$ such that $\kappa(z) \le R$ for all $z \in E$. Then $E = A_R$ is a closed subset of $\mathbb{C}$.

2.2.8 Corollary. *If E is not closed then κ is unbounded. More generally, if $(z_n)_{n\in\mathbb{N}}$ is a sequence in E such that $z_n \to z_0 \notin E$ as $n \to \infty$ then $\kappa(z_n) \to \infty$.*

Proof: Assume that $\kappa(z_n) \nrightarrow \infty$ as $n \to \infty$. Then there exists a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ and $R > 0$ such that $\kappa(z_{n_k}) \le R$ for all k. Thus $z_{n_k} \in A_R$ for all k. As A_R is closed, this implies $z_0 \in A_R \subset E$ and yields a contradiction. $\square$

Remark. If E is closed, κ still may be unbounded. In 4.1.3 and 4.1.4 we will be able to construct examples where E is a countable compact set and κ is unbounded.

2.2.9 Proposition. *Suppose $E^\circ \neq \varnothing$ and set $G := \bigcup_{k\in\mathbb{N}} A_k^\circ \subset E^\circ$.*

Then G is dense in E° (in particular, $G \neq \varnothing$) and all $f \in \mathcal{H}(K)$ are holomorphic in G.

Proof: Take $z_0 \in E^\circ$ and let F be a closed disk centered at z_0 and contained in E. Then $F = F \cap \bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} (F \cap A_k)$.
According to Baire category theory, applied to the complete metric space F, there exists $k \in \mathbb{N}$ such that $F \cap A_k^\circ \neq \varnothing$ and, in particular, $F \cap G \neq \varnothing$.
As z_0 and F were chosen arbitrarily, this proves that G is dense in E°.

If $A_k^\circ \neq \varnothing$ for some k then, by 2.2.3, all $f \in \mathcal{H}(K)$ are holomorphic in A_k°. Therefore, all $f \in \mathcal{H}(K)$ are holomorphic in $\bigcup_{k\in\mathbb{N}} A_k^\circ \subset E^\circ$. $\square$

It remains an open question whether the case $G \neq E^\circ$ can occur. We will show that κ obeys a maximum principle in E° and that if $z_0 \in E^\circ \setminus G$ exists then κ must be unbounded in every neighborhood of z_0.

2.2.10 Corollary. *In the situation of* 2.2.9, *suppose that $E^\circ \setminus G \neq \varnothing$. Then, for every $z_0 \in E^\circ \setminus G$, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in G such that $z_n \to z_0$ and $\kappa(z_n) \to \infty$ as $n \to \infty$. In particular, κ is not continuous at z_0.*

Proof: Choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n > 0$ for all n, $\varepsilon_n \to 0$ as $n \to \infty$, and $U_n := \{z \in \mathbb{C} : |z - z_0| < \varepsilon_n\} \subset E^\circ$ for all n.
Now, for fixed n, assume that $U_n \subset A_n$. This implies $U_n \subset A_n^\circ \subset G$ in contradiction to $z_0 \notin G$. Thus $U_n \cap (\mathbb{C} \setminus A_n) \neq \varnothing$. As this is an open set contained in E°, it must have non-empty intersection with the dense set G.
Hence there exists $z_n \in U_n \cap (\mathbb{C} \setminus A_n) \cap G$ for every $n \in \mathbb{N}$.
By construction, the sequence $(z_n)_{n\in\mathbb{N}}$ has the asserted properties, the discontinuity of κ at z_0 is an immediate consequence. $\square$

2.2.11 Lemma. *For $R > 0$, we have $\kappa(z) < R$ whenever $z \in A_R^\circ$.*

Proof: Assume $A_R^\circ \neq \varnothing$ and fix $z_0 \in A_R^\circ$.

Let U be an open disk centered at z_0 such that $\overline{U} \subset A_R$. Then K_{z_0} is continuous in $\overline{U}$ and holomorphic in U. Using the maximum principle we find $z_1 \in \partial U$ such that $|K_{z_0}(z)| < |K_{z_0}(z_1)|$ for all $z \in U$ and hence $\kappa(z_0) = K_{z_0}(z_0) < |K_{z_0}(z_1)|$.

Analogously, we now apply the maximum principle to K_{z_1} and find z_2 in ∂U such that $|K_{z_1}(z_0)| < |K_{z_1}(z_2)|$.

Finally,

$$\begin{aligned}\kappa(z_0) < |K_{z_0}(z_1)| = |K_{z_1}(z_0)| < |K_{z_1}(z_2)| \\ = |\langle K_{z_2}, K_{z_1} \rangle| \leq \|K_{z_2}\| \cdot \|K_{z_1}\| = \kappa(z_1)^{\frac{1}{2}} \kappa(z_2)^{\frac{1}{2}} \leq R\,.\end{aligned}$$

Thus $\kappa(z) < R$ for all $z \in A_R^\circ$. □

2.2.12 Maximum Principle for $\boldsymbol{\kappa}$.
In E°, κ does not attain a local maximum.

Proof: Assume that there exists a local maximum $M := \kappa(z_0)$ at $z_0 \in E^\circ$, i.e. there exists an open disk $U \subset E^\circ$ centered at z_0 such that $\kappa(z) \leq M$ for all $z \in U$ which implies $U \subset A_M$ and thus $z_0 \in A_M^\circ$. Now 2.2.11 implies $\kappa(z_0) < M$ which is a contradiction. □

2.2.13 Lemma. *Let $f \in \mathcal{H}(K)$ and $R > 0$. The restriction $f|A_R$ is continuous on A_R.*

Proof: In analogy to the proof of 2.2.3, we can find a sequence $(q_n)_n$ in $\mathbb{C}[z]$ such that $q_n \to f$ in $\mathcal{H}(K)$ as $n \to \infty$ and $q_n|A_R \to f|A_R$ uniformly on A_R. As the uniform limit of a sequence of continuous functions is continuous, the proof is complete. □

Remark. Note that we cannot conclude that f is continuous on E. In 4.2.10 we will see that $\mathcal{H}(K)$ may contain functions which are not continuous on E.

In 3.5.6 we will be able to show that G defined in 2.2.9 is precisely the largest open set contained in E° where all $f \in \mathcal{H}(K)$ are holomorphic. We will then give characterizations of that set by other means as well.

As for now, we conclude this section by a different approach to finding subsets of E where all $f \in \mathcal{H}(K)$ are holomorphic.

2.2.14 Theorem. *Let γ be a simply closed rectifiable curve in A_R for some $R > 0$ and denote by V the bounded component of $\mathbb{C} \setminus \gamma$.*

Then $\overline{V} \subset E$ and all $f \in \mathcal{H}(K)$ are holomorphic in V and continuous in $\overline{V}$.

Proof: Fix $z_0 \in V$. Then there exists $d > 0$ such that $|z - z_0| \geq d$ for all $z \in \gamma$. Now let $a = (a_k)_k \in \ell^2$. Using the Cauchy integral formula, for $m > n$ we have

$$\Big|\sum_{k=n}^{m} a_k P_k(z_0)\Big| = \Big|\frac{1}{2\pi \mathrm{i}} \int\limits_\gamma \frac{1}{z-z_0} \sum_{k=n}^{m} a_k P_k(z)\, \mathrm{d}z\Big| \leq \frac{1}{2\pi} \int\limits_\gamma \frac{1}{d}\Big|\sum_{k=n}^{m} a_k P_k(z)\Big|\; |\mathrm{d}z|$$
$$\leq \frac{1}{2\pi d} \int\limits_\gamma \Big(\sum_{k=n}^{m} |a_k|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=n}^{m} |P_k(z)|^2\Big)^{\frac{1}{2}} |\mathrm{d}z| \leq \Big(\sum_{k=n}^{m} |a_k|^2\Big)^{\frac{1}{2}} \frac{\sqrt{R}}{2\pi d} \cdot l(\gamma)$$

where $l(\gamma)$ denotes Euclidean length of γ. This shows that $\sum\limits_{k=0}^{\infty} a_k P_k(z_0)$ exists, and

$$\Big|\sum_{k=0}^{\infty} a_k P_k(z_0)\Big| \leq \|a\|_{\ell^2} \cdot \frac{\sqrt{R}}{2\pi d} \cdot l(\gamma)$$

implies that $a \mapsto \sum\limits_{k=0}^{\infty} a_k P_k(z_0)$ is a continuous linear functional; thus $\big(P_k(z_0)\big)_k \in \ell^2$. In other words, $z_0 \in E$. Therefore, $\overline{V} = V \cup \gamma \subset E$.

For $f \in \mathcal{H}(K)$, according to 2.1.5, $f = \sum\limits_{k \geq 0} \langle\, P_k \,,\, f \,\rangle_{\mathcal{H}(K)} P_k$. Set

$$f_n := \sum_{k=0}^{n} \langle\, P_k \,,\, f \,\rangle_{\mathcal{H}(K)} P_k$$

for $n \in \mathbb{N}$. The sequence $(f_n)_n$ is convergent in $\mathcal{H}(K)$, hence pointwisely in V and uniformly, as γ is contained in A_R (see 1.4.4), on $\partial V = \gamma$. It is a well known fact (see [Re, 8.5.4], for instance) that then the limit is holomorphic in V and continuous in $\overline{V}$. □

The previous theorem can also be found in [StSz3, Theorem 8]. If we do not require κ to be bounded but at least integrable on γ, we can prove a similar result:

2.2.15 Theorem. *Let γ be a simply closed rectifiable curve in E such that $\int\limits_\gamma \kappa(z)\, |\mathrm{d}z| < \infty$. Denote by V the bounded component of $\mathbb{C} \setminus \gamma$.*
Then $V \subset E$ and all $f \in \mathcal{H}(K)$ are holomorphic in V.

Proof: For any compact set $K \subset V$ there exists $d > 0$ such that $|z - w| \geq d$ for all $z \in K$ and all $w \in \gamma$. Then

$$\int\limits_\gamma \frac{1}{|\zeta - z|^2}\, |\mathrm{d}\zeta| \leq \frac{l(\gamma)}{d^2} \qquad \text{for all } z \in K$$

where $l(\gamma)$ denotes Euclidean length of γ. Now, for $n \in N$ and $z \in K$, we have

$$|P_n(z)|^2 = \Big|\frac{1}{2\pi \mathrm{i}} \int\limits_\gamma \frac{P_n(\zeta)}{\zeta - z}\, \mathrm{d}\zeta\Big|^2 \leq \frac{1}{4\pi^2}\Big(\int\limits_\gamma \Big|\frac{P_n(\zeta)}{\zeta - z}\Big|\, |\mathrm{d}\zeta|\Big)^2$$
$$\leq \frac{1}{4\pi^2} \int\limits_\gamma |P_n(\zeta)|^2\, |\mathrm{d}\zeta| \cdot \int\limits_\gamma \frac{1}{|\zeta - z|^2}\, |\mathrm{d}\zeta| \leq \frac{l(\gamma)}{4\pi^2 d^2} \int\limits_\gamma |P_n(\zeta)|^2\, |\mathrm{d}\zeta|\ .$$

Using dominated convergence, we obtain

$$\kappa(z) = \sum_{n\geq 0} |P_n(z)|^2 \leq \frac{l(\gamma)}{4\pi^2 d^2} \int_\gamma \sum_{n\geq 0} |P_n(\zeta)|^2 \, |\mathrm{d}\zeta| = \frac{l(\gamma)}{4\pi^2 d^2} \int_\gamma \kappa(\zeta) \, |\mathrm{d}\zeta| =: c_K$$

for all $z \in K$. Thus $K \subset E$. Moreover, by 2.2.3, all $f \in \mathcal{H}(K)$ are holomorphic in V. $\square$

Note that in 2.2.15 we do not have to require $\gamma \subset E$. There might exist a nullset (with respect to line measure) where $\sum_{n\geq 0} |P_n(z)|^2 = \infty$.

2.3 Analytic Functions Forming Subspaces of L^2_μ

In this section we construct RKHS whose elements are analytic functions which are square integrable with respect to some suitable measure μ.

If all polynomials in $\mathbb{C}[z]$ are square integrable with respect to μ and $\mu(\mathbb{C}) = 1$ then we can use Gram-Schmidt orthonormalization to obtain a system of polynomials as in 1.1.1 such that μ is an om.

Hence, in this case, we will be able to embed $\mathcal{H}(K)$, defined as in 2.1.8, into L^2_μ.

2.3.1 The Space A^2_μ. Let μ be a measure on $\mathfrak{B}(\mathbb{C})$, absolutely continuous with respect to Lebesgue measure, $\mu = \alpha\lambda$, and U be an open subset of $\mathbb{C}$.

Furthermore, assume that $\int_B \frac{1}{\alpha} \, \mathrm{d}\lambda < \infty$ for all compact $B \subset U$.

Now define

$$A^2_\mu := \left\{ f \text{ holomorphic in } U : \int_U |f(z)|^2 \, \mathrm{d}\mu < \infty \right\}.$$

Obviously, A^2_μ is a vector space and inherits an inner product from L^2_μ. We will show that A^2_μ is an RKHS which is a well-known fact but sometimes proved under stronger conditions, see [Hall, Theorem 2.2], for example, where α is supposed to be continuous. See also [Ga, Chapter I] or [Con2, §29] where U is bounded and μ is normalized Lebesgue measure on U. In this case, A^2_μ is often referred to as **Bergman space**. For a generalization of Bergman spaces to unbounded domains of finite Lebesgue measure, see [CJK].

In the following, for $z \in \mathbb{C}$ and $r > 0$, set $B_r(z) := \{\zeta \in \mathbb{C} : |\zeta - z| \leq r\}$.

2.3.2 Lemma. *For every $z \in U$, there exists $c_z > 0$ such that*

$$|f(z)| \leq c_z \, \|f\|_{L^2_\mu} \qquad \text{for all } f \in A^2_\mu\,.$$

Proof: Let $z \in U$. Then there exists $r > 0$ such that $B := B_r(z) \subset U$.
Now we have

$$\infty > \int_B \frac{1}{\alpha}\, \mathrm{d}\lambda = \int_U \left|\mathbf{1}_B \cdot \frac{1}{\alpha}\right|^2 \cdot \alpha\, \mathrm{d}\lambda = \int_U \left|\mathbf{1}_B \cdot \frac{1}{\alpha}\right|^2 \mathrm{d}\mu$$

which implies $\mathbf{1}_B \cdot \frac{1}{\alpha} \in L^2_\mu$. Using the mean value theorem, for $f \in A^2_\mu$, we get

$$\left\langle \mathbf{1}_B \cdot \frac{1}{\alpha}, f \right\rangle_{L^2_\mu} = \int_U \mathbf{1}_B \cdot \frac{1}{\alpha} \cdot f \cdot \alpha\, \mathrm{d}\lambda = \int_B f\, \mathrm{d}\lambda = r^2\pi\, f(z)$$

and Cauchy-Schwarz yields $|f(z)| \le \frac{1}{r^2\pi} \left\|\mathbf{1}_B \cdot \frac{1}{\alpha}\right\|_{L^2_\mu} \cdot \|f\|_{L^2_\mu}$.
Therefore, set $c_z := \frac{1}{r^2\pi} \left\|\mathbf{1}_B \cdot \frac{1}{\alpha}\right\|_{L^2_\mu}$ to conclude the proof. □

2.3.3 Lemma. *If $B_r(z_0) \subset U$ then there exists $c > 0$ such that $|f(z)| \le c\, \|f\|_{L^2_\mu}$ for all $f \in A^2_\mu$ and all $z \in B_r(z_0)$.*

Proof: There exists $R > r$ such that $B_r(z_0) \subset B_R(z_0) \subset U$.
Set $\rho := R - r$, then

$$B_\rho(z) \subset B_R(z_0) \qquad \text{for all } z \in B_r(z_0)\,.$$

Furthermore, $\mathbf{1}_{B_r(z_0)} \cdot \frac{1}{\alpha} \in L^2_\mu$, and, for arbitrary $z \in B_r(z_0)$, we have

$$\mathbf{1}_{B_\rho(z)}(w) \cdot \frac{1}{\alpha(w)} \le \mathbf{1}_{B_R(z_0)}(w) \cdot \frac{1}{\alpha(w)} \qquad \text{for all } w \in U$$

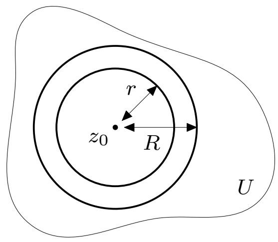


Figure 1.
On the proof of 2.3.3

implying

$$\left\|\mathbf{1}_{B_\rho(z)} \cdot \frac{1}{\alpha}\right\|_{L^2_\mu} \le \left\|\mathbf{1}_{B_R(z_0)} \cdot \frac{1}{\alpha}\right\|_{L^2_\mu} \qquad \text{for all } z \in B_r(z_0)\,.$$

In analogy to 2.3.2, we get

$$|f(z)| \le \frac{1}{\rho^2\pi} \left\|\mathbf{1}_{B_\rho(z)} \cdot \frac{1}{\alpha}\right\|_{L^2_\mu} \cdot \|f\|_{L^2_\mu} \le \underbrace{\frac{1}{\rho^2\pi} \left\|\mathbf{1}_{B_R(z_0)} \cdot \frac{1}{\alpha}\right\|_{L^2_\mu}}_{=:\, c} \cdot \|f\|_{L^2_\mu}$$

for all $f \in A^2_\mu$ and all $z \in B_r(z_0)$. □

2.3.4 Proposition. *The space A^2_μ is an RKHS.*

Proof: According to 2.3.2, point evaluation in A^2_μ is continuous. Regarding A^2_μ as a subspace of L^2_μ, it only remains to show that A^2_μ is closed in L^2_μ.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in A^2_μ.
For any closed disk $B \subset U$, due to 2.3.3, there exists $c > 0$ such that

$$|f_n(z) - f_m(z)| \le c\, \|f_n - f_m\|_{L^2_\mu} \qquad \text{for all } z \in B \text{ and } m, n \in \mathbb{N}\,.$$

Hence $(f_n)_n$ is uniformly convergent on every closed disk contained in U. Denote the limit by f and note that $f : U \to \mathbb{C}$ is holomorphic.
Therefore, A^2_μ is a Hilbert space. Using 1.4.3, we see that A^2_μ is an RKHS. $\square$

2.3.5 Orthogonal Polynomials in $\boldsymbol{A^2_\mu}$. Suppose now that $\mu(U) = 1$ and that all $p \in \mathbb{C}[z]$ are square integrable with respect to μ. Then we can regard $\mathbb{C}[z]$ as a subspace of A^2_μ. Furthermore, Gram-Schmidt orthonormalization applied to $\{1, z, z^2, \ldots\}$ yields an orthonormal system $(P_n)_{n\geq 0}$ of polynomials with $P_0 = 1$ and $\deg P_n = n$ for $n \geq 1$.
There exists an onb $(e_k)_{k\in I}$ of A^2_μ, I a suitable index set, that extends $(P_n)_{n\geq 0}$, i.e. for every $n \in \mathbb{N}_0$, there exists $k_n \in I$ such that $P_n = e_{k_n}$.
Let $\widetilde{K}$ be the kernel of A^2_μ, then $\widetilde{K}(z, w) = \sum_{k\in I} \overline{e_k(z)}\, e_k(w)$ by 1.4.6. In particular,

$$\sum_{n\geq 0} |P_n(z)|^2 \leq \sum_{k\in I} |e_k(z)|^2 < \infty \qquad \text{for all } z \in U\,.$$

Therefore, $U \subset E$.

Remark. For bounded U and continuous α a proof of the last statement, $U \subset E$, is also given in [GTV, Proposition 2].

Note that A^2_μ is a proper subspace of L^2_μ, as L^2_μ contains elements which do not have a holomorphic representative. In particular, $(P_n)_n$ is not an onb in L^2_μ.

2.3.6 Lemma. *In the situation of* 2.3.5*,* $(P_n)_n$ *is an onb in* $\mathcal{H}(K)$*.*

Proof: Let $m \in \mathbb{N}_0$. According to 2.1.5, $P_m = \sum_{n\geq 0} \langle P_n, P_m \rangle_{\mathcal{H}(K)} P_n$ holds in $\mathcal{H}(K)$ and, by (2.3),

$$P_m(z) = \sum_{n\geq 0} \langle P_n, P_m \rangle_{\mathcal{H}(K)} P_n(z) \qquad \text{for all } z \in E\,.$$

In particular, $\big(\langle P_n, P_m \rangle_{\mathcal{H}(K)}\big)_n \in \ell^2$; thus $f := \sum_{n\geq 0} \langle P_n, P_m \rangle_{\mathcal{H}(K)} P_n \in A^2_\mu$ exists.
As point evaluation in A^2_μ is continuous, $f(z) = \sum_{n\geq 0} \langle P_n, P_m \rangle_{\mathcal{H}(K)} P_n(z)$ holds for all $z \in U$.

Using $U \subset E$, we obtain $f(z) = P_m(z)$ for all $z \in U$; hence $f = P_m$ in A^2_μ.
As $(P_n)_n$ is an onb in A^2_μ, this implies $\langle P_n, P_m \rangle_{\mathcal{H}(K)} = \delta_{mn}$, completing the proof. $\square$

2.3.7 Example (Bergman space on the unit disk). Denote by $\mathbb{D}$ the open unit disk and let μ be normalized Lebesgue measure on $\mathbb{D}$.
A short calculation shows that the polynomials $P_n(z) := \sqrt{n+1}\, z^n$, $n \in \mathbb{N}_0$, form an orthonormal system in L^2_μ. Clearly, $\mathbb{C}[z] \subset A^2_\mu$. It is well-known that here $P^2_\mu = A^2_\mu$ holds, see [Halm, Problem 25], for instance. According to 2.3.6, $\mathcal{H}(K)$ is isometrically isomorphic to A^2_μ via $P_n \mapsto P_n$. Moreover,

$$K(z, w) = \sum_{n\geq 0} (n+1)(\,\overline{z}w)^n = \tfrac{1}{(1-\overline{z}w)^2}\,.$$

In the following, we will turn to a similar situation where μ is not supported on an open set but on its boundary.

2.3.8 A^2_μ-spaces on a Curve. In the remainder of this section, suppose that $\gamma : [0,1] \to \mathbb{C}$ is a simply closed piecewise continuously differentiable curve. For abbreviation, we denote $\gamma([0,1])$ by γ again. Let μ be absolutely continuous w.r.t. line measure on γ with density α and assume

$$\int\limits_\gamma \frac{1}{\alpha(z)}\,|\mathrm{d}z| < \infty\,.$$

Now $\quad \infty < \int\limits_\gamma \frac{1}{\alpha(z)}\,|\mathrm{d}z| = \int \left|\frac{1}{\alpha}\right|^2 \mathrm{d}\mu \quad$ yields $\frac{1}{\alpha} \in L^2_\mu$.

Moreover, let V be the bounded component of $\mathbb{C} \setminus \gamma$.

2.3.9 Lemma. *For any compact $L \subset V$, there exists $c > 0$ such that*

$$\big|f(z)\big| \le c \cdot \big\|f\big\|_{L^2_\mu}$$

for all $z \in L$ and all functions f which are holomorphic in an open set containing $\overline{V}$ and satisfy $\int |f|^2\,\mathrm{d}\mu < \infty$.

Proof: Clearly, $d := \inf\big\{|z-w| : z \in L, w \in \gamma\big\} > 0$.
Let now $z \in L$. Then, using the Cauchy integral formula, we get

$$\begin{aligned}
\big|f(z)\big| &= \left|\frac{1}{2\pi\mathrm{i}}\int\limits_0^1 \frac{f(\gamma(t))}{\gamma(t)-z}\,\gamma'(t)\,\mathrm{d}t\right| \le \frac{1}{2\pi d}\int\limits_0^1 \big|f\big(\gamma(t)\big)\big| \cdot \big|\gamma'(t)\big|\,\mathrm{d}t = \frac{1}{2\pi d}\int\limits_\gamma \big|f(\zeta)\big|\,|\mathrm{d}\zeta| \\
&= \frac{1}{2\pi d}\int\limits_\gamma \big|f(\zeta)\big|\,\frac{1}{\alpha(\zeta)}\,\alpha(\zeta)\,|\mathrm{d}\zeta| = \frac{1}{2\pi d}\Big\langle\, |f|\,, \frac{1}{\alpha}\,\Big\rangle_{L^2_\mu} \le \frac{1}{2\pi d}\,\Big\|\frac{1}{\alpha}\Big\|_{L^2_\mu}\,\big\|f\big\|_{L^2_\mu}
\end{aligned}$$

for all $z \in L$. In particular, the constant $c := \frac{1}{2\pi d}\,\big\|\frac{1}{\alpha}\big\|_{L^2_\mu}$ is independent of f. $\square$

2.3.10 Lemma. *Let $U \supset \overline{V}$ be open and $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ such that*

$$\int |f_n|^2\,\mathrm{d}\mu < \infty \qquad \textit{for all } n$$

and $(f_n)_n$ is convergent w.r.t. the L^2_μ-norm.
Then $f : V \to \mathbb{C}$, $f(z) := \lim\limits_{n\to\infty} f_n(z)$, exists and is holomorphic in V.

Proof: For any compact set $L \subset V$, according to 2.3.9, there exists $c > 0$ such that

$$\big|f_n(z) - f_m(z)\big| \le c\big\|f_n - f_m\big\|_{L^2_\mu} \qquad \text{for all } z \in L\,.$$

Hence $f_n \to f$ uniformly in every compact subset of V which implies that f is holomorphic in V. $\square$

Note that if, in particular, $\mu(\gamma) = 1$ then every $p \in \mathbb{C}[z]$ is square integrable w.r.t. μ and, applying Gram-Schmidt in L^2_μ to the monomials, we obtain an orthonormal system $(P_n)_n$ such that $P_0 \equiv 1$ and $\deg(P_n) = n$ for all n.

2.3.11 Theorem. *In the situation of* 2.3.8, *assume that* μ *is an om for* $(P_n)_{n\geq 0}$. *Then* $\sum\limits_{n\geq 0} |P_n(z)|^2 < \infty$ *for all* $z \in V$ *and, for any* $a = (a_n)_{n\geq 0} \in \ell^2$,

$$q(z) := \sum_{n\geq 0} a_n P_n(z)$$

defines a holomorphic function $q : V \to \mathbb{C}$.

Proof: Clearly, $q_m : \mathbb{C} \to \mathbb{C}$, $q_m(z) := \sum\limits_{n=0}^{m} a_n P_n(z)$ is an entire function for every $m \in \mathbb{N}$. Moreover, as $(P_n)_n$ is an orthonormal system in L^2_μ, we have $q_m \to q$ in L^2_μ as $m \to \infty$ and, according to 2.3.10, $q : V \to \mathbb{C}$, $q(z) := \sum\limits_{n\geq 0} a_n P_n(z)$, is holomorphic in V.

Now fix $z \in V$. By 2.3.9, there exists $c > 0$ such that $|q_m(z)| \leq c\,\|q_m\|_{L^2_\mu}$ for all m. This yields

$$\Big|\sum_{n\geq 0} a_n P_n(z)\Big| \leq c\,\Big\|\sum_{n\geq 0} a_n P_n\Big\|_{L^2_\mu} = c\,\|a\|_{\ell^2}$$

which shows that $a \mapsto \sum\limits_{n\geq 0} a_n P_n(z)$ is a continuous linear functional.

Therefore, by the Riesz representation theorem, we obtain $\big(P_n(z)\big)_n \in \ell^2$ to complete the proof. $\square$

Remark. For continuous α, this result can be found in [Sze, Theorem 16.3] and is also part of [GTV, Proposition 2].

Note that, by 2.3.11, every $f \in P^2_\mu$ can in a natural way be defined pointwisely in V and represents a holomorphic function there. One can show that this gives rise to an RKHS whose domain contains V. We will do this in a more general situation in section 3.6; in particular, see 3.6.8. See also 3.3.7 where we can already conclude that $P^2_\mu \subsetneqq L^2_\mu$ and section 4.2 for the special case that γ is the unit circle.

3 Relations between the Spaces $\mathcal{H}$, $\mathcal{H}(K)$, and L^2_μ

Throughout this chapter, $(P_n)_{n\geq 0}$ always denotes a sequence of polynomials as in 1.1.1; the Hilbert space $\mathcal{H}$ and the multiplication operator D are defined as in 1.2.1.
Furthermore, recall $E = \left\{z \in \mathbb{C} : \sum_{n\geq 0} |P_n(z)|^2 < \infty\right\}$ and define $\mathcal{H}(K)$ as in 2.1.8.

By construction, $(P_n)_n$ is an onb of $\mathcal{H}$. Moreover, these polynomials are total in $\mathcal{H}(K)$ but need not be orthogonal there, see section 2.1. If they are orthogonal then $\mathcal{H}$ and $\mathcal{H}(K)$ are isometrically isomorphic via the identity map on $\mathbb{C}[z]$.

Finally, if an om μ exists then let P^2_μ be the closure of $\mathbb{C}[z]$ in L^2_μ and P^2_μ is isometrically isomorphic to $\mathcal{H}$ via $P_n \mapsto P_n$, too. Note that in this case D is subnormal and the multiplication operator M_μ is a minimal normal extension of D, see 1.3.7.

As mentioned in 1.4.3, a Hilbert space consisting of functions $E \to \mathbb{C}$ is an RKHS if and only if point evaluation is a bounded linear functional for every $z \in E$; therefore, if $(P_n)_n$ is an onb in $\mathcal{H}(K)$ – as well as in P^2_μ, here by definition – the elements of E are commonly referred to as the **bounded point evaluations** of L^2_μ, see e.g. [Con1] or [Tr].

3.1 Preliminaries

We start with some definitions concerning linear operators in Hilbert space. For more details, see chapter A.1 in the appendix.

3.1.1 Definitions. Let A be a densely defined linear operator in a Hilbert space. We denote by $\mathrm{dom}(A)$, $\mathrm{ran}(A)$, and $\mathcal{N}(A)$ its domain, range, and null-space, respectively. Let $\sigma(A)$ be the spectrum of A, and $\rho(A)$ its resolvent set, $\rho(A) = \mathbb{C} \setminus \sigma(A)$. If A is closable then let $\overline{A}$ denote its closure.

A point $z \in \mathbb{C}$ is called a **regular** value of A, if there exists a constant $c > 0$ such that $\|(A - z\,\mathrm{id})f\| \geq c\,\|f\|$ for all $f \in \mathrm{dom}(A)$. The set $\mathrm{reg}(A)$ of all regular points of A is the **domain of regularity** of A.

The space $\mathrm{ran}(A - z\,\mathrm{id})^\perp$ is called **deficiency space** of A in z, its dimension is the **deficiency index** of A in z.

The spectrum $\sigma(A)$ can be decomposed into three disjoint subsets,

(i) the point spectrum $\sigma_p(A) := \{z \in \mathbb{C} : (A - z\,\mathrm{id}) \text{ is not one-to-one.}\}$,

(ii) the continuous spectrum

$$\sigma_c(A) := \{z \in \mathbb{C} : (A - z\,\mathrm{id})^{-1} \text{ is densely defined and not continuous}\}\,,$$

(iii) the residual spectrum $\sigma_r(A) := \{z \in \mathbb{C} : (A - z\,\mathrm{id})^{-1} \text{ is not densely defined}\}$;

in (ii) and (iii) the operator $(A - z\,\mathrm{id})$ is supposed to be one-to-one, i.e. $(A - z\,\mathrm{id})^{-1}$ is well-defined with $\mathrm{dom}(A - z\,\mathrm{id})^{-1} = \mathrm{ran}(A - z\,\mathrm{id})$.

Note that at every $z \in \sigma_r(A)$ the deficiency index is different from 0 and that at every $z \in \rho(A) \cup \sigma_c(A)$ it is 0.

The following properties are also well-known. A proof can be found in any textbook concerning Hilbert space theory, such as [AG], [EE], or [W], for example.

3.1.2 Proposition. *Let A be a densely defined linear operator in a Hilbert space. Then its adjoint A^* exists and* $\mathrm{ran}(A - z\,\mathrm{id})^\perp = \mathcal{N}(A^* - \overline{z}\,\mathrm{id})$.
The domain of regularity is an open subset of $\mathbb{C}$ and $\rho(A) \subset \mathrm{reg}(A) \subset \rho(A) \cup \sigma_r(A)$. In particular, if A is normal then $\sigma_r(A) = \varnothing$ and $\rho(A) = \mathrm{reg}(A)$.
If A is closable then $\mathrm{reg}(A) = \mathrm{reg}(\overline{A})$.
Moreover, in any connected subset of $\mathrm{reg}(A)$*, the deficiency index is constant.*
For $z \in \mathrm{reg}(A)$, denote by P_z the orthogonal projection onto $\mathrm{ran}(A - z\,\mathrm{id})^\perp$*. If $(z_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathrm{reg}(A)$ with limit $z_* \in \mathrm{reg}(A)$ then $\left\|P_{z_n} - P_{z_*}\right\| \to 0$ as $n \to \infty$.*
Let now A be symmetric. Then $\mathbb{C} \setminus \mathbb{R} \subset \rho(A)$ and the deficiency indices in the upper and lower half-planes are constant. Thus one can restrict to the deficiency indices at $\pm\mathrm{i}$. The operator A has a self-adjoint extension in $\mathcal{H}$ if and only if these are equal and is essentially self-adjoint if and only if they are 0.

Concerning the following theorem, see also [Kl2, 2.1.5] or [StSz3, Proposition 6]. However, we will state a proof here, too, as it provides an essential connection between the operator D in $\mathcal{H}$ and the domain E of $\mathcal{H}(K)$.

For $z \in E$, we can regard

$$\sum_{n\geq 0} \overline{P_n(z)}\, P_n$$

as $K_z \in \mathcal{H}(K)$ or as an element of the abstract space $\mathcal{H}$. We will denote the latter by k_z.

3.1.3 Theorem. *For $z \in \mathbb{C}$, the deficiency index of D in z is either* 0 *or* 1*. In particular,*

$$\mathrm{ran}(D - z\,\mathrm{id})^\perp = \mathcal{N}(D^* - \overline{z}\,\mathrm{id}) = \begin{cases} \{0\} & \textit{for } z \notin E\,, \\ \mathbb{C}\cdot k_z & \textit{for } z \in E\,. \end{cases}$$

Proof: The first equality is due to 3.1.2.

Now fix $z \in \mathbb{C}$ and let $f \in \mathcal{N}(D^* - \overline{z}\,\mathrm{id})$. Using (1.1), for any $n \in \mathbb{N}_0$, we get the recursion formula

$$z\langle f, P_n \rangle = \langle \overline{z} f, P_n \rangle = \langle D^* f, P_n \rangle = \langle f, DP_n \rangle = \sum_{i=0}^{n+1} d_{in} \langle f, P_i \rangle .$$

Starting with $c := \langle f, P_0 \rangle$, again by (1.1), we recursively obtain $\langle f, P_n \rangle = c\, P_n(z)$ for all $n \in \mathbb{N}_0$. Now the Parseval equation yields

$$f = \sum_{n\geq 0} \langle P_n, f \rangle P_n = \sum_{n\geq 0} \overline{c\, P_n(z)}\, P_n .$$

Hence $\big(c\, P_n(z)\big)_n \in \ell^2$. For $z \notin E$, this implies $c = 0$ and, therefore, $f = 0$.

If $z \in E$ then $f = \overline{c} \sum_{n\geq 0} \overline{P_n(z)}\, P_n = \overline{c}\, k_z$. Hence $\mathcal{N}(D^* - \overline{z}\,\mathrm{id}) = \mathbb{C} \cdot k_z$. $\square$

3.1.4 Corollary. *Let $E_{reg} := E \cap \mathrm{reg}(D)$. This is an open subset of $\mathbb{C}$.*

Proof: The case $E_{reg} = \varnothing$ is trivial. Otherwise, consider $z_0 \in E_{reg}$. As $\mathrm{reg}(D)$ is open, there exists an open disk U containing z_0 such that $U \subset \mathrm{reg}(D)$. According to 3.1.2, the deficiency index of D is constant in any connected subset of $\mathrm{reg}(D)$. Note that, by 3.1.3, the deficiency index in z is equal to 1 if and only if $z \in E$. This implies $U \subset E$; hence $U \subset E_{reg}$. Thus E_{reg} is open. $\square$

Note that $\langle k_z, P_n \rangle = P_n(z)$ for all $n \in \mathbb{N}_0$. By linearity, we get the following analogon to the reproducing kernel property in $\mathcal{H}(K)$.

3.1.5 Corollary. *Let $z \in E$. Then $\langle k_z, p \rangle_{\mathcal{H}} = p(z)$ for all $p \in \mathbb{C}[z]$.*

Note that $\langle K_z, f \rangle_{\mathcal{H}(K)} = f(z)$ for all $f \in \mathcal{H}(K)$ while in 3.1.5 we can only speak of polynomials as, in general, elements of $\mathcal{H}$ need not be functions. However, to any $f \in \mathcal{H}$ we can assign a function $f^K : E \to \mathbb{C}$ satisfying $f^K(z) = \langle k_z, f \rangle_{\mathcal{H}}$. We will take care of this construction in section 3.6.

3.1.6 Theorem. *If D is bounded then E is bounded. More precisely, if D is bounded then $E \subset \{z \in \mathbb{C} : |z| \leq \|D\|\}$.*

Proof: It is well-known that if D is bounded then D^* is bounded, too, and $\|D\| = \|D^*\|$, see [W, Satz 4.14], for instance.

For $z \in E$, using 3.1.3, we get $|z|\, \|k_z\| = \|\overline{z}\, k_z\| = \|D^* k_z\| \leq \|D^*\|\, \|k_z\|$ implying $\|D^*\| \geq |z|$ for all $z \in E$. Thus $|z| \leq \|D\|$ for all $z \in E$. $\square$

Remark. In the case of the Hermite polynomials, see 4.5.11, D is unbounded and $E = \varnothing$.

Suppose now that μ is an om for $(P_n)_{n\geq 0}$ and define $\Lambda_\mu := \big\{z \in \mathbb{C} : \mu(\{z\}) > 0\big\}$.
Note that Λ_μ is precisely the set of eigenvalues of the multiplication operator M_μ in L^2_μ and $M_\mu \mathbf{1}_{\{z\}} = z\,\mathbf{1}_{\{z\}}$ for all $z \in \Lambda_\mu$. In particular, $\Lambda_\mu = \sigma_p(M_\mu)$.
Furthermore, for $z \in E$, define $\kappa(z) := K(z,z)$.

3.1.7 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$. Then*

$$\mu(\{z\}) \leq \kappa(z)^{-1} \qquad \textit{for all } z \in \mathbb{C} \tag{3.1}$$

where $\kappa(z)^{-1} := 0$ if $z \notin E$. In particular, $\Lambda_\mu \subset E$.
In (3.1) equality holds if and only if $\mathbf{1}_{\{z\}} \in P^2_\mu$. Moreover, if $(P_n)_n$ is an onb in L^2_μ then equality holds for all $z \in \Lambda_\mu$.

Proof: Obviously, (3.1) holds for $z \notin \Lambda_\mu$.

Let now $z \in \Lambda_\mu$. Then $\mathbf{1}_{\{z\}} \in L^2_\mu \setminus \{0\}$. Since $\langle\, f\,, \mathbf{1}_{\{z\}}\,\rangle_{L^2_\mu} = \overline{f(z)}\cdot \mu(\{z\})$, point evaluation at z is well-defined for all $f \in L^2_\mu$.

Choose an onb $(e_i)_{i\geq 0}$ of L^2_μ such that for every $n \in \mathbb{N}_0$ there exists $i_n \in \mathbb{N}_0$ with $P_n = e_{i_n}$.
For $k \in \mathbb{N}$, set

$$s_k := \sum_{i=0}^{k} \langle\, e_i\,, \mathbf{1}_{\{z\}}\,\rangle e_i = \mu(\{z\}) \sum_{i=0}^{k} \overline{e_i(z)}\, e_i\,.$$

Then $s_k \to \mathbf{1}_{\{z\}}$ as $k \to \infty$. Thus there exists a subsequence $(s_{k_l})_{l\in\mathbb{N}}$ such that $s_{k_l} \to \mathbf{1}_{\{z\}}$ μ-almost everywhere as $l \to \infty$. In particular,

$$\lim_{l\to\infty} \mu(\{z\}) \sum_{i=0}^{k_l} \overline{e_i(z)}\, e_i(w) = \mathbf{1}_{\{z\}}(w) \qquad \text{for all } w \in \Lambda_\mu\,.$$

With $w = z$ we get

$$\frac{1}{\mu(\{z\})}\, \mathbf{1}_{\{z\}}(z) = \frac{1}{\mu(\{z\})} = \lim_{l\to\infty} \sum_{i=0}^{k_l} |e_i(z)|^2 = \sum_{i\geq 0} |e_i(z)|^2 \geq \sum_{n\geq 0} |P_n(z)|^2 = \kappa(z) \tag{3.2}$$

where equality holds if and only if $\mathbf{1}_{\{z\}} \in P^2_\mu$. If, in particular, $(P_n)_n$ is an onb in L^2_μ then equality holds in (3.2) for every $z \in \Lambda_\mu$.

To complete the proof, note that, for $z \in \Lambda_\mu$, in (3.1) equality holds if and only if equality holds in (3.2). $\square$

In general, however, $\Lambda_\mu \subsetneqq E$. Let $P_n(z) := z^n$ for $n \in \mathbb{N}_0$. Recall 2.1.11 where we have found $E = \{z \in \mathbb{C} : |z| < 1\}$ and 1.2.2 where we have seen that, up to a constant factor, Lebesgue measure on the unit circle is an om, hence $\Lambda_\mu = \varnothing$.

3.1.8 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ and denote by P the orthogonal projection in L^2_μ onto P^2_μ. Then $P\mathbf{1}_{\{z\}} = \mu(\{z\}) \cdot k_z$ for all $z \in \Lambda_\mu$.*

Proof: As $(P_n)_n$ is an onb of P^2_μ, we obtain

$$P\mathbf{1}_{\{z\}} = \sum_{n\geq 0} \langle\, P_n\,,\mathbf{1}_{\{z\}}\,\rangle P_n = \sum_{n\geq 0} \mu(\{z\})\,\overline{P_n(z)}\,P_n = \mu(\{z\})\cdot k_z\,. \qquad \square$$

Let us now have a short look at the case that $P^2_\mu = L^2_\mu$ and μ is supported in $\mathbb{R}$. Then M_μ is self-adjoint, D is symmetric, and its deficiency indices in the upper and lower half-planes are equal, see also section 4.1 for more details. Combining this with 3.1.3, we get the following.

3.1.9 Proposition. *Suppose μ is an om for $(P_n)_{n\geq 0}$ with* $\operatorname{supp}(\mu) \subset \mathbb{R}$ *and $P^2_\mu = L^2_\mu$. Then either $\mathbb{C}\setminus\mathbb{R} \subset E$ or $(\mathbb{C}\setminus\mathbb{R}) \cap E = \varnothing$. The latter is the case if and only if D is essentially self-adjoint.*

In 4.1.6 we will show that if D is symmetric and not essentially self-adjoint then $E = \mathbb{C}$ and there exists an om μ such that $P^2_\mu = L^2_\mu$ as well as an om ν such that $P^2_\nu \neq L^2_\nu$.

3.2 Subnormal Hessenberg Operators

As already mentioned in 1.3.7, there exists an om for a given sequence $(P_n)_{n\geq 0}$ of polynomials if and only if the Hessenberg operator D is subnormal. In general, however, there is no canonical method to construct normal extensions of a linear operator, it might even be difficult to determine whether there exist any normal extensions at all.

3.2.1 Lemma. *Let N be a normal operator in a Hilbert space $\mathcal{L}$ which extends a densely defined operator A, i.e.* $\operatorname{dom}(A) \subset \operatorname{dom}(N)$ *and $Ax = Nx$ for all $x \in$* $\operatorname{dom}(A)$. *Then A is formally normal.*

Proof: Fix $x \in \operatorname{dom}(N^*)$. Then, for all $y \in \operatorname{dom}(A)$, we have $\langle\, x\,, Ay\,\rangle = \langle\, x\,, Ny\,\rangle = \langle\, N^*x\,, y\,\rangle$ showing $x \in \operatorname{dom}(A^*)$ and $A^*x = N^*x$.
Therefore, $\operatorname{dom}(A) \subset \operatorname{dom}(N) = \operatorname{dom}(N^*) \subset \operatorname{dom}(A^*)$ and $\|Ax\| = \|Nx\| = \|N^*x\| = \|A^*x\|$ for all $x \in \operatorname{dom}(A)$. Hence A is formally normal. $\square$

Note that, if A is subnormal but has no normal extension in the space $\mathcal{L}$ but in a larger space $\mathcal{K} \supset \mathcal{L}$ then A need not be formally normal and A^* need not extend N^*. An example for a subnormal but not formally normal operator is the unilateral shift, see 1.3.8 for more details.

