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Integrated Risk Management When the Stock Price Follows an Exponential Lévy Process

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Abstract

It is well known that the success of an insurance company depends not only on its insurance business, but also on how well the company invests its reserve. The risk such a company faces arises both from potential losses on the financial market and from unexpectedly high insurance claims. Consequently, an integrated risk model incorporating the dynamics of the financial markets and the insurance portfolio as well as the interaction between them is needed.

We consider a stochastic model for the wealth of an insurance company which has the possibility to invest into a risky and a riskless asset under a constant mix strategy. The total insurance claim amount is modeled by a compound Poisson process and the price of the risky asset follows a general exponential Lévy process. We derive the integrated risk process and calculate certain quantities as characteristic functions and moments. We investigate the distribution of the integrated risk process over a fixed time period. We show that this distribution satisfies a partial integro-differential equation and provide a numerical solution to it.

Our main goal is a stable assessment of the capital reserve needed to prevent a negative outcome of the integrated risk process with a high probability. Following long tradition in insurance, we work with discounted losses and investigate the corresponding discounted net loss process. We provide conditions for its stationarity and derive the left and the right tail behaviour of the resulting stationary distribution. This opens up a way to define as a risk measure the Value-at-Risk in the framework of our integrated model. Our results indicate that the model carries a high risk, which may originate either from large insurance claims or from the risky investment. Furthermore, we provide an approximation of the optimal investment strategy, which maximizes the expected wealth of the insurance company under a risk constraint on the Value-at-Risk.

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Contents

Chapter 1

Introduction

1.1 Risk theory

Collective risk theory, as a part of insurance mathematics, has developed over a long period of time but mainly has its roots at the beginning of the 20th century. It was inaugurated by Filip Lundberg [45] who in his thesis of 1903 introduced the collective risk model. Since then Lundberg's model has attracted a lot of attention by mathematicians and actuaries. About 50 years later Harold Cramér [11] incorporated Lundberg's ideas into the theory of stochastic processes.

In the insurance models resulting from these and other contributions, the occurrence of the claims is described by a point process and the amount of money to be paid by the company at each claim by a sequence of random variables. The company receives a certain amount of premium to cover its liability. The difference between the premium income and the average cost of the claims is called safety loading. Furthermore, the company is assumed to have a certain initial capital at its disposal.

The *classical risk model*, referred often to as *the Cramér-Lundberg model* can be described roughly as follows:

1. The point process counting the claims is a Poisson process.

2. The claim sizes are described by a sequence of independent and identically distributed random variables.

3. The point process (1) and the random variables (2) are independent.

4. The premiums are described by a constant and deterministic premium rate.

One important problem in collective risk theory is to investigate the "infinite time ruin probability", i.e. the probability that the insurance risk process ever becomes negative. The famous Cramér-Lundberg Theorem for small claims states that the infinite time ruin probability can be bounded from above by an exponential function of the initial capital with an explicitly given exponent, the so called Lundberg exponent. Important also is the case when the claims have heavier than exponential tail. In such models, the ruin is typically driven by a single large claim, and the infinite time ruin probability can be approximated using the tail behaviour of the claim sizes, see e.g. Embrechts, Klüppelberg and Mikosch [15], Section 1.3.

1.2 Portfolio theory

For over 40 years, the modern portfolio theory (MPT) has been an important portfolio management tool commonly used by the financial institutions. In one sentence, MPT can be described as a set of quantitative methods, designed to help an investor to find the optimal trade-off between a high expected value of the portfolio's returns and a constraint on the portfolio's risk. The earliest approach to this issue was first introduced by Markowitz [47], who received for his work also the Nobel prize in economic sciences in 1990. The Markowitz notion of risk is indeed quite abstract; for the sake of convenience he defined the risk mathematically as the variance of the portfolio's return. The resulting mean-variance portfolio optimization approach is still popular nowadays at the risk departments of the banks, since it can be applied with a basic knowledge on the underlying stochastic models.

The most popular stochastic model describing the development of a financial portfolio is the classical Black-Scholes model (1973). In this model, the investor has the opportunity to invest in a riskless asset (bond) and in (several) risky asset(s) (stock(s)). The following further assumptions are often made:

1. The price of the stock is modeled by a geometric Brownian motion and the bond has a constant interest rate.

2. The portfolio is self-financing, i.e. the investor has a certain initial capital and does not receive any external capital outside of the portfolio.

3. The investor follows the so called constant mix strategy. Under such a strategy the investor holds, at each instant of time, a constant fraction of the wealth in the stock and the rest in the bond.

4. Shortselling is not allowed.

Typically the expected price of the risky asset is higher than the corresponding price of the bond. Therefore, the investor has to look for a trade-off between the (potentially more profitable) risky investment in stocks and the less profitable, but riskless investment in the bond, i.e. to optimize the portfolio according to some risk-aversion criteria. A classical risk-aversion criteria is to set an upper bound on the portfolio's variance, see e.g. Korn [40]. Then the investor may find an investment strategy which maximizes the expected return of the portfolio subject to this risk constraint. Another opportunity, often used in contemporary risk management, is to use a downside risk measure like Value-at-Risk, see e.g. Fishburn [20]. In the framework of the classical Black-Scholes model, however, the mean-variance and the mean-Value-at-Risk optimization problems lead to identical results.

1.3 Integrated risk management

It is well known that the success of an insurance company depends both on the insurance business and on its investment skills. Consequently, one important generalization is to consider a model in which the insurance company, additionally to the linear premium income, has the opportunity to invest its reserve. Such generalizations have been considered since the 90's of the last century and have gained more and more interest in the last few years. The main object of interest, as in classical risk theory, remains the infinite time ruin probability.

One possibility is to consider a model in which the insurer invests its reserve into a bond with a constant interest rate. Such a model is considered for example by Klüppelberg and Stadtmüller [38]. They show that, when the claim distribution has regularly varying tail, the ruin probability decays with the same rate as the tail of the claims, for large initial capital. Asmussen [1] derives the same result in the case when the claim size distribution belongs to the more general set of subexponential distributions. In the case of exponentially fast decreasing tail of the claim size distribution, Sund and Teugels [58] derive bounds for the ruin probability and asymptotics for large initial capital.

A lot of attention attracts the more complicated generalization – allowing for investment in a risky asset, usually modeled by a geometric Brownian motion. In this setting Frolova, Kabanov and Pergamenchtchikov [21] show that for an insurer, who invests a constant fraction of the wealth in the risky asset and when the claims are exponentially distributed, then depending on the model parameters the ruin probability is either 1 for all initial capital reserves or decreases asymptotically for large initial capital like a negative power function. Gjessing and Paulsen [53] and Kalashnikov and Norberg [35] generalize this result for light-tailed claim size distributions (a nice survey can be found also in Paulsen [51]). The case of regularly varying claim size distribution is analysed by Gaier and Grandits [23].

One frequently considered optimization problem in the framework of the integrated risk models is: "what is the minimal ruin probability that the insurer can obtain". Browne [7] investigates first such a problem, but under the assumption that the insurance risk process follows a Brownian motion (the so called 'diffusion approximation'). In this simpler setting, the investment strategy which minimizes the ruin probability consists in holding a constant amount of wealth in the risky asset, and the corresponding minimal ruin probability is given by an exponential function. Hipp and Plum [31] investigate the general problem with the compound Poisson process for an insurance risk model and the geometric Brownian motion for a risky asset model. They derive the corresponding Hamilton-Jacobi-Belman (HJB) equation for the maximal survival probability. This nonlinear second order integro-differential equation is in general hard to solve. Hipp and Plum [31] present a special example with exponentially distributed claims, where the solution can be explicitly calculated. Remarkably, in this example the solution decreases as an exponential function of the initial capital, but with a greater exponent than the classical Lundberg exponent without investment.

Gaier, Grandits and Schachermayer [24] show that in the case of exponential claims the minimal ruin probability for an insurer with a risky investment possibility can be bounded from above and from below by an exponential function with a greater exponent than the classical Lundberg exponent without investment. Further, they show that there exists an investment strategy, which is holding a fixed amount of the wealth invested in the risky asset, such that the corresponding ruin probability is between the derived bounds. This strategy can be calculated explicitly. In the case of regularly varying claim size distribution Gaier and Grandits [23] show that the minimal ruin probability is also a regularly varying function of the initial capital, with the same index as that of the claims.

With respect to the investment model, an important issue is whether the geometric Brownian motion appropriately describes the development of the prices of the risky assets. Many empirical studies of the stock markets indicate that the log returns of various risky assets exhibit a number of features which contradict the normality assumption, like skewness and heavy tails. In fact the empirical distribution of real data is often leptokurtic, which means that there are more values close to the mean than a normal law would suggest and at the same time a lot of extremes, indicating semi-heavy tails (see e.g. Eberlein and Keller [13]). In other words, the prices of many stocks have sudden downward (or upward) jumps, which cannot be explained by the continuous geometric Brownian motion.

One way to handle this problem is to model the price of the risky asset by a more general exponential Lévy process with jumps. In the context of investment portfolio optimization, this approach has been applied by Emmer and Klüppelberg [16]. The Lévy processes retain part of the flexibility of their special case – the Brownian motion. This makes it possible to find explicit solutions to the mean-variance portfolio optimization problem and semi-explicit solutions to the mean-Value-at-Risk problem in Emmer and Klüppelberg [16]. In the context of integrated risk models, Paulsen [52] investigates the asymptotic behaviour for large initial capital of the infinite time ruin probability when the investment process is a general exponential Lévy process. The results indicate that the ruin probability behaves like a Pareto function of the (large) initial capital. The Pareto exponent depends on the interaction between the insurance claims and the investment process.

In this thesis we consider an integrated risk model. The insurerance company invests its reserve both in a bond and in a stock under a constant mix strategy. The risky asset is modeled by a general exponential Lévy process and the bond brings a constant interest rate. In contrast to e.g. Hipp and Plum [31], our integrated portfolio is self-financing and short-selling is not allowed. We call the resulting model for the wealth of the insurance company Integrated Risk Process (IRP).

We derive the characteristic function and the moments of the IRP and investigate its distribution over a fixed time period. We show that this distribution satisfies a partial integro-differential equation and provide a numerical solution to it.

Our main goal is a stable assessment of the capital reserve needed to prevent a negative outcome of the IRP with a high probability. Following long tradition in insurance, we work with discounted losses and investigate the corresponding discounted net loss process. We provide conditions for its stationarity and derive the left and the right tail behaviour of the resulting stationary distribution. This opens up a way to define the risk measure Value-at-Risk in the framework of our integrated model. Our results indicate that the model carries a high risk (heavy tails of the stationary distribution), which may originate either from large insurance claims or from the risky investment. Furthermore, we provide an approximation of the optimal investment strategy, which maximizes the expected wealth of the insurance company under a risk constraint on the Value-at-Risk. We conclude with some illustrating examples.

The outline of the thesis is as follows. We start with some preliminary results on Lévy processes, stochastic calculus and stochastic recurrence equations in Chapter 2. In Chapter 3 we introduce the investment process and the integrated risk process and derive basic properties like characteritic functions and moments. In Chapter 4 we derive a partial integro-differential equation for the integrated risk process over a fixed time period and we solve it numerically. In Chapter 5 we define the discounted net loss process. We investigate its properties and derive conditions for its stationarity. A key result in this chapter is stated in Section 5.3, where we show that the stationary distribution of the discounted net loss process has a Pareto-like tail behaviour. These theoretical results enable the definition and the approximation of the risk measure Value-at-Risk (VaR) which we present in Chapter 6. We indicate the application towards optimal investment and investigate the impact of the different model regimes on the resulting optimal strategy.

Chapter 2

Preliminaries

In this chapter we introduce the most important preliminary results which we are going to apply in the sequel. The covered topics are Lévy processes, stochastic calculus, stochastic recurrence equations and some basics from extreme value theory.

Throughout this thesis we use the following notation. By $\mathbb{N}_0$ we denote the set of the natural numbers plus the zero and $\mathbb{R}^+$ is the set of the positive real numbers. For $a \in \mathbb{R}$ we set $a^+ = \max(0,\, a)$ and $a^- = \max(0,\, -a)$; we also define $\log^+ a = \max(0,\, \log a)$ for $a > 0$. Furthermore, we write $\int_a^b := \int_{(a,b]}$ for $a < b$, $a,\, b \in \mathbb{R}$. We also denote $\lceil x \rceil = \min\{n \in \mathbb{N} : x \leq n\}$ for $x > 0$ and recall that for $x > y$ we estimate $x - y - 1 < \lceil x \rceil - \lceil y \rceil < x - y + 1$.

2.1 Lévy processes

Our aim in this section is to introduce the class of Lévy processes and to state some of their basic properties which will be used in the sequel. For general Lévy process theory we refer to the monographs by Bertoin [4], Cont and Tankov [10] or Sato [56].

Throughout this thesis let $(\Omega,\, \mathcal{F},\, (\mathcal{F}_t)_{t\geq 0},\, P)$ be a filtered complete probability space on which all stochastic quantities are defined. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and all stochastic processes to be defined in this thesis are adapted. We start with the concept of infinite divisibility of a distribution function (df) or of a random variable (rv). For a df F on $[0, \infty)$ we define the n-fold convolution F^{*n} as

$$F^{*n}(x) = \int_0^x F^{*(n-1)}(x-u)\, dF(u)\ , \quad n \geq 1\,,$$

where $F^{*0}(x) = 1$ for $x \geq 0$ and 0 elsewhere. Then the notion of infinite divisibility is as follows.

Definition 2.1.1. *A df F or a real-valued rv X with df F is said to be* infinitely divisible*, if for each $n \in \mathbb{N}$ there is a probability distribution F_n such that $F = F_n^{*n}$ or equivalently $X \stackrel{d}{=} X_1 + \cdots + X_n$, where $(X_j)_{j=1}^n$ are independent and identically distributed (iid) random variables (rv's) with common df F_n.* □

Here "$\stackrel{d}{=}$" means equality in distribution. Assume now that X is an infinitely divisible rv. Then there is a unique continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ such that $\Psi(0) = 0$ and for $s \in \mathbb{R}$,

$$E\left[e^{isX}\right] = e^{\Psi(s)}\,,$$

see for example Sato [56], Section 7. According to the *Lévy-Khintchine formula*, see e.g. Theorem 8.1 in Sato [56], a continuous function Ψ is the logarithm of the characteristic function (chf) of an infinitely divisible rv on the real line if and only if it may be written in the form

$$\Psi(s) = is\gamma - \frac{\sigma^2}{2}s^2 + \int_{-\infty}^{+\infty} \left(e^{isx} - 1 - isx\mathbf{1}_{\{|x|\leq 1\}}\right)\,\nu(dx)\,, \tag{2.1}$$

where $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$ and $\sigma \geq 0$. Here ν is a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\nu(dx) < \infty\,,$$

where $(a \wedge b) = \min(a, b)$. The measure ν is called *Lévy measure.* Note that the term corresponding to $x1_{\{|x|\leq 1\}}$ in (2.1) represents centering without which the integral may not converge.

It is then possible to construct a strong Markov process $L = (L(t))_{t\geq 0}$ with *stationary independent increments* such that $L(0) = 0$ almost surely (a.s.) and for $t \geq 0$, $s \in \mathbb{R}$,

$$E\left[e^{isL(t)}\right] = e^{t\Psi(s)}\,.$$

The process $(L(t))_{t\geq 0}$ is called *Lévy process.* The function Ψ is called *characteristic exponent* and the triplet (γ, σ^2, ν) is referred to as *characteristic triplet* of the Lévy process. The process is cádlág, i.e. the sample path functions belong to the space $\mathbb{D} = \mathbb{D}[0, \infty)$ of real-valued functions on $[0, \infty)$, right-continuous on $[0, \infty)$ with left limits on $(0, \infty)$.

To understand the structure of the general Lévy processes we follow Sato [56], Chapter 4. First for each $\omega \in \Omega$ denote by $\Delta L(t, \omega) = L(t, \omega) - L(t-, \omega)$ the jump of the process L at time $t > 0$. For each Borel set $B \subset [0, \infty) \times \mathbb{R} \setminus \{0\}$ set

$$M(B, \omega) = \#\{(t, \Delta L(t, \omega)) \in B\}\,.$$

Lévy's theory says that M is a Poisson random measure with intensity

$$m(dt, dx) = dt\nu(dx)\,,$$

where ν is the Lévy measure of the process L. Note that m is σ- finite and $M(B, \cdot) = \infty$ a.s. when $m(B) = \infty$.

Now take $B = [t_1, t_2] \times A$, where $0 \leq t_1 < t_2 < \infty$ and A is a Borel set in $\mathbb{R} \setminus \{0\}$. Then

$$M(B, \omega) = \#\{(t, \Delta L(t, \omega))\,:\, t_1 \leq t \leq t_2,\, \Delta L(t, \omega) \in A\}$$

counts the jumps of size in A which happen in the time interval $[t_1, t_2]$. Hence, $M(B, \omega)$ is a Poisson rv with mean $(t_2 - t_1)\nu(A)$.

With this notation, the Lévy-Khintchine representation (2.1) corresponds to the following representation for the Lévy process L for $t \geq 0$:

$$L(t) = \gamma t + \sigma W(t) + \sum_{0<s\leq t} \Delta L(s) 1_{\{|\Delta L(s)>1|\}} + \int_0^t \int_{|x|\leq 1} x(M(ds,\, dx) - \nu(dx)\, ds)\,.$$

In the case of finite variation of the jumps, i.e. when $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, the last representation reduces to

$$L(t) = \gamma_0 t + \sigma W(t) + \sum_{0<s\leq t} \Delta L(s)\,,\ t \geq 0\,,$$

where $\gamma_0 = \gamma - \int_{|x|\leq 1} x\nu(dx)$. This means that in this case L is the independent sum of a drift term, a Gaussian component and a pure jump part represented by a process of finite variation. For instance, standard Brownian motion is obtained if we choose in (2.1) $\gamma = 0$ and $\nu = 0$. A homogeneous compound Poisson process $\sum_{j=1}^{N(t)} Y_j$, $t \geq 0$, with intensity $\lambda > 0$ of the Poisson process N has Lévy measure $\nu(dx) = \lambda F(dx)$, where F is the common df of the iid sequence of rv's $(Y_j)_{j\in\mathbb{N}}$, and constants $\gamma = \sigma = 0$.

The Lévy process L has *finite mean* if $\int_{|x|>1} |x|\,\nu(dx) < \infty$. Then

$$E[L(t)] = \gamma_1 t\,, \quad \text{where} \quad \gamma_1 = \gamma + \int_{\mathbb{R}} x(1 - \mathbf{1}_{\{|x|\leq 1\}})\nu(dx)\,,$$

see for example Sato [56] E25.12, p.163.

We will need also the next lemma. The proof can be found for example in Sato [56], Proposition 11.10.

Lemma 2.1.2. *Let L be a Lévy process with characteristic triplet (γ, σ, ν) and $a \in \mathbb{R}$. Then aL is again a Lévy process with characteristic triplet $(\gamma_a, \sigma_a^2, \nu_a)$ given by*

$$\begin{aligned}
\gamma_a &= a\gamma + \int_{\mathbb{R}} ax\left(\mathbf{1}_{\{|ax|\leq 1\}} - \mathbf{1}_{\{|x|\leq 1\}}\right)\,\nu(dx)\,,\\
\sigma_a^2 &= (a\sigma)^2\,,\\
\nu_a(A) &= \nu\left(\{x\in\mathbb{R} \,:\, ax \in A\}\right) \textit{ for any Borel set } A \in \mathbb{R}\,.
\end{aligned}$$

□

2.2 Itô's formula and stochastic exponential

In this section we give some theorems from stochastic calculus that will be used later. For details see for example Protter [54], Chapter 2, or Cont and Tankov [10], Chapter 8. Throughout the section let X denote a semimartingale and H – a predictable (cádlág) process. We use directly the notion of a *stochastic integral of H with respect to X*,

$$(H\cdot X)_t := \int_0^t H(s)\, dX(s) = \int_{[0,t]} H(s)\, dX(s)\,,\, t\geq 0.$$

We say that two processes Y and Z are indistinguishable if

$$P(\omega : t \to Y(t,\omega) \text{ and } t \to Z(t,\omega) \text{ are the same functions}) = 1\,.$$

The next result can be found for example in Protter [54], Chapter 2, Theorem 13.

Theorem 2.2.1. *The jump process $(\Delta(H\cdot X)_t)_{t\geq 0}$ is indistinguishable from the process $(H(t)(\Delta X(t)))_{t\geq 0}$.*

□

We need also the notion of quadratic (co)variation of a semimartingale. If X and Y are two semimartingales, the *quadratic variation process of* X, denoted $[X,X] = ([X,X]_t)_{t\geq 0}$, is defined by

$$[X,X]_t = X^2(t) - 2\int_0^t X(s-)\,dX(s)\,,$$

and the *quadratic covariation of* X *and* Y, denoted $[X,Y] = ([X,Y]_t)_{t\geq 0}$, is defined by

$$[X,Y]_t = X(t)Y(t) - \int_0^t X(s-)\,dY(s) - \int_0^t Y(s-)\,dX(s)\,,$$

if they exist, see Protter [54], Chapter 2, for details.

Further, we denote by $[X,X]^c$ the path by path continuous part of $[X,X]$. Then we can write for $t > 0$

$$[X,X]_t = X^2(0) + [X,X]^c_t + \sum_{0<s\leq t} (\Delta X_s)^2\,.$$

A semimartingale X is called *quadratic pure jump* if $[X,X]^c = 0$. We need also the following three theorems, for proofs see for example Protter [54], Chapter 2, Section 6.

Theorem 2.2.2. ***Integration by parts formula***
Let X and Y be two semimartingales. Then XY is a semimartingale and

$$d(X(t)Y(t)) = X(t-)dY(t) + Y(t-)dX(t) + d[X,Y]_t\,, \quad t > 0\,.$$

□

Theorem 2.2.3. *Let X be a quadratic pure jump semimartingale. Then for every semimartingale Y we have*

$$[X,Y]_t = X(0)Y(0) + \sum_{0<s\leq t} \Delta X(s)\Delta Y(s)\,.$$

□

Theorem 2.2.4. *Let X and Y be two semimartingales, and let H and K be two predictable processes. Then*

$$[H\cdot X, K\cdot Y]_t = \int_0^t H(s)K(s)\,d[X,Y]_s\,,\ \ t\geq 0\,,$$

and, in particular,

$$[H\cdot X, H\cdot X]_t = \int_0^t H(s)^2\,d[X,X]_s\,,\ \ t\geq 0\,.$$

□

Now we state Itô's formula. We give also its general version for multidimensional semimartingales.

Proposition 2.2.5. ***Itô's formula for multidimensional semimartingales***
Let $X = (X_1, \ldots, X_d)$ be a d-dimensional semimartingale and let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function. Then $f(X)$ is again a semimartingale, and the following formula holds:

$$\begin{aligned} f(X(t)) - f(X(0)) &= \sum_{j=1}^{d} \int_0^t \frac{\partial f}{\partial x_j}(X(s-))\, dX_j(s) + \frac{1}{2} \int_{0+}^t \sum_{j,k=1}^{d} \frac{\partial^2 f}{\partial x_j \partial x_k}(X(s))\, d[X_j,\, X_k]_s^c \\ &\quad + \sum_{0<s\le t} \left[f(X(s)) - f(X(s-)) - \sum_{j=1}^{d} \Delta X_j(s) \frac{\partial f}{\partial x_j}(X_j(s-)) \right] . \quad (2.2) \end{aligned}$$

Proof. See for example Jacod and Shiryaev [34], 4.57 and 4.58. For $d = 1$ see Protter [54], Chapter 2, Section 7, Theorem 32. □

For a one-dimensional Lévy process $L = (L(t))_{t\ge 0}$ with characteristic triplet (γ, σ^2, ν) we have that $d[L,\, L]_s^c = \sigma^2 ds$, hence (2.2) reduces to (2.3) in the next proposition.

Proposition 2.2.6. ***Itô's formula for Lévy processes***
Let $(L(t))_{t\ge 0}$ be a one-dimensional Lévy process with characteristic triplet (γ, σ^2, ν) and $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then

$$\begin{aligned} f(L(t)) &= f(0) + \frac{\sigma^2}{2} \int_{0+}^t f''(L(s))\, ds + \int_0^t f'(L(s-))\, dL(s) \\ &\quad + \sum_{0<s\le t} \left[f(L(s)) - f(L(s-)) - \Delta L(s) f'(L(s-)) \right] . \quad (2.3) \end{aligned}$$

□

To the end of this section we will consider Lévy processes, but everything can be done also for semimartingales.

Let L be a Lévy process with characteristic triplet (γ, σ^2, ν) and define the stochastic process $X = (X(t))_{t\ge 0}$ by

$$X(t) = e^{L(t)}\,, \;\; t \ge 0\,,$$

i.e. X is the (ordinary) exponential of L. Then by Itô's formula we have

$$dX(t) = X(t-) \left(dL(t) + \frac{\sigma^2}{2} dt + \exp(\Delta L(t)) - 1 - \Delta L(t) \right) .$$

Now the question arises how does the solution to the more simple, yet important and non-trivial, stochastic differential equation – the above stochastic differential equation without the Itô term $\frac{\sigma^2}{2} dt + \exp(\Delta L(t)) - 1 - \Delta L(t)$ – differ from the process X. The next theorem, which is an application of Itô's formula, answers this question.

Theorem 2.2.7. *Let $L = (L(t))_{t\geq 0}$ be a Lévy process with characteristic triplet (γ, σ^2, ν). Then there exists an unique cádlág process $Z = (Z(t))_{t\geq 0}$ such that*

$$dZ(t) = Z(t-)dL(t)\,, \;\; Z(0) = 1\,,$$

and Z is given by

$$Z(t) = e^{L(t)-\frac{\sigma^2}{2}t} \prod_{0<s\leq t} (1 + \Delta L(s))e^{-\Delta L(s)}\,, \;\; t \geq 0\,. \tag{2.4}$$

Further, if $\int_{|x|\leq 1} |x|\, \nu(dx) < \infty$, i.e. the jumps of L have finite variation, then

$$Z(t) = e^{L^c(t)-\frac{\sigma^2}{2}t} \prod_{0<s\leq t} (1 + \Delta L(s))\,, \;\; t \geq 0\,,$$

where L^c denotes the continuous part of L.

Proof. See for example Cont and Tankov [10], Proposition 8.21.

□

The process Z in Theorem 2.2.7 is called *the stochastic exponential* or *the Doléans-Dade exponential of* L and is denoted by $Z = \mathcal{E}(L)$. Notice that one can define it for an arbitrary semimartingale, not only for a Lévy process.

From the last theorem it is clear that the ordinary exponential and the stochastic exponential of a Lévy process are two different notions, corresponding to two different stochastic processes. In fact, contrarily to the ordinary exponential, which is always a positive process, the stochastic exponential is not necessarily positive. Indeed, from (2.4) follows that the process Z is always non-negative only if $\Delta L(t) > -1$, for each $t \geq 0$, or equivalently, $\nu((-\infty, -1]) = 0$. The next result, due to Goll and Kallsen [30], show that if $Z > 0$ is the stochastic exponential of some Lévy process, then it is also the ordinary exponential of another Lévy process and vice versa.

Proposition 2.2.8. *(a) Let $L = (L(t))_{t\geq 0}$ be a real-valued Lévy process with characteristic triplet (γ, σ^2, ν). Then there exists a Lévy process $\widehat{L} = (\widehat{L}(t))_{t\geq 0}$ such that $\exp(L(t)) = \mathcal{E}(\widehat{L}(t))$, $t \geq 0$, and $\widehat{L}$ has characteristic triplet $(\widehat{\gamma}, \widehat{\sigma}^2, \widehat{\nu})$ given by*

$$\begin{aligned} \widehat{\gamma} &= \gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left((e^x - 1)\mathbf{1}_{\{|e^x-1|\leq 1\}} - x\mathbf{1}_{\{|x|\leq 1\}} \right) \nu(dx)\,, \\ \widehat{\sigma}^2 &= \sigma^2\,, \\ \widehat{\nu}(A) &= \nu\left(\{x \in \mathbb{R} \,:\, e^x - 1 \in A\}\right) \textit{ for every Borel set } A \in \mathbb{R}\,. \end{aligned}$$

(b) Conversely, let $\widehat{L}$ be a real-valued Lévy process with characteristic triplet $(\widehat{\gamma}, \widehat{\sigma}^2, \widehat{\nu})$. Then there exists a Lévy process L such that $\mathcal{E}(\widehat{L}(t)) = \exp(L(t))$, $t \geq 0$, and L has characteristic triplet (γ, σ^2, ν) given by

$$\begin{aligned} \gamma &= \widehat{\gamma} - \frac{\widehat{\sigma}^2}{2} + \int_{\mathbb{R}} \left(\log(1+x)\mathbf{1}_{\{|\log(1+x)|<1\}} - x\mathbf{1}_{\{|x|<1\}} \right) \widehat{\nu}(dx)\,, \\ \sigma^2 &= \widehat{\sigma}^2\,, \\ \nu(A) &= \widehat{\nu}\left(\{x \in \mathbb{R} \,:\, \log(1+x) \in A\}\right) \textit{ for every Borel set } A \in \mathbb{R}\,. \end{aligned}$$

□

We also need the following lemma. The proof can be found for example in Cont and Tankov [10], Lemma 15.1.

Lemma 2.2.9. *For every $t > 0$ and left continuous function $f : [0, t] \to \mathbb{R}$ and a Lévy process Z with characteristic exponent Ψ_Z, holds*

$$E\left[\exp\left(i\int_0^t f(v)\,dZ(v)\right)\right] = \exp\left(\int_0^t \Psi_Z(f(v))\,dv\right). \tag{2.5}$$

□

2.3 Stochastic recurrence equations

The aim of this section is to give the concepts and some results for stochastic recurrence equations, connected with the discrete time accumulation and discounting techniques, that we will need in the sequel. More background is to be found in Embrechts, Klüppelberg and Mikosch [15], Section 8.4 or in Goldie and Grübel [28].

We start with one of the most popular stochastic recurrence equations, namely the *forward stochastic recurrence equation (FSRE)* given by

$$X_k = X_0 \prod_{j=1}^{k} B_j + \sum_{m=1}^{k} A_m \prod_{j=m+1}^{k} B_j\,, \quad k \in \mathbb{N}\,, \tag{2.6}$$

where X_0 is a rv and $((A_k, B_k))_{k\in\mathbb{N}}$ is a sequence of iid bivariate rv's. This FSRE can be thought of successively applying the *random affine mapping* $\phi_k(x) = A_k + B_k x$, such that $X_k = \phi_k(X_{k-1})$, $k \in \mathbb{N}$. The latter relation is also called "outer iteration"; see e.g. Embrechts and Goldie [14]. In the insurance context, $(X_k)_{k\in\mathbb{N}}$ as defined by (2.6) can be interpreted as the value of a *perpetuity*: the payments A_k are made at the beginning of each period and the accumulated payments X_{k-1} are subject to interest.

The reverse stochastic recurrence equation related to the discounting problem is the *backward stochastic recurrence equation (BSRE)* given by

$$Y_k = Y_0 \prod_{j=1}^{k} C_j + \sum_{m=1}^{k} A'_m \prod_{j=1}^{m-1} C_j\,, \quad k \in \mathbb{N}\,, \tag{2.7}$$

where Y_0 is a rv and $((A'_k, C_k))_{k\in\mathbb{N}}$ a sequence of iid bivariate rv's. For the interpretation of (2.7) as an "inner iteration" of random affine mapping see Embrechts and Goldie [14]. Note that the value Y_0, which may be viewed as the final (at time k) down-payment, is unimportant when we are interested in the behaviour of Y_k for large k. In fact, under weak assumptions, the first term in (2.7) can be shown to converge to 0 a.s. as k goes to infinity.

When we look at the form of (2.6) and (2.7), we conclude that the structure of the discounted and the accumulated sequences $(Y_k)_{k\in\mathbb{N}}$ and $(X_k)_{k\in\mathbb{N}}$ is very similar. This also concerns the distribution of these rv's. Assume that X_0 is independent of the iid sequence

$((A_k,\, B_k))_{k\in\mathbb{N}}$ and Y_0 is independent of the iid sequence $((A'_k,\, C_k))_{k\in\mathbb{N}}$. Now observe that for every $k \in \mathbb{N}$

$$(X_0,\, ((A_j,\, B_j))_{1\le j\le k}) \stackrel{d}{=} (X_0,\, ((A_{k-j+1},\, B_{k-j+1}))_{1\le j\le k}) \;,$$

which implies that

$$\begin{aligned} X_k &= X_0 \prod_{j=1}^{k} B_j + \sum_{m=1}^{k} A_m \prod_{j=m+1}^{k} B_j \\ &\stackrel{d}{=} X_0 \prod_{j=1}^{k} B_j + \sum_{m=1}^{k} A_m \prod_{j=1}^{m-1} B_j \,. \end{aligned}$$

From this immediately follows, that if $X_0 \stackrel{d}{=} Y_0$ and $(A_1,\, B_1) \stackrel{d}{=} (A'_1,\, C_1)$, then

$$Y_k \stackrel{d}{=} X_k \,, \;\; k \in \mathbb{N} \,.$$

Throughout this section we assume that $X_0 = 0$ and that we have an iid sequence of bivariate rv's $((A_k,\, B_k))_{k\in\mathbb{N}}$. Further we denote by $(U_k)_{k\in\mathbb{N}}$ the sequence defined by the BSRE associated to the FSRE (2.6), i.e.

$$U_k = \sum_{m=1}^{k} A_m \prod_{j=1}^{m-1} B_j \,, \;\; k \in \mathbb{N} \,. \tag{2.8}$$

From the discussion above it follows that for all $k \in \mathbb{N}$ every statement about the distribution of X_k is also about the distribution of U_k. In the next proposition, we consider the stationarity of $(U_k)_{k\in\mathbb{N}}$.

Proposition 2.3.1. *Let $(U_k)_{k\in\mathbb{N}}$ be the stochastic process defined by (2.8) and assume that*

$$E\left[\log^+ |A|\right] < \infty \quad \text{and} \quad -\infty \le E\left[\log |B|\right] < 0 \,. \tag{2.9}$$

Then $U_k \stackrel{\text{a.s.}}{\to} U$ when $k \to \infty$ for some rv U, where

$$U = \sum_{m=1}^{\infty} A_m \prod_{j=1}^{m-1} B_j \,. \tag{2.10}$$

The rhs of (2.10) converges absolutely with probability 1. Moreover, the rv U satisfies the identity in law

$$U \stackrel{d}{=} A + BU \,, \tag{2.11}$$

where U and (A, B) are independent .

Proof. The proof follows the proof of Proposition 8.4.3 (a) and (b) in Embrechts, Klüppelberg and Mikosch [15], Section 8.4.1. Note that in our case we have set $X_0 = 0$.

First we have to show a.s. convergence of the series on the rhs of (2.10). By the SLLN and (2.9) we have:

$k^{-1}\sum_{j=1}^{k}\log|B_j| \overset{\text{a.s.}}{\to} E\log|B| < 0\,,\ k\to\infty;$
$m^{-1}\sum_{k=1}^{m}\log^+|A_k| \overset{\text{a.s.}}{\to} E\log^+|A| < \infty\,,\ m\to\infty$, hence $m^{-1}\log^+|A_m| \overset{\text{a.s.}}{\to} 0\,,\ m\to\infty\,.$
Then we can write

$$\left|A_m\prod_{j=1}^{m-1}B_j\right| \le \exp\left(m\left(\frac{1}{m}\log^+|A_m| + \frac{1}{m}\sum_{j=1}^{m-1}\log|B_j|\right)\right) \le e^{-am}$$

for large m with probability 1, where $a\in(0,|E[\log|B|]|)$. This assures that the series under consideration converges a.s., i.e. we have that

$$U_k \overset{\text{a.s.}}{\to} U\,,\ \ k\to\infty\,,$$

where $U=\sum_{m=1}^{\infty}A_m\prod_{j=1}^{m-1}B_j$ is a finite rv.