3.2.2 Corollary. *Let μ be an om for $(P_n)_{n\geq 0}$. If $P^2_\mu = L^2_\mu$ then D is formally normal.*

Proof: As $(P_n)_n$ is an onb in L^2_μ, we can regard D as a densely defined operator in L^2_μ. Moreover, the multiplication operator M_μ is a normal extension of D, see 1.3.4. Now 3.2.1 completes the proof. $\square$

Remark. Although we know that there exist formally normal operators which are not subnormal, see [Sch] or [Cod], we do not know whether this is possible in the case of Hessenberg operators.

3.2.3 Lemma. *Let A be a formally normal operator in a Hilbert space $\mathcal{L}$.*

(i) *A is closable and its closure $\overline{A}$ is formally normal, too.*

(ii) *If A is bounded then A is essentially normal.*

Proof: Due to $\operatorname{dom}(A) \subset \operatorname{dom}(A^*)$, the adjoint operator A^* is densely defined in $\mathcal{L}$. It is well-known, see e.g. [W, Satz 5.3], that a densely defined operator is closable if and only if its adjoint is densely defined; moreover, then $A^{**} = \overline{A}$ and $\overline{A}^* = A^*$.

To see that $\overline{A}$ is formally normal, we follow [Kl1, Satz 1.1.3]. Let $x \in \operatorname{dom}(\overline{A})$. Then there exists a sequence $(x_n)_n$ in $\operatorname{dom}(A)$ such that $x_n \to x$ and $Ax_n = \overline{A}x_n \to \overline{A}x$ as $n \to \infty$. Moreover, as A is formally normal, $\|A^*(x_n - x_m)\| = \|A(x_n - x_m)\|$ for all $n, m \in \mathbb{N}$. This shows that $(A^*x_n)_n$ is a Cauchy sequence and, as A^* is closed, this yields $x \in \operatorname{dom}(A^*)$ as well as $A^*x_n \to A^*x$. We now obtain

$$\|\overline{A}x\| = \lim_{n\to\infty} \|Ax_n\| = \lim_{n\to\infty} \|A^*x_n\| = \|A^*x\| \ ;$$

hence $\overline{A}$ is formally normal.

Let now A be bounded. Then $\operatorname{dom}(\overline{A}) = \operatorname{dom}(\overline{A}^*) = \mathcal{L}$. As we have just seen, $\overline{A}$ is formally normal. Hence here $\overline{A}$ is normal and A is essentially normal. $\square$

3.2.4 Corollary. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials such that the Hessenberg operator D is continuous and formally normal. Then there exists a unique om μ,* $\operatorname{supp}(\mu)$ *is compact, and $(P_n)_n$ is an onb in L^2_μ.*

Proof: According to 3.2.3, D is essentially normal. Now 1.3.7 yields existence and uniqueness of μ as well as $(P_n)_n$ being an onb in L^2_μ. Moreover, $\overline{D} = M_\mu$ is continuous, too, which implies $\sigma(M_\mu) = \operatorname{supp}(\mu)$ is compact. $\square$

We will now have a closer look at the matrix representation of the Hessenberg operator D to determine a necessary condtition for D being subnormal as well as a necessary and sufficient condition for D being formally normal.

According to 1.2.1, D is a densely defined operator in $\mathcal{H}$ with $\operatorname{dom}(D) = \mathbb{C}[z]$ where $\mathcal{H}$ is the abstract completion of $\mathbb{C}[z]$ with respect to the inner product defined by $\langle P_n , P_m \rangle_{\mathcal{H}} := \delta_{n,m}$. Note that (1.1) yields $d_{ij} = \langle P_i , DP_j \rangle_{\mathcal{H}}$ for $i, j \in \mathbb{N}_0$. Recall that $d_{ij} = 0$ whenever $i > j + 1$.

For $i \in \mathbb{N}_0$, let e_i be the i^{th} standard unit vector of ℓ^2 and denote by $\beta : \ell^2 \to \mathcal{H}$ the isometric isomorphism given by $\beta(e_i) := P_i$. We define an operator in ℓ^2 by $D_2 := \beta^{-1} D \beta$ with $\operatorname{dom}(D_2) = \{(x_n)_{n\geq 0} : x_i \neq 0 \text{ for finitely many } i\} =: \ell_0$.

Clearly, $\left\langle e_i \,, D_2 e_j \right\rangle_{\ell^2} = d_{ij}$ for all $i, j \in \mathbb{N}_0$ and $(D_2 x)_i = \sum\limits_{j \geq i-1} d_{ij} x_j$.

3.2.5 Lemma. *The domain of D_2^* is given by*

$$\operatorname{dom}(D_2^*) = \left\{ (y_n)_n \in \ell^2 : \left(\sum_{k=0}^{\infty} \overline{d_{kj}}\, y_k \right)_j \in \ell^2 \right\}.$$

Moreover, $\left(D_2^* y\right)_j = \left\langle e_j \,, D_2^* y \right\rangle_{\ell^2} = \sum\limits_{k=0}^{\infty} \overline{d_{kj}}\, y_k = \sum\limits_{k=0}^{j+1} \overline{d_{kj}}\, y_k$ *for all $y \in \operatorname{dom}(D_2^*)$.*

Proof: Fix $y \in \operatorname{dom}(D_2^*)$ and set $z := D_2^* y$. Then $\overline{z_j} = \langle z \,, e_j \rangle_{\ell^2} = \left\langle y \,, D_2 e_j \right\rangle_{\ell^2}$ and

$$\left\langle y \,, D_2 e_j \right\rangle_{\ell^2} = \left\langle \sum_{k=0}^{\infty} y_k e_k \,, D_2 e_j \right\rangle_{\ell^2} = \sum_{k=0}^{\infty} \overline{y_k} \left\langle e_k \,, D_2 e_j \right\rangle_{\ell^2} = \sum_{k=0}^{\infty} d_{kj}\, \overline{y_k}\,. \tag{3.3}$$

Therefore, $\left(D_2^* y\right)_j = z_j = \sum\limits_{k=0}^{\infty} \overline{d_{kj}}\, y_k$ and $\left(\sum\limits_{k=0}^{\infty} \overline{d_{kj}}\, y_k \right)_j \in \ell^2$ for all $y \in \operatorname{dom}(D_2^*)$.

To show the converse inclusion, let $y \in \ell^2$ such that $z := \left(\sum\limits_{k=0}^{\infty} \overline{d_{kj}}\, y_k \right)_j \in \ell^2$. For fixed j, we can use (3.3) to obtain

$$\sum_{k=0}^{\infty} \overline{d_{kj}}\, y_k = \left\langle D_2 e_j \,, y \right\rangle_{\ell^2}.$$

Now $\langle e_j \,, z \rangle_{\ell^2} = z_j = \sum\limits_{k=0}^{\infty} \overline{d_{kj}}\, y_k$ yields $\langle e_j \,, z \rangle_{\ell^2} = \left\langle D_2 e_j \,, y \right\rangle_{\ell^2}$ for all j and, by linearity,

$$\langle x \,, z \rangle_{\ell^2} = \left\langle D_2 x \,, y \right\rangle_{\ell^2} \qquad \text{for all } x \in \ell_0\,.$$

Thus $y \in \operatorname{dom}(D_2^*)$ and $z = D_2^* y$.

The last equality in the assertion is due to $d_{kj} = 0$ for $k > j+1$. $\square$

3.2.6 Theorem. *For fixed $i \in \mathbb{N}_0$, $P_i \in \operatorname{dom}(D^*)$ if and only if $\sum\limits_{j=0}^{\infty} |d_{ij}|^2 < \infty$.*
Furthermore, if D is formally normal then $\sum\limits_{j=0}^{\infty} |d_{ij}|^2 < \infty$ for all $i \in \mathbb{N}_0$.

Proof: Clearly, $P_i \in \operatorname{dom}(D^*)$ if and only if $e_i \in \operatorname{dom}(D_2^*)$ and, according to 3.2.5,

$$e_i \in \operatorname{dom}(D_2^*) \iff \left(\sum_{k=0}^{\infty} \overline{d_{kj}}\, \delta_{ik} \right)_j \in \ell^2 \iff \left(\overline{d_{ij}}\right)_j \in \ell^2 \iff \sum_{j=0}^{\infty} |d_{ij}|^2 < \infty\,.$$

If D is formally normal then, by definition, $P_i \in \operatorname{dom}(D^*)$ for all i. Thus the second assertion is an immediate consequence. $\square$

3.2.7 Matrix Representation of D_2 and D. Let d be the infinite matrix $d := (d_{ij})_{i,j\geq 0}$ and d^* its Hermitian $d^* = (d^*_{ij})_{i,j\geq 0}$ where $d^*_{ij} := \overline{d_{ji}}$.

For $i \in \mathbb{N}_0$, denote by $c_i := (d_{ki})_k$ the i^{th} column of the matrix d which contains only finitely many entries different from 0, i.e. $c_i \in \ell_0$. Thus the formal matrix product d^*d is well-defined and

$$(d^*d)_{ij} = \sum_{k=0}^{\infty} d^*_{ik}\, d_{kj} = \sum_{k=0}^{\infty} \overline{d_{ki}}\, d_{kj} = \sum_{k=0}^{\min\{i,j\}+1} \overline{d_{ki}}\, d_{kj} = \langle\, c_i \,,\, c_j \,\rangle_{\ell^2} \,.$$

Moreover, for $i, j \in \mathbb{N}_0$,

$$\Big\langle\, DP_i \,,\, DP_j \,\Big\rangle_{\mathcal{H}} = \Big\langle\, D_2 e_i \,,\, D_2 e_j \,\Big\rangle_{\ell^2} = \sum_{k=0}^{\infty} \Big\langle\, D_2 e_i \,,\, e_k \,\Big\rangle_{\ell^2} \Big\langle\, e_k \,,\, D_2 e_j \,\Big\rangle_{\ell^2} = \sum_{k=0}^{\infty} \overline{d_{ki}}\, d_{kj} = (d^*d)_{ij}$$

and if $P_i, P_j \in \operatorname{dom}(D^*)$ then

$$\begin{aligned}\Big\langle\, D^*P_i \,,\, D^*P_j \,\Big\rangle_{\mathcal{H}} &= \Big\langle\, D_2^* e_i \,,\, D_2^* e_j \,\Big\rangle_{\ell^2} = \sum_{k=0}^{\infty} \Big\langle\, D_2^* e_i \,,\, e_k \,\Big\rangle_{\ell^2} \Big\langle\, e_k \,,\, D_2^* e_j \,\Big\rangle_{\ell^2} \\ &= \sum_{k=0}^{\infty} \Big\langle\, e_i \,,\, D_2 e_k \,\Big\rangle_{\ell^2} \Big\langle\, D_2 e_k \,,\, e_j \,\Big\rangle_{\ell^2} = \sum_{k=0}^{\infty} d_{ik}\, \overline{d_{jk}} = \sum_{k=0}^{\infty} d_{ik}\, d^*_{kj} \,.\end{aligned}$$

If, in particular, $\mathbb{C}[z] \subset \operatorname{dom}(D^*)$ then the matrix product dd^* is well-defined, too.

This leads to a necessary and sufficient criterion for formal normality of the Hessenberg operator D in terms of the coefficients d_{ij}; we just prove a short lemma in advance.

3.2.8 Lemma. *Let V and W be complex vector spaces and $X \subset V$ such that* $\operatorname{lin} X = V$. *Furthermore, let $\langle \cdot \,,\, \cdot \rangle$ be an inner product in W and A, B linear mappings $V \to W$. Then*

$$\|Av\| = \|Bv\| \quad \textit{for all } v \in V \qquad \Longleftrightarrow \qquad \langle\, Ax \,,\, Ay \,\rangle = \langle\, Bx \,,\, By \,\rangle \quad \textit{for all } x, y \in X\,.$$

where $\|\cdot\|$ is the norm on W induced by the inner product.

Proof: Using the polarisation identity, one can easily see that $\|Av\| = \|Bv\|$ for all $v \in V$ implies $\langle\, Ax \,,\, Ay \,\rangle = \langle\, Bx \,,\, By \,\rangle$ for all $x, y \in V$.

To prove the converse implication, assume $\langle\, Ax \,,\, Ay \,\rangle = \langle\, Bx \,,\, By \,\rangle$ for all $x, y \in X$. For arbitrary $v \in V$, there exist $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{C}$, and $x_1, \ldots, x_n \in X$ such that

$$v = \sum_{k=1}^{n} a_k\, x_k$$

and we obtain $\|Av\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a_i}\, a_j \,\langle\, Ax_i \,,\, Ax_j \,\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a_i}\, a_j \,\langle\, Bx_i \,,\, Bx_j \,\rangle = \|Bv\|^2$. □

3.2.9 Theorem. *The operator D is formally normal if and only if $\sum_{j=0}^{\infty} |d_{ij}|^2 < \infty$ for all $i \in \mathbb{N}_0$ and $d^*d = dd^*$.*

Proof: Let D be formally normal. Then $\mathbb{C}[z] \subset \mathrm{dom}(D^*)$ and 3.2.6 yields $\sum_{j=0}^{\infty} |d_{ij}|^2 < \infty$ for all $i \in \mathbb{N}_0$.

Now use 3.2.8 with $V = \mathbb{C}[z]$, $X = \{P_n : n \geq 0\}$, $W = \mathcal{H}$, $A = D$, and $B = D^*$ to conclude $\langle DP_i , DP_j \rangle_{\mathcal{H}} = \langle D^*P_i , D^*P_j \rangle_{\mathcal{H}}$ for all i, j, and the calculations in 3.2.7 show $d^*d = dd^*$.

As to the converse implication, note that, by 3.2.6, $\sum_{j=0}^{\infty} |d_{ij}|^2 < \infty$ implies $P_i \in \mathrm{dom}(D^*)$. Therefore $\mathbb{C}[z] \subset \mathrm{dom}(D^*)$.

Now 3.2.7 shows $\langle DP_i , DP_j \rangle_{\mathcal{H}} = \langle D^*P_i , D^*P_j \rangle_{\mathcal{H}}$ for all i, j. Use 3.2.8 again to conclude that $\|Dp\| = \|D^*p\|$ for all $p \in \mathbb{C}[z]$; hence D is formally normal. $\square$

Note the following consequence. If D is formally normal then

$$\sum_{k=0}^{\infty} |d_{jk}|^2 = \sum_{i=0}^{j+1} |d_{ij}|^2 \qquad \text{for all } j \in \mathbb{N}_0 \, . \tag{3.4}$$

For bounded D, we have the following stronger result concerning formal normality.

3.2.10 Theorem. *Let μ ba a compactly supported om for $(P_n)_{n\geq 0}$ and define $j \in L^2_\mu$ by $j(z) := \overline{z}$. The following properties are equivalent.*

(i) $P^2_\mu = L^2_\mu$,

(ii) $j \in P^2_\mu$,

(iii) $\sum_{j=1}^{\infty} |d_{0j}|^2 = |d_{10}|^2$,

(iv) *D is formally normal.*

(v) *D is essentially normal.*

Moreover, μ is the only om for $(P_n)_n$.

Proof: Note that, as $\mathrm{supp}(\mu) = \sigma(M_\mu)$ is compact, M_μ and hence D is bounded. Clearly, $j \in L^2_\mu$ and (i)$\Rightarrow$(ii) is obvious.

Now assume $j \in P^2_\mu$. As D is continuous, $\mathrm{dom}(\overline{D}) = P^2_\mu$ and P^2_μ is an invariant subspace of M_μ. Therefore, $(P^2_\mu)^\perp$ is an invariant subspace of M^*_μ. Moreover, $p \cdot j = p(M_\mu)j = p(D)j \in P^2_\mu$ for all $p \in \mathbb{C}[z]$. Let $f \in (P^2_\mu)^\perp$. Then

$$\langle M_\mu f , p \rangle = \int \overline{f(z)}\, \overline{z}\, p(z) \; \mathrm{d}\mu(z) = \langle f , p(D)j \rangle = 0 \qquad \text{for all } p \in \mathbb{C}[z] \, .$$

Thus $M_\mu f \in (P^2_\mu)^\perp$ and hence $(P^2_\mu)^\perp$ is an invariant subspace of M_μ as well. It is a well-known fact, see e.g. [W, Aufgabe 5.39] that, as M_μ is bounded and $(P^2_\mu)^\perp$ is an invariant subspace of M_μ as well as of M^*_μ, the space $(P^2_\mu)^\perp$ and also P^2_μ itself reduce M_μ. According to 1.3.7, M_μ is a minimal normal extension of D, implying $P^2_\mu = L^2_\mu$. Thus (ii)$\Rightarrow$(i).

We next show (ii) $\iff$ (iii).
By the parseval equation, we have $j \in P^2_\mu \iff \|j\|^2 = \sum_{n\geq 0} |\langle\, j\,, P_n\,\rangle|^2$. Note that

$$\|j\|^2 = \|\overline{j}\|^2 = \|DP_0\|^2 = |d_{00}|^2 + |d_{10}|^2\ .$$

Furthermore, $\langle\, j\,, P_n\,\rangle = \int z\, P_n(z)\ \mathrm{d}\mu(z) = \langle\, P_0\,, DP_n\,\rangle = \left\langle\, P_0\,, \sum_{i=0}^{n+1} d_{in}P_i\,\right\rangle = d_{0n}$. Hence

$$\|j\|^2 = \sum_{n\geq 0} |\langle\, j\,, P_n\,\rangle|^2 \iff \sum_{j=1}^{\infty} |d_{0j}|^2 = |d_{10}|^2\ .$$

The implication (iv)$\Rightarrow$(iii) is an immediate consequence of (3.4) and (i)$\Rightarrow$(iv) is due to 3.2.2.
For (iv)$\Rightarrow$(v), see 3.2.3, and (v)$\Rightarrow$(i) is due to 1.3.7.
Finally, uniqueness of μ has already been proved in 3.2.4. □

For more equivalences, see also 3.5.9. Furthermore, if supp(μ) is compact then μ is unique even if $P^2_\mu \neq L^2_\mu$, see 3.2.14.

A weaker condition than (3.4) holds if D is subnormal. Remember that there exist formally normal operators which are not subnormal as well as subnormal operators which are not formally normal.

3.2.11 Theorem. *If D is subnormal then*

$$\sum_{k=0}^{\infty} |d_{jk}|^2 \leq \sum_{i=0}^{j+1} |d_{ij}|^2 \qquad \textit{for all } j \in \mathbb{N}_0\ . \tag{3.5}$$

Proof: As D is subnormal, there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a normal operator N in $\mathcal{K}$ such that $Dp = Np$ for all $p \in \mathbb{C}[z]$.
Fix $j \in \mathbb{N}_0$. Then $d_{jk} = \langle\, P_j\,, DP_k\,\rangle_{\mathcal{H}} = \langle\, P_j\,, NP_k\,\rangle_{\mathcal{K}} = \langle\, N^*P_j\,, P_k\,\rangle_{\mathcal{K}}$ for all $k \in \mathbb{N}_0$ and

$$\sum_{k=0}^{\infty} |d_{jk}|^2 = \sum_{k=0}^{\infty} \left|\langle\, N^*P_j\,, P_k\,\rangle_{\mathcal{K}}\right|^2 < \infty\ ,$$

as $(P_n)_n$ is an orthonormal system in $\mathcal{K}$. Moreover, using the Bessel inequality, we obtain

$$\sum_{k=0}^{\infty} \left|\langle\, N^*P_j\,, P_k\,\rangle_{\mathcal{K}}\right|^2 \leq \|N^*P_j\|^2_{\mathcal{K}} = \|NP_j\|^2_{\mathcal{K}} = \|DP_j\|^2_{\mathcal{H}} = \left\|\sum_{i=0}^{j+1} d_{ij}\, P_i\right\|^2_{\mathcal{H}} = \sum_{i=0}^{j+1} |d_{ij}|^2$$

to complete the proof. □

We point out that (3.4) and (3.5) are necessary but not sufficient for formal normality or subnormality of the Hessenberg operator D. However, they will come in handy, see 4.4.5 where we will deduce a necessary condition for the existence of an om when D is a weighted shift, and 4.5.3 or 4.5.7 in order to prove that D is not formally normal or not subnormal, respectively.

3.2.12 Formally Normal Extension of $\boldsymbol{D}$. Let us now canonically extend the multiplication operator D to $\mathbb{C}[z,\overline{z}] := \big\{z \mapsto q(z,\overline{z}) : q \in \mathbb{C}[z,w]\big\}$.

Given $(P_n)_{n\geq 0}$ as in 1.1.1, we can also extend the inner product on $\mathbb{C}[z]$ – with respect to which $(P_n)_n$ is an orthonormal sequence – to a sesquilinear form $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathbb{C}[z,\overline{z}]$ via

$$\langle\,\overline{u}v\,,w\,\rangle = \langle\,u\,,vw\,\rangle \qquad \text{for all } u,v,w \in \mathbb{C}[z,\overline{z}] \tag{3.6}$$

where $\overline{u}$ is defined by $\overline{u}(z) := \overline{u(z)}$. It is the matter of a lengthy but not difficult calculation, see [Kl2, 1.1.4], that this is indeed well-defined and leads to a uniquely determined sesquilinear extension of the inner product on $\mathbb{C}[z]$. Yet, it may happen that $\langle\,u\,,u\,\rangle < 0$ for some $u \in \mathbb{C}[z,\overline{z}]$, see [Kl2, 1.1.5].

In the case that $\langle\,u\,,u\,\rangle \geq 0$ for all $u \in \mathbb{C}[z,\overline{z}]$, let $\mathcal{N} := \{u \in \mathbb{C}[z,\overline{z}] : \langle\,u\,,u\,\rangle = 0\}$. Now $\langle\,\cdot\,,\cdot\,\rangle$ becomes an inner product on the quotient space $\mathbb{C}[z,\overline{z}]/\mathcal{N}$ whose abstract completion, in analogy to the completion $\mathcal{H}$ of $\mathbb{C}[z]$ defined in section 1.2, we will denote by $\mathcal{K}$. Regard $\mathcal{H}$ as a closed subspace of $\mathcal{K}$. Then the canonical extension of the multiplication operator D to $\mathbb{C}[z,\overline{z}]$ leads to a well-defined operator F in $\mathcal{K}$ with $\operatorname{dom}(F) = \mathbb{C}[z,\overline{z}]/\mathcal{N}$, see [Kl2, 1.3.12], for instance.

Assume that there exists an om μ for $(P_n)_n$. Clearly, every $u \in \mathbb{C}[z,\overline{z}]$ is square integrable with respect to μ. As shown in [Kl2, 1.1.6], when we regard $u \in \mathbb{C}[z,\overline{z}]$ as a member of L^2_μ, the inner product on L^2_μ extends $\langle\,\cdot\,,\cdot\,\rangle$ constructed in (3.6) and

$$\mathcal{N} = \Big\{u \in \mathbb{C}[z,\overline{z}] \;:\; \int |u|^2 \,\mathrm{d}\mu = 0\Big\}\,.$$

Now $\mathcal{K}$ can be canonically embedded into L^2_μ and the operator M_μ in L^2_μ is a minimal normal extension not only of D but also of F. In particular, every normal extension of D is a normal extension of F as well. Note that, if F is essentially normal then the unique minimal normal extension of F is its closure $\overline{F}$ which must also be the unique minimal normal extension of D. However, in this situation D need not be essentially normal.

Finally, we point out that F^n is formally normal for all $n \in \mathbb{N}$. To see that, set $u(z) := z^n$ for fixed n; now (3.6) yields

$$\langle\,v\,,F^n w\,\rangle = \langle\,v\,,uw\,\rangle = \langle\,\overline{u}v\,,w\,\rangle \qquad \text{for all } v,w \in \mathbb{C}[z,\overline{z}]\,,$$

hence $\mathbb{C}[z,\overline{z}] \subset \operatorname{dom}\big[(F^n)^*\big]$ and $\big[(F^n)^*v\big](z) = (\overline{u}v)(z) = \overline{z^n}v(z)$; using (3.6) once again, we obtain

$$\|F^n v\|^2_{\mathcal{K}} = \langle\,uv\,,uv\,\rangle = \langle\,\overline{u}v\,,\overline{u}v\,\rangle = \|(F^n)^*v\|^2_{\mathcal{K}}\,.$$

3.2.13 Example. Let μ be an om for $(P_n)_{n\geq 0}$ such that $\operatorname{supp}(\mu) \subset \{z \in \mathbb{C} : |z| = 1\}$; set $q(z,w) := 1 - zw$, and define $u \in \mathbb{C}[z,\overline{z}]$ by $u(z) := q(z,\overline{z}) = 1 - |z|^2$.

Then $u(z) = 0$ whenever $z \in \operatorname{supp}(\mu)$ and hence $u \in \mathcal{N}$.

The operator F is studied in depth in [Kl2]; we have only mentioned here some of its most important properties in order to give a short proof of the following fundamental fact concerning uniqueness of compactly supported om.

3.2.14 Proposition. *If there exists a compactly supported om μ for $(P_n)_{n\geq 0}$ then μ is the only om at all.*

Proof: As $\operatorname{supp}(\mu) = \sigma(M_\mu)$ is compact, M_μ is bounded. Thus the operator F defined in 3.2.12 is bounded, too, and 3.2.3 implies that F is essentially normal. Therefore, F has a unique minimal extension $\overline{F}$. As we have seen in 3.2.12, every normal extension of D is a normal extension of F. Thus M_μ is the unique minimal normal extension of D, and 1.3.7 implies uniqueness of the orthonormalizing measure. □

3.2.15 Definition. Let S be a (not necessarily densely defined) linear operator in a Hilbert space $\mathcal{H}$ with invariant domain, i.e. $\operatorname{ran}(S) \subset \operatorname{dom}(S)$. Then S is said to obey the **Halmos condition** (see [Kl2, 1.3.20], [Br, Introduction], or [StSz2, p.156 (H)], for instance), if

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \langle\, S^j q_i \,,\, S^i q_j \,\rangle_{\mathcal{H}} \geq 0$$

for all $n \in \mathbb{N}_0$ and all sequences $(q_i)_i$ in $\operatorname{dom}(S)$.

3.2.16 Proposition. *If D is bounded then the following properties are equivalent.*

(i) *There exists an om,*

(ii) *there exists a unique om,*

(iii) *there exists a compactly supported om,*

(iv) $\langle\, u \,,\, u \,\rangle \geq 0$ *for all* $u \in \mathbb{C}[z, \overline{z}]$ *with* $\langle\, \cdot \,,\, \cdot \,\rangle$ *defined as in* 3.2.12*,*

(v) *D obeys the Halmos condition.*

For a proof we refer to [Kl2, 1.3.21 and 1.5.4].

The main effort to prove this, is to see that F is bounded whenever D is bounded and obeys the Halmos condition which is, for example, shown in [Kl2, 1.5.1] following an essay by Bram [Br].

3.3 Spectral Properties of the Multiplication Operator

In this section, we will prove more detailed relations between the set E, the spectrum of D, and – if an om μ exists – the support of μ. Recall that in 3.1.3 we have seen that E is precisely the set of numbers where the deficiency index of D is equal to 1 and, according to 3.1.7, the discrete mass points of an om must belong to E.

Some statements in the following theorem can be found in [StSz3, Th. 1 and Cor. 12] and (v) extends [Halm, Problem 158].

3.3.1 Theorem. *Let μ be an om for $(P_n)_{n\geq 0}$. Regard D as a densely defined operator in P^2_μ and M_μ as a minimal normal extension of D acting in L^2_μ (see also* 1.3.7*). Then*

(i) $\sigma(M_\mu) \subset \sigma(\overline{D})$,

(ii) $\sigma_p(\overline{D}) \subset \sigma_p(M_\mu) = \Lambda_\mu \subset E$ *and* $\sigma_p(\overline{D}) \subset E \setminus E_{reg}$,

(iii) $E = \sigma_p(\overline{D}) \cup \sigma_r(\overline{D})$ *and* $\sigma_c(\overline{D}) = \sigma(\overline{D}) \setminus E$,

(iv) $\sigma_c(\overline{D}) \subset \sigma_c(M_\mu)$,

(v) *if $A \subset \rho(M_\mu)$ is connected then either $A \subset \rho(\overline{D})$ or $A \subset E_{reg}$,*

(vi) $\sigma(M_\mu) \setminus E = \sigma(\overline{D}) \setminus E = \sigma_c(\overline{D})$,

(vii) $E \cup \sigma(M_\mu) = \sigma(\overline{D})$.

Proof: First note that D is subnormal and M_μ is a minimal normal extension of D. In particular, D is closable.

(i) Equivalently, we show $\rho(\overline{D}) \subset \rho(M_\mu)$. The following is taken from [StSz3, Th. 1 - 9°]. Let $\lambda \in \rho(\overline{D})$. Then $(\overline{D} - \lambda\,\mathrm{id})^{-1}$ is a continuous linear operator on $\mathcal{H}$. Now choose $\varepsilon > 0$ such that $\varepsilon\,\|(\overline{D} - \lambda\,\mathrm{id})^{-1}\| < 1$; define $\Delta := \{z \in \mathbb{C} : |z - \lambda| < \varepsilon\}$ and $h := \mathbf{1}_\Delta \in L^2_\mu$. Then $h \in \mathrm{dom}(M_\mu - \lambda\,\mathrm{id})^{*n}$ and

$$\begin{aligned}
\mu(\Delta) &= \int h \cdot P_0 \ \mathrm{d}\mu = \big|\langle\, h\,, (\overline{D} - \lambda\,\mathrm{id})^n (\overline{D} - \lambda\,\mathrm{id})^{-n} P_0 \,\rangle\big| \\
&= \big|\langle\, h\,, (M_\mu - \lambda\,\mathrm{id})^n (\overline{D} - \lambda\,\mathrm{id})^{-n} P_0 \,\rangle\big| = \big|\langle\, (M_\mu - \lambda\,\mathrm{id})^{*n} h\,, (\overline{D} - \lambda\,\mathrm{id})^{-n} P_0 \,\rangle\big| \\
&\leq \|(M_\mu - \lambda\,\mathrm{id})^{*n} h\| \cdot \|(D - \lambda\,\mathrm{id})^{-1}\|^n \cdot \|P_0\| = \Big(\int_\Delta \big|\overline{z} - \overline{\lambda}\big|^{2n} \,\mathrm{d}\mu \Big)^{\frac{1}{2}} \|(D - \lambda\,\mathrm{id})^{-1}\|^n \\
&\leq \big(\varepsilon^{2n} \mu(\Delta)\big)^{\frac{1}{2}} \ \|(D - \lambda\,\mathrm{id})^{-1}\|^n = \mu(\Delta)^{\frac{1}{2}} \ \big(\varepsilon\,\|(D - \lambda\,\mathrm{id})^{-1}\|\big)^n
\end{aligned}$$

for all $n \in \mathbb{N}$. This implies $\mu(\Delta) = 0$. Thus Δ is an open μ-nullset and, therefore, $\Delta \subset \mathbb{C} \setminus \mathrm{supp}(\mu) = \rho(M_\mu)$. In particular, $\lambda \in \rho(M_\mu)$.

(ii) $\sigma_p(\overline{D}) \subset \sigma_p(M_\mu)$ is obvious. For $\sigma_p(M_\mu) = \Lambda_\mu \subset E$ see 3.1.7 and, by definition, an eigenvalue of $\overline{D}$ cannot be a regular value of $\overline{D}$. Thus $\sigma_p(\overline{D}) \subset E \setminus E_{reg}$.

(iii) Due to 3.1.3, $E = \{\lambda \in \mathbb{C} : \mathrm{ran}(D - \lambda\,\mathrm{id})$ is not dense in $P^2_\mu\}$. Note that $\mathrm{ran}(D - \lambda\,\mathrm{id})$ is not dense in P^2_μ if and only if $\mathrm{ran}(\overline{D} - \lambda\,\mathrm{id})$ is not dense in P^2_μ. Hence $E \subset \sigma(\overline{D})$.

Furthermore, by definition of the residual and continuous spectrum, we have $\sigma_r(\overline{D}) \subset E$ and $\sigma_c(\overline{D}) \cap E = \varnothing$.

Finally, $\sigma_p(\overline{D}) \subset E$, according to (ii).

As $\sigma(\overline{D})$ is the union of the disjoint sets $\sigma_r(\overline{D}) \cup \sigma_c(\overline{D}) \cup \sigma_p(\overline{D})$, we get $E = \sigma_p(\overline{D}) \cup \sigma_r(\overline{D})$ and $\sigma_c(\overline{D}) = \sigma(\overline{D}) \setminus E$.

(iv) If $\lambda \in \sigma_c(\overline{D})$ then $(\overline{D} - \lambda\,\mathrm{id})^{-1}$ is densely defined in P^2_μ and not continuous. Suppose $\lambda \in \rho(M_\mu)$. Then $(M_\mu - \lambda\mathrm{id})^{-1}$ is a continuous linear operator in L^2_μ extending $(\overline{D} - \lambda\mathrm{id})^{-1}$ which is a contradiction. Hence $\sigma_c(\overline{D}) \subset \sigma(M_\mu)$.

Moreover, $\sigma_c(\overline{D}) \cap E = \varnothing$ according to (iii). Now (ii) implies $\sigma_c(\overline{D}) \cap \sigma_p(M_\mu) = \varnothing$. As M_μ is normal, $\sigma_r(M_\mu) = \varnothing$. Thus $\sigma_c(\overline{D}) \subset \sigma_c(M_\mu)$.

(v) Obviously, if λ is not a regular value of $\overline{D}$ then it is not a regular value of M_μ, either. As M_μ is normal, $\mathrm{reg}(M_\mu) = \rho(M_\mu)$. Thus $\rho(M_\mu) \subset \mathrm{reg}(\overline{D})$.

Let now $A \subset \rho(M_\mu)$ be connected. According to 3.1.2, the deficiency index is constant in every connected subset of $\mathrm{reg}(D) = \mathrm{reg}(\overline{D})$. Hence either $A \subset E$ or $A \cap E = \varnothing$.

In the first case we obtain $A \subset E_{reg}$.

Consider now $A \subset \mathbb{C} \setminus E$. By (iii) we have $\mathbb{C} \setminus E = \mathbb{C} \setminus \big(\sigma_p(\overline{D}) \cup \sigma_r(\overline{D})\big) = \rho(\overline{D}) \cup \sigma_c(\overline{D})$. Moreover, $\sigma_c(\overline{D}) \subset \sigma_c(M_\mu)$, see (iv). Using $A \subset \rho(M_\mu)$ this yields $A \cap \sigma_c(\overline{D}) = \varnothing$.

Hence only $A \subset \rho(\overline{D})$ remains.

(vi) Using the previous results, we get

$$\sigma(M_\mu) \setminus E \overset{\text{(i)}}{\subset} \sigma(\overline{D}) \setminus E \overset{\text{(iii)}}{=} \sigma_c(\overline{D}) = \sigma_c(\overline{D}) \setminus E \overset{\text{(iv)}}{\subset} \sigma(M_\mu) \setminus E\,,$$

thus $\sigma(M_\mu) \setminus E = \sigma(\overline{D}) \setminus E = \sigma_c(\overline{D})$.

(vii) Using (iii) and (vi), we see $\sigma(M_\mu) \cup E = \big(\sigma(M_\mu) \setminus E\big) \cup E = \sigma_c(\overline{D}) \cup \sigma_p(\overline{D}) \cup \sigma_r(\overline{D}) = \sigma(\overline{D})$. $\square$

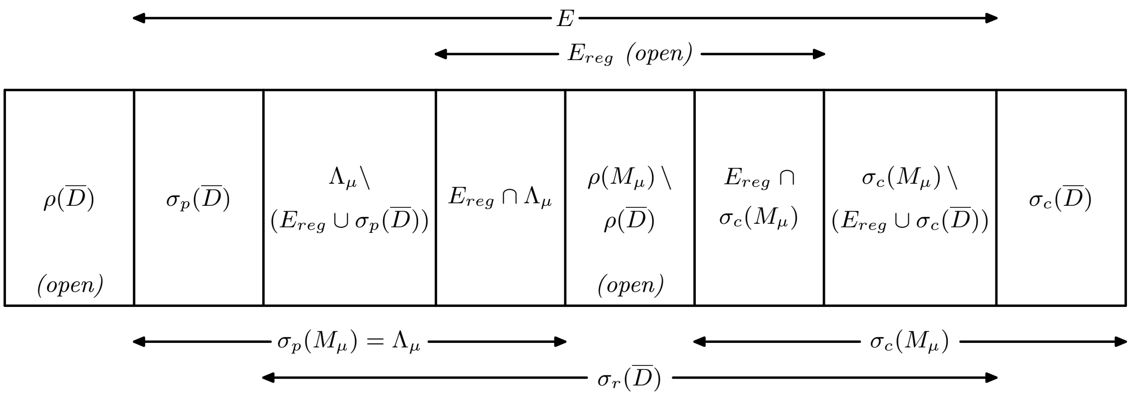


Figure 2.

This graphically summarizes the assertions of 3.3.1.

We have $\rho(\overline{D}) \subset \rho(M_\mu)$ and $\sigma_p(\overline{D}) \subset \sigma_p(M_\mu)$ as well as $\sigma_c(\overline{D}) \subset \sigma_c(M_\mu)$, while $\sigma_r(\overline{D})$ can contain parts of $\sigma_p(M_\mu)$, $\sigma_c(M_\mu)$, and also of $\rho(M_\mu)$.

Note that $\rho(M_\mu)$ and hence all its connected components are open. As shown in 3.3.1(v), every connected component of $\rho(M_\mu)$ is contained in either $\rho(\overline{D})$ or E_{reg}. Therefore, $\rho(M_\mu) \setminus \rho(\overline{D}) \subset E_{reg}$ is open.

Recall that, according to 3.1.4, E_{reg} is open, too.

If D is essentially normal, i.e. $\overline{D}$ is normal, we obtain the following.

3.3.2 Corollary. *If D is essentially normal then there exists a unique om μ, $\overline{D} = M_\mu$, and $E = \Lambda_\mu = \{z \in \mathbb{C} : \mu(\{z\}) > 0\}$.*

Proof: According to 1.3.7, there exists a unique om μ and $\overline{D} = M_\mu$ is the only minimal normal extension of D.
As the residual spectrum of a normal operator is empty, 3.3.1(ii) and (iii) yield $\sigma_p(M_\mu) = E = \Lambda_\mu$. □

Note that E can be finite or even empty.

Note also that if, in 3.3.1, D is bounded then $\sigma(\overline{D})$ and hence also E is bounded. In 3.1.6, however, we have already proved that D bounded implies E bounded, even if there exists no om.

Remark. If M_μ is bounded then D is bounded, too. On the other hand, if D is bounded and there exists an om μ then 3.3.1(i) shows that D bounded implies M_μ bounded. Hence D is bounded if and only if M_μ is bounded, provided that there exists an om μ at all.

The following is an immediate consequence of 3.3.2 in connection with 3.2.10.

3.3.3 Proposition. *Let μ be an om for $(P_n)_{n\geq 0}$ and $\operatorname{supp}(\mu)$ be compact. If $(P_n)_n$ is an onb in L^2_μ (in other words, $P^2_\mu = L^2_\mu$) then D is essentially normal, $E = \Lambda_\mu$, and μ is unique.*

Remark. Recall that in 3.2.14 we have seen that if there exists a compactly supported om then it is uniquely determined – even if $P^2_\mu \neq L^2_\mu$.
To prove uniqueness in this case, we could also use the fact that, according to [Con1, II.2.7], a bounded subnormal operator has a unique – up to unitary equivalence – minimal normal extension.

Let us now have a closer look at the boundary of E,

$$\partial E := \{z \in \mathbb{C} : U \cap E \neq \varnothing \text{ and } U \cap (\mathbb{C} \setminus E) \neq \varnothing \text{ for all open sets } U \text{ containing } z\}\,.$$

We start with a short lemma.

3.3.4 Lemma. *If there exists an om μ for $(P_n)_{n\geq 0}$ then $\mathbb{C} \setminus \operatorname{supp}(\mu) \subset \operatorname{reg}(D)$.*

Proof: As M_μ is normal, $\rho(M_\mu) = \operatorname{reg}(M_\mu)$. Clearly, $z \notin \operatorname{reg}(D)$ implies $z \notin \operatorname{reg}(M_\mu)$. Now $\mathbb{C} \setminus \operatorname{supp}(\mu) = \rho(M_\mu)$ completes the proof. □

3.3.5 Theorem. *If there exists an om μ for $(P_n)_{n\geq 0}$ then $\partial E \subset \operatorname{supp}(\mu)$.*

Proof: In the cases $\partial E = \varnothing$ or $\operatorname{supp}(\mu) = \mathbb{C}$, the assertion is trivial.

Now assume that there exists $z \in \partial E \cap \big(\mathbb{C} \setminus \operatorname{supp}(\mu)\big)$. As $\mathbb{C} \setminus \operatorname{supp}(\mu) = \rho(M_\mu)$ is open, we can find an open disk U containing z such that $U \subset \rho(M_\mu)$. By 3.3.1(v), either $U \subset \rho(\overline{D})$ or $U \subset E_{reg}$. Moreover, $\rho(\overline{D}) \subset (\mathbb{C} \setminus E)$, see 3.3.1(iii), and $E_{reg} \subset E$. Thus either $U \subset (\mathbb{C} \setminus E)$ or $U \subset E$ which contradicts $z \in \partial E$. Hence $\partial E \cap \big(\mathbb{C} \setminus \operatorname{supp}(\mu)\big) = \varnothing$ as asserted. $\square$

3.3.6 Corollary. *Let G be a simply connected bounded open subset of $\mathbb{C}$ and μ an om for $(P_n)_n$ such that $\operatorname{supp}(\mu) \subset \partial G$. If $E \setminus \operatorname{supp}(\mu) \neq \varnothing$ then*

(i) $G \subset E \subset G \cup \partial G$ *and* $G = E^\circ = E_{reg}$,

(ii) $\operatorname{supp}(\mu) = \partial G$,

(iii) $P^2_\mu \neq L^2_\mu$ *and* D *is not essentially normal.*

Proof: (i) We first show $E \subset \overline{G}$.

Let $U := \mathbb{C} \setminus \overline{G}$ and assume that there exists $z \in E \cap U$. Note that, as G and hence also $\overline{G}$ is simply connected and bounded, U is connected. By assumption, $U \cap \operatorname{supp}(\mu) = \varnothing$, thus $U \subset \rho(M_\mu)$ and 3.3.1(v) implies $U \subset E_{reg}$. In particular, E is not bounded.

On the other hand, as $\operatorname{supp}(\mu)$ is bounded, the multiplication operator M_μ and, consequently, D is bounded. Now, by 3.1.6, E is bounded and we have a contradiction.

Therefore, $E \cap U = \varnothing$, in other words, $E \subset \overline{G}$.

Now we show $G \subset E$.

Note that $\partial E \subset \operatorname{supp}(\mu)$ and $E \setminus \operatorname{supp}(\mu) \neq \varnothing$ imply $E^\circ \neq \varnothing$. Hence $E^\circ \cap G \neq \varnothing$ and 3.3.1(v) implies $G \subset E_{reg}$.

Now $E^\circ = G \subset E_{reg}$ is an immediate consequence. According to 3.1.4, E_{reg} is open; hence $E_{reg} \subset E^\circ$ and, finally, we obtain $G = E^\circ = E_{reg}$.

(ii) Furthermore, $G \subset E \subset G \cup \partial G$ implies $\partial E = \partial G$ and, again using $\partial E \subset \operatorname{supp}(\mu)$, we get $\partial G \subset \operatorname{supp}(\mu)$. Given $\operatorname{supp}(\mu) \subset \partial G$, we now have $\partial G = \operatorname{supp}(\mu)$.

(iii) If D is essentially normal or $P^2_\mu = L^2_\mu$ then 3.3.2 and 3.3.3, respectively, imply $E = \Lambda_\mu$ which contradicts $E \setminus \operatorname{supp}(\mu) \neq \varnothing$. $\square$

Remark. If $\operatorname{supp}(\mu)$ is a proper subset of ∂G then 3.3.6(ii) implies $E \subset \operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu)$ is an α-set in accordance to section 4.3. In that case, 4.3.4 shows that D is essentially normal. However, it is also possible that D is essentially normal if $\operatorname{supp}(\mu) = \partial G$, as we will see in section 4.2, where G is the open unit disk.

3.3.7 Example. Let $\gamma : [0,1] \to \mathbb{C}$ be a simply closed piecewise continuously differentiable curve and μ be absolutely continuous with respect to line measure on γ, having density α, such that

$$\int\limits_{\gamma} \frac{1}{\alpha(z)} \,|\,\mathrm{d}z| < \infty$$

(see 2.3.8 for the details).
Moreover, assume $\mu(\gamma) = 1$, hence μ is an om for some $(P_n)_n$, and denote by G the bounded component of $\mathbb{C} \setminus \gamma$.

Then, according to 2.3.11, $G \subset E$. Thus 3.3.6 is applicable to this situation.

Remark. If $\mathrm{supp}(\mu)$ contains an open set, a similar integrability condition concerning the absolutely continuous (with respect to two-dimensional Lebesgue mesaure) part of μ implies $P^2_\mu \neq L^2_\mu$, see 3.6.5.

3.3.8 Theorem. *Let μ be an om for $(P_n)_{n\geq 0}$ and denote by P the orthogonal projection in L^2_μ onto P^2_μ. For $a \in E$ and $b \in E \setminus \mathrm{supp}(\mu)$, define $f_{a,b} := (M_\mu - a)^*[(M_\mu - b)^{-1}]^* k_a$.*

Then $P f_{a,b} = \langle\, f_{a,b}\,, P_0 \,\rangle\, k_b$.

Moreover, $f_{a,b} = 0$ if and only if $a \in \Lambda_\mu$ and $\mathbf{1}_{\{a\}} \in P^2_\mu$.

Proof: As $b \notin \mathrm{supp}(\mu)$, $(M_\mu - b)^{-1}$ is a bounded linear operator in L^2_μ. Furthermore, $\varphi(z) := \frac{z-a}{z-b}$ defines a bounded function on $\mathrm{supp}(\mu)$.

Hence $(M_\mu - a)^*[(M_\mu - b)^{-1}]^* f = \overline{\varphi} \cdot f \in L^2_\mu$ for all $f \in L^2_\mu$ and $f_{a,b}$ is well-defined.

For arbitrary $p \in \mathbb{C}[z]$, define $q \in \mathbb{C}[z]$ by $q(z) := p(z) - p(b)$. Note that $\langle\, k_b\,, q \,\rangle = q(b) = 0$ and $\langle\, k_b\,, P_0 \,\rangle = P_0(b) = 1$. This yields

$$\begin{aligned}
&\langle\, f_{a,b} - \langle\, f_{a,b}\,, P_0 \,\rangle k_b\,, p \,\rangle \\
&\qquad = \langle\, f_{a,b}\,, q \,\rangle + \langle\, f_{a,b}\,, p(b)P_0 \,\rangle - \langle\, f_{a,b}\,, P_0 \,\rangle\langle\, k_b\,, q \,\rangle - \langle\, f_{a,b}\,, P_0 \,\rangle\langle\, k_b\,, p(b)P_0 \,\rangle \\
&\qquad = \langle\, f_{a,b}\,, q \,\rangle = \langle\, k_a\,, (M_\mu - b)^{-1}(M_\mu - a)q \,\rangle = 0\,,
\end{aligned}$$

as $(M_\mu - b)^{-1}(M_\mu - a)q \in \mathbb{C}[z]$ has a zero at a.

Thus $f_{a,b} - \langle\, f_{a,b}\,, P_0 \,\rangle k_b \in (P^2_\mu)^\perp$ and, as $k_b \in P^2_\mu$, we obtain $P f_{a,b} = \langle\, f_{a,b}\,, P_0 \,\rangle\, k_b$.

Note that $\overline{\varphi} \cdot f = 0$ if and only if $f = c\mathbf{1}_{\{a\}}$ for some $c \in \mathbb{C}$. According to 3.1.8, if $a \in \Lambda_\mu$ and $\mathbf{1}_{\{a\}} \in P^2_\mu$ then $\mathbf{1}_{\{a\}} = \mu(\{a\})\, k_a$; hence $f_{a,b} = 0$. As to the converse implication, $0 = f_{a,b} = \overline{\varphi} \cdot k_a$ implies $k_a = c\mathbf{1}_{\{a\}}$ showing $a \in \Lambda_\mu$ and $\mathbf{1}_{\{a\}} \in P^2_\mu$. $\square$

3.4 Denseness of $\mathbb{C}[z]$ in L^2_μ

Let μ be an om for $(P_n)_{n\geq 0}$. We now consider the case that $\mathbb{C}[z]$ is dense in L^2_μ. In other words, $P^2_\mu = L^2_\mu$.

Assume that these polynomials form a basis of $\mathcal{H}(K)$, too. If $E^\circ \neq \varnothing$ then there exists an open set G which is dense in E° and all $f \in \mathcal{H}(K)$ are holomorphic in G, see 2.2.9. On the other hand, in general, L^2_μ contains elements which do not have a holomorphic representative.

Thus, if $P^2_\mu = L^2_\mu$, one can suppose that G – except for some discrete points – is a μ-nullset. We will prove that, in this case, $\mu(E \setminus \Lambda_\mu) = 0$ even if the $(P_n)_n$ are not orthogonal in $\mathcal{H}(K)$.

As for now, we start with a more general situation.

3.4.1 Definitions. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space and $(h_n)_{n\geq 0}$ be an onb in L^2_μ. Define

$$\pi : \Omega \to [0, \infty]\,, \qquad \pi(\omega) := \Big(\sum_{n\geq 0} \big|h_n(\omega)\big|^2 \Big)^{\frac{1}{2}}$$

and $E_\pi := \{\omega \in \Omega : \pi(\omega) < \infty\}$. Then π is μ-measurable and $E_\pi \in \mathcal{A}$ (note that π and E_π depend on the particular choice of the representatives of h_n).

Moreover, define $K : E_\pi \times E_\pi \to \mathbb{C}$, $K(\omega', \omega) := \sum\limits_{n\geq 0} \overline{h_n(\omega')}\, h_n(\omega)$.

A set $B \in \mathcal{A}$ is called a **block** (or **atom**) if $\mu(B) > 0$ and

$$\mu(C) = 0 \quad \text{or} \quad \mu(B \setminus C) = 0 \qquad \text{for all } C \in \mathcal{A}\,,\ C \subset B\,.$$

Note that if B is a block and N a nullset then $B \cup N$ and $B \setminus N$ are blocks, too.

A (possibly finite and) at most countable family $(A_j)_j$ of mutually disjoint sets in $\mathcal{A}$ such that $\bigcup\limits_j A_j = A$ is called a **partition** of A.

3.4.2 Lemma. *If $A \in \mathcal{A}$, $A \subset E_\pi$, $\mu(A) < \infty$, and $\int\limits_A \pi \,\mathrm{d}\mu < \infty$, then*

$$\mu(A) = \int\limits_{A\times A} K \,\mathrm{d}\,\mu\otimes\mu\,.$$

Proof: For $N \in \mathbb{N}$, $\Big|\sum\limits_{n=0}^{N} \overline{h_n(\omega')}\, h_n(\omega)\Big| \leq \pi(\omega')\,\pi(\omega)$, and

$$\sum_{n=0}^{N} |\langle\, \mathbf{1}_A\,, h_n \,\rangle|^2 = \sum_{n=0}^{N} \Big(\int\limits_A \overline{h_n} \,\mathrm{d}\mu \Big) \Big(\int\limits_A h_n \,\mathrm{d}\mu \Big) = \sum_{n=0}^{N} \int\limits_{A\times A} \overline{h_n} \times h_n \,\mathrm{d}\,\mu\otimes\mu$$

$$= \int\limits_{A\times A} \sum_{n=0}^{N} \overline{h_n} \times h_n \,\mathrm{d}\,\mu\otimes\mu\,.$$

As $\pi \times \pi$ is integrable on $A \times A$, dominated convergence yields

$$\mu(A) = \|\mathbf{1}_A\|^2 = \sum_{n=0}^{\infty} |\langle \mathbf{1}_A , h_n \rangle|^2 = \int_{A\times A} K \,\mathrm{d}\,\mu\otimes\mu\,. \qquad \square$$

Now denote by $\mathcal{Z}(A)$ the class of all partitions of A; for $\mathcal{P} = (A_j)_j \in \mathcal{Z}(A)$ define $s(\mathcal{P}) := \sup_j \mu(A_j)$ and set $s_A := \inf_{\mathcal{P}\in\mathcal{Z}(A)} s(\mathcal{P})$.

3.4.3 Lemma. *Let A be as in 3.4.2. If, in addition, $\mu(A) > 0$ then $s_A > 0$.*

Proof: Assume $s_A = 0$. Then, for $n \in \mathbb{N}$, there exists $\mathcal{P}_n = (A_j^n)_j \in \mathcal{Z}(A)$ such that $s(\mathcal{P}_n) < \frac{1}{n\,\mu(A)}$.

Without restriction we can assume $\mathcal{P}_{n+1} \subset \mathcal{P}_n$ for all n, i.e. for every $B \in \mathcal{P}_{n+1}$ there exists $A_j^n \in \mathcal{P}_n$ such that $B \subset A_j^n$.

For abbreviation, set $C_n := \bigcup_j A_j^n \times A_j^n$. We have

$$\mu\otimes\mu(C_n) = \sum_j \mu\otimes\mu(A_j^n \times A_j^n) = \sum_j \big(\mu(A_j^n)\big)^2 \le \frac{1}{n\,\mu(A)} \sum_j \mu(A_j^n) = \frac{1}{n}$$

and, using 3.4.2, we obtain $\mu(A) = \sum_j \mu(A_j^n) = \sum_j \int_{A_j^n\times A_j^n} K \,\mathrm{d}\,\mu\otimes\mu = \int_{C_n} K \,\mathrm{d}\,\mu\otimes\mu$.

Due to $\mu\otimes\mu(C_n) = \frac{1}{n}$ and $C_{n+1} \subset C_n$ for all n, the integral on the right hand side tends to 0 as $n \to \infty$. This implies $\mu(A) = 0$ which is a contradiction. Hence $s_A > 0$. $\square$

3.4.4 Lemma. *Let A be as in 3.4.2. If, in addition, $0 < \mu(A) < \infty$ then there exists a block $B \subset A$.*

Proof: According to 3.4.3, $s := s_A > 0$. Note that $s < \infty$, as $\mu(A) < \infty$. Thus we can choose a partition $\mathcal{P} \in \mathcal{Z}(A)$ with $s(\mathcal{P}) < \frac{6}{5}\,s$. Now we can find $A_1 \in \mathcal{P}$ such that

$$\mu(C) \ge \tfrac{4}{5}\,s \quad \text{or} \quad \mu(A_1 \setminus C) \ge \tfrac{4}{5}\,s \qquad \text{for all } C \subset A_1\,,\ C \in \mathcal{A}$$

because otherwise we could construct $\mathcal{P}' \in \mathcal{Z}(A)$ with $s(\mathcal{P}') \le \frac{4}{5}\,s$.

Note that $\mu(A_1 \setminus C) \ge \frac{4}{5}\,s$ implies $\mu(C) < \frac{2}{5}\,s$, as $\mu(A_1) < \frac{6}{5}\,s$. Thus

$$\mu(C) \ge \tfrac{4}{5}\,s \quad \text{or} \quad \mu(C) < \tfrac{2}{5}\,s \qquad \text{for all } C \subset A_1\,,\ C \in \mathcal{A}\,. \tag{3.7}$$

Let $C_i \in \mathcal{A}$, $C_i \subset A_1$, and $\mu(C_i) \ge \frac{4}{5}\,s$ for $i \in \{1,2\}$. Assume $\mu(C_1 \cap C_2) < \frac{2}{5}\,s$. Then $\mu(C_1 \cup C_2) = \mu(C_1) + \mu(C_2) - \mu(C_1 \cap C_2) \ge \frac{8}{5}\,s - \frac{2}{5}\,s = \frac{6}{5}\,s$ in contradiction to $\mu(A_1) < \frac{6}{5}\,s$.

Thus (3.7) yields $\mu(C_1 \cap C_2) \ge \frac{4}{5}\,s$. $\quad(*)$

Now set $s_1 := \inf\{\mu(C) : C \in \mathcal{A}\,,\ C \subset A_1\,,\ \mu(C) \ge \frac{4}{5}\,s\}$ and choose a sequence $(C_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $C_n \subset A_1$ and $\mu(C_n) \ge \frac{4}{5}\,s$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \mu(C_n) = s_1$.

For $n \in \mathbb{N}$, let $\widetilde{C}_n := \bigcap_{i=1}^{n} C_i$.

Note that $\mu(\widetilde{C}_n)$ is monotonically decreasing as $n \to \infty$ and, in analogy to $(*)$, inductively, we obtain $\mu(\widetilde{C}_n) \geq \frac{4}{5}\, s$ while, by construction, $\mu(\widetilde{C}_n) \leq \mu(C_n)$ for all n. Hence

$$\tfrac{4}{5}\, s \leq \lim_{n\to\infty} \mu(\widetilde{C}_n) \leq \lim_{n\to\infty} \mu(C_n) = s_1 \,. \tag{3.8}$$

Define $B := \bigcap_{n\in\mathbb{N}} \widetilde{C}_n$.

Then $B \in \mathcal{A}$, $B \subset A_1$, and (3.8) reads $\frac{4}{5}\, s \leq \mu(B) \leq s_1$. On the other hand, $\mu(B) \geq \frac{4}{5}\, s$ implies $\mu(B) \geq s_1$, simply by definition of s_1. Thus only $\mu(B) = s_1$ remains.

Finally, we show that B is a block. Assume that there exists $C \in \mathcal{A}$, $C \subset B$ such that $0 < \mu(C) < \mu(B) = s_1$. Recall the definition of s_1 again to see that $\mu(C) \geq \frac{4}{5}\, s$ or $\mu(B \setminus C) \geq \frac{4}{5}\, s$ would imply $\mu(C) \geq s_1$ or $\mu(B \setminus C) \geq s_1$, respectively, and yield a contradiction. Now (3.7) shows that $\mu(C) < \frac{2}{5}\, s$ and also $\mu(B \setminus C) < \frac{2}{5}\, s$ in contradiction to $\mu(B) \geq \frac{4}{5}\, s$, completing the proof. □

3.4.5 Lemma. *Let A be as in* 3.4.2*. Then A is the union of at most countably many blocks $A = \bigcup_n B_n$ and $\mu(B_i \cap B_j) = 0$ for $i \neq j$.*

Proof: If $B_1, B_2 \subset A$ are blocks such that $\mu(B_1 \cap B_2) > 0$ then $\mu(B_1) = \mu(B_1 \cap B_2) = \mu(B_2)$ and hence $\mu\big((B_1 \cup B_2) \setminus (B_1 \cap B_2)\big) = 0$. In this case B_1 and B_2 are said to be *equivalent* (note that $\mu(B_1 \cap B_2) > 0$ and $\mu(B_2 \cap B_3) > 0$ for blocks $B_1, B_2, B_3 \subset A$ implies $\mu(B_1 \cap B_3) > 0$).

Assume that there exist $n \in \mathbb{N}$ and a sequence $(B_k)_{k\in\mathbb{N}}$ of mutually non-equivalent blocks in A such that $\mu(B_k) > \frac{1}{n}$ for all $k \in \mathbb{N}$.
As $\mu(B_i \cap B_j) = 0$ for $i \neq j$, this implies $\mu\big(\bigcup_{k\in\mathbb{N}} B_k\big) = \sum_{k\in\mathbb{N}} \mu(B_k) = \infty$ in contradiction to $\mu(A) < \infty$.
Hence, for every $n \in \mathbb{N}$, there exist only finitely many mutually non-equivalent blocks whose measure exceeds $\frac{1}{n}$. Therefore, there exist at most countably many classes of equivalence.
Now let $(B_i)_i$ be a sequence of blocks in A which contains exactly one element of each equivalence class and set $B := \bigcup_i B_i$. Then $A \setminus B$ does not contain a block and, by 3.4.4, $\mu(A \setminus B) = 0$.

Clearly, $B_1 \cup (A \setminus B)$ is a block, too, equivalent to B_1. Thus $A = \big(\bigcup_{i\neq 1} B_i\big) \cup \big(B_1 \cup (A \setminus B)\big)$ completes the proof. □

3.4.6 Theorem. *In the situation of* 3.4.1*, E_π is the union of at most countably many blocks.*

Proof: As μ is σ-finite, there exists a partition $(A_n)_n$ of Ω such that $\mu(A_n) < \infty$ for all n. Now, for $m \in \mathbb{N}$ set

$$A_{nm} := A_n \cap \{\omega \in \Omega : \pi(\omega) < m\} \,.$$

If $\mu(A_{nm}) > 0$ then, by construction, A_{nm} fulfils the premises of 3.4.5 and hence is the union of at most countably many blocks. As the union af a block and a nullset is a block, too, E_π is the union of at most countably many blocks as well. $\square$

We now have a closer look at what blocks look like in $\mathbb{R}^d$ with respect to a σ-finite measure defined on the Borel sets.

3.4.7 Theorem. *Denote by $\mathfrak{B}(\mathbb{R}^d)$ the Borel σ-algebra in $\mathbb{R}^d$ and let μ be a σ-finite measure on $\mathfrak{B}(\mathbb{R}^d)$.*
If $B \in \mathfrak{B}(\mathbb{R}^d)$ is a block with respect to μ then there exists $b \in B$ such that $\mu(\{b\}) > 0$ and $\mu(B \setminus \{b\}) = 0$.

Proof: For $m \in \mathbb{N}$, let $\mathbb{R}^d = \bigcup\limits_{n\in\mathbb{N}} C_n^m$ where each C_n^m is a closed ball with radius $\frac{1}{m}$.
For an arbitrary block B, in particular, $\mu(B \cap C_n^1) = 0$ or $\mu(B \setminus C_n^1) = 0$. Assume $\mu(B \cap C_n^1) = 0$ for all n. Then $\mu(B) = 0$ which is a contradiction. Thus there exists $n_1 \in \mathbb{N}$ such that $\mu(B \setminus C_{n_1}^1) = 0$.
Set $B_1 := B \cap C_{n_1}^1$. Then $\mu(B_1) = \mu(B)$. In analogy to the above, we can find $n_2 \in \mathbb{N}$ such that $\mu(B_1 \setminus C_{n_2}^2) = 0$ and with $B_2 := B_1 \cap C_{n_2}^2$ we get $\mu(B_2) = \mu(B_1)$. Repeating this, we obtain a sequence $(B_j)_{j\in\mathbb{N}}$ satisfying $B_{j+1} \subset B_j$ and $\mu(B_j) = \mu(B)$ for all $j \in \mathbb{N}$. Furthermore, each B_j is contained in a ball with radius $\frac{1}{j}$. Therefore, $B_* := \bigcap\limits_{j\in\mathbb{N}} B_j$ contains at most one point.
Moreover, $\mu(B_*) = \lim\limits_{j\to\infty} \mu(B_j) = \mu(B) > 0$, hence $B_* \neq \varnothing$.
Thus $B_* = \{b\}$ for some $b \in \mathbb{R}^d$ and $\mu(\{b\}) = \mu(B) > 0$, $\mu(B \setminus \{b\}) = 0$. $\square$

3.4.8 Corollary. *Let μ be an om such that $P_\mu^2 = L_\mu^2$. Then $\mu(E \setminus \Lambda_\mu) = 0$.*

Proof: $\Lambda_\mu \subset E$ is due to 3.1.7 and, by 3.4.6, E is the union of at most countably many blocks, $E = \bigcup\limits_n B_n$, say.
According to 3.4.7, for each of these blocks there exists $b_n \in B_n$ such that $\mu(\{b_n\}) > 0$ and $\mu(B_n \setminus \{b_n\}) = 0$. Hence $b_n \in \Lambda_\mu$ and $B_n \setminus \{b_n\} \cap \Lambda_\mu = \varnothing$ for all n.
Therefore, $\mu(E \setminus \Lambda_\mu) = \mu\big(\bigcup\limits_n (B_n \setminus \{b_n\})\big) = 0$. $\square$

Remark. We will return to this situation in 3.5.7, where we will be able to prove that E_{reg} does not contain a limit point of Λ_μ.
If D is essentially normal then $E = \Lambda_\mu$, see 3.3.2. Thus, in that case, 3.4.8 is trivial.
Note that, if D is continuous and has a normal extension in $\mathcal{H}$ (i.e. $P_\mu^2 = L_\mu^2$) then D is essentially normal, as $\overline{D}$ is defined on the whole space L_μ^2.

Now consider the case that $P_\mu^2 = L_\mu^2$ and D is not essentially normal, hence M_μ is a proper extension of $\overline{D}$ in the same space. In particular, then D is not continuous.

3.4.9 Lemma. *Let S and T be closed linear operators in a Hilbert space $\mathcal{H}$ such that* $\mathrm{dom}(S) \subset \mathrm{dom}(T)$ *and $Tf = Sf$ for all $f \in \mathrm{dom}(S)$. If $\rho(S) \cap \rho(T) \neq \varnothing$ then $S = T$.*

Proof: Let $z \in \rho(S) \cap \rho(T)$. Then $(S - z\,\mathrm{id})^{-1}$ and $(T - z\,\mathrm{id})^{-1}$ are continuous linear operators in $\mathcal{H}$ and for arbitrary $f \in \mathcal{H}$,

$$(T - z\,\mathrm{id})(T - z\,\mathrm{id})^{-1}f = f = (S - z\,\mathrm{id})(S - z\,\mathrm{id})^{-1}f = (T - z\,\mathrm{id})(S - z\,\mathrm{id})^{-1}f\,.$$

As $(T - z\,\mathrm{id})$ is one-to-one, this implies $(T - z\,\mathrm{id})^{-1}f = (S - z\,\mathrm{id})^{-1}f$. Hence $(T - z\,\mathrm{id})^{-1} = (S - z\,\mathrm{id})^{-1}$.

Let now $g \in \mathrm{dom}(T)$. Then $g = (T - z\,\mathrm{id})^{-1}(T - z\,\mathrm{id})g = (S - z\,\mathrm{id})^{-1}(T - z\,\mathrm{id})g \in \mathrm{dom}(S)$. Thus $\mathrm{dom}(T) \subset \mathrm{dom}(S)$ which completes the proof. □

3.4.10 Theorem. *Let $(P_n)_{n\geq 0}$ be as in* 1.1.1. *Assume that there exists an om μ such that $P^2_\mu = L^2_\mu$ and the multiplication operator D is not essentially normal. Then*

(i) $\sigma(\overline{D}) = \mathbb{C}$,

(ii) $\mathrm{reg}(D) = E_{reg}$,

(iii) $\rho(M_\mu) \subset E_{reg}$ *and* $E_{reg} \cup \mathrm{supp}(\mu) = \mathbb{C}$,

(iv) $E^\circ \setminus \overline{\Lambda_\mu} = \rho(M_\mu)$.

Proof: (i) According to 3.3.1(i), we have $\rho(\overline{D}) \subset \rho(M_\mu)$. By 3.4.9, $\rho(\overline{D}) \cap \rho(M_\mu) \neq \varnothing$ would imply $\overline{D} = M_\mu$ in contradiction to D being not essentially normal. Hence only $\rho(\overline{D}) = \varnothing$ remains.

(ii) Recall $\mathrm{reg}(D) = \mathrm{reg}(\overline{D}) \subset \rho(\overline{D}) \cup \sigma_r(\overline{D})$, see 3.1.2, and $\sigma_r(D) \subset E$, see 3.3.1(iii). Thus here $\mathrm{reg}(D) \subset E$ and, therefore, $\mathrm{reg}(D) = \mathrm{reg}(D) \cap E = E_{reg}$.

(iii) As $\rho(\overline{D}) = \varnothing$, the inclusion $\rho(M_\mu) \subset E_{reg}$ is an immediate consequence of 3.3.1(v). Now $\mathbb{C} = \rho(M_\mu) \cup \sigma(M_\mu) \subset E_{reg} \cup \sigma(M_\mu)$ implies $E_{reg} \cup \sigma(M_\mu) = \mathbb{C}$.

(iv) $E^\circ \setminus \overline{\Lambda_\mu}$ is open and, by 3.4.8, a μ-nullset. Thus $E^\circ \setminus \overline{\Lambda_\mu} \subset \mathbb{C} \setminus \mathrm{supp}(\mu) = \rho(M_\mu)$. For the converse inclusion note that $\Lambda_\mu = \sigma_p(M_\mu)$, hence $\rho(M_\mu) \cap \Lambda_\mu = \varnothing$. By (iii), $\rho(M_\mu) \subset E$. Thus $\rho(M_\mu) \subset E \setminus \Lambda_\mu$. As $\rho(M_\mu)$ is open, this yields $\rho(M_\mu) \subset E^\circ \setminus \overline{\Lambda_\mu}$. □

3.4.11 Corollary. *If, in the situation of* 3.4.10, $E^\circ = \varnothing$ *then* $\mathrm{supp}(\mu) = \mathbb{C}$.

Proof: According to 3.1.4, E_{reg} is open; therefore, $E^\circ = \varnothing$ implies $E_{reg} = \varnothing$. Now, using 3.4.10(iii), we obtain $\mathrm{supp}(\mu) = \mathbb{C}$ as asserted. □

Remark. In 3.6.5 we will see that the density of the absolutely continuous (w.r.t. Lebesgue measure) part of μ must be of rather peculiar appearance (or vanish) in order to have $P^2_\mu = L^2_\mu$. See there for details.

3.5 Open Subsets of E

If E contains a non-void open set then there exists an open set G which is dense in E° such that all $f \in \mathcal{H}(K)$ are holomorphic in G, see 2.2.9.
As we shall prove in this section, if there exists an om then E_{reg} is the largest open set contained in E where all functions of $\mathcal{H}(K)$ are holomorphic.

Define $G_K := \bigcup \big\{ G \subset E : \text{all } \ f \in \mathcal{H}(K) \ \text{ are holomorphic in } \ G \big\}$.

As the property "holomorphic" is only defined in open sets, G_K is the union of open sets and hence open. Moreover, all $f \in \mathcal{H}(K)$ are holomorphic in G_K and there exists no proper superset $G' \supsetneqq G_K$ such that all $f \in \mathcal{H}(K)$ are holomorphic in G'.

3.5.1 Lemma. *The map $z \mapsto K(z,z)$ is continuous on E_{reg}.*

Proof: For $z \in E$, define $h_z := \frac{k_z}{\|k_z\|_{\mathcal{H}}} = \frac{1}{\sqrt{K(z,z)}}\, k_z$.
Moreover, for $z \in E_{reg}$ let Q_z be the orthogonal projection in $\mathcal{H}$ onto $\operatorname{ran}(D - z\,\mathrm{id})^\perp$. Taking into account that $\operatorname{ran}(D - z\,\mathrm{id})^\perp = \mathbb{C} \cdot h_z$ (see 3.1.3) and $\|h_z\|_{\mathcal{H}} = 1$, we get

$$Q_z f = \langle\, h_z \,, f \,\rangle_{\mathcal{H}}\, h_z \qquad \text{for all } f \in \mathcal{H}\, .$$

Let now $(z_n)_{n\in\mathbb{N}}$ be a convergent sequence in $\operatorname{reg}(D)$ such that $\lim\limits_{n\to\infty} z_n =: z_* \in \operatorname{reg}(D)$.
Recall that, by 3.1.2, $\|Q_{z_n} - Q_{z_*}\| \to 0$ as $n \to \infty$. Thus

$$(Q_{z_n} - Q_{z_*})h_{z_*} = \langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}\, h_{z_n} - \langle\, h_{z_*} \,, h_{z_*} \,\rangle_{\mathcal{H}}\, h_{z_*} = \langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}\, h_{z_n} - h_{z_*} \tag{3.9}$$

implies $\langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}\, h_{z_n} \to h_{z_*}$ as $n \to \infty$. Due to $\|h_{z_n}\|_{\mathcal{H}} = 1$ for all n, we obtain $|\langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}| \to 1$. Moreover, (3.9) yields

$$\begin{aligned} \Big\langle\, (Q_{z_n} - Q_{z_*})h_{z_*} \,, P_0 \,\Big\rangle_{\mathcal{H}} &= \langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}\, \langle\, h_{z_n} \,, P_0 \,\rangle_{\mathcal{H}} - \langle\, h_{z_*} \,, P_0 \,\rangle_{\mathcal{H}} \\ &= \langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}\, \frac{1}{\sqrt{K(z_n,z_n)}} - \frac{1}{\sqrt{K(z_*,z_*)}}\, . \end{aligned}$$

Using $|\langle\, h_{z_n} \,, h_{z_*} \,\rangle_{\mathcal{H}}| \to 1$, this implies $\frac{1}{\sqrt{K(z_n,z_n)}} \to \frac{1}{\sqrt{K(z_*,z_*)}}$ and, finally,

$$K(z_n, z_n) \to K(z_*, z_*) \qquad \text{as } n \to \infty\, ,$$

showing continuity of $z \mapsto K(z,z)$. $\square$

3.5.2 Theorem. *If $E_{reg} \neq \varnothing$ then all $f \in \mathcal{H}(K)$ are holomorphic in E_{reg}.*

Proof: According to 3.5.1, the map $\kappa : E \to [1, \infty)$, $\kappa(z) := K(z,z)$ is continuous in E_{reg}. Thus, for every compact $K \subset E_{reg}$ there exists $c_K > 0$ such that $\kappa(z) \leq c_K$ for all $z \in K$. Now 2.2.3 yields the assertion. $\square$

The following corollary, given there exists an om μ, characterizes $E \setminus \operatorname{supp}(\mu)$ as an open set in which all $f \in \mathcal{H}(K)$ are holomorphic. Note that we do not require $\operatorname{supp}(\mu) \subset E$.

3.5.3 Corollary. *Let μ be an om for $(P_n)_{n\geq 0}$. Then $E \setminus \operatorname{supp}(\mu)$ is open. Moreover, if $E \setminus \operatorname{supp}(\mu) \neq \varnothing$ then all $f \in \mathcal{H}(K)$ are holomorphic in $E \setminus \operatorname{supp}(\mu)$.*

Proof: According to 3.3.4, $\mathbb{C} \setminus \operatorname{supp}(\mu) \subset \operatorname{reg}(D)$. This implies $E \setminus \operatorname{supp}(\mu) \subset E_{reg}$. Now $\partial E \subset \operatorname{supp}(\mu)$, see 3.3.5, yields that $E \setminus \operatorname{supp}(\mu)$ is open and, as all $f \in \mathcal{H}(K)$ are holomorphic in E_{reg}, see 3.5.2, the proof is complete. $\square$

In 3.5.2 we have seen that $E_{reg} \subset G_K$. To prove the converse inclusion, we use the following lemma.

3.5.4 Lemma. *Suppose $(f_n)_n$ is a sequence in $\mathcal{H}(K)$, $M > 0$, and $\lambda \in E$ such that $\|f_n\|_{\mathcal{H}(K)} \leq M$ for all n and $\lim\limits_{n\to\infty} f_n(z)$ exists for all $z \in E \setminus \{\lambda\}$.*

Then there exist a subsequence $(f_{n_k})_k$ and $f \in \mathcal{H}(K)$ such that

$$\lim_{k\to\infty} \langle f_{n_k}, h \rangle_{\mathcal{H}(K)} = \langle f, h \rangle_{\mathcal{H}(K)} \qquad \text{for all } h \in \mathcal{H}(K)$$

and, in particular, $\lim\limits_{k\to\infty} f_{n_k}(z) = f(z)$ for all $z \in E$.

Proof: As $\left|f_n(\lambda)\right| = \left|\langle K_\lambda, f_n \rangle_{\mathcal{H}(K)}\right| \leq \|K_\lambda\|_{\mathcal{H}(K)} \|f_n\|_{\mathcal{H}(K)} \leq M \|K_\lambda\|_{\mathcal{H}(K)}$ for all n, we can find a subsequence $(f_{n_k})_k$ such that $\lim\limits_{k\to\infty} f_{n_k}(\lambda)$ exists.

As $(f_{n_k})_k$ is a bounded and pointwisely convergent sequence in an RKHS, it is weakly convergent, see A.3.2; in other words, $\lim\limits_{k\to\infty} \langle f_{n_k}, h \rangle_{\mathcal{H}(K)} = \langle f, h \rangle_{\mathcal{H}(K)}$ for all $h \in \mathcal{H}(K)$ where $f(z) := \lim\limits_{k\to\infty} f_{n_k}(z)$ and $f \in \mathcal{H}(K)$. $\square$

3.5.5 Theorem. *If there exists an om μ for $(P_n)_{n\geq 0}$ then $G_K \subset E_{reg}$.*

Proof: The case $G_K = \varnothing$ is trivial.

Let now $\lambda \in G_K$. According to 2.2.2, $z \mapsto \kappa(z) := K(z, z)$ is continuous in G_K. Hence there exist $c > 0$ and $r > 0$ such that $\kappa(z) \leq c$ for all $z \in U := \{z \in \mathbb{C} : |z - \lambda| < r\}$.

Assume $\lambda \notin \operatorname{reg}(D)$. Then there exists a sequence $(q_n)_n$ in $\mathbb{C}[z]$ such that $\|q_n\|_{\mathcal{H}} = 1$ for all n and $\|(D - \lambda \operatorname{id})q_n\|_{\mathcal{H}} \to 0$ as $n \to \infty$. By (2.4), $\|p\|_{\mathcal{H}(K)} \leq \|p\|_{\mathcal{H}}$ for all $p \in \mathbb{C}[z]$; hence $\|(D - \lambda \operatorname{id})q_n\|_{\mathcal{H}(K)} \to 0$ and $\|q_n\|_{\mathcal{H}(K)} \leq 1$ for all n.

Now, by 1.4.4, $(z - \lambda)q_n(z) \to 0$ for all $z \in E$ which implies $q_n(z) \to 0$ for all $z \in E \setminus \{\lambda\}$ and we can apply 3.5.4 to obtain a subsequence weakly convergent in $\mathcal{H}(K)$. We tacitly assume that $(q_n)_n$ is already such a subsequence and denote the weak limit by q. As the weak limit is also the pointwise limit and $q \in \mathcal{H}(K)$ is holomorphic in an open neighborhood of λ, we get $q(\lambda) = 0$.

As μ is an om,

$$
\begin{aligned}
1 = \left\|q_n\right\|_{\mathcal{H}}^2 = \left\|q_n\right\|_{L^2_\mu}^2 &= \int\limits_{\mathbb{C}\setminus U} \tfrac{|z-\lambda|^2}{|z-\lambda|^2} \left|q_n(z)\right|^2 \mathrm{d}\mu + \int\limits_U \left|q_n(z)\right|^2 \mathrm{d}\mu \\
&\leq \tfrac{1}{r^2} \int\limits_{\mathbb{C}\setminus U} |z-\lambda|^2 \left|q_n(z)\right|^2 \mathrm{d}\mu + \int\limits_U \left|q_n(z)\right|^2 \mathrm{d}\mu \\
&\leq \tfrac{1}{r^2} \left\|(D-\lambda\,\mathrm{id})q_n\right\|_{L^2_\mu}^2 + \int\limits_U \left|q_n(z)\right|^2 \mathrm{d}\mu \qquad \text{for all } n\,. \qquad (3.10)
\end{aligned}
$$

Furthermore, for $z \in U$, we have $\left|q_n(z)\right| \leq \sqrt{\kappa(z)}\ \|q_n\|_{\mathcal{H}(K)} \leq \sqrt{c}$.
Using dominated convergence and $\lim\limits_{n\to\infty} q_n(z) = 0$, we obtain $\lim\limits_{n\to\infty} \int\limits_U \left|q_n(z)\right|^2 \mathrm{d}\mu = 0$.
Together with $\|(D-\lambda\,\mathrm{id})q_n\|_{\mathcal{H}} \to 0$, this yields

$$
\lim_{n\to\infty} \Big(\tfrac{1}{r^2} \left\|(D-\lambda\,\mathrm{id})q_n\right\|_{\mathcal{H}}^2 + \int\limits_U \left|q_n(z)\right|^2 \mathrm{d}\mu \Big) = 0
$$

in contradiction to (3.10).
Therefore, we get $\lambda \in \mathrm{reg}(D)$ to complete the proof. □

We are now able to characterize the set G_K by varoius means. For $R > 0$, recall $A_R = \{z \in E : \kappa(z) \leq R\}$, see 2.2.7.

3.5.6 Theorem. *The set G_K can be characterized as follows.*

$$
\begin{aligned}
G_K &= \big\{z \in E^\circ : \kappa \text{ is continuous in a neighborhood of } z\big\} \\
&= \big\{z \in E^\circ : \kappa \text{ is bounded in a neighborhood of } z\big\} \\
&= \big\{z \in E^\circ : \sum_{n\geq 0} \left|P_n(z)\right|^2 \text{ is uniformly convergent in a neighborhood of } z\big\} \\
&= \bigcup_{k\in\mathbb{N}} A_k^\circ\,.
\end{aligned}
$$

If, in addition, there exists an om for $(P_n)_n$ then $G_K = E_{reg}$ and $E_{reg} \neq \varnothing \iff E^\circ \neq \varnothing$.

Proof: Recall that, by construction, G_K is open and $G_K \subset E^\circ$.

(a) Let $z_0 \in G_K$. By 2.2.2, κ is continuous in G_K. In particular, κ is continuous in a neighborhood of z_0.

(b) Let now $z_0 \in E^\circ$ such that κ is continuous in a neighborhood of z_0. Then clearly there exists a neighborhood of z_0 where κ is bounded.

(c) Let $z_0 \in E^\circ$ such that κ is bounded in a neighborhood of z_0. Then there exist an open disk U centered at z_0 and $m \in \mathbb{N}$ such that $\kappa(z) \leq m$ for all $z \in U$, in other words, $U \subset A_m$. As U is open, this implies $z_0 \in A_m^\circ \subset \bigcup\limits_{k\in\mathbb{N}} A_k^\circ$.

(d) The inclusion $\bigcup\limits_{k\in\mathbb{N}} A_k^\circ \subset G_K$ is due to 2.2.9.

(e) We show $G_K = \big\{ z \in E^\circ : \sum_{n\geq 0} \big|P_n(z)\big|^2$ is uniformly convergent in a neighborhood of $z \big\}$.

Due to (a) and (b), for $z_0 \in G_K$, we can find $m \in \mathbb{N}$ such that $z_0 \in A^\circ_m$ and a compact disc $C \subset A^\circ_m$ centered at z_0 such that κ is continuous on C. Now Dini's theorem (see [Kö, 15.7], for example) yields uniform convergence of $\sum_{n\geq 0} \big|P_n(z)\big|^2$ on C.

For the converse inclusion, use that the limit of a uniformly convergent sequence of continuous functions is continuous. Suppose that $\sum_{n\geq 0} \big|P_n(z)\big|^2$ is uniformly convergent in a neighborhood U of z_0.

This implies that κ is continuous in U; using (b), (c), and (d), we see $z_0 \in G_K$.

(f) If there exists an om for $(P_n)_{n\geq 0}$ then $G_K = E_{reg}$ is an immediate consequence of 3.5.2 and 3.5.5. Moreover, by 2.2.9, G_K is dense in E°; hence $G_K \neq \varnothing \iff E^\circ \neq \varnothing$. □

Remark. In the case that there exists a compactly supported om, the characterization

$$G_K = \big\{ z \in E^\circ : \kappa \text{ is continuous in a neighborhood of } z \big\}$$

is proved in [Con1, II.7.6]. In the more special case that $\operatorname{supp}(\mu)$ is contained in the closed unit disk, the equality $G_K = E_{reg}$ is the subject of [Tr, Theorem 1.1].

3.5.7 Theorem. *Let μ be an om such that $P^2_\mu = L^2_\mu$. Then $\mu(E \setminus \Lambda_\mu) = 0$ and $E_{reg} \cap \Lambda_\mu$ does not have a limit point within E_{reg}. In particular, $E_{reg} \cap \Lambda_\mu$ is a discrete set.*

Proof: In 3.4.8, we have already shown $\mu(E \setminus \Lambda_\mu) = 0$.

Assume now that there exists a convergent sequence $(z_n)_n$ of mutually different points in $E_{reg} \cap \Lambda_\mu$ such that $z_* := \lim_{n\to\infty} z_n \in E_{reg}$.

Recall that, by 3.1.7, $\mu(\{z_n\}) = \frac{1}{\kappa(z_n)}$ for all n and note that $\mu(\{z_n\}) \to 0$ as $n \to \infty$. According to 3.5.6, κ is continuous on E_{reg} which yields a contradiction.

Therefore, a limit point of $E_{reg} \cap \Lambda_\mu$ cannot belong to E_{reg}. □

Remark. Note that we did not have to require $z_* \in \Lambda_\mu$.

Furthermore, recall that if D is essentially normal then $E = \Lambda_\mu$, see 3.3.2; hence in that case we have $E_{reg} = \varnothing$ and 3.5.7 is trivial. Moreover, note that if $\operatorname{supp}(\mu)$ is compact and $P^2_\mu = L^2_\mu$ then D is essentially normal, see 3.2.10.

3.5.8 Open Questions.

(i) Is $G_K \neq E^\circ$ possible ?

(ii) Given an om μ such that $P^2_\mu \neq L^2_\mu$, is E° always non-empty ?

According to [Con1, II.7.11], (i) is an open question even if there exists a compactly supported om. Regarding orthonormalizing measures supported on the closed unit disk, this question also appears – and remains unanswered – in [Tr, Remark after Cor. 1.3]. In 4.5.4 we will give an example where $E = \{z \in \mathbb{C} : |z| < 1\} \cup \{1\}$ and κ is not continuous at $z = 1$ but still continuous in E°.

Assume now that there exists an om μ for $(P_n)_n$ such that $G_K \neq E^\circ$. As shown in 3.5.6, $G_K = E_{reg}$, hence $\lambda \in E^\circ \backslash G_K$ is not a regular value of D and, therefore, not a regular value of M_μ, either. As M_μ is normal, $\mathrm{reg}(M_\mu) = \rho(M_\mu)$, implying $E^\circ \backslash G_K \subset \sigma(M_\mu) = \mathrm{supp}(\mu)$. Moreover, G_K does not contain "holes" in the sense that the interior of a domain, whose boundary is a simply connected rectifiable curve in G_K, is completely contained in G_K, see 2.2.14; furthermore, as shown in 2.2.9, G_K is dense in E°.

For any $f \in \mathcal{H}(K)$, there exists a sequence $(q_n)_n$ of polynomials convergent to f w.r.t. the norm in $\mathcal{H}(K)$, hence pointwisely on E and, in particular, uniformly on every compact subset of G_K. Thus, if $E^\circ \setminus G_K \neq \varnothing$ then there exists a sequence of polynomials convergent pointwisely on E and uniformly on any compact subset of G_K whose limit is not holomorphic in (at least some subset of) $E^\circ \setminus G_K$.

Given a pointwisely, on some open set $A \subset \mathbb{C}$, convergent sequence of polynomials, the question what those domains look like where the limit function is not holomorphic, was already discussed in a paper by Hartogs and Rosenthal [HR1] published in 1928. The answer is rather complicated; more recently, that topic appears in a paper by Davie [Da]. Yet, we still cannot give an answer to question (i) above.

It is worth mentioning that Beardon and Minda [BM] show how to construct a sequence $(q_n)_n$ of polynomials such that $q_n(z) \to 1$ whenever z is a non-negative real number while, otherwise, $q_n(z) \to 0$. They also present a sequence of polynomials convergent to 0 on the whole complex plane except for one point where the limit is 1.

Concerning question (ii), Conway proves [Con1, VIII.4.3] that if there exists a compactly supported om μ such that $P^2_\mu \neq L^2_\mu$ then $E^\circ \neq \varnothing$, using an idea by Thomson [Th]. In a later paper [CY, sec. 4] he calls that method "excellent but complicated", supposing that there might exist a somewhat simpler proof. However, we add the following.

3.5.9 Theorem. *Let μ be a compactly supported om for $(P_n)_{n\geq 0}$. The following properties are equivalent.*

(i) $P^2_\mu = L^2_\mu$,

(ii) *D is formally normal,*

(iii) *D is essentially normal,*

(iv) $(z \mapsto \overline{z}) \in P^2_\mu$,

(v) $\sum_{j=1}^{\infty} |d_{0j}|^2 = |d_{10}|^2$,

(vi) $E = \Lambda_\mu$,

(vii) $E^\circ = \varnothing$.

(viii) $E_{reg} = \varnothing$.

Proof: We have already proved equivalence of the first five properties in 3.2.10.
The implication (iii)⇒(vi) is due to 3.3.2 and (vi)⇒(vii) is clear. By 3.5.6, we have (vii) $\iff$ (viii), and, finally, (vii) implies (i) by [Con1, VIII.4.3]. □

Remark. The implications (iii)⇒(i), (iii)⇒(vi), and (i)⇒(ii) as well as the equivalence (vii) $\iff$ (viii) hold even if $\operatorname{supp}(\mu)$ is unbounded, but we still do not know whether, in the unbounded case, $E^\circ = \varnothing$ implies $P^2_\mu = L^2_\mu$.