Now as $X_k \overset{d}{=} U_k$ we have that $X_k \overset{d}{\to} U$, $k\to\infty$. Moreover, for $k\in\mathbb{N}$ we have that $X_k = A_k + X_{k-1}B_k$, with X_{k-1} independent of $(A_k,\,B_k)$, hence

$$(A_k,\,B_k,\,X_k) \quad \overset{d}{\to} \quad (A,\,B,\,U)\,,\ k\to\infty\,,$$

with $(A,\,B)$ independent of U. This, together with the continuous mapping theorem, implies (2.11). □

Now let the rv U satisfy the random equation

$$U \overset{d}{=} A + BU\,, \tag{2.12}$$

where the bivariate rv $(A,\,B)$ is independent of U. The next results, which we are going to use in the sequel, are due to Goldie [27]. For the more general multivariate case we refer to Kesten [36]. The treatment is expository. We start with some preliminary results on the rv B.

Lemma 2.3.2. *[Goldie [27], Lemma 2.2]*
Let B be a rv such that, for some $\kappa>0$, $E[B^\kappa]=1$, $E\left[|B|^\kappa\log^+|B|\right]<\infty$, and the conditional law of $\log|B|$ given $B\neq 0$ is nonarithmetic, i.e. is not concentrated on $\{nh:\,n\in\mathbb{Z}\}$ for some $h>0$. Then

$$-\infty \le E[\log|B|] < 0\,,$$

and

$$m := E[|B|^\kappa\log|B|] \in (0,\,\infty)\,. \tag{2.13}$$

□

The next theorem gives the tail behaviour of the solution to the random equation (2.12).

Theorem 2.3.3. *[Goldie [27]], Theorem 4.1]*
Let A and B be rv's and suppose that B satisfies the conditions of Lemma 2.3.2 and that

$$E\left[|A|^\kappa\right] < \infty\,.$$

Then there is a unique law for U satisfying (2.12) such that

$$P(U > x) \sim C_+ x^{-\kappa}\,, \;\; x \to \infty\,,$$

$$P(U < -x) \sim C_- x^{-\kappa}\,, \;\; x \to \infty\,,$$

where if $B \geq 0$ a.s. then

$$C_+ = \frac{1}{\kappa m} E\left[\left((A+UB)^+\right)^\kappa - \left((UB)^+\right)^\kappa\right]\,,$$

$$C_- = \frac{1}{\kappa m} E\left[\left((A+UB)^-\right)^\kappa - \left((UB)^-\right)^\kappa\right]\,,$$

where m is given in (2.13). Moreover, $C_+ + C_- > 0$ if and only if for each fixed $c \in \mathbb{R}$, $P(A = (1-B)c) < 1$.

□

Apart from the formulae for C_+ and C_-, this is Theorem 5 in Kesten [36]. The concrete form of C_+ and C_- is due to Goldie [27]. We continue with a result quantifying the rate of approach of the functions $x^\kappa P(U > x)$ and $x^\kappa P(U < -x)$ to C_+ and C_-, respectively.

Theorem 2.3.4. *[Goldie [27]], Theorem 4.7]*
Let A and B be rv's and suppose that B satisfies the conditions of Lemma 2.3.2. Let $B \geq 0$ a.s. and let for some $\gamma > 0$

$$E\left[|A|^{\kappa+\gamma}\right] < \infty\,, \quad E\left[B^{\kappa+\gamma}\right] < \infty\,.$$

Suppose also that B satisfies the technical conditions in Theorem 3.1. in Goldie [27]. Then there is a unique law for U satisfying (2.12) and some $\beta \in (0,\gamma)$ such that

$$P(U > x) \sim C_+ x^{-\kappa} + O(x^{-(\kappa+\beta/2)})\,, \;\; x \to \infty\,,$$

$$P(U < -x) \sim C_- x^{-\kappa} + O(x^{-(\kappa+\beta/2)})\,, \;\; x \to \infty\,,$$

where C_+ and C_- are as in Theorem 2.3.3.

□

The result is a direct consequence of Theorem 4.7 in Goldie [27], where it is possible to choose the parameter $\beta > 0$ in Theorem 3.2 of Goldie [27] small enough such that the contour integral in formulae (3.8) and (3.10) vanishes (note that we can always choose the same β for the left and right tail). For more details see also the discussions after Theorems 3.1 and 3.2 in Goldie [27]. The conditions of Theorems 3.2 are satisfied: (3.3) and (3.4) hold by the choice of β and the assumtions in Theorem 2.3.4.

2.4 Extreme value theory

The concept of regular variation plays a crucial role in the study of extreme events. In this section we give some classical results from the one-dimensional extreme value theory, see Embrechts, Klüppelberg and Mickosch [15] for a detailed exposition. We first recall the definition of regular variation for one-dimensional rv's.

Definition 2.4.1. *The (non-degenerate) rv X is said to be* regularly varying *with* tail index $\alpha > 0$ *if for all* $x > 0$

$$\lim_{t\to\infty} \frac{P(X > tx)}{P(X > t)} = x^{-\alpha} .$$

□

In the one-dimensional case, the extremal behaviour of a sequence of rv's can be illustrated by the behaviour of their maxima. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of rv's and denote by $M_n = \max(X_1, \dots, X_n)$. The following result is the basis of the classical extreme value theory.

Theorem 2.4.2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of iid rv's with non-degenerate df. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$, $n \in \mathbb{N}$, and some non-degenerate rv M such that*

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} M \, , n \to \infty \ , \tag{2.14}$$

then the df of M belongs to the type of one of the following three df's:

Frechet*:* $\Phi_\alpha(x) = \begin{cases} 0 & x \le 0 \, , \\ \exp\left(-x^{-\alpha}\right) & x > 0 \, , \quad \alpha > 0 \, ; \end{cases}$

Weibull*:* $\Psi_\alpha(x) = \begin{cases} \exp\left(-(-x)^{\alpha}\right) & x \le 0 \, , \quad \alpha > 0 \, , \\ 1 & x > 0 \, ; \end{cases}$

Gumbel*:* $\Lambda(x) = \exp\left(-e^{-x}\right) \ x \in \mathbb{R}$. □

Details of the proof are for instance to be found in Resnick [55], Proposition 0.3. The three types of df's in Theorem 2.4.2 are called *extreme value distributions.*

Definition 2.4.3. *The df of the rv X is said to belong to the* maximum domain of attractition *of the extreme value distribution H, if there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that (2.14) holds and M has df H.* □

The concept of regular variation is crucial when one has to determine to which of the three types of extreme value df's converges the centered and normalized maxima of an iid sequence of rv's with a given df F, i.e. the domain of attraction of F.

Of particular interest in financial applications are the distributions in the domain of attraction of the Frechet distribution, see for instance Embrechts, Klüppelberg and Mickosch [15], Chapter 6. The next proposition charaterizes the distributions in this domain.

Proposition 2.4.4. *The df F belongs to the maximim domain of attraction of Φ_α, $\alpha > 0$, if and only if for the tail $\overline{F} = 1 - F$ holds*

$$\overline{F}(x) = x^{-\alpha} l(x)\,, \;\; x > 0\,,$$

where l is a slowly varying function, i.e. for all $x > 0$

$$\lim_{t\to\infty} \frac{l(tx)}{l(t)} = 1\,.$$

□

For a proof see Embrechts, Klüppelberg and Mickosch [15], Theorem 3.3.7. If the rv X has df F in the domain of attraction of Φ_α, then X is regularly varying and its tail decreases quite slowly (roughly said at a polynomial rate). Note that this implies, for instance, that $E[X^\beta] = \infty$ for every $\beta \geq \alpha$. Thus, X is a 'heavy-tailed' rv. Heavy-tailed features are often to be seen in financial and insurance data. The extreme value theory provides a set of methods for detecting such features and approximating the tails and the quantiles of the rv's. The next proposition concerns the asymptocis behaviour of the quantiles of the heavy-tailed rv's.

Proposition 2.4.5. *[Bingham, Goldie and Teugels [5], Theorem 1.5.12]*
Let the df F belong to the maximum domain of attraction of Φ_α, $\alpha > 0$. Then for its generalized inverse function $F^{\leftarrow}(x) = \inf\{y \in \mathbb{R} : \overline{F}(y) \leq x\}$, $x \in (0,1)$, holds

$$F^{\leftarrow}(x) = (1-x)^{-1/\alpha} L^{\leftarrow}(1/(1-x)), \;\; x \in (0,1)\,,$$

where $L^{\leftarrow}$ is a slowly varying function. □

For details of $L^{\leftarrow}$ see Bingham, Goldie and Teugels [5].

Chapter 3

The model

In this chapter we introduce the main model in the thesis. We start with the classical insurance model in Section 3.1. The investment process is introduced in Section 3.2 and its key properties are investigated. The integrated risk process is defined in Section 3.3. Basic quantities related to the integrated risk process like characteristic function and moment functions are derived in Section 3.4. We give some examples in Section 3.5.

3.1 Insurance model

In this section we define the *insurance risk reserve process* $U = (U(t))_{t\geq 0}$ as in the Cramér-Lundberg model by

$$U(t) = u + ct - S(t)\,, \quad t \geq 0\,, \tag{3.1}$$

where $u > 0$ is the *initial capital reserve,* $c > 0$ is the constant *premium rate* and the process $S = (S(t))_{t\geq 0}$ is the *total claim amount process*, defined by the compound Poisson process $S(t) = \sum_{j=1}^{N(t)} Y_j$, $t \geq 0$. The claim sizes $(Y_j)_{j\in\mathbb{N}}$ are positive iid rv's with common df F and finite mean μ. The claims arrive at random time points $0 < T_1 < T_2 < \ldots$ and the *claim arrival process* $N = (N(t))_{t\geq 0}$ defined by

$$N(t) \;=\; \begin{cases} \#\{k \geq 1 : T_k \leq t\} & t > 0\,, \\ 0 & t = 0\,, \end{cases}$$

is a homogeneous Poisson process with intensity $\lambda > 0$. Finally, N and $(Y_j)_{j\in\mathbb{N}}$ are independent processes.

Such models are very well studied, see, for example, Asmussen [2], Chapter 3, or Embrechts, Klüppelberg and Mikosch [15], Chapter 1.

Note that $E[U(t)] = u + (c - \lambda\mu)t$, $t \geq 0$, hence when the time goes to infinity, the expectataion of the risk reserve tends to $-\infty$ or $+\infty$ depending on the sign of the difference $c - \lambda\mu$. It is natural to assume that the positive safety loading condition holds, i.e. $c - \lambda\mu > 0$, see e.g. Grandell [25], Chapter 1.

3.2 Investment model

The classical risk model, introduced in the previous section, is extended by allowing for investment of the risk reserve. We consider an insurance company investing into a Black-Scholes type market consisting of a *bond* and some *stock*, modeled by an exponential Lévy process. Their respective price processes follow the equations

$$X_0(t) = e^{\delta t} \quad \text{and} \quad X_1(t) = e^{L(t)}\,, \quad t \geq 0\,. \tag{3.2}$$

The constant $\delta > 0$ is the *riskless interest rate*. The process $L = (L(t))_{t\geq 0}$ is a Lévy process with characteristic triplet (γ, σ^2, ν) and characteristic exponent Ψ, i.e. for $t \geq 0$

$$E[e^{isL(t)}] = e^{t\Psi(s)}\,, \;\; s \in \mathbb{R}\,,$$

where Ψ has Lévy-Khintchine representation as given in (2.1).

For allocation of the reserve among the riskless and the risky asset we use the so-called *constant mix strategy;* i.e. the initial proportions which are invested into bond and stock remain constant over a predetermined planning horizon; see e.g. Emmer, Klüppelberg and Korn [17], Section 2. Such a strategy is dynamic in the sense that it requires at any instance of time a rebalancing of the portfolio depending on the corresponding price changes. We denote by $\theta \in [0,1]$ the fraction of the reserve invested into the risky asset; we call θ the *investment strategy*.

To derive the investment process we follow the calculations in Emmer and Klüppelberg [16] and Emmer, Klüppelberg and Korn [17]. We state first the corresponding stochastic differetial equations (SDEs) for the price processes, where we use Itô's formula, see Proposition 2.2.6,

$$\begin{aligned}
dX_0(t) &= \delta X_0\,(t)\,dt\,, \quad t > 0\,, \quad X_0(0) = 1\,,\\
dX_1(t) &= X_1(t-)\,d\widehat{L}(t)\\
&= X_1(t-)\left(dL(t) + \frac{\sigma^2}{2}\,dt + e^{\Delta L(t)} - 1 - \Delta L(t)\right)\,, \;\; t > 0\,, \;\; X_1(0) = 1\,,
\end{aligned}$$

where $\Delta L(t,\omega) = L(t,\omega) - L(t-,\omega)$ for each $\omega \in \Omega$ denotes the jump of L at time $t > 0$. The process $\widehat{L}$ is such that $e^{L(t)} = \mathcal{E}(\widehat{L}(t))$, $t \geq 0$, where $\mathcal{E}$ denotes the stochastic exponential of a process, see Section 2.2.

Definition 3.2.1. *For $\theta \in [0,1]$ we define the* investment process $X_\theta = (X_\theta(t))_{t\geq 0}$ *as the solution to the SDE*

$$dX_\theta(t) = X_\theta(t-)\left((1-\theta)\delta dt + \theta d\widehat{L}(t)\right)\,, \;\; t > 0\,, \;\; X_\theta(0) = 1\,, \tag{3.3}$$

where δ is the riskless interest rate, $\mathcal{E}(\widehat{L}) = e^L$, and L is the Lévy process describing the log returns of the risky asset given in (3.2).

□

As $\theta \leq 1$, this approach is based on self-financing portfolios and hence classical in financial portfolio optimization; see Korn [40], Section 2.1. The following is a consequence of Itô's formula.

Lemma 3.2.2. *The SDE* (3.3) *has the solution*

$$X_\theta(t) \;=\; \mathcal{E}(\widehat{L}_\theta(t)) = e^{L_\theta(t)}\,, \quad t \geq 0\,, \tag{3.4}$$

where $\widehat{L}_\theta(t) = (1-\theta)\delta t + \theta\widehat{L}(t)$ *and the process* $L_\theta = (L_\theta(t))_{t\geq 0}$ *is such that* $\mathcal{E}(\widehat{L}_\theta(t)) = e^{L_\theta(t)}$, $t \geq 0$.

In the next lemma we specify the characteristic triplet of the resulting Lévy process L_θ in (3.4) in terms of the characteristic triplet of the original Lévy process L for the log ruturns of the risky asset.

Lemma 3.2.3. *The process* L_θ *in (3.4) is a Lévy process with characteristic exponent* Ψ_θ *and the characteristic triplet* $(\gamma_\theta, \sigma_\theta^2, \nu_\theta)$ *is given by*

$$\begin{aligned}
\gamma_\theta &= \gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) \\
&\quad + \int_{\mathbb{R}} (\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\leq 1\}} - \theta x 1_{\{|x|\leq 1\}})\nu(dx)\,, \\
\sigma_\theta^2 &= (\sigma\theta)^2\,, \\
\nu_\theta(A) &= \nu\left(\{x\in\mathbb{R} \,:\, \log(1+\theta(e^x-1)) \in A\}\right) \text{ for every Borel set } A \subset \mathbb{R}\,.
\end{aligned}$$

Proof. Denote by $(\widehat{\gamma},\, \widehat{\sigma}^2,\, \widehat{\nu})$ and by $(\widehat{\gamma}_\theta,\, \widehat{\sigma}_\theta^2,\, \widehat{\nu}_\theta)$ the characteristic triplets of the Lévy processes $\widehat{L}$ and $\theta\widehat{L}$, respectively. By Lemma 2.1.2 and Proposition 2.2.8(a) we obtain the drift term of $\theta\widehat{L}$:

$$\begin{aligned}
\widehat{\gamma}_\theta &= \widehat{\gamma}\theta + \int_{\mathbb{R}} \theta x \left(\mathbf{1}_{\{|\theta x|\leq 1\}} - \mathbf{1}_{\{|x|\leq 1\}}\right) \widehat{\nu}(dx) \\
&= \left(\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left((e^x-1)\mathbf{1}_{\{|e^x-1|\leq 1\}} - x\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(dx)\right)\theta \\
&\quad + \int_{\mathbb{R}} \theta(e^x-1)\left(\mathbf{1}_{\{|\theta(e^x-1)|\leq 1\}} - \mathbf{1}_{\{|e^x-1|\leq 1\}}\right)\nu(dx) \\
&= \theta\left(\gamma + \frac{\sigma^2}{2}\right) + \int_{\mathbb{R}} \left(\theta(e^x-1)\mathbf{1}_{\{|\theta(e^x-1)|\leq 1\}} - \theta x\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(dx)\,.
\end{aligned}$$

For the Gausian component and the Lévy measure of $\theta\widehat{L}$ we get

$$\begin{aligned}
\widehat{\sigma}_\theta^2 &= (\widehat{\sigma}\theta)^2 = (\sigma\theta)^2\,, \\
\widehat{\nu}_\theta(A) &= \widehat{\nu}(\{x\in\mathbb{R} \,:\, \theta x\in A\}) = \nu(\{x\in\mathbb{R} \,:\, \theta(e^x-1)\in A\})\,, \text{ for any Borel set } A\,.
\end{aligned}$$

Finally by (3.3) and (3.4) and applying Propositon 2.2.8(b) for the characteristic triplet of L_θ we get

$$\begin{aligned}
\gamma_\theta &= \widehat{\gamma}_\theta + (1-\theta)\delta - \frac{\widehat{\sigma}_\theta^2}{2} + \int_{\mathbb{R}} \left(\log(1+x)\mathbf{1}_{\{|\log(1+x)|\le 1\}} - x\mathbf{1}_{\{|x|\le 1\}}\right)\widehat{\nu}_\theta(dx) \\
&= \gamma\theta + \frac{\sigma^2}{2}\theta + \int_{\mathbb{R}} \left(\theta(e^x-1)\mathbf{1}_{\{|\theta(e^x-1)|\le 1\}} - \theta x\mathbf{1}_{\{|x|\le 1\}}\right)\nu(dx) + (1-\theta)\delta - \frac{\sigma^2}{2}\theta^2 \\
&\quad + \int_{\mathbb{R}} \left(\log(1+\theta(e^x-1))\mathbf{1}_{\{|\log(1+\theta(e^x-1))|\le 1\}} - \theta(e^x-1)\mathbf{1}_{\{|\theta(e^x-1)|\le 1\}}\right)\nu(dx) \\
&= (1-\theta)\delta + \theta(\gamma + \frac{\sigma^2}{2}) - \frac{\sigma^2}{2}\theta^2 \\
&\quad + \int_{\mathbb{R}} \left(\log(1+\theta(e^x-1))\mathbf{1}_{\{|\log(1+\theta(e^x-1))|\le 1\}} - \theta x\mathbf{1}_{\{|x|\le 1\}}\right)\nu(dx)\,; \\
\sigma_\theta^2 &= (\sigma\theta)^2\,; \\
\nu_\theta &= \nu(\{x\in\mathbb{R} : \log(1+\theta(e^x-1))\in A\})\,, \ \text{for any Borel set } A\,.
\end{aligned}$$

□

Remark 3.2.4. (i) Besides the characteristic exponents Ψ and Ψ_θ of the processes L and L_θ respectively, we shall also need the *Laplace exponents* given by

$$\varphi(s) = \Psi(is) = \log E[e^{-sL(1)}] \quad \text{and} \quad \varphi_\theta(s) = \Psi_\theta(is) = \log E[e^{-sL_\theta(1)}]\,, \tag{3.5}$$

provided they exist. If $\varphi(s) < \infty$, then $E\left[\exp(-sL(t))\right] = \exp(t\varphi(s)) < \infty$ for all $t \ge 0$, see Sato [56], Theorem 25.17. As we show in Lemma 3.2.5(c), $\varphi_\theta(s) < \infty$ for all $\theta \in [0,1]$ provided $\varphi(s) < \infty$.

(ii) A jump of size ΔL of L leads to a jump of size $\exp(\Delta L)-1$ of $\widehat{L}$ and to a jump of size $\Delta L_\theta = \log(1+\theta(e^{\Delta L}-1)) > \log(1-\theta)$ of L_θ. In other words, ν_θ is the image measure of ν under the transformation $x \mapsto \log(1+\theta(e^x-1))$. This explains the requirement $\theta \le 1$.

(iii) If L is a process with finite variation of the jumps, i.e. $\int_{|x|\le 1}|x|\nu(dx) < \infty$, then L_θ is as well. Indeed,

$$\begin{aligned}
\int_{|x|\le 1}|x|\nu_\theta(dx) &= \int_{|\log(1+\theta(e^x-1))|\le 1} |\log(1+\theta(e^x-1))|\nu(dx) \\
&\le \int_{-\infty}^{-1} |\log(1+\theta(e^x-1))|\nu(dx) + \int_{-1}^{p} |\log(1+\theta(e^x-1))|\nu(dx)\,,
\end{aligned}$$

where $p = \log(1+\theta^{-1}(e-1)) > 0$. Then

$$\int_{-\infty}^{-1} |\log(1+\theta(e^x-1))|\nu(dx) \le |\log(1-\theta)| \int_{-\infty}^{-1}\nu(dx) < \infty$$

and, because of the finite variation of L, also

$$\int_{-1}^{p} |\log(1+\theta(e^x-1))|\nu(dx) \le \int_{-1}^{p}|x|\nu(dx) < \infty$$

holds. □

The next lemma is a consequence of Lemma 3.2.3 and conserns the connection between the expectation and the Laplace exponent of L_θ and those of L. The last part of the lemma states, that under natural condition for the risky investment, the Laplace exponent of the resulting Lévy process L_θ is strictily convex function in the investment strategy θ. This result will turn out to be very usefull in the sequel.

Lemma 3.2.5. *Let $\theta \in [0,1]$ and φ and φ_θ be the Laplace exponents of L and L_θ, respectively. Then the following hold.*
(a) If $E[L(1)] < \infty$, then also $E[L_\theta(1)] < \infty$.
(b) If $E[L(1)] > 0$, then also $E[L_\theta(1)] > 0$.
(c) If $\varphi(s) < \infty$, then $\varphi_\theta(s) < \infty$.
(d) If $\delta < \varphi(-1)$, then for fixed $s > 0$ the function $\varphi_\theta(s)$ is strictly convex in θ.

Proof. (a) $E[L(1)] < \infty$ is equivalent to $\int_{\mathbb{R}} x1_{\{|x|>1\}}\,\nu(dx) < \infty$. Note that this formulation is chosen as a particular way to control the large jumps of the process. The cut-off points -1 and 1 can be chosen arbitrarily and do not need to have the same modulus. By Remark 3.2.4(ii), the large jumps are of the form $\log(1+\theta(e^{\Delta L(1)}-1))$. Since $\log(1+\theta(e^x-1)) \geq \log(1-\theta)$, i.e. negative jumps are bounded below, we only need to control large positive jumps. Note that by l'Hospital's rule

$$\lim_{x\to\infty} \frac{\log(1+\theta(e^x-1))}{x} = 1\,.$$

This implies that for large enough $h > 0$

$$\int_h^\infty \log(1+\theta(e^x-1))\nu(dx) \leq \int_h^\infty (x+\varepsilon)\nu(dx) < \infty\,,$$

where the last holds due to the bounded large positive jumps of L.
(b) First recall that, whenever the expectations are finite, then $E[L(1)] = \gamma + \int_{\mathbb{R}} x1_{\{|x|>1\}}\,\nu(dx)$ and $E[L_\theta(1)] = \gamma_\theta + \int_{\mathbb{R}} x1_{\{|x|>1\}}\,\nu_\theta(dx)$ (see Section 2.1). Now assume that $E[L(1)] > 0$. By Lemma 3.2.3, setting $(1-\theta)(\delta + \frac{\sigma^2}{2}\theta) =: a > 0$, we obtain

$$\begin{aligned} E[L_\theta(1)] &= \gamma\theta + a + \int_{\mathbb{R}} \log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|>1\}}\nu(dx) \\ &\quad + \int_{\mathbb{R}} (\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\leq 1\}} - \theta x 1_{\{|x|\leq 1\}})\nu(dx) \\ &= a + \theta E[L(1)] + \int_{\mathbb{R}} (\log(1+\theta(e^x-1)) - \theta x)\,\nu(dx) > 0\,, \end{aligned}$$

since the integrand is also positive.

(c) Recall that

$$\varphi(s) = \Psi(is) = -\gamma s + \frac{\sigma^2}{2}s^2 + \int_{\mathbb{R}} (e^{-sx} - 1 + sx1_{\{|x|\leq 1\}})\nu(dx)$$

and we assume that $\varphi(s) < \infty$, equivalently, the integral being finite. We consider the corresponding integral for $\varphi_\theta(s)$.

$$\begin{aligned} h(\theta) &:= \int_{\mathbb{R}} (e^{-sx} - 1 + sx1_{\{|x|\le 1\}})\nu_\theta(dx) \\ &= \int_{\mathbb{R}} \left((1+\theta(e^x-1))^{-s} - 1 - s\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\le 1\}}\right)\nu(dx)\,. \end{aligned}$$

Now the function $h(\theta)$ is continuous in $\theta \in [0,1]$. Moreover, $h(0) = 0$ and

$$h(1) = \int_{\mathbb{R}} \left(e^{sx} - 1 - sx1_{\{|x|\le 1\}}\right)\nu(dx) < \infty\,,$$

hence $h(\theta)$ is finite for all $\theta \in [0,1]$.

(d) We consider the function $\varphi_\theta(s) =: \varphi(\theta,\, s)$ for $\theta \in [0,\,1]$ and $s > 0$ as a function in two variables. Then

$$\begin{aligned} \varphi(\theta,\, s) &:= -\left(\delta + \theta(\gamma + \frac{\sigma^2}{2} - \delta)\right)s + \frac{\sigma^2}{2}s(s+1)\theta^2 \\ &\quad + \int_{\mathbb{R}} \left((1+\theta(e^x-1))^{-s} - 1 + s\theta x1_{\{|x|\le 1\}}\right)\nu(dx)\,, \end{aligned}$$

and we investigate $\varphi(\theta,\, s)$ as a function of θ. First notice that

$$\frac{\partial}{\partial\theta}\varphi(\theta,\, s) = -\left(\gamma + \frac{\sigma^2}{2} - \delta\right)s + \sigma^2 s(s+1)\theta - s\int_{\mathbb{R}}\left(\frac{(e^x-1)}{(1+\theta(e^x-1))^{s+1}} - x1_{\{|x|\le 1\}}\right)\nu(dx)\,.$$

As $\delta < \varphi(-1)$, we have $\frac{\partial}{\partial\theta}\varphi(0,\, s) = -(\varphi(-1)-\delta)s < 0$. Secondly,

$$\frac{\partial^2}{\partial\theta^2}\varphi(\theta,\, s) = s(s+1)\left(\sigma^2 + \int_{\mathbb{R}} \frac{(e^x-1)^2}{(1+\theta(e^x-1))^{s+2}}\nu(dx)\right) > 0\,,$$

i.e. the function $\varphi(\theta,\, s)$ is strictly convex in θ. □

We will also need the following result, giving conditions for the process L, under which the resulting Lévy process L_θ for $\theta > 0$ has negative jumps with positive probability.

Lemma 3.2.6. *If $\sigma > 0$ or $\nu((-\infty,0)) > 0$, then for all $\theta \in (0,1]$ holds $P(L_\theta(1) < 0) > 0$.*

Proof. If $\sigma > 0$, then by Lemma 3.2.3 also $\sigma_\theta > 0$, and the Gaussian component guarantees the result. On the other hand, if $\nu((-\infty,0)) > 0$, then Remark 3.2.4(ii) ensures also downwards jumps of L_θ, giving again the result. For more details we refer to Sato [56], Section 24. □

In the next lemma we represent $\varphi_\theta(-1)$ and $\varphi_\theta(-2)$, needed for the calculation of the mean and the variance function of the process X_θ, by means of $\varphi(-1)$ and $\varphi(-2)$.

Lemma 3.2.7. *Let $\theta \in [0,1]$ and φ and φ_θ be the Laplace exponents of L and L_θ, respectively. Then the following hold.*
(a) If $\varphi(-1) < \infty$, then

$$\varphi_\theta(-1) \;=\; \delta + \theta(\varphi(-1) - \delta) < \infty\,.$$

(b) If $\varphi(-2) < \infty$, then

$$\varphi_\theta(-2) \;=\; 2\delta + 2\theta(\varphi(-1) - \delta) + \theta^2\left(\varphi(-2) - 2\varphi(-1)\right) < \infty\,.$$

Proof. Applying Lemma 3.2.3 we obtain (a)

$$\begin{aligned}
\varphi_\theta(-1) &= \gamma_\theta + \frac{\sigma_\theta^2}{2} + \int_{\mathbb{R}} \left(e^x - 1 - x\mathbf{1}_{\{|x|\le 1\}}\right)\nu_\theta(dx) \\
&= (1-\theta)\delta + \theta(\gamma + \frac{\sigma^2}{2}) - \frac{\sigma^2}{2}\theta^2 + \frac{(\sigma\theta)^2}{2} \\
&\quad + \int_{\mathbb{R}} (\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\le 1\}} - \theta x 1_{\{|x|\le 1\}})\nu(dx) \\
&\quad + \int_{\mathbb{R}} \left(e^{\log(1+\theta(e^x-1))} - 1 - \log(1+\theta(e^x-1))\mathbf{1}_{\{|\log(1+\theta(e^x-1))|\le 1\}}\right)\nu(dx) \\
&= (1-\theta)\delta + \theta(\gamma + \frac{\sigma^2}{2}) + \int_{\mathbb{R}} (\theta(e^x-1) - \theta x 1_{\{|x|\le 1\}})\nu(dx) \\
&= (1-\theta)\delta + \theta\Psi(-i) = \delta + \theta(\varphi(-1) - \delta)\,.
\end{aligned}$$

Similarly for (b) we calculate

$$\begin{aligned}
\varphi_\theta(-2) &= 2\gamma_\theta + 2\sigma_\theta^2 + \int_{\mathbb{R}} \left(e^{2x} - 1 - 2x\mathbf{1}_{\{|x|\le 1\}}\right)\nu_\theta(dx) \\
&= 2(1-\theta)\delta + 2\theta(\gamma + \frac{\sigma^2}{2}) - \sigma^2\theta^2 + 2(\sigma\theta)^2 \\
&\quad + 2\int_{\mathbb{R}} (\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\le 1\}} - \theta x 1_{\{|x|\le 1\}})\nu(dx) \\
&\quad + \int_{\mathbb{R}} \left(e^{2\log(1+\theta(e^x-1))} - 1 - 2\log(1+\theta(e^x-1))\mathbf{1}_{\{|\log(1+\theta(e^x-1))|\le 1\}}\right)\nu(dx) \\
&= 2\delta + 2\theta(\varphi(-1) - \delta) + \sigma^2\theta^2 - 2\theta\int_{\mathbb{R}} (e^x - 1 - x1_{\{|x|\le 1\}})\nu(dx) \\
&\quad + \int_{\mathbb{R}} ((1+\theta(e^x-1))^2 - 1 - 2\theta x 1_{\{|x|\le 1\}})\nu(dx) \\
&= 2\delta + 2\theta(\varphi(-1) - \delta) + \theta^2\left(\sigma^2 + \int_{\mathbb{R}} (e^x-1)^2\,\nu(dx)\right)\,,
\end{aligned}$$

where in the last equality we use the simple fact that

$$\varphi(-2) - 2\varphi(-1) \;=\; \sigma^2 + \int_{\mathbb{R}} (e^x-1)^2\,\nu(dx)\,.$$

□

From Lemma 3.2.5(c) follows that if $\varphi(-s) < \infty$ for some $s \in \mathbb{R}$, then the s-th moment of $X_\theta(t) = \exp(L_\theta(t))$, $t \geq 0$, exists. From the Lévy-Khintchine representation we have

$$E\left[X_\theta^s(t)\right] = e^{t\Psi_\theta(-is)} = e^{(s\gamma_\theta + s^2\sigma_\theta^2/2)t} e^{t\Delta_s} \quad , \; t \geq 0\,, \tag{3.6}$$

where γ_θ and σ_θ are given in Lemma 3.2.3 and

$$\Delta_s = \int_{\mathbb{R}} \left((1+\theta(e^x-1))^s - 1 - s\log(1+\theta(e^x-1))\mathbf{1}_{\{|\log(1+\theta(e^x-1))|\leq 1\}}\right)\nu(dx)\,.$$

The next lemma is an application of Lemma 3.2.7. It gives conditions for existence and expressions for the moment functions and the autocovariance function of the investment process $X_\theta = \exp(L_\theta)$ in terms of the Laplace exponent of the original process L.

Lemma 3.2.8. *Let $\theta \in [0,1]$ and φ be the Laplace exponents of L. Then the following hold.*
(a) If $\varphi(-1) < \infty$, then for $t \geq 0$, $E\left[X_\theta(t)\right]$ exists and

$$E\left[X_\theta(t)\right] \;=\; e^{t(\delta+\theta(\varphi(-1)-\delta))}\,. \tag{3.7}$$

(b) If $\varphi(-2) < \infty$, then for $t \geq 0$, $E\left[X_\theta^2(t)\right]$ and $\operatorname{var}(X_\theta(t))$ exist and

$$E\left[X_\theta^2(t)\right] \;=\; e^{t\left(2\delta+2\theta(\varphi(-1)-\delta)+\theta^2(\varphi(-2)-2\varphi(-1))\right)}\,, \tag{3.8}$$

$$\operatorname{var}(X_\theta(t)) \;=\; e^{2t(\delta+\theta(\varphi(-1)-\delta))}\left(e^{t\theta^2(\varphi(-2)-2\varphi(-1))} - 1\right) \tag{3.9}$$

and for $0 \leq v < t$,

$$\begin{aligned} \operatorname{cov}(X_\theta(t),\, X_\theta(v)) &= E\left[X_\theta(t-v)\right]\operatorname{var}(X_\theta(v)) \\ &= e^{(t+v)(\delta+\theta(\varphi(-1)-\delta))}\left(e^{v\theta^2(\varphi(-2)-2\varphi(-1))} - 1\right)\,. \end{aligned} \tag{3.10}$$

Proof. Apllying Lemma 3.2.7 to $E[X_\theta(t)] = \exp(t\varphi_\theta(-1))$ and $E[X_\theta^2(t)] = \exp(t\varphi_\theta(-2))$ we obtain (3.7) and (3.8). Then the formula for the variance of X_θ is direct. For the autocovariance function of X_θ, using the independent and stationary increments of the Lévy process, we obtain for $0 < v \leq t$,

$$\begin{aligned} \operatorname{cov}(X_\theta(t),\, X_\theta(v)) &= \operatorname{cov}(e^{L_\theta(v)},\, e^{L_\theta(t)-L_\theta(v)+L_\theta(v)}) \\ &= \operatorname{cov}(e^{L_\theta(v)},\, e^{L_\theta(t-v)}e^{L_\theta(v)}) \\ &= E[e^{L_\theta(t-v)}]\operatorname{var}(e^{L_\theta(v)}) \\ &= E\left[X_\theta(t-v)\right]\operatorname{var}(X_\theta(v))\,. \end{aligned}$$

□

For the mean and the variance function of the investment process in the multidimentional case see Emmer and Klüppelberg [16], Lemma 2.6.