3.6 Point Evaluation on $\mathbb{C}[z]$

While in an RKHS point evaluation always is well defined and continuous, point evaluation in L^2_μ in general is not defined. However, point evaluation exists on $\mathbb{C}[z]$ and if μ is an om for $(P_n)_{n\geq 0}$ then we can establish relations between the members of P^2_μ and the members of $\mathcal{H}(K)$. In particular, we will see that every element in P^2_μ has a representative that pointwisely on E coincides with a function in $\mathcal{H}(K)$.

3.6.1 Lemma. *Let μ be a measure on $\mathfrak{B}(\mathbb{C})$ such that $\mu(\mathbb{C}) = 1$, and $\mathbb{C}[z] \subset L^2_\mu$. Fix $a \in \mathbb{C}$ and define $\eta_a := \inf\{\|p\|_{L^2_\mu} : p \in \mathbb{C}[z]\,,\ p(a) = 1\}$.*

There exists a constant $c_a > 0$ such that

$$\big|p(a)\big| \leq c_a \|p\|_{L^2_\mu} \qquad \text{for all } p \in \mathbb{C}[z] \tag{3.11}$$

if and only if $\eta_a > 0$.
Moreover, if $\eta_a > 0$ then $c_a = \frac{1}{\eta_a}$ is the smallest constant satisfying (3.11).

Proof: Suppose that there exists $c_a > 0$ such that $\big|p(a)\big| \leq c_a \|p\|_{L^2_\mu}$ for all $p \in \mathbb{C}[z]$ and assume $\eta_a = 0$. Then there exists a sequence $(p_n)_n$ in $\mathbb{C}[z]$ such that $p_n(a) = 1$ for all n and $\|p_n\|_{L^2_\mu} \to 0$ as $n \to \infty$. On the other hand, $1 = \big|p_n(a)\big| \leq c_a \|p_n\|_{L^2_\mu}$ for all n which is a contradiction. Hence $\eta_a > 0$.

For the converse, suppose that $\eta_a > 0$.
Now choose arbitrary $p \in \mathbb{C}[z]$ with $p(a) \neq 0$, and set $\widetilde{p}(z) := \frac{p(z)}{p(a)}$. Then $\widetilde{p}(a) = 1$ and

$$\|p\|^2_{L^2_\mu} = \int \big|p(z)\big|^2 \,\mathrm{d}\mu = \big|p(a)\big|^2 \int \big|\widetilde{p}(z)\big|^2 \,\mathrm{d}\mu = \big|p(a)\big|^2 \ \|\,\widetilde{p}\,\|^2_{L^2_\mu} \geq \eta_a^2 \,\big|p(a)\big|^2 \,.$$

Thus $\left|p(a)\right| \le \frac{1}{\eta_a}\,\|p\|_{L^2_\mu}$. Note that this is also valid if $p(a)=0$.

It remains to show that $c_a = \frac{1}{\eta_a}$ is the smallest constant satisfying (3.11).
By definition of η_a, there exists a sequence $(p_n)_n$ in $\mathbb{C}[z]$ such that $p_n(a)=1$ for all n and $\|p_n\|_{L^2_\mu} \to \eta_a$ as $n\to\infty$. Let now $c>0$ satisfy (3.11). Then $1 = \left|p_n(a)\right| \le c\,\|p_n\|_{L^2_\mu} \to c\eta_a$ as $n\to\infty$, which implies $\frac{1}{\eta_a} \le c$ and the proof is complete. ☐

3.6.2 Theorem. *Let μ be as in* 3.6.1 *and set $A := \{a\in\mathbb{C} : \eta_a > 0\}$. For $f \in P^2_\mu$, there exists a sequence $(p_n)_n$ in $\mathbb{C}[z]$, convergent in L^2_μ with limit f. Moreover, the function*

$$f^K : A \to \mathbb{C}\,, \qquad f^K(z) := \lim_{n\to\infty} p_n(z)\,,$$

is well-defined, i.e. the limit exists and does not depend on the particular choice of $(p_n)_n$.

Proof: Let $(p_n)_n$ and $(q_n)_n$ be sequences in $\mathbb{C}[z]$, both convergent to f in L^2_μ (clearly, such sequences exist). Fix $a\in A$. Now 3.6.1 yields $\left|p_n(a)-p_m(a)\right| \le \frac{1}{\eta_a}\,\|p_n-p_m\|_{L^2_\mu}$ for all $m,n\in\mathbb{N}$. Thus there exists $f_a := \lim_{n\to\infty} p_n(a)$. Analogously, $g_a := \lim_{n\to\infty} q_n(a)$ exists.
The sequence $p_1, q_1, p_2, q_2, \ldots$ converges to f in L^2_μ, too, and the inequality above yields $f_a = g_a$. ☐

3.6.3 Corollary. *In the situation of* 3.6.2*,*

$$\left|f^K(a)\right| \le \frac{1}{\eta_a}\,\|f\|_{L^2_\mu} \qquad \textit{for all}\, f\in P^2_\mu \textit{ and all}\, a\in A\,. \tag{3.12}$$

Moreover, if $(f_n)_n$ is a convergent sequence in P^2_μ with limit f, then f_n^K is pointwisely on A convergent with limit f^K.

Proof: According to 3.6.2, $f^K(a) = \lim_{n\to\infty} p_n(a)$ where $(p_n)_n$ is a sequence of polynomials convergent to f in L^2_μ.
Due to 3.6.1, we have $\left|p_n(a)\right| \le \frac{1}{\eta_a}\,\|p_n\|_{L^2_\mu}$. Letting $n\to\infty$, we obtain (3.12).
Let now $(f_n)_n$ be a sequence in P^2_μ, $f_n \to f$ in L^2_μ. Then $f\in P^2_\mu$ and (3.12) yields

$$\left|f_n^K(a) - f^K(a)\right| \le \frac{1}{\eta_a}\big\|f_n - f\big\|_{L^2_\mu} \qquad \text{for all } n\in\mathbb{N}$$

and, with $n\to\infty$, the assertion follows. ☐

We can now apply this to orthonormalizing measures.

3.6.4 Lemma. *Let μ be an om for $(P_n)_{n\ge 0}$ and define η_a as in* 3.6.1*.*
Then $\eta_a > 0$ if and only if $a\in E$. Furthermore, if $a\in E$ then $\eta_a = \frac{1}{\sqrt{K(a,a)}}$.

Proof: We start with $\eta_a > 0$. Fix $b = (b_n)_n \in \ell^2$. For $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\Big(\sum_{k=n}^{m} |b_k|^2\Big)^{\frac{1}{2}} < \varepsilon$ for all $n, m \in \mathbb{N}$ with $m > n > N$ and 3.6.1 yields

$$\Big|\sum_{k=n}^{m} b_k P_k(a)\Big| \le \frac{1}{\eta_a}\Big\|\sum_{k=n}^{m} b_k P_k\Big\|_{L^2_\mu} < \frac{1}{\eta_a}\,\varepsilon\,,$$

showing that $\sum_{k=0}^{\infty} b_k P_k(a)$ exists. Using 3.6.1 again, we obtain

$$\Big|\sum_{n=0}^{M} b_n P_n(a)\Big| \le \frac{1}{\eta_a}\Big\|\sum_{n=0}^{M} b_n P_n\Big\|_{L^2_\mu} \le \frac{1}{\eta_a}\;\|b\|_{\ell^2} \qquad \text{for all } M \in \mathbb{N}\,.$$

Therefore, $b \mapsto \sum_{n=0}^{\infty} b_n\, P_n(a)$ is a continuous linear functional on ℓ^2.
Now the Riesz representation theorem implies $\big(P_n(a)\big)_n \in \ell^2$; in other words, $a \in E$.

For the converse, suppose $a \in E$. Let $p \in \mathbb{C}[z]$. Then $p = \sum_{n=0}^{N} a_n\, P_n$ where $N = \deg p$ and $a_0, \dots, a_N \in \mathbb{C}$. Moreover,

$$\big|p(a)\big| = \Big|\sum_{n=0}^{N} a_n\, P_n(a)\Big| \le \Big(\sum_{n=0}^{N} |a_n|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{N}\big|P_n(a)\big|^2\Big)^{\frac{1}{2}} \le \|p\|_{L^2_\mu}\;\sqrt{K(a,a)}\,.$$

Now 3.6.1 implies $\eta_a > 0$ and $\frac{1}{\eta_a} \le \sqrt{K(a,a)}$. It remains to show that $\frac{1}{\eta_a} \ge \sqrt{K(a,a)}$. By 3.6.1, $\big|p(a)\big| \le \frac{1}{\eta_a}\,\|p\|_{L^2_\mu}$ for all $p \in \mathbb{C}[z]$. Thus

$$\begin{aligned}\sum_{n=0}^{M}\big|P_n(a)\big|^2 &= \Big(\sum_{n=0}^{M} \overline{P_n(a)}\, P_n\Big)(a) \le \frac{1}{\eta_a}\Big\|\sum_{n=0}^{M} \overline{P_n(a)}\, P_n\Big\|_{L^2_\mu} = \frac{1}{\eta_a}\Big(\sum_{n=0}^{M}\big|P_n(a)\big|^2\Big)^{\frac{1}{2}}\\ &\le \frac{1}{\eta_a}\sqrt{K(a,a)} \qquad \text{for all } M \in \mathbb{N}\,.\end{aligned}$$

For $M \to \infty$ this yields $K(a,a) \le \frac{1}{\eta_a}\sqrt{K(a,a)}$, hence $\frac{1}{\eta_a} \ge \sqrt{K(a,a)}$ as asserted. $\square$

As we have just seen, $E = \{a \in \mathbb{C} : \eta_a > 0\}$ is another characterization of the set E if an om exists.

3.6.5 Theorem. *Let $\mu = \alpha\lambda + \mu_\perp$ be an om with $\alpha\lambda$ and $\mu_\perp$ its absolutely continuous and singular parts w.r.t. 2-dimensional Lebesgue measure, respectively.*
If there exists an open set U such that $\int_U \frac{1}{\alpha}\,\mathrm{d}\lambda < \infty$ then $U \subset E$ and $P^2_\mu \ne L^2_\mu$.

Proof: For $\nu := \alpha\lambda|U$, define A^2_ν as in 2.3.1. Clearly, $\mathbb{C}[z] \subset A^2_\nu$ and $\|p\|_{A^2_\nu} \le \|p\|_{L^2_\mu}$ for all $p \in \mathbb{C}[z]$. Let $a \in U$. Using 2.3.2, we find $c_a > 0$ such that $|p(a)| \le c_a\,\|p\|_{A^2_\nu} \le c_a\,\|p\|_{L^2_\mu}$ for all $p \in \mathbb{C}[z]$ and, according to 3.6.1, $\eta_a > 0$ which, by 3.6.4, is equivalent to $a \in E$. Thus $U \subset E$.

Assume $P^2_\mu = L^2_\mu$. Then 3.4.8 implies $\mu(E \setminus \Lambda_\mu) = 0$ and hence $\mu(U \setminus \Lambda_\mu) = 0$.
Thus α vanishes λ-almost everywhere on U in contradiction to $\int\limits_U \frac{1}{\alpha}\, \mathrm{d}\lambda < \infty$.
Therefore, $P^2_\mu \neq L^2_\mu$. $\square$

Recall, for $a \in E$, $k_a = \sum\limits_{n\geq 0} \overline{P_n(a)}\, P_n \in P^2_\mu$. The following lemma extends 3.1.5.

3.6.6 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ and $E \neq \varnothing$. Then $\langle\, k_a\,, f\,\rangle_{L^2_\mu} = f^K(a)$ for all $a \in E$ and all $f \in P^2_\mu$.*
If $\mu(\{z\}) > 0$ for some z then $z \in E$ and $f^K(z) = f(z)$ for all $f \in P^2_\mu$.

Proof: Let $f \in P^2_\mu$. Note that $\sum\limits_{n=0}^{N} \langle\, P_n\,, f\,\rangle_{L^2_\mu} P_n \to f$ in L^2_μ as $N \to \infty$. For $a \in E$, 3.6.2 yields

$$f^K(a) = \sum_{n=0}^{\infty} P_n(a)\, \langle\, P_n\,, f\,\rangle_{L^2_\mu} = \Big\langle \sum_{n\geq 0} \overline{P_n(a)}\, P_n\,, f\,\Big\rangle_{L^2_\mu} = \langle\, k_a\,, f\,\rangle_{L^2_\mu} .$$

If $\mu(\{z\}) > 0$ then, according to 3.1.7, $z \in E$. Moreover, the expression $f(z)$ is well-defined for all $f \in L^2_\mu$.
By 3.1.8, $P\mathbf{1}_{\{z\}} = \mu(\{z\}) \cdot k_z$, where P is the orthogonal projection in L^2_μ onto P^2_μ. This yields

$$f(z) = \tfrac{1}{\mu(\{z\})} \langle\, \mathbf{1}_{\{z\}}\,, f\,\rangle_{L^2_\mu} = \langle\, k_z\,, f\,\rangle_{L^2_\mu} = f^K(z) . \qquad \square$$

3.6.7 Proposition. *Let μ be an om for $(P_n)_{n\geq 0}$ and $E \neq \varnothing$. For $f \in P^2_\mu$, define $\alpha(f) := f^K$. Then α is a partial isometry $P^2_\mu \to \mathcal{H}(K)$.*

Proof: Let $\iota : P^2_\mu \to \ell^2$ be the isomorphism given by $\iota(f) := \big(\langle\, P_n\,, f\,\rangle_{L^2_\mu}\big)_{n\geq 0}$.
Now let $\beta : \ell^2 \to \mathcal{H}(K)$ as in 2.1.8,

$$\beta(b) = \sum_{n\geq 0} b_n P_n \qquad \text{where } b = (b_n)_n \in \ell^2$$

and recall that β is a partial isometry. Therefore, $\beta \circ \iota$ is a partial isometry $P^2_\mu \to \mathcal{H}(K)$ and

$$\beta \circ \iota(f) = \sum_{n\geq 0} \langle\, P_n\,, f\,\rangle_{L^2_\mu} P_n = f^K = \alpha(f) .$$

Hence $\alpha : P^2_\mu \to \mathcal{H}(K)$, $\alpha(f) := f^K$, is a partial isometry and, in particular, $f^K \in \mathcal{H}(K)$ for all $f \in P^2_\mu$. $\square$

Note that α maps $P_n \in P^2_\mu$ to $P_n \in \mathcal{H}(K)$.

3.6.8 Example. Let $\gamma : [0,1] \to \mathbb{C}$ be a simply closed piecewise continuously differentiable curve and μ absolutely continuous w.r.t. line measure on γ, having density α, such that

$$\int_\gamma \frac{1}{\alpha(z)} \, |\mathrm{d}z| < \infty \,,$$

see 2.3.8 for the details; see also 3.3.7.
Moreover, assume $\mu(\gamma) = 1$, hence μ is an om for some $(P_n)_n$, and let V be the bounded component of $\mathbb{C} \setminus \gamma$.
Recall 2.3.11, where we have shown $V \subset E$ and also, to $f \in P^2_\mu$, have assigned a holomorphic function $V \to \mathbb{C}$ which, as we know now, coincides with $f^K|V$.
Using 3.3.6 and 3.3.7, we also see $E \subset \overline{V} = V \cup \partial V = V \cup \gamma$; in general, we do not know whether E and γ are disjoint or not.

3.6.9 Corollary. *Let μ be an om for $(P_n)_{n\geq 0}$ and $E \neq \varnothing$. The following properties are equivalent.*

(i) $(P_n)_n$ *is* **not** *an onb in $\mathcal{H}(K)$.*

(ii) *There exists $f \in P^2_\mu$, $f \neq 0$, such that $f^K = 0$.*

(iii) *There exists $f \in P^2_\mu$, $f \neq 0$, such that $\langle k_a \,, f \rangle_{L^2_\mu} = 0$ for all $a \in E$.*

(iv) *There exists $b = (b_n)_{n\geq 0} \in \ell^2$, $b \neq 0$, such that $\sum_{n\geq 0} b_n P_n(z) = 0$ for all $z \in E$.*

Proof: The equivalence (ii) $\Longleftrightarrow$ (iii) is an immediate consequence of 3.6.6; the equivalence (i) $\Longleftrightarrow$ (iv) is due to 2.1.9. As to (iii) $\Longleftrightarrow$ (iv), note that, for $z \in E$, we have $\langle k_z \,, f \rangle_{L^2_\mu} = \sum_{n\geq 0} b_n P_n(z)$ whenever $f = \sum_{n\geq 0} b_n P_n \in P^2_\mu$. $\square$

If we do not know whether there exists an om then, for $f \in \mathcal{H}$, we can define

$$f^K : E \to \mathbb{C} \,, \qquad f^K(z) := \langle k_z \,, f \rangle_{\mathcal{H}} = \sum_{n\geq 0} P_n(z) \, \langle P_n \,, f \rangle_{\mathcal{H}}$$

and Cauchy-Schwarz yields $\left|f^K(z)\right| \leq \|k_z\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}} = \sqrt{K(z,z)} \cdot \|f\|_{\mathcal{H}}$; matching (3.12) if an om exists. Using this definition for f^K and replacing L^2_μ and P^2_μ by $\mathcal{H}$ in the proofs of 3.6.7 and 3.6.9, we get the following proposition.

3.6.10 Proposition. *Let $(P_n)_{n\geq 0}$ such that $E \neq \varnothing$. For $f \in \mathcal{H}$ define $\alpha(f) := f^K$. Then α is a partial isometry $\mathcal{H} \to \mathcal{H}(K)$.*
Furthermore, the following properties are equivalent.

(i) $(P_n)_n$ *is* **not** *an onb in $\mathcal{H}(K)$.*

(ii) *There exists $f \in \mathcal{H}$, $f \neq 0$, such that $f^K = 0$.*

(iii) *There exists $f \in \mathcal{H}$, $f \neq 0$, such that $\langle k_z \,, f \rangle_{\mathcal{H}} = 0$ for all $z \in E$.*

(iv) *There exists $b = (b_n)_{n\geq 0} \in \ell^2$, $b \neq 0$, such that $\sum_{n\geq 0} b_n P_n(z) = 0$ for all $z \in E$.*

3.6.11 Summary. Let W be the closure of the linear span of $\{(\overline{P_n(z)})_{n\geq 0} : z \in E\}$ in ℓ^2, see also 2.1.8, and note that the canonical isomorphism $\ell^2 \to \mathcal{H}$ maps $W^\perp$ onto $\{f \in \mathcal{H} : f^K = 0\} = \mathcal{N}(\alpha)$.
Taking into account that $\operatorname{ran}(D - a\,\mathrm{id})^\perp = \mathbb{C} \cdot k_a$ for all $a \in E$, see 3.1.3, which implies $\{k_a\}^\perp = \overline{\operatorname{ran}(D - a\,\mathrm{id})}$ in $\mathcal{H}$, we obtain
$\{f \in \mathcal{H} : f^K = 0\} = \mathcal{N}(\alpha) = \{f \in \mathcal{H} : \langle\, k_a\,, f\,\rangle_{\mathcal{H}} = 0 \text{ for all } a \in E\} = \bigcap_{a\in E} \overline{\operatorname{ran}(D - a\,\mathrm{id})}\,.$

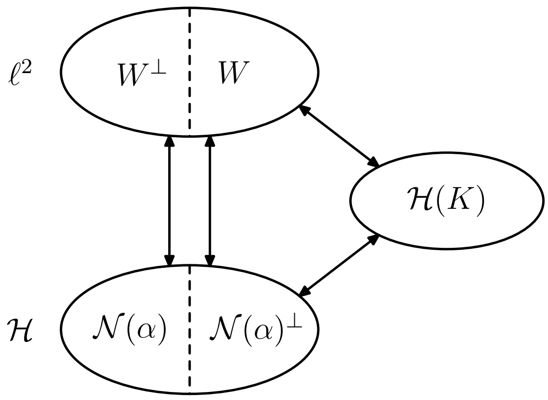


Figure 3.
The arrows denote isometric isomorphisms.
The canonical isomorphism $\ell^2 \to \mathcal{H}$ given by

$$(b_n)_{n\geq 0} \mapsto \sum_{n\geq 0} b_n\, P_n \qquad (*)$$

maps W and $W^\perp$ onto $\mathcal{N}(\alpha)^\perp$ and $\mathcal{N}(\alpha)$, respectively. We can also regard $(*)$ as the map $\beta : \ell^2 \to \mathcal{H}(K)$ defined in 2.1.8 which is a partial isometry with initial space W; hence its restricton to W is an isometric isomorphism $W \to \mathcal{H}(K)$.

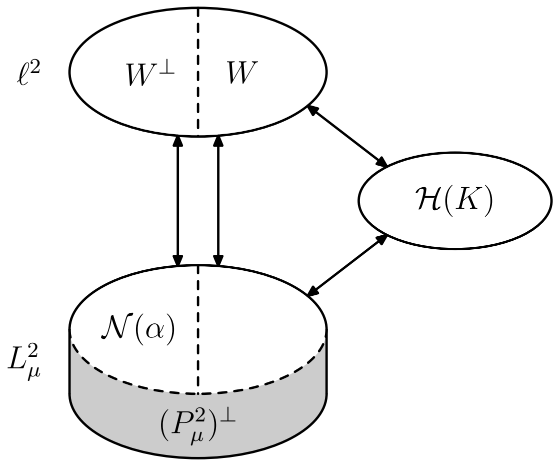


Figure 4.
If there exists an om for $(P_n)_{n\geq 0}$ then we can embed $\mathcal{H}$ isometrically into L^2_μ.

3.7 Orthonormal Bases in $\mathcal{H}$ and $\mathcal{H}(K)$

In this section, we will study conditions for $(P_n)_n$ being an onb in $\mathcal{H}(K)$. If $(P_n)_n$ is an onb in $\mathcal{H}(K)$ then $\mathcal{H}$ and $\mathcal{H}(K)$ are isometrically isomorphic via the identity map on $\mathbb{C}[z]$. Recall that $(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if $\{(P_n(z))_n : z \in E\}$ is total in ℓ^2 or, using the notations from 2.1.8 and 3.6.11, $W^\perp = \{0\}$, i.e. $W = \ell^2$. See also 3.6.9.

3.7.1 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$. If there exist $b = (b_n)_{n\geq 0} \in \ell^2$, $b \neq 0$, and $M \in \mathfrak{B}(\mathbb{C})$ such that*

$$\sum_{n=0}^{\infty} b_n\, P_n(z) = 0 \qquad \textit{for all } z \in M$$

then $\mu(\mathbb{C} \setminus M) > 0$.

Proof: For $\mathbf{f} := \sum\limits_{n\geq 0} b_n\, P_n \in L^2_\mu \setminus \{0\}$, there exists a sequence $(f_k)_{k\in\mathbb{N}}$,

$$f_k(z) := \sum_{j=0}^{n_k} b_j\, P_j(z)\;,$$

which is μ-a.e. convergent to a representative f of $\mathbf{f}$ where $(n_k)_k$ is a monotonically increasing sequence of integers.
Define $A := \big\{z \in M : f(z) = \lim\limits_{k\to\infty} f_k(z)\big\}$. Hence $M \setminus A$ is a μ-nullset.
Note that $\lim\limits_{k\to\infty} f_k(z) = 0$ for all $z \in M$. Therefore, $f(z) = 0$ for all $z \in A$. Moreover,

$$0 \neq \|\mathbf{f}\|^2_{L^2_\mu} = \int\limits_{A} |f|^2 \,\mathrm{d}\mu \;+ \int\limits_{M\setminus A} |f|^2 \,\mathrm{d}\mu \;+ \int\limits_{\mathbb{C}\setminus M} |f|^2 \,\mathrm{d}\mu\;.$$

The first integral vanishes, as $f(z) = 0$ for all $z \in A$; the second integral vanishes, too, as $M \setminus A$ is a μ-nullset. Thus the last integral must be different from 0 which implies $\mu(\mathbb{C} \setminus M) > 0$. □

Remark. Note that we did not have to require $\big(P_n(z)\big)_{n\geq 0} \in \ell^2$ for $z \in M$ here.

Recall that E is an F_σ-set (see 2.2.7) and, therefore, measurable.

3.7.2 Theorem. *If there exists an om μ for $(P_n)_{n\geq 0}$ such that $\mu(E) = 1$ then $(P_n)_n$ is an onb in $\mathcal{H}(K)$.*

Proof: Note that $\mu(\mathbb{C}) = 1$; hence here $\mathbb{C} \setminus E$ is a μ-nullset.
We have to show $W^\perp = \{0\}$. Choose $b = (b_n)_n \in W^\perp$. In other words,

$$\sum_{n\geq 0} b_n\, P_n(z) = 0 \qquad \text{for all } z \in E\;.$$

If $b \neq 0$ then 3.7.1 implies $\mu(\mathbb{C} \setminus E) > 0$ which is a contradiction. Thus $b = 0$ and $W^\perp = \{0\}$ as asserted. □

The converse is not true. Our most famous example, see also 1.2.2, 1.3.8, and 2.1.11, shows that even $\mu(E) = 0$ and E open is possible while $(P_n)_n$ is an onb in $\mathcal{H}(K)$:

3.7.3 Example. Let $P_n(z) := z^n$ for $n \in \mathbb{N}_0$. Then one-dimensional Lebesgue measure on the unit circle is, up to a constant factor, an om. For the moment denote this measure by μ. Here, E is the interior of the unit disk and hence $\mu(E) = 0$. However, $(P_n)_n$ is an onb in $\mathcal{H}(K)$. Note that, according to 3.2.14, μ is the only om for $(P_n)_n$.

3.7.4 Theorem. *Let μ be a compactly supported om for $(P_n)_{n\geq 0}$ and $E \neq \varnothing$. If $(P_n)_n$ is not an onb in $\mathcal{H}(K)$ then* $\dim(W^\perp) = \infty$.

Proof: As $(P_n)_n$ is not an onb in $\mathcal{H}(K)$, there exists $h \in P^2_\mu$, $h \neq 0$, such that $h^K = 0$, see 3.6.9.

As $\mathrm{supp}(\mu)$ is compact, the multiplication operator M_μ is continuous; hence the Hessenberg operator D is continuous, too, and its closure is defined on the whole space $\mathcal{H}$ which we identify with P^2_μ. Moreover, $0 \neq (M_\mu)^j h \in P^2_\mu$ for all $j \in \mathbb{N}$. In particular,

$$\begin{aligned}
\sum_{n\geq 0} \left\langle P_n\,, (M_\mu)^j h \right\rangle_{L^2_\mu} P_n(z) = \left[(M_\mu)^j h\right]^K(z) &\overset{3.6.6}{=} \left\langle k_z\,, (M_\mu)^j h \right\rangle_{L^2_\mu} = \left\langle k_z\,, \overline{D}^j h \right\rangle_{\mathcal{H}} \\
&= \left\langle (D^*)^j k_z\,, h \right\rangle_{\mathcal{H}} \overset{3.1.3}{=} \left\langle \overline{\lambda}^j k_z\,, h \right\rangle_{\mathcal{H}} \\
&= \lambda^j \langle k_z\,, h \rangle_{\mathcal{H}} = \lambda^j \langle k_z\,, h \rangle_{P^2_\mu} \overset{3.6.6}{=} \lambda^j\, h^K(z) = 0
\end{aligned}$$

for all $z \in E$ and all $j \in \mathbb{N}$. Thus $\alpha^{(j)} := (\alpha^{(j)}_n)_n := \big(\langle P_n\,, (M_\mu)^j h \rangle_{L^2_\mu}\big)_n \in W^\perp$.

Assume now that $\dim(W^\perp) = k \in \mathbb{N}$. Then $\{\alpha^{(j)} : j = 0, \ldots, k\}$ cannot be linearly independent in $W^\perp$.

Hence there exist $c_0, \ldots, c_k \in \mathbb{C}$, $c_i \neq 0$ for some i, such that $\sum_{j=0}^{k} c_j\, \alpha^{(j)} = 0$ in ℓ^2 which means

$$0 = \sum_{j=0}^{k} c_j \left\langle P_n\,, (M_\mu)^j h \right\rangle_{L^2_\mu} = \left\langle P_n\,, \sum_{j=0}^{k} c_j (M_\mu)^j h \right\rangle_{L^2_\mu} \qquad \text{for all } n\,.$$

As $(P_n)_n$ is an onb in P^2_μ and $\sum_{j=0}^{k} c_j (M_\mu)^j h \in P^2_\mu$, we get $\sum_{j=0}^{k} c_j (M_\mu)^j h = 0$; in other words,

$$0 = \left\| \sum_{j=0}^{k} c_j (M_\mu)^j h \right\|^2_{L^2_\mu} = \int\ |ph|^2\ \mathrm{d}\mu$$

where $p(z) := c_0 + c_1 z + \cdots + c_k z^k$. Thus $ph = 0$ μ-almost everywhere.

If $\mu(\{z_0\}) > 0$ for some z_0 then $z_0 \in E$, see 3.1.7, and, by 3.6.6, $h(z_0) = h^K(z_0) = 0$.

As p has only finitely many zeros, we finally obtain $h = 0$ μ-almost everywhere in contradiction to $h \neq 0$ in P^2_μ. Thus $\dim(W^\perp) = \infty$. $\square$

Due to 2.1.9, the following is an immediate consequence.

3.7.5 Corollary. *If, for a sequence of polynomials $(P_n)_{n\geq 0}$, the Hessenberg operator D is continuous and $0 < \dim(W^\perp) < \infty$ then there exists no om.*

For $a \in E_{reg}$, define $V_a := \bigcap_{n\in\mathbb{N}} \mathrm{ran}\big((\overline{D} - a\,\mathrm{id})^n\big)$. Clearly, $V_a \subset \mathcal{H}$.

3.7.6 Lemma. *For $(P_n)_{n\geq 0}$ such that $E_{reg} \neq \varnothing$ and fixed $a \in E_{reg}$, denote by Z the connected component of E_{reg} containing a. Then $V_a = \big\{f \in \mathcal{H} : f^K|Z = 0\big\}$.*

Proof: If $f \in V_a$ then, for every $n \in \mathbb{N}$, there exists $f_n \in \operatorname{dom}(\overline{D}) \subset \mathcal{H}$ such that $f = (\,\overline{D} - a\,\mathrm{id})^n f_n$. According to 3.6.10, for $z \in E$, we obtain

$$\begin{aligned} f^K(z) &= \langle\, k_z\,, f\,\rangle_{\mathcal{H}} = \big\langle\, k_z\,, (\,\overline{D} - a\,\mathrm{id})^n f_n\,\big\rangle_{\mathcal{H}} = \big\langle\, (D^* - \overline{a}\,\mathrm{id})^n k_z\,, f\,\big\rangle_{\mathcal{H}} \\ &= (z-a)^n \langle\, k_z\,, f_n\,\rangle_{\mathcal{H}} = (z-a)^n\, f_n^K(z) \end{aligned}$$

where f^K and f_n^K are members of $\mathcal{H}(K)$ which, see 3.5.2, are holomorphic in E_{reg}. Thus, for every $n \in \mathbb{N}$, f^K has a zero of order $\geq n$ at a. As f^K is holomorphic in Z, and Z is connected by definition, this yields $f^K(z) = 0$ for all $z \in Z$; in other words, $f^K|Z = 0$.

To prove the converse inclusion, let $f \in \mathcal{H}$ such that $f^K|Z = 0$. Choose a sequence $(p_n)_n$ in $\mathbb{C}[z]$ such that $p_n \to f$ w.r.t. $\|\cdot\|_{\mathcal{H}}$ as $n \to \infty$. Thus $p_n^K \to f^K$ in $\mathcal{H}(K)$ which yields $p_n(z) \to f^K(z)$ for all $z \in E$; in particular $p_n(a) \to 0$. Define

$$q_n : \mathbb{C} \to \mathbb{C}\,, \qquad q_n(z) := \tfrac{p_n(z) - p_n(a)}{z-a} \ \text{ for } \ z \neq a\,, \qquad q_n(a) := p_n'(a)\,.$$

Clearly, $q_n \in \mathbb{C}[z]$ and $(\,\overline{D} - a\,\mathrm{id})q_n = p_n - p_n(a)P_0 \to f$ in $\mathcal{H}$. Now $a \in \operatorname{reg}(D) = \operatorname{reg}(\overline{D})$ implies that $(q_n)_n$ is convergent in $\mathcal{H}$ to f_1, say. As $(\,\overline{D} - a\,\mathrm{id})$ is a closed operator, we obtain $(\,\overline{D} - a\,\mathrm{id})q_n \to (\,\overline{D} - a\,\mathrm{id})f_1$, hence $f = (\,\overline{D} - a\,\mathrm{id})f_1 \in \operatorname{ran}(\,\overline{D} - a\,\mathrm{id})$.
Moreover, an analogous calculation as above yields $f^K(z) = (z-a)f_1^K(z)$ for all $z \in E$ and, as $f_1^K|Z$ is holomorphic, $f_1^K|Z = 0$.
Starting with f_1 instead of f, we can see that $f_1 \in \operatorname{ran}(\,\overline{D} - a\,\mathrm{id})$ and obtain $f_2 \in \mathcal{H}$ such that $(\,\overline{D} - a\,\mathrm{id})f_2 = f_1$ and $f_2^K|Z = 0$ and, by induction, finally $f \in \bigcap\limits_{n\in\mathbb{N}} \operatorname{ran}\big((\,\overline{D} - a\,\mathrm{id})^n\big)$ as asserted. $\square$

Now let $A \subset E_{reg}$ such that A contains exactly one point of each connected component of E_{reg} and define $V_\infty := \bigcap\limits_{a\in A} V_a = \{f \in \mathcal{H} : f^K|E_{reg} = 0\}$.
As we have just seen, V_∞ does not depend on the particular choice of A. Moreover, if $E = E_{reg}$ then $V_\infty = \{f \in \mathcal{H} : f^K = 0\}$.

3.7.7 Corollary. *If $E_{reg} \neq \varnothing$ and $V_\infty = \{0\}$ then $(P_n)_{n\geq 0}$ is an onb in $\mathcal{H}(K)$.*

Proof: Assume that $(P_n)_n$ is not an onb in $\mathcal{H}(K)$. According to 3.6.10, there exists $f \in \mathcal{H}$, $f \neq 0$, such that $f^K = 0$. This implies $f \in V_\infty$; hence $V_\infty \neq \{0\}$ in contradiction to the premises. Thus $(P_n)_n$ is an onb in $\mathcal{H}(K)$. $\square$

Recall that if an om exists then $E_{reg} = G_K$ is dense in E° and $E_{reg} \neq \varnothing \iff E^\circ \neq \varnothing$, see section 3.5.

3.7.8 Corollary. *Let μ be a compactly supported om for $(P_n)_{n\geq 0}$ such that $E^\circ \neq \varnothing$. If $\dim(V_\infty) < \infty$ then $(P_n)_n$ is an onb in $\mathcal{H}(K)$.*

Proof: If $(P_n)_n$ is not an onb in $\mathcal{H}(K)$ then 3.7.4 implies $\dim(W^\perp) = \infty$. Furthermore, recall that the canonical isomorphism $\ell^2 \to \mathcal{H}$ maps $W^\perp$ onto $\{f \in \mathcal{H} : f^K = 0\}$, see 3.6.11, and $\{f \in \mathcal{H} : f^K = 0\} \subset V_\infty$. This yields $\dim(V_\infty) = \infty$. $\square$

Note that if $E^\circ = \varnothing$ then V_∞ remains undefined. If, for example, D is essentially normal then $E = \Lambda_\mu = \big\{z \in \mathbb{C} : \mu(\{z\}) > 0\big\}$ where μ is the unique om for $(P_n)_{n\geq 0}$, see 3.3.2; in this case $E^\circ = \varnothing$.

Note also that if μ is an om such that $P^2_\mu = L^2_\mu$ but D is not essentially normal then $E \cup \operatorname{supp}(\mu) = \mathbb{C}$, see 3.4.10(iii). If, in particular, $\operatorname{supp}(\mu) \neq \mathbb{C}$ then $E^\circ \neq \varnothing$.

In section 4.2, concerning om supported on the unit circle, we will encounter the space V_∞ from another point of view.

3.7.9 Theorem. *Let μ be an om for $(P_n)_{n\geq 0}$ such that $E^\circ \neq \varnothing$ and assume that there exists $\lambda \in \mathbb{C}$ satisfying $\overline{D}f = \lambda f$ for some $f \in \operatorname{dom}(\overline{D})$, $f \neq 0$. Then $f = c \cdot k_\lambda \in V_\infty$ with some $c \in \mathbb{C}$.*

Proof: Identify $\mathcal{H}$ with P^2_μ and note that $M_\mu f = \lambda f$. Hence $\mu(\{\lambda\}) > 0$ and $f = c' \cdot \mathbf{1}_{\{\lambda\}}$ for some $c' \neq 0$. By assumption, $f \in P^2_\mu$, and 3.1.8 yields $\mathbf{1}_{\{\lambda\}} = \mu(\{\lambda\})k_\lambda$; in particular, $\lambda \in E$. Hence $f = c \cdot k_\lambda$ for some $c \in \mathbb{C}$.

Clearly, $\lambda \notin \operatorname{reg}(D)$. Therefore, $\lambda - a \neq 0$ for $a \in E_{reg}$ and $(\overline{D} - a\,\mathrm{id})k_\lambda = (\lambda - a)k_\lambda$. This shows that $k_\lambda \in \operatorname{ran}(\overline{D} - a\,\mathrm{id})^n$ for all $n \in \mathbb{N}$ and all $a \in E_{reg}$. Hence $k_\lambda \in V_\infty$. □

Remark. According to 3.7.6, $V_\infty = \{f \in \mathcal{H} : f^K | E_{reg} = 0\}$. Therefore, in the situation of 3.7.9, $\langle\, k_a \,, k_\lambda \,\rangle_{L^2_\mu} = 0$ for all $a \in E_{reg}$. Moreover,

$$0 = \langle\, k_a \,, k_\lambda \,\rangle_{L^2_\mu} = \sum_{n\geq 0} \overline{P_n(\lambda)}\, P_n(a) = \langle\, K_a \,, K_\lambda \,\rangle_{\mathcal{H}(K)} \qquad \text{for all } a \in E_{reg}$$

regardless of $(P_n)_n$ being an onb in $\mathcal{H}(K)$ or not.
If, in particular, λ is a boundary point of E_{reg} then $\mathcal{H}(K)$ contains a function which is not continuous at λ, namely K_λ. In 4.2.10 we will see that this can actually occur.

3.7.10 Discrete om. Suppose now that there exists a discrete om μ for $(P_n)_{n\geq 0}$, i.e. there are mutually different complex numbers z_i, $i \in \mathbb{N}$, and

$$\mu = \sum_{i\in\mathbb{N}} g_i \delta_{z_i}$$

where δ_{z_i} denotes Dirac measure at z_i and g_i are positive numbers such that their sum is equal to 1. Now consider $\Lambda_\mu := \big\{z \in \mathbb{C} : \mu(\{z\}) > 0\big\} = \{z_1, z_2, \ldots\}$. By construction, $\mu(\Lambda_\mu) = 1$ and, according to 3.1.7, $\Lambda_\mu \subset E$. This implies $\mu(E) = 1$ and, using 3.7.2, we conclude that $(P_n)_{n\geq 0}$ is an onb in $\mathcal{H}(K)$.

3.7.11 Proposition. *Let μ be a discrete om for $(P_n)_{n\geq 0}$ such that Λ_μ is closed. Then the following properties hold.*

(i) *Either $E = \mathbb{C}$ or $E = \Lambda_\mu$.*

(ii) *If, in addition, Λ_μ is bounded then $E = \Lambda_\mu$ and μ is the unique om for $(P_n)_n$.*

Proof: As Λ_μ is closed, $\Lambda_\mu = \operatorname{supp}(\mu) = \sigma(M_\mu)$. As $\mathbb{C} \setminus \Lambda_\mu$ is connected, 3.3.1(v) implies either $\mathbb{C} \setminus \Lambda_\mu \subset E$ or $(\mathbb{C} \setminus \Lambda_\mu) \cap E = \varnothing$. Combining this with $\Lambda_\mu \subset E$, see 3.1.7, we get either $E = \mathbb{C}$ or $E = \Lambda_\mu$.

If, in addition, Λ_μ is bounded then M_μ, and hence D, is bounded. Now 3.1.6 implies that E is bounded. Thus only the case $E = \Lambda_\mu$ remains. Moreover, according to 3.2.14, μ is the unique om for $(P_n)_n$. □

We now turn to the situation that $(P_n)_n$ is an onb in $\mathcal{H}(K)$ as well as in L^2_μ.

First consider the case that D is essentially normal, i.e. its closure $\overline{D}$ is the multiplication operator M_μ.

3.7.12 Theorem. *Suppose that D is essentially normal. Then there exists a unique om μ, $P^2_\mu = L^2_\mu$, and $E = \Lambda_\mu$. Moreover, $(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if μ is discrete.*

Proof: Due to 1.3.7, there exists a unique om μ and $P^2_\mu = L^2_\mu$; by 3.3.2, $E = \Lambda_\mu$. According to 3.1.8, $\mathbf{1}_{\{z\}} = \mu(\{z\}) \cdot k_z$ for all $z \in \Lambda_\mu$.

Let now $(P_n)_n$ be an onb in $\mathcal{H}(K)$ and assume that μ is not discrete, i.e. there exists $0 \neq f \in L^2_\mu$ such that $\langle \mathbf{1}_{\{z\}}, f \rangle_{L^2_\mu} = 0$ for all $z \in \Lambda_\mu$. As $(P_n)_n$ is an onb in L^2_μ, too, there exists an isometry $\iota : L^2_\mu \to \mathcal{H}(K)$ which is the identity on $\mathbb{C}[z]$. Therefore, $\langle K_z, \iota(f) \rangle_{\mathcal{H}(K)} = 0$ for all $z \in \Lambda_\mu = E$. Furthermore, by 2.1.4, $\{K_z : z \in E\}$ is total in $\mathcal{H}(K)$ implying $\iota(f) = 0$ which is a contradiction. Hence μ is discrete.

As to the converse implication, note that if there exists a discrete om then $(P_n)_n$ is an onb in $\mathcal{H}(K)$, see 3.7.10. □

Now suppose that D is not essentially normal but has a minimal normal extension in the same space P^2_μ, i.e. there exists an om μ such that $P^2_\mu = L^2_\mu$ and the multiplication operator M_μ is a proper extension of $\overline{D}$.

3.7.13 Theorem. *Let μ be an om for $(P_n)_{n\geq 0}$ such that $P^2_\mu = L^2_\mu$. If $\mu(E) = 1$ then μ is discrete. Furthermore, $(P_n)_n$ is an onb in $\mathcal{H}(K)$, too.*

Proof: According to 3.7.2, $\mu(E) = 1$ implies that $(P_n)_n$ is an onb in $\mathcal{H}(K)$.

Moreover, due to 3.4.8, $\mu(E \setminus \Lambda_\mu) = 0$ which implies $\mu(\Lambda_\mu) = \mu(E) = 1$. Hence μ is discrete. □

3.7.14 Definitions. Let $\mathcal{H}(K)$ be an RKHS with domain E.

A set $\Delta \subset E$ is called **domain of determinacy** for $\mathcal{H}(K)$ if $f(z) = 0$ for all $z \in \Delta$ implies $f = 0$.

If, for example, $\mathcal{H}(K)$ is an RKHS with a connected domain $E \subset \mathbb{C}$ such that all $f \in \mathcal{H}(K)$ are holomorphic in E then every subset of E having a limit point is a domain of determinacy for $\mathcal{H}(K)$.

A set $\Delta \subset E$ is called **diagonal** if $K(z, w) = 0$ for all $z, w \in \Delta$, $z \neq w$.

Note that, for $\mathcal{H}(K)$ as in 2.1.8, $K(z, w) = 0$ if and only if $\big(P_n(z)\big)_n \perp \big(P_n(w)\big)_n$ in ℓ^2. Thus $\Delta \subset E$ is diagonal if and only if $P(\Delta) := \big\{\big(P_n(z)\big)_n : z \in \Delta\big\}$ is an orthogonal system in ℓ^2.

3.7.15 Theorem. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials as in* 1.1.1*, define $\mathcal{H}(K)$ as in* 2.1.8*, and let $\Delta \subset E$. The following properties are equivalent.*

(i) $(P_n)_n$ *is an onb in L^2_μ where $\mu = \sum_{z\in\Delta} \frac{1}{K(z,z)}\,\delta_z$ with δ_z the Dirac measure at z,*

(ii) $P(\Delta)$ *is an orthogonal basis of ℓ^2,*

(iii) Δ *is a diagonal domain of determinacy for $\mathcal{H}(K)$ and $(P_n)_n$ is an onb in $\mathcal{H}(K)$.*

Proof: We start with (i)⇒(ii).

According to 3.1.8, $\mathbf{1}_{\{z\}} = \mu(\{z\})\, k_z$ for all $z \in \Delta = \Lambda_\mu \subset E$.

In 3.7.10 we have seen that if a discrete om exists then $(P_n)_n$ is an onb in $\mathcal{H}(K)$. Moreover, here L^2_μ and $\mathcal{H}(K)$ are isometrically isomorphic via $P_n \mapsto P_n$ and for $z, w \in \Delta$, $z \neq w$, we obtain

$$0 = \Big\langle \frac{1}{\mu(\{w\})}\mathbf{1}_{\{w\}}\,,\, \frac{1}{\mu(\{z\})}\mathbf{1}_{\{z\}} \Big\rangle_{L^2_\mu} = \langle\, K_w\,, K_z\,\rangle_{\mathcal{H}(K)} = K(z, w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w)\,.$$

Hence $P(\Delta)$ is an orthogonal system in ℓ^2. Now choose $a = (a_n)_n \in \ell^2$ such that $\sum_{n\geq 0} \overline{a_n}\, P_n(z) = 0$ for all $z \in \Delta$ and set $f := \sum_{n\geq 0} \overline{a_n}\, P_n \in L^2_\mu$. Then

$$0 = \sum_{n\geq 0} a_n\, \overline{P_n(z)} = \langle\, f\,, k_z\,\rangle_{L^2_\mu} = \Big\langle\, f\,, \frac{1}{\mu(\{z\})}\mathbf{1}_{\{z\}} \Big\rangle_{L^2_\mu} \qquad \text{for all } z \in \Delta$$

which implies $f = 0$ and hence $a = 0$. Thus $P(\Delta)$ is an orthogonal basis in ℓ^2.

Next we prove (ii)⇒(iii).

Obviously, if $P(\Delta)$ is an orthogonal system in ℓ^2 then Δ is diagonal. As here $P(\Delta)$ is total in ℓ^2, 2.1.9 implies that $(P_n)_n$ is an onb in $\mathcal{H}(K)$.

Let now $f := \sum_{n\geq 0} a_n\, P_n \in \mathcal{H}(K)$ with $a = (a_n)_{n\geq 0} \in \ell^2$ such that $f(z) = 0$ for all $z \in \Delta$.

This means

$$0 = f(z) = \langle\, K_z \,, f \,\rangle_{\mathcal{H}(K)} = \sum_{n\geq 0} a_n\, P_n(z) \qquad \text{for all } z \in \Delta$$

and, again because $P(\Delta)$ is total in ℓ^2, we get $a = 0$ and hence $f = 0$.
Thus Δ is a domain of determinacy for $\mathcal{H}(K)$.

It remains to show that (iii) implies (i).
As Δ is diagonal, $\langle\, K_w \,, K_z \,\rangle_{\mathcal{H}(K)} = K(z,w) = 0$ for $z, w \in \Delta$, $z \neq w$. Thus $\{K_z : z \in \Delta\}$ is an orthogonal system in $\mathcal{H}(K)$. Furthermore, since Δ is a domain of determinacy then $0 = f(z) = \langle\, K_z \,, f \,\rangle_{\mathcal{H}(K)}$ for all $z \in \Delta$ implies $f = 0$. Hence $\{\|K_z\|^{-1} K_z : z \in \Delta\}$ is an onb in $\mathcal{H}(K)$. Recall that $\|K_z\|^{-1} = \frac{1}{\sqrt{K(z,z)}}$.

Clearly, Δ is countable. For $z \in \Delta$, let δ_z denote Dirac measure at z and define

$$\mu := \sum_{z\in\Delta} \frac{1}{K(z,z)}\, \delta_z \,.$$

By construction, $\left\{ \sqrt{K(z,z)}\; \mathbf{1}_{\{z\}} : z \in \Delta \right\}$ is an onb in L^2_μ.
Thus we can define an isometric isomorphism $\beta : \mathcal{H}(K) \to L^2_\mu$ by $\beta(K_z) := K(z,z)\, \mathbf{1}_{\{z\}}$ for $z \in \Delta$. Then $\big(\beta(P_n)\big)_n$ is an onb in L^2_μ since $(P_n)_n$ is an onb in $\mathcal{H}(K)$. The members of L^2_μ are defined pointwisely for $z \in \Delta$ and

$$\beta(P_n)(z) = \left\langle\, \tfrac{1}{K(z,z)}\, \mathbf{1}_{\{z\}} \,, \beta(P_n) \,\right\rangle_{L^2_\mu} = \langle\, K_z \,, P_n \,\rangle_{\mathcal{H}(K)} = P_n(z) \qquad \text{for all } z \in \Delta \,.$$

Thus β maps $P_n \in \mathcal{H}(K)$ to $P_n \in L^2_\mu$ and we are done. $\square$

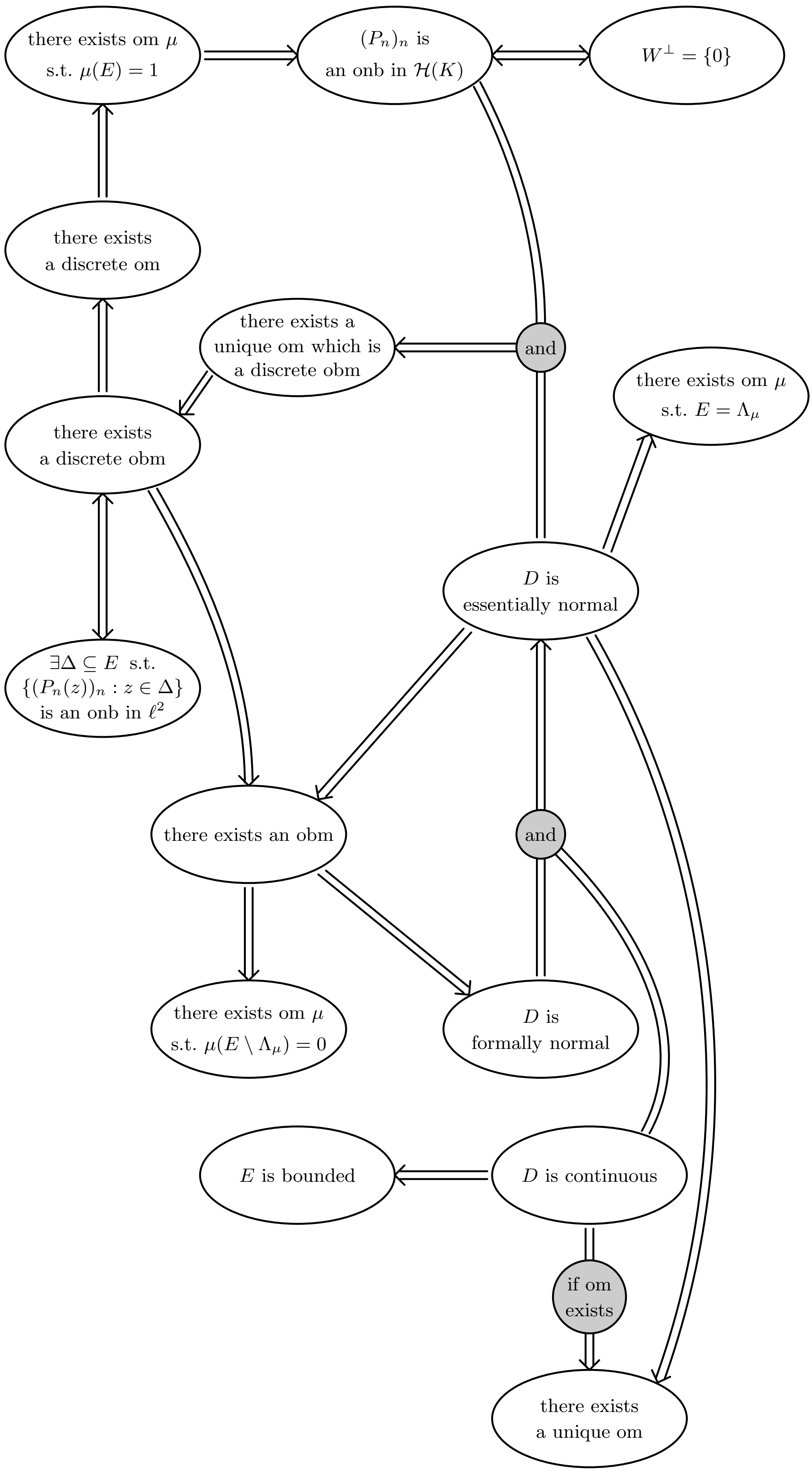
3.8 Concluding Remarks

Before we turn to applications of the theory developed so far, we graphically summarize some remarkable properties and implications in the following figure.
If μ is an om for $(P_n)_{n\geq 0}$ such that $P^2_\mu = L^2_\mu$, we will, for abbreviation, call μ an obm.

Figure 5. (opposite page)

The diagram shows some of the implications proved so far.

Note, for example, that if μ is a compactly supported (hence D is continuous) discrete obm for $(P_n)_n$ then all of the shown statements apply.


there exists om μ s.t. $\mu(E) = 1$
$(P_n)_n$ is an onb in $\mathcal{H}(K)$
$W^\perp = \{0\}$
there exists a discrete om
there exists a unique om which is a discrete obm
and
there exists om μ s.t. $E = \Lambda_\mu$
there exists a discrete obm
D is essentially normal
$\exists \Delta \subseteq E$ s.t. $\{(P_n(z))_n : z \in \Delta\}$ is an onb in ℓ^2
there exists an obm
and
there exists om μ s.t. $\mu(E \setminus \Lambda_\mu) = 0$
D is formally normal
E is bounded
D is continuous
if om exists
there exists a unique om

4 Applications and Examples

Given a sequence of polynomials $(P_n)_{n\geq 0}$ as in 1.1.1, it seems, in general, be rather difficult to determine whether there exists an om, and it is even more difficult to find an om. In some cases, however, we can quite easily conclude that there exists no om, see 4.5.2 or 4.5.4, for instance.

Before we turn to a collection of various examples, we will have a look at orthogonal polynomials on (possibly subsets of) the real line or the unit circle. In the former case, as already mentioned in 3.1.9, we can restrict to the theory of self-adjoint extensions. In the latter case, the Hessenberg operator D must be isometric because here the multiplication operator M_μ in L^2_μ is isometric, as $|z| = 1$ for all $z \in \operatorname{supp}(\mu)$.

4.1 Orthogonal Polynomials on the Real Line

Recall that the support of an om for $(P_n)_{n\geq 0}$ is precisely the spectrum of a minimal normal extension of the Hessenberg operator D. Thus, if there exists an om μ such that $\operatorname{supp}(\mu) \subset \mathbb{R}$ then D must have a normal extension with spectrum in $\mathbb{R}$, i.e. a self-adjoint extension. In this case, D is symmetric. Thus its matrix representation is of the form

$$\begin{pmatrix} a_0 & c_0 & 0 & 0 & \cdots \\ \overline{c_0} & a_1 & c_1 & 0 & \cdots \\ 0 & \overline{c_1} & a_2 & c_2 & \\ 0 & 0 & \overline{c_2} & a_3 & \ddots \\ 0 & \vdots & & \ddots & \ddots \end{pmatrix}$$

where $c_i \in \mathbb{C} \setminus \{0\}$ and $a_i \in \mathbb{R}$ for all i. A matrix of this form is called a **Jacobi matrix**. Accordingly, D is called **Jacobi operator** and (1.1) can be written as a 3-term recurrence,

$$z\, P_n(z) = c_{n-1}P_{n-1}(z) + a_nP_n(z) + \overline{c_n}P_{n+1}(z) \tag{4.1}$$

where $c_{-1} := 0$, $P_{-1} :\equiv 0$.

Note that the orthogonal polynomials associated to a particular om μ are uniquely determined up to a factor of absolute value 1. For given μ we can obtain these by the

Gram-Schmidt algorithm applied to $\{1, z, z^2, \ldots\}$. Performing Gram-Schmidt in L^2_μ where $\operatorname{supp}(\mu) \subset \mathbb{R}$, we can always achieve that these polynomials have real coefficients. On the other hand, given $(P_n)_{n\geq 0}$ such that D is symmetric, we can explicitly calculate polynomials $(Q_n)_{n\geq 0}$ such that $Q_n \in \mathbb{R}[z]$ having positive leading coefficient and $Q_n = \alpha_n P_n$ with $|\alpha_n| = 1$ for all $n \in \mathbb{N}_0$, as we shall see now.

4.1.1 Lemma. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials as in* 1.1.1 *such that D is symmetric. Using the notations in* (4.1)*, for $n \in \mathbb{N}$ define*

$$\alpha_n := \prod_{k=0}^{n-1} \frac{\overline{c_k}}{|c_k|}$$

and $Q_n := \alpha_n P_n$, $Q_0 := P_0 \equiv 1$. Then $|\alpha_n| = 1$ and $Q_n \in \mathbb{R}[z]$ with positive leading coefficient for all $n \in \mathbb{N}_0$.

Proof: Clearly, $|\alpha_n| = 1$ for all n. With $\alpha_{-1} := \alpha_0 := 1$ and $Q_{-1} :\equiv 0$, (4.1) yields

$$\begin{aligned} Q_{n+1}(z) = \alpha_{n+1} P_{n+1}(z) &= \frac{\alpha_{n+1}}{\overline{c_n}} \Big((z - a_n) P_n(z) - c_{n-1} P_{n-1}(z) \Big) \\ &= \frac{\alpha_{n+1}}{\alpha_n} \cdot \frac{1}{\overline{c_n}} (z - a_n)\, Q_n(z) - \frac{\alpha_{n+1}}{\alpha_{n-1}} \cdot \frac{c_{n-1}}{\overline{c_n}}\, Q_{n-1}(z) \end{aligned}$$

for all $n \geq 0$. Moreover, by

$$\frac{\alpha_{n+1}}{\alpha_n} \cdot \frac{1}{\overline{c_n}} = \frac{\overline{c_n}}{|c_n|} \cdot \frac{1}{\overline{c_n}} = \frac{1}{|c_n|} \qquad \text{and} \qquad \frac{\alpha_{n+1}}{\alpha_{n-1}} \cdot \frac{c_{n-1}}{\overline{c_n}} = \frac{\overline{c_n c_{n-1}}}{|c_n c_{n-1}|} \cdot \frac{c_{n-1}}{\overline{c_n}} = \frac{|c_{n-1}|}{|c_n|}$$

we can inductively conclude the assertions. □

Note that we did not have to require that an om for $(P_n)_n$ exists.

Note also that, if $p \in \mathbb{R}[z]$ then $p(\overline{z}) = \overline{p(z)}$. Therefore, if D is symmetric then $z \in E$ if and only if $\overline{z} \in E$. As the deficiency indices of the symmetric operator D in the upper and lower half-planes $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$, respectively, are constant, this yields either $\mathbb{C} \setminus \mathbb{R} \subset E$ or $E \subset \mathbb{R}$. It is well-known that a symmetric operator in a Hilbert space $\mathcal{H}$ is closable, having self-adjoint extensions in $\mathcal{H}$ if and only if the deficiency indices in the upper and lower half-planes are equal, see [Ru2, 13.20], for example. Thus the question whether there exists an om for $(P_n)_{n\geq 0}$ with support in $\mathbb{R}$ can be answered with comparatively little effort. The theory of real orthogonal polynomials has been studied intensely for many years; as standard references we mention the textbooks by Chihara [Chi] and Szegő [Sze] as well as Borwein and Erdélyi [BE].

The assertions in this section are already well-known. However, we will consider the situation of real orthogonal polynomials as a special case of complex polynomials. We point out that if all the coefficients of the polynomials $(P_n)_n$ (and hence all entries in the matrix representation of D) are real and D is not symmetric, there still might exist an om μ with $\operatorname{supp}(\mu) \cap (\mathbb{C} \setminus \mathbb{R}) \neq \varnothing$ which, for example, is the case for the Newton Polynomials, see 4.5.7, and the "modified" Hermite polynomials in 4.5.12.

4.1.2 Theorem. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials as in* 1.1.1 *such that D is symmetric. The operator D is essentially self-adjoint if and only if* $\mathrm{ran}(D \pm \mathrm{i}\cdot\mathrm{id})^\perp = \{0\}$. *Moreover, if D is essentially self-adjoint then*

(i) *there exists a unique om μ with* $\mathrm{supp}(\mu) \subset \mathbb{R}$ *and* $P^2_\mu = L^2_\mu$,

(ii) $E = \Lambda_\mu$,

(iii) *$(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if μ is discrete.*

Proof: It is a well-known fact that any densely defined symmetric operator S in a Hilbert space is essentially self-adjoint if and only if $\mathrm{ran}(S - \mathrm{i}\cdot\mathrm{id})^\perp = \mathrm{ran}(S + \mathrm{i}\cdot\mathrm{id})^\perp = \{0\}$, see also [Ru2, 13.20].

Taking into account that the spectrum of a self-adjoint operator is contained in the real line, and that a self-adjoint operator is normal, the remainder of this proposition is just a special case of 3.7.12. □

4.1.3 Example. We start with a sequence $(g_n)_{n\geq 0}$ of positive reals such that $\sum\limits_{n=0}^{\infty} g_n = 1$ and an infinite countable set $\{z_0, z_1, z_2, \ldots\} \subset [0, 1]$. Then

$$\mu := \sum_{n=0}^{\infty} g_n \delta_{z_n} ,$$

where δ_z denotes Dirac measure at z, defines a discrete om for some $(P_n)_n$. The operator D is bounded and symmetric, hence essentially self-adjoint. Due to 4.1.2, μ is the unique om for $(P_n)_n$, $E = \Lambda_\mu = \{z_0, z_1, z_2, \ldots\}$, and $(P_n)_n$ is an onb in $\mathcal{H}(K)$.

Remark. If, in addition, $\Lambda_\mu = \{z_0, z_1, z_2, \ldots\}$ is closed (i.e. contains all its limit points) then we could also use 3.7.11 to conclude uniqueness of μ and $E = \Lambda_\mu$.

4.1.4 Example. Again, choose a sequence $(g_n)_{n\geq 0}$ of positive reals such that $\sum\limits_{n=0}^{\infty} g_n = 1$ and an infinite countable set $\{z_0, z_1, z_2, \ldots\} \subset [0, 1]$.

Let λ denote Lebesgue measure on $[0, 1]$ and δ_z be Dirac measure at the point z. Then

$$\mu := \tfrac{1}{2}\Big(\lambda + \sum_{n=0}^{\infty} g_n \delta_{z_n}\Big)$$

is an om for some $(P_n)_n$ with $\mathrm{supp}(\mu) = [0, 1]$. As in the previous example, the operator D is bounded and symmetric and hence essentially self-adjoint. Now 4.1.2 shows that μ is the unique om for $(P_n)_n$, $E = \Lambda_\mu = \{z_0, z_1, z_2, \ldots\}$, and $(P_n)_n$ is **not** an onb in $\mathcal{H}(K)$.

Remark. As usual, for $z \in \Lambda_\mu$, we define $\kappa(z) := K(z, z)$. According to 3.1.7, $\mu(\{z_n\}) = \kappa(z_n)^{-1}$ for all $z \in \Lambda_\mu$. For arbitrary $\varepsilon > 0$, in 4.1.3 and 4.1.4 we have $g_n < \varepsilon$ for all but at most finitely many n. This implies $\kappa(z_n) > \frac{1}{\varepsilon}$ for all but at most finitely many n. In particular, κ is unbounded.

Recall that, by 2.2.8, if κ is bounded then E is closed. The converse is not true: In 4.1.3 or 4.1.4, $E = \Lambda_\mu = \{z_0, z_1, z_2, \ldots\}$ can be chosen to be closed while κ is unbounded.

We now turn to the case that D is symmetric but not essentially self-adjoint. As the proof of the following lemma requires some lengthy calculations rather than operator theoretical means, we will not give a proof here but refer to [Ak, Theorem 1.3.2], or [Bay, Satz 2.2.12 and Lemma 2.2.1], for instance.

4.1.5 Lemma. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials in $\mathbb{R}[z]$ satisfying a symmetric 3-term recurrence* (4.1)*. If there exists $z \in \mathbb{C}\setminus\mathbb{R}$ such that $\sum\limits_{n\geq 0} |P_n(z)|^2 < \infty$ then*

$$\sum_{n\geq 0} |P_n(z)|^2 < \infty \qquad \text{for all } z \in \mathbb{C}\,.$$

Moreover, the series is uniformly convergent in any compact subset of $\mathbb{C}$.

4.1.6 Theorem. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials as in* 1.1.1 *such that D is symmetric. If D is not essentially self-adjoint then*

(i) $E = E_{reg} = \mathbb{C}$,

(ii) *$(P_n)_n$ is an onb in $\mathcal{H}(K)$ and all members of $\mathcal{H}(K)$ are entire functions,*

(iii) *there exists an om μ such that $(P_n)_n$ is an onb in L^2_μ (i.e. $P^2_\mu = L^2_\mu$),*

(iv) *there exists an om ν such that $P^2_\nu \neq L^2_\nu$,*

(v) $\operatorname{supp}(\mu) \subset \mathbb{R}$ *for all om μ.*

Proof: As D is symmetric and not essentially self-adjoint, there exists $\lambda \in \mathbb{C}\setminus\mathbb{R}$ such that $\operatorname{ran}(D-\lambda\,\mathrm{id})^\perp \neq \{0\}$, in other words, see 3.1.3, $\lambda \in E$.
Without restriction, see 4.1.1, we can assume $P_n \in \mathbb{R}[z]$; 4.1.5 yields $E = \mathbb{C}$ and uniform convergence of the series $\kappa(z) = \sum\limits_{n\geq 0} |P_n(z)|^2$ in any compact subset of $\mathbb{C}$.
According to 3.5.6, all $f \in \mathcal{H}(K)$ are entire functions and, in particular, $E_{reg} = \mathbb{C}$, provided there exists an om.
If μ is an om then $\mu(E) = \mu(\mathbb{C}) = 1$ and 3.7.2 shows that $(P_n)_n$ is an onb in $\mathcal{H}(K)$. Moreover, $E = \mathbb{C}$ implies $\dim(\operatorname{ran}(D - \mathrm{i}\cdot\mathrm{id})^\perp) = \dim(\operatorname{ran}(D + \mathrm{i}\cdot\mathrm{id})^\perp) = 1$; hence there exists a self-adjoint extension of D in $\mathcal{H}$ which we can regard as the multiplication operator M_μ in L^2_μ for some om μ. In particular, $(P_n)_n$ is an onb in L^2_μ.
Every symmetric operator which is not essentially self-adjoint also has self-adjoint extensions in a larger space, see [AG, IX.111, Satz 1]. These correspond to om ν such that $(P_n)_n$ is not an onb in L^2_ν, hence $P^2_\nu \neq L^2_\nu$.
Let now μ be an om and regard D as an operator in P^2_μ. Note that $P_0(z) = 1$, $DP_0(z) = z$, and $D^2P_0(z) = z^2$. Therefore,

$$\begin{aligned}\int |z-\overline{z}|^2 \,\mathrm{d}\mu &= \int \left(2\,|z|^2 - z^2 - \overline{z}^2\right) \mathrm{d}\mu = 2\,\|DP_0\|^2_{L^2_\mu} - \langle\, P_0\,, D^2P_0\,\rangle_{L^2_\mu} - \langle\, D^2P_0\,, P_0\,\rangle_{L^2_\mu}\\ &= 2\,\|DP_0\|^2_{L^2_\mu} - \langle\, DP_0\,, DP_0\,\rangle_{L^2_\mu} - \langle\, DP_0\,, DP_0\,\rangle_{L^2_\mu} = 0\end{aligned}$$

because D is symmetric. This implies that $\mathbb{C}\setminus\mathbb{R}$ is a μ-nullset. As $\mathbb{C}\setminus\mathbb{R}$ is open, we obtain $\operatorname{supp}(\mu) \subset \mathbb{R}$. $\square$

Remark. The last step of the proof, in particular, shows that a symmetric Hessenberg operator cannot have a minimal normal extension which is not self-adjoint.

Combining the preceding results, we see that in the symmetric case there always exists an om.

4.1.7 Corollary. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials as in* 1.1.1*. There exists an om μ with* $\operatorname{supp}(\mu) \subset \mathbb{R}$ *if and only if D is symmetric.*

Proof: Clearly, if there exists an om supported in $\mathbb{R}$ then D is symmetric.

On the other hand, if D is symmetric then either 4.1.2 or 4.1.6 is applicable and in both cases existence of an om supported on the real line follows. □

4.1.8 N-Extremal Measures. We have just seen that if D is symmetric then there always exists an om. In particular, there always exists an om μ such that $P_\mu^2 = L_\mu^2$. These measures are called **Nevanlinna-extremal** or **N-extremal** for short, see [BCh] and also [Ak, Theorem 2.3.3]. It is known that N-extremal measures corresponding to a not essentially self-adjoint Jacobi operator are discrete. We can prove that, too, using results from chapter 3. Before we do so, note the following proposition concerning self-adjoint extensions with particularly desired eigenvalues.

4.1.9 Proposition. *Let S be a symmetric operator in a Hilbert space $\mathcal{K}$ with equal deficiency indices* $\dim(\operatorname{ran}(S - \mathrm{i}\cdot\mathrm{id})^\perp) = \dim(\operatorname{ran}(S + \mathrm{i}\cdot\mathrm{id})^\perp) = m$ *where* $0 < m < \infty$*. For any $\lambda \in \mathbb{R} \cap \operatorname{reg}(S)$, there exists a self-adjoint extension T of S within the space $\mathcal{K}$ having λ as an eigenvalue of multiplicity m.*

For a proof, see [AG, VIII.105, Satz 3].

4.1.10 Theorem. *Assume that D is symmetric but not essentially self-adjoint.*

If μ is an om for $(P_n)_{n\geq 0}$ satisfying $P_\mu^2 = L_\mu^2$ then μ is discrete and every bounded interval contains at most finitely many points of $\operatorname{supp}(\mu)$*.*

For arbitrary $x \in \mathbb{R}$, there exists a unique discrete om ν such that $\nu(\{x\}) > 0$ and $L_\nu^2 = P_\nu^2$. Moreover, then $K(x, w) = 0 \iff w \in \operatorname{supp}(\nu) \setminus \{x\}$.

Proof: By 4.1.6, we have $E = \mathbb{C}$ which implies $\mu(E) = 1$ and, according to 3.7.13, μ is discrete. In particular, with $\Lambda_\mu = \{z \in \mathbb{C} : \mu(\{z\}) > 0\}$ and δ_z the Dirac measure at z, using 3.1.7, we obtain

$$\mu = \sum_{z\in\Lambda_\mu} \frac{1}{K(z,z)}\, \delta_z \,.$$

By 3.7.15, $P(\Lambda_\mu)$ is an onb in ℓ^2, implying $K(z, w) = 0$ for all $z, w \in \Lambda_\mu$, $z \neq w$. Assume that there exists a bounded interval containing infinitely many points of Λ_μ. Thus there exists a limit point of Λ_μ. Let $z \in \Lambda_\mu$ and recall that K_z is a member of $\mathcal{H}(K)$ and, by 4.1.6(ii), an entire function. On the other hand, $K(z, w) = 0$ for $w \in \Lambda_\mu \setminus \{z\}$ implies that the zeros of K_z have a limit point and therefore K_z must vanish identically which is a contradiction. Hence every bounded interval contains at most finitely many points of Λ_μ. In particular, Λ_μ is closed and $\operatorname{supp}(\mu) = \Lambda_\mu$.

Now fix $x \in \mathbb{R}$ and use that $E_{reg} = \mathbb{C}$, see 4.1.6 again. According to 4.1.9, there exists a self-adjoint extension T of D in $\mathcal{H}$ such that x is an eigenvalue of T. By identifying $\mathcal{H}$ with $L^2_\nu = P^2_\nu$ as usual, T corresponds to the multiplication operator M_ν whose eigenvalues are exactly the elements of Λ_ν; hence $\nu(\{x\}) > 0$.
Assume that there exists $b \in \mathbb{C} \setminus \operatorname{supp}(\nu)$ such that $K(x,b) = 0$. Hence $\langle\, k_b\,, k_x \,\rangle_{L^2_\nu} = 0$. Recall that ν is discrete and $\nu(\{x\})\, k_x = \mathbf{1}_{\{x\}}$ in L^2_ν. Thus $(M_\nu - b)^{-1}(M_\nu - a)k_x = \frac{x-a}{x-b}\, k_x$. According to 3.3.8, for arbitrary $a \in \mathbb{C}$, we have $f_{a,b} - \langle\, f_{a,b}\,, P_0 \,\rangle_{L^2_\nu} k_b = 0$ where $f_{a,b} = (M_\nu - a)^*[(M_\nu - b)^{-1}]^* k_a$. This yields

$$0 = \langle\, f_{a,b}\,, k_x \,\rangle_{L^2_\nu} = \langle\, k_a\,, (M_\nu - b)^{-1}(M_\nu - a)k_x \,\rangle_{L^2_\nu} = \tfrac{x-a}{x-b}\, K(x,a)$$

for all $a \in \mathbb{C}$. Hence the entire function K_x vanishes on $\mathbb{C} \setminus \{a\}$ implying $K_x = 0$ which is a contradiction. Thus $K(x,b) \neq 0$ for all $b \in \mathbb{C} \setminus \operatorname{supp}(\nu)$.

As we have already shown, $K(x,b) = 0$ whenever $b \in \operatorname{supp}(\nu) \setminus \{x\}$. Therefore, $\operatorname{supp}(\nu) = \Lambda_\nu = \{x\} \cup \{b \in \mathbb{C} : K(x,b) = 0\}$ and $\nu = \sum_{z \in \Lambda_\nu} \frac{1}{K(z,z)}\, \delta_z$ is uniquely determined. $\square$

Note that we could also use 3.5.7 to prove that every bounded interval contains only finitely many points of Λ_μ.

4.1.11 Corollary. *Let D be a Jacobi operator in a Hilbert space $\mathcal{H}$ which is not essentially self-adjoint. Then there exist self-adjoint extensions of D in the space $\mathcal{H}$ as well as in a larger space $\mathcal{K} \supsetneqq \mathcal{H}$. The spectrum of any self-adjoint extension in $\mathcal{H}$ is discrete. Moreover, for every $x \in \mathbb{R}$, there exists a self-adjoint extension T in $\mathcal{H}$ such that $x \in \sigma(T)$ which is uniquely determined up to unitary equivalence.*

This was just a re-formulation of the previous theorem.

Remark. In terms of the Hamburger moment problem, cf. 1.1.3, the case where there exists a unique om is referred to as the *determinate* case while otherwise one speaks of the *indeterminate* case. The property $E = \mathbb{C}$ in the indeterminate case, see 4.1.6(i), can be found in [ShTa, Cor. 2.7], for example. For more details, see also [BCh].

4.1.12 Orthogonal polynomials on an arbitrary line. It is not difficult to generalize the case of ortonormalizing measures supported in $\mathbb{R}$ to om with support contained in an arbitrary straight line given by $\{a^{-1}(t-b) : t \in \mathbb{R}\}$ where $a, b \in \mathbb{C}$ and $|a| = 1$. Replacing D by $aD + b\,\mathrm{id}$, one can prove analogous results to the above. In particular, D is still tri-diagonal. More generally, a formally normal Hessenberg operator whose matrix representation only has finitely many non-vanishing entries in every row, is tri-diagonal. For details, we refer to [Kl2, Kap. 3.1] and [CaKl1], for example.

4.2 Orthogonal Polynomials on the Unit Circle

Let $\mathbb{D}$ denote the open unit disk, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. In this section we will deal with om μ such that $\operatorname{supp}(\mu) \subset \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Note that then, for $f \in \operatorname{dom}(M_\mu)$,

$$\left\|M_\mu f\right\|_{L^2_\mu}^2 = \int \left|z\, f(z)\right|^2 \,\mathrm{d}\mu(z) = \int \left|f(z)\right|^2 \,\mathrm{d}\mu(z) = \|f\|_{L^2_\mu}^2 \ .$$

Hence M_μ is isometric. Therefore, D must be an isometry, too. Note that $\overline{z} = z^{-1}$ for $z \in \partial\mathbb{D}$ and M_μ^* is multiplication by z^{-1} in L^2_μ. This shows $M_\mu M_\mu^* = \mathrm{id}_{L^2_\mu} = M_\mu^* M_\mu$, thus M_μ is unitary.

We will apply some of our results to this situation. However, orthogonal polynomials on the unit circle have been widely studied in the past few years. A two-volume textbook by Simon [Si] presents a vast treatise on this topic. For a proof of the following proposition, see also [Kl2, Satz 3.3.2], for example.

4.2.1 Proposition. *If D is isometric then there exists a unique om μ. Moreover, $\operatorname{supp}(\mu) \subset \partial\mathbb{D}$.*

The monomials $\{1, z, z^2, \ldots\}$ may serve as a first example. They form an orthonormal system in L^2_μ where μ is the normalized Lebesgue measure on $\partial\mathbb{D}$, see 1.2.2, 1.3.8, and also 2.1.11.

Recall that, whenever there exists an om μ for $(P_n)_{n\geq 0}$, the boundary of the set E is contained in $\operatorname{supp}(\mu)$. Moreover, for $\operatorname{supp}(\mu) \subset \partial\mathbb{D}$, the following holds.

4.2.2 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ such that $\operatorname{supp}(\mu) \subset \partial\mathbb{D}$. The following properties are equivalent.*

(i) $P^2_\mu = L^2_\mu$,

(ii) *D is essentially normal,*

(iii) $E = \Lambda_\mu$,

(iv) $E \subset \operatorname{supp}(\mu)$,

(v) $\sum_{n\geq 0} |P_n(0)|^2 = \infty$,

Moreover, given these properties are satisfied, $(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if μ is discrete.

Proof: For (i) $\iff$ (ii) $\iff$ (iii) see 3.5.9; (iii)$\Rightarrow$(iv)$\Rightarrow$(v) is obvious.
According to 3.1.3, (v) implies $\operatorname{ran}(D)^\perp = \{0\}$. As D is isometric, $\operatorname{ran}(\overline{D})$ is a closed subspace of $\mathcal{H}$. Therefore, $\operatorname{ran}(\overline{D}) = \mathcal{H}$, hence $\overline{D}$ is unitary and D is essentially normal.

For the last assertion, see 3.7.12. □

4.2.3 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ such that* $\operatorname{supp}(\mu) \subset \partial\mathbb{D}$. *The following properties are equivalent.*

(i) $P^2_\mu \neq L^2_\mu$,

(ii) $E \setminus \operatorname{supp}(\mu) \neq \varnothing$,

(iii) $\mathbb{D} \subset E \subset \mathbb{D} \cup \partial\mathbb{D}$,

(iv) $\sum_{n\geq 0} |P_n(0)|^2 < \infty$.

Moreover, if these properties are satisfied then $E_{reg} = E^\circ = \mathbb{D}$ *and* $\operatorname{supp}(\mu) = \partial\mathbb{D}$.

Proof: The implication (ii)⇒(iii) is due to 3.3.6, the implications (iii)⇒(iv) and (iv)⇒(ii) are obvious, and, by 4.2.2, (i) $\iff$ (iv).

For the last assertion, see 3.3.6 again. □

Remark. We could also use $E \subset \left\{z \in \mathbb{C} : |z| \leq \|D\|\right\}$, see 3.1.6, and $\|D\| = 1$ to prove $E \subset \mathbb{D} \cup \partial\mathbb{D}$.

4.2.4 Example. For

$$\Lambda := \bigcup_{n\in\mathbb{N}} \{z \in \mathbb{C} : z^{2^n} = 1\}\,,$$

define a measure μ such that $\mu(\Lambda) = 1$ and $\mu(\{\lambda\}) > 0$ for all $\lambda \in \Lambda$. Thus $\Lambda = \Lambda_\mu$ and $\operatorname{supp}(\mu) = \partial\mathbb{D}$. Clearly, $\mathbb{C}[z] \subset L^2_\mu$, μ is an om for some $(P_n)_n$. Moreover, $\Lambda \subset E$.

For $n \in \mathbb{N}$, define $q_n \in \mathbb{C}[z]$ by $q_n(z) := z^{2^n}$. Note that $\lim_{n\to\infty} q_n(z) = 1$ for all $z \in \Lambda$.

By dominated convergence, $\lim_{n\to\infty} \|q_n - P_0\|_{L^2_\mu} = 0$; (2.4) implies $\lim_{n\to\infty} \|q_n - P_0\|_{\mathcal{H}(K)} = 0$.

Recall that convergence in the $\mathcal{H}(K)$-norm implies pointwise convergence on E. Therefore, $\lim_{n\to\infty} q_n(z) = 1$ for all $z \in E$. In particular, $0 \notin E$.

Now 4.2.2 shows that $E = \Lambda$ and $P^2_\mu = L^2_\mu$. Hence here $\overline{D}$ is a unitary operator with pure point spectrum dense in $\partial\mathbb{D}$.

We will now have a closer look at the case that $\mathbb{C}[z]$ is not total in L^2_μ.

4.2.5 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ supported on $\partial\mathbb{D}$ and suppose that $P^2_\mu \neq L^2_\mu$. Then $\kappa := \sum_{n\geq 0} |P_n(0)|^2 < \infty$. Moreover,*

$$h := \frac{1}{\sqrt{\kappa}} \sum_{n\geq 0} \overline{P_n(0)}\, P_n \in P^2_\mu$$

and $\{(M_\mu)^n h : n \in \mathbb{Z}\}$ is an orthonormal system in L^2_μ.

Finally, let $\widehat{\lambda}$ denote normalized one-dimensional Lebesgue measure on $\partial\mathbb{D}$.

Then $\widehat{\lambda}(\{h = 0\}) = 0$ and $\widehat{\lambda}(\{f \neq 0\}) = 0$ for all $f \in P^2_\mu \cap (h \cdot \mathbb{C}[z])^\perp$.

Proof: According to 4.2.2, $\kappa = \infty$ is equivalent to $P^2_\mu = L^2_\mu$. Hence here $\kappa < \infty$ and h is well-defined. By construction, $h \in P^2_\mu$, $\|h\|_{L^2_\mu} = 1$.

Regard D and $\overline{D}$ as bounded operators in P^2_μ. Clearly, $\mathrm{dom}(\overline{D}) = P^2_\mu$ and M_μ is a normal extension of D as well as $\overline{D}$, acting in the larger space L^2_μ.

Note that $h \in P^2_\mu$ and, for $m, n \in \mathbb{Z}$, $m > n$, we have $(M_\mu)^{m-n} h = \overline{D}^{\,m-n} h \in P^2_\mu$. As M_μ is unitary, $\big\|(M_\mu)^n h\big\|_{L^2_\mu} = 1$ and $M^*_\mu M_\mu = \mathrm{id}_{L^2_\mu}$. Now we get

$$\big\langle (M_\mu)^n h \,, (M_\mu)^m h \big\rangle_{L^2_\mu} = \big\langle (M^*_\mu)^n (M_\mu)^n h \,, (M_\mu)^{m-n} h \big\rangle_{L^2_\mu} = \big\langle h \,, (M_\mu)^{m-n} h \big\rangle_{L^2_\mu}$$
$$= \big\langle h \,, \overline{D}^{\,m-n} h \big\rangle_{P^2_\mu} = \big\langle (D^*)^{m-n} h \,, h \big\rangle_{P^2_\mu} .$$

By 3.1.3, $D^* h = 0$, hence $\big\langle (M_\mu)^n h \,, (M_\mu)^m h \big\rangle_{L^2_\mu} = 0$. Therefore, $\{(M_\mu)^n h : n \in \mathbb{Z}\}$ is an orthonormal system in L^2_μ.

In particular, for $m, n \in \mathbb{N}_0$,

$$\delta_{mn} = \big\langle (M_\mu)^n h \,, (M_\mu)^m h \big\rangle_{L^2_\mu} = \int \overline{z}^n \, z^m \, \big|h(z)\big|^2 \,\mathrm{d}\mu(z)$$

shows that $|h|^2\mu$ is an om for the monomials which, according to 3.2.14, is uniquely determined. Note that $\widehat{\lambda}$ is an om for $(z^n)_n$; thus $|h|^2\mu = \widehat{\lambda}$ and we obtain $\widehat{\lambda}(\{h = 0\}) = 0$.

Let now $f \in P^2_\mu \cap (h \cdot \mathbb{C}[z])^\perp$, i.e. $f \in P^2_\mu$ and, for $n \in \mathbb{N}_0$,

$$0 = \int \overline{f(z)} \, z^n h(z) \,\mathrm{d}\mu(z) = \int \overline{\big(f(z)/h(z)\big)} \, z^n \big|h(z)\big|^2 \,\mathrm{d}\mu(z) = \int \overline{\big(f(z)/h(z)\big)} \, z^n \,\mathrm{d}\widehat{\lambda}(z) .$$

Furthermore,

$$\int \overline{f(z)} \, z^{-n} h(z) \,\mathrm{d}\mu(z) = \big\langle f \,, (M^*_\mu)^n h \big\rangle_{L^2_\mu} = \big\langle (M_\mu)^n f \,, h \big\rangle_{L^2_\mu} = \big\langle \overline{D}^n f \,, h \big\rangle_{P^2_\mu}$$
$$= \big\langle f \,, (D^*)^n h \big\rangle_{P^2_\mu} = \langle f \,, 0 \rangle_{P^2_\mu} = 0 \qquad \text{for all } n \in \mathbb{N} .$$

Hence $0 = \int \overline{f(z)} \, z^n h(z) \,\mathrm{d}\mu(z) = \int \overline{\big(f(z)/h(z)\big)} \, z^n \,\mathrm{d}\widehat{\lambda}(z)$ holds for all $n \in \mathbb{Z}$.

As $\{z^n : n \in \mathbb{Z}\}$ is an onb in $L^2_{\widehat{\lambda}}$, we obtain $f/h = 0$ in $L^2_{\widehat{\lambda}}$ and $\widehat{\lambda}(\{f \neq 0\}) = 0$. $\square$

4.2.6 Lebesgue Decomposition. Let μ be a Borel measure on $\partial\mathbb{D}$. We denote its absolutely continuous, discrete, and singularly continuous parts by μ_{ac}, μ_{d}, and μ_{sc}, respectively. Moreover, let λ denote Lebesgue measure on $\partial\mathbb{D}$. Then $\mu_{\mathrm{ac}} = \alpha\lambda$ with measurable non-negative α. For abbreviation, set $\mu_\perp := \mu_{\mathrm{d}} + \mu_{\mathrm{sc}}$.

Note that μ_{ac}, μ_{d}, and μ_{sc} are mutually orthogonal, i.e. there exist mutually disjoint subsets A_{ac}, A_{d}, and A_{sc} of $\partial\mathbb{D}$ such that

$$\mu_{\mathrm{sc}}(A_{\mathrm{ac}}) = 0 = \mu_{\mathrm{d}}(A_{\mathrm{ac}}) \,, \qquad \mu_{\mathrm{ac}}(A_{\mathrm{d}}) = 0 = \mu_{\mathrm{sc}}(A_{\mathrm{d}}) \,, \qquad \mu_{\mathrm{ac}}(A_{\mathrm{sc}}) = 0 = \mu_{\mathrm{d}}(A_{\mathrm{sc}}) \,,$$

and $\mu(A_{\mathrm{ac}}) = \mu_{\mathrm{ac}}(A_{\mathrm{ac}}) > 0$, $\mu(A_{\mathrm{d}}) = \mu_{\mathrm{d}}(A_{\mathrm{d}}) > 0$, $\mu(A_{\mathrm{sc}}) = \mu_{\mathrm{sc}}(A_{\mathrm{sc}}) > 0$. The set A_{d} is at most countable while A_{ac} and A_{sc} are uncountable (or empty).

For more details, we refer to [El, VII. §4], for example.

4.2.7 Proposition. *In the situation of* 4.2.5, *let* $\mu = \alpha\widehat{\lambda} + \mu_\perp$.
Then $\alpha = \frac{1}{|h|^2}$ *holds* $\widehat{\lambda}$*-almost everywhere and* $\mu_\perp(\{h \neq 0\}) = 0$.

Proof: Recall that $\widehat{\lambda} = |h|^2\mu = |h|^2\mu_\perp + |h|^2\mu_{\mathrm{ac}}$.
Therefore, $|h|^2\mu_\perp = 0$ and $\mu_{\mathrm{ac}} = \frac{1}{|h|^2}\,\widehat{\lambda}$; hence $\mu_\perp(\{h \neq 0\}) = 0$ and $\alpha = \frac{1}{|h|^2}$. □

For $f \in P^2_\mu$, let f^K be as in 3.6.2 and recall that $f^K(z) = \langle\, k_z\,, f\,\rangle_{L^2_\mu}$ for all $z \in E$, see 3.6.6.