3.3 Integrated risk model

Now we are ready to define the integrated risk process as the total risk reserve, i.e. the result of the insurance business and the net gains of the investment.

Definition 3.3.1. *With the quantities as introduced in Section 3.1 and Section 3.2 we define for* $\theta \in [0,1]$ *the* integrated risk process (IRP) *as the solution to the SDE*

$$dU_\theta(t) = c\,dt - dS(t) + U_\theta(t-)\Big((1-\theta)\delta\,dt + \theta d\widehat{L}(t)\Big)\,, \quad t>0\,, \quad U_\theta(0)=u\,. \tag{3.11}$$

□

Recall that the process $\widehat{L}$ is such that $e^{L(t)} = \mathcal{E}(\widehat{L}(t))$, $t \geq 0$, where $\mathcal{E}$ denotes the stochastic exponential of a process. In the next lemma we give the solution to the SDE of the IRP.

Lemma 3.3.2. *If the investment process L is independent of the insurance process S, then the SDE (3.11) has the solution*

$$U_\theta(t) = e^{L_\theta(t)}\left(u + \int_0^t e^{-L_\theta(v)}\,(cdv - dS(v))\right)\,, \quad t \geq 0\,. \tag{3.12}$$

Proof. Define

$$Z(t) = \int_0^t e^{-L_\theta(v-)}(cdv - dS(v)) = \int_0^t e^{-L_\theta(v)}\,(cdv - dS(v))\,, \quad t\geq 0\,. \tag{3.13}$$

Equality holds as the independent processes L_θ and S have no common jumps a.s. (see e.g. Cont and Tankov [10], Proposition 5.3). The integration by parts formula (see Theorem 2.2.2) gives

$$d(X_\theta(t)Z(t)) = X_\theta(t-)dZ(t) + Z(t-)dX_\theta(t) + d[X_\theta, Z]_t\,, \quad t>0\,,$$

where $[X_\theta, Z]$ is the quadratic variation process of X_θ and Z, see Section 2.2. Let us show that $d[X_\theta, Z]_t = 0$, $t > 0$. First, note that $[Z,Z]^c = 0$, i.e. Z is a quadratic pure jump semimartingale. To show this we decompose $Z = Z_2 + Z_1$, where Z_1 and Z_2 are defined by

$$Z_1(t) = -\int_0^t e^{-L_\theta(v-)}\,dS(v) \ \text{ and } \ Z_2(t) = c\int_0^t e^{-L_\theta(v-)}\,dv\,, \ \ t\geq 0\,.$$

Then from Theorem 2.2.4 we have

$$[Z,Z]_t = [Z_2,Z_2]_t + 2[Z_1,Z_2]_t + [Z_1,Z_1]_t\,,\ t\geq 0\,,$$

where $[Z_2,Z_2]_t = 0$ and also $[Z_1,Z_1]^c_t = 0$. From the last and Theorem 2.2.3 we have that

$$[Z_1,Z_2]_t = 0 + \sum_{0<s\leq t}\Delta Z_1(s)\Delta Z_2(s)\,,\ t\geq 0\,,$$

hence also $[Z,Z]_t = 0 + jumps$, $t \geq 0$. From this and from Theorem 2.2.3, as $X_\theta(0)Z(0) = 0$ we conclude that

$$[X_\theta, Z]_t = \sum_{0<s\leq t} \Delta X_\theta(s)\Delta Z(s) \ , t \geq 0\,. \tag{3.14}$$

Further, by Theorem 2.2.1 we have that $\Delta Z(t) = \exp(-L_\theta(t-))\Delta S(t)$. Then, using again that the processes L_θ and S have no common jumps a.s., we conclude that

$$\begin{aligned}\Delta X_\theta(t)\Delta Z(t) &= \left(e^{L_\theta(t-)+\Delta L_\theta(t)} - e^{L_\theta(t-)}\right) e^{-L_\theta(t-)}\Delta dS(t) \\ &= (e^{\Delta L_\theta(t)} - 1)\Delta S(t) = 0\,.\end{aligned}$$

From this and (3.14) follows that $[X_\theta, Z]_t \equiv 0$. Thus, as $X_\theta(t-)dZ(t) = cdt - dS(t)$, we have

$$\begin{aligned}d\,(X(t)Z(t)) &= X_\theta(t-)dZ(t) + dX_\theta(t)Z(t-) \\ &= cdt - dS(t) + dX_\theta(t)\left(\int_0^{t-} e^{-L_\theta(v-)}(cdv - dS(v))\right)\ , \ \ t > 0\,.\end{aligned}$$

Finally from the last equality and from (3.3), (3.12) and (3.13) we get for $t > 0$

$$\begin{aligned}dU_\theta(t) &= udX_\theta(t) + cdt - dS(t) + dX_\theta(t)\left(\int_0^{t-} e^{-L_\theta(v-)}(cdv - dS(v))\right) \\ &= cdt - dS(t) + X_\theta(t-)\left(u + \int_0^{t-} e^{-L_\theta(v)}(cdv - dS(v))\right)\frac{dX_\theta(t)}{X_\theta(t-)} \\ &= cdt - dS(t) + U_\theta(t-)\left((1-\theta)\delta\,dt + \theta d\widehat{L}(t)\right)\,.\end{aligned}$$

□

Lemma 3.3.2 shows that the IRP U_θ fits into the framework of *generalized Ornstein-Uhlenbeck (OU) processes*, which have recently attracted much attention, see e.g. Lindner and Maller [44] or Carmona, Petit and Yor [9].

In the insurance framework, similar models have been investigated for example by Paulsen [52] and Kalashnikov and Norberg [35], and, in the special case of a geometric Brownian motion as an investment process, by Gaier and Grandits [23], Gaier, Grandits and Schachermayer [24] and Frolova, Kabanov and Pergamenshchikov [21].

Hipp and Plum [31, 32] analyse a model when the insurance company invests into risky assets, not necessarily financed from the risk reserve. In contrast to that, in our model the trading strategy θ is constant and $\theta \in [0,1]$, i.e. short selling is not allowed and the portfolio is self-financing.

3.4 Properties of the IRP

3.4.1 Characteristic function and moments

We start with the characteristic function of the IRP.

Lemma 3.4.1. *For $t \geq 0$ denote by $\widehat{u}_{\theta,t}(s) = E[\exp(isU_\theta(t))]$ and $\widehat{f}(s) = E[\exp(isY)]$, $s \in \mathbb{R}$. Then*

$$\widehat{u}_{\theta,t}(s) = E\left[\exp\left(is\left(ue^{L_\theta(t)} + c\int_0^t e^{L_\theta(v)}\,dv\right) + \lambda\int_0^t\left(\widehat{f}(-se^{L_\theta(v)}) - 1\right)\,dv\right)\right]. \quad (3.15)$$

Proof. We use equation (3.12) and Lemma 2.2.9. Setting $Z(t) = ct - S(t)$, $t \geq 0$, we obtain $\Psi_Z(s) = ics + \lambda(\widehat{f}(-s) - 1)$, $s \in \mathbb{R}$. Conditioning on the sample path of L up to time t, and hence on those of L_θ, and using the notation $E_L\,[E\,[\cdot]] = E\,[E\,[\cdot \mid L(v),\, v \in (0,\,t]]]$ for $t \geq 0$, by independence of L_θ and S, we have for $s \in \mathbb{R}$,

$$\begin{aligned}
&E[\exp(isU_\theta(t))] \\
&= E\left[\exp\left(is\left(ue^{L_\theta(t)} + e^{L_\theta(t)}\int_0^t e^{-L_\theta(v)}\,(cdv - dS(v))\right)\right)\right] \\
&= E_L\left[E\left[\exp\left(isue^{L_\theta(t)}\right)\right] E\left[\exp\left(is\int_0^t e^{L_\theta(t)-L_\theta(v)}\,(cdv - dS(v))\right)\right]\right] \\
&= E\left[\exp\left(is\left(ue^{L_\theta(t)}\right)\right)\exp\left(\int_0^t\left(\lambda(\widehat{f}(-se^{L_\theta(t)-L_\theta(v)}) - 1) + isce^{L_\theta(t)-L_\theta(v)}\right)\,dv\right)\right] \\
&= E\left[\exp\left(is\left(ue^{L_\theta(t)}\right)\right)\exp\left(\int_0^t\left(isce^{L_\theta(v)} + \lambda(\widehat{f}(-se^{L_\theta(v)}) - 1)\right)\,dv\right)\right].
\end{aligned}$$

In the last equality we have used the stationary and independent property of the Lévy process L_θ. □

From the characteristic function of the IRP U_θ given in (3.15) we can calculate all moment functions, provided they exist. In the next lemma we give conditions for existence and formulae for the mean, the variance and the autocovariance function of U_θ. For shortness we use the notation $X_\theta = e^{L_\theta}$ for the investment process. Recall the notation φ for the Laplace exponent of the Lévy process L.

Lemma 3.4.2. *Let the IRP U_θ be given by (3.12). Recall that $E[Y] = \mu < \infty$.*
(a) Assume that $\varphi(-1) < \infty$. Then for $t \geq 0$, $E\,[U_\theta(t)]$ exists and

$$E\,[U_\theta(t)] \;=\; uE\,[X_\theta(t)] + (c - \lambda\mu)\int_0^t E\,[X_\theta(v)]\,dv\,. \quad (3.16)$$

(b) Assume that $\varphi(-2) < \infty$ and $E[Y^2] = \mu_2 < \infty$. Then for $t \geq 0$, $\mathrm{var}(U_\theta(t))$ exists and

$$\begin{aligned}
\mathrm{var}(U_\theta(t)) \;=\;& u^2\mathrm{var}(X_\theta(t)) + 2u(c - \lambda\mu)\int_0^t \mathrm{cov}(X_\theta(t),\, X_\theta(v))\,dv \quad (3.17)\\
&+\lambda\mu_2\int_0^t E\left[X_\theta^2(v)\right]\,dv + (c - \lambda\mu)^2\int_0^t\int_0^t \mathrm{cov}(X_\theta(v), X_\theta(w))\,dw\,dv\,.
\end{aligned}$$

(c) Assume that $\varphi(-2) < \infty$ and $E[Y^2] = \mu_2 < \infty$. Then for $0 \leq y < t$, $\mathrm{cov}(U_\theta(y),\, U_\theta(t))$ exists and

$$\mathrm{cov}(U_\theta(y),\, U_\theta(t)) = \mathrm{var}(U_\theta(y))e^{(t-y)(\delta+\theta(\varphi(-1)-\delta))}\,. \quad (3.18)$$

The expressions for $E\left[X_\theta(t)\right]$, $E\left[X_\theta^2(t)\right]$, $\operatorname{var}\left(X_\theta(t)\right)$ *and* $\operatorname{cov}\left(X_\theta(t),\, X_\theta(v)\right)$ *by means of the Laplace exponent* φ *of the process* L *are given in Lemma 3.2.8.*

Proof. Using that $\widehat{u}'_{\theta,t}(0) = iE\left[U_\theta(t)\right]$ and $\widehat{f}'(0) = i\mu$ we obtain (a). To show (b) we use that $\widehat{u}''_{\theta,t}(0) = i^2E\left[U_\theta^2(t)\right]$ and $\widehat{f}''(0) = i^2\mu_2$. We first calculate the second moment of U_θ

$$\begin{aligned} E\left[U_\theta^2(t)\right] &= E\left[\left(uX_\theta(t) + (c-\lambda\mu)\int_0^t X_\theta(v)\,dv\right)^2\right] + \lambda\mu_2\int_0^t E[X_\theta^2(v)]\,dv \\ &= u^2E[X_\theta^2(t)] + 2u(c-\lambda\mu)E\left[X_\theta(t)\int_0^t X_\theta(v)\,dv\right] \\ &\quad + (c-\lambda\mu)^2E\left[\left(\int_0^t X_\theta(v)\,dv\right)^2\right] + \lambda\mu_2\int_0^t E[X_\theta^2(v)]\,dv\,. \end{aligned}$$

From this and (3.16) and using that

$$\begin{aligned} E\left[\left(\int_0^t X_\theta(v)\,dv\right)^2\right] &= \int_0^t\int_0^t E\left[X_\theta(v)X_\theta(w)\right]\,dw\,dv\,, \\ \left(\int_0^t E\left[X_\theta(v)\right]\,dv\right)^2 &= \int_0^t\int_0^t E\left[X_\theta(v)\right]E\left[X_\theta(w)\right]\,dw\,dv\,, \end{aligned}$$

we invoke (3.17).
(c) Let $0 \le y < t$ and consider

$$\operatorname{cov}(U_\theta(t),\, U_\theta(y)) = E\left[U_\theta(t)U_\theta(y)\right] - E\left[U_\theta(t)\right]E\left[U_\theta(y)\right]\,.$$

Conditioning on $\mathcal{F}_y$, we can write

$$E[U_\theta(t)U_\theta(y)] = E[E[U_\theta(t)U_\theta(y)\,|\,\mathcal{F}_y]\,] = E[U_\theta(y)E[U_\theta(t)\,|\,\mathcal{F}_y]\,]\,. \tag{3.19}$$

Now we calculate the conditional expectation in the last expresion.

$$\begin{aligned} & E\left[U_\theta(t)\,|\,\mathcal{F}_y\right] \\ =\; & E\left[e^{L_\theta(t)+L_\theta(y)-L_\theta(y)}\left(u + \left(\int_0^y + \int_y^t\right)e^{-L_\theta(v)}\,(c\,dv - dS(v))\right)\,|\,\mathcal{F}_y\right] \\ =\; & e^{L_\theta(y)}\left(u + \int_0^y e^{-L_\theta(v)}\,(c\,dv - dS(v))\right)E\left[e^{L_\theta(t)-L_\theta(y)}\,|\,\mathcal{F}_y\right] \\ & + E\left[e^{L_\theta(t)-L_\theta(y)}\left(u + \int_y^t e^{-(L_\theta(v)-L_\theta(y))}\,(c\,dv - dS(v))\right)\,|\,\mathcal{F}_y\right] \\ =\; & U_\theta(y)E\left[e^{L_\theta(t-y)}\right] + E\left[e^{L_\theta(t-y)}\left(u + \int_y^t e^{-L_\theta(v)}\,(c\,dv - dS(v))\right)\right] \\ =\; & U_\theta(y)e^{(t-y)\varphi_\theta(-1)} + A(t,\,y)\,, \end{aligned} \tag{3.20}$$

where $A(t,\,y)$ is a non-random function and $\varphi_\theta(-1)$ is given in Lemma 3.2.7. Taking expectation in (3.20), we obtain

$$E\left[U_\theta(t)\right] = E\left[U_\theta(y)\right]e^{(t-y)\varphi_\theta(-1)} + A(t,\,y).$$

Hence, adding and substracting the term $E\left[U_\theta(y)\right] e^{(t-y)\varphi_\theta(-1)}$ in (3.20) we obtain

$$E\left[U_\theta(t) \,|\, \mathcal{F}_y\right] = \left(U_\theta(y) - E\left[U_\theta(y)\right]\right) e^{(t-y)\varphi_\theta(-1)} + E\left[U_\theta(t)\right] \,.$$

Plugging this in (3.19) we obtain

$$\begin{aligned} E\left[U_\theta(t)U_\theta(y)\right] &= E\left[U_\theta(y)\left(\left(U_\theta(y) - E\left[U_\theta(y)\right]\right) e^{(t-y)\varphi_\theta(-1)} + E\left[U_\theta(t)\right]\right)\right] \\ &= E\left[U_\theta(t)\right] E\left[U_\theta(y)\right] + e^{(t-y)\varphi_\theta(-1)} \mathrm{var}(U_\theta(t)) \,, \end{aligned}$$

which invokes (c). □

In the general case when L_θ and S are not necessarilly independent general Lévy processes, Lindner and Maller [44] have derived the same formula for the autocovariance function of U_θ, see [44], Theorem 4.3.

Remark 3.4.3. The mean and the variance functions of the IRP U_θ are determined by the mean and the variance of the insurance claims and the mean and the variance functions of the exponential Lévy process e^L describing the price of the risky asset. In general, the mean and the variance functions of the log returns of the risky asset L are not sufficient to determine the moments of the IRP. □

Of particular interest for us is the behaviour of the moment functions of the IRP U_θ with respect to the investment strategy θ.

Lemma 3.4.4. *Let the safety loading condition $c - \lambda\mu > 0$ hold and $\varphi(-1) > \delta$. Then the mean, the variance and the autocovariance function of the process U_θ are increasing functions in $\theta \in [0, 1]$.*

Proof. Note that from $\varphi(-1) > \delta$ follows that $\varphi_\theta(-1)$ is increasing function of θ, see Lemma 3.2.7. Then, for all $t > 0$, the function $E[X_\theta(t)] = E[\exp(L_\theta(t))] = \exp(t\varphi_\theta(-1))$ is increasing in θ . Hence, as $c - \lambda\mu > 0$, for all $t > 0$, the function $E\left[U_\theta(t)\right]$ given in (3.16) is increasing in θ.

To show that $\mathrm{var}(U_\theta(t))$ given in (3.17) is increasing function in θ we note that the moment functions and the autocovariance function of X_θ given in Lemma 3.2.8 are increasing in θ as $\varphi(-1) > \delta$ and $\varphi(-2) - 2\varphi(-1) > 0$. Then, as $c - \lambda\mu > 0$, the variance function of U_θ is increasing in θ. Finally, from (3.18) it is straightforward that the autocovariance function of the process U_θ is also increasing in θ. □

3.4.2 Markov and pathwise properties

As we have already mentioned, the IRP U_θ belongs to the class of the generalized OU processes and as such, it is a time-homogeneous Markov process. Though this is a known result, shown for example in Carmona, Petit and Yor [9], Section 3 or in Lindner and Maller [44], Lemma 6.2, we state it for the IRP U_θ and give a short proof.

Lemma 3.4.5. *The IRP U_θ is a time-homogeneous Markov process. More precisely, for $t \geq 0$ and $s \geq 0$ define*

$$\begin{aligned} M_\theta(t,\,s) &= e^{L_\theta(t+s)-L_\theta(t)}\,, \\ N_\theta(t,\,s) &= e^{L_\theta(t+s)-L_\theta(t)} \int_{t+}^{t+s} e^{-(L_\theta(v)-L_\theta(t))}\, d(cv - S(v))\,. \end{aligned}$$

Then for $t \geq 0$ the bivariate process $((M_\theta(t,\,s),\,N_\theta(t,\,s)))_{s\geq 0}$ is independent of $\mathcal{F}_t = \sigma(L_\theta(v),\,S(v),\,v \leq t)$, $(M_\theta(t+h,\,s+h),\,N_\theta(t+h,\,s+h)) \stackrel{d}{=} (M_\theta(t,\,s),\,N_\theta(t,\,s))$ for $h \geq 0$ and

$$U_\theta(t+s) = M_\theta(t,\,s)U_\theta(s) + N_\theta(t,\,s)\,. \tag{3.21}$$

Proof. With M_θ and N_θ defined as in the lemma and by equation (3.12) we have

$$\begin{aligned} &M_\theta(t,\,s)U_\theta(s) + N_\theta(t,\,s) \\ &= e^{L_\theta(t+s)} \left(u + \int_0^t e^{-L_\theta(v)}\, d(cv - S(v)) \right) \\ &\quad + e^{L_\theta(t+s)-L_\theta(t)} \int_{t+}^{t+s} e^{-(L_\theta(v)-L_\theta(t))}\, d(cv - S(v)) \\ &= e^{L_\theta(t+s)} \left(u + \int_0^{t+s} e^{-L_\theta(v)}\, d(cv - S(v)) \right) \;=\; U_\theta(t+s)\,, \end{aligned}$$

which is equation (3.21). The fact that for fixed $t \geq 0$ the bivariate process

$$(L_\theta(t+s) - L_\theta(t),\, S(t+s) - S(s))_{s\geq 0}$$

is independent of $(L_\theta(s),\,S(s))_{0\leq s\leq t}$ and has the same distribution as $(L_\theta(s),\,S(s))_{s\geq 0}$ completes the proof. □

In the analysis of the pathwise properties of the IRP, we restrict ourselves to the jump structure.

Lemma 3.4.6. *The jumps of the IRP U_θ, for $t > 0$, are given by*

$$\Delta U_\theta(t) = \Delta S(t) + U_\theta(t-) \log(1 + \theta(e^{\Delta L(t)} - 1))\,.$$

Proof. Using SDE (3.11) and Lemma 3.2.3 we obtain immediatelly the required result. □

From the above proposition we may conclude, that the jumps of the IRP have two independent sources. An insurance claim leads immediatelly to a jump of the IRP of the same size as that of the claim. A jump of the risky asset process leads also to a jump of the IRP. However, the size of this jump is determined not only by the jump of the risky asset, but also by the level of the IRP before the jump, i.e. by $U_\theta(t-)$. Consequently, whenever the wealth of the insurance company is large, a jump of the risky asset process has a high impact on the IRP.

3.5 Examples

We give some examples to illustrate the properties of the IRP.

Example 3.5.1. [Geometric Brownian motion with jumps]
Assume that the log returns of the risky asset are modeled by

$$L(t) = \xi t + \sigma W(t) + C(t)\,, \quad t \geq 0\,,$$

where $\xi \in \mathbb{R}$, $\sigma > 0$, $W = (W(t))_{t\geq 0}$ is a standard Brownian motion, and $C(t) = \sum_{j=1}^{M(t)} Z_j$, $t \geq 0$, is a compound Poisson process with intensity $\eta > 0$ and jump size represented by the generic rv Z. The Laplace exponent of L is given by

$$\varphi(s) = -\xi s + \sigma^2 \frac{s^2}{2} + \eta(E[e^{-sZ}] - 1)\,.$$

Note that L has drift $\gamma = E[L(1)] = \xi + \eta E[Z]$.
By Lemma 3.2.3

$$L_\theta(t) = \xi_\theta t + \sigma_\theta W(t) + C_\theta(t)\,, \quad t \geq 0\,,$$

where C_θ is a compound Poisson process with the same jump intensity η as that of the compound Poisson process C and jump size $\log(1 + \theta(e^Z - 1))$. Moreover,

$$\xi_\theta = \xi\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) \quad \text{and} \quad \sigma_\theta^2 = (\sigma\theta)^2\,. \tag{3.22}$$

The Laplace exponent of L_θ is given by

$$\varphi_\theta(s) = -\xi_\theta s + \sigma_\theta^2 \frac{s^2}{2} + \int_{-\infty}^{\infty} (e^{-sx} - 1)\nu_\theta(dx) = -\xi_\theta s + \sigma_\theta^2 \frac{s^2}{2} + \eta(E\left[(1 + \theta(e^Z - 1))^{-s}\right] - 1)\,,$$

and L_θ has drift

$$\gamma_\theta = \gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) + \eta(E[\log(1 + \theta(e^Z - 1))] - EZ)\,.$$

In the case of the classical geometric Brownian motion model with drift, i.e. when $C \equiv 0$, as

$$L(t) = \gamma t + \sigma W(t)\,,\ t \geq 0\,,$$

then $\widehat{L}(t) = (\gamma + \sigma^2/2)t + \sigma W(t)$, $t \geq 0$. Then the SDE (3.11) for U_θ reduces to ($U_\theta(0) = u$)

$$dU_\theta(t) = c dt - dS(t) + U_\theta(t-)\Big(\big((1-\theta)\delta + \theta(\gamma + \frac{\sigma^2}{2})\big)dt + \theta\sigma dW(t)\Big)\,, t > 0\,. \tag{3.23}$$

Furthermore, the solution to this SDE is given by (3.12), where L_θ is again a Brownian motion with drift with Laplace exponent

$$\varphi_\theta(s) = -\gamma_\theta s + \frac{\sigma_\theta^2}{2} s^2\,, \tag{3.24}$$

where, due to Lemma 3.2.3,

$$\gamma_\theta = \theta\gamma + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) \quad \text{and} \quad \sigma_\theta^2 = (\sigma\theta)^2\,. \tag{3.25}$$

□

In the sequel we will often come back to the example of the geometric Brownian motion as a model for the risky asset to give explicit illustration of our results.

Monte Carlo simulation is a widely applied technique in the contemporary risk management. One of the problems in the simulation of continuous time stochastic processes is that one has typically to introduce a discrete grid and to simulate the process on it.

We start, however, with an algorithm for exact simulation of a compound Poisson process. Sample paths of such processes are piecewise linear and there is a finite number of jumps in every bounded interval. Hence we can simulate the sample path directly (without any discretization error) using a finite number of operations.

Algorithm 1 Simulation of the compound Poisson process S with jump intensity λ, claim size distribution F and drift c over some time interval $[0,T]$.

1. Simulate a Poisson rv N with parameter λT. N gives the total number of jumps in the time interval $[0,T]$.

2. Simulate N iid uniformly distributed rv's U_i, $i=1,\ldots,N$, on the time interval $[0,T]$. The order statistics of these variables correspond to the jump arrival times T_i, $i = 1,\ldots,N$.

3. Simulate jump sizes: N iid rv's Y_i, $i=1,\ldots,N$, with df F.

The sample path is given by

$$S(t) \;=\; ct + \sum_{i=1}^{N} 1_{\{U_i \le t\}} Y_i\,, \quad t \in [0,T]\,.$$

□

The above algorithm enables the simulation of the insurance process, which is an important ingredient of the the IRP. The next algorithm is an extension of it, it enables the simulation of an investment process as in Example 3.5.1.

Algorithm 2 Simulation of a Brownian motion with jumps over some time interval $[0,T]$.

Assume that

$$L(t) = \xi t + \sigma W(t) + C(t)\,, \quad t \ge 0\,,$$

where $\xi \in \mathbb{R}$, $\sigma > 0$, $(W(t))_{t\ge 0}$ is standard Brownian motion, and $C(t) = \sum_{j=1}^{M(t)} Z_j$, $t \ge 0$, is a compound Poisson process with intensity $\eta > 0$ and jump size represented by the generic rv Z.

1. Apply Algorithm 1 to simulate the compound Poisson process C with drift.

2. Select a finite discrete grid $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$, which contains also the jump arrival times from step 1.

3. Simulate $n+1$ standard normal iid rv's $N_1,\ldots,N_{n+1}$.

4. Set $\Delta W_i = \sigma N_i \sqrt{t_i - t_{i-1}}$, $i = 1,\ldots,n+1$.

The sample path of L on the discrete grid is given by $L(0) = 0$,

$$L(t_i) \quad = \quad \xi t_i + \sum_{k=1}^{i} \Delta W_k + C(t_i)\,, \quad i = 1, \dots, n+1\,.$$

□

Applying Donsker's invariance prinsiple, see [5], for $n \to \infty$ the simulated process in the suggested algorithm converges to a Brownian motion with jumps. Since in the case of Example 3.5.1, the investment process $X_\theta = \exp(L_\theta)$ is again an exponential Brownian motion with jumps, and the parameters of L_θ can be computed explicitly, Algorithm 2 enables the simulation of the investment process for every $\theta \in [0, 1]$.

The next algorithm is for the sumulation of the IRP given in (3.12).

Algorithm 3 Simulation of the IRP U_θ on some time interval $[0, T]$.

1. Apply Algorithm 1 and simulate exactly the total claim amount process

$$S(t) = \sum_{i=1}^{N(t)} Y_i\,, \quad t \in [0, T]\,.$$

2. Select a discrete grid $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T$, which includes the jump arrival times from step 1.

3. Apply Algorithm 2 and simulate L_θ on the discrete grid from step 2.

An approximate sample path of the IRP U_θ is given by

$$U_\theta(t_i) = e^{L_\theta(t_i)} \left(u + \sum_{j=1}^{i} c(t_j - t_{j-1}) e^{-L_\theta(t_j)} - \sum_{j=1}^{N(t_i)} Y_j e^{-L_\theta(T_j)} \right)\,,$$

for $i = 1, \dots, n+1$, where T_j, $j = 1, \dots N(T)$ denote the claim arrival times from step 1.
□

Example 3.5.2. [Simulation of the IRP with geometric Brownian motion as risky investment process; continuation of Example 3.5.1]
Let the risky asset be modeled by a geometric Brownian motion with drift $\gamma = 0.15$ and volatility $\sigma = 0.2$ and the riskless interest rate be $\delta = 0.05$. We consider an insurance model with premium rate $c = 1$, an intensity of the Poisson claim counting process $\lambda = 9$ and exponentially distributed insurance claims with mean $\mu = 0.1$. The initial capital is $u = 10$ and the time horizon is $T = 1$ year.

Using the fact that the resulting Lévy process L_θ is again a Brownian motion with parameters γ_θ and σ_θ given in (3.25), we can apply Algorithm 3 for all investment strategies $\theta \in [0, 1]$. In Figure 3.1 we have ploted sample paths of the IRP for three different investment strategies – the pure bond strategie ($\theta = 0$), half in bond and half in stock ($\theta = 0.5$) and pure stock strategie ($\theta = 1$). All sample paths are based on the same random seed. In this simulation the value of the wealth at the end of the time period is

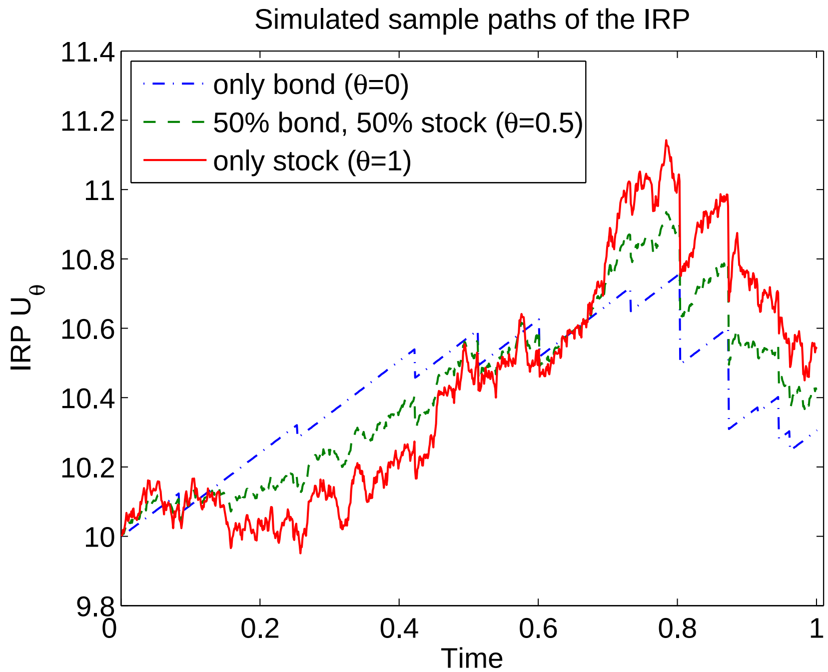

Figure 3.1: Sample paths of the IRP U_θ for three investment strategies – $\theta = 0$, $\theta = 0.5$ and $\theta = 1$, based on the same random seed. The parameters are as in Example 3.5.2. For the simulation we have applied Algorithm 3.

greatest in the case of pure stock strategy. Indeed, recall that under natural conditions on the model, which in this example are satisfied, the expected value of the IRP is an increasing function of the investment strategy θ, see Lemma 3.4.4. However, the question which is the best strategy stays open (notice the intersections of the three sample paths of the IRP during the time interval). We state this problem precisely in Chapter 6. $\square$

It has been well observed in numerous empirical studies, that the Brownian motion model is not realistic for various stock prices, as these often exhibit sudden downward jumps and the distribution of the returns has heavier tails than the normal distribution, see e.g. Madan and Seneta [46]. In such cases it would be natural to model the stock prices by a more general exponential Lévy process with jumps. Unfortunately, such models lead to less explicit results, as the jump measure should be taken into account.

Example 3.5.3. [VG Lévy process as risky investment process]
The variance gamma (VG) process, suggested by Madan and Seneta [46], is a normal mixture model, i.e. obtained by time changing of an independent Brownian motion. The time changing process is a gamma Lévy process S_Γ, where $S_\Gamma(1) \stackrel{d}{=} \Gamma(\eta, r)$, i.e. the density is given by $f_\Gamma(x) = r^\eta x^{\eta-1} e^{-rx}/\Gamma(\eta)$, $x > 0$, for parameters $r, \eta > 0$. The characteristic triplet of S_Γ is $(0, 0, \nu_\Gamma)$ where $\nu_\Gamma(dx) = 1_{\{x>0\}} \eta x^{-1} e^{-rx}\, dx$. A non-symmetric VG model is given by

$$L(t) = \xi t + W_{a,b}(S_\Gamma(t)), \quad t \geq 0,$$

where $\xi > 0$ and $W_{a,b}$ is a Brownian motion with drift $a < 0$ and variance b^2. This makes it possible to model the usually observed positive drift in combination with downward jumps of the price process. The mean and the variance of $L(1)$ are given by $\gamma = E[L(1)] = \xi + a\eta/r$ and $\mathrm{var}(L(1)) = b^2\eta/r + a^2\eta/r^2$. For the Laplace exponent of L we have

$$\varphi(s) = -\xi s - \eta \log\left(1 - \frac{1}{r}\big(b^2\frac{s^2}{2} - sa\big)\right), \quad s \in \mathbb{R}. \tag{3.26}$$

The Lévy measure of L is given by

$$\nu(dx) = \frac{r^2}{\eta|x|}\exp\Big(\frac{a}{b^2}x - \frac{\sqrt{a^2+2b^2r^2/\eta}}{b^2}|x|\Big)dx\,, \tag{3.27}$$

so that the VG process is a pure jump process with finite variation but of infinite activity (with infinitely many jumps in every compact interval) with drift.

If $\theta < 1$, the characteristic triplet of L_θ calculated by Lemma 3.2.3 shows that L_θ is no longer a VG Lévy process. However, as L is of finite variation, also L_θ is (see Remark 3.2.4(iii)) and its Laplace exponent is given by

$$\varphi_\theta(s) = -\xi_\theta s + \int_{\mathbb{R}}(e^{-sx}-1)\nu_\theta(dx) = -\xi_\theta s + \int_{x>\log(1-\theta)}\big((1+\theta(e^x-1))^{-s}-1\big)\nu(dx)\,,$$

where $\xi_\theta = \theta\xi + (1-\theta)\delta$ and ν is as in (3.27). We refer to Cont and Tankov [10], Section 4, for more details. $\square$

Unfortunatelly, for most Lévy processes the law of the increments is not known explicitly. This makes it more difficult to simulate a path of a general Lévy process. On the other hand, it is still a possible task when the Lévy process is a subordinated Brownian motion, provided we can simulate the subordinator.

Algorithm 4 Simulation of a subordinated Brownian motion L on a discrete grid $0 = t_0 < t_1 \leq \ldots \leq t_n < t_{n+1} = T$.

Assume that

$$L(t) = \xi t + W_{a,b}(C(t))\,, \quad t \geq 0\,,$$

where $\xi \in \mathbb{R}$, $W_{a,b}$ is a Brownian motion with mean $a \in \mathbb{R}$ and standard deviation $b > 0$ and C is a strictly positive subordinator process, e.g. the Gamma Lévy process from Example 3.5.3.

1. Simulate $n+1$ standard normal iid rv's $N_1, \ldots, N_{n+1}$.
2. Simulate the increments of the subordinator $\Delta C_i = C(t_i) - C(t_{i-1})$, $i = 1, \ldots, n+1$.
3. Set $\Delta W_i = a\Delta C_i + b\sqrt{\Delta C_i}N_i$, $i = 1, \ldots, n+1$.

The sample path on the discrete grid is given by

$$L(t_i) = \xi t_i + \sum_{k=1}^{i}\Delta W_k\,, \quad i = 1, \ldots, n+1\,.$$

As noted in Example 3.5.3, for $\theta < 1$ the process L_θ is no longer of the same type as L. Therefore, for infinite activity models like the VG model, the simulation of the investment process $X_\theta = \exp(L_\theta)$ by means of Algorithm 4 is possible only when $\theta = 1$. In some cases, it is possible to approximate the large jumps of such a process by a compound Poisson process and the small jumps by a Brownian motion, see Asmussen and Rosinski [3]. Then we may apply Algorithm 2 to get an approximate sample path of L_θ, for $\theta < 1$.