4.2.8 Theorem. *In the situation of* 4.2.5,

$$V_\infty = \bigcap_{n\in\mathbb{N}} \mathrm{ran}\big(\overline{D}^n\big) = \big\{f \in P^2_\mu : f^K|\mathbb{D} = 0\big\} = \big\{f \in P^2_\mu : \widehat{\lambda}(\{f \neq 0\}) = 0\big\}\,.$$

Proof: First note that, due to 4.2.3, we have $E_{reg} = E^\circ = \mathbb{D}$ which is a connected set and 3.7.6 yields $V_\infty = \bigcap\limits_{n\in\mathbb{N}} \mathrm{ran}\big(\overline{D}^n\big) = \big\{f \in P^2_\mu : f^K|\mathbb{D} = 0\big\}$.

Let now $f \in \bigcap\limits_{n\in\mathbb{N}} \mathrm{ran}\big(\overline{D}^n\big) \subset P^2_\mu$.

Then, for every $m \in \mathbb{N}$, there exists $f_m \in P^2_\mu$ such that $f = \overline{D}^m f_m = (M_\mu)^m f_m$. Thus

$$\begin{aligned}\big\langle\, (M_\mu)^n h\,, f\,\big\rangle_{L^2_\mu} &= \big\langle\, (M_\mu)^n h\,, (M_\mu)^{n+1} f_{n+1}\,\big\rangle_{L^2_\mu} = \big\langle\, (M^*_\mu)^n (M_\mu)^n h\,, M_\mu f_{n+1}\,\big\rangle_{L^2_\mu}\\ &= \big\langle\, h\,, M_\mu f_{n+1}\,\big\rangle_{L^2_\mu} = \big\langle\, h\,, \overline{D} f_{n+1}\,\big\rangle_{P^2_\mu} = \big\langle\, D^* h\,, f_{n+1}\,\big\rangle_{P^2_\mu} = 0\end{aligned}$$

for all $n \in \mathbb{N}_0$ and 4.2.5 yields $\widehat{\lambda}(\{f \neq 0\}) = 0$.

As to the converse inclusion, let $f \in P^2_\mu$ such that $\widehat{\lambda}(\{f \neq 0\}) = 0$. Then

$$\langle\, h\,, f\,\rangle_{L^2_\mu} = \int \overline{h}\, f\ \mathrm{d}\mu = \int \overline{h}\, f\ \mathrm{d}\mu_\perp\,.$$

Note that, by 4.2.7, $\mu_\perp(\{h \neq 0\}) = 0$. Hence the above integral vanishes and we obtain $\langle\, h\,, f\,\rangle_{L^2_\mu} = 0$.
As $\overline{D}$ is an isometry, $\mathrm{ran}(\overline{D})$ is a closed subspace of P^2_μ; 3.1.3 implies $\mathrm{ran}(\overline{D}) = \{h\}^\perp$. Therefore, $f \in \mathrm{ran}(\overline{D})$ and there exists $f_1 \in P^2_\mu$ such that $f = \overline{D} f_1 = M_\mu f_1$. This shows $f_1 = (M_\mu)^{-1} f = M^*_\mu f$. As $f \neq 0$ only on a null-set w.r.t. Lebesgue measure, so is f_1.
Now we can apply this result to f_1 instead of f. Repeating that n times, we see that $f \in \mathrm{ran}(\overline{D}^n)$ for arbitrary n. □

4.2.9 Theorem. *Let* μ *be an om for* $(P_n)_{n\geq 0}$ *such that* $\mathrm{supp}(\mu) \subset \partial\mathbb{D}$. *For* $z \in E$, *recall* $k_z = \sum\limits_{n\geq 0} \overline{P_n(z)}\, P_n \in P^2_\mu$.

We have $E \cap \partial\mathbb{D} = \Lambda_\mu$ *and* $\mathbf{1}_{\{z\}} = \mu(\{z\})\, k_z \in P^2_\mu$ *for all* $z \in \Lambda_\mu$.

Proof: In the case that $(P_n)_n$ is an onb in L^2_μ, 4.2.2 shows $E = \Lambda_\mu$. The other assertion follows easily using 3.1.8.

Otherwise, 4.2.3 shows $\mathbb{D} \subset E \subset \mathbb{D} \cup \partial\mathbb{D}$ and, according to 4.2.5, $\{\overline{D}^n h : n \in \mathbb{N}_0\} = \{M_\mu^n h : n \in \mathbb{N}_0\}$ is an orthonormal system in P^2_μ.
This yields $\big(\langle f , \overline{D}^n h \rangle_{P^2_\mu}\big)_n \in \ell^2$ for all $f \in P^2_\mu$.

Now let $a \in E \cap \partial\mathbb{D}$. Then $k_a := \sum\limits_{n \geq 0} \overline{P_n(a)}\, P_n \in P^2_\mu$ and $D^* k_a = \overline{a} k_a$. This implies

$$\langle k_a , \overline{D}^n h \rangle_{P^2_\mu} = \langle (D^*)^n k_a , h \rangle_{P^2_\mu} = a^n \langle k_a , h \rangle \qquad \text{for all } n \in \mathbb{N}$$

and, as $|a| = 1$, we obtain an ℓ^2-sequence only if $\langle k_a , h \rangle_{P^2_\mu} = 0$.

Once again we use that $\operatorname{ran}(\overline{D}) = \{h\}^\perp$. Hence $k_a \in \operatorname{ran}(\overline{D})$ and, as $\overline{D}D^*$ is the orthogonal projection in P^2_μ onto $\operatorname{ran}(\overline{D})$, we finally see $a\, k_a = \overline{D} D^* a\, k_a = \overline{D}\overline{a}\, a\, k_a = \overline{D} k_a = M_\mu k_a$.

As the eigenvalues of the multiplication operator M_μ are precisely the indicator functions $\mathbf{1}_{\{z\}}$ for $z \in \Lambda_\mu$, we obtain $\mathbf{1}_{\{a\}} = c \cdot k_a \in P^2_\mu$ with some $c \neq 0$. Thus $E \cap \partial\mathbb{D} \subset \Lambda_\mu$. The converse inclusion is clear, as $\Lambda_\mu \subset E$.

Now, according to 3.1.8, $\mathbf{1}_{\{z\}} = \mu(\{z\})\, k_z \in P^2_\mu$ for all $z \in \Lambda_\mu$. □

Remark. This theorem is not new. Our proof, however, mainly uses the theory of subnormal Hessenberg operators. The more general statement 3.1.8, consequently, cannot be found in works concerning om supported on the unit circle only. For other ways proving 4.2.9, see e.g. [Si, 2.7.3 and 4.3.15(a)].

So far, in this section we have not yet had a look at the reproducing kernel space $\mathcal{H}(K)$. In the case that $(P_n)_n$ is an onb in L^2_μ, we know that $E = \Lambda_\mu$ and $\mathcal{H}(K)$ is isometrically isomorphic to the closed linear span of $\{\mathbf{1}_{\{z\}} : z \in \Lambda_\mu\}$ via $\mathbf{1}_{\{z\}} \mapsto \mu(\{z\})\, K_z$. Otherwise, we have $\mathbb{D} \subset E$; here the connections between L^2_μ and $\mathcal{H}(K)$ are much more interesting.

4.2.10 Theorem. *Let μ be an om for $(P_n)_{n\geq 0}$ supported on $\partial\mathbb{D}$ such that $P^2_\mu \neq L^2_\mu$. The following properties hold.*

(i) $E = \mathbb{D} \cup \Lambda_\mu$,

(ii) *all $f \in \mathcal{H}(K)$ are holomorphic in $\mathbb{D}$,*

(iii) $\langle k_\lambda , k_a \rangle_{L^2_\mu} = \langle K_\lambda , K_a \rangle_{\mathcal{H}(K)} = 0$ *for all $\lambda \in \Lambda_\mu$ and all $a \in \mathbb{D}$,*

(iv) $k_\lambda \in V_\infty$ *for all $\lambda \in \Lambda_\mu$,*

(v) *if $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $\lim\limits_{n\to\infty} z_n =: z_* \in \partial\mathbb{D}$ then*

$$\kappa(z_n) \to \infty \qquad \textit{as } n \to \infty$$

where $\kappa(z) = K(z,z) = \sum\limits_{n\geq 0} |P_n(z)|^2$ for $z \in E$.

Proof: Assertion (i) is an immediate consequence of 4.2.9 in connection with 4.2.3(iii).

As to (ii), according to 4.2.3, we have $E_{reg} = \mathbb{D}$ and, by 3.5.2, all members of $\mathcal{H}(K)$ are holomorphic in $\mathbb{D}$.

Concerning (iii) and (iv), let $\Lambda_\mu \neq \varnothing$. For any $\lambda \in \Lambda_\mu$, we have $\overline{D}k_\lambda = \lambda k_\lambda$, see 4.2.9, which implies $k_\lambda \in V_\infty$. Now 4.2.8 yields $k_\lambda^K|\mathbb{D} = 0$, in other words, $\langle\, k_\lambda \,, k_a \,\rangle_{\mathcal{H}} = 0$ for all $a \in \mathbb{D}$. Isometrically embedding $\mathcal{H}$ into L^2_μ as usual, we obtain

$$0 = \langle\, k_\lambda \,, k_a \,\rangle_{L^2_\mu} = \sum_{n\geq 0} \overline{P_n(\lambda)}\, P_n(a) = \langle\, K_\lambda \,, K_a \,\rangle_{\mathcal{H}(K)} \qquad \text{for all } \lambda \in \Lambda_\mu \text{ and all } a \in \mathbb{D}\,.$$

For the proof of (v), assume that there exists $R > 0$ such that $\kappa(z_n) \leq R$ for all $n \in \mathbb{N}$, i.e. $\{z_n \,:\, n \in \mathbb{N}\} \subset A_R = \{z \in E \,:\, \kappa(z) \leq R\}$. As A_R is closed, see 2.2.7, we obtain $z_* \in A_R$ and 2.2.13 yields that $K_{z_*}|A_R$ is continuous. Note that $K_{z_*}(z_*) \neq 0$ and $K_{z_*}(z_n) = \langle\, K_{z_n} \,, K_{z_*} \,\rangle_{\mathcal{H}(K)} = 0$ for all $n \in \mathbb{N}$ by (iii). This is a contradiction; therefore, there exists no such R and $\kappa(z_n) \to \infty$. $\square$

Note that in (ii) and (iii) we do not require $(P_n)_n$ be an onb in $\mathcal{H}(K)$, see also the remark following 3.7.9. If $\Lambda_\mu \neq \varnothing$, here $\mathcal{H}(K)$ contains functions which are not continuous.

In order to determine a necessary and sufficient condition for $(P_n)_n$ being an onb in $\mathcal{H}(K)$, we will now have a closer look at the space V_∞.

4.2.11 Von Neumann-Wold Decomposition. *Let S be an isometric operator in a Hilbert space $\mathcal{H}$, $\mathcal{H}_0 := \operatorname{ran}(S)^\perp$, and $V_\infty := \bigcap_{n\in\mathbb{N}} \operatorname{ran}(S^n)$.*

Then V_∞ is a closed subspace of $\mathcal{H}$ which reduces S and the restriction of S to V_∞ is unitary. Moreover, $V_\infty^\perp$ is the orthogonal sum

$$V_\infty^\perp = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}_n$$

of spaces $\mathcal{H}_n$ where $S\mathcal{H}_{n-1} = \mathcal{H}_n$ for all $n \in \mathbb{N}$.

For a proof we refer to [Con1, I.3.6].

Note that S is unitary if and only if $V_\infty^\perp = \{0\}$. Furthermore, recall that in the case of an om μ supported on the unit circle, we have $P^2_\mu = L^2_\mu$ if and only if $\overline{D}$ is unitary. Thus either $V_\infty = P^2_\mu = L^2_\mu$ or $V_\infty \subsetneqq P^2_\mu \subsetneqq L^2_\mu$.

In the latter case, to avoid confusion with orthogonal complements taken in L^2_μ, denote the orthogonal complement of V_∞ in P^2_μ by V_+, hence we have $P^2_\mu = V_\infty \oplus V_+$. Moreover, the orthogonal complement of $\operatorname{ran}(\overline{D})$ in P^2_μ is given by $\mathcal{H}_0 = \mathbb{C} \cdot k_0$ where

$$k_0 = \sum_{n\geq 0} \overline{P_n(0)}\, P_n\,,$$

see also 3.1.3; normalizing this by $h := \frac{k_0}{\|k_0\|}$, we obtain

$$V_+ = \bigoplus_{n\in\mathbb{N}_0} \overline{D}^n \mathcal{H}_0 = \overline{\mathrm{lin}\{\overline{D}^n h : n \in \mathbb{N}_0\}} = \overline{\mathrm{lin}\{(M_\mu)^n h : n \in \mathbb{N}_0\}}$$

and $\{\overline{D}^n h : n \in \mathbb{N}_0\}$ is an orthonormal basis of V_+.

Note that we already know that $\{\overline{D}^n h : n \in \mathbb{N}_0\}$ is an onb of V_+ from 4.2.5.

4.2.12 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ supported on $\partial\mathbb{D}$ such that $P^2_\mu \neq L^2_\mu$. Then $k_\lambda \in V_\infty$ for all $\lambda \in \Lambda_\mu$ and $k_a \in V_+$ for all $a \in \mathbb{D}$.*

Proof: The first assertion is due to 4.2.10(iv).

Denote by P the orthogonal projection in P^2_μ onto V_+ and, for $a \in \mathbb{D}$, set $j_a := Pk_a$. Then

$$j_a = \sum_{n\geq 0} \left\langle \overline{D}^n h \,, k_a \right\rangle \overline{D}^n h = \sum_{n\geq 0} \left\langle h \,, (D^*)^n k_a \right\rangle \overline{D}^n h = \sum_{n\geq 0} \langle h \,, k_a \rangle \, \overline{a}^n \, \overline{D}^n h$$

and, using $D^*h = 0$, $D^*\overline{D}h = h$, we obtain

$$D^* j_a = \sum_{n\geq 0} \langle h \,, k_a \rangle \, \overline{a}^n \, D^*\overline{D}^n h = \sum_{n\geq 1} \langle h \,, k_a \rangle \, \overline{a}^n \, \overline{D}^{n-1} h = \sum_{n\geq 0} \langle h \,, k_a \rangle \, \overline{a}^{n+1} \, \overline{D}^n h = \overline{a} \, j_a \, .$$

Hence $j_a \in \mathcal{N}(D^* - \overline{a}\,\mathrm{id})$ and, therefore, $j_a = c_a k_a$ with some $c_a \in \mathbb{C}$. As $j_a = Pk_a$ with P an orthogonal projection, only $c_a = 0$ or $c_a = 1$ is possible. Note that $c_a = 0$ means $j_a = Pk_a = 0$ and, therefore, $k_a \in V_\infty$ which implies $k^K_a|\mathbb{D} = 0$ in contradiction to $k^K_a|\mathbb{D}\,(a) = \langle k_a \,, k_a \rangle \neq 0$.

Thus $j_a = Pk_a = k_a$ and $k_a \in V_+$. □

In order to show that V_∞ reduces not only $\overline{D}$ but also M_μ we need the following.

4.2.13 Lemma. *Let U be a unitary operator in a Hilbert space $\mathcal{H}$ and $\mathcal{L}$ be a closed subspace of $\mathcal{H}$. Then $\mathcal{L}$ reduces U if and only if $U\mathcal{L} = \mathcal{L}$.*

Proof: Let P be the orthogonal projection in $\mathcal{H}$ onto $\mathcal{L}$.

Assume that $\mathcal{L}$ reduces U. Then $PU = UP$ and, for any $f \in \mathcal{L}$, we have $Uf = UPf = PUf \in \mathcal{L}$; hence $U\mathcal{L} \subset \mathcal{L}$. Moreover, $f = PUU^*f = UPU^*f$ shows $U\mathcal{L} \supset \mathcal{L}$.

As to the converse implication, from $U^* = U^{-1}$ and $U\mathcal{L} = \mathcal{L}$ we obtain $U^*\mathcal{L} = U^{-1}\mathcal{L} = \mathcal{L}$. Then $UP = PUP$ and $U^*P = PU^*P$. The latter implies $PU = PUP$ and altogether we have $UP = PU$ showing that $\mathcal{L}$ reduces U. □

4.2.14 Corollary. *Let μ be an om for $(P_n)_{n\geq 0}$ supported on the unit circle. Then V_∞ reduces M_μ.*

Proof: According to 4.2.11, the restriction of $\overline{D}$ to V_∞ is unitary. This implies $\overline{D}V_\infty = V_\infty$ and, as M_μ extends $\overline{D}$, also $M_\mu V_\infty = V_\infty$. Now we can apply 4.2.13 to conclude that V_∞ reduces M_μ. □

Recall that, according to 3.6.9, $(P_n)_n$ is not an onb in $\mathcal{H}(K)$ if and only if there exists $f \in P^2_\mu$, $f \neq 0$, such that $f^K = 0$. Therefore, we define $W_K := \{f \in P^2_\mu : f^K = 0\} = \{f \in P^2_\mu : \langle\, k_a \,,\, f \,\rangle = 0 \text{ for all } a \in E\}$ and obtain that $(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if $W_K = \{0\}$. Note that $W_K \subset \{f \in P^2_\mu : f^K|\mathbb{D} = 0\} = V_\infty$, see also 4.2.8, and if we define $W := \overline{\operatorname{lin}\{k_a : a \in E\}}$, we get $P^2_\mu = W_K \oplus W$.

Finally, let $V_d := \overline{\operatorname{lin}\{k_a : a \in \Lambda_\mu\}}$.

4.2.15 Lemma. *Using the notations above, we have*

$$V_\infty = V_d \oplus W_K \qquad \text{and} \qquad V_+ = \overline{\operatorname{lin}\{k_a : a \in \mathbb{D}\}}\,.$$

Moreover, V_d and W_K reduce M_μ.

Proof: Let $X := \overline{\operatorname{lin}\{k_a : a \in \mathbb{D}\}}$. Then $W = V_d \cup X$.
According to 4.2.12, $V_d \subset V_\infty$ and $X \subset V_+$. In connection with $P^2_\mu = V_\infty \oplus V_+$, this yields $W = V_d \oplus X$. Using $P^2_\mu = W_K \oplus W$, we get $P^2_\mu = W_K \oplus V_d \oplus X$.
With $W_K \oplus V_d \subset V_\infty$ and $X \subset V_+$, we obtain $V_d \oplus W_K = V_\infty$ and $X = V_+$.

Now we show that both V_d and W_K are invariant subspaces for M_μ. As V_d is the closure of the linear span of the eigenvectors of M_μ, clearly $M_\mu f \in V_d$ for all $f \in V_d$. Let now $f \in W_K$. Then, for all $a \in E$,

$$0 = a\langle\, k_a \,,\, f \,\rangle = \left\langle\, \overline{a}\, k_a \,,\, f \,\right\rangle = \left\langle\, D^* k_a \,,\, f \,\right\rangle = \left\langle\, k_a \,,\, \overline{D} f \,\right\rangle$$

and, therefore, $M_\mu f = \overline{D} f \in W_K$.
By 4.2.13, $M_\mu V_\infty = V_\infty$. As $V_\infty = V_d \oplus W_K$, we obtain $M_\mu V_d = V_d$ as well as $M_\mu W_K = W_K$ and can use 4.2.13 to conclude that V_d and W_K reduce M_μ. □

Remark. We can also deduce $M_\mu V_d = V_d$ from the fact that V_d is the closure of the linear span of the eigenvectors of M_μ and 0 is not an eigenvalue.

4.2.16 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ supported on $\partial\mathbb{D}$ such that $P^2_\mu \neq L^2_\mu$ and define $V_- := \overline{\operatorname{lin}\{(M_\mu)^{-n} h : n \in \mathbb{N}\}}$. Then $V_- = \left(P^2_\mu\right)^\perp$ and*

$$L^2_\mu = V_\infty \oplus V_+ \oplus V_-\,.$$

Moreover, $V_s := V_+ \oplus V_- = \overline{\operatorname{lin}\{(M_\mu)^{-n} h : n \in \mathbb{Z}\}} = \{\mathbf{1}_{\{h\neq 0\}} f \,:\, f \in L^2_\mu\}$ and $V_\infty = \{\mathbf{1}_{\{h=0\}} f : f \in L^2_\mu\}$.

Proof: Using 4.2.5, we see that $\{(M_\mu)^n h \,:\, n \in \mathbb{N}_0\}$ and $\{(M_\mu)^{-n} h \,:\, n \in \mathbb{N}\}$ are orthonormal bases of V_+ and V_-, respectively. For arbitrary $n \in \mathbb{N}$ and $p \in P^2_\mu$, we obtain

$$\left\langle\, (M_\mu)^{-n} h \,,\, p \,\right\rangle_{L^2_\mu} = \left\langle\, h \,,\, (M_\mu)^n p \,\right\rangle_{L^2_\mu} = \left\langle\, h \,,\, \overline{D}^n p \,\right\rangle_{P^2_\mu} = \left\langle\, (D^*)^n h \,,\, p \,\right\rangle_{P^2_\mu} = 0\,.$$

This yields $V_- \subset \left(P^2_\mu\right)^\perp$ and, in particular, $V_\infty \perp V_-$. As V_+ was defined as the orthogonal complement of V_∞ in P^2_μ, we now have $V_\infty \perp V_s$.

Clearly, $M_\mu V_s = V_s$ and, according to 4.2.13, V_s reduces M_μ. Thus $V_\infty \oplus V_s$ also reduces M_μ. Furthermore, $P^2_\mu \subset V_\infty \oplus V_s$ and, as M_μ is a minimal normal extension of $\overline{D}$, the only reducing subspace of M_μ containing P^2_μ is the whole space L^2_μ.

Therefore, $L^2_\mu = V_\infty \oplus V_s$ and $V_- = \left(P^2_\mu\right)^\perp$.

Now we use that $L^2_\mu = \{\mathbf{1}_{\{h\neq 0\}} f : f \in L^2_\mu\} \oplus \{\mathbf{1}_{\{h=0\}} f : f \in L^2_\mu\}$.

Assume $g \in V_\infty \cap \{\mathbf{1}_{\{h\neq 0\}} f : f \in L^2_\mu\}$. According to 4.2.8, $\widehat{\lambda}(\{g \neq 0\}) = 0$ and, according to 4.2.7, $\mu_\perp(\{h \neq 0\}) = 0$ which implies $\mu_\perp(\{g \neq 0\}) = 0$. Altogether, we now have $\mu(\{g \neq 0\}) = 0$ and hence $g = 0$.

Thus $\{\mathbf{1}_{\{h\neq 0\}} f : f \in L^2_\mu\} \perp V_\infty$.

On the other hand, for any $f \in L^2_\mu$ and $n \in \mathbb{Z}$,

$$\left\langle \mathbf{1}_{\{h=0\}} f \,, (M_\mu)^n h \right\rangle = \int\limits_{\{h=0\}} \overline{f(z)}\, z^n\, h(z)\, \mathrm{d}\mu(z) = 0$$

showing $\{\mathbf{1}_{\{h=0\}} f : f \in L^2_\mu\} \perp V_s$.

Now we finally can conclude $V_\infty = \{\mathbf{1}_{\{h=0\}} f : f \in L^2_\mu\}$ and $V_s = \{\mathbf{1}_{\{h\neq 0\}} f : f \in L^2_\mu\}$. □

4.2.17 Theorem. *Let $\mu = \alpha\widehat{\lambda} + \mu_d + \mu_{sc}$ be an om for $(P_n)_{n\geq 0}$ supported on $\partial\mathbb{D}$ such that $P^2_\mu \neq L^2_\mu$. Then*

(i) $\mu_d(\{h \neq 0\}) = 0 = \mu_{sc}(\{h \neq 0\})$,

(ii) $\widehat{\lambda}(\Lambda_\mu) = 0 = \mu_{sc}(\Lambda_\mu)$,

(iii) $\widehat{\lambda}(\{h = 0\} \setminus \Lambda_\mu) = 0 = \mu_d(\{h = 0\} \setminus \Lambda_\mu)$,

and $L^2_\mu = V_s \oplus V_d \oplus W_K$ where

- $V_s = \{\mathbf{1}_{\{h\neq 0\}} f : f \in L^2_\mu\}$,
- $V_d = \{\mathbf{1}_{\Lambda_\mu} f : f \in L^2_\mu\}$,
- $W_K = \{\mathbf{1}_{\{h=0\}\setminus\Lambda_\mu} f : f \in L^2_\mu\}$.

Furthermore, $\mathcal{H}(K)$ is isometrically isomorphic to $V_+ \oplus V_d$ via $P_n \mapsto P_n$ and $(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if $\mu_{sc} = 0$.

Moreover, if $\mu_{sc} \neq 0$ then $P_n \notin V_d \oplus V_+$ for all n.

Proof: (i) is due to 4.2.7; (ii) is clear because Λ_μ is countable; $\Lambda_\mu = \{z \in \mathbb{C} : \mu(\{z\}) > 0\}$ also proves the second equality in (iii). Finally, according to 4.2.5, $\widehat{\lambda}(\{h = 0\} \setminus \Lambda_\mu) = 0$.

The decomposition $L^2_\mu = V_s \oplus V_d \oplus W_K$ was the subject of 4.2.16 where we have already proved $V_s = \{\mathbf{1}_{\{h\neq 0\}} f : f \in L^2_\mu\}$.

From 4.2.16 we also know that $V_\infty = \{\mathbf{1}_{\{h=0\}} f : f \in L^2_\mu\}$ and, by 4.2.15, $V_\infty = V_d \oplus W_K$. As V_d is spanned by the eigenvectors of M_μ which are the $\mathbf{1}_{\{\lambda\}}$ for $\lambda \in \Lambda_\mu$, we see $V_d = \{\mathbf{1}_{\Lambda_\mu} f : f \in L^2_\mu\}$. Now $W_K = \{\mathbf{1}_{\{h \neq 0\} \setminus \Lambda_\mu} f : f \in L^2_\mu\}$ is clear, too.

Note that W_K is the null-space of the partial isometry $\alpha : P^2_\mu \to \mathcal{H}(K)$, $f \mapsto f^K$, defined in 3.6.7; hence α isometrically maps $V_d \oplus V_+$ onto $\mathcal{H}(K)$. Moreover, $(P_n)_n$ is an onb in $\mathcal{H}(K)$ if and only if $W_K = \{0\}$. By the characterizations above, this is the case if and only if $\mu_{\mathrm{sc}} = 0$.

Concerning the last assertion, assume $P_n \in V_d \oplus V_+$ for some n. Hence $\int |P_n|^2 \, \mathrm{d}\mu_{\mathrm{sc}} = 0$. This implies $P_n(z) = 0$ for μ_{sc}-almost all z in contradiction to P_n having only finitely many zeros. $\square$

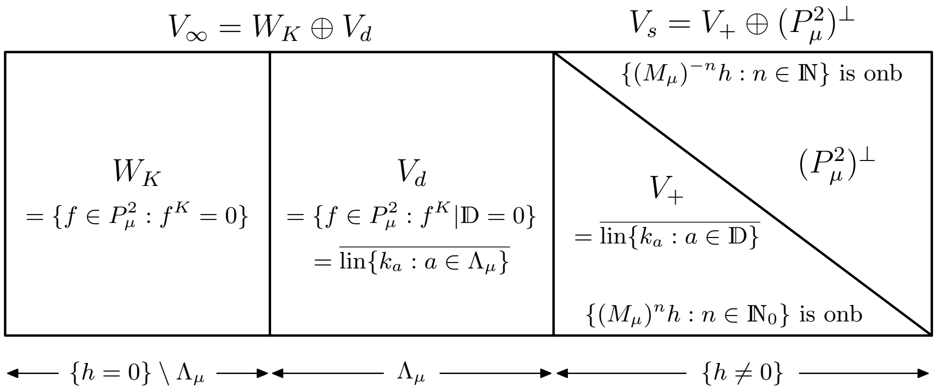


Figure 6.
In the case of an om μ supported on $\partial\mathbb{D}$ such that $P^2_\mu \neq L^2_\mu$, the space L^2_μ can be decomposed into $L^2_\mu = W_K \oplus V_d \oplus V_+ \oplus (P^2_\mu)^\perp$.
The spaces W_K, V_d, and $V_s = V_+ \oplus (P^2_\mu)^\perp$ reduce M_μ and can be characterized by the sets $\{h = 0\} \setminus \Lambda_\mu$, Λ_μ, and $\{h \neq 0\}$ as well as by the singularly continuous, discrete, and absolutely continuous parts of μ, respectively.

As mentioned before, om supported on $\partial\mathbb{D}$ have been widely studied. In the remainder of this section we will give a short overview on other remarkable aspects concerning orthogonal polynomials on the unit circle.

4.2.18 Definition. A Borel measure $\mu = \alpha\lambda + \mu_{\mathrm{d}} + \mu_{\mathrm{sc}}$ on $\partial\mathbb{D}$ is said to obey the **Szegő condition** if

$$\int \ln \alpha \, \mathrm{d}\lambda > -\infty .$$

Note that μ cannot obey the Szegő condition if $\mathrm{supp}(\mu) \subsetneqq \partial\mathbb{D}$.

Let μ be an om for $(P_n)_{n \geq 0}$. It is well-known, see [Sze, Chapter XI] and also [Con1, III.12.9] or [Si, 2.7.15], for example, that μ obeys the Szegő condition if and only if $P^2_\mu \neq L^2_\mu$.

4.2.19 Example. Let $\mu = \alpha\widehat{\lambda}$ be an om supported on $\partial\mathbb{D}$ such that

$$\int \tfrac{1}{\alpha}\,\mathrm{d}\widehat{\lambda} < \infty\,.$$

Recall 2.3.11 which implies that in this case $\mathbb{D} \subset E$. By 4.2.3, then $P^2_\mu \neq L^2_\mu$; hence μ must obey the Szegő condition. We will verify that by a short calculation.

First note that the zeros of α form a Lebesgue null-set because otherwise the integral above would not be finite.

Clearly, $0 \leq \ln(x) < x$ for $x \geq 1$ and, for $0 < x < 1$, we have $\bigl|\ln(x)\bigr| = \ln\bigl(\frac{1}{x}\bigr) \leq \frac{1}{x} - 1 = \frac{1-x}{x} \leq \frac{1}{x}$. Thus, for $A := \{0 < \alpha < 1\}$ and $B := \{\alpha \geq 1\}$, the integrals

$$\int_A -\ln(\alpha)\,\mathrm{d}\lambda \;\leq\; \int_A \tfrac{1}{\alpha}\,\mathrm{d}\lambda \qquad \text{and} \qquad \int_B \ln(\alpha)\,\mathrm{d}\lambda \;<\; \int_B \alpha\,\mathrm{d}\lambda$$

are finite and, consequently, $\int \ln\alpha\;\mathrm{d}\widehat{\lambda}$ is finite as well.

One may ask if in 2.3.11 the condition $\frac{1}{\alpha} \in \mathcal{L}^1$ can be weakened, too.

4.2.20 Corollary. *Let μ be an om for $(P_n)_{n\geq 0}$ such that* $\operatorname{supp}(\mu) \subset \partial\mathbb{D}$*. The following properties are equivalent.*

(i) $P^2_\mu \neq L^2_\mu$,

(ii) $\sum_{n\geq 0} |P_n(0)|^2 < \infty$,

(iii) $\sum_{n\geq 0} |P_n(0)|^2 < \infty$ *and* $L^2_\mu = W_K \oplus V_d \oplus V_+ \oplus \bigl(P^2_\mu\bigr)^\perp$,

(iv) $E = \mathbb{D} \cup \Lambda_\mu$,

(v) μ *obeys the Szegő condition,*

(vi) $p \notin V_\infty$ *for all* $p \in \mathbb{C}[z] \setminus \{0\}$.

Proof: Combining 4.2.3, 4.2.10, and 4.2.17, we obtain (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

Furthermore, we have already mentioned (i) $\Longleftrightarrow$ (v) in 4.2.18.

According to 4.2.11, $P^2_\mu = L^2_\mu$ is equivalent to $V_\infty = P^2_\mu$. Hence (vi) implies (i).

For the converse, assume that $P^2_\mu \neq L^2_\mu$ and there exists $0 \neq p \in \mathbb{C}[z] \cap V_\infty$. Then, by 4.2.17, $\int |p|^2\,\alpha\;\mathrm{d}\widehat{\lambda} = 0$, implying $p(z)\alpha(z) = 0$ Lebesgue-almost everywhere and hence $\alpha(z) = 0$ for $\widehat{\lambda}$-almost all z. Therefore, α does not obey the Szegő condition, implying $P^2_\mu = L^2_\mu$ which is a contradiction. $\square$

We can now conclude that, given an om μ supported on the unit circle such that $P^2_\mu \neq L^2_\mu$ and all its parts μ_{ac}, μ_{d}, and μ_{sc} are non-trivial, there must exist an uncountable Lebesgue-nullset B, which then can be chosen $B = \{h = 0\} \setminus \Lambda_\mu$, having positive measure.

4.2.21 Corollary. *Let μ be a measure on $\partial\mathbb{D}$ with $\mu(\partial\mathbb{D}) = 1$ obeying the Szegő condition and having non-trivial singularly continuous part. Then there exists a sequence $(q_n)_n$ in $\mathbb{C}[z]$ with the following properties.*

(i) $\lim\limits_{n\to\infty} q_n(z) = 0$ *for all $z \in \mathbb{D}$,*

(ii) $\lim\limits_{n\to\infty} q_n(z) = 0$ *for Lebesgue-almost all $z \in \partial\mathbb{D}$,*

(iii) $\lim\limits_{n\to\infty} q_n(z) = 1$ *for uncountably many $z \in \partial\mathbb{D}$.*

(again, we refer to one-dimensional Lebesgue measure on $\partial\mathbb{D}$).

Proof: Let $(P_n)_{n\geq 0}$ be orthonormal polynomials in L^2_μ satisfying $P_0 \equiv 1$ and $\deg(P_n) = n$ for all n. Then, according to 4.2.20, $P^2_\mu \neq L^2_\mu$ and h exists.
With $B := \{h = 0\} \setminus \Lambda_\mu$, 4.2.17 yields $\widehat{\lambda}(B) = 0 = \mu_{\mathrm{d}}(B)$ and $\mu(B) = \mu_{\mathrm{sc}}(B) > 0$. Hence B is an uncountable Lebesgue-nullset.
Now define the sequence $(p_k)_k$ by

$$p_k(z) := \sum_{i=0}^{k} \langle\, \mathbf{1}_B \,, P_i \,\rangle P_i(z) \,,$$

then there exists a subsequence $(q_n)_n$ of $(p_k)_k$ which is μ-a.e. convergent to $\mathbf{1}_B$. Moreover, as $\mathbf{1}_B \in W_K$, we obtain $q_n(z) = \langle\, k_z \,, q_n \,\rangle \to \langle\, k_z \,, \mathbf{1}_B \,\rangle = 0$ for $z \in \mathbb{D}$.
Hence $(q_n)_n$ has the asserted properties. $\square$

Remark. The result of [Si, Th. 2.5.1] is similar to 4.2.21 but does not specify the set where the limit is different from 0. In 4.5.13 and 4.5.16 we will construct such sets using a Cantor-like method. In particular, we will see how to construct a singularly continuous measure supported on a given uncountable compact set.

We conclude this section with another characterization of isometric Hessenberg operators which Simon [Si] calls the GGT representation.

4.2.22 Proposition. *Let $G = (g_{ij})_{i,j\geq 0}$ be an isometric Hessenberg matrix. Then there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ of complex numbers satsifying $|\alpha_n|^2 + |\beta_n|^2 = 1$ for all n and*

$$g_{ij} = \begin{cases} -\overline{\alpha_j}\,\alpha_{i-1} \prod\limits_{k=i}^{j-1} \overline{\beta_j} & \text{for } 0 \leq i \leq j \,, \\ \beta_j & \text{for } i = j+1 \,, \\ 0 & \text{for } i > j+1 \,, \end{cases}$$

that is,

$$G = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\,\overline{\beta_0} & \overline{\alpha_2}\,\overline{\beta_0}\,\overline{\beta_1} & \overline{\alpha_3}\,\overline{\beta_0}\,\overline{\beta_1}\,\overline{\beta_2} & \cdots \\ \beta_0 & -\overline{\alpha_1}\,\overline{\alpha_0} & -\overline{\alpha_2}\,\alpha_0\,\overline{\beta_1} & -\overline{\alpha_3}\,\alpha_0\,\overline{\beta_1}\,\overline{\beta_2} & \cdots \\ 0 & \beta_1 & -\overline{\alpha_2}\,\alpha_1 & -\overline{\alpha_3}\,\alpha_1\,\overline{\beta_2} & \cdots \\ 0 & 0 & \beta_2 & -\overline{\alpha_3}\,\alpha_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} .$$

For a proof see [Si, Prop. 4.1.2].

Note that multiplication with a complex number of absolute value 1 does not affect orthogonality in L^2_μ; therefore, if G represents the multiplication operator D, we can assume $0 < \beta_n = \sqrt{1-|\alpha_n|^2}$ for all n.

4.2.23 Proposition. *Let μ be an om for $(P_n)_{n\geq 0}$ such that* $\operatorname{supp}(\mu) \subset \partial\mathbb{D}$ *and $P^2_\mu \neq L^2_\mu$. For $n \in \mathbb{N}$ define $\mathcal{H}_{-n} := (M^*_\mu)^n P^2_\mu$ and let Q_{-n} be the orthogonal projection in L^2_μ onto $\mathcal{H}_{-n}$, Q_0 the orthogonal projection onto P^2_μ.*
Then with $\varphi_{-n} := Q_{1-n}(M^*_\mu)^n P_0$ and $P_{-n} := \frac{\varphi_{-n}}{\|\varphi_{-n}\|_{L^2_\mu}}$, we obtain an onb of L^2_μ given by $\{P_n : n \in \mathbb{Z}\}$.
In terms of this basis, M_μ has the matrix representation $M = (m_{ij})_{i,j\in\mathbb{Z}}$ where

$$m_{ij} = \begin{cases} g_{ij} & \textit{for } i,j \geq 0\,, \\ \delta_{i+1,j} & \textit{for } i,j \leq -1\,, \\ -\alpha_{i-1}\prod_{k=i}^{\infty}\beta_k & \textit{for } i \geq 0, j=-1\,, \\ 0 & \textit{otherwise}\,, \end{cases}$$

with g_{ij} as in 4.2.22.

This construction is discussed in detail in [Si, Section 4.1], see there for a proof.

Remark. The coefficients α_n in 4.2.22 and 4.2.23 also appear in the following context. For any polynomial $p \in \mathbb{C}[z]$, $p(z) = b_0 + b_1 z + \ldots = b_m z^m$, we define the reversed polynomial by $p^*(z) := b_m + b_{m-1}z + \ldots + b_0 z^m$. Furthermore, for $n \in \mathbb{N}$, let $\Phi_n = c_n P_n$ with $c_n \in \mathbb{C}$ such that the leading coefficient of Φ_n is 1. Then, according to [Si, Theorem 1.5.2],

$$\Phi_{n+1}(z) = z\,\Phi_n(z) + \overline{\alpha_n}\,\Phi^*_n(z) \qquad \text{and} \qquad \Phi^*_{n+1}(z) = \Phi^*_n(z) - z\,\alpha_n\,\Phi_n(z)\,.$$

Simon [Si] calls these α_n the *Verblunsky Coefficients* and, amongst many other things, also shows [Si, Theorem 2.7.15] that $P^2_\mu \neq L^2_\mu$ if and only if $(\alpha_n)_n \in \ell^2$.

For details on isometric Hessenberg operators, we also refer to [Kl2, 3.3].

4.3 Orthogonal Polynomials on α-sets

Given a probability measure μ supported on some subset of $\mathbb{C}$, without further assumptions on $\operatorname{supp}(\mu)$ it might be rather intricate to calculate an orthonormal sequence of polynomials in L^2_μ via the Gram-Schmidt algorithm, or to check whether $P^2_\mu = L^2_\mu$ or not. The following definition is due to [HR1, §6].

4.3.1 Definition. A set $K \subset \mathbb{C}$ is called an α**-set** if, for any continuous function $f : K \to \mathbb{C}$, there exists a sequence $(q_n)_{n\in\mathbb{N}}$ in $\mathbb{C}[z]$ such that $q_n \to f$ uniformly on K as $n \to \infty$.

Note that, according to the Weierstrass approximation theorem, every compact subset of $\mathbb{R}$ is an α-set.

4.3.2 Proposition. *A set $K \subset \mathbb{C}$ is an α-set if and only if K is compact, its interior is empty, and its complement in $\mathbb{C}$ is connected.*

This characterization is the subject of [HR2], see there for a proof.

4.3.3 Lemma. *Let μ be a Borel measure such that $K := \operatorname{supp}(\mu)$ is an α-set and $\mu(K) < \infty$. Then $P^2_\mu = L^2_\mu$.*

Proof: It is a well-known fact (see [Ru1, 3.14], for instance) that, as $\mu(A) < \infty$ for every compact $A \subset \mathbb{C}$, the space $C_c(\mathbb{C})$ of all compactly supported (complex-valued) functions on $\mathbb{C}$ is dense in L^2_μ. Clearly, $f|K \in C(K)$ whenever $f \in C_c(\mathbb{C})$ and $C(K)$, in a canonical way, is a dense subspace of L^2_μ as well. Moreover, as K is an α-set, any $f \in C(K)$ can uniformly on K be approximated by a sequence of polynomials. Note that, in connection with $\mu(K) < \infty$, uniform convergence on K implies convergence with respect to the L^2_μ-norm. Hence $\mathbb{C}[z]$ is a dense (w.r.t. $\|\cdot\|_{L^2_\mu}$) subspace of $C(K)$.

Thus $\mathbb{C}[z]$ is a dense subspace of L^2_μ, too. In other words, $P^2_\mu = L^2_\mu$. $\square$

4.3.4 Corollary. *Let μ be an om for $(P_n)_{n\geq 0}$ such that* $\operatorname{supp}(\mu)$ *is an α-set. Then $(P_n)_n$ is an onb in L^2_μ, the multiplication operator D is essentially normal, and $E = \Lambda_\mu$.*

Proof: According to 4.3.3, $P^2_\mu = L^2_\mu$, hence $(P_n)_n$ is an onb in L^2_μ. By the characterization 4.3.2, $\operatorname{supp}(\mu)$ is compact; now 3.3.3 shows the remaining assertions. $\square$

Remark. Using 4.3.2, we see that, in particular, every compact proper subset of the unit circle, is an α-set. More generally, if $G \subset \mathbb{C}$ is open, bounded, simply connected, and $\operatorname{supp}(\mu) \subsetneqq \partial G$ then $\operatorname{supp}(\mu)$ is an α-set. In that case, 4.3.4 gives rise to another proof of 3.3.6(ii).

4.4 Weighted Shifts

In the case that $P_n(z) = b_n z^n$ the Hessenberg operator D is a weighted shift with respect to the onb $(P_n)_n$, see also 2.1.11 and 2.2.4.

4.4.1 Definition. A linear operator S in a Hilbert space $\mathcal{H}$ is called a **weighted shift** if there exists an onb $(e_n)_{n\geq 0}$ such that $e_n \in \operatorname{dom}(S)$ and there exist $a_n \in \mathbb{C}$ satisfying $e_{n+1} = a_{n+1} S e_n$ for all n. Clearly, $a_n \neq 0$ for all n and it is easy to see that S is bounded if and only if $\sup\limits_{n\in\mathbb{N}} \left|\frac{1}{a_n}\right| < \infty$ and then this supremum equals $\|S\|$.

Subormality of weighted shifts is quite well-understood and is the subject of a variety of papers as [Sta], [BCZ], and [StSz3, pp. 132ff]; see also [Con1, II. §6]. We will give a brief survey here.

4.4.2 Stieltjes Moment Problem. Let $(m_n)_{n\geq 0}$ be a sequence of positive real numbers. We look for a Borel measure τ supported on $[0,\infty)$ such that

$$\int\limits_0^\infty t^n \, \mathrm{d}\tau(t) = m_n$$

for all $n \in \mathbb{N}_0$. If such τ exists then $(m_n)_n$ is called a **Stieltjes moment sequence**.

4.4.3 Proposition. *Let $P_n(z) = b_n z^n$ where $b_0 = 1$ and $b_n \neq 0$ for all $n \in \mathbb{N}$. There exists an om μ if and only if $\left(|b_n|^{-2}\right)_{n\geq 0}$ is a Stieltjes moment sequence.*

This is a well-known fact, see [Kl2, 4.2.2] or [StSz2, Theorem 4] for a proof.

Note that $(P_n)_n$ is an orthogonal system with respect to $\widehat{\lambda}$ (normalized Lebesgue measure on the unit circle). The following is part of [Kl2, 4.2.2], too. See also [BT].

4.4.4 Proposition. *Let μ be a rotation invariant om for $(P_n)_{n\geq 0}$. Then $P_n(z) = b_n z^n$ where $b_0 = 1$ and $b_n \neq 0$ for all $n \in \mathbb{N}$.*

Conversely, assume that, in the situation of 4.4.3, $\left(|b_n|^{-2}\right)_{n\geq 0}$ is a Stieltjes moment sequence and let τ be a solution of the associated moment problem.

Then $\mu := \tau \otimes \widehat{\lambda}$ is an om for $(P_n)_n$.

4.4.5 Lemma. *Let μ be an om for $(P_n)_{n\geq 0}$ as in 4.4.3. Then*

$$|b_{n-1} b_{n+1}| \leq |b_n|^2 \qquad \textit{for all } n \in \mathbb{N}\,. \tag{4.2}$$

Moreover, $P^2_\mu \neq L^2_\mu$.

Proof: As $DP_n = \frac{b_n}{b_{n+1}} P_{n+1}$, the matrix representation of D is of the form $d_{n+1,n} = \frac{b_n}{b_{n+1}}$ and $d_{ij} = 0$ whenever $i \neq j+1$. Now (3.5) yields

$$\left|\tfrac{b_{j-1}}{b_j}\right|^2 = |d_{j,j-1}|^2 = \sum_{k=0}^{\infty} |d_{jk}|^2 \leq \sum_{i=0}^{j+1} |d_{ij}|^2 = |d_{j+1,j}|^2 = \left|\tfrac{b_j}{b_{j+1}}\right|^2 \tag{4.3}$$

for all $j \in \mathbb{N}$ and (4.2) is an immediate consequence.

Now assume $P^2_\mu = L^2_\mu$. Then, according to 3.2.2, D is formally normal. However, D cannot satisfy (3.4), as in the 0-th row of its matrix representation all entries vanish while the 0-th column contains one element different from 0. Hence we have a contradiction and, therefore, $P^2_\mu \neq L^2_\mu$. □

Note that, according to (4.3), we can define a monotonically decreasing sequence by $a_n := \left|\frac{b_{n+1}}{b_n}\right|$ and obtain $\sum\limits_{n\geq 0} |P_n(z)|^2 = \sum\limits_{n\geq 0} |b_n|^2 \, |z^2|^n < \infty$ if $|z| < \left(\lim\limits_{n\to\infty} a_n\right)^{-1}$.

4.4.6 Lemma. *If, in the situation of* 4.4.3, *an om exists then E is an open disk centered at the origin (including the case $E = \mathbb{C}$). More precisely, with $a := \lim\limits_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$,*

$$E = \begin{cases} \left\{z \in \mathbb{C} : |z| < \frac{1}{a}\right\} & \text{if } a > 0\,, \\ \mathbb{C} & \text{if } a = 0\,. \end{cases}$$

Proof: It only remains to show that $z \notin E$ if $a > 0$ and $|z| = \frac{1}{a}$.
As $(a_n)_n$ is monotonically decreasing, we observe that

$$|b_n|^2 \, \frac{1}{(a^2)^n} \geq |b_n|^2 \, \frac{1}{a_{n-1}^2} \, \frac{1}{(a^2)^{n-1}} = |b_{n-1}|^2 \, \frac{1}{(a^2)^{n-1}} \qquad \text{for all } n \in \mathbb{N}$$

implying $\sum\limits_{n\geq 0} |b_n|^2 \left(\frac{1}{a^2}\right)^n = \infty$ and hence $\sum\limits_{n\geq 0} |P_n(z)|^2 = \infty$ whenever $|z| = \frac{1}{a}$. $\square$

4.4.7 Summary on Weighted Shifts. Let $P_n(z) = b_n z^n$ where $b_0 = 1$ and $b_n \in \mathbb{C}\setminus\{0\}$ for all $n \in \mathbb{N}$. If there is an om μ for $(P_n)_n$ then $|b_{n-1}b_{n+1}| \leq |b_n|^2$ for all n and $a := \lim\limits_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$ exists.

Note that D is bounded with $\|D\| = \frac{1}{a}$ if and only if $a > 0$. Recall that, according to 3.2.16, if D is bounded and there exists an om then it is unique.

Moreover, $E = \left\{z \in \mathbb{C} : |z| < \frac{1}{a}\right\}$ or $E = \mathbb{C}$ if $a > 0$ or $a = 0$, respectively, and, see 2.1.11 and 2.2.4, $(P_n)_n$ is an onb in the reproducing kernel space $\mathcal{H}(K)$, consisting of holomorphic functions defined on E, whose kernel is given by

$$K : E \times E \to \mathbb{C}\,, \qquad K(z,w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w) = \sum_{n\geq 0} |b_n|^2 \, (\overline{z}\, w)^n\,.$$

Hence $\mathcal{H}(K)$ is, via $P_n \mapsto P_n$, isometrically isomorphic to P_μ^2 which is a proper subspace of L_μ^2. Note that, by 3.5.6, $E = E_{reg}$.

Our favorite example, $P_n(z) := z^n$, see 1.2.2, 1.3.8, and 2.1.11, fits into this situation. Here $E = \{z \in \mathbb{C} : |z| < 1\}$ and $(P_n)_n$ is an onb in $\mathcal{H}(K)$ where $K(z,w) = \frac{1}{1-\overline{z}w}$ and $\widehat{\lambda}$ is the unique om.

4.4.8 Corollary. *Whenever, for $(P_n)_{n\geq 0}$ as in* 4.4.7, *E is not an open disk (and $\neq \mathbb{C}$) then there exists no om.*

This is an immediate consequence of the above.

In the following two examples, D is a non-subnormal weighted shift.

4.4.9 Example. Let $P_n(z) := b_n z^n$ where $b_0 = 1$ and $b_{2n-1} := b_{2n} := \frac{1}{\sqrt{(n-1)!}}$ for $n \geq 1$. For $n > 2$ we have $|b_{2n-2}\, b_{2n}| = \frac{1}{\sqrt{n-1}\cdot(n-2)!} > \frac{1}{(n-1)!} = |b_{2n-1}|^2$ and, according to 4.4.5, there exists no om.

However,

$$\sum_{n\geq 0} |P_n(z)|^2 = 1 + \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\big(|z|^{2(2k-1)} + |z|^{2\cdot 2k}\big) = 1 + |z|^2\left(1+|z|^2\right)\exp\big(|z|^4\big) < \infty$$

for all $z \in \mathbb{C}$. Hence, see 2.2.4, $(P_n)_n$ is an onb in $\mathcal{H}(K)$ which consists of entire functions and an analogous calculation to the above yields $K(z,w) = 1 + \overline{z}\,w\big(1+\overline{z}\,w\big)\exp\big((\overline{z}\,w)^2\big)$.

4.4.10 Example. Let $P_n(z) := n!\, z^n$ for $n \in \mathbb{N}_0$. Here $\sum_{n\geq 0} |P_n(z)|^2 < \infty \iff z = 0$. Hence $E = \{0\}$ is not an open disk and, according to 4.4.8, there exists no om.
Note that, as E only consists of a single point, the question whether $(P_n)_n$ is an onb in $\mathcal{H}(K)$, is easily answered.

It is worth mentioning that for a subnormal weighted shift the case $E = \mathbb{C}$ can occur, as we shall see now.

4.4.11 Gauss Measure. It is the matter of an easy calculation that the polynomials $(P_n)_{n\geq 0}$, given by $P_n(z) := \frac{1}{\sqrt{n!}}\, z^n$, satisfy

$$\int_{\mathbb{C}} \overline{P_n(x+\mathrm{i}y)}\, P_m(x+\mathrm{i}y)\, \frac{1}{\pi}\, \exp\big(-(x^2+y^2)\big)\,\mathrm{d}x\,\mathrm{d}y = \delta_{nm} \qquad (z = x+\mathrm{i}y)\,.$$

Hence here we have an om μ such that $\mathrm{supp}(\mu) = \mathbb{C}$. Moreover,

$$\sum_{n\geq 0} |P_n(z)|^2 = \sum_{n=0}^{\infty} \frac{1}{n!}\left(|z|^2\right)^n = \exp\big(|z|^2\big)$$

and therefore, $E = \mathbb{C}$. Clearly, the kernel of $\mathcal{H}(K)$ is $K(z,w) = \exp(\overline{z}w)$.

Recall the spectral properties given in 3.3.1 as well as 3.1.3. Accordingly, this is an example for an unbounded subnormal operator $\overline{D}$ such that $\sigma(\overline{D}) = \sigma_r(\overline{D}) = \mathbb{C} = \sigma(D^*) = \sigma_p(D^*)$ which has a minimal normal extension M_μ acting in a larger space with $\sigma(M_\mu) = \sigma_c(M_\mu) = \mathbb{C}$.

Remark. A characterization of the spectra of unbounded subnormal weighted shift operators can also be found in [StSz3, p. 136].

4.5 Various Examples

In addition to the examples already discussed in the previous sections and chapters, we will now present some more sequences of polynomials and associated orthonormalizing measures as well as such sequences which do not have an om at all.

The following theorem provides a nice tool for proving the non-existence of orthonormalizing measures.

4.5.1 Theorem. *Let $(P_n)_{n\geq 0}$ be a sequence of polynomials as in* 1.1.1 *such that $E = \mathbb{C}$. If there exists $(a_n)_{n\geq 0} \in \ell^2 \setminus \{0\}$ satisfying*

$$\sum_{n=0}^{\infty} a_n P_n(z) = 0 \qquad \text{for all } z \in \mathbb{C}$$

then there exists no om for $(P_n)_n$.

Proof: According to 2.1.9, $(P_n)_n$ is not an onb in $\mathcal{H}(K)$.
Now assume that there exists an om μ. Then $\mu(E) = \mu(\mathbb{C}) = 1$ and 3.7.2 yields that $(P_n)_n$ is an onb in $\mathcal{H}(K)$; thus we have a contradiction.
Hence there exists no om. $\square$

4.5.2 Example (Continuation of 2.1.12). Let $P_0 \equiv 1$ and $P_n(z) = \frac{1}{\sqrt{n!}} z^{n-1}(z-n)$ for $n \geq 1$.
In 2.1.12 we have already seen that $\sum\limits_{n\geq 0} |P_n(z)|^2 < \infty$ and $\sum\limits_{n=0}^{\infty} \frac{1}{\sqrt{n!}} P_n(z) = 0$ for all $z \in \mathbb{C}$.
Therefore, by 4.5.1, there exists no om for $(P_n)_n$.

An apparently slight modification of the above leads to the following example which looks very similar but where 4.5.1 is not applicable.

4.5.3 Example. Let $P_0 :\equiv 1$ and $P_n(z) := \frac{1}{\sqrt{(n-1)!}} z^{n-1}(z-1)$ for $n \geq 1$.

It is not difficult to see that $\sum\limits_{n\geq 0} |P_n(z)|^2 = 1 + |z-1|^2 \exp\big(|z|^2\big)$ for all $z \in \mathbb{C}$, hence $E = \mathbb{C}$.

Now let $x = (x_n)_{n\geq 0} \in \ell^2$ such that

$$0 = \sum_{n=0}^{\infty} x_n P_n(z) = x_0 + (z-1) \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{(n-1)!}} z^{n-1} \qquad \text{for all } z \in \mathbb{C}\,.$$

For $|z| < 1$ we obtain

$$\sum_{n=1}^{\infty} \frac{x_n}{\sqrt{(n-1)!}} z^{n-1} = \frac{x_0}{1-z} = x_0 \sum_{n=1}^{\infty} z^{n-1}$$

and uniqueness of power series implies $x_n = \sqrt{(n-1)!} \cdot x_0$ for all $n \in \mathbb{N}$. As $x \in \ell^2$, only

$x = 0$ remains and 2.1.9 implies that $(P_n)_n$ is an onb in $\mathcal{H}(K)$. Furthermore,

$$K(z,w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w) = 1 + (\overline{z}-1)(w-1)\, \exp\bigl(\overline{z}\, w\bigr)\, .$$

In particular, $(P_n)_n$ does not satisfy the premises of 4.5.1.

Moreover, we do not know whether there exists an om for $(P_n)_n$ at all. Yet we see that the multiplication operator D is unbounded by 3.1.6 since $E = \mathbb{C}$; a simple calculation shows that the matrix representation of D is given by

$$(d_{ij})_{i,j\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & 0 & \sqrt{2} & 0 & \cdots \\ & \vdots & 0 & \sqrt{3} & \cdots \\ & & \vdots & & \ddots \end{pmatrix}$$

which does not obey (3.4). Hence D is not formally normal and therefore, if there exists an om μ then, according to 3.2.2, $P^2_\mu \neq L^2_\mu$.

Note that the matrix coefficients d_{ij} obey (3.5) which, however, is just a necessary criterion for existence of an om; it remains an open question if there exists an om for these $(P_n)_n$.

The following example looks similar to the previous one but leads to a situation where we can use (3.5) to conclude that there exists no orthonormalizing measure. Moreover, we will see that it is possible that E is connected and $\mathcal{H}(K)$ contains functions which are not continuous on all of E. However, this will not give an answer to the question whether all $f \in \mathcal{H}(K)$ have to be holomorphic on E° in general.

4.5.4 Example. Let $P_0 :\equiv 1$ and $P_n(z) := n\, z^{n-1}(z-1)$ for $n \geq 1$. Now

$$\sum_{n=0}^{N} |P_n(z)|^2 = 1 + |z-1|^2 \sum_{n=1}^{N} n^2 \bigl(|z|^2\bigr)^{n-1}$$

shows that $\sum_{n\geq 0} |P_n(z)|^2 < \infty \iff |z| < 1$ or $z = 1$.

We can construct the reproducing kernel space $\mathcal{H}(K)$ via the kernel $K : E \times E \to \mathbb{C}$,

$$K(z,w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w) = \begin{cases} 1 \quad \text{if } z = 1 \text{ or } w = 1 & \text{and} \\ 1 + (\overline{z}-1)(w-1) \sum\limits_{n=1}^{\infty} n^2 (\overline{z}\, w)^{n-1} & \text{otherwise}\, , \end{cases}$$

where $E = \{z \in \mathbb{C} : |z| < 1\} \cup \{1\}$.

Note that $K(z,z)$ is bounded on every compact subset of $E^\circ = \{z \in \mathbb{C} : |z| < 1\}$ and hence, by 2.2.3, all $f \in \mathcal{H}(K)$ are holomorphic in E°.

Now let $x = (x_n)_n \in \ell^2$ such that $\sum\limits_{n=0}^{\infty} x_n P_n(z) = 0$ for all $z \in E$.

For $z = 1$, we immediately obtain $x_0 = 0$. Then

$$0 = \sum_{n=1}^{\infty} x_n P_n(z) = (z-1)\sum_{n=1}^{\infty} x_n\, n\, z^{n-1} = -x_1 + \sum_{n=1}^{\infty}\big(n\, x_n - (n+1)\, x_{n+1}\big)\, z^n \quad \text{for all } z \in E$$

implies $x_1 = 0$ and $x_{n+1} = \frac{n}{n+1}\, x_n$ for all $n \geq 1$. This yields $x_n = 0$ for all n and, according to 2.1.9, $(P_n)_n$ is an onb in $\mathcal{H}(K)$.

We next show that $\mathcal{H}(K)$ contains a function which is not continuous at $z = 1$.
Set $a_0 := 1$ and $a_n := \frac{1}{n}$ for $n \in \mathbb{N}$. Clearly, $a := (a_n)_n \in \ell^2$ and $f := \sum\limits_{n \geq 0} a_n P_n \in \mathcal{H}(K)$ is well-defined. In particular,

$$f(z) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n\, P_n(z) = \lim_{N\to\infty}\Big(1 + \sum_{n=1}^{N} \tfrac{1}{n}\, n\, z^{n-1}(z-1)\Big) = \lim_{N\to\infty}\Big(1 + \sum_{n=1}^{N}\big(z^n - z^{n-1}\big)\Big)$$

$$= \lim_{N\to\infty} z^N = \begin{cases} 1 & \text{for } z = 1\,, \\ 0 & \text{for } z \in E \setminus \{1\}\,. \end{cases}$$

Thus we have found a member of $\mathcal{H}(K)$ which is not continuous at $z = 1$. However, all $f \in \mathcal{H}(K)$ are holomorphic in E°.

Having in mind 4.2.10, one could expect to find an om supported on the unit circle having positive point mass at $z = 1$. To see that this is not the case here, let us examine the Hessenberg operator D more closely. We have $zP_0(z) = z - 1 + 1 = P_0(z) + P_1(z)$ and $zP_n(z) = \frac{n}{n+1} P_{n+1}(z)$ for $n \geq 1$. A look at the associated Hessenberg matrix

$$(d_{ij})_{i,j\geq 0} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ \vdots & 0 & \frac{2}{3} & 0 & \cdots \\ & \vdots & 0 & \frac{3}{4} & \cdots \\ & & \vdots & & \ddots \end{pmatrix}$$

shows that $\sum\limits_{k=0}^{\infty} |d_{1k}|^2 = 1 > \frac{1}{4} = \sum\limits_{i=0}^{2} |d_{i1}|^2$ and, due to 3.2.11, there exists no om.

Finally, note that, for arbitrary $p = \sum\limits_{k=0}^{n} c_k\, P_k \in \mathbb{C}[z]$,

$$\|Dp\|^2 = \Big\| c_0(P_0 + P_1) + \sum_{k=1}^{n} c_k\, \frac{k}{k+1}\, P_{k+1} \Big\|^2 \leq 2\, |c_0|^2 + \sum_{k=1}^{n} |c_k|^2 \leq 2\, \|p\|^2\ ,$$

showing that D is continuous.

4.5.5 Polynomial Sequences of Finite Bandwidth. In [AM] Adams and McGuire define a sequence $(f_k)_{k\geq 0}$ in $\mathbb{C}[z]$ to be of bandwidth j if $f_k(z) = z^k p_k(z)$ where every p_k is of degree j. Moreover, an RKHS $\mathcal{H}(K^A)$ with domain $E \subset \mathbb{C}$ is said to be of bandwidth j if it has an onb $(f_k)_k$ of polynomials of bandwitdth j and $E = \big\{z \in \mathbb{C} : \sum_{k\geq 0} |f_k(z)|^2 < \infty\big\}$. According to 1.4.6, the kernel of $\mathcal{H}(K^A)$ is then given by

$$K^A(z,w) = \sum_{k\geq 0} \overline{f_k(z)}\, f_k(w)\,.$$

If $(f_k)_{k\geq 0}$ is of bandwidth j then we can construct a sequence $(P_n)_{n\geq 0}$ as in 1.1.1 by defining $P_n := f_{n-j}$ for $n \geq j$ and arbitrarily adding $P_0, \ldots, P_{j-1}$. Clearly, the spaces $\mathcal{H}(K)$ and $\mathcal{H}(K^A)$ have the same domain and

$$K(z,w) = \sum_{n\geq 0} \overline{P_n(z)}\, P_n(w) = \sum_{n=0}^{j-1} \overline{P_n(z)}\, P_n(w) + K^A(z,w)\,.$$

We point out that, for $(P_n)_{n\geq 0}$ such that $(P_{n+j})_n$ is of bandwidth j, it is possible that $(P_{n+j})_n$ is an onb in $\mathcal{H}(K^A)$ while $(P_n)_n$ is not an onb in $\mathcal{H}(K)$. As an example, recall 2.1.12 where $P_0 \equiv 1$ and $P_n(z) = \frac{1}{\sqrt{n!}}\, z^{n-1}(z-n)$ for $n \geq 1$. We know that the kernel of $\mathcal{H}(K)$ is given by

$$K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\,, \qquad K(z,w) = \big(1 + (1-\overline{z})(1-w)\big)\, \exp(\overline{z}\, w)$$

and that $(P_n)_n$ is not an onb in $\mathcal{H}(K)$.
Leaving out P_0, we obtain a sequence of bandwith $j = 1$ and $\mathcal{H}(K^A)$ with kernel

$$K^A(z,w) = \sum_{n=1}^{\infty} \overline{P_n(z)}\, P_n(w) = K(z,w) - 1\,.$$

Let $a = (a_n)_{n\geq 1}$ such that $\sum_{n=1}^{\infty} a_n\, P_n(z) = 0$ for all $z \in \mathbb{C}$. Then

$$0 = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\, a_n z^{n-1}(z-n) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\, a_n z^n - \sum_{n=1}^{\infty} \frac{n}{\sqrt{n!}}\, a_n z^{n-1}$$

yields $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\, a_n z^n = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{n!}}\, a_{n+1} z^n$ for all $z \in \mathbb{C}$ and we obtain $a_n = 0$ for all $n \in \mathbb{N}$.
Now 2.1.4 shows that $(P_{n+1})_n$ is an onb in $\mathcal{H}(K^A)$.

Remark. According to [AM, Theorem 1], the domain of any RKHS of bandwidth j is the union of an open disk and at most j points off that disk.

Note that, when we leave out P_0 in the examples 4.5.3 or 4.5.4, we obtain a sequence of bandwidth $j = 1$. Moreover, the following lemma shows that they are orthonormal bases in the corresponding $\mathcal{H}(K^A)$-spaces.

4.5.6 Lemma. *Let $(P_n)_{n\geq 0}$ be a sequence as in* 1.1.1 *and define E and $\mathcal{H}(K)$ as in* 2.1.8. *Fix $j \in \mathbb{N}$, then*

$$K^A : E \times E \to \mathbb{C}\,, \qquad K^A(z,w) := \sum_{n\geq j} \overline{P_n(z)}\, P_n(w)$$

is the kernel of an RKHS $\mathcal{H}(K^A)$ and if $(P_n)_n$ is an onb in $\mathcal{H}(K)$ then $(P_{n+j})_n$ is an onb in $\mathcal{H}(K^A)$.

Proof: Clearly, K^A is well defined on $E \times E$; the existence of $\mathcal{H}(K^A)$ is due to 2.1.1.

Now choose $a = (a_n)_{n\geq 0} \in \ell^2$ such that $\sum_{n=0}^{\infty} a_n\, P_{n+j}(z) = 0$ for all $z \in E$. Define $b = (b_n)_{n\geq 0}$ by

$$b_n := \begin{cases} a_{n-j} & \text{for } n \geq j\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\sum_{n=0}^{\infty} b_n\, P_n(z) = 0$ for all $z \in E$. As $(P_n)_n$ is an onb in $\mathcal{H}(K)$, 2.1.4 implies $b = 0$. Thus $a = 0$ and, using 2.1.4 again, we see that $(P_{n+j})_n$ is an onb in $\mathcal{H}(K^A)$. $\square$

4.5.7 Newton Polynomials. For $n \in \mathbb{N}_0$, let

$$P_n(z) := (-1)^n \binom{z-1}{n} = \frac{(-1)^n}{n!}\,(z-1)(z-2)\cdots(z-n)\,.$$

Due to $z\,P_n(z) = (z-n-1)P_n(z) + (n+1)P_n(z) = -(n+1)P_{n+1}(z) + (n+1)P_n(z)$, the matrix representation of D is given by

$$(d_{ij})_{i,j\geq 0} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 2 & 0 & \cdots \\ 0 & -2 & 3 & \cdots \\ \vdots & 0 & -3 & \cdots \\ & \vdots & & \ddots \end{pmatrix}$$

which, as one can easily see, does not obey (3.4) but obeys (3.5). Hence D is not formally normal but may be subnormal. Indeed, it is subnormal, as we shall see below. If there exists an om μ then 3.2.2 implies $P^2_\mu \neq L^2_\mu$.

Clearly, $\big(P_n(z)\big)_n \in \ell^2$ if z is a positive integer, as in these cases only finitely many entries are different from 0. Thus $\mathbb{N} \subset E$. More precisely, if $k \in \mathbb{N}$ then $P_n(k) = 0$ for all $n \geq k$ and $P_{k-1}(k) \neq 0$ which shows that $\big\{\big(P_n(k)\big)_n : k \in \mathbb{N}\big\}$ is total in ℓ^2 and, by 2.1.9, $(P_n)_n$ is an onb in $\mathcal{H}(K)$.

The space $\mathcal{H}(K)$ is known explicitly. According to [Kst, Theorem 1(ii)], here

$$E = \Big\{z \in \mathbb{C} : \sum_{n\geq 0} |P_n(z)|^2 < \infty\Big\} = \Big\{z \in \mathbb{C} : \mathrm{Re}\,(z) > \tfrac{1}{2}\Big\}\,,$$

$K(z,w) = \sum_{n\geq 0} \binom{\overline{z}-1}{n}\binom{w-1}{n} = \frac{\Gamma(\overline{z}+w-1)}{\Gamma(\overline{z})\Gamma(w)}$, and all $f \in \mathcal{H}(K)$ are holomorphic in E.

Moreover, see [Kst, Theorem 1(iv)], there exists an om μ which is given by

$$\mathrm{d}\mu(x,y) = \frac{1}{2\pi}\,\frac{|\Gamma(x+\mathrm{i}y)|^2}{\Gamma(2x)}\;\mathrm{d}y\;\mathrm{d}\gamma(x) \qquad (z = x+\mathrm{i}y)\,,$$

where γ is the discrete measure having unit mass at the points $x = \frac{n+1}{2}$, $n = 0,1,2,\ldots$

Note that here we have an example for a sequence $(P_n)_{n\geq 0}$ with the remarkable properties

(i) D is an unbounded subnormal operator which is not formally normal,

(ii) ∂E is a *proper* subset of $\operatorname{supp}(\mu)$,

(iii) $E \cap \operatorname{supp}(\mu) \neq \varnothing$, $E \cap \big(\mathbb{C} \setminus \operatorname{supp}(\mu)\big) \neq \varnothing$, and $(\mathbb{C} \setminus E) \cap \operatorname{supp}(\mu) \neq \varnothing$,

(iv) all $f \in \mathcal{H}(K)$ are holomorphic in E,

(v) all P_n have real coefficients but $\operatorname{supp}(\mu)$ is not contained in the real line.

More generally, we will now examine sequences $(P_n)_n$ where the zeros of any P_n are zeros of P_{n+1}, too.

4.5.8 Example. Let $a_1, a_2, \ldots$ be mutually different complex numbers and $c_1, c_2, \ldots \in \mathbb{C} \setminus \{0\}$. Define $(P_n)_{n\geq 0}$ by $P_0 :\equiv 1$ and

$$P_n(z) := c_n(z-a_1)(z-a_2)\cdots(z-a_n)$$

for $n \geq 1$. One can easily check that $A := \{a_1, a_2, \ldots\} \subset E$ and that $\big\{\big(P_n(a)\big)_n : a \in A\big\}$ is total in ℓ^2. Hence 2.1.9 imples that $(P_n)_n$ is an onb in $\mathcal{H}(K)$.
For $n \in \mathbb{N}$, we obtain $z\,P_n(z) = (z-a_{n+1})P_n(z) + a_{n+1}P_n(z) = \frac{c_n}{c_{n+1}}\,P_{n+1}(z) + a_{n+1}P_n(z)$ and $z\,P_0(z) = \frac{1}{c_1}P_1(z) + a_1P_0(z)$. Thus the matrix representation of D is given by

$$(d_{ij})_{i,j\geq 0} = \begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots \\ \frac{1}{c_1} & a_2 & 0 & 0 & \cdots \\ 0 & \frac{c_1}{c_2} & a_3 & 0 & \cdots \\ 0 & 0 & \frac{c_2}{c_3} & a_4 & \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}$$

which, as $|a_1|^2 + \left|\frac{1}{c_1}\right|^2 \neq |a_1|^2$, does not obey (3.4) and hence D is not formally normal. Therefore, if an om μ exists then $P_\mu^2 \neq L_\mu^2$ by 3.2.2 and, furthermore, (3.5) implies $\left|\frac{c_{n-1}}{c_n}\right|^2 + |a_{n+1}|^2 \leq |a_{n+1}|^2 + \left|\frac{c_n}{c_{n+1}}\right|^2$ which finally yields $|c_{n-1}c_{n+1}| \leq |c_n|^2$ for all $n \geq 1$.

Note that we have an analogous necessary condition (4.2) for existence of an om in the weighted shift case. Note also that the Newton polynomials fit into this situation and, in contrary to the weighted shift case, here E need not be a disk.

While a point which is a zero of all but finitely many P_n must belong to E, it is possible that there exists $z \in \mathbb{C}$ which is a zero of infinitely many P_n and yet does not belong to E, as we will see in the following.

4.5.9 Example. Choose $a, b \in \mathbb{C}$ and define

$$P_n(z) := \begin{cases} (z-a)^n & \text{for even } n\,, \\ \frac{1}{n!}\,(z-b)^n & \text{for odd } n\,. \end{cases}$$

As $\sum\limits_{n\geq 0} |P_{2n}(z)|^2 < \infty \iff |z-a| < 1$ and $\sum\limits_{n\geq 0} |P_{2n+1}(z)|^2 < \infty$ for all $z \in \mathbb{C}$, we obtain

$$E = \{z \in \mathbb{C} : |z-a| < 1\}\,.$$

If, in particular, $|b-a| \geq 1$ then $b \notin E$.

As we have already (and not just once) pointed out that polynomials orthogonal with respect to a measure supported on the real line are quite well understood, we should not go without mentioning the probably best-known class of orthogonal polynomials.

4.5.10 Jacobi Polynomials. For $\alpha, \beta > -1$, let $(P_n^{(\alpha,\beta)})_{n\geq 0}$ be (real) polynomials such that $\deg(P_n) = n$, satisfying the orthogonality relation

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,(1-x)^\alpha (1+x)^\beta \,\mathrm{d}x = 0 \qquad \text{whenever } n \neq m\,. \tag{4.4}$$

In other words, $(P_n^{(\alpha,\beta)})_n$ is a sequence of polynomials having an om μ, defined via the above integral, where $\operatorname{supp}(\mu)$ is the interval $[-1,1]$. It is well known that the Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n\,n!}(1-z)^{-\alpha}(1+z)^{-\beta}\,\frac{\mathrm{d}^n}{\mathrm{d}z^n}\big((1-z)^\alpha(1+z)^\beta(1-z^2)^n\big)\,,$$

satisfy (4.4); a short calculation shows that indeed each $P_n^{(\alpha,\beta)}$ is a polynomial of degree n and square integrable with respect to μ. See [Sze, Chapter IV] for more details; see also [Chi, V.2(A)] or [BE, 2.3 Exercise 5].

Note that, to match 1.1.1, we still have to normalize the Jacobi polynomials. According to [Sze, (4.3.4)],

$$P_n := \left(\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}\,\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right)^{\frac{1}{2}} P_n^{(\alpha,\beta)}\,,$$

yields an orthonormal system in L^2_μ.

Note also that, as $\operatorname{supp}(\mu)$ is contained in the real line, the multiplication operator D is symmetric and the Jacobi polynomials obey a 3-term recurrence (4.1) which is explicitly known, see [Sze, (4.5.1)] or [BE, 2.3 Exercise 5], for example.

Now we have several means to show $P^2_\mu = L^2_\mu$, for example, using the Stone-Weierstrass theorem. Another way goes via the operator D which is symmetric and bounded and hence essentially self-adjoint. According to 1.3.7 and 4.1.2, we obtain

(i) $P^2_\mu = L^2_\mu$, i.e. the (normalized) Jacobi Polynomials form an onb in L^2_μ,

(ii) μ is the unique om,

(iii) $\sum_{n\geq 0} |P_n(z)|^2 = \infty$ for all $z \in \mathbb{C}$; in other words, $E = \varnothing$.

Remark. The Jacobi polynomials in the case $\alpha = \beta$ are called *ultraspherical polynomials* and sometimes referred to as *Gegenbauer polynomials*, too. In particular, for $\alpha = \beta = -\frac{1}{2}$ or $\alpha = \beta = \frac{1}{2}$, we obtain the so called *Chebyshev polynomials* or *Chebyshev polynomials of the second kind*, respectively. The simplest case is $\alpha = \beta = 0$ where we get the *Legendre polyonomials* with normalized Lebesgue measure on $[-1, 1]$ as (the unique) om.

In the following example, D is an unbounded essentially self-adjoint operator.

4.5.11 Hermite Polynomials. One can easily verify that, for $n \in \mathbb{N}_0$,

$$\widetilde{H}_n(z) := (-1)^n \exp(z^2)\, \frac{\mathrm{d}^n}{\mathrm{d}z^n}\big[\exp(-z^2)\big]$$

is a polynomial of degree n and, due to

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\big[\exp(-z^2)\big] = -2z\frac{\mathrm{d}^n}{\mathrm{d}z^n}\big[\exp(-z^2)\big] - 2n\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\big[\exp(-z^2)\big]\ ,$$

we obtain the 3-term recurrence $\widetilde{H}_{n+1}(z) = 2z\, \widetilde{H}_n(z) - 2n\, \widetilde{H}_{n-1}(z)$ for $n \geq 1$.
Re-normalizing this sequence by $H_n := \frac{1}{\sqrt{2^n\, n!}}\, \widetilde{H}_n$ yields

$$\begin{aligned}\left(\tfrac{n+1}{2}\right)^{\frac{1}{2}} H_{n+1}(z) + \left(\tfrac{n}{2}\right)^{\frac{1}{2}} H_{n-1}(z) &= \left(\tfrac{1}{2^{n+2}\, n!}\right)^{\frac{1}{2}} \widetilde{H}_{n+1}(z) + \left(\tfrac{n}{2^n\,(n-1)!}\right)^{\frac{1}{2}} \widetilde{H}_{n-1}(z)\\ &= \left(\tfrac{1}{2^n\, n!}\right)^{\frac{1}{2}} z\, \widetilde{H}_n(z) = z\, H_n(z)\ . \end{aligned} \tag{4.5}$$

Note that $z\, H_0(z) = z = \frac{1}{\sqrt{2}}\, H_1(z)$ and D is symmetric.
As (4.5) implies $H_{n+1}(0) = \left(\frac{n}{n+1}\right)^{\frac{1}{2}} H_{n-1}(0)$, we inductively obtain

$$H_{2k}(0) = \left(\tfrac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\right)^{\frac{1}{2}} \qquad \text{and} \qquad H_{2k+1}(0) = 0 \qquad \text{for all } k \in \mathbb{N}_0\ .$$

In particular, $\sum_{n\geq 0} |H_n(0)|^2 = \infty$, i.e. $0 \notin E$.
Moreover, D is essentially self-adjoint because otherwise 4.1.6 would imply $E = \mathbb{C}$ in contradiction to the above. Now 4.1.2 shows that there exists a unique om μ supported on the real line satisfying $P^2_\mu = L^2_\mu$. In particular, 3.1.7 yields $\mu(\{0\}) = 0$.

Note that one can conclude these facts without knowledge of the measure itself. Indeed, the om for $(H_n)_n$ is explicitly known. As shown in [BE, 2.3 Exercise 6], for instance, the Hermite polynomials satisfy

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\, \frac{\exp(-x^2)}{\sqrt{\pi}}\, \mathrm{d}x = \delta_{nm}\ .$$

A treatise on the Hermite polynomials can also be found in [HS, (16.25)]. The fact that they are a basis in L^2_μ, i.e. $P^2_\mu = L^2_\mu$, can be proved using knowledge on analytic functions; see [HS, (21.64)] for a sketch of proof which is far from being trivial.

Finally, using 3.3.2, we see $E = \varnothing$.

The following example is taken from [EM].

4.5.12 Modified Hermite Polynomials. Keeping with the notations in 4.5.11 (which slightly differ from those in [EM]), for $0 < A < 1$, define

$$H_n^A := \left(\tfrac{1-A}{1+A}\right)^{\frac{n}{2}} H_n \,.$$

According to [EM, Theorem 3.2], the set of functions $(\Psi_n^A)_{n\geq 0}$ defined by

$$\Psi_n^A(z) := \left(\tfrac{1-A}{\pi\sqrt{A}}\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}z^2\right) H_n^A(z)$$

is an onb in a Hilbert space X_A consisting precisely of all entire functions φ satisfying

$$\int \; |\varphi(x+\mathrm{i}y)|^2 \exp\left(Ax^2 - \tfrac{1}{A}\,y^2\right) \;\mathrm{d}x\,\mathrm{d}y < \infty$$

with inner product $\langle\, \varphi \,, \psi \,\rangle_{X_A} = \int \overline{\varphi(x+\mathrm{i}y)}\, \psi(x+\mathrm{i}y)\; \exp\left(Ax^2 - \tfrac{1}{A}\,y^2\right)\; \mathrm{d}x\,\mathrm{d}y$.
Now

$$\int \overline{H_n^A(x+\mathrm{i}y)}\, H_m^A(x+\mathrm{i}y)\, \tfrac{1-A}{\pi\sqrt{A}} \;\exp\left(-(1-A)x^2 - \left(\tfrac{1}{A}-1\right)y^2\right)\; \mathrm{d}x\,\mathrm{d}y$$

$$= \int \overline{\Psi_n^A(x+\mathrm{i}y)}\, \Psi_m^A(x+\mathrm{i}y)\; \exp\left(Ax^2 - \tfrac{1}{A}\,y^2\right)\; \mathrm{d}x\,\mathrm{d}y = \delta_{mn}$$

is the matter of a simple calculation; hence we have found an om μ^A for $(H_n^A)_n$ with $\mathrm{supp}(\mu) = \mathbb{C}$.
Moreover, as shown in [EM, Corollary 3.3], X_A is an RKHS with kernel

$$K_{X_A}(z,w) = \sum_{n=0}^{\infty} \overline{\Psi_n^A(z)}\, \Psi_n^A(w) = \tfrac{1-A^2}{2\pi A}\; \exp\left(-\tfrac{1+A^2}{4A}\,(\overline{z}^2 + w^2) + \tfrac{1-A^2}{2A}\,\overline{z}w\right)$$

which shows that $\sum\limits_{n\geq 0} |H_n^A(z)|^2 < \infty$ for all $z \in \mathbb{C}$.
Thus we can now define an RKHS $\mathcal{H}(K^A)$ with domain $E = \mathbb{C}$ and kernel

$$K^A(z,w) = \sum_{n=0}^{\infty} \overline{H_n^A(z)}\, H_n^A(w) = \tfrac{\pi\sqrt{A}}{1-A}\; \exp\left(\tfrac{1}{2}\left(\overline{z}^2 + w^2\right)\right) \sum_{n=0}^{\infty} \overline{\Psi_n^A(z)}\, \Psi_n^A(w)$$

$$= \tfrac{1+A}{2\sqrt{A}}\; \exp\left(-\tfrac{(1-A)^2}{4A}\,(\overline{z}^2 + w^2) + \tfrac{1-A^2}{2A}\,\overline{z}w\right)$$

$$= \tfrac{1+A}{2\sqrt{A}}\; \exp\left(\tfrac{1-A}{4A}\left(A(\overline{z}+w)^2 - (\overline{z}-w)^2\right)\right) .$$

Clearly, $K^A(z,z)$ is bounded on every compact subset of $\mathbb{C}$ and, see 2.2.3, all members of $\mathcal{H}(K^A)$ are entire functions. Note that $(H_n^A)_n$ have an om μ^A such that $\mu^A(E) = 1$. Thus, by 3.7.2, $(H_n^A)_n$ is an onb in the space $\mathcal{H}(K^A)$ which we now can identify with $P^2_{\mu^A}$ and 3.7.13 yields $P^2_{\mu^A} \neq L^2_{\mu^A}$. Note that (4.5) yields

$$z\, H_n^A(z) = \left(\tfrac{n+1}{2}\right)^{\frac{1}{2}} \left(\tfrac{1+A}{1-A}\right)^{\frac{1}{2}} H_{n+1}^A(z) + \left(\tfrac{n}{2}\right)^{\frac{1}{2}} \left(\tfrac{1-A}{1+A}\right)^{\frac{1}{2}} H_{n-1}^A(z)\,,$$

showing that the multiplication operator D in this case is not symmetric but still tri-diagonal and we obtain (4.5) when letting $A \to 0$.

Moreover, D is not formally normal, as (3.4) is not satisfied here, and 3.2.2 implies that there exists no om ν for $(H_n^A)_n$ such that $P^2_\nu = L^2_\nu$.

However, we do not know whether μ^A is the only om for $(H_n^A)_n$.

4.5.13 Measures on Fractal Sets. The method described below is a standard tool, see [F, Prop. 1.7], for example, to construct measures on Cantor-like fractal sets.

We start with a sequence $(E_k)_{k\in\mathbb{N}_0}$ of subsets of a metric space (X,d) such that each E_k, $k>0$, is the union of finitely many non-empty disjoint sets

$$E_k = \bigcup_{i=1}^{m_k} U_{k,i}$$

where every $U_{k,j}$ contains some $U_{k+1,i}$ and is contained in one $U_{k-1,j}$. Hence $E_k \subset E_{k-1}$ for all $k \in \mathbb{N}$.

The sets $U_{k,i}$ will be referred to as **basic sets**; for fixed k, the sets $U_{k,1},\dots,U_{k,m_k}$ are called the basic sets of **level** k. Let $\mathfrak{U}$ be the collection of all basic sets and define $F := \bigcap_{k\in\mathbb{N}_0} E_k$. Note that, in general, F may be empty.

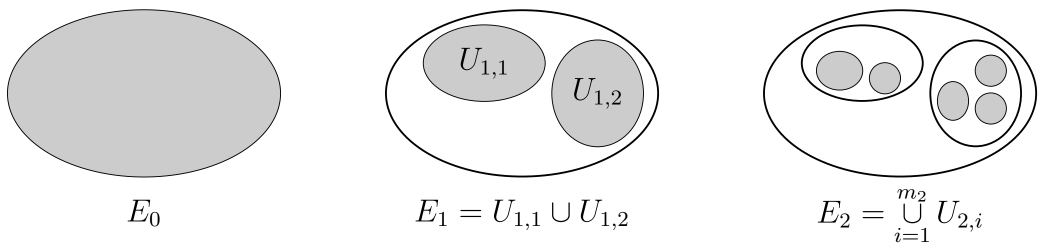


Figure 7.
An example to illustrate the construction of Cantor-like sets.

Furthermore, we define the maximum diameter d_k of E_k by

$$d_k := \max_{i=1,\dots,m_k} \Big(\sup\big\{d(x,y) : x,y \in U_{k,i}\big\}\Big)$$

and assume $d_k \to 0$ as $k \to \infty$. We assign a mass distribution ρ to the sets $U_{k,i}$ such that

$$0 < \rho(U_{k,i}) < \infty \qquad \text{and} \qquad \rho(U_{k,i}) = \sum_j \rho(U_{k+1,j}) \qquad \text{for } k>0\,,$$

the sum to be taken over all j satisfying $U_{k+1,j} \subset U_{k,i}$.

Now, for $A \subset X$, define

$$\mu(A) := \inf\Big\{\sum_j \rho(U_j) : (A \cap F) \subset \bigcup_j U_j \text{ where } U_j \in \mathfrak{U} \text{ for all } j\Big\} . \tag{4.6}$$

It is the matter of an almost straightforward calculation that $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \subset X$ with $d(A, B) > 0$. Therefore, by Carathéodory's lemma, the restriction of μ to the Borel subsets of X is a measure, see [Ma, 4.1 and 4.2] or [El, 9.2 and 9.3] for details.

4.5.14 Theorem. *In the situation of* 4.5.13*, let X as well as all basic sets be compact and $\rho(E_0) = 1$. Then $\mu(U) = \rho(U)$ for all $U \in \mathfrak{U}$ and $\mu(F) = 1$.*

Proof: Compactness of the basic sets yields $F \neq \varnothing$ and $V \cap F = \bigcap_{i \in \mathbb{N}} V \cap E_i \neq \varnothing$ for all $V \in \mathfrak{U}$.

Next we show that $V \cap F$ is relatively open w.r.t. F for any $V \in \mathfrak{U}$. To see that, let V be a basic set of level k. By construction,

$$O := X \setminus \bigcup \big\{U : U \text{ is basic set of level } k\ ,\ U \neq V\big\}$$

is open in X and $O \cap F = V \cap F$. Hence V is relatively open as asserted.

Let now U be a basic set of level p and $\mathfrak{V}$ a family of basic sets satisfying $(U \cap F) \subset \bigcup_{V \in \mathfrak{V}} V$. In order to see that for $A = U$ in (4.6) the infimum is attained when we simply cover $U \cap F$ with U, we have to show $\rho(U) \leq \sum_{V \in \mathfrak{V}} \rho(V)$.