Chapter 4

The distribution of the IRP

In this chapter we derive a partial integro-differential equation for the distribution of the IRP over a fixed time horizon and explain how to solve it numerically.

4.1 PIDE for the IRP

In Chapter 3 the IRP U_θ given in (3.12) was introduced and its basic properties were investigated. In the previous section we described a simulation algorithm for the IRP, which enables the computation of its distribution at a fixed time horizon. In the present section we aim at applying numerical methods to find this distribution. All results are based on Brokate, Klüppelberg, Kostadinova, Maller and Seydel [6].

We shall be interested in the *net loss process* $Q_\theta = (Q_\theta(t))_{t\geq 0}$, which we define by the following transformation of the IRP

$$Q_\theta(t) = ue^{L_\theta(t)} - U_\theta(t) = \int_{(0,t]} e^{L_\theta(t)-L_\theta(v)} \left(dS(v) - c\,dv\right), \quad t \geq 0\,. \tag{4.1}$$

Note that the process Q_θ does not take the risk reserve into account, but simply calculates the balance sheet of the integrated risk model.

4.1.1 Derivation of the PIDE

Denote by Y a typical random claim size. The following is the main result of this section.

Theorem 4.1.1. *Define*

$$H(x,t) = P(Q_\theta(t) > x)\,, \quad t \geq 0\,,\, x \in \mathbb{R}, \tag{4.2}$$

and assume that the following conditions hold.

(1) The Lévy measure ν of L satisfies $\int_{|x|>1} e^{2|x|}\nu(dx) < \infty$.

(2) The partial derivative $\partial_t H(x,\cdot)$ exists for each $x \in \mathbb{R}$.

(3) For each $t > 0$, the first and second partial derivatives $\partial_x H(\cdot,t)$ and $\partial_{xx} H(\cdot,t)$ exist, and are continuous and bounded, on $\mathbb{R}$.

Then H is the solution to the PIDE

$$\begin{aligned}
&\partial_t H(x,t) - \lambda\left(EH(x-Y,t) - H(x,t)\right) \\
= \;& \frac{\sigma_\theta^2}{2}(\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)x) - \gamma_\theta \partial_x H(x,t)x \\
&+ \int \left(H(xe^z,t) - H(x,t) - z\partial_x H(x,t)x1_{\{|z|\le 1\}}\right)\nu_\theta(-dz) + c\partial_x H(x,t) \qquad (4.3)
\end{aligned}$$

with boundary condition $H(\cdot,0) = \mathbf{1}_{(-\infty,0)}(\cdot)$.

Proof. For fixed $x \in \mathbb{R}$ and $t > 0$, take $s > 0$ small and consider the probability

$$P\left(Q_\theta(t+s) > x\right) = P\left(\int_{(0,t+s]} e^{L_\theta(t+s)-L_\theta(v)}\left(dS(v) - c\,dv\right) > x\right). \qquad (4.4)$$

We introduce the process $(\overline{L}_\theta(v))_{v\ge 0} := (L_\theta(t+v) - L_\theta(t))_{v\ge 0}$. Due to the independent increments property of Lévy processes, $(\overline{L}_\theta(v))_{v\ge 0}$ is an independent copy of L_θ, independent of $\mathcal{F}_t$, where $(\mathcal{F}_t)_{t\ge 0}$ is the filtration generated by $(X_\theta(t))_{t\ge 0}$. By definition we have

$$\begin{aligned}
Q_\theta(t+s) &= \int_{(0,t+s]} e^{L_\theta(t+s)-L_\theta(v)}\left(dS(v) - c\,dv\right) \\
&= e^{\overline{L}_\theta(s)}\left(\int_{(0,t]} + \int_{(t,t+s]}\right) e^{-(L_\theta(v)-L_\theta(t))}\left(dS(v) - c\,dv\right) \\
&= e^{\overline{L}_\theta(s)}\left(Q_\theta(t) + \int_{(0,s]} e^{-\overline{L}_\theta(u)}\left(d\overline{S}(u) - c\,du\right)\right),
\end{aligned}$$

where in the last line we have set $u = v - t$. Furthermore, we have denoted by $\overline{S}(u) := S(t+u) - S(t)$ the sum of the claims in the interval $(t, t+u]$.

In order to derive a formula for the tail of $Q_\theta(t+s)$ we condition on the number of claims in a small interval $(t, t+s]$. From the total probability formula we have

$$\begin{aligned}
P\left(Q_\theta(t+s) > x\right) &= (1 - \lambda s + o(s))P\left(Q_\theta(t+s) > x \,|\, N(t+s) = N(t)\right) \qquad (4.5)\\
&\quad + \lambda s P\left(Q_\theta(t+s) > x \,|\, N(t+s) = N(t) + 1\right) + o(s).
\end{aligned}$$

Consider first the case with no claims in $(t, t+s]$. Then using that $\int_{(t,t+s]} e^{-\overline{L}_\theta(u)}\,d\overline{S}(u) = 0$ and that $\overline{L}_\theta$ is independent of $\mathcal{F}_t$, we get

$$\begin{aligned}
I_0(s) &:= P\left(Q_\theta(t+s) > x \,|\, N(t+s) = N(t)\right) \\
&= P\left(e^{\overline{L}_\theta(s)}\left(Q_\theta(t) - c\int_{[0,s)} e^{-\overline{L}_\theta(u)}\,du\right) > x\right).
\end{aligned}$$

If there is one claim in the interval $(t, t+s]$, we have that $\int_{(t,t+s]} e^{-\overline{L}_\theta(u)}\,d\overline{S}(u) = Ye^{-\overline{L}_\theta(\overline{T})}$, where $\overline{T}$ is the jump time and Y the jump size of $\overline{S}$ in $(0, s]$. As the Poisson process $\overline{S}$ is independent of $\mathcal{F}_t$, so are $\overline{T}$ and Y. Moreover, due to the order statistics property of the

Poisson process, the rv $\left(\overline{T}\,|\,\overline{T}\in(t,t+s]\right)\stackrel{d}{=}U_1$ is uniformly distributed in the interval $[0,s]$. Hence

$$\begin{aligned} I_1(s) &:= P\left(Q_\theta(t+s)>x\,|\,N(t+s)=N(t)+1\right) \\ &= P\left(e^{\overline{L}_\theta(s)}\left(Q_\theta(t)+Ye^{-\overline{L}_\theta(\overline{T})}-c\int_{[0,s)}e^{-\overline{L}_\theta(u)}\,du\right)>x\;\big|\,\overline{T}\in(0,s]\right) \\ &= P\left(e^{\overline{L}_\theta(s)}\left(Q_\theta(t)+Ye^{-\overline{L}_\theta(U_1)}-c\int_{[0,s)}e^{-\overline{L}_\theta(u)}\,du\right)>x\right). \end{aligned}$$

Now we want to study equation (4.5) for $s\to 0$, which we rewrite as

$$\frac{P\left(Q_\theta(t+s)>x\right)-P\left(Q_\theta(t)>x\right)}{s}=\lambda I_1(s)-\lambda I_0(s)+\frac{I_0(s)-P\left(Q_\theta(t)>x\right)}{s}+\frac{o(s)}{s}\,.$$

We have

1. $\lim_{s\to 0}I_1(s)=P\left(Q_\theta(t)+Y>x\right)$. Indeed, as a Lévy process is cádlág process, we have that $\lim_{s\to 0}\overline{L}_\theta(s)=\overline{L}_\theta(0)=0$ a.s. Also $\lim_{s\to 0}\int_{[0,s)}e^{-\overline{L}_\theta(v)}\,dv=0$ a.s. Further we have $U_1\to 0$ a.s. when $s\to 0$, hence also $\lim_{s\to 0}\overline{L}_\theta(U_1)=0$ a.s.

2. Similarly as for $I_1(s)$, $\lim_{s\to 0}I_0(s)=P\left(Q_\theta(t)>x\right)$.

Since $P\left(Q_\theta(t)>x\right)=H(x,t)$, assuming that the limit below exists, from the equation above we obtain the following PIDE for H:

$$\partial_t H(x,t)=\lambda\left(EH(x-Y,t)-H(x,t)\right)+\lim_{s\to 0}\frac{1}{s}\left(I_0(s)-H(x,t)\right)\,, \tag{4.6}$$

with boundary condition $H(\cdot,0)=\mathbf{1}_{(-\infty,0)}(\cdot)$.

We now calculate the last term in (4.6). For $s>0$ we have

$$\begin{aligned} & I_0(s)-H(x,t) \\ =\; & \left(P\left(Q_\theta(t)>xe^{-\overline{L}_\theta(s)}\right)-P(Q_\theta(t)>x)\right) \\ & -\left(P\left(Q_\theta(t)>xe^{-\overline{L}_\theta(s)}\right)-P\left(Q_\theta(t)>xe^{-\overline{L}_\theta(s)}+c\int_{[0,s)}e^{-\overline{L}_\theta(v)}\,dv\right)\right) \\ =:\; & J_1(s)-J_2(s)\,,\ \text{say.} \end{aligned} \tag{4.7}$$

First consider $J_1(s)$, separately for $x>0$ and $x<0$; note that for $x=0$ we have $J_1\equiv 0$. For $x>0$, we set $y=\ln x$ and $g(y)=H(e^y,t)$. Then by the independence of the investment process and the insurance risk process we can write

$$\begin{aligned} J_1(s) &= EH(xe^{-L_\theta(s)},t)-H(x,t) \\ &= Eg(y-L_\theta(s))-g(y). \end{aligned}$$

Under Assumption (3) of Theorem 4.1.1, $g(y)$ is continuous and bounded, and has continuous and bounded first and second derivatives, for $y\in\mathbb{R}$. So we can apply, e.g., Gihman and Skorohod [26], p. 292, to deduce that

$$\lim_{s\to 0}\frac{1}{s}J_1(s)\;=\;\lim_{s\to 0}\frac{1}{s}\left(Eg(y-L_\theta(s))-g(y)\right)=\mathcal{A}g(y)\,,\quad y\in\mathbb{R}\,, \tag{4.8}$$

where $\mathcal{A}$ is the infinitesimal generator of the Lévy process $-L_\theta$.

Calculating the partial derivatives implicit in the infinitesimal generator, we can check that (4.8) implies, for $x > 0$,

$$\begin{aligned} \lim_{s\to 0}\frac{1}{s}J_1(s) &= \frac{\sigma_\theta^2}{2}(\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)x) - \gamma_\theta \partial_x H(x,t)x \\ &\quad + \int \left(H(xe^z,t) - H(x,t) - z\partial_x H(x,t)x1_{\{|z|\le 1\}}\right)\nu_\theta(-dz)\,. \qquad (4.9) \end{aligned}$$

Here and in what follows we always take the integral over the support of the corresponding Lévy measure.

For $x < 0$, we set $y = \ln|x|$ and $g(y) = H(-e^y,t)$. Then an analogous calculation gives (4.9) again.

It remains to estimate the second term in (4.7). Define $R(s) = c\int_{[0,s)} e^{-L_\theta(v)}\,dv$. Then we can write, using a Taylor expansion,

$$\begin{aligned} J_2(s) &= P\left(Q_\theta(t) > xe^{-L_\theta(s)}\right) - P\left(Q_\theta(t) > xe^{-L_\theta(s)} + c\int_{[0,s)} e^{-L_\theta(v)}\,dv\right) \\ &= \int\int \left(P\left(Q_\theta(t) > xe^{-y}\right) - P\left(Q_\theta(t) > xe^{-y} + r\right)\right) dP(L(s)\le y, R(s)\le r) \\ &= \int\int \left(H(xe^{-y},t) - H(xe^{-y}+r,t)\right) dP(L_\theta(s)\le y, R(s)\le r) \\ &= -E\left[R(s)\partial_x H(\xi(s),t)\right]\,. \qquad (4.10) \end{aligned}$$

Here $\xi(s) \in [xe^{-L_\theta(s)}, xe^{-L_\theta(s)} + R(s)]$.

Now let

$$T(s) := -\frac{1}{s}R(s)\partial_x H(\xi(s),t)\,, \quad s > 0\,.$$

Then $T(s) \ge 0$ a.s. and $J_2(s)/s = ET(s)$. We have $R(s) \stackrel{\text{a.s.}}{\to} 0$ and also $R(s)/s \stackrel{\text{a.s.}}{\to} c$, as $s \to 0$, so $\xi(s) \stackrel{\text{a.s.}}{\to} x$, a.s., and consequently,

$$T := \lim_{s\to 0} T(s) = -\lim_{s\to 0}\frac{1}{s}R(s)\partial_x H(\xi(s),t) = -c\partial_x H(x,t), \text{ a.s.}$$

We will have L_1 convergence and deduce that $\lim_{s\to 0} J_2(s)/s = \lim_{s\to 0} ET(s) = T$ if we show that $(T(s))_{s>0}$ is uniformly integrable as $s \to 0$. To see this, recall Assumption (3) in Theorem 4.1.1. Take $\zeta > 0$ and consider

$$E(T(s)1_{\{T(s)>\zeta\}}) \le \sqrt{E(T^2(s))P(T(s)>\zeta)}.$$

Let $K_t = \sup_{x\in\mathbb{R}}(-\partial_x H(x,t))$, which is finite by Assumption (3). Now by Theorem 25.3 of Sato [56] and Lemma 3.2.5(c), $\int_{|x|>1} e^{2|x|}\nu(dx) < \infty$ implies that $Ee^{-2L_\theta(u)} < \infty$ for all $u \in \mathbb{R}$. Using the notation $Ee^{-sL_\theta(t)} = e^{-t\Psi_\theta(s)}$ for $t \ge 0$ and for all $s \in \mathbb{R}$ such that the expectation is finite, we conclude

$$\begin{aligned}
E[T^2(s)] &\leq \frac{c^2K_t^2}{s^2}E\left(\int_0^s e^{-L_\theta(v)}dv\right)^2 \\
&= \frac{2c^2K_t^2}{s^2}E\int_0^s\int_u^s e^{-(L_\theta(v)-L_\theta(u))}e^{-2L_\theta(u)}dvdu \\
&= \frac{2c^2K_t^2}{s^2}\int_0^s\int_u^s e^{-(v-u)\Psi_\theta(1)}e^{-u\Psi_\theta(2)}dvdu \\
&\leq \frac{2c^2K_t^2}{s^2}\left(\int_0^s e^{u(\Psi_\theta(1)-\Psi_\theta(2))}du\right)\left(\int_0^s e^{-v\Psi_\theta(1)}dv\right) \\
&= O(1)\,, \quad s\to 0\,.
\end{aligned}$$

Since $\lim_{s\to 0} P(T(s)>\zeta) = P(-c\partial_x H(x,t)>\zeta)$ equals 0 for large enough ζ, $(T(s))_{s>0}$ is uniformly integrable, as asserted, and it follows that $T(s)\xrightarrow{L_1} T$ as $s\to 0$. Hence, via (4.10), the second term of (4.7) tends to $c\partial_x H(x,t)$ a.s. as $s\to 0$.

Plugging this into (4.6), we obtain (4.3). □

4.1.2 Jump diffusion investment model

In this case

$$L(t) = \gamma\, t + \sigma W(t) + \sum_{j=1}^{M(t)} Z_j\,, \quad t\geq 0\,, \tag{4.11}$$

for $\gamma\in\mathbb{R}$, $\sigma>0$ and Z_j iid, independent of a Poisson process M with intensity $\eta>0$. The process L_θ has a similar representation given by

$$L_\theta(t) = \gamma_\theta\, t + \sigma_\theta W(t) + \sum_{j=1}^{M(t)} Z_j^{(\theta)}\,, \quad t\geq 0\,, \tag{4.12}$$

for $\gamma_\theta = \delta+\theta(\gamma-\delta-\sigma^2/2)$, $\sigma_\theta=\theta\sigma$ and $Z_j^{(\theta)} = \ln(1+\theta(e^{Z_j}-1))$ iid, independent of the Poisson process M. This means that the Lévy measure $\nu(z) = \eta P(Z\leq z) = F_Z(z)$ of L is transformed into

$$\nu_\theta(z) = \eta P(\ln(1+\theta(e^Z-1))\leq z) = \eta P(Z^{(\theta)}\leq z) = \eta F_Z(\ln(1+(e^z-1)/\theta)). \tag{4.13}$$

Recall that L and L_θ jump at the same time and that a jump of size Z of L leads to a jump of size $\ln(1+\theta(e^Z-1)) > \ln(1-\theta)$ of L_θ.

In this case, it is not necessary to compensate the small jumps in the Lévy-Khinchine representation and the PIDE in (4.3) reduces to

$$\begin{aligned}
&\partial_t H(x,t) - \lambda\,(EH(x-Y,t)-H(x,t)) \\
&= \frac{\sigma_\theta^2}{2}(\partial_{xx}H(x,t)x^2+\partial_x H(x,t)x) - \gamma_\theta\partial_x H(x,t)x \\
&\quad + \int H(xe^z,t)\nu_\theta(-dz) - \eta H(x,t) + c\partial_x H(x,t)\,.
\end{aligned} \tag{4.14}$$

We can rewrite this as

$$\begin{aligned}&\partial_t H(x,t) - \lambda E H(x-Y,t) + (\lambda+\eta)H(x,t)\\ = &\frac{\sigma_\theta^2}{2}\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)((\frac{\sigma_\theta^2}{2}-\gamma_\theta)x + c) + \int H(xe^z,t)\nu_\theta(-dz)\,.\end{aligned}$$

This formula further simplifies, since

$$\int H(xe^z,t)\nu_\theta(-dz) \;=\; \eta\int H(xe^{-z},t)F_\theta(dz) \;=\; \eta E H(xe^{-Z^{(\theta)}},t)\,.$$

Then

$$\begin{aligned}&\partial_t H(x,t) - \lambda E H(x-Y,t) + (\lambda+\eta)H(x,t)\\ = &\frac{\sigma_\theta^2}{2}\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)\left((\frac{\sigma_\theta^2}{2}-\gamma_\theta)\,x + c\right) + \eta E H(xe^{-Z^{(\theta)}},t)\,.\end{aligned} \tag{4.15}$$

4.1.3 Numerical solution

For the PIDE (4.15) we present a numerical solution using a finite difference (FD) method.

Let us first emphasize that it is not known *a priori* whether a sufficiently smooth (or classical) solution exists; for more details on existence and uniqueness see Seydel [57].

For a numerical solution, we shall assume that the insurance claim Y and the market jump $Z^{(\theta)}$ are absolutely continuous with densities f_Y and f_θ, respectively. By (4.13) we can express f_θ in terms of the density f of a market jump Z of L:

$$f_\theta(z) = \begin{cases} f\left(\ln\left(1+(e^z-1)/\theta\right)\right)\dfrac{e^z}{e^z-1+\theta}, & z > \ln(1-\theta)\,,\\ 0\,, & z \le \ln(1-\theta)\,.\end{cases}$$

Rewriting (4.15) we have to solve the following initial value problem:

$$\begin{aligned}&\partial_t H(x,t) - \lambda\int_{-\infty}^{x} f_Y(x-y)H(y,t)dy + (\lambda+\eta)H(x,t)\\ = &\frac{\sigma_\theta^2}{2}\partial_{xx}H(x,t)x^2 + \partial_x H(x,t)\left((\frac{\sigma_\theta^2}{2}-\gamma_\theta)\,x + c\right) + \eta\int_{\ln(1-\theta)}^{\infty} f_\theta(z)H(xe^{-z},t)dz\,,\end{aligned} \tag{4.16}$$

with the initial condition $H(\cdot,0) = 1_{(-\infty,0)}$. With a further substitution $u = xe^{-z}$ in the market jump integral, we are able to apply numerical schemes to our problem.

The basic idea is to apply the FD method as for a standard initial value problem (or parabolic PDE). That is, we discretize the derivatives using standard finite differences. For the integrals (they integrate across space for a constant time), we substitute an integration formula, for instance the composite trapezoidal rule (a formula that is of order 2). For stability considerations, we discretize in time such that we obtain an implicit numerical scheme.

The infinite domains of integration require specific numerical treatment. We restrict the computation to the domain $(-R,R)\times(0,T)$ for some $R>0$. We use boundary conditions $H(-R,t) = 1$ and $H(R,t) = 0$ and approximate those parts of the integrals in (4.16) outside $(-R,R)$ for $x > 0$ by

$$\int_{-\infty}^{-R} f_Y(x-y)H(y,t)dy \approx \overline{F}_Y(x+R)$$

and for $x < 0$ and $-R > x/(1-\theta)$ (where we interpret $x/(1-\theta) = -\infty$ for $\theta = 1$) by

$$\int_{x/(1-\theta)}^{-R} -f_\theta(\ln(x/u))\frac{1}{u}H(u,t)du \approx \int_{\ln(1-\theta)}^{\ln(-x/R)} f_\theta(u)du = F\left(\ln\left(1+(-\frac{x}{R}-1)/\theta\right)\right) .$$

The localization error can be easily derived; see [57] for details.

The result of this discretization is a sequence of linear systems $AH^{(i+1)} = H^{(i)} + b$, $i = 0,\ldots,n$ for some $n \in \mathbb{N}$ with $H^{(0)} = 1_{(-\infty,0)}$. In contrast to an ordinary parabolic PDE, A is not a sparse but a dense matrix filled with entries from the two integrals.

Further details and extensions of the method (for instance an improved method of order 2 using a BDF2 discretization in time) can be found in Seydel [57]. We computed any of the illustrative results of Figure 4.1 using this improved FD method, comparing it with the results of a Monte Carlo simulation for verification.

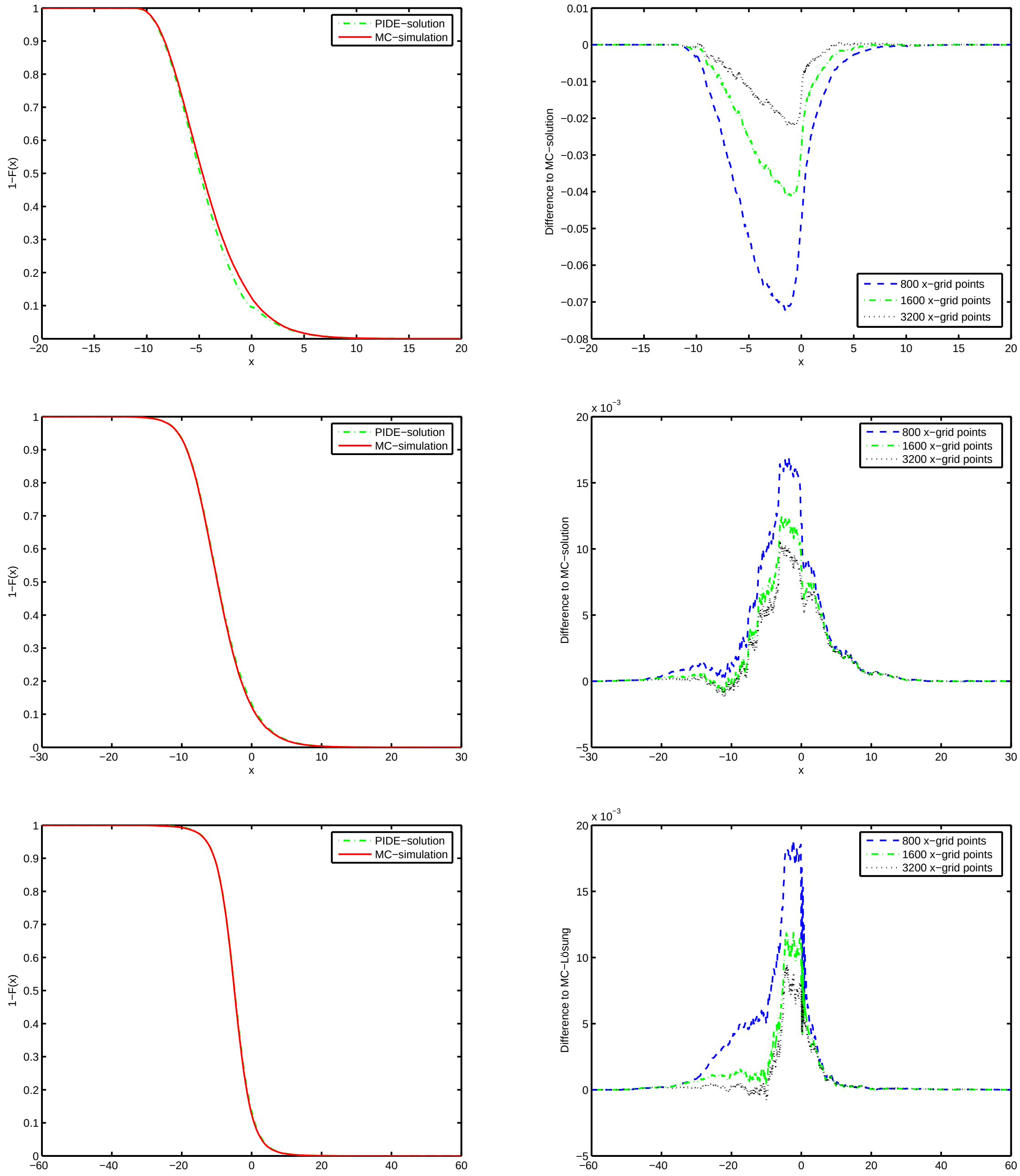

Figure 4.1: Numerical solution of $H(\cdot, T) = P(Q_\theta(T) > \cdot)$ for $T = 1$ in comparison to a Monte Carlo simulation (left: both solutions plotted, right: difference of both solutions) for three values of θ (first line: $\theta = 0.1$, middle line: $\theta = 0.5$ and last line: $\theta = 0.9$. The following set of parameters has been used. Insurance model: premium rate $c = 10$, standard exponential claim size Y, claims intensity $\lambda = 5$. Investment model: $\gamma = 0.2$, $\sigma = 0.4$, the jump intensity is $\eta = 3$, a jump Z is centered normal with variance 0.09. For the finite difference method we have used 800 x-grid points and 100 t-grid points.

Chapter 5

Discounted net loss process

Following long trandition in insurance, in this chapter we work with discounted losses. From a mathematical point of view we want to work with a stationary process aiming at a reasonable statistical risk assessment. Taking all this into account we introduce in Section 5.1 the discounted net loss process. This process describes the total net loss (both from insurance and investment) of the company, discounted to time 0. We derive certain quantities like characteristic function and moments, and discuss the Markov structure and the techniques for the simulation of the process. A key advantage of the process lies in the fact that it has a natural embedded discrete-time skeleton. This enables us, in Section 5.2, to give conditions under which the process has a stationary a.s. limit. In Section 5.3 we investigate the tail behaviour of the stationary distribution. We find out that the model carries a significant risk (heavy tails), driven eighter by large insurance claims or by the investment losses.

5.1 Definition and basic properties

Definition 5.1.1. *With the quantities introduced in Chapter 3 we define the* discounted net loss process (DNLP) *by*

$$V_\theta(t) = u - e^{-L_\theta(t)} U_\theta(t) = \int_0^t e^{-L_\theta(v)} \left(dS(v) - c\, dv \right), \quad t \geq 0\,. \tag{5.1}$$

□

5.1.1 Characteristic function and moments of the DNLP

First we calculate the characteristic function, the moment functions and the autocovariance function of the DNLP V_θ.

Lemma 5.1.2. *For $t \geq 0$ denote by $\widehat{v}_{\theta,t}(s) = E[\exp(isV_\theta(t))]$ and $\widehat{f}(s) = E[e^{isY}]$, $s \in \mathbb{R}$. Then*

$$\widehat{v}_{\theta,t}(s) \;=\; E\left[\exp\left(\int_0^t \left(\lambda(\widehat{f}(se^{-L_\theta(v)}) - 1) - icse^{-L_\theta(v)}\right) dv\right)\right].$$

Moreover, the following moment representations hold.
(a) Assume that $\varphi(1) < \infty$*. Then, for* $t \geq 0$*,* $E[V_\theta(t)]$ *exists and*

$$E[V_\theta(t)] = (\lambda\mu - c)\int_0^t E\left[e^{-L_\theta(v)}\right]\,dv = \begin{cases} = \frac{c-\lambda\mu}{\varphi_\theta(1)}(1 - e^{t\varphi_\theta(1)}) & \text{if } \varphi_\theta(1) \neq 0\,, \\ = (c - \lambda\mu)t & \text{if } \varphi_\theta(1) = 0\,. \end{cases} \tag{5.2}$$

(b) Assume that $\varphi(2) < \infty$ *and* $E[Y^2] = \mu_2 < \infty$*. Then for* $t \geq 0$*,* $\mathrm{var}(V_\theta(t))$ *exists and*

$$\mathrm{var}(V_\theta(t)) = \lambda\mu_2\int_0^t E\left[e^{-2L_\theta(v)}\right]\,dv + (c - \lambda\mu)^2\int_0^t\int_0^t \mathrm{cov}(e^{-L_\theta(v)}, e^{-L_\theta(w)})\,dw\,dv\,. \tag{5.3}$$

(c) Assume that $\varphi(2) < \infty$ *and* $E[Y^2] = \mu_2 < \infty$*. Then for* $0 \leq y \leq t$*,* $\mathrm{cov}(V_\theta(y),\, V_\theta(t))$ *exists and*

$$\mathrm{cov}(V_\theta(y),\, V_\theta(t)) = \mathrm{var}(V_\theta(y)) + (\lambda\mu - c)^2\int_0^{t-y} E\left[e^{-L_\theta(v)}\right]dv\int_0^y \mathrm{cov}(e^{-L_\theta(v)},\, e^{-L_\theta(y)})\,dv\,.$$

Proof. To obtain the chf of the DNLP we apply Lemma 2.2.9. Setting $Z(t) = S(t) - ct$, $t \geq 0$, we obtain $\Psi_Z(s) = \lambda(\widehat{f}(s) - 1) - ics$, $s \in \mathbb{R}$. Conditioning on the sample path of L up to time t, and using the notation $E_L[E[\cdot]] = E[E[\cdot \mid L(v),\, v \in (0,\, t]]]$ for $t \geq 0$, we have by independence of L and S for $s \in \mathbb{R}$,

$$\begin{aligned} \widehat{v}_{\theta,\,t}(s) &= E_L\left[E\left[\exp\left(is\int_0^t e^{-L_\theta(v)}dZ(v)\right)\right]\right] \\ &= E\left[\exp\left(\int_0^t \Psi_Z(se^{-L_\theta(v)})dv\right)\right] \\ &= E\left[\exp\left(\int_0^t \left(\lambda(\widehat{f}(se^{-L_\theta(v)}) - 1) - icse^{-L_\theta(v)}\right)dv\right)\right]. \end{aligned}$$

Then the moments of the process V_θ (if they exist, see Lemma 3.2.5(c)) can be obtained by taking derivatives of the chf in 0. For the autocovariance function we also need $E\left[V_\theta(y)e^{-L_\theta(y)}\right]$. We apply (2.5) again and obtain

$$\begin{aligned} &E\left[\exp(isV_\theta(y)e^{-L_\theta(y)})\right] \\ &= E\left[\exp\left(\int_0^y \left(\lambda\left(\widehat{f}(se^{-L_\theta(v)-L_\theta(y)}) - 1\right) - isce^{-L_\theta(v)-L_\theta(y)}\right)dv\right)\right]. \end{aligned}$$

Taking the first derivative of this chf in 0 we obtain

$$E\left[V_\theta(y)e^{-L_\theta(y)}\right] = (\lambda\mu - c)E\left[\int_0^y e^{-L_\theta(v)-L_\theta(y)}\,dv\right].$$

For the autocovariance function we calculate for $0 \leq y < t$

$$\begin{aligned} \mathrm{cov}(V_\theta(t),\, V_\theta(y)) &= E\left[V_\theta(t)V_\theta(y)\right] - E\left[V_\theta(t)\right]E\left[V_\theta(y)\right] \\ &= E\left[V_\theta(y)E\left[V_\theta(t) \mid \mathcal{F}_y\right]\right] - E\left[E\left[V_\theta(t) \mid \mathcal{F}_y\right]\right]E\left[V_\theta(y)\right]. \end{aligned}$$

We calculate the conditional expectation

$$\begin{aligned} E[V_\theta(t) \,|\, \mathcal{F}_y] &= E\left[V_\theta(y) + e^{-L_\theta(y)} \int_y^t e^{-(L_\theta(v)-L_\theta(y))}\, dZ(v) \,|\, \mathcal{F}_y\right] \\ &= V_\theta(y) + e^{-L_\theta(y)} E\left[\int_y^t e^{-(L_\theta(v)-L_\theta(y))}\, dZ(v)\right], \end{aligned}$$

where the last equality holds by the independent increments of L. By the stationarity increments property of L and Z we obtain

$$\int_y^t e^{-(L_\theta(v)-L_\theta(y))}\, dZ(v) \stackrel{d}{=} \int_0^{t-y} e^{-L_\theta(v)}\, dZ(v) \stackrel{d}{=} V_\theta(t-y)\,,$$

where the rv $V_\theta(t-y)$ is independent of $\mathcal{F}_y$. Hence we can write

$$\begin{aligned} &\operatorname{cov}(V_\theta(t),\, V_\theta(y)) \\ &= \operatorname{var}(V_\theta(y)) + E\left[V_\theta(t-y)\right]\left(E\left[V_\theta(y)e^{-L_\theta(y)}\right] - E\left[V_\theta(y)\right]E\left[e^{-L_\theta(y)}\right]\right) \\ &= \operatorname{var}(V_\theta(y)) + (\lambda\mu - c)^2 E\left[V_\theta(t-y)\right] \\ &\quad \times \int_0^y \left(E\left[X_\theta^{-1}(v)X_\theta^{-1}(y)\right] \,-\, E\left[X_\theta^{-1}(v)\right]E\left[X_\theta^{-1}(y)\right]\right)\, dv\,, \end{aligned}$$

which implies (c). □

Remark 5.1.3. Note that for $\varphi_\theta(1) < 0$ we have $\lim_{t\to\infty} EV_\theta(t) = (\lambda\mu - c)/|\varphi_\theta(1)|$. Under the net profit condition $c - \lambda\mu > 0$ the right hand side is negative. This can be interpreted that in this situation the mean profit is positive. □

In Section 5.2 we will consider in more details the limit distribution of the DNLP when the time horizon t goes to infinity.

5.1.2 Discrete time skeleton of the DNLP

Note that by (5.1) the SDE for the DNLP V_θ is

$$dV_\theta(t) = e^{-L_\theta(t)}\left(dS(t) - cdt\right)\,, \;\; t > 0, \quad V_\theta(0) = 0\,,$$

and by Theorem 2.2.1 and (3.4) for $t > 0$ for the jumps of the DNLP we have

$$\Delta V_\theta(t) = e^{L_\theta(t)} \Delta S(t)\,.$$

Hence, V_θ is not a time-homogeneous Markov process.

Anyway, it can be shown that the bivariate process $(V_\theta(t),\, L_\theta(t))_{t\geq 0}$ is a time-homogeneous Markov process.

Lemma 5.1.4. *Let the process V_θ be defined as in (5.1) and L_θ is the Lévy process defined in (3.4). Then the bivariate process $(V_\theta(t),\, L_\theta(t))_{t\geq 0}$ is a time-homogeneous Markov process.*

Proof. For a fixed time $t \geq 0$ consider the bivariate process $Z = (Z(s))_{s\geq 0}$ defined by

$$\begin{aligned} Z(s) &= (V_\theta(t+s) - V_\theta(t),\, L_\theta(t+s) - L_\theta(t)) \\ &= \left(\int_{t+}^{t+s} e^{-L_\theta(v)}\, d(S(v) - cv),\, L_\theta(s+t) - L_\theta(t) \right) \\ &= \left(e^{-L_\theta(t)} \int_{t+}^{t+s} e^{-(L_\theta(v) - L_\theta(t))}\, d(S(v) - cv),\, L_\theta(s+t) - L_\theta(t) \right) . \end{aligned}$$

Due to the stationary and independence increments of the Lévy processes we obtain

$$Z(s) \stackrel{d}{=} \left(e^{-L_\theta(t)} \int_0^s e^{-\widetilde{L}_\theta(v)}\, d(S(v) - cv),\, \widetilde{L}_\theta(s) \right) ,$$

where $\widetilde{L}_\theta$ is an independent copy of L_θ. Therefore Z and $\mathcal{F}_t = \sigma(L_\theta(v),\, S(v),\, v \leq t)$ are conditionally independent given $(V_\theta(t),\, L_\theta(t))$, which proves the Markov property. The time-homogenity follows from the fact that the Lévy processes L_θ and S are time-homogeneous. □

An interesting property of the DNLP is its diskrete time skeleton, which will prove useful in the sequel. Let $T_j \stackrel{d}{=} \sum_{k=1}^{j} E_k$, $j \in \mathbb{N}$, be the claim arrival times, where $(E_k)_{k\in\mathbb{N}}$ is a sequence of iid exponentially distributed rv's with parameter λ. We denote by E a generic rv of $(E_k)_{k\in\mathbb{N}}$. Recall that Y is a generic claim size. This allows us to introduce a natural discretization of the continuous time process V_θ given by $(V_\theta(T_k))_{k\in\mathbb{N}_0}$. We denote by

$$(A_\theta,\, B_\theta) = \left(Y e^{-L_\theta(E)} - c \int_0^E e^{-L_\theta(v)}\, dv \,,\, e^{-L_\theta(E)} \right) . \tag{5.4}$$

Proposition 5.1.5. *Set $T_0 = 0$ and note that $N(T_k) = k$. For $k \in \mathbb{N}$ define*

$$\begin{aligned} A_{\theta,k} &= \int_{T_{k-1}}^{T_k} e^{-(L_\theta(v) - L(T_{k-1}))} (dS(v) - c dv) \,, \\ B_{\theta,k} &= e^{-(L_\theta(T_k) - L_\theta(T_{k-1}))} \,. \end{aligned}$$

(a) Then $((A_{\theta,k}, B_{\theta,k}))_{k\in\mathbb{N}}$ is a sequence of iid bivariate rv's with the same distribution as the vector in (5.4).

(b) Define $(V_{\theta,k})_{k\in\mathbb{N}_0}$ by the following backward stochastic recurrence equation

$$V_{\theta,0} = 0 \quad \textit{and} \quad V_{\theta,k} = \sum_{m=1}^{k} A_{\theta,m} \prod_{j=1}^{m-1} B_{\theta,j} \,, \quad k \in \mathbb{N} \,. \tag{5.5}$$

Then, $V_\theta(T_k) = V_{\theta,k}$ for all $k \in \mathbb{N}$.

Proof. (a) is an immediate consequence of the stationary and independent increments of Lévy processes.

(b) For $k \in \mathbb{N}$ we have

$$\begin{aligned} V_\theta(T_k) &= \int_0^{T_k} e^{-L_\theta(v)}\,(dS(v) - cdv) \\ &= \int_0^{T_{k-1}} e^{-L_\theta(v)}\,(dS(v) - cdv) + \int_{T_{k-1}}^{T_k} e^{-L_\theta(v)}\,(dS(v) - cdv) \\ &= V_\theta(T_{k-1}) + e^{-L_\theta(T_{k-1})} \int_{T_{k-1}}^{T_k} e^{-(L_\theta(v) - L_\theta(T_{k-1}))}\,(dS(v) - cdv) \\ &= V_\theta(T_{k-1}) + \prod_{j=1}^{k-1} e^{-(L_\theta(T_j) - L_\theta(T_{j-1}))} \int_{T_{k-1}}^{T_k} e^{-(L_\theta(v) - L_\theta(T_{k-1}))}\,(dS(v) - cdv) \\ &= V_\theta(T_{k-1}) + A_{\theta,k} \prod_{j=1}^{k-1} B_{\theta,j}\,. \end{aligned}$$

We have used that for any Lévy process the stationary and independent increments property also holds for the random time intervals defined by $(T_j)_{j\in\mathbb{N}}$. Equation (5.5) follows then by iteration. □

The first application of the discrete time skeleton of the DNLP is towards the simulation of the process. Due to Proposition 5.1.5, we can simulate the DNLP at random discrete times, provided we can simulate the bivariate rv (A_θ, B_θ) defined in (5.4). This can be done only if we are able to simulate a sample path of the Lévy process L_θ, which may not always be possible, see Section 3.5.

We use the following algorithm:

1. Simulate an exponential random variable E with parameter λ – the claim interarrival time.

2. Simulate a claim Y.

3. Compute

$$(A_\theta,\, B_\theta) = \left(Y e^{-L_\theta(E)} - c \int_0^E e^{-L_\theta(v)}\,dv\,,\, e^{-L_\theta(E)} \right). \tag{5.6}$$

□

Repeating the above algorithm, we may simulate the discrete time skeleton of the DNLP using its backward stochastic recurrence equation (5.5). This method will allow us to draw an approximate sample path of the process. Whenever we are not interested in simulating a sample path, but rather in distributional properties of the DNLP, we may use the corresponding forward stochastic recurrence equation. We apply this method in the following example.

Example 5.1.6. [Simulation of the DNLP]
We aim at investigating some distributional properties of the DNLP given by

$$V_\theta(t) = \int_0^t e^{-L_\theta(v)}\,(dS(v) - cdv)\,,\;\; t \ge 0\,,$$

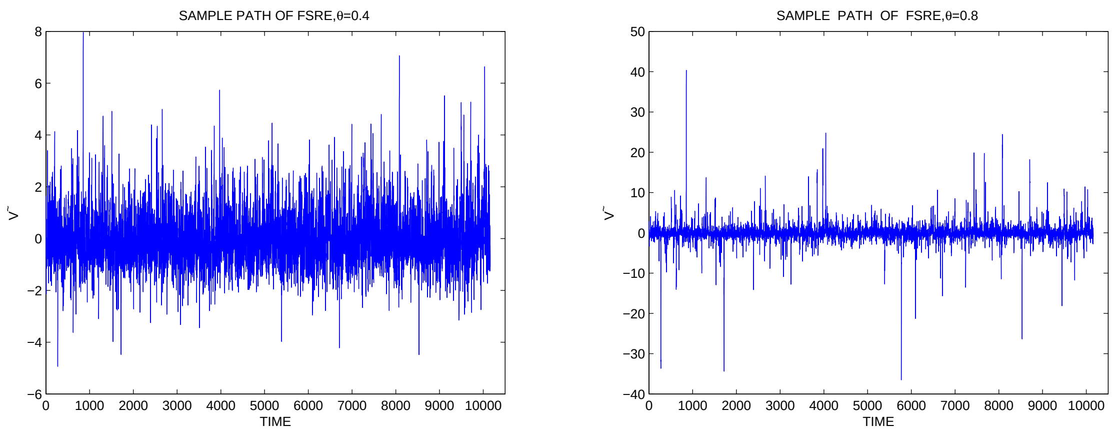

Figure 5.1: Sample path of $\widetilde{V}_\theta$ for $\theta = 0.4,\ 0.8$, the parameters are from Example 5.1.6.

with the following parameters:

- total insurance claims process: $S(t) = \sum_{j=1}^{N(t)} Y_j$, $Y \sim$ LOGN $(-0.01{,}0.124)$ (lognormally distriburted claims), claim intensity $\lambda = 0.1$ and premium rate $c = 0.1$;

- risky investment: L - Brownian motion with mean $\gamma = 0.08$ and standard deviation $\sigma = 0.35$;

- riskless interest rate $\delta = 0.04$;

We simulate $(\widetilde{V}_\theta^k)_{k=1}^n$ with $n = 10000$ using the FSRE $(\widetilde{V}_\theta^k \stackrel{d}{=} V_\theta(T_k))$

$$\widetilde{V}_\theta^0 = 0\,, \quad \widetilde{V}_\theta^k = \sum_{m=1}^{k} A_\theta^m \prod_{j=m+1}^{k} B_\theta^j\,, \ \ k \in \mathbb{N}\,.$$

In Figure 5.1 we show sample paths of $(\widetilde{V}_\theta)_{k\in\mathbb{N}_0}$ for $\theta = 0.4,\ 0.8$. Note that the higher the investment strategy θ (the more risky strategy) leads to larger jumps of the process. This is demonstrated also in Figure 5.2, where histograms of the simulated data for $\theta = 0,\ 0.4,\ 0.6$ and 1 are compared.

□

5.2 Stationarity of the DNLP

We are interested in possible stationarity of the discounted net loss process V_θ defined in 5.1. The following example is well-known in the case of $c = 0$. For a pure bond strategy, i.e. when $\theta = 0$, the DNLP converges to a rv with finite left endpoint, when the time goes to infinity. In particular, when the insurance claims are exponentially distributed, the discounted net loss process converges to a gamma distribution.

Example 5.2.1. [Pure bond strategy]

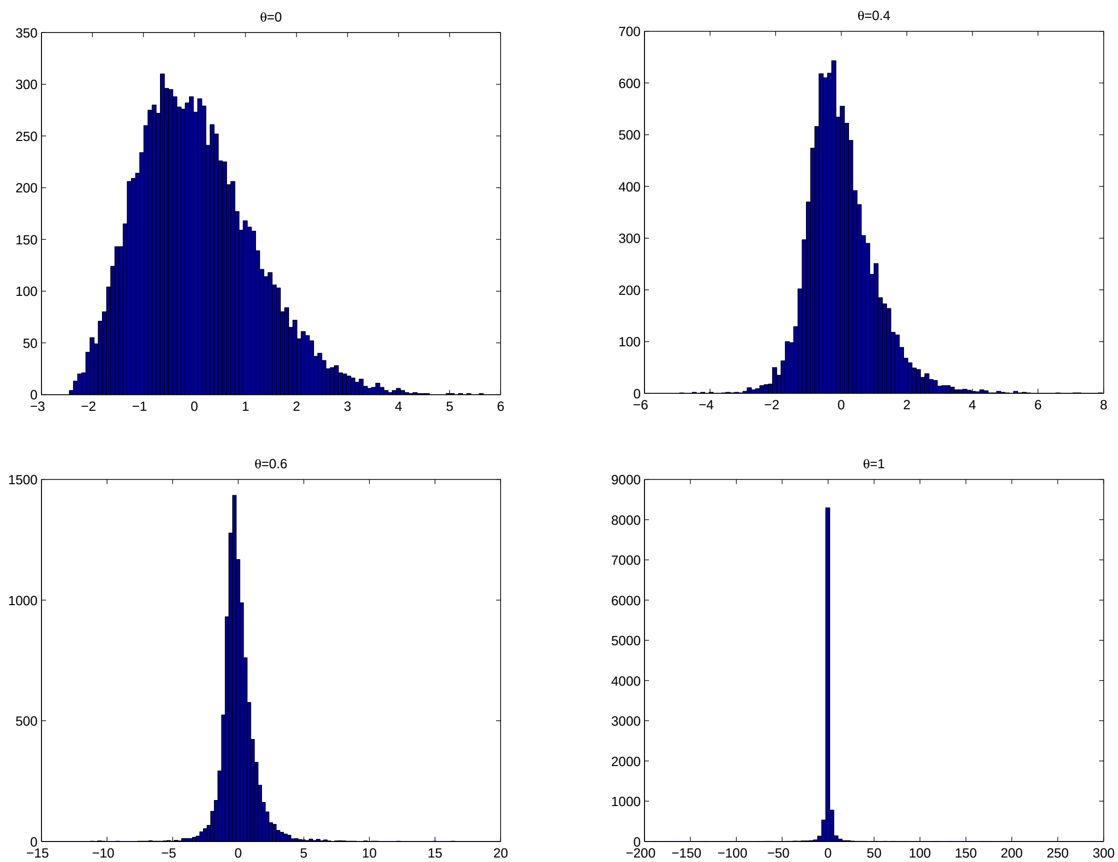

Figure 5.2: Histogram of $\widetilde{V}_\theta$ for $\theta = 0,\ 0.4,\ 0.6,\ 1$, the parameters are from Example 5.1.6.

For $\theta = 0$ we have $L_\theta(t) = \delta t$. Then for $s \in \mathbb{R}$ we get

$$\begin{aligned} E\left[e^{isV_0(t)}\right] &= \left[\exp\left(\lambda \int_0^t \left(E\exp\left(ise^{-\delta v}Y\right) - 1\right) dv\right) \exp\left(-isc\int_0^t e^{-\delta v}\, dv\right)\right] \\ &= \left[\exp\left(\frac{\lambda}{\delta}\int_{e^{-\delta t}}^1 \left(E\exp\left(ise^{-\delta v}Y\right) - 1\right)\frac{1}{y}\, dv\right) \exp\left(isc\,\frac{e^{-\delta t}-1}{\delta}\right)\right] \\ &\to \exp\left(\frac{\lambda}{\delta}\int_0^1 \left(Ee^{isyY} - 1\right)\frac{1}{y}\, dy\right) e^{-isc/\delta}, \quad t \to \infty . \end{aligned}$$

Denote by V_0^∞ the stationary rv. From the limit result above follows that V_0^∞ can be decomposed to $V_0^\infty = V_{0,+}^\infty - c/\delta$ where the random variable $V_{0,+}^\infty$ is a.s. positive. From this follows that the stationary rv V_0^∞ has no left tail and its left endpoint is $-c/\delta$.

Assume that the claims are exponentially distributed with density $f(y) = e^{-y/\mu}/\mu$, $y > 0$, and chf $\widehat{f}(s) = Ee^{isY} = (1 - is\mu)^{-1}$, $s \in \mathbb{R}$; then we get for $s \in \mathbb{R}$

$$\begin{aligned} \lim_{t\to\infty} E\left[e^{isV_0(t)}\right] &= \exp\left(\frac{\lambda}{\delta}\int_0^1 \left(\frac{1}{1 - is\mu y} - 1\right)\frac{1}{y}\, dy\right) e^{-isc/\delta} \\ &= e^{-isc/\delta}\left(1 - is\mu\right)^{-\lambda/\delta} . \end{aligned}$$

We recognise $(1 - is\mu)^{-\lambda/\delta}$ as the chf of a gamma distributed rv $X \stackrel{d}{=} \Gamma(\frac{\lambda}{\delta}, \frac{1}{\mu})$ with density $f_X(x) = \mu^{-\lambda/\delta} x^{-1+(\lambda/\delta)} e^{-x/\mu}/\Gamma(\lambda/\delta)$, $x > 0$. Consequently, we have shown that

$$V_0(t) \quad \stackrel{d}{\to} \quad V_0^\infty \stackrel{d}{=} \Gamma(\frac{\lambda}{\delta}, \frac{1}{\mu}) - \frac{c}{\delta}, \quad t \to \infty .$$

For $c = 0$ this is a well known result, see e.g. the introduction in Nilsen and Paulsen [48] and references therein. □

We now turn to the discounted net loss process for investment strategies $\theta > 0$. As this process is an exponential functional of a Lévy process and fits in the framework of generalized OU processes, and L_θ and S are independent processes, the NASCs of Proposition 2.4 of Lindner and Maller [44] apply to our situation. Whenever $L(t) \to \infty$ a.s. and the tail $\overline{F}(x) = 1 - F(x)$, $x > 0$, of the claim size distribution decreases to 0 not too slowly, then there exists a finite rv $V_\theta^{\infty,c}$ such that

$$V_\theta(t) \quad \stackrel{\text{a.s.}}{\to} \quad V_\theta^{\infty,c}, \quad t \to \infty . \tag{5.7}$$

Unfortuntely, for very few examples the stationary distribution is known. The following examples can be found in Carmona, Petit and Yor [9]. We present them in terms of our insurance application.

Example 5.2.2. [Geometric Brownian motion as risky investment process and small claims; continuation of Example 3.5.1]
Let the risky asset be modeled by a geometric Brownian motion. Then, according to Example 3.5.1, the resulting investment process is also geometric Brownian motion with parameters γ_θ and σ_θ given in (3.25). When the claims of a portfolio are sufficiently small,

it is possible to approximate the total claim amount process by Brownian motion. We consider this situation and take $(S(t) - ct)_{t\geq 0}$ as Brownian motion with drift $\lambda\mu - c < 0$ and variance $\lambda\mu$. Then $V_\theta^{\infty,c}$ follows a Pearson type IV distribution with density

$$f(x) = const.\,(1+x^2)^{-(\gamma_\theta/\sigma_\theta^2)+1/2} \exp\left(-\frac{2}{\sigma_\theta}\frac{c-\lambda}{\sqrt{\lambda\mu}}\arctan x\right), \quad x \in \mathbb{R}\,. \qquad \square$$

Example 5.2.3. [Geometric Brownian motion as risky investment process, exponential claims and no premiums; continuation of Example 3.5.1]
Let the risky asset be modeled by a geometric Brownian motion and γ_θ and σ_θ be given by (3.25). Assume that $c = 0$ and that the insurance claims are exponentially distributed with mean μ. Then it is shown in Nilsen and Paulsen [48] that

$$V_\theta^{\infty,c} \stackrel{d}{=} \frac{X}{Z}\,,$$

where $X \sim \Gamma(b, \frac{1}{\mu})$ with density $f_X(x) = \mu^{-b}x^{b-1}e^{-x/\mu}/\Gamma(b)$, $x > 0$, and is independent of Z, which is beta distributed with density

$$f_Z(x) = \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b+1)}x^{a-1}(1-x)^{b-1}\,, \quad 0 < x < 1\,,$$

where

$$a = \frac{2\gamma_\theta}{\sigma_\theta^2} \quad \text{and} \quad b = \frac{\gamma_\theta}{\sigma_\theta^2}\left(\sqrt{1+\frac{2\lambda\sigma_\theta^2}{\gamma_\theta^2}} - 1\right).$$

Straightforward calculations show that the density of $1/Z$

$$f_{1/Z}(x) \sim \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b+1)}x^{-a-1}\,, \quad x \to \infty\,.$$

Hence the corresponding distribution tail

$$\overline{F}_{1/Z}(x) \sim \frac{\Gamma(a+b+1)}{a\,\Gamma(a)\Gamma(b+1)}x^{-a}\,, \quad x \to \infty\,.$$

On the other hand, the rv X has light right tail and is independent of Z. Consequently, by Breiman's classical result

$$P(V_\theta^{\infty,c} > x) \sim const.\,x^{-a}\,, \quad x \to \infty\,,$$

with a as above. This will be confirmed by our result in Theorem 5.3.6; see also Example 5.3.8 below. $\square$

For more general models the theory of discrete and continuous time perpetuities can provide at least the tail behaviour of such models. The advantage of our model lies in the fact that it has a natural discrete time skeleton, given by the sequence $(V_{\theta,k})_{k\in\mathbb{N}_0}$ as introduced in Proposition 5.1.5. This discretization of the DNLP allows us to apply standard methods from the theory of stochastic recurrence equations, see for example Kesten [36] and Goldie [27].

For the the discrete time process given by the sequence $(V_{\theta,k})_{k\in\mathbb{N}_0}$ as defined in (5.5) Goldie and Maller [29] derive NASCs for stationarity; see their Theorem 2.1. In our insurance context it is, however, more natural to work with moment conditions, which are slightly weaker. They are stated and discussed in Corollary 4.1 of [29], where also precise references to earlier work can be found, see also conditions (2.9) in Proposition 2.3.1. In the next lemma we show that these conditions are satisfied in our model under weak conditions. Recall the notation for the Laplace exponents φ and φ_θ in (3.5).

Lemma 5.2.4. *Assume that $E[Y] = \mu < \infty$, $E[L(1)] > 0$ and $\varphi_\theta(1) < \lambda$. Then for all $\theta \in [0, 1]$ for the rv's A_θ and B_θ defined in* (5.4) *holds*

(a) $E\left[\log^+ |A_\theta|\right] \leq \dfrac{\lambda\mu + c}{\lambda - \varphi_\theta(1)} < \infty\,;$

(b) $-\infty \leq E\left[\log |B_\theta|\right] = -\frac{1}{\lambda} E[L_\theta(1)] < 0\,.$

Proof. We first prove (b). From Lemma 3.2.5(b) we know that $E[L_\theta(1)] > 0$. Further, as $E[L_\theta(1)] < \infty$, then $E[L_\theta(t)] = tE[L_\theta(1)]$ (see Sato [56], E25.12, Formula (25.7) at p. 163). Then we obtain

$$E[\log |B_\theta|] = -E[L_\theta(E)] = -\lambda \int_0^\infty E[L_\theta(z)] e^{-\lambda z}\, dz = -\frac{E[L_\theta(1)]}{\lambda} < 0\,.$$

For the proof of (a) we use the fact that for any a.s. positive rv X holds $\log X < X$ a.s. and $\max(0, \log X) \leq X$; hence $E[\max(0, \log X)] \leq E[X]$. Then we estimate

$$\begin{aligned} E[\log^+ |A_\theta|] &= E\left[\log^+ \left| Y e^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}\, dv\right|\right] \\ \leq E\left[\left| Y e^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}\, dv\right|\right] &\leq \mu E\left[e^{-L_\theta(E)}\right] + cE\left[\int_0^E e^{-L_\theta(v)}\, dv\right]\,. \end{aligned}$$

Now for the first summand we calculate

$$E\left[e^{-L_\theta(E)}\right] = \lambda \int_0^\infty e^{\varphi_\theta(1) z} e^{-\lambda z}\, dz = \frac{\lambda}{\lambda - \varphi_\theta(1)} < \infty\,,$$

as $\varphi_\theta(1) < \lambda$. For the second summand we write

$$E\left[\int_0^E e^{-L_\theta(v)}\, dv\right] = \lambda \int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)} dv\right) e^{-\lambda z}\, dz\,.$$

If $\varphi_\theta(1) = 0$, then the last term is equal to $1/\lambda < \infty$. If $\varphi_\theta(1) \neq 0$, then, as $\varphi_\theta(1) < \lambda$, we have

$$\lambda \int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)} dv\right) e^{-\lambda z}\, dz = \frac{\lambda}{\varphi_\theta(1)} \int_0^\infty e^{-z(\lambda - \varphi_\theta(1))} dz - \frac{1}{\varphi_\theta(1)} = \frac{1}{\lambda - \varphi_\theta(1)} < \infty\,.$$

□

With Lemma 5.2.4 we obtain the following result for the convergence of the DNLP as the time goes to infinity.

Theorem 5.2.5. *Assume that* $E[Y] = \mu < \infty$*,* $E[L(1)] > 0$ *and* $\varphi_\theta(1) < \lambda$*. Let the discrete time process* $(V_{\theta,k})_{k\in\mathbb{N}_0}$ *be defined as in* (5.5).
(a) Then

$$V_{\theta,k} \quad \overset{\text{a.s.}}{\to} \quad V_\theta^\infty = \sum_{m=1}^{\infty} A_{\theta,m} \prod_{j=1}^{m-1} B_{\theta,j}\,, \quad k \to \infty\,, \tag{5.8}$$

where the series on the rhs converges absolutely with probability 1.
Moreover, V_θ^∞ *satisfies the identity in law*

$$V_\theta^\infty \overset{d}{=} A_\theta + B_\theta V_\theta^\infty\,, \tag{5.9}$$

where V_θ^∞ *and* $(A_\theta,\, B_\theta)$ *are independent.*
(b) Let the discounted net loss process $(V_\theta(t))_{t\geq 0}$ *be defined by equation* (5.1)*. Then* $V_\theta(t)$ *converges a.s. if and only if* $V_{\theta,k}$ *does and*

$$V_\theta^\infty = V_\theta^{\infty,c} \quad a.s.\,. \tag{5.10}$$

Proof. (a) Stationarity of the discrete time process $(V_{\theta,k})_{k\in\mathbb{N}}$ is usually proved via the corresponding *backward stochastic recurrence equation*, see Proposition 2.3.1. In order to prove (5.9) we introduce the rv's $\widetilde{V}_{\theta,k}$ for $k \in \mathbb{N}_0$ invoking the same iid sequence $((A_{\theta,k}, B_{\theta,k}))_{k\in\mathbb{N}}$ as above:

$$\widetilde{V}_{\theta,0} = 0 \quad \text{and} \quad \widetilde{V}_{\theta,k} \;=\; A_{\theta,k} + \widetilde{V}_{\theta,k-1} B_{\theta,k} = \sum_{m=1}^{k} A_{\theta,m} \prod_{j=m+1}^{k} B_{\theta,j}\,, \quad k \in \mathbb{N}\,.$$

We observe that for every $k \in \mathbb{N}$

$$((A_{\theta,j}, B_{\theta,j}))_{1\leq j\leq k} \overset{d}{=} ((A_{\theta,k-j+1}, B_{\theta,k-j+1}))_{1\leq j\leq k}\,,$$

implying that

$$\sum_{m=1}^{k} A_{\theta,m} \prod_{j=1}^{m-1} B_{\theta,j} \overset{d}{=} \sum_{m=1}^{k} A_{\theta,m} \prod_{j=m+1}^{k} B_{\theta,j}\,,$$

hence $V_{\theta,k} \overset{d}{=} \widetilde{V}_{\theta,k}$ for all $k \in \mathbb{N}$. The result goes back to Kesten [36] (see his Theorem 5); see also Proposition 2.3.1 which states the result with proof.

(b) Consider the continuous time process V_θ.

$$\begin{aligned} V_\theta(t) \;&\overset{a.s.}{=}\; \int_0^{T_{N(t)}} e^{-L_\theta(v)}(dS(v) - cdv) + \int_{T_{N(t)}}^{t} e^{-L_\theta(v)}(dS(v) - cdv) \\ &=\; V_{\theta,N(t)} + e^{-L_\theta(T_{N(t)})} \int_{T_{N(t)}}^{t} e^{-(L_\theta(v) - L_\theta(T_{N(t)}))}\,(dS(v) - cdv)\,, \end{aligned}$$

where in the last line the integral is independent of the first summand. As $N(t) \overset{\text{a.s.}}{\to} \infty$ when $t \to \infty$, we know from part (a) that $V_{\theta,N(t)} \overset{\text{a.s.}}{\to} V_\theta^\infty$ when $t \to \infty$.Moreover, as

$E[L(1)] > 0$, we have by Lemma 3.2.5(b) that $E[L_\theta(1)] > 0$ and hence $e^{-L_\theta(T_{N(t)})} \stackrel{\text{a.s.}}{\to} 0$ when $t \to \infty$. Finally, as $t - T_{N(t)} \stackrel{d}{=} E$, the last integral is a finite random variable. This implies (5.10). $\square$

5.3 Tail behaviour of the a.s. limit of the DNLP

From now on in most of our results we exclude the pure bond strategy and assume that $\theta \in (0, 1]$. Moreover, we assume that the conditions of Theorem 5.2.5 hold. Then the stationary random variable V_θ^∞ exists and satisfies the fix point equation (5.9). As we are interested in distributional properties of V_θ^∞ we can work with the continuous time process or with the discrete skeleton process as they both lead to the same a.s. limit.

Our next goal is the tail behaviour of V_θ^∞. To this end we start with some preliminary results on Laplace transforms.

Lemma 5.3.1. *Let $\theta \in (0, 1]$ and assume that $0 < E[L(1)] < \infty$, and either $\sigma > 0$ or $\nu((-\infty, 0)) > 0$. Define $\mathcal{V}_\infty = \{s \geq 0 : \varphi_1(s) < \infty\}$.*

(a) For every $\theta \in (0, 1)$, then there exists a unique positive $\kappa = \kappa(\theta) > 0$ such that $\varphi_\theta(\kappa) = 0$. Moreover, $\varphi'_\theta(\kappa) \in (0, \infty)$ and

$$\kappa(\theta) \begin{cases} > 1 & \text{if} \varphi_\theta(1) < 0\,, \\ = 1 & \text{if } \varphi_\theta(1) = 0\,, \\ < 1 & \text{if } \varphi_\theta(1) \in (0, \lambda)\,. \end{cases} \tag{5.11}$$

If in addition $v_1^* = \sup \mathcal{V}_\infty \notin \mathcal{V}_\infty$, then there exists a unique positive $\kappa = \kappa(1) > 0$ such that $\varphi_1(\kappa) = \varphi(\kappa) = 0$.

(b) Assume that $\delta < \varphi(-1)$. Then the function $\kappa(\theta)$ as defined in (a) is decreasing in θ.

Proof. (a) For $\theta < 1$ we set $p = \log((1 + \theta^{-1}(e - 1)) > 0$, $q = -\infty$, if $\theta^{-1}(1 - e^{-1}) \geq 1$, and $q = \log(1 + \theta^{-1}(e^{-1} - 1)) < 0$, if $\theta^{-1}(1 - e^{-1}) < 1$. Then

$$\begin{aligned} \int_{|x|\geq 1} e^{-sx}\nu_\theta(dx) &= \int_{|\log(1+\theta(e^x-1))|\geq 1} (1 + \theta(e^x - 1))^{-s}\nu(dx) \\ &= \int_{-\infty}^{q} (1 + \theta(e^x - 1))^{-s}\nu(dx) + \int_p^\infty (1 + \theta(e^x - 1))^{-s}\nu(dx) \\ &\leq (1 - \theta)^{-s} \int_{-\infty}^{q} \nu(dx) + e^{-s} \int_p^\infty \nu(dx) \,<\, \infty\,. \end{aligned}$$

By Proposition 3.14 in Cont and Tankov [10] follows that for all $\theta < 1$ $\varphi_\theta(s) < \infty$ for all $s \in \mathbb{R}^+$. On the other hand, for $\theta \in (0, 1]$ we have

$$\begin{aligned} E\left[e^{-sL_\theta(1)}\right] &= P(L_\theta(1) < 0)E[e^{-sL_\theta(1)} \mid L_\theta(1) < 0] + P(L_\theta(1) \geq 0)E[e^{-sL_\theta(1)} \mid L_\theta(1) \geq 0] \\ &\geq P(L_\theta(1) < 0)E[e^{-sL_\theta(1)} \mid L_\theta(1) < 0)]\,. \end{aligned}$$

Note that $\lim_{s\to\infty} E[e^{-sL_\theta(1)} \,|\, L_\theta(1) < 0] = \infty$. Since by Lemma 3.2.6 holds $P(L_\theta(1) < 0) > 0$, for $\theta \in (0,1]$ we have

$$\lim_{s\to\infty} E\left[e^{-sL_\theta(1)}\right] = \lim_{s\to\infty} e^{\varphi_\theta(s)} = \infty\,. \tag{5.12}$$

Then the existence of $\kappa(\theta)$ for $\theta \in (0,1)$ follows from the convexity of $\varphi_\theta(s)$ in s and the fact that $\varphi_\theta(0) = 0$, $\varphi'_\theta(0) = -E[L_\theta(1)] \in (-\infty, 0)$ (see Lemma 3.2.5 (a) and (b)) and $\varphi_\theta(s) < \infty$ for all $s \in \mathbb{R}^+$.
The same arguements guarantee the existence of $\kappa(1)$ (the case when $\theta = 1$) if $v^* = \infty$. If $v^* < \infty$, the existence of $\kappa(1)$ follows straightforward from the convexity of $\varphi_1(s)$ and the fact that $\lim_{s\to v^*} \varphi_1(s) = \infty$.

(b) First recall that by Lemma 3.2.5(d) for any fixed $s > 0$ the function $\varphi(\theta, s) = \varphi_\theta(s)$ is convex in θ. Consider $0 \le \theta_1 < \theta_2 \le 1$. We shall show that $\kappa(\theta_1) > \kappa(\theta_2) > 0$. To this end fix $s = \kappa(\theta_2)$ and consider its corresponding value $\theta_*(s) = \operatorname{argmin}_\theta \varphi(\theta, s)$. Assume first that $\theta_*(\kappa(\theta_2)) \ge \theta_2$. Then $\varphi(\theta,\, \kappa(\theta_2))$ is decreasing in $[0,\, \theta_2)$ and hence

$$0 = \varphi(\theta_2,\, \kappa(\theta_2)) \le \varphi(0,\, \kappa(\theta_2)) = -\delta\kappa(\theta_2) < 0\,,$$

which is a contradiction. Hence $\theta_*(\kappa(\theta_2)) < \theta_2$. So for $\theta \in [0,\, \theta_*(\kappa(\theta_2))\,)$, the function $\varphi(\theta,\, \kappa(\theta_2))$ is decreasing in θ and for $\theta \in (\theta_*(\kappa(\theta_2)),\, 1]$, the function $\varphi(\theta,\, \kappa(\theta_2))$ is increasing in θ. Next consider $\theta_1 < \theta_*(\kappa(\theta_2))$. This implies

$$\varphi(\theta_1,\, \kappa(\theta_2)) < \varphi(0,\, \kappa(\theta_2)) < 0 = \varphi(\theta_1,\, \kappa(\theta_1))\,,$$

which – by convexity of $\varphi(\theta,\, s)$ in s (for fixed θ) – implies that $\kappa(\theta_1) > \kappa(\theta_2)$. Assume now that $\theta_*(\kappa(\theta_2)) \le \theta_1 < \theta_2$, where $\varphi(\theta,\, s)$ is increasing in θ. Then

$$\varphi(\theta_1,\, \kappa(\theta_2)) < \varphi(\theta_2,\, \kappa(\theta_2)) = 0 = \varphi(\theta_1,\, \kappa(\theta_1))\,,$$

hence, again by convexity of $\varphi(\theta,\, s)$ in s, we obtain $\kappa(\theta_1) > \kappa(\theta_2)$. This completes the proof. □

5.3.1 Claims with finite moment of order κ

To the end of Section 5.3 we fix the investment strategy $\theta \in (0,1]$. The next lemma concerns properties of the following Laplace exponent

$$l_\theta(s) = \log E[e^{-sL_\theta(E)}] = \log E[B_\theta^s] = \log \frac{\lambda}{\lambda - \varphi_\theta(s)}\,. \tag{5.13}$$

First note that $l_\theta(s) < \infty$ on $\mathcal{S}_\theta = \{v \ge 0 \,:\, \varphi_\theta(v) < \lambda\}$ and $\sup \mathcal{S}_\theta \notin \mathcal{S}_\theta$.

Lemma 5.3.2. *Let the conditions of Lemma 5.3.1 hold and $\kappa = \kappa(\theta) \in (0,\infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Then the following hold.*
(a) l_θ is strictly convex and continuously differentiable on the interior of $\mathcal{S}_\theta$ and $l_\theta(\kappa) = 0$.
(b) There exists $\beta = \beta(\theta) > 0$ such that $l_\theta(\kappa + \beta) < \infty$.
(c) $l'_\theta(\kappa) \in (0,\, \infty)$ and $P(B_\theta > 1) > 0$.

Proof. (a) follows as in Lemma 5.3.1 and by definition of κ.
For (b) we note that from $\varphi_\theta(0) = \varphi_\theta(\kappa) = 0$ and strict convexity of $\varphi_\theta(s)$ in s follows that $\varphi_\theta(s) < 0$ for $s \in (0, \kappa)$. As $\lambda > 0$, there exist $b \in (0, \lambda)$ and $\beta > 0$, such that $\varphi_\theta(\kappa + \beta) = b < \lambda$. Hence, for this $\beta > 0$ we have $l_\theta(\kappa + \beta) < \infty$.
The first part of (c) follows from $l'_\theta(\kappa) = \lambda^{-1}\varphi'_\theta(\kappa) \in (0, \infty)$ as $\varphi'_\theta(\kappa) \in (0, \infty)$, see Lemma 5.3.1(a).
For the second part of (c) we use continuity in probability of Lévy processes. Since $P(L_\theta(1) < 0) > 0$, there exist $\eta, \epsilon > 0$, such that for $t \in (1 - \epsilon, 1 + \epsilon)$ we have $P(L_\theta(t) < 0) \geq \eta$. Then

$$P(B_\theta > 1) = P(L_\theta(E) < 0) \geq \lambda \int_{1-\epsilon}^{1+\epsilon} P(L_\theta(z) < 0)e^{-\lambda z}dz \geq \eta\lambda \int_{1-\epsilon}^{1+\epsilon} e^{-\lambda z}dz > 0\,.$$

□

Now we can show the following result, which is needed to apply Theorem 2.3.3.

Lemma 5.3.3. *Let the conditions of Lemma 5.3.1 be satisfied and $\kappa = \kappa(\theta) \in (0, \infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Then*
(a) $E\left[B_\theta^\kappa\right] = 1$;
(b) $E\left[B_\theta^\kappa \log^+ B_\theta\right] < \infty$;
(c) if $E\left[Y^q\right] < \infty$ for some $q \geq 1$, then $E\left[\,|A_\theta|^{\min(q,\kappa)}\,\right] < \infty$.

Proof. From (5.13) we know that $E\left[B_\theta^s\right] = e^{l_\theta(s)}$, hence (a) follows directly from the definition of $\kappa(\theta)$ as in Lemma 5.3.2.
To prove (b) first note that

$$E\left[B_\theta^\kappa \log^+ B_\theta\right] = E\left[e^{-\kappa L_\theta(E)} \max(0, \log e^{-L_\theta(E)})\right] = -E\left[L_\theta(E)e^{-\kappa L_\theta(E)} 1_{\{L_\theta(E) \leq 0\}}\right]\,.$$

Using Lemma 5.3.2(c) and the fact that $E\left[B_\theta^\kappa\right] = 1$, we calculate

$$-E\left[L_\theta(E)e^{-\kappa L_\theta(E)}\right] = \frac{d}{ds}E[B_\theta^s]_{|s=\kappa} = \frac{d}{ds}\log E[B_\theta^s]_{|s=\kappa} = l'_\theta(\kappa) < \infty\,.$$

On the other hand

$$E\left[L_\theta(E)e^{-\kappa L_\theta(E)}\right] = E\left[L_\theta(E)e^{-\kappa L_\theta(E)} 1_{\{L_\theta(E) \leq 0\}}\right] + E\left[L_\theta(E)e^{-\kappa L_\theta(E)} 1_{\{L_\theta(E) > 0\}}\right]\,.$$

So we can write

$$-E\left[L_\theta(E)e^{-\kappa L_\theta(E)} 1_{\{L_\theta(E) \leq 0\}}\right] = l'_\theta(\kappa) + E\left[L_\theta(E)e^{-\kappa L_\theta(E)} 1_{\{L_\theta(E) > 0\}}\right]$$

Now as $\kappa > 0$, for a positive rv X we have that $\kappa X \leq e^{\kappa X}$ a.s., or $Xe^{-\kappa X} \leq \kappa^{-1}$ a.s., hence

$$E\left[L_\theta(E)e^{-\kappa L_\theta(E)} 1_{\{L_\theta(E) > 0\}}\right] \leq \frac{1}{\kappa}P(L_\theta(E) > 0) \leq \frac{1}{\kappa} < \infty\,.$$

To prove (c) we consider two cases.
Assume first that $\kappa \le 1 \le q$ and observe that then $\varphi_\theta(1) \ge 0$. As the function $f(x) = x^\kappa$ is concave on $\mathbb{R}^+$, $|x+y|^\kappa \le |x|^\kappa + |y|^\kappa$ for any $x,\, y \in \mathbb{R}$. Hence we estimate

$$\begin{aligned} E\left[|A_\theta|^\kappa\right] &= E\left[\left|Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}\,dv\right|^\kappa\right] \\ &\le E\left[\left(Ye^{-L_\theta(E)}\right)^\kappa\right] + c^\kappa E\left[\left(\int_0^E e^{-L_\theta(v)}\,dv\right)^\kappa\right]. \end{aligned}$$

The first term on the rhs of the inequality is finite as $E[Y^\kappa] < \infty$ and $E[e^{-\kappa L_\theta(E)}] = 1$ and both rv's are independent. For the second term Jensen's inequality yields

$$\begin{aligned} E\left[\left(\int_0^E e^{-L_\theta(v)}\,dv\right)^\kappa\right] &= \lambda\int_0^\infty E\left[\left(\int_0^z e^{-L_\theta(v)}\,dv\right)^\kappa\right] e^{-\lambda z}\,dz \\ \le \lambda\int_0^\infty \left(\int_0^z E\left[e^{-L_\theta(v)}\right]\,dv\right)^\kappa e^{-\lambda z}\,dz &= \lambda\int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)}\,dv\right)^\kappa e^{-\lambda z}\,dz\,. \end{aligned}$$

If $\varphi_\theta(1) = 0$ then the last term is equal to $E[E^\kappa] < \infty$ as E is exponentially distributed. If $\varphi_\theta(1) \ne 0$, then we have

$$\begin{aligned} \lambda\int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)}\,dv\right)^\kappa e^{-\lambda z}\,dz &= \frac{\lambda}{\varphi_\theta^\kappa(1)}\int_0^\infty \left(e^{z\varphi_\theta(1)} - 1\right)^\kappa e^{-\lambda z}\,dz \\ &\le \frac{\lambda}{\varphi_\theta^\kappa(1)}\int_0^\infty e^{-z(\lambda - \kappa\varphi_\theta(1))}\,dz < \infty\,, \end{aligned}$$

provided that $\varphi_\theta(1) < \lambda/\kappa$, which is satisfied for $\varphi_\theta(1) < \lambda$ as $\kappa \le 1$.

Now assume that $g = \min(\kappa, q) > 1$. Then the function $f(x) = x^g$ is convex. First use in the second inequality below Jensen's inequality

$$\begin{aligned} E\left[|A_\theta|^g\right] &= E\left[\left|Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}dv\right|^g\right] \\ &\le E\left[\left(Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}dv\right)^g\right] \\ &\le 2^{g-1}\left(E\left[Y^g e^{-gL_\theta(E)}\right] + c^g E\left[\left(\int_0^E e^{-L_\theta(v)}dv\right)^g\right]\right). \end{aligned} \tag{5.14}$$

Now again from Jensen's inequality for the second expectation in (5.14) we have

$$\begin{aligned} E\left[\left(\int_0^E e^{-L_\theta(v)}dv\right)^g\right] &= \lambda\int_0^\infty E\left[\left(\int_0^z e^{-L_\theta(v)}dv\right)^g\right] e^{-\lambda z}dz \\ \le \lambda\int_0^\infty z^{g-1}\int_0^z E\left[e^{-gL_\theta(v)}\right] dv\, e^{-\lambda z}dz &\le \lambda\int_0^\infty z^{g-1} z\, e^{-\lambda z}dz = E\left[E^g\right] < \infty\,, \end{aligned} \tag{5.15}$$

as $E[e^{-gL_\theta(v)}] \le 1$. For the first expectation in (5.14) we have

$$E\left[Y^g e^{-gL_\theta(E)}\right] = E[Y^g]E[B_\theta^g] < \infty$$

by part (a), and the proof is completed. □

Theorem 2.3.3 guarantees under natural conditions, which hold by Lemma 5.3.3, that V_θ^∞ has a heavy left **or** right tail. In the context of risk management, however, only the right tail is of prime interest. Invoking the theory of large deviations as suggested in the context of ruin theory by Nyrhinen [49] gives us a method to decide about right and left tails separately. Lemma 5.3.4 is a large deviations result. We largely follow Nyrhinen [49] with adaptations to our situation.

Lemma 5.3.4. *Let the conditions of Lemma 5.3.1 hold and $\kappa = \kappa(\theta) \in (0, \infty)$ be the unique positive value satisfying $\varphi_\theta(\kappa) = 0$. Assume also that Y has unbounded support and that $E[Y] < \infty$. Then*

$$\liminf_{x\to\infty} \frac{\log P(V_\theta^\infty > x)}{\log x} \geq -\kappa(\theta) \quad \textit{and} \quad \liminf_{x\to\infty} \frac{\log P(V_\theta^\infty < -x)}{\log x} \geq -\kappa(\theta)\,. \tag{5.16}$$

□

Before starting with the proof, we introduce some notation first. Set

$$m(\theta) = l'_\theta(\kappa(\theta))\,. \tag{5.17}$$

For $d \in (0,\, 1/m(\theta))$, $\epsilon' > 0$ and $n \in \mathbb{N}$ define the subsets $D_n = D_n(d,\, \epsilon')$ and $E_n = E_n(d,\, \epsilon')$ of Ω by

$$\begin{aligned} D_n &= \Big\{\omega \in \Omega : \Big|\frac{1}{n}\sum_{j=1}^{\lceil \alpha n\rceil} \log B_{\theta,j} - \alpha m(\theta)\Big| \leq \epsilon'\,,\ 0 < \alpha \leq 1/m(\theta) - d\Big\}\,, \\ E_n &= \big\{\omega \in \Omega : |A_{\theta,j}| \leq e^{\epsilon' n}\,,\ j = 1, \ldots, \lceil (1/m(\theta) - d)n\rceil\big\}\,. \end{aligned} \tag{5.18}$$

The following lemma is the key for the proof of Lemma 5.3.4.

Lemma 5.3.5. *Let the conditions of Lemma 5.3.1 hold and $\kappa = \kappa(\theta) \in (0, \infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Let also $EY < \infty$. Then for any $d \in (0,\, 1/m(\theta))$ there exists some $\epsilon' > 0$ such that*

$$\liminf_{n\to\infty} \frac{\log P(D_n(d,\, \epsilon') \cap E_n(d,\, \epsilon'))}{n} \quad \geq \quad -\kappa(\theta)\,. \tag{5.19}$$

Proof. Recall that under the probability measure P the sequence $((A_{\theta,k},\, B_{\theta,k}))_{k\in\mathbb{N}}$ consists of iid random vectors all distributed like $(A_\theta,\, B_\theta)$ as defined in (5.4). Define a new probability measure Q by

$$dQ(y_1,\, y_2) = y_2^{\kappa(\theta)} dP(y_1,\, y_2)\,,$$

for $(y_1,\, y_2) \in \mathbb{R}^2$ and such that $((A_{\theta,k},\, B_{\theta,k}))_{k\in\mathbb{N}}$ is again a sequence of iid random vectors with respect to Q. We denote by E_Q the expectation under Q. Then, for $k \in \mathbb{N}$ and any measurable function $f : \mathbb{R}^{2k} \to \mathbb{R}$ we have

$$\begin{aligned} &E\left[f((A_{\theta,1},\, B_{\theta,1}), \ldots, (A_{\theta,k},\, B_{\theta,k}))\right] \\ &= \int_{\mathbb{R}^k}\int_{(0,\infty)^k} f((y_1^1,\, y_2^1), \ldots, (y_1^k,\, y_2^k))\, dP(y_1^1,\, y_2^1)\cdots dP(y_1^k,\, y_2^k) \\ &= \int_{\mathbb{R}^k}\int_{(0,\infty)^k} f((y_1^1,\, y_2^1), \ldots, (y_1^k,\, y_2^k))\, (y_2^1)^{-\kappa(\theta)} dQ(y_1^1,\, y_2^1)\, \cdots\, (y_2^k)^{-\kappa(\theta)} dQ(y_1^k,\, y_2^k) \\ &= E_Q\left[\Big(\prod_{j=1}^k B_{\theta,j}\Big)^{-\kappa(\theta)} f((A_{\theta,1},\, B_{\theta,1}), \ldots, (A_{\theta,k},\, B_{\theta,k}))\right]\,. \end{aligned} \tag{5.20}$$

Take $\epsilon'' \in (0, \epsilon')$. Then, using (5.20) and the definition of D_n we estimate

$$\begin{aligned}
& P(D_n(d, \epsilon') \cap E_n(d, \epsilon')) \\
& \geq P(D_n(d, \epsilon'') \cap E_n(d, \epsilon'')) = E\left[1_{\{D_n(d,\epsilon'')\cap E_n(d,\epsilon'')\}}\right] \\
& = E_Q\left[e^{\kappa(\theta)(-\log B_{\theta,1}-\cdots-\log B_{\theta,\lceil(1/m(\theta)-d)n\rceil})}1_{\{D_n(d,\epsilon'')\cap E_n(d,\epsilon'')\}}\right] \\
& \geq e^{-\kappa(\theta)n(\epsilon''+1-m(\theta)d)}Q(D_n(d, \epsilon'') \cap E_n(d, \epsilon''))\,.
\end{aligned}$$

As $0 < m(\theta)d < 1$, from the above inequality follows that

$$\frac{\log P(D_n(d, \epsilon') \cap E_n(d, \epsilon'))}{n} \quad \geq \quad -\kappa(\theta)(1+\epsilon'') + \frac{\log Q(D_n(d, \epsilon'') \cap E_n(d, \epsilon''))}{n}\,.$$

Then, if we show that

$$\lim_{n\to\infty} Q(D_n(d, \epsilon'') \cap E_n(d, \epsilon'')) \quad = \quad 1\,, \tag{5.21}$$

we obtain (5.19) after letting $\epsilon'' \to 0$. For the proof of (5.21) it is sufficient to show that

$$\lim_{n\to\infty} Q(D_n(d, \epsilon'')) = 1 \quad \text{and} \quad \lim_{n\to\infty} Q(E_n(d, \epsilon'')) = 1\,. \tag{5.22}$$

We start with the lhs of (5.22). Note that by Lemma 5.3.2(b) there exists some s in a neighborhood of 0 such that

$$E_Q\left[B_\theta^s\right] = E\left[B_\theta^{s+\kappa(\theta)}\right] < \infty\,.$$

This implies for such s

$$\begin{aligned}
E_Q\left[B_\theta^s\right] \quad &= \quad \frac{d}{ds}\left(\log E_Q\left[B_\theta^s\right]\right)_{|s=0} = \frac{d}{ds}\left(\log E\left[B_\theta^{s+\kappa(\theta)}\right]\right)_{|s=0} \\
&= \quad \frac{d}{ds}\left(\log E\left[B_\theta^s\right]\right)_{|s=\kappa(\theta)} = l'_\theta(\kappa(\theta)) = m(\theta)\,.
\end{aligned}$$

From the above follows that for the sum $S_n = \log B_{\theta,1} + \cdots + \log B_{\theta,n}$ the SLLN holds under the measure Q, i.e.

$$\frac{S_n}{n} \stackrel{P_Q\,a.s.}{\to} m(\theta)\,, \quad n \to \infty\,.$$

For $x > 0$ we have $xn/\lceil xn\rceil \to 1$ as $n \to \infty$, hence for $0 < \alpha \leq 1/m(\theta) - d$ we obtain

$$\frac{S_{\lceil \alpha n\rceil}}{n} \stackrel{P_Q\,a.s.}{\to} \alpha m(\theta)\,, \quad n \to \infty\,,$$

from which follows the lhs of (5.22), i.e.

$$Q(D_n(d, \epsilon'')) = Q\left(\left|S_{\lceil\alpha n\rceil}/n - \alpha m(\theta)\right| \leq \epsilon''\,,\ 0 < \alpha \leq 1/m(\theta) - d\right) \to 1\,, \quad n \to \infty\,.$$

To show the rhs of (5.22) first note that, if $E[Y] < \infty$, then Lemma 5.3.3(c) implies that $E[|A_\theta|^{\min(1,\kappa)}] < \infty$. Second, by Hölder's inequality, for $p,\, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ we get for some $s > 0$

$$E_Q\left[|A_\theta|^s\right] \quad = \quad E\left[B_\theta^\kappa\,|A_\theta|^s\right] \leq \left(E\left[B_\theta^{\kappa p}\right]\right)^{1/p}\left(E\left[|A_\theta|^{sq}\right]\right)^{1/q}\,.$$

We can choose $p > 1$ such that $\kappa p < \kappa + \beta$, where $\beta > 0$ is as in Lemma 5.3.2(b) and $s' > 0$ such that $s'q < \min(\kappa, 1)$. In other words, Hölder's inequality guarantees the existence of some $s' > 0$, such that

$$E_Q[|A_\theta|^{s'}] < \infty\,.$$

Then for this $s' > 0$ we estimate

$$E_Q\left[|A_\theta|^{s'}\right] \geq E_Q\left[|A_\theta|^{s'}\, 1\{|A_\theta| \geq e^{\epsilon'' n}\}\right] \geq e^{s'\epsilon'' n} Q(|A_\theta| \geq e^{\epsilon'' n})\,. \tag{5.23}$$

Furthermore, for this $s' > 0$, using that $(A_{\theta,j})_{j\in\mathbb{N}}$ is a sequence of iid rv's and by (5.23), we have

$$\begin{aligned} Q(E_n(d,\, \epsilon'')) &= Q\left(|A_{\theta,j}| \leq e^{\epsilon'' n}\,,\ j = 1, \dots, \lceil (1/m(\theta) - d)n \rceil\right) \\ &= 1 - \lceil (1/m(\theta) - d)n \rceil\, Q(|A_\theta| > e^{\epsilon'' n}) \\ &\geq 1 - \lceil (1/m(\theta) - d)n \rceil\, e^{-s'\epsilon'' n} E_Q\left[|A_\theta|^{s'}\right]\,. \end{aligned}$$

Now, as $E_Q[|A_\theta|^{s'}] < \infty$, after letting in the last expression $n \to \infty$, we get the rhs of (5.22). This completes the proof. □

Proof of Lemma 5.3.4. By Lemma 5.3.2(c) we have that $P(B_\theta > 1) > 0$, from which follows the existence of some $b > 1$ satisfying $P(|B_\theta - b| < \epsilon) > 0$ for $\epsilon \in (0, b-1)$. Then

$$0 < P(|B_\theta - b| < \epsilon) = \lambda \int_0^\infty P\left(\log(b-\epsilon) < -L_\theta(z) < \log(b+\epsilon)\right) e^{-\lambda z}\, dz\,,$$

and, therefore, there exists some $t > 0$ such that $P(\log(b-\epsilon) < -L_\theta(t) < \log(b+\epsilon)) > 0$. Applying Theorem 24.4 in Sato [56] we know that $L_\theta(v)$ has unbounded support, hence, for this $t > 0$ we have

$$P(\log(b-\epsilon) < -L_\theta(v) < \log(b+\epsilon) \text{ for } v \in (0, t]) > 0 \tag{5.24}$$

Then we have

$$\begin{aligned} q &:= P(A_\theta > 0,\, |B_\theta - b| < \epsilon) \\ &= P(Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)} dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon) \\ &\geq P(Y(b-\epsilon) - c\int_0^E e^{-L_\theta(v)} dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon) \\ &= P\left(Y(b-\epsilon) - c\int_0^E e^{-L_\theta(v)} dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon \mid Y > \frac{b+\epsilon}{b-\epsilon}\, c\, y\right) P\left(Y > \frac{b+\epsilon}{b-\epsilon}\, c\, y\right)\,, \end{aligned}$$

where $y > 0$ is arbitrary. As $q_1 := P(Y > \frac{b+\epsilon}{b-\epsilon}\, c\, y) > 0$ due to the unbounded support of

Y, we estimate

$$\begin{aligned} q &\geq q_1 P\left(y(b+\epsilon) - \int_0^E e^{-L_\theta(v)}dv > 0\,,\ |e^{-L_\theta(E)} - b| < \epsilon\right) \\ &= q_1\lambda \int_0^\infty P\left(y(b+\epsilon) - \int_0^z e^{-L_\theta(v)}dv > 0\,,\ |e^{-L_\theta(z)} - b| < \epsilon\right) e^{-\lambda z}\,dz \\ &\geq q_1\lambda \int_0^\infty P\left(y(b+\epsilon) - \int_0^z e^{-L_\theta(v)}dv > 0\,,\ |e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\,dz \\ &\geq q_1\lambda \int_0^\infty P\left(y - z > 0,\ |e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\,dz \\ &\geq q_1\lambda \int_0^y P\left(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\,dz\,. \end{aligned}$$

Therefore, $q \geq q_1\lambda P(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,y))P(E < y)$. The probability

$$P(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,y)) > 0$$

is selected in such a way that (5.24) is satisfied. Consequently, we may choose numbers $b > 1$ and $\epsilon \in (0,\, b-1)$, such that

$$q = P(|B_\theta - b| < \epsilon,\ A_\theta > 0) > 0\,. \tag{5.25}$$

To prove our result we take some $d \in (0,\, 1/m(\theta))$, where $m(\theta)$ is as in (5.17), and some small number $\epsilon' > 0$, which we shall fix later. Recall the sets $D_n = D_n(d,\,\epsilon')$ and $E_n = E_n(d,\,\epsilon')$ in (5.18) and set $m = 1 + \lceil \alpha n \rceil$ for $0 < \alpha < 1/m(\theta) - d$. Then for $\omega \in D_n$ we have (cf. (5.18))

$$\log B_{\theta,1} + \cdots + \log B_{\theta,m-1} \leq (\epsilon' + \alpha m(\theta))n \leq \big(\epsilon' + \frac{m-1}{n} m(\theta)\big)n \leq (\epsilon' + 1 - m(\theta)d)n\,.$$

For $m \in \mathbb{N}$ set $\Pi_m = \prod_{j=1}^m B_{\theta,j}$ and $\Pi_0 = 1$. For sufficiently large $n \in \mathbb{N}$ and $\omega \in D_n \cap E_n$ we estimate, starting with the definition in (5.5),

$$\begin{aligned} V_{\theta,\lceil(1/m(\theta)-d)n\rceil} &= \sum_{m=1}^{\lceil(1/m(\theta)-d)n\rceil} A_{\theta,m}\Pi_{m-1} \\ &\geq -e^{\epsilon' n} \sum_{m=1}^{\lceil(1/m(\theta)-d)n\rceil} \exp\left(\sum_{j=1}^{m-1} \log B_{\theta,j}\right) \\ &\geq -\lceil(1/m(\theta) - d)n\rceil\, e^{\epsilon' n} e^{n(\epsilon'+1-dm(\theta))} \\ &> -e^{(3\epsilon'+1-m(\theta)d)n}\,. \end{aligned} \tag{5.26}$$

The last inequality holds as for all $\epsilon' > 0$ and sufficiently large $n \in \mathbb{N}$ we have

$$\lceil(1/m(\theta) - d)n\rceil < e^{\epsilon' n}\,.$$

Let $d' \in (0,\, d)$. For $n \in \mathbb{N}$ introduce the following subset of Ω

$$F_n = \{\omega \in \Omega \,|\, A_{\theta,j} > 0,\ |B_{\theta,j} - b| < \epsilon,\ j = \lceil(1/m(\theta) - d)n\rceil + 1, \ldots, \lceil(1/m(\theta) - d')n\rceil\}\ .$$

As the index sets are disjoint, F_n is independent of $D_n \cap E_n$. From (5.25) we conclude

$$P(F_n) \geq q^{(d-d')n+1} . \tag{5.27}$$

Further, for sufficiently large $n \in \mathbb{N}$ and $\omega \in F_n$ we consider the increment (recall that $b - \epsilon > 1$)

$$V_{\theta,\lceil(1/m(\theta)-d')n\rceil} - V_{\theta,\lceil(1/m(\theta)-d)n\rceil} = \sum_{m=\lceil(1/m(\theta)-d)n\rceil+1}^{\lceil(1/m(\theta)-d')n\rceil} A_{\theta,m}\Pi_{m-1} \; > \; 0 . \tag{5.28}$$

Next we define for $n \in \mathbb{N}$ one more subset of Ω:

$$G_n = \left\{\omega \in \Omega \,|\, A_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} > 1\right\} ,$$

From $x_F = \infty$ follows that A_θ has infinite right endpoint; hence

$$P(G_n) = P(A_\theta > 1) = r > 0 . \tag{5.29}$$

Finally for sufficiently large $n \in \mathbb{N}$ we consider for $\omega \in D_n \cap E_n \cap F_n \cap G_n$

$$\begin{aligned} V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} &= \left(V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} - V_{\theta,\lceil(1/m(\theta)-d')n\rceil}\right) \\ &\quad + \left(V_{\theta,\lceil(1/m(\theta)-d')n\rceil} - V_{\theta,\lceil(1/m(\theta)-d)n\rceil}\right) + V_{\theta,\lceil(1/m(\theta)-d)n\rceil} \\ &> \Pi_{\lceil(1/m(\theta)-d')n\rceil} - e^{n(1-m(\theta)d+3\epsilon')} , \end{aligned} \tag{5.30}$$

where we have used that

1. $V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} - V_{\theta,\lceil(1/m(\theta)-d')n\rceil} \geq \Pi_{\lceil(1/m(\theta)-d')n\rceil}$ for $\omega \in G_n$;
2. $V_{\theta,\lceil(1/m(\theta)-d')n\rceil} - V_{\theta,\lceil(1/m(\theta)-d)n\rceil} \geq 0$ for $\omega \in F_n$ from (5.28);
3. $V_{\theta,\lceil(1/m(\theta)-d)n\rceil} > -e^{n(1-m(\theta)d+3\epsilon')}$ for $\omega \in D_n \cap E_n$ from (5.26).

Further, for $\omega \in D_n \cap F_n$ we estimate the product in (5.30) using the definitions of D_n and F_n

$$\begin{aligned} \Pi_{\lceil(1/m(\theta)-d')n\rceil} &= \exp\left(\sum_{j=1}^{\lceil(1/m(\theta)-d)n\rceil} \log B_{\theta,j}\right) \times \exp\left(\sum_{\lceil(1/m(\theta)-d)n\rceil+1}^{\lceil(1/m(\theta)-d')n\rceil} \log B_{\theta,j}\right) \\ &\geq e^{n(-\epsilon'+1-m(\theta)d)}(b-\epsilon)^{(d-d')n-1} , \end{aligned} \tag{5.31}$$

where $b - \epsilon > 1$. By fixing ϵ' such that $5\epsilon' = (d - d')\log(b - \epsilon)$ we obtain the following lower bound in (5.31)

$$\Pi_{\lceil(1/m(\theta)-d')n\rceil} \geq \frac{1}{b-\epsilon} e^{n(4\epsilon'+1-m(\theta)d)} .$$

Using this in (5.30) we obtain the following inequality

$$\begin{aligned} V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} &\geq e^{n(1-m(\theta)d)}\left(\frac{1}{b-\epsilon}e^{n4\epsilon'} - e^{n3\epsilon'}\right) \\ &> e^{n(1-m(\theta)d)} , \end{aligned} \tag{5.32}$$

where for the last inequality we have used that for sufficiently large $n \in \mathbb{N}$ holds

$$e^{4\epsilon' n} > (b-\epsilon)(e^{3\epsilon' n}-1)\,.$$

We derived inequality (5.32) for sufficiently large $n \in \mathbb{N}$ for $\omega \in (D_n \cap E_n) \cap F_n \cap G_n$, where $D_n \cap E_n$, F_n and G_n are mutually independent. Hence, together with (5.27) and (5.29), taking logarithm and dividing by n, we obtain the following inequality

$$\begin{aligned} &\frac{\log P\left(V_{\theta,\lceil (1/m(\theta)-d')n\rceil+1} > e^{n(1-m(\theta)d)}\right)}{n} \\ &= \frac{\log P(D_n \cap E_n)}{n} + \frac{\log P(G_n)}{n} + \frac{\log P(F_n)}{n} \\ &\geq \frac{\log P(D_n \cap E_n)}{n} + \frac{\log r}{n} + (d - d' + \frac{1}{n}) \log q\,. \end{aligned}$$

Now we let $n \to \infty$ and make use of (5.19) resulting into

$$\liminf_{n\to\infty} \frac{\log P\left(V_{\theta,\lceil (1/m(\theta)-d')n\rceil+1} > \exp\left((1-m(\theta)d)n\right)\right)}{n} \geq -\kappa(\theta) + (d-d')\log q\,.$$

Finally, letting $d' \to d$ and $d \to 0$ and substituting $n = \log x$, we obtain

$$\liminf_{x\to\infty} \frac{\log P(V_{\theta,\lceil \log x/m(\theta)\rceil+1} > x)}{\log x} \geq -\kappa(\theta)\,. \tag{5.33}$$

Denote now $k := k(x) = \lceil \log x/m(\theta)\rceil + 1$ and note that, due to the iid increments property of the Lévy processes,

$$\begin{aligned} V_\theta^\infty &= \int_0^\infty e^{-L_\theta(v)}(dS(v) - c dv) \\ &= V_\theta(T_k) + \int_{T_k}^\infty e^{-L_\theta(v)}(dS(v) - c dv) \\ &= V_\theta(T_k) + e^{-L_\theta(T_k)} \int_{T_k}^\infty e^{-(L_\theta(v)-L_\theta(T_k))}(dS(v) - c dv) \\ &\stackrel{d}{=} V_\theta(T_k) + e^{-L_\theta(T_k)} \widetilde{V}_\theta^\infty\,, \end{aligned}$$

where $\widetilde{V}_\theta^\infty$ is copy of V_θ^∞, independent of $\mathcal{F}_{T_k}$. Furthermore, recalling from Proposition 5.1.5(b) that $V_{\theta,k} = V_\theta(T_k)$ we can write

$$\frac{\log P(V_\theta^\infty > x)}{\log x} \geq \frac{\log P(V_{\theta,k} > x)}{\log x} + \frac{\log P(\widetilde{V}_\theta^\infty \geq 0)}{\log x}\,.$$

Letting x to infinity and making use of (5.33) gives the lhs of (5.16).

To prove the rhs of (5.16) it suffices to show (5.25) and (5.29) for the rv $-A_\theta$. Again we take $b>1$ such that $P(|B_\theta - b|<\epsilon)>0$ for all $\epsilon\in(0,b-1)$.

$$\begin{aligned} q &= P(-A_\theta>0,\,|B_\theta-b|<\epsilon) \;=\; P\left(c\int_0^E e^{-L_\theta(v)}dv - Ye^{-L_\theta(E)}>0\,,\,|e^{-L_\theta(E)}-b|<\epsilon\right)\\ &\ge P\left(c\int_0^E e^{-L_\theta(v)}dv - Y(b+\epsilon)>0\,,\,|e^{-L_\theta(E)}-b|<\epsilon\right)\\ &= P\left(c\int_0^E e^{-L_\theta(v)}dv - Y(b+\epsilon)>0\,,\,|e^{-L_\theta(E)}-b|<\epsilon\,|\,Y<\frac{b-\epsilon}{b+\epsilon}\,c\,y\right)P\left(Y<\frac{b-\epsilon}{b+\epsilon}\,c\,y\right), \end{aligned}$$

where $y>0$ is arbitrary. Denote $q_1 := P(Y<\frac{b-\epsilon}{b+\epsilon}\,c\,y)>0$ due to the unbounded support of Y. Therefore

$$\begin{aligned} q &\ge q_1 P\left(\int_0^E e^{-L_\theta(v)}dv - y(b-\epsilon)>0\,,\,|e^{-L_\theta(E)}-b|<\epsilon\right)\\ &= q_1\lambda\int_0^\infty P\left(\int_0^z e^{-L_\theta(v)}dv - y(b-\epsilon)>0\,,\,|e^{-L_\theta(z)}-b|<\epsilon\right)e^{-\lambda z}\,dz\\ &\ge q_1\lambda\int_0^\infty P\left(\int_0^z e^{-L_\theta(v)}dv - y(b-\epsilon)>0\,,\,|e^{-L_\theta(v)}-b|<\epsilon \text{ for } v\in(0,z]\right)\;e^{-\lambda z}\,dz\\ &\ge q_1\lambda\int_0^\infty P\left(z-y>0,\,|e^{-L_\theta(v)}-b|<\epsilon \text{ for } v\in(0,z]\right)e^{-\lambda z}\,dz\\ &\ge q_1\lambda\int_y^\infty P\left(|e^{-L_\theta(v)}-b|<\epsilon \text{ for } v\in(0,z]\right)e^{-\lambda z}\,dz \;>\; 0\,. \end{aligned}$$

Then the proof of the rhs of (5.16) follows by repetition of all steps of the proof of the lhs of (5.16), replacing A_θ and $V_{\theta,k}$ for $k\in\mathbb{N}$ by $-A_\theta$ and $-V_{\theta,k}$, $k\in\mathbb{N}$, respectively. To this end we still have to show that

$$r \;=\; P(-A_\theta>1)>0 \tag{5.34}$$

Indeed, from the infinite right end point of the exponentially distributed rv E follows

$$\begin{aligned} &P(c\int_0^E e^{-L_\theta(v)}dv - Ye^{-L_\theta(E)}>1)\\ &\ge \int_0^\infty P(c\int_0^E e^{-L_\theta(v)}dv - ye^{-L_\theta(E)}>0\,,\,\left|e^{-L_\theta(v)}-b\right|<\epsilon \text{ for } v\in(0,E))\,dF(y)\\ &\ge \int_0^\infty P(E>\frac{y(b+\epsilon)}{c(b-\epsilon)})\,dF(y)\;>\;0\,. \end{aligned}$$

□

In the following result we show that, under moment conditions for the claims, the a.s. limit of the DNLP V_θ^∞ has heavy left **and** right tail. It is an application of Theorem 4.7 of Goldie [27] in combination with Lemma 5.3.4.

Theorem 5.3.6. *Assume that the conditions of Theorem 5.2.5 and Lemma 5.3.1 hold. Let $\kappa=\kappa(\theta)\in(0,\infty)$ be the unique value satisfying $\varphi_\theta(\kappa)=0$. Assume also that the claim* *size distribution Y has unbounded support. Let β be as in Lemma 5.3.2(b) and assume that*

$$E[Y^{\kappa+\beta}] < \infty\,. \tag{5.35}$$

Then there exist constants $C_\pm$ such that for $x \to \infty$

$$P(V_\theta^\infty > x) = C_+\, x^{-\kappa} + O(x^{-(\kappa+\beta/2)}) \quad \text{and} \quad P(V_\theta^\infty < -x) = C_-\, x^{-\kappa} + O(x^{-(\kappa+\beta/2)})\,. \tag{5.36}$$

Moreover,

$$C_\pm = C_\pm(\theta) = \frac{1}{\kappa\, m} E\left[\left((A_\theta + B_\theta V_\theta^\infty)^\pm\right)^\kappa - \left((B_\theta V_\theta^\infty)^\pm\right)^\kappa\right] > 0\,, \tag{5.37}$$

where

$$m = m(\theta) = \frac{1}{\lambda}\varphi_\theta'(\kappa(\theta)) \in (0,\infty)\,. \tag{5.38}$$

Proof. Lemma 5.3.2 guarantees that $m = \varphi_\theta'(\kappa)/\lambda = l_\theta'(\kappa) \in (0,\infty)$. Then the rate result (5.36) holds by Lemma 5.3.2(b) and Theorem 2.3.4.

Furthermore, $E[|A_\theta|^{\kappa+\beta}] < \infty$ by (5.35) and Lemma 5.3.3(c). Define the probability law $\eta(dx) = e^{\kappa x} P(\log B_\theta \in dx)$. This is spread out as $\log B_\theta = -L_\theta(E)$ is. The corresponding first moment is positive as $\varphi_\theta'(\kappa) > 0$ by Lemma 5.3.1(a), and the second moment is finite, since the moment generating function exists in a neighbourhood of 0. Finally, $\widetilde{\eta}(\beta) = \varphi_\theta(\kappa+\beta) < \infty$ by Lemma 5.3.2(b).

To prove that $C_+(\theta) > 0$ we apply Lemma 5.3.4. Assume that $C_+(\theta) = 0$. Then from (5.36) follows that there exists some constant $M \in (0,\infty)$ and some x_0 such that

$$P(V_\theta^\infty > x) \le M x^{-(\kappa+\beta/2)}\,, \quad x > x_0\,,$$

which implies, taking logarithms,

$$\frac{\log P(V_\theta^\infty > x)}{\log x} \le \frac{\log M}{\log x} - \kappa - \frac{\beta}{2}\,.$$

Now letting $x \to \infty$ and making use of the lhs of (5.16) we get the following inequality chain

$$-\kappa \le \liminf_{x\to\infty} \frac{\log P(V_\theta^\infty > x)}{\log x} \le \lim_{x\to\infty} \frac{\log P(V_\theta^\infty > x)}{\log x} \le -\kappa - \frac{\beta}{2}\,,$$

which is a contradiction to $\beta > 0$. Hence $C_+ > 0$.

To prove that $C_- > 0$ note that $P(V_\theta^\infty < -x) = P(-V_\theta^\infty > x)$. Moreover, $-V_\theta^\infty$ is the almost sure limit of the random recurrence equation

$$-V_{\theta,0} = 0 \quad \text{and} \quad -V_{\theta,n} = \sum_{m=1}^n (-A_{\theta,m}) \prod_{j=1}^{m-1} B_{\theta,j}\,, \quad n \in \mathbb{N}\,,$$

with $(A_{\theta,k},\, B_{\theta,k})$ as defined in (5.5). Hence, Lemma 5.3.4 applies. □

Remark 5.3.7. Theorem 5.3.6 says that V_θ^∞ has left **and** right Pareto-like tails. By Lemma 5.3.1(c) the Pareto index $\kappa = \kappa(\theta)$ is decreasing in θ. This can be interpreted that the more we invest into the risky asset the heavier the tail of the stationary DNLP becomes. More risky investment increases the risk. □

Example 5.3.8. [Dangerous investment]
In this example we demonstrate that investment into risky stock can be dangerous, although the insurance claims are moderate. Assume for simplicity that the claims have moments of all order. Let the conditions on the investment process in Theorem 5.2.5 be satisfied so that there exists an a.s. limit V_θ^∞ of the DNLP. Let also the condition of Lemma 5.3.1 be satisfied, so that there exists an unique positive value $\kappa = \kappa(\theta)$ such that $\varphi_\theta(\kappa) = 0$. Then Theorem 5.3.6 gives

$$P(V_\theta^\infty > x) \sim C_+(\theta)x^{-\kappa}\,, \quad x \to \infty\,, \tag{5.39}$$

where κ is determined by the investment process only. Intuitively, in this case the extremes of the investment process dominate the extremes of the resulting integrated risk process. This is illustrated in Figure 5.3.

The parameter κ can be calculated explicitly, only if the price process of the risky asset is geometric Brownian motion; see second part of Example 3.5.1. Then the investment process is again geometric Brownian motion given by

$$X_\theta(t) = \exp(\gamma_\theta t + \sigma_\theta W(t))\,, \quad t \ge 0\,,$$

with γ_θ and σ_θ as in (3.25). The value κ is the unique positive solution to

$$\varphi_\theta(s) = -\gamma_\theta s + \frac{\sigma_\theta^2}{2}s^2 = 0$$

given by

$$\kappa = \kappa(\theta) = \frac{2\gamma_\theta}{\sigma_\theta^2} = \frac{2}{\sigma^2\theta^2}\left(\gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta)\right)\,.$$

In the case of Brownian motion with jumps with distribution Z (first part of Example 3.5.1), κ is given as the unique positive solution to

$$\varphi_\theta(s) = -\xi_\theta s + \sigma_\theta^2\frac{s^2}{2} + \eta(E[(1+\theta(e^Z-1))^{-s}] - 1) = 0\,,$$

where ξ_θ and σ_θ are given in (3.22). Even in this simple case $\kappa(\theta)$ can only be found by numerical methods. The problem becomes even more difficult for a VG Lévy process (Example 3.5.3) or any other process with infinite jump activity.

In Figure 5.4 we have plotted the value $\kappa(\theta)$ as a function of the investment strategy θ for three different models for the risky asset. Recall that by Lemma 5.3.1(b) the function $\kappa(\theta)$ is decreasing in θ for all Lévy models. This means that in all models more investment into the risky asset leads to a heavier tail of V_θ^∞; i.e. more risky investment yields a higher risk.

We compare a Brownian motion model with two different VG models. The parameters are chosen such that mean and variance of the log returns of the risky asset are the same in all models. As we can see in Figure 5.4, jumps in the model yield a smaller κ, corresponding to a heavier tail of V_θ^∞. Higher intensity of large negative jumps yields also a smaller κ. $\square$

5.3.2 Regularly varying claims

In this section we consider claim size distributions satisfying $\overline{F}(x) = x^{-\alpha}\ell(x)$, $x > 0$, where $\lim_{x\to\infty} \ell(xt)/\ell(x) = 1$ for all $t > 0$; i.e. $\overline{F}$ is regularly varying with index $\alpha > 1$ and we require throughout that $\alpha < \kappa$. Here $\kappa = \kappa(\theta) \in (1, \infty)$ is the unique value satisfying $\varphi_\theta(\kappa) = 0$ for some fixed $\theta \in (0, 1]$ as defined in Lemma 5.3.1. In this case $E[Y^\kappa] = \infty$, hence this is a different situation than in Section 5.3.1. In the next proposition we shall see that in this case the tail of the stationary rv V_θ^∞ is determined by the tail behaviour of the claim size distribution Y.

Theorem 5.3.9. *Let V_θ^∞ be the stationary solution to the backward stochastic recurrence equation (5.5). Let $\kappa = \kappa(\theta) \in (1, \infty)$ be the unique value satisfying $\varphi_\theta(v) = 0$. Assume that the claim size Y has distribution with regularly varying tail for some $\alpha \in (1, \kappa(\theta))$. Then the following assertions hold.*

(a) Right tail. *V_θ^∞ has also regularly varying tail with index α, more precisely,*

$$P(V_\theta^\infty > x) \sim \frac{\lambda}{|\varphi_\theta(\alpha)|} P(Y > x)\,, \quad x \to \infty\,. \tag{5.40}$$

(b) Left tail. *Assume that $\sigma > 0$ or $\nu(-\infty, 0) > 0$. In the case when L is of finite variation, assume that either the drift is non-zero, or that for no $r > 0$ the support of the Lévy measure ν_θ is concentrated on $r\mathbb{Z}$. Then*

$$\limsup_{x\to\infty} \frac{\log P(V_\theta^\infty < -x)}{\log x} = -\kappa\,.$$

In particular, the left tail of V_θ^∞ decreases faster than the right one, i.e.

$$\lim_{x\to\infty} \frac{P(V_\theta^\infty < -x)}{P(V_\theta^\infty > x)} = 0\,.$$

Proof. (a) Recall that $\varphi_\theta(0) = \varphi_\theta(\kappa) = 0$ and φ_θ is strictly convex in s; i.e. $\varphi_\theta(s) < 0$ for all $0 < s < \kappa$. As $\alpha < \kappa$ we have $\varphi_\theta(\alpha) < 0$. Hence

$$E[B_\theta^\alpha] = \frac{\lambda}{\lambda - \varphi_\theta(\alpha)} < 1\,,$$

and there exists some $\beta > 0$ such that $E[B_\theta^{\alpha+\beta}] < \infty$. Then, if we can show that A_θ is regularly varying with index α, it follows directly from Proposition 2.4 in Konstantinides and Mikosch [39] that

$$P(V_\theta^\infty > x) \sim \frac{1}{(1 - E[B_\theta^\alpha])} P(A_\theta > x)\,, \quad x \to \infty\,.$$

As there exists some $\beta > 0$, such that $E[B_\theta^{\alpha+\beta}] < \infty$, from Breiman's classical result, see Lemma 2.2 in [39], follows that

$$P(Ye^{-L_\theta(E)} > x) = P(YB_\theta > x) \sim E[B_\theta^\alpha]P(Y > x)\,, \quad x \to \infty\,. \tag{5.41}$$

Define $\xi = Ye^{-L_\theta(E)}$ and $\eta = c\int_0^E e^{-L_\theta(v)}\,dv$, then both rv's ξ and η are a.s. positive. On the one hand we estimate

$$P(A_\theta > x) = P(\xi - \eta > x) \le P(\xi > x)\,. \tag{5.42}$$

On the other hand, for any $\epsilon > 0$ we calculate

$$\begin{aligned} P(\xi - \eta > x) + P(\eta > \epsilon x) &\ge P(\xi - \eta > x\,,\, \eta \le \epsilon x) + P(\eta > \epsilon x) \\ &\ge P(\xi > (1+\epsilon)x,\, \eta \le \epsilon x) + P(\eta > \epsilon x,\, \xi > (1+\epsilon)x) \\ &= P(\xi > (1+\epsilon)x)\,. \end{aligned}$$

This implies

$$P(A_\theta > x) = P(\xi - \eta > x) \ge P(\xi > (1+\epsilon)x) - P(\eta > \epsilon x)\,. \tag{5.43}$$

As $1 < \alpha < \kappa$, by (5.15) we know that $E[\eta^\alpha] < \infty$. As a consequence, for every $\epsilon > 0$ follows $\lim_{x\to\infty} x^\alpha P(\eta > \epsilon x) = 0$. This together with (5.41) and inequalities (5.42) and (5.43) implies the following estimates for the tail of A_θ as $x \to \infty$:

$$E[B_\theta^\alpha]\frac{P(Y > (1+\epsilon)x)}{P(Y > x)} \sim \frac{P(\xi > (1+\epsilon)x)}{P(Y > x)} \le \frac{P(A_\theta > x)}{P(Y > x)} \le \frac{P(\xi > x)}{P(Y > x)} \;\to\; E[B_\theta^\alpha]\,.$$

Letting $x \to \infty$ on the left hand side, and then $\epsilon \to 0$ gives (5.40).
(b) First notice that

$$P(V_\theta^\infty < -x) \le P\left(c\int_0^\infty \exp(-L_\theta(v))dv > x\right)\,. \tag{5.44}$$

The rv $V_\theta^{\infty,-} = c\int_0^\infty \exp(-L_\theta(v))dv$ satisfies the fix point equation

$$V_\theta^{\infty,-} \stackrel{d}{=} A_\theta^- + B_\theta V_\theta^{\infty,-}\,,$$

for $A_\theta^- = c\int_0^E \exp(-L_\theta(v))dv > 0$ and $B_\theta = \exp(-L_\theta(E)) > 0$ a.s.. By (5.15), setting $g = \kappa > 1$, $E[|A_\theta^-|^\kappa] < \infty$. Therefore we may apply Theorem 2.3.3 and Lemma 2.3.2and we get for some constant $C > 0$

$$P\Big(c\int_0^\infty \exp(-L_\theta(v))dv > x\Big) \sim C\,x^{-\kappa} \quad x \to \infty\,.$$

Inequality (5.44) ensures that

$$\limsup_{x\to\infty} \frac{\log P(V_\theta^\infty < -x)}{\log x} \le -\kappa\,.$$

From this follows that for every $\varepsilon > 0$ there exists some $x_0 = x_0(\varepsilon)$ such that

$$P(V_\theta^\infty < -x) \le x^{-\kappa+\varepsilon}$$

holds for all $x \ge x_0$; on the other hand, due to (5.40), also

$$P(V_\theta^\infty > x) \ge x^{-\alpha+\varepsilon/2}$$

for all $x \ge x_0$. Since $\alpha < \kappa$, for ε small enough, for all $x > x_0$ we get

$$\frac{P(V_\theta^\infty < -x)}{P(V_\theta^\infty > x)} \le x^{-(\kappa-\alpha-\varepsilon/2)} \to 0\,, \quad x \to \infty\,.$$

□

Remark 5.3.10. Theorem 5.3.9(a) gives a Pareto-like right tail of the a.s. limit V_θ^∞ of the DNLP. In the context of risk management this is the important tail as it describes the likelihood of large losses. From Lemma 3.2.5(d) follows that there is a unique investment strategy θ minimizing the right tail of the stationary DNLP; cf. Figure 5.6. □

Example 5.3.11. [Dangerous claims]
In this example we demonstrate how large insurance claims may dominate the extremes in the integrated risk process. Let the claims have Pareto-like tail with exponent $\alpha > 1$, i.e. $P(Y > x) \sim C_Y\, x^{-\alpha}$, $x \to \infty$, for some constant $C_Y > 0$. Then the claims have finite moments up to order α, including a finite mean, but no moments of order larger than α. Let the conditions on the investment process in Theorem 5.2.5 be satisfied. Then there exists an a.s. limit V_θ^∞ of the DNLP. Further, let the conditions of Lemma 5.3.1 be satisfied and $\kappa(\theta)$ be the unique positive value such that $\varphi_\theta(\kappa(\theta)) = 0$ and assume that $\kappa(1) > \alpha$. Then Theorem 5.3.9 applies: recall first that by Lemma 5.3.1(b) if $\kappa(1) > \alpha$, then $\kappa(\theta) > \alpha$ for all $\theta \in (0, 1]$. In this case, for all $\theta \in (0, 1]$ holds

$$P(V_\theta^\infty > x) \sim C(\theta)x^{-\alpha}\,, \quad x \to \infty\,. \tag{5.45}$$

The investment process enters only into the constant $C(\theta) = \lambda\mu C_Y/|\varphi_\theta(\alpha)|$. Intuitively, in this case the large insurance claims dominate the extremes of the resulting IRP. This is illustrated in Figure 5.5.

The constant $C(\theta)$ can be calculated explicitly for models such that $\varphi_\theta(\alpha)$ can be calculated. In principle this holds for the geometric Brownian motion model, and also for special cases of the geometric Brownian motion with jumps (see Example 3.5.1). For processes with infinite jump activity (Example 3.5.3), the constant $C(\theta)$ has to be computed numerically.

In Figure 5.6 we have plotted the Pareto constant $C(\theta)$ as a function of the investment strategy θ for three different models for the risky asset, chosen as in Example 5.3.8. □

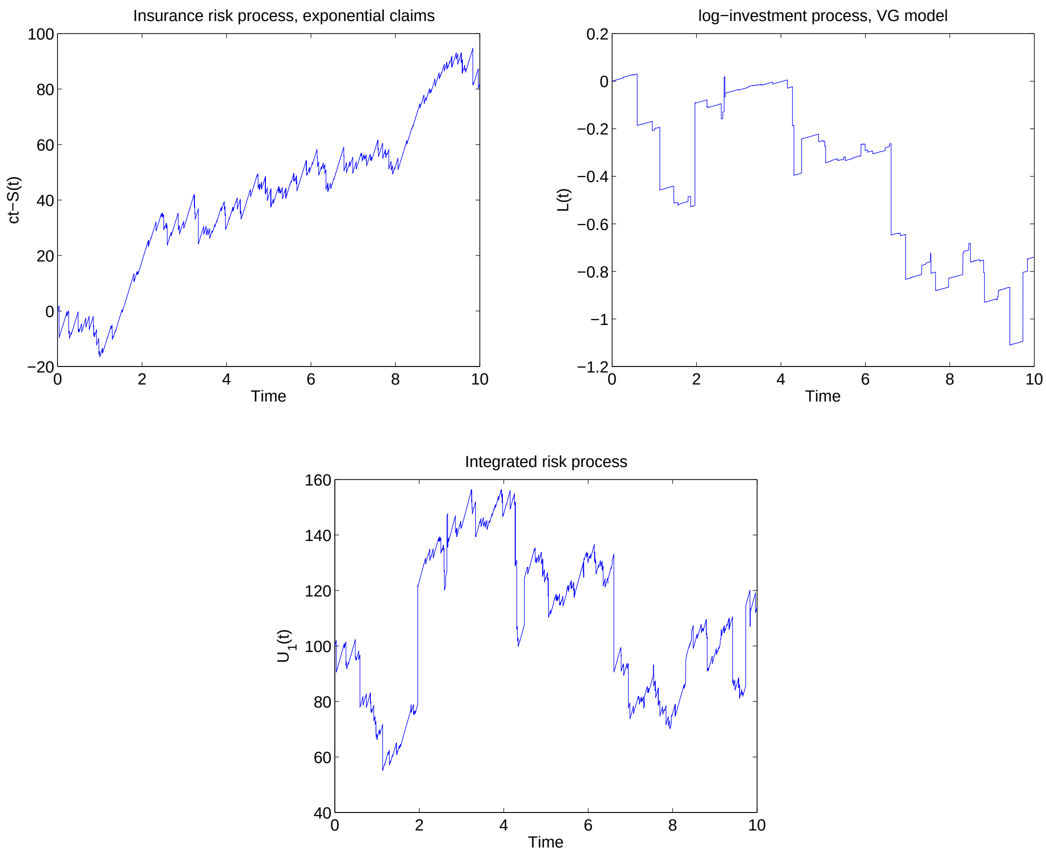

Figure 5.3: *Upper left plot:* sample path of the insurance tisk process with premium rate $c = 50$, intensity of the Poisson claim arrival process $\lambda = 20$ and exponentially distributed claims with mean $\mu = 2$, i.e. $\overline{F}(x) = e^{-x/2}$, $x > 0$. *Upper right plot:* sample path of the log-investment process for investment strategy $\theta = 1$ (pure stock investment) and the VG process $L(t) = qt + W_{a,b}(S_\Gamma(t))$, $t > 0$, with parameters $q = 0.05$. $W_{a,b}$ is Brownian motion with drift $a = -0.01$, variance $b^2 = 0.04 - a^2$ and $\text{var}(\mathrm{S}_\Gamma) = 1$. *Lower plot:* sample path of the resulting IRP with initial capital $u = 100$. The time horizon is $T = 10$. It is clearly seen that the large jumps of the IRP are dominated by the large jumps of the investment process.

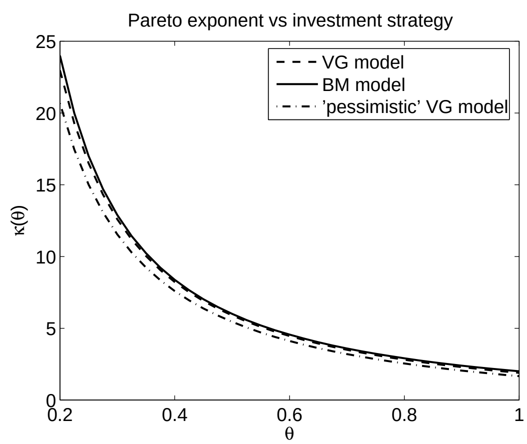

Figure 5.4: The Pareto exponent $\kappa(\theta)$ in (5.39) as a function of the investment strategy θ. We compare the following investment models: Brownian motion model with drift 0.04 and volatility 0.2, a VG model with parameters as in Figure 5.3 and a more pessimistic VG model of the form $L(t) = qt + W_{a,b}(S_\Gamma(t))$ where $q = 0.14$, $a = -0.1$ and $b^2 = 0.04 - a^2$ (more large negative jumps, which are compensated by a larger deterministic drift q).

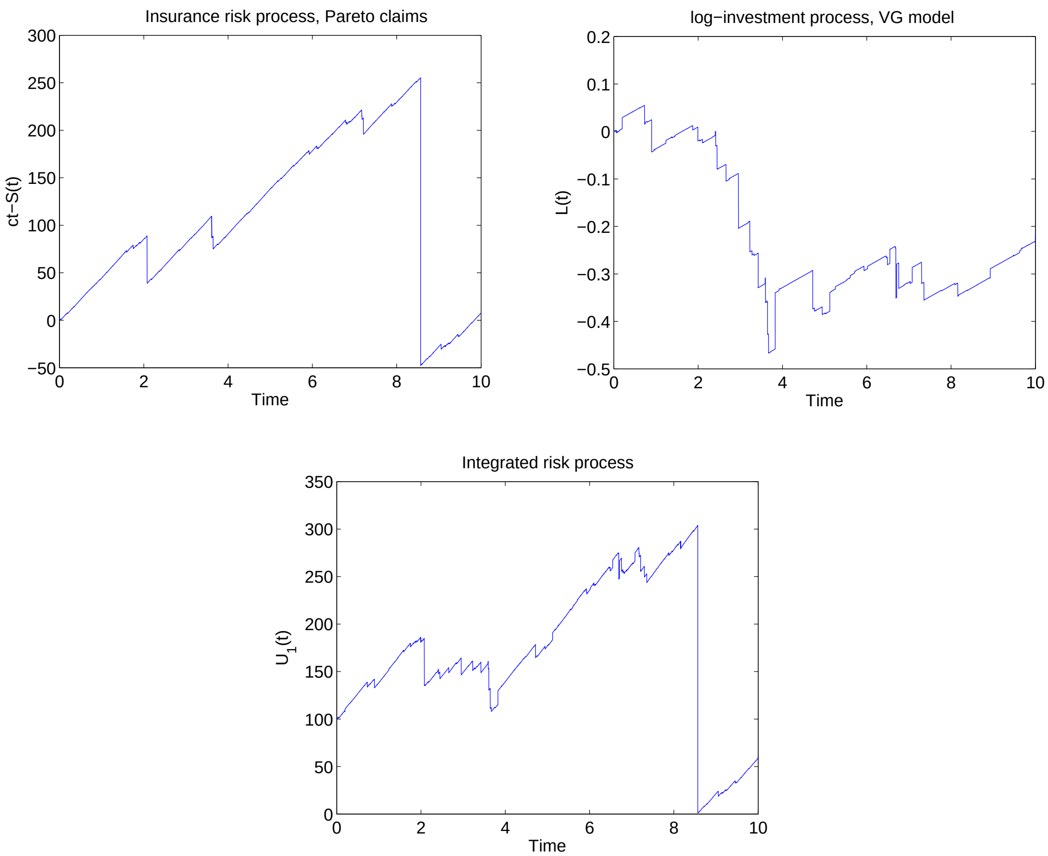

Figure 5.5: *Upper left plot:* sample path of the insurance process with premium rate $c = 50$, intensity of the Poisson claim arrival process $\lambda = 20$ and Pareto distributed claims with mean $\mu = 2$ and Pareto exponent $\alpha = 1.1$, i.e. $\overline{F}(x) = (\frac{0.2}{0.2+x})^{-1.1}$, $x > 0$. *Upper right plot:* sample path of the investment process for investment strategy $\theta = 1$ (pure stock investment) and log returns of the risky asset modeled by a VG process with parameters as in Figure 5.3. *Lower plot:* sample path of the resulting IRP with initial capital $u = 100$. The time horizon is $T = 10$. It is clearly seen that the large jumps of the IRP are dominated by the large insurance claims.

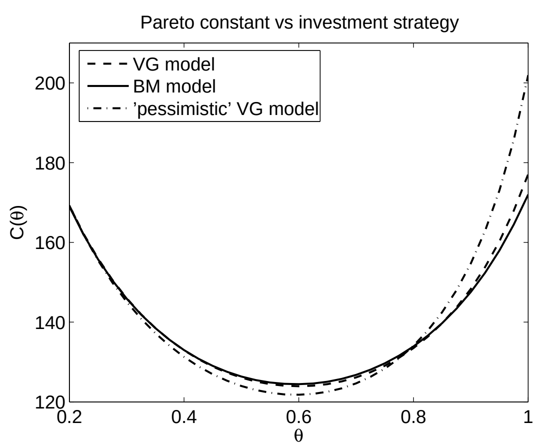

Figure 5.6: The Pareto constant $C(\theta)$ in (5.45) as a function of the investment strategy θ. We compare a Brownian motion and two VG models. The parameters of the models are as in Figure 5.4. Note that the more risky the investment model, the larger is the difference between the minimal and the maximal value of $C(\theta)$; i.e. between the minimal and the maximal value of the tail of V_θ^∞.

Chapter 6

Optimal investment for insurers

This chapter applies the theoretical results for the integrated risk model of an insurance company investing into bond and stock, obtained in Chapter 3 and Chapter 5. A risk measure frequently used in practice – Value-at-Risk (VaR), is defined in the framework of the integrated risk model. We provide and compare several methods for an approximation of the optimal investment strategy, which maximizes the expected wealth of the insurance company under a risk constraint on the Value-at-Risk. We conclude with some examples.

6.1 Risk measurement

The Value-at-Risk has become a standard risk measure for the insurance and banking industry. It is related to the capital reserve, which the financial institution needs to hold, in order to prevent (at a sufficiently high confidence level) insolvency due to an extremely negative development of the risks in its portfolio. Mathematically, the VaR is defined as some high quantile of the corresponding loss distribution.

In this section we provide a definition of the VaR for the integrated risk model. We aim at a stationary loss distribution. Following long tradition in insurance, we work with discounted losses represented by the discounted net loss process (DNLP) as defined in Chapter 5 by the following transformation of the IRP U_θ

$$V_\theta(t) = u - e^{-L_\theta(t)} U_\theta(t) = \int_0^t e^{-L_\theta(v)} \left(dS(v) - c\, dv \right), \quad t \geq 0\,. \tag{6.1}$$

Recall that $c > 0$ is the constant premium rate, $S(t) = \sum_{j=1}^{N(t)} Y_j$ is a compound Poisson process with intensity λ describing the total claim amount, where $(Y_j)_{j \in \mathbb{N}}$ is a sequence of positive iid rv's with generic rv Y and mean μ. The Lévy process L_θ is obtained by mixing the riskless interest rate δ and the risky asset, whose log returns are described by a Lévy process L, see Section 3.2. The constant θ, denoting the fraction invested into the risky asset, is the investment strategy. The process V_θ describes the total net loss (both from insurance and investment) of the insurance company, (randomly) discounted to time 0. An important relation between the IRP and the DNLP is

$$P(U_\theta(t) < 0 \,|\, U_\theta(0) = u) = P(V_\theta(t) > u)\,, \;\; t \geq 0\,. \tag{6.2}$$

As we saw in Chapter 5, the advantage of this approach lies in the fact that the DNLP has a natural embedded discrete time skeleton, see Proposition 5.1.5. This allowed us to apply standard methods from the theory of stochastic recurrence equations, see for example Kesten [36] and Goldie [27]. The next proposition is a consequence of Theorem 5.2.5 and gives conditions, under which the DNLP defined in (6.1) has an a.s. limit as time tends to infinity. Recall the notation for the Laplace exponents φ and φ_θ of the Lévy processes L and L_θ , respectively.

Proposition 6.1.1. *Let $E[Y] = \mu < \infty$, $0 < E[L(1)] < \infty$ and $\delta < \varphi(-1)$.*

(a) If $\varphi(1) < \lambda$, then, for every $\theta \in [0, 1]$,

$$V_\theta(t) \stackrel{\text{a.s.}}{\to} V_\theta^\infty\,, \;\; t \to \infty\,, \tag{6.3}$$

where V_θ^∞ is a finite rv.

(b) If $\varphi(1) \geq \lambda$, then (6.3) holds for every $\theta \in [0, \theta_u)$, where $\theta_u \in (0, 1]$ is the unique strictly positive solution to the equation $\varphi_\theta(1) = \lambda$.

Proof. We apply Theorem 5.2.5. By Lemma 3.2.5(d) we have that for a fixed $s > 0$ the function $\varphi_\theta(s)$ is convex in $\theta \in [0, 1]$. Furthermore, $\varphi_0(1) = -\delta < 0 < \lambda$, hence we have two cases. In the first case, if $\varphi_1(1) = \varphi(1) < \lambda$, then $\varphi_\theta(1) < \lambda$ for all $\theta \in [0, 1]$, which proves (a). In the second case, if $\varphi_1(1) = \varphi(1) \geq \lambda$, then θ_u as in (b) exists due to the convexity of $\varphi_\theta(1)$ in θ, see Lemma 3.2.5(d). Then $\varphi_\theta(1) < \lambda$ for all $\theta \in [0, \theta_u)$, which proves (b). □

Remark 6.1.2. Note that the conditions in Proposition 6.1.1 are quite natural. Indeed, $E[Y] < \infty$ is seen as a prerequisite for any insurance, $E[L(1)] > 0$ is a prerequisite for any investment and $\delta < \log E[\exp(L(1))] = \varphi(-1)$ guarantees that the expected value of the risky investment is larger than the riskless investment. □

The distribution of the a.s. limit V_θ^∞ in Proposition 6.1.1 is of central interest in the present chapter. In particular, it enables us to measure the risk in a stationary way.

Definition 6.1.3. *Let the conditions of Proposition 6.1.1 be satisfied. Denote by $\Theta \subseteq [0, 1]$ the non-empty interval of investment strategies θ for which (6.3) holds for some finite rv V_θ^∞. For $\theta \in \Theta$ we define*

$$\mathbf{VaR}_\alpha(V_\theta^\infty) = \inf\{x \in \mathbb{R} \,:\, P(V_\theta^\infty > x) \leq \alpha\}\,,$$

where $\alpha \in (0, 1)$ is some (typically small) probability. □

For risky assets, for which $\varphi(1) \geq \lambda$, the constant θ_u in Proposition 6.1.1(b) gives an upper bound for the reasonable investment strategies θ. Above this upper bound we cannot guarantee an a.s. limit of the DNLP and, hence, no reasonable statistical risk assesment is possible. This is illustrated in the following example.

Example 6.1.4. [Continuation of Example 3.5.1]
Consider the geometric Brownian motion as a model for the risky asset as in the second part of Example 3.5.1. Recall the Laplace exponent φ_θ in (3.24). Applying Proposition 6.1.1, straightforward calculations show that, if the volatility of the risky asset is

small enough, i.e. $\sigma^2 < 2(\gamma+\lambda)$, then the set Θ in Definition 6.1.3 is the whole interval $[0,1]$. Otherwise, if $\sigma^2 \geq 2(\gamma+\lambda)$, then $\Theta = [0, \theta_u)$, where

$$\theta_u = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma+\sigma^2/2-\delta)^2 + 4\sigma^2(\delta+\lambda)}}{2\sigma^2} \leq 1 \tag{6.4}$$

is the unique strictly positive solution to the equation $\varphi_\theta(1) = \lambda$. Hence, for Brownian motion models with a very large volatility, i.e. $\sigma^2 \geq 2(\gamma+\lambda)$, no investment strategies greater or equal than θ_u given in (6.4) should be allowed. □

Insurance companies usually review their success at predetermined times, for example every year or every quarter of a year. Hence, on the one hand, it seems reasonable to choose an investment strategy, which maximizes the wealth of the company at the end of the planing period. On the other hand, there are certain regulatory or financial bounds on the amount of risk, which an insurance company may take on. The following optimization problem is based on these considerations:

$$\max_{\theta\in\Theta} E[U_\theta(t)] \quad \text{subject to} \quad \mathbf{VaR}_\alpha(V_\theta^\infty) \leq C\,, \tag{6.5}$$

for a given constraint $C > 0$ on the risk, some fixed time period $t > 0$ and a given small probability α. The set Θ of reasonable investment strategies is as in Definition 6.1.3. Such problems are typical for the financial industry, see e.g Korn [40]. Our goal is to provide explicit solutions to (6.5).

The use of $\mathbf{VaR}_\alpha(V_\theta^\infty)$ as a risk measure in the portfolio optimization problem is explained by the fact that this quantity is equal to the capital reserve required to prevent insolvency with a sufficiently high probability $1-\alpha$ for a long time horizon, see (6.2). Note that in our definition the VaR does not depend on the initial capital and on the time t, but only on the selected investment strategy θ and on the stochastic properties of the insurance and the investment processes. On the other hand, due to Lemma 3.4.4, under natural conditions for the insurance and for the investment process, the expectation of the wealth of the company is an increasing function of the investment strategy θ for every fixed time period $t > 0$ and initial capital $u > 0$. Consequently, the portfolio optimization problem (6.5) is equivalent to

$$\max\left\{\theta \in \Theta \,:\, \mathbf{VaR}_\alpha(V_\theta^\infty) \leq C\right\}\,, \tag{6.6}$$

which depends only on the risk measure itself. This is, from a mathematical point of view, no surprise, as the a.s. limit V_θ^∞ of the DNLP, which is independent of u and t, is the basis for the risk measure. For an economic interpretation, recall that the investment strategy takes extreme risks into account during time intervals, where all parameters of the insurance model and the investment model are fixed. Only changes in these parameters would indicate that the investment strategy should be reconsidered.

6.2 Analytic results

In order to find the solution of the optimization problem (6.6), we need a method to compute $\mathbf{VaR}_\alpha(V_\theta^\infty)$ as a function of the investment strategy θ. As it is hard or even impossible to do this analytically, we approximate the df of V_θ^∞. We are interested in approximating its far out upper tail, as $\mathbf{VaR}_\alpha(V_\theta^\infty)$ is defined as a high $(1-\alpha)$-quantile.

We start with the mean and the variance of V_θ^∞. Recall the notation for the Laplace transforms φ and φ_θ in (3.5).

Lemma 6.2.1. *Let the conditions of Proposition 6.1.1 hold.*

(a) $\varphi(2) < 0$, *then for every* $\theta \in [0,1]$

$$E\left[V_\theta^\infty\right] = \frac{c-\lambda\mu}{\varphi_\theta(1)} < \infty\,, \tag{6.7}$$

and, provided that $E[Y^2] = \mu_2 < \infty$,

$$\operatorname{var}(V_\theta^\infty) = \frac{2\varphi_\theta(1)-\varphi_\theta(2)}{\varphi_\theta^2(1)\varphi_\theta(2)}(c-\lambda\mu)^2 - \frac{\lambda\mu_2}{\varphi_\theta(2)} < \infty\,. \tag{6.8}$$

(b) If $\varphi(2) \geq 0$ *and* $\varphi(1) < 0$, *then (6.7) holds for every* $\theta \in [0,1]$ *and (6.8) holds for every* $\theta \in [0,\theta_2)$, *where* $\theta_2 \in (0,1]$ *is the unique positive value such that* $\varphi_{\theta_2}(2) = 0$.

(c) If $0 \leq \varphi(1) < \lambda$, *then (6.7) holds for every* $\theta \in [0,\theta_1)$, *where* θ_1 *is the unique positive value such that* $\varphi_{\theta_1}(1) = 0$ *and (6.8) holds for every* $\theta \in [0,\theta_2)$, *where* θ_2 *is as in (b).*
In this case $0 < \theta_2 < \theta_1 \leq 1$.

(d) If $\varphi(1) \geq \lambda$, *then (6.7) holds for every* $\theta \in [0,\theta_1)$ *and (6.8) holds for every* $\theta \in [0,\theta_2)$, *where* θ_1 *and* θ_2 *are as in (c).*
In this case $0 < \theta_2 < \theta_1 < \theta_u \leq 1$, *where* θ_u *is given in Proposition 6.1.1(b).*

Proof. We use the formulae for the moment functions $E[V_\theta(t)]$ and $\operatorname{var}(V_\theta(t))$, $t \geq 0$ given in Lemma 5.1.2. The calculations below show that $E\left[V_\theta^\infty\right]$ is finite, whenever $\varphi_\theta(1) < 0$, and $\operatorname{var}\left(V_\theta^\infty\right)$ is finite, whenever $\varphi_\theta(2) < 0$.

The formula for the expectation of V_θ^∞ follows directly from (5.2) letting the time to go to infinity. To derive the formula for the variance of V_θ^∞ we first calculate the double integral of the autocovariance function of $e^{-L_\theta(t)}$ in (5.3). For the Lévy process L_θ we have that for $0 \leq v < t$ holds $\operatorname{cov}(e^{L_\theta(t)}, e^{L_\theta(v)}) = E[e^{L_\theta(t-v)}]var(e^{L_\theta(v)})$. We devide the double integral in two parts:

$$\int_0^t \int_0^t \operatorname{cov}(e^{-L_\theta(v)}, e^{-L_\theta(w)})\,dw\,dv \quad = \quad I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_0^t \int_0^v E[e^{-L(v-w)}]\operatorname{var}(e^{-L(w)})\,dw\,dv = \int_0^t \int_0^v e^{(v-w)\varphi_\theta(1)}(e^{w\varphi_\theta(2)} - e^{2w\varphi_\theta(1)})\,dw\,dv$$

and

$$I_2 = \int_0^t \int_v^t E[e^{-L_\theta(w-v)}]\operatorname{var}(e^{-L_\theta(v)})\,dw\,dv = \int_0^t e^{-v\varphi_\theta(1)}(e^{v\varphi_\theta(2)} - e^{2v\varphi_\theta(1)}) \int_v^t e^{w\varphi_\theta(1)}\,dw\,dv\,.$$

Under the assumption that $\varphi_\theta(2) < 0$, hence also $\varphi_\theta(1) < 0$, letting t to infinity, we obtain

$$I_1(t) + I_2(t) \to \frac{2}{\varphi_\theta(1)\varphi_\theta(2)} - \frac{1}{\varphi_\theta^2(1)} = \frac{2\varphi_\theta(1) - \varphi_\theta(2)}{\varphi_\theta(2)\varphi_\theta^2(1)} (c - \lambda\mu)^2 \ , \ t \to \infty, ,$$

and, hence, letting t to infinity in (5.3) we obtain

$$\mathrm{var}(V_\theta^\infty) \quad = \quad \frac{2\varphi_\theta(1) - \varphi_\theta(2)}{\varphi_\theta(2)\varphi_\theta^2(1)} (c - \lambda\mu)^2 - \frac{\lambda\mu_2}{\varphi_\theta(2)} \, .$$

On the other hand by Lemma 3.2.5(c) we have that $\varphi_\theta(s) < \infty$ for every $\theta \in [0, 1]$ whenever $\varphi(s) < \infty$. Finally, recall that by Lemma 3.2.5(d), for fixed $s > 0$, the function $\varphi_\theta(s)$ is convex in $\theta \in [0, 1]$. Since $\varphi_0(s) = -\delta s < 0$ for $s > 0$, similar arguments as in the proof of Proposition 6.1.1(b) imply the required results. □

Remark 6.2.2. Using Lemma 6.2.1, a more prudent regulator or insurance company may derive a stricter upper bound for the investment strategies than the upper bound introduced in Proposition 6.1.1(b), see the comments after Definition 6.1.3. For instance, let $\varphi(2) \geq 0$, $\varphi(1) < 0$ and $E[Y^2] < \infty$. Then the a.s. limit V_θ^∞ of the DNLP exists and has finite mean for every $\theta \in [0, 1]$, see Proposition 6.1.1 and Lemma 6.2.1(b). However, V_θ^∞ does not have a finite second moment for investment strategies larger than θ_2 as defined in Lemma 6.2.1(b). Therefore, a risk averse insurance company may avoid investment strategies $\theta \geq \theta_2$. □

Example 6.2.3. [Continuation of Example 3.5.1]
Consider the geometric Brownian motion as a model for the risky asset as in Example 3.5.1. For simplicity assume that $E[Y^2] = \mu_2 < \infty$. We apply Lemma 6.2.1. Straightforward calculations show that we have the following cases.
(a) If $\sigma^2 < \gamma$, then $E\left[V_\theta^\infty\right] < \infty$ and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0, 1]$.
(b) If $\gamma \leq \sigma^2 < 2\gamma$, then $E\left[V_\theta^\infty\right] < \infty$ for every $\theta \in [0, 1]$ and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0, \theta_2)$, where

$$\theta_2 = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma + \sigma^2/2 - \delta)^2 + 6\sigma^2\delta}}{3\sigma^2} \in (0, 1] \, . \tag{6.9}$$

(c) If $2\gamma \leq \sigma^2 < 2(\gamma + \lambda)$, then $E\left[V_\theta^\infty\right] < \infty$ for every $\theta \in [0, \theta_1)$, where

$$\theta_1 = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma + \sigma^2/2 - \delta)^2 + 4\sigma^2\delta}}{2\sigma^2} \, , \tag{6.10}$$

and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0, \theta_2)$, where θ_2 is as in (6.9). In this case we have $0 < \theta_2 < \theta_1 \leq 1$.
(d) If $\sigma^2 \geq 2(\gamma + \lambda)$, then $E\left[V_\theta^\infty\right] < \infty$ for every $\theta \in [0, \theta_1)$ and $\mathrm{var}(V_\theta^\infty) < \infty$ for every $\theta \in [0, \theta_2)$, where θ_1 and θ_2 are as in (6.10) and (6.9) respectively. In this case $0 < \theta_2 < \theta_1 < \theta_u \leq 1$, where θ_u is as in (6.4). □

Since knowing the mean and the variance of a rv is not sufficient to compute its extreme quantiles, some additional analysis is needed. We make use of the fact that the

DNLP has a natural embedded discrete time skeleton, namely the process sampled at the claim arrival times. Using stochastic recurrence equations we achieve two goals, see Chapter 5 for details:

(1) explicit and easy to check in practice conditions for the existence of an a.s. limit V_θ^∞ of the DNLP as in Propisition 6.1.1;

(2) conditions for deriving the tail behaviour of V_θ^∞.

The next theorem combines the results for the behaviour of the right tail of V_θ^∞ in Theorem 5.3.6 and Theorem 5.3.9. As we have shown, there are two main regimes to consider, based on the interaction between the insurance claims and the investment process. We clarify this in the next result. Recall the notation for the set Θ in Definition 6.1.3.

Theorem 6.2.4. *Let the conditions in Proposition 6.1.1 hold and denote $\overline{\theta} = \sup \Theta$. For $\theta \in (0, \overline{\theta}]$ denote by $\kappa(\theta)$ the unique strictly positive solution in s to $\varphi_\theta(s) = 0$.*

(a) Dangerous investment*: Assume that Y has moments of every order. Then, for $\theta \in \Theta \setminus \{0\}$, there exists $C_+(\theta) > 0$, such that*

$$P(V_\theta^\infty > x) \sim C_+(\theta) x^{-\kappa(\theta)}\,, \quad x \to \infty\,. \tag{6.11}$$

(b) Dangerous claims*: Assume that $P(Y > x) \sim C_Y x^{-\rho}$, $x \to \infty$, for some constants $C_Y > 0$ and $\rho > 1$. If $\rho < \kappa(\overline{\theta})$, then for $\theta \in \Theta$,*

$$P(V_\theta^\infty > x) \sim \frac{\lambda}{|\varphi_\theta(\rho)|} C_Y x^{-\rho}\,, \quad x \to \infty\,. \tag{6.12}$$

If $\rho > \kappa(\overline{\theta})$, then (6.12) holds for $\theta \in [0, \theta_\rho)$ and (6.11) holds for $\theta \in (\theta_\rho, 1] \cap \Theta \neq \emptyset$, where $\theta_\rho \in (0, \overline{\theta})$ is the unique positive solution in θ to the equation $\varphi_\theta(\rho) = 0$.

Proof. (a) is a direct consequence of Theorem 5.3.6. To show (b), we use Theorem 5.3.9(a). We know that (6.12) holds for every θ such that $\rho < \kappa(\theta)$ and for $\theta = 0$. On the other hand, if $\rho > \kappa(\theta)$, then $E[Y^{\kappa(\theta)+\beta}] < \infty$ for some $\beta > 0$. Hence, if $\rho > \kappa(\theta)$, then (6.11) holds by Theorem 5.3.6. By Lemma 5.3.1(b) the function $\kappa(\theta)$ is strictly decreasing in $\theta \in \Theta \setminus \{0\}$. Therefore, if $\rho < \kappa(\overline{\theta})$, then $\rho < \kappa(\theta)$ for every $\theta \in \Theta \setminus \{0\}$.

Let now $\rho > \kappa(\overline{\theta})$. Note that $\varphi_0(\rho) = -\delta\rho < 0$ and that by Lemma 3.2.5(d), $\varphi_\theta(\rho)$ is convex in θ. Therefore, to show that $\varphi_\theta(\rho) = 0$ has a unique solution $\theta_\rho \in (0, \overline{\theta})$, it suffices to show that $\varphi_{\overline{\theta}}(\rho) > 0$. There are two cases. First, if we are in case (a) of Proposition 6.1.1, then $\overline{\theta} = 1$. By definition, $\varphi_1(\kappa(1)) = 0$. Since $\varphi_1(s)$ is convex in s and $\rho > \kappa(1)$, we have $\varphi_1(\rho) > \varphi_1(\kappa(1)) = 0$. Second, assume that we are in case (b) of Proposition 6.1.1, i.e. $\overline{\theta} = \theta_u$. By definition, $\varphi_{\theta_u}(1) = \lambda > 0$. Since $\varphi_{\theta_u}(s)$ is convex in s and $\rho > 1$, we have $\varphi_{\theta_u}(\rho) > \varphi_{\theta_u}(1) = \lambda > 0$. Finally, note that by definition $\kappa(\theta_\rho) = \rho$. Since $\kappa(\theta)$ is strictly decreasing in $\theta \in \Theta \setminus \{0\}$, we get that $\rho > \kappa(\theta)$ for $\theta > \theta_\rho$ and $\rho < \kappa(\theta)$ for $\theta < \theta_\rho$. This implies the required result. □

Remark 6.2.5. (i) Note that by Lemma 5.3.1(b) the Pareto index $\kappa(\theta)$ is a decreasing function in the investment strategy θ. This means that, whenever (6.11) holds, the more risky investment we choose, the heavier tail of the DNLP we get, which is quite natural.
(ii) The constant $C_+(\theta)$ in (6.11) cannot be computed analytically; see (5.37).
(iii) As for every fixed $s > 0$ the function $\varphi_\theta(s)$ is convex in θ (see Lemma 3.2.5(d)), there exists an investment strategy, minimizing the rhs of (6.12). □

Example 6.2.6. [Continuation of Example 3.5.1]
Consider the geometric Brownian motion as a model for the risky asset as in Example 3.5.1. Recall that in this example the investment process $\exp(L_\theta)$ is again a geometric Brownian motion with drift γ_θ and volatility σ_θ as in (3.25). For simplicity assume that $\sigma^2 < 2(\gamma + \lambda)$, so that $\Theta = [0, 1]$, see Example 6.1.4. The value $\kappa(\theta)$ is given by

$$\kappa(\theta) = \frac{2\gamma_\theta}{\sigma_\theta^2} = \frac{2}{\sigma^2\theta^2}\left(\gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta)\right),$$

see Example 5.3.8. Note that $\kappa(\theta) \to \infty$ as $\theta \to 0$. Therefore, when the claims have moments of every order, and when the fraction invested in the risky asset tends to 0, the tail of V_θ^∞ approaches a tail, which is no longer a Pareto tail. The limit case $\theta = 0$ is treated in detail in Example 5.2.1; see also Sundt and Teugels [58], and, in the case of Pareto claims, Klüppelberg and Stadtmüller [38].

When the insurance claims have Pareto tail with a small tail index $\rho < 2\gamma/\sigma^2$, then the tail of the a.s. limit V_θ^∞ of the DNLP is similar to that of the claims. In other words, the claims dominate the integrated risk process, regardless of the selected investment strategy. On the other hand, if $\rho > 2\gamma/\sigma^2$, then the claims dominate for the less risky strategies $\theta \in [0, \theta_\rho)$, whereas the investment process dominates for the more risky investment strategies $\theta \in (\theta_\rho, 1]$. We can compute θ_ρ from Theorem 6.2.4(b) as

$$\theta_\rho = \frac{\gamma + \sigma^2/2 - \delta + \sqrt{(\gamma + \sigma^2/2 - \delta)^2 + 2\sigma^2\delta(\rho+1)}}{(\rho+1)\sigma^2} \in (0, 1).$$

The investment strategy θ_ρ plays the role of a change-point strategy between the dangerous claims regime and the dangerous investment regime. □

6.3 Examples

Recall the optimization problem (6.6) considering the maximal investment strategy θ, such that a risk constraint is satisfied. To solve this problem we need a method to compute the quantile of V_θ^∞ (the VaR) as a function of the investment strategy θ. Unfortunately, apart from the very few special cases considered in Section 5.2 the distribution of V_θ^∞ is not known. In general we can compute the moments of V_θ^∞, if they exist, see Lemma 6.2.1. Further, we know from Theorem 6.2.4 that V_θ^∞ has a Pareto tail. The Pareto index depends on the interaction between the insurance claims and the investment process, see also Example 4.6 and Example 4.8 in [37]. We distinguish between two different regimes.

(a) Dangerous investment: insurance claims have moments of a sufficiently large order; then the Pareto index of V_θ^∞ is determined only by the investment process.

(b) Dangerous claims: insurance claims have a Pareto tail with a sufficiently small Pareto index; then the Pareto index of V_θ^∞ is the same as that of the claims.

6.3.1 Dangerous investment

First we consider the dangerous investment regime, i.e. when the investment process dominates the integrated risk process. In what follows we assume that there exists an a.s. limit V_θ^∞ of the DNLP with finite mean and variance for all investment strategies $\theta \in [0,1]$. This is satisfied when $\varphi(2) < 0$ and the insurance claims have finite second moment ($E[Y^2] < \infty$), see Lemma 6.2.1.

A crude and often used approximation of the $(1-\alpha)$-quantile of V_θ^∞ (the VaR) can be achieved by the $(1-\alpha)$-quantile of a normal rv with the same mean and variance.

Normal approximation algorithm, dangerous investment

Let V_θ^N be a normal rv with mean and variance as those of V_θ^∞ and let $q_\alpha(\theta)$ be its $(1-\alpha)$-quantile. Then we have $P(V_\theta^N > x) = P(E[V_\theta^\infty] + \sqrt{\mathrm{var}(V_\theta^\infty)}N(0,1) > x)$, where $N(0,1)$ is a standard normal rv. Therefore, we obtain that

$$q_\alpha(\theta) = E[V_\theta^\infty] + \Phi^{-1}(1-\alpha)\sqrt{\mathrm{var}(V_\theta^\infty)}\,,$$

where Φ^{-1} is the quantile function of the standard normal distribution. □

Assuming that the df of V_θ^N approximates the df of V_θ^∞ (and hence $q_\alpha(\theta)$ approximates $\mathbf{VaR}_\alpha(V_\theta^\infty)$), we replace the optimization problem (6.6) by

$$\max\{\theta \in [0,1] \,:\, q_\alpha(\theta) \le C\}\ . \tag{6.13}$$

Note that the moments $E[V_\theta^\infty]$ and $\mathrm{var}(V_\theta^\infty)$ can be computed by Lemma 6.2.1 and, hence, the optimization problem (6.13) can be solved by numerical methods. However, it is well known that the normal approximation does not take into account interesting properties of the original distribution as skewness or heavy tails. Hence, for our model it will presumably underestimate the risk considerably. We demonstrate this in the following example.

Example 6.3.1. [Exponential claims]
We consider an insurance model with a premium rate $c = 2.1$, an intensity of the Poisson claim counting process $\lambda = 1$ and exponential claims with a mean $E[Y] = \mu = 2$. We assume also that the price of the risky asset follows a geometric Brownian motion with a drift $\gamma = 0.06$ and a volatility $\sigma = 0.2$. In this example we analyze the pure stock strategy $\theta = 1$ only.

We simulate 10 000 copies of the rv V_1^∞. As we do not know the distribution of V_1^∞, we invoke the forward stochastic recurrence equation corresponding to the discrete time skeleton of the DNLP, see Section 5.1.

In Figure 6.1, left plot, the histogram of the simulated data is compared to the corresponding normal density of the rv V_1^N with a mean $E[V_1^\infty] = -2.5$ and a variance $\mathrm{var}(V_1^\infty) = 106.25$, as computed by Lemma 6.2.1. We see more values close to the mean in the simulated data than the normal approximation suggests.

In Figure 6.1, right plot, we compare the empirical quantiles of the simulated data to the normal quantiles. We see that in this example, when α is around 2.5%, the normal approximation works quite well. However, when we go further in the tail, for α =1% or 0.5%, the normal approximation underestimates the risk significantly. □

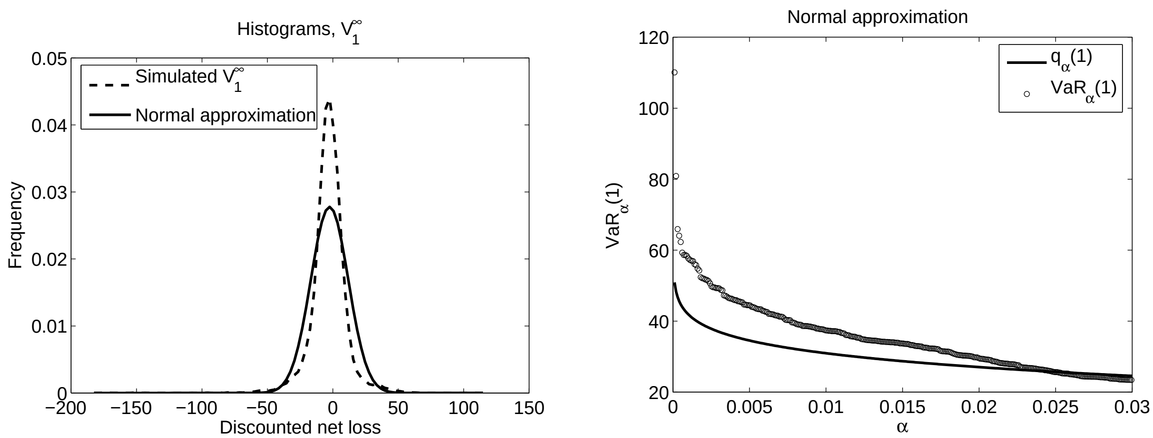

Figure 6.1: Histograms and quantiles of V_1^∞ for simulated data versus the normal approximation. The parameters are as in Example 6.3.1. Left plot: histogram of 10 000 simulated copies of V_1^∞ compared to the density of a normal rv with the same mean and variance as V_1^∞. Right plot: empirical quantiles of the simulated data compared to the normal quantiles.

Example 6.3.1 demonstrates that the normal approximation of the quantile of the rv V_1^∞ is not very satisfactory, when one is interested in extreme quantiles. This happens despite the fact, that we have a light-tailed input (i.e. exponentially distributed insurance claims and a geometric Brownian motion for the stock price). In this case Theorem 6.2.4(a) applies and V_θ^∞ has a Pareto tail, for all $\theta \in (0,1]$.

From now on we assume for simplicity that the claims have finite moments of every order, so that we are in the dangerous investment regime for all $\theta \in (0,1]$. In the next approximation method we make extensive use of Theorem 6.2.4(a). Unfortunately, a straightforward approximation of the tail of V_θ^∞ with asymptotic as in (6.11) is not possible, since the constant $C_+(\theta)$ cannot be computed, see Remark 6.2.5(ii). As a remedy we combine the normal approximation with the Pareto tail behaviour in the following algorithm.

Pareto approximation algorithm, dangerous investment

If (6.11) holds for V_θ^∞, then it also holds for the centered rv:

$$P(V_\theta^\infty - E[V_\theta^\infty] > x) \sim C_+(\theta)(x + E[V_\theta^\infty])^{-\kappa(\theta)} \sim C_+(\theta) x^{-\kappa(\theta)}\,, \;\; x \to \infty\,. \qquad (6.14)$$

Denote by $G_\theta^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} \,:\, P(V_\theta^\infty - E[V_\theta^\infty] > x) \le \alpha\}$, $\alpha \in (0,1)$, the generalized inverse function of the df of the centered rv. Then, by (6.14) and Theorem 2.4.5, we have

$$\frac{G_\theta^{\leftarrow}(\alpha)}{G_\theta^{\leftarrow}(\beta)} \sim \left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)}\,, \;\; \alpha < \beta, \;\; \alpha \to 0,\, \beta \to 0\,. \qquad (6.15)$$

We select some small probability $\beta > \alpha$, where α is the given probability of interest in Definition 6.1.3. Using (6.15) we approximate

$$G_\theta^{\leftarrow}(\alpha) \approx G_\theta^{\leftarrow}(\beta) \left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)}\,.$$

Then we apply the normal approximation to $G_\theta^{\leftarrow}(\beta)$ to get

$$G_\theta^{\leftarrow}(\alpha) \approx \Phi^{-1}(1-\beta)\sqrt{\mathrm{var}(V_\theta^\infty)}\left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)} .$$

Since $G_\theta^{\leftarrow}(\alpha) = \mathbf{VaR}_\alpha(V_\theta^\infty) - E[V_\theta^\infty]$, we approximate $\mathbf{VaR}_\alpha(V_\theta^\infty)$ by

$$p_\alpha(\theta) = E[V_\theta^\infty] + \Phi^{-1}(1-\beta)\sqrt{\mathrm{var}(V_\theta^\infty)}\left(\frac{\alpha}{\beta}\right)^{-1/\kappa(\theta)} . \tag{6.16}$$

Note that, when $\alpha = \beta$, then $p_\alpha(\theta) = q_\alpha(\theta)$. □

Using the above algorithm, we replace the optimization problem (6.6) by

$$\max\{\theta \in [0,1] \,:\, p_\alpha(\theta) \le C\} , \tag{6.17}$$

where $p_\alpha(\theta)$ is as in (6.16). In the next example we investigate the accuracy of the Pareto approximation applied to the model in Example 6.3.1.

Example 6.3.2. [Continuation of Example 6.3.1]
Consider the model with light-tailed input as in Example 6.3.1. In Figure 6.2 we compare the normal approximation to the suggested Pareto approximation of $\mathbf{VaR}_\alpha(V_\theta^\infty)$ for two ranges for α. In the left plot we show the VaR for comparatively large probabilities $\alpha \in(0.5\%, 3\%)$ based on $\beta = 0.03$, and in the right plot – for very small probabilities $\alpha < 0.5\%$ based on $\beta = 0.005$. In both cases the Pareto approximation provides a better fit to the empirical quantiles than the normal approximation, in particular for $\alpha \in (0.5\%, 1.5\%)$ in the left plot and for $\alpha < 0.2\%$ in the right plot. Note that the Pareto and the normal approximation are equal when $\alpha = \beta$, which explains the gap between the empirical quantiles and the approximations at $\alpha = \beta = 0.5\%$ in the right plot of Figure 6.2, see also Figure 6.1, right plot. □

6.3.2 Dangerous insurance claims

We consider the dangerous insurance claims regime, i.e. when the insurance process dominates the integrated risk process. In what follows we assume that there exists an a.s. limit V_θ^∞ of the DNLP with finite mean for all investment strategies $\theta \in [0,1]$. This is satisfied when $\varphi(1) < 0$, see Lemma 6.2.1. Moreover, we assume that the claims have a Pareto distribution, i.e. for some $\rho > 1$, $l > 0$,

$$P(Y > x) = \left(\frac{l}{l+x}\right)^\rho , \; x > 0 .$$

From now on we assume for simplicity that $\rho < \kappa(1)$ in all considered models for the risky investment. In this case (6.12) holds and V_θ^∞ has Pareto tail with Pareto index ρ for all investment strategies $\theta \in [0,1]$. We suggest the following approximation algorithm:

Pareto approximation algorithm, dangerous claims

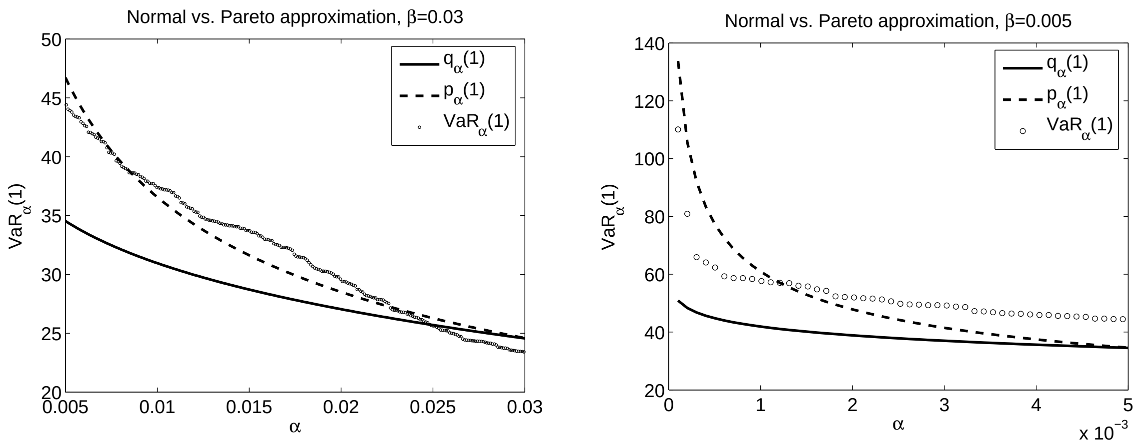

Figure 6.2: Comparison of the normal approximation, the Pareto approximation and the empirical quantiles of the simulated data. The parameters are as in Example 6.3.1. Left plot: in the Pareto approximation algorithm we have chosen $\beta = 0.03$ and we are interested in $\alpha \in (0.005, 0.03)$. Right plot: in the Pareto approximation algorithm we have chosen $\beta = 0.005$ and we are interested in $\alpha \in (0, 0.005)$.

Similar to the approximation algorithm in Section 6.3.1, if (6.12) holds for V_θ^∞, then it also holds for the centered rv:

$$P(V_\theta^\infty - E[V_\theta^\infty] > x) \sim \frac{l^\rho \lambda}{|\varphi_\theta(\rho)|} x^{-\rho}\,, \ \ x \to \infty\,.$$

Hence, for the generalized inverse function $G_\theta^{\leftarrow}(\alpha)$ of the df of the centered rv holds

$$G_\theta^{\leftarrow}(\alpha) \sim \alpha^{-1/\rho} \left(\frac{\lambda l^\rho}{|\varphi_\theta(\rho)|} \right)^{1/\rho}, \quad \alpha \to 0\,. \tag{6.18}$$

Using (6.18) and the fact that $G_\theta^{\leftarrow}(\alpha) = \mathbf{VaR}_\alpha(V_\theta^\infty) - E[V_\theta^\infty]$, we approximate $\mathbf{VaR}_\alpha(V_\theta^\infty)$, for a small α, by

$$r_\alpha(\theta) = E[V_\theta^\infty] + \alpha^{-1/\rho} \left(\frac{\lambda l^\rho}{|\varphi_\theta(\rho)|} \right)^{1/\rho}. \tag{6.19}$$

□

Using the above algorithm, we replace the optimization problem (6.6) by

$$\max\left\{\theta \in [0,1] \,:\, r_\alpha(\theta) \le C\right\}, \tag{6.20}$$

where $r_\alpha(\theta)$ is as in (6.19). We investigate the accuracy of the suggested Pareto approximation for the dangerous claims regime in the next example.

Example 6.3.3. [Pareto claims]
We consider an insurance model with a premium rate $c = 2.1$, an intensity of the claim counting process $\lambda = 1$ and Pareto claims with $\rho = 1.1$ and $l = 0.2$. The parameters of

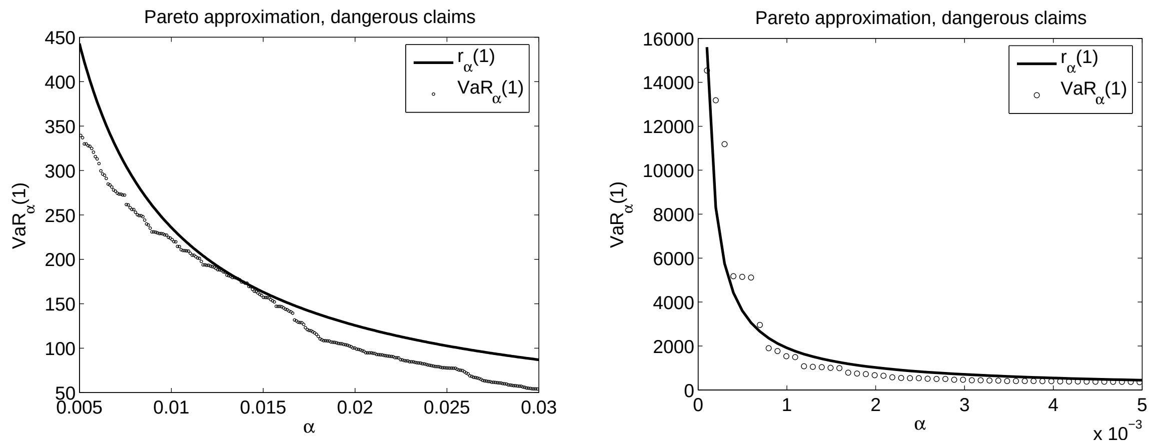

Figure 6.3: Comparison of the Pareto approximation and the empirical quantiles of the simulated data. The parameters are from Example 6.3.3. Left plot: we are interested in $\alpha \in (0.005, 0.03)$ Right plot: we are interested in extreme quantiles for $\alpha < 0.005$.

the investment model are the same as in Example 6.3.1. In this example we analyze the pure stock strategy $\theta = 1$ only.

We simulate 10 000 copies of the rv V_1^∞ using the same method as in Example 6.3.1.

In Figure 6.3 we compare the suggested Pareto approximation to the empirical quantiles of the simulated data. We see that in the tail, i.e. for probabilities less than 2%, the suggested approximation is quite accurate. □

6.3.3 Comparison of the models

In the previous two sections we have discussed methods to approximate the VaR in the dangerous investment and in the dangerous claims regime. This enables us to find in each of the two regimes an (approximate) solution to the optimization problem (6.6) by numerical methods. Up to now for simplicity we have considered only examples where the risky asset is modeled by a geometric Brownian motion. In this section we focus on the impact of different models for both the insurance claims and the risky asset on the optimal investment strategy. From one side we discuss the difference between the optimal investment strategy in the dangerous claims regime and in the dangerous investment regime. From another side, we compare the magnitude of the influence of different models for the risky asset on the optimal investment strategy within each of the regimes.

Example 6.3.4. [Comparison of the models]
We compare the Brownian motion model from Example 6.3.1 to a variance gamma (VG) model for the risky asset, see Example 3.5.3. We consider a VG process with paramethers $\xi = 0.16$, $a = -0.1$, $b^2 = 0.04 - a^2$ and $\eta = r = 1$.

In order to allow for a comparison, we have selected the parameters of the Brownian motion and of the VG process in such a way, that the mean and the variance of $L(1)$, i.e. of the log returns of the stock price, coincide in both models. However, the VG model

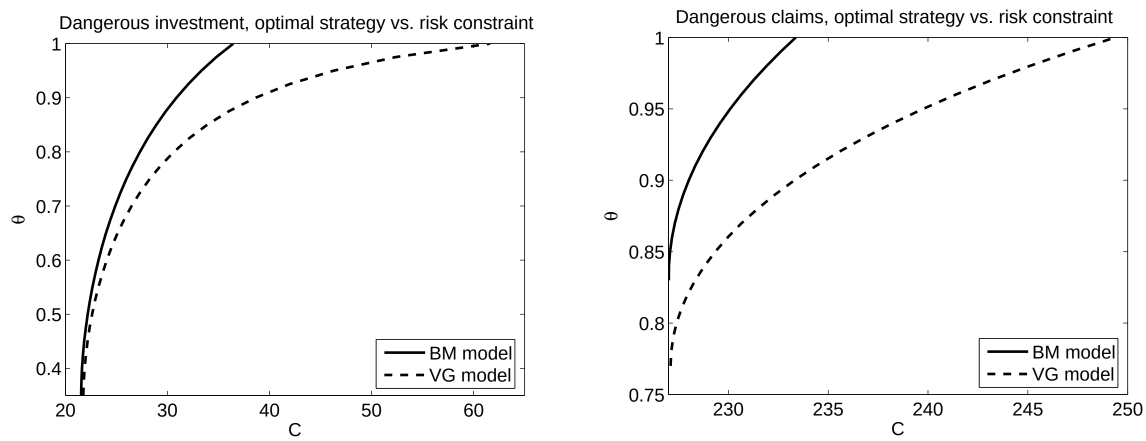

Figure 6.4: The solution to optimization problem (6.6) for different risk constraints. The parameters are from Example 6.3.4. Left plot: Dagerous investment regime. The claims are modeled as in Example 6.3.1. Right plot: Dangerous claims regime. The claims are modeled as in Example 6.3.3

has jumps, taken here with a negative mean, corresponding to (downward) jumps of the stock price. Hence, one would consider it as a more 'dangerous' investment model.

For the insurance business we use the parameters from Example 6.3.1 (dangerous investment regime) and Example 6.3.3 (dangerous claims regime). We set the riskless interest rate to $\delta = 0.01$. We solve (6.17) for the dangerous investment regime and (6.20) for the dangerous claims regime, using a number of risk constraints C. We have selected $\alpha = 1\%$ in the definition of the VaR.

Let us first fix the regime (i.e. the distribution of the claim sizes) and compare the impact of the stock price model on the VaR and on the optimal investment strategy.

In Figure 6.4, left plot, we show the optimal investment strategy in the dangerous investment regime (i.e. the solution to the optimization problem (6.17)), vs. the risk constraint C for the VaR. We observe that the more risky VG model affects significantly the VaR, in particular for more risky investment strategies $\theta > 80\%$. For a fixed risk constraint C, it allows for less investment in the risky asset than the Brownian motion model. Hence, we may conclude that the more risky model for the stock price leads to more conservative investment strategies.

In Figure 6.4, right plot, we show the optimal investment strategy in the dangerous claims regime (i.e. the solution to the optimization problem (6.20)), vs. the risk constraint C for the VaR. Again, we observe that the more risky VG model implies more conservative investment strategies. However, in the dangerous claims regime this impact is weaker compared to the dangerous investment regime; compare the right and the left plot of Figure 6.4 and note the difference in the scales of the horizontal axes. For instance, at investment strategy $\theta = 1$, changing the investment model from a Brownian motion to a VG leads to about 90% increase of the VaR in the dangerous investment regime, whereas this change leads to only 10% increase of the VaR in the dangerous claims regime.

Let us fix the model of the stock price and investigate the impact of the claim size distribution. Note that in the dangerous claims and in the dangerous investment regime we have the same interest rate, premium rate, claim arrival intensity and mean claim size. This implies that, holding the stock price model fixed, for every fixed investment strategy $\theta \in [0,1]$, the mean of the a.s. limit V_θ^∞ of the DNLP in both regimes is the same. However, for both stock price models, the Pareto claims from Example 6.3.3 lead to higher risk compared to the exponential claims from Example 6.3.1. Indeed, in the dangerous claims regime, the risk constraint C has to be set much higher than in the dangerous investment regime (almost 10 times), in order to obtain a solution to the optimization problem at all, notice again the difference in the scales of the horizontal axes in Figure 6.4. Furthermore, the difference between the light- and the heavy-tailed claims model is much more severe than between the two stock price models. For instance, at investment strategy $\theta = 1$, changing the insurance claims model from exponential to Pareto leads to almost 9 times increase of the VaR in the Brownian motion case (compare the two plots in Figure 6.4), while changing the investment model from a Brownian motion to a VG leads to only about 90% increase of the VaR in the exponential claims case (Figure 6.4, left plot).

Recall that, in the dangerous claims regime, the Pareto index of V_θ^∞ is the same as that of the insurance claims. In contrast to that, in Example 6.3.1, the investment process determines the Pareto index $\kappa(\theta)$ of V_θ^∞ regardless of the insurance process. Therefore, the choice of the investment strategy is much more important in the case of Example 6.3.1 than in Example 6.3.3. In other words, the VaR is less sensitive to the investment strategy in the dangerous claims regime than it is in the dangerous investment regime. For instance, within the Brownian motion model, increasing the investment strategy from 0.85 to 1 leads to an increase of about 3% of the VaR in the dangerous claims regime (Figure 6.4, right plot), while