First we note that $(U \cap F) \subset \bigcup_{V \in \mathfrak{V}} V$ yields $(U \cap F) \subset \bigcup_{V \in \mathfrak{V}} (V \cap F)$.

As $U \cap F$ is compact and all $V \cap F$ are relatively open, there exist finitely many $V_1, \dots, V_n$ such that

$$U \cap F \subset (V_1 \cup \dots \cup V_n) \cap F \subset V_1 \cup \dots \cup V_n\ . \tag{4.7}$$

It is no restriction to assume that $V_1, \dots, V_n$ all are of level $\leq q$ with some $q > p$.

Now, for $i \in \{1, \dots, n\}$, let $W_{i,1}, \dots, W_{i,k_i}$ be te basic sets of level q contained in V_i and let $U_1, \dots, U_{k_0}$ be the basic sets of level q contained in U. Then

$$\rho(V_i) = \sum_{j=1}^{k_i} \rho(W_{i,j}) \qquad \text{and} \qquad \rho(U) = \sum_{l=1}^{k_0} \rho(U_l)\ .$$

For fixed $l_* \in \{1, \dots, k_0\}$, choose $x \in U_{l_*} \cap F$. By (4.7), there exist i_* and j_* such that $x \in W_{i_*,j_*}$. As two basic sets of the same level either have empty intersection or coincide, this yields $U_{l_*} = W_{i_*,j_*}$. Therefore,

$$\rho(U) = \sum_{l=1}^{k_0} \rho(U_l) \leq \sum_{i=1}^{n} \sum_{j=1}^{k_i} \rho(W_{i,j}) = \sum_{i=1}^{n} \rho(V_i) \leq \sum_{V \in \mathfrak{V}} \rho(V)$$

and hence $\mu(U) = \rho(U)$ for all basic sets U.

Finally, note that $\rho(E_k) = \mu(E_k) = 1$ for all k and hence $\mu(F) = \lim_{k \to \infty} \mu(E_k) = 1$. $\square$

4.5.15 Example. Let now K be a compact uncountable Lebesgue-nullset[3] contained in $[0,1]$ and define $\mu(K) := 1$. Then there exists $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$ such that $x \notin K$. Set $U_{1,1} := [0,x] \cap K$, $U_{1,2} := [x,1] \cap K$, and $E_1 := U_{1,1} \cup U_{1,2}$. Note that these sets are disjoint and compact. If both $U_{1,1}$ and $U_{1,2}$ are uncountable, then let $\mu(U_{1,1}) := \mu(U_{1,2}) := \frac{1}{2}$. If $U_{1,1}$ is at most countable then $U_{1,2}$ must be uncountable and we define $\mu(U_{1,1}) := 0$ and $\mu(U_{1,2}) := 1$. Otherwise, we set $\mu(U_{1,1}) := 1$ and $\mu(U_{1,2}) := 0$.

As K is a Lebesgue-nullset, any interval $[a,b]$ with $a < b$ intersects the complement of K. Therefore, repeating the technique above, we can recursively define $E_2, E_3, \ldots$ such that $d_k \to 0$ as $k \to \infty$. On the other hand, $E_k = K$ for all k, and hence $K = \bigcap_{k \in \mathbb{N}_0} E_k$.
Note that some of the $U_{k,i}$ defined in this way may be empty which, however, does not affect the soundness of this construction; we can simply leave them out.

According to 4.5.14, μ can be extended to a Borel measure with $\mu(K) = 1$.

Clearly, one can analogously construct a Borel measure supported on an arbitrary uncountable compact set $K \subset \partial\mathbb{D}$ which is s nullset w.r.t. one-dimensional Lebesgue measure on the unit circle. This gives rise to the following remarkable example.

4.5.16 Example. Let $\widehat{\lambda}$ denote normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$ and $K \subset \partial\mathbb{D}$ be an uncountable compact set such that $\widehat{\lambda}(K) = 0$. Now construct a Borel measure ν with and $\nu(K) = 1$ in analogy to 4.5.15.

Then the measure $\mu := \frac{1}{2}(\widehat{\lambda} + \nu)$ is an om for a sequence $(P_n)_{n \geq 0}$ of polynomials as usual, satisfying the Szegő condtition 4.2.18, and $\frac{1}{2}\nu$ is the singularly continuous part of μ.
According to 4.2.20, $P^2_\mu \neq L^2_\mu$ and $E = \mathbb{D}$. Moreover, K, up to a μ-nullset, coincides with the set B in the proof of 4.2.21 and $\mathbf{1}_K \in P^2_\mu$. Hence there exists a sequence $(q_n)_n$ of polynomials satisfying

(i) $\lim_{n \to \infty} q_n(z) = 0$ for all $z \in \mathbb{D}$,

(ii) $\lim_{n \to \infty} q_n(z) = 0$ for Lebesgue-almost all $z \in \partial\mathbb{D}$,

(iii) $\lim_{n \to \infty} q_n(z) = 1$ for μ-almost all and hence uncountably many $z \in K$.

Remark. According to [HS, (10.55)], every uncountable borel set in a complete metric space contains a non-void perfect set which, see [HS, (6.61)] for the definition, is uncountable and compact.

4.5.17 Summary. The table on the following page presents a survey on (almost) all of the examples mentioned so far. For abbreviation, we will call μ an obm if it is an om such that $P^2_\mu = L^2_\mu$. Moreover, set $H_r := \{z \in \mathbb{C} : \mathrm{Re}\,(z) > \frac{1}{2}\}$.
Recall that, according to 3.5.6, if there exists an om μ for a given sequence $(P_n)_n$ of polynomials then E_{reg} is precisely the largest subset of E where all members of $\mathcal{H}(K)$ are holomorphic. Note that we do not know an example where $E_{reg} \neq E^\circ$.

[3] As an example, one may think of the middle-third Cantor set.

	where to find	E	E_{reg}	$(P_n)_n$ onb in $\mathcal{H}(K)$	$\exists$ om	$\exists$ obm	om is unique
$P_n(z) = \frac{1}{\sqrt{n!}} z^{n-1}(z-n)$	2.1.12 4.5.2	$\mathbb{C}$	$\mathbb{C}$	no	no	no	n/a
$P_n(z) = \frac{1}{\sqrt{(n-1)!}} z^{n-1}(z-1)$	4.5.3	$\mathbb{C}$	$\mathbb{C}$	yes	?	no	?
$P_n(z) = n\, z^{n-1}(z-1)$	4.5.4	$\mathbb{D} \cup \{1\}$	$\mathbb{D}$	yes	no	no	n/a
$P_n(z) = b_n z^n$ where $b_{2n-1} = b_{2n} = \frac{1}{\sqrt{(n-1)!}}$	4.4.9	$\mathbb{C}$	$\mathbb{C}$	yes	no	no	n/a
$P_n(z) = z^n$	1.2.2 1.3.8 2.1.11 4.4.7	$\mathbb{D}$	$\mathbb{D}$	yes	yes	no	yes
$P_n(z) = \sqrt{n+1}\, z^n$	2.3.7 4.4.7	$\mathbb{D}$	$\mathbb{D}$	yes	yes	no	yes
D symmetric but not essentially self-adjoint	4.1.6	$\mathbb{C}$	$\mathbb{C}$	yes	yes	yes	no
Jacobi polynomials	4.5.10	$\varnothing$	$\varnothing$	n/a	yes	yes	yes
Hermite polynomials $P_n(z) = \frac{(-1)^n}{\sqrt{2^n n!}} \exp(z^2) \cdot \frac{\mathrm{d}^n}{\mathrm{d}z^n}[\exp(-z^2)]$	4.5.11	$\varnothing$	$\varnothing$	n/a	yes	yes	yes
modified Hermite polynomials	4.5.12	$\mathbb{C}$	$\mathbb{C}$	yes	yes	no	?
Newton polynomials $P_n(z) = \frac{(-1)^n}{n!} \prod_{k=1}^{n} (z-k)$	4.5.7	H_r	H_r	yes	yes	no	?
discrete om μ where Λ_μ is compact	3.7.11	Λ_μ	$\varnothing$	yes	yes	yes	yes
om μ supported on $[0,1]$; not discrete but having some discrete mass points	4.1.4	Λ_μ	$\varnothing$	no	yes	yes	yes
om μ supported on $\partial\mathbb{D}$; not discrete but having some discrete mass points and not obeying the Szegő condition	4.2.2 4.2.18	Λ_μ	$\varnothing$	no	yes	yes	yes
om μ supported on $\partial\mathbb{D}$; obeying the Szegő condition and $\mu_{\mathrm{sc}} = 0$	4.2.17 4.2.20	$\mathbb{D} \cup \Lambda_\mu$	$\mathbb{D}$	yes	yes	no	yes
om μ supported on $\partial\mathbb{D}$; obeying the Szegő condition and $\mu_{\mathrm{sc}} \neq 0$	4.2.17 4.2.20 4.2.21 4.5.16	$\mathbb{D} \cup \Lambda_\mu$	$\mathbb{D}$	no	yes	no	yes

Appendix

A.1 The Spectral Theorem for Normal Operators

In a complex Hilbert space $\mathcal{H}$ we denote the inner product and norm by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively. We will omit the index if it is clearly understood. Note that the inner product is linear in the *second* argument. Moreover, let $\mathscr{B}(\mathcal{H})$ be the set of bounded linear operators in $\mathcal{H}$.
We will denote the spectrum of a linear operator A by $\sigma(A)$. For standard definitions and theorems concerning (possibly unbounded) linear operators in a Hilbert space we refer to [AG], [W], or [Ru2, Ch. 13], for instance.
In particular, the spaces L^2_μ, where μ is an om for a given sequence of polynomials, are of special interest here. For mesaure theoretical details see e.g. [Bau] or [El].

In this section we will state the spectral theorem for normal operators which plays a central role in the theory of orthogonal polynomials, as the orthonormalizing measures for a given sequence of polynomials correspond to the spectral measures of certain normal operators, see 1.3.7. Furthermore, we will quote some facts from the theory of subnormal operators and we will also have a look at fundamental properties of Reproducing Kernel Hilbert Spaces (RKHS) including proofs of the most important statements cited in section 1.4.

A densely defined linear operator N in a Hilbert space $\mathcal{H}$ is **normal**, if $\mathrm{dom}(N) = \mathrm{dom}(N^*)$ and $\|Nx\| = \|N^*x\|$ for all $x \in \mathrm{dom}(N)$. In particular, every self-adjoint operator is normal.

A.1.1 Proposition. *Let N be a densely defined closed operator in a Hilbert space $\mathcal{H}$. The following properties are equivalent.*

(i) *N is normal.*

(ii) *N^* is normal.*

(iii) *$N^*N = NN^*$.*

For a proof see e.g. [W, 5.6, Folgerung 2].

A.1.2 Resolutions of the Identity. The following definitions are standard tools of spectral theory of linear operators in a Hilbert space.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ a σ-algebra on a set Ω. A map $E : \mathcal{A} \to \mathscr{B}(\mathcal{H})$ is called a **resolution of the identity** or **projection valued measure** if it satisfies the following properties.

(i) $E(\Delta)$ is an orthogonal projection for every $\Delta \in \mathcal{A}$.

(ii) $E(\Omega) = \mathrm{id}$.

(iii) If $(\Delta_n)_{n\in\mathbb{N}}$ is a sequence of mutually disjoint sets in $\mathcal{A}$ then

$$E\big(\bigcup_{n=1}^{\infty} \Delta_n\big)x = \sum_{n=1}^{\infty} E(\Delta_n)x \qquad \text{for all } x \in \mathcal{H}\,.$$

As an almost immediate consequence, we get $E(\varnothing) = 0$ and $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \mathcal{A}$.

For $x \in \mathcal{H}$ we define $E_x : \mathcal{A} \to \mathbb{R}$, $E_x(\Delta) := \langle\, x\,, E(\Delta)x\,\rangle = \|E(\Delta)x\|^2$. Then E_x is a measure on $\mathcal{A}$ and $E_x(\Omega) = \|x\|^2$.

For an elementary function $u : \Omega \to \mathbb{C}$, i.e. the image $u(\Omega)$ is finite and $u^{-1}(\{a\}) \in \mathcal{A}$ for all $a \in u(\Omega)$, we define the integral

$$\int u\,\mathrm{d}E := \sum_{a\in u(\Omega)} a\,E\big(u^{-1}(\{a\})\big)$$

which clearly is a bounded linear operator in $\mathcal{H}$.

Let now $f : \Omega \to \mathbb{C}$ be measurable, i.e. $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathfrak{B}(\mathbb{C})$. Then one can define a linear operator by

$$\mathrm{dom}\Big(\int f\,\mathrm{d}E\Big) = \big\{x \in \mathcal{H} : \int\, |f|^2\,\mathrm{d}E_x < \infty\big\}\,, \qquad \Big(\int f\,\mathrm{d}E\Big)x := \lim_{n\to\infty}\Big(\int u_{x,n}\,\mathrm{d}E\Big)x$$

where $(u_{x,n})_n$ is a sequence of elementary functions such that $\int\, |u_{x,n} - f|^2\,\mathrm{d}E_x \to 0$ as $n \to \infty$.

It is well known that $\Psi(f) := \int f\,\mathrm{d}E$ is a densely defined linear operator in $\mathcal{H}$ and

$$\big\|\Psi(f)x\big\|^2 = \int\, |f|^2\,\mathrm{d}E_x \qquad \text{for all } x \in \mathrm{dom}\big(\Psi(f)\big)\,.$$

A.1.3 Spectral Theorem for Normal Operators. *Let N be a normal operator in $\mathcal{H}$. Then there exists a unique resolution of the identity E on $\mathfrak{B}(\mathbb{C})$ such that*

$$N = \int \mathrm{id}_{\mathbb{C}}\,\mathrm{d}E\,.$$

In particular, $E(\Delta) = 0$ whenever $\Delta \cap \sigma(N) = \varnothing$.

Moreover, $E(\Delta)S = SE(\Delta)$ for all $\Delta \in \mathfrak{B}(\mathbb{C})$ and all $S \in \mathscr{B}(\mathcal{H})$ satisfying $SN \subset NS$.

A proof of the spectral theorem can be found in [Ru2, 13.33] or [W, Satz 7.32], for instance. Note that $E\big(\sigma(N)\big) = \mathrm{id}_{\mathcal{H}}$ and $E(\Delta) = E\big(\sigma(N) \cap \Delta\big)$ for all $\Delta \in \mathfrak{B}(\mathbb{C})$. Therefore, E is also referred to as the **spectral decomposition** or the **spectral measure** of N.

It is not difficult to see that, for a σ-finite measure μ on $\mathfrak{B}(\mathbb{C})$, the spectral decomposition of the multiplication operator M_μ in L^2_μ (which is normal, see 1.3.2) is given by $E(\Delta) = M_\Delta$ where M_Δ denotes multiplication with $\mathbf{1}_\Delta$ in L^2_μ.

A.1.4 Lemma. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and T a bounded linear operator in L^2_μ that commutes with multiplication by $\mathbf{1}_\Delta$ for every $\Delta \in \mathcal{A}$, i.e. $T(\mathbf{1}_\Delta \cdot f) = \mathbf{1}_\Delta \cdot Tf$ for all $f \in L^2_\mu$. Then T acts as multiplication with an L^∞_μ-function.*

Proof: As μ is finite, we have $\mathbf{1}_\Delta \in L^2_\mu$ for all $\Delta \in \mathcal{A}$.
Set $g := T\mathbf{1}_\Omega$ and note that $g \in L^2_\mu$ implies $g \in L^1_\mu$ for finite μ. Define $\gamma \in L^2_\mu$ by

$$\gamma(z) := \begin{cases} \frac{|g(z)|}{g(z)} & \text{if } g(z) \neq 0\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

We obtain

$$\int_\Delta |g| \ \mathrm{d}\mu = \int_\Delta \gamma \cdot g \ \mathrm{d}\mu \leq \big|\langle \overline{\gamma} \cdot \mathbf{1}_\Delta \,, T\mathbf{1}_\Omega \rangle\big| \leq \|\overline{\gamma} \cdot \mathbf{1}_\Delta\| \ \|T\| \ \|\mathbf{1}_\Omega\| = \mu(\Delta) \ \|T\| \ \|\mathbf{1}_\Omega\|$$

and, therefore, $|g(z)| \leq \|T\| \ \|\mathbf{1}_\Omega\|$ for μ-almost all z. This shows $g \in L^\infty_\mu$. Moreover,

$$T\mathbf{1}_\Delta = T(\mathbf{1}_\Delta \cdot \mathbf{1}_\Omega) = \mathbf{1}_\Delta \cdot T\mathbf{1}_\Omega = \mathbf{1}_\Delta \cdot g$$

for all $\Delta \in \mathcal{A}$. As $\{\mathbf{1}_\Delta : \Delta \in \mathcal{A}\}$ is total in L^2_μ, this yields $Tf = f \cdot g$ for all $f \in L^2_\mu$. $\square$

A.1.5 Symbolic Calculus for Normal Operators. Let N be a normal operator in a Hilbert space $\mathcal{H}$ and E its spectral decomposition. For measurable $f : \sigma(N) \to \mathbb{C}$, we define

$$\Psi(f) := \int f \, \mathrm{d}E$$

as above. This is well-defined as the integral only depends on the values of f on $\sigma(N)$. For arbitrary $p \in \mathbb{C}[z]$, one can show that $\Psi(p) = p(N)$ where $\mathrm{dom}\big(\Psi(f)\big)$ is to be chosen as the canonical domain of $p(N)$. For more details, see e.g. [Ru2, 12.24]. In particular,

$$\big\|p(N)x\big\|^2 = \int \ |p|^2 \ \mathrm{d}E_x \qquad \text{for all } x \in \mathrm{dom}\big(p(N)\big)\,. \tag{A.1}$$

A.2 Normal Extensions of Hessenberg Operators

A densely defined linear operator F in a Hilbert space $\mathcal{H}$ is **formally normal**, if $\mathrm{dom}(F) \subset \mathrm{dom}(F^*)$ and $\|Fx\| = \|F^*x\|$ for all $x \in \mathrm{dom}(F)$. Consequently, every normal operator is formally normal.
A densely defined linear operator S in $\mathcal{H}$ is **subnormal**, if there exist a Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H}$ can be embedded isometrically into $\mathcal{K}$ and $Fx = Nx$ for all $x \in \mathrm{dom}(F)$. In particular, every symmetric operator is subnormal, as if S is a symmetric operator in $\mathcal{H}$ then there always exists a self-adjoint operator extending S acting in a possibly larger space $\mathcal{K}$, see [AG, IX. 111. Satz 1], for example.

The textbook by Conway [Con1] provides extensive information on the theory of bounded subnormal operators. Concerning the theory of unbounded subnormal operators, we will in particular refer to a trilogy of papers by Stochel and Szafraniec, [StSz1], [StSz2], and [StSz3]. As mentioned before, see 1.3.4 and 1.3.7 (for both of which we will present a proof here), subnormal Hessenberg operators play an important role in the theory of orthogonal polynomials. Therefore, subnormality is one of the main topics of any work dealing with questions about orthonormalizing measures, as [CaKl1], [CaKl2], or [Kl2], to name just a few. The following can be found in [CaKl2, Theorem 1(iii)], for instance.

A.2.1 (Proof of 1.3.4). Let N be a normal extension of D in a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and let E denote the spectral measure of N. For $\Delta \in \mathfrak{B}(\mathbb{C})$ define $\mu(\Delta) := E_{P_0}(\Delta) = \left\langle P_0 \,, E(\Delta)P_0 \right\rangle_{\mathcal{K}}$. By construction, $\mathrm{supp}(\mu) \subset \sigma(N)$.
The assignment $\mathbf{1}_\Delta \mapsto E(\Delta)P_0$ gives rise to a linear map $\beta : L^2_\mu \to \mathcal{K}$ which is an isometry since

$$\begin{aligned}\left\langle \mathbf{1}_\Delta \,, \mathbf{1}_{\Delta'} \right\rangle_{L^2_\mu} &= \int \overline{\mathbf{1}_\Delta}\, \mathbf{1}_{\Delta'} \,\mathrm{d}\mu = \mu(\Delta \cap \Delta') = \left\langle P_0 \,, E(\Delta \cap \Delta')P_0 \right\rangle_{\mathcal{K}} \\ &= \left\langle P_0 \,, E(\Delta)E(\Delta')P_0 \right\rangle_{\mathcal{K}} = \left\langle E(\Delta)P_0 \,, E(\Delta')P_0 \right\rangle_{\mathcal{K}}\end{aligned}$$

for $\Delta, \Delta' \in \mathfrak{B}(\mathbb{C})$. Consequently, $\mathcal{L} := \beta(L^2_\mu)$ is the closure of the linear span of $\{E(\Delta)P_0 : \Delta \in \mathfrak{B}(\mathbb{C})\}$ in $\mathcal{K}$ and $\beta : L^2_\mu \to \mathcal{L}$ is an isomorphism. If

$$u = \sum_{j=1}^{k} \alpha_j \mathbf{1}_{\Delta_j}$$

is an elementary function where $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ and, without loss of generality, $\Delta_1, \dots, \Delta_k \in \mathfrak{B}(\mathbb{C})$ are mutually disjoint then

$$\beta u = \sum_{j=1}^{k} \alpha_j E(\Delta_j) P_0 = \Big(\int u \,\mathrm{d}E\Big) P_0 \,.$$

Considering arbitrary $p \in \mathbb{C}[z]$ as a member of $\mathcal{K}$, we have $p = p(N)P_0 = p(D)P_0$ and (A.1) yields

$$\left\|p(N)P_0\right\|^2_{\mathcal{K}} = \int \; |p|^2 \;\mathrm{d}E_{P_0} = \int \; |p|^2 \;\mathrm{d}\mu$$

showing that p is square integrable with respect to μ. We now show that β maps $p \in L^2_\mu$ to $p \in \mathcal{H}$.

There exists a sequence $(u_n)_n$ of elementary functions such that $\|u_n - p\|_{L^2_\mu} \to 0$ which implies $\|\beta u_n - \beta p\|_{\mathcal{K}} \to 0$ as $n \to \infty$. On the other hand,

$$\beta u_n = \Big(\int u_n \,\mathrm{d}E\Big) P_0 \to \Big(\int p \,\mathrm{d}E\Big) P_0$$

with respect to $\|\cdot\|_{\mathcal{K}}$. Using A.1.5, we obtain

$$\Big(\int p \,\mathrm{d}E\Big) P_0 = \Psi(p) P_0 = p(N) P_0 = p(D) P_0 = p \in \mathcal{H}\,.$$

Therefore, $\beta p = p$ for all $p \in \mathbb{C}[z]$; in particular, $(P_n)_n$ is an orthonormal system in L^2_μ. Hence μ is an om for $(P_n)_n$.

Denote by F the spectral measure of M_μ and recall that $F(\Delta)$ is multiplication with $\mathbf{1}_\Delta$ for $\Delta \in \mathfrak{B}(\mathbb{C})$. Thus

$$\beta\big(F(\Delta)\mathbf{1}_{\Delta'}\big) = \beta \mathbf{1}_{\Delta\cap\Delta'} = E(\Delta\cap\Delta')P_0 = E(\Delta)E(\Delta')P_0 = E(\Delta)\beta\mathbf{1}_{\Delta'}$$

for all $\Delta, \Delta' \in \mathfrak{B}(\mathbb{C})$ which shows $\beta(M_\mu f) = N(\beta f)$ for $f \in \mathrm{dom}(M_\mu)$. Therefore, $\beta M_\mu \beta^{-1} = N|\mathcal{L} \cap \mathrm{dom}(N)$ and $D \subset N|\mathcal{L} \cap \mathrm{dom}(N) \subset N$.

Let P^2_μ be the completion of $\mathbb{C}[z]$ w.r.t. $\|\cdot\|_{L^2_\mu}$. Then β maps P^2_μ onto $\mathcal{H}$ and we can regard D as a densely defined subnormal operator in P^2_μ with normal extension M_μ acting in the possibly larger space L^2_μ. Clearly, $\mathrm{supp}(\mu) = \sigma(M_\mu)$. $\square$

A.2.2 Lemma. *Let N be a normal operator in a Hilbert space $\mathcal{K}$ and $\mathcal{L}$ be a closed subspace of $\mathcal{K}$. Then $\mathcal{L}$ reduces N if and only if the restriction of N to $\mathcal{L} \cap \mathrm{dom}(N)$ is a normal operator in $\mathcal{L}$.*

For self-adjoint operators this result is well-known. The generalization to normal operators uses that the real and imaginary parts of a normal operator are self-adjoint, see [Kl2, Satz 1.3.9] or [CaKl2, Theorem 17] for a detailed proof.

It is also well-known, see e.g. [W, 5.6 - Aufgabe 5.39], that $\mathcal{L}$ reduces N if and only if $\mathcal{L} \cap \mathrm{dom}(N) + \mathcal{L}^\perp \cap \mathrm{dom}(N) = \mathrm{dom}(N)$ and both $\mathcal{L}$ as well as $\mathcal{L}^\perp$ are invariant under N, i.e. N maps $\mathcal{L} \cap \mathrm{dom}(N)$ and $\mathcal{L}^\perp \cap \mathrm{dom}(N)$ into $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively.

A.2.3 Theorem. *Let μ be a finite measure on the Borel σ-algebra in $\mathbb{C}$. A closed subspace $\mathcal{L} \subset L^2_\mu$ reduces M_μ if and only if $\mathcal{L} = \{\mathbf{1}_\Delta f : f \in L^2_\mu\}$ with some Borel set Δ.*

Proof: For an arbitrary Borel set Δ, define $\mathcal{L} = \{\mathbf{1}_\Delta f : f \in L^2_\mu\}$.
Clearly, $\mathcal{L}^\perp = \{\mathbf{1}_{\mathbb{C}\setminus\Delta} f : f \in L^2_\mu\}$ and $\mathcal{L}$ as well as $\mathcal{L}^\perp$ are closed subspaces of L^2_μ, invariant under M_μ. Hence $\mathcal{L}$ (and also $\mathcal{L}^\perp$) reduce M_μ.

Now let $\mathcal{M}$ reduce M_μ, i.e. $PM_\mu \subset M_\mu P$ where P denotes the orthogonal projection in L^2_μ onto $\mathcal{M}$. Now the spectral theorem A.1.3 shows $PE(\Delta) = E(\Delta)P$ for all $\Delta \in \mathfrak{B}(\mathbb{C})$ where E is the spectral measure of M_μ. As $E(\Delta)$ is multiplication with $\mathbf{1}_\Delta$, we can apply A.1.4 and see that P acts as multiplication with an L^∞_μ-function g. Note that $P^2 = P$ which implies that g can only attain the values 0 or 1. Hence $g = \mathbf{1}_{\Delta'}$ for some $\Delta' \in \mathfrak{B}(\mathbb{C})$ and $Pf = \mathbf{1}_{\Delta'} f$ for all $f \in L^2_\mu$. Therefore, $\mathcal{M} = \{\mathbf{1}_{\Delta'} f : f \in L^2_\mu\}$ as asserted. $\square$

A.2.4 (Proof of 1.3.7). Let N be a minimal normal extension of D acting in a Hilbert space $\mathcal{K} \supset \mathcal{H}$. In analogy to A.2.1 we can define an om μ and embed L^2_μ isometrically into $\mathcal{K}$. Then $\beta M_\mu \beta^{-1}$ is normal in $\mathcal{L}$ and A.2.2 implies that $\mathcal{L}$ reduces N; by definition of minimality we obtain $\mathcal{L} = \mathcal{K}$. Hence β is an isomorphism and $\beta^{-1} N \beta = M_\mu$.

As to the converse, let μ be an om. Then M_μ is a normal extension of D; we have to show that it is minimal.

Denote by E be the spectral measure of M_μ. Assume that $\mathcal{M}$ is a closed subspace of L^2_μ reducing M_μ and $P_0 \in \mathcal{M}$. According to A.2.3, there exists $\Delta \in \mathfrak{B}(\mathbb{C})$ such that $\mathcal{M} = \{\mathbf{1}_\Delta f : f \in L^2_\mu\}$ while on the other hand, $P_0 = \mathbf{1}_\mathbb{C}$. Therefore, only $\Delta = \mathbb{C}$ remains. Hence $\mathcal{M} = L^2_\mu$ and, therefore, M_μ is minimal.

For the last assertion, let D be essentially normal, i.e. the closure $\overline{D}$ of D is a normal operator in $\mathcal{H}$. Then any normal extension of D also extends $\overline{D}$ and as an immediate consequence of A.2.2, $\overline{D}$ is the only minimal extension of D. Hence there exists a unique om μ and we can identify $\overline{D}$ with M_μ. In particular, $\mathcal{H}$ is isometrically isomorphic to L^2_μ via $P_n \mapsto P_n$ and $(P_n)_n$ is an onb in L^2_μ. $\square$

A.3 Reproducing Kernel Hilbert Spaces

The theory of Reproducing Kernel Hilbert Spaces (RKHS) is treated in a very detailed way by Aronszajn whose article [Ar] may be given as standard reference on this topic. Other references are a textbook by Meschkowski [Me] and also [Do, Chapter X].
We will here construct an RKHS such that a given positive matrix (in accordance to 1.4.5) becomes its kernel and also give a proof of the RKHS-test 1.4.7 as well as provide some examples of RKHS.

The following is well-known, see [Ar, I.4], for example.

A.3.1 (Proof of 1.4.5). Let E be an arbitrary non-empty set and $K : E \times E \to \mathbb{C}$ a map satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i}\, c_j\, K(z_i, z_j) \geq 0 \tag{A.2}$$

for all finite sets $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$.

We immediately see that $K(z, z) \geq 0$ for all $z \in E$ and a short calculation yields $K(z, w) = \overline{K(w, z)}$ for all $z, w \in E$.

For $z \in E$ we set $K_z : E \to \mathbb{C}$, $K_z(w) := K(z,w)$.

Let now $\mathcal{H}_0$ be the linear span of $\{K_z : z \in E\}$, i.e. every $u \in \mathcal{H}_0$ can be written as a finite(!) sum

$$u = \sum_k a_k \, K_{z_k} \tag{A.3}$$

where $a_k \in \mathbb{C}$ and $z_k \in E$. We define a map $\langle \, , \, \rangle : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{C}$ as follows. For u as above and $v = \sum_j b_j \, K_{w_j}$ set $\langle \, u \, , v \, \rangle := \sum_k \sum_j \overline{a_k} \, b_j \, K(w_j, z_k)$. Taking into account that

$$\sum_j \overline{u(w_j)} \, b_j = \langle \, u \, , v \, \rangle = \sum_k \overline{a_k} \, v(z_k) \, ,$$

we observe that $\langle \, u \, , v \, \rangle$ is well-defined although the representations of u and v as in (A.3) may not be unique. In particular, for $u = K_z$ we get $\langle \, K_z \, , v \, \rangle = v(z)$.

Obviously, $\langle \, u \, , v \, \rangle = \overline{\langle \, v \, , u \, \rangle}$ and $\langle \, , \, \rangle$ is linear in the second argument. Moreover, (A.2) yields $\langle \, u \, , u \, \rangle \geq 0$. Therefore, we can use Cauchy-Schwarz to show

$$|u(z)|^2 = |\langle \, K_z \, , u \, \rangle|^2 \leq \langle \, u \, , u \, \rangle \cdot \langle \, K_z \, , K_z \, \rangle \, .$$

Hence $\langle \, u \, , u \, \rangle = 0$ implies $u(z) = 0$ for all $z \in E$, i.e. $u = 0$ in $\mathcal{H}_0$. Thus $\langle \, , \, \rangle$ is a positive definite sesquilinear form on $\mathcal{H}_0$.

Let $\widetilde{\mathcal{H}}$ denote the abstract completion of $\mathcal{H}_0$ with respect to the norm $\|u\| := \langle \, u \, , u \, \rangle^{\frac{1}{2}}$.

We now define a map $\iota : \widetilde{\mathcal{H}} \to \mathcal{F}(E)$ as follows (here $\mathcal{F}(E)$ denotes the set of all maps $E \to \mathbb{C}$). For $\widetilde{f} \in \widetilde{\mathcal{H}}$ there exists a Cauchy sequence $(u_n)_n$ in $\mathcal{H}_0$ with $u_n \to \widetilde{f}$ as $n \to \infty$. Set $f : E \to \mathbb{C}$, $f(z) := \lim_{n\to\infty} \langle \, K_z \, , u_n \, \rangle$ and $\iota(\widetilde{f}) := f$.

Note that ι is well-defined and its restriction to $\mathcal{H}_0$ is the canonical embedding into $\mathcal{F}(E)$. As $\mathcal{H}_0$ is a dense subspace of $\widetilde{H}$, the range $\operatorname{ran}(\iota)$ is a Hilbert space isometrically isomorphic to $\widetilde{\mathcal{H}}$.

Thus $\mathcal{H}(K) := \operatorname{ran}(\iota)$ is a completion of $\mathcal{H}_0$ and $f(z) = \lim_{n\to\infty} \langle \, K_z \, , u_n \, \rangle_{\widetilde{\mathcal{H}}} = \langle \, K_z \, , \widetilde{f} \, \rangle_{\widetilde{\mathcal{H}}} = \langle \, K_z \, , f \, \rangle_{\mathcal{H}(K)}$.

According to 1.4.1, $\mathcal{H}(K)$ is an RKHS with kernel $K(z,w) := K_z(w) = \langle \, K_w \, , K_z \, \rangle$. □

From pointwise convergence, in general, we can not conclude norm convergence in $\mathcal{H}(K)$. The following lemma shows that pointwise convergence of a bounded sequence implies weak convergence in $\mathcal{H}(K)$.

A.3.2 Lemma. *Let $(f_n)_n$ be a sequence in $\mathcal{H}(K)$ and $M > 0$ such that $\|f_n\| \leq M$ for all n and $f(z) := \lim_{n\to\infty} f_n(z)$ exists for all $z \in E$. Then*

$$f \in \mathcal{H}(K) \qquad \textit{and} \qquad \lim_{n\to\infty} \langle \, f_n \, , h \, \rangle = \langle \, f \, , h \, \rangle \quad \textit{for all } h \in \mathcal{H}(K) \, .$$

Proof: Choose arbitrary $F \in \mathcal{H}(K)$ and define $\mathcal{H}_0$ as in A.3.1.
For $\varepsilon > 0$ there exists $g \in \mathcal{H}_0$, $g = \sum_{j=1}^{k} \alpha_j K_{z_j}$ say, such that $\|F - g\| < \varepsilon$. Then

$$\langle g , f_n \rangle = \Big\langle \sum_{j=1}^{k} \alpha_j K_{z_j} , f_n \Big\rangle = \sum_{j=1}^{k} \alpha_j \, f_n(z_j)$$

and, in particular, $\lim_{n\to\infty} \langle g , f_n \rangle$ exists.
Let now $N \in \mathbb{N}$ such that $\big|\langle g , f_n \rangle - \langle g , f_m \rangle\big| < \varepsilon$ for all $m, n \geq N$. We obtain

$$\begin{aligned}\big|\langle F , f_n \rangle - \langle F , f_m \rangle\big| &\leq \big|\langle F - g , f_n - f_m \rangle\big| + \big|\langle g , f_n - f_m \rangle\big| \\ &\leq \|F - g\| \; \|f_n - f_m\| + \big|\langle g , f_n \rangle - \langle g , f_m \rangle\big| \leq \varepsilon \cdot 2M + \varepsilon\end{aligned}$$

for all $m, n \geq N$. Hence $\big(\langle F , f_n \rangle\big)_n$ is convergent.
Therefore, $L : \mathcal{H}(K) \to \mathbb{C}$, $Lh := \lim_{n\to\infty} \langle f_n , h \rangle$, is a well-defined linear functional. Moreover,

$$\big|\langle f_n , h \rangle\big| \leq \|f_n\| \; \|h\| \leq M \; \|h\| \qquad \text{for all } n \in \mathbb{N} \text{ and all } h \in \mathcal{H}(K)\,.$$

Hence L is bounded and the Riesz representation theorem provides $\widetilde{f} \in \mathcal{H}(K)$ such that $Lh = \langle \widetilde{f} , h \rangle$. Furthermore,

$$\widetilde{f}(z) = \langle K_z , \widetilde{f} \rangle = \overline{L\,K_z} = \overline{\lim_{n\to\infty} \langle f_n , K_z \rangle} = \lim_{n\to\infty} f_n(z) \qquad \text{for all } z \in E\,.$$

Thus $f = \widetilde{f} \in \mathcal{H}(K)$ and $\lim_{n\to\infty} \langle f_n , h \rangle = Lh = \langle f , h \rangle$ as asserted. □

Given an RKHS with domain E, we would like to be able to decide whether a function $f : E \to \mathbb{C}$ belongs to this space.

A.3.3 Lemma. *Let $\mathcal{H}(K)$ be an RKHS with domain E. If g is a complex-valued function defined on a non-empty set $F \subset E$ such that*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i} c_j \, K(z_i, z_j) \geq \Big|\sum_{i=1}^{n} c_i \, g(z_i)\Big|^2 \tag{A.4}$$

for all finite sets $\{z_1, \ldots, z_n\} \subset F$ and $c_1, \ldots, c_n \in \mathbb{C}$.
Then there exists $f \in \mathcal{H}(K)$ satisfying $f(z) = g(z)$ for all $z \in F$ and (A.4) *remains valid for f instead of g and all $\{z_1, \ldots, z_n\} \subset E$.*

Proof: Let $\mathcal{H}_0(F)$ be the linear span of $\{K_z : z \in F\}$, in analogy to (A.3).
For $z \in F$ set $\varphi(K_z) := \overline{g(z)}$. In order to show that φ can be extended to a linear form on $\mathcal{H}_0(F)$, suppose

$$\sum_{k=1}^{m} a_k \, K_{v_k} = 0\,.$$

Then (A.4) yields

$$0 = \Big\langle \sum_{k=1}^{m} a_k\, K_{v_k}\,,\, \sum_{k=1}^{m} a_k\, K_{v_k} \Big\rangle = \sum_{i=1}^{m}\sum_{j=1}^{m} \overline{a_i} a_j\, \langle\, K_{v_i}\,,\, K_{v_j}\,\rangle$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{m} \overline{a_i} a_j\, K(v_j, v_i) \ge \Big|\sum_{i=1}^{m} \overline{a_i}\, g(v_i)\Big|^2 .$$

Thus for $h = \sum_{i=1}^{n} c_i\, K_{z_i} \in \mathcal{H}_0(F)$ the expression

$$\varphi(h) := \sum_{i=1}^{n} c_i\, \overline{g(z_i)}$$

is well-defined. Using (A.4) once more, we get

$$|\varphi(h)|^2 = \Big|\sum_{i=1}^{n} \overline{c_i}\, g(z_i)\Big|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} c_i\, \overline{c_j}\, K(z_i, z_j) = \Big\|\sum_{k=1}^{n} c_k\, K_{z_k}\Big\|^2 = \|h\|^2 .$$

According to the Hahn-Banach theorem, φ can be extended to a continuous linear functional on $\mathcal{H}(K)$ with $|\varphi(h)| \le \|h\|$ for all $h \in \mathcal{H}(K)$. By the Riesz representation theorem there exists $f \in \mathcal{H}(K)$ such that $\varphi(h) = \langle\, f\,,h\,\rangle$ for all $h \in \mathcal{H}(K)$ and $\|f\| \le 1$.
For $z \in F$ then $f(z) = \langle\, K_z\,, f\,\rangle = \overline{\varphi(K_z)} = g(z)$ and for arbitrary $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$, finally,

$$\Big|\sum_{i=1}^{n} c_i\, f(z_i)\Big|^2 = \Big|\sum_{i=1}^{n} c_i \langle\, K_{z_i}\,, f\,\rangle\Big|^2 = \Big|\Big\langle \sum_{i=1}^{n} \overline{c_i}\, K_{z_i}\,, f\,\Big\rangle\Big|^2$$
$$\le \Big\|\sum_{i=1}^{n} \overline{c_i}\, K_{z_i}\Big\|^2 \cdot \|f\|^2 \le \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j\, K(z_i, z_j) \qquad \square$$

A.3.4 (Proof of 1.4.7). Let $\mathcal{H}(K)$ be an RKHS with domain E and $f : E \to \mathbb{C}$.

(i) Suppose there exists $r > 0$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j\, K(z_i, z_j) \ge r\, \Big|\sum_{i=1}^{n} c_i\, f(z_i)\Big|^2$$

for all finite sets $\{z_1, \ldots, z_n\} \subset E$ and $c_1, \ldots, c_n \in \mathbb{C}$. Then $g := r \cdot f$ obeys (A.4) with $F = E$. Thus $g \in \mathcal{H}(K)$ and, obviously, $f \in \mathcal{H}(K)$.

(ii) Suppose now $f \in \mathcal{H}(K) \setminus \{0\}$. Take $\{z_1, \ldots, z_n\} \subset E$; using Cauchy-Schwarz, we get

$$\Big|\sum_{i=1}^{n} c_i\, f(z_i)\Big|^2 = \Big|\Big\langle \sum_{i=1}^{n} \overline{c_i}\, K_{z_i}\,, f\,\Big\rangle\Big|^2 \le \Big\|\sum_{i=1}^{n} \overline{c_i}\, K_{z_i}\Big\|^2 \cdot \|f\|^2 \le \|f\|^2 \cdot \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j\, K(z_i, z_j) .$$

Setting $r := \frac{1}{\|f\|^2}$ completes the proof. $\square$

The following theorem is also well-known, see [Ar, I.5], for example.

A.3.5 Theorem. *Let $\mathcal{H}(K)$ be an RKHS with domain E und suppose F is a non-empty subset of E. Then $L := K|F \times F$ is the kernel of an RKHS which we will denote by $\mathcal{H}(L)$. In particular, $\mathcal{H}(K) = \mathcal{F} \oplus \mathcal{F}^\perp$, where $\mathcal{F} := \{f \in \mathcal{H}(K) : f(z) = 0$ for all $z \in F\}$, and $\mathcal{F}^\perp$ is isometrically isomorphic to $\mathcal{H}(L)$ via $f \mapsto f|F$.*

Proof: Note that $\mathcal{F}$ is a closed linear subspace of $\mathcal{H}(K)$, as convergence in $\mathcal{H}(K)$-norm implies pointwise convergence on E. Consequently, $\mathcal{H}(K) = \mathcal{F} \oplus \mathcal{F}^\perp$.

Define a linear map β from $\mathcal{H}(K)$ onto a set $\mathfrak{L}$ say, such that $\beta(f) := f|F$, i.e. $\mathfrak{L}$ is a subset of the set of all functions $F \to \mathbb{C}$. Then $\beta(f) = 0$ if and only if $f \in \mathcal{F}$.

In order to show that $\varphi := \beta|\mathcal{F}^\perp$ is one-to-one, take $f_1, f_2 \in \mathcal{F}^\perp$ such that $\varphi(f_1) = \varphi(f_2)$. Then, on the one hand, $0 = \varphi(f_1 - f_2) = \beta(f_1 - f_2)$ which implies $f_1 - f_2 \in \mathcal{F}$ but, on the other hand, $f_1 - f_2 \in \mathcal{F}^\perp$. Thus $f_1 - f_2 = 0$.

Now define an inner product on $\mathfrak{L}$ by $\langle f , g \rangle_{\mathfrak{L}} := \langle \varphi^{-1}(f) , \varphi^{-1}(g) \rangle_{\mathcal{H}(K)}$. Then $\mathfrak{L}$, with respect to this inner product, is a Hilbert space isometrically isomorphic to $\mathcal{F}^\perp$.

For $z \in F$ let $L_z := \varphi(K_z)$, then

$$\langle L_z , f \rangle_{\mathfrak{L}} = \langle K_z , \varphi^{-1}(f) \rangle_{\mathcal{H}(K)} = f(z) \qquad \text{for all} f \in \mathfrak{L} \text{ and all } z \in F \, .$$

Hence $\mathfrak{L}$ is an RKHS with domain F and kernel $L := K|F \times F$. □

A.3.6 Positivity of $K(z, z)$. Let $\mathcal{H}(K)$ be an RKHS with domain E and suppose there is $E_0 \subset E$ such that $K(z, z) = 0$ for all $z \in E_0$.

Then $|f(z)| = |\langle K_z , f \rangle| \le \|K_z\| \cdot \|f\| = K(z, z) \cdot \|f\| = 0$ for all $z \in E_0$. Thus, if $f \in \mathcal{H}(K)$ vanishes on $F := E \setminus E_0$ it is the zero function. Using the notation of A.3.5, we have $\mathcal{F}^\perp = \mathcal{H}(K)$.

Therefore, without loss of generality, we always assume $K(z, z) > 0$ for all $z \in E$.

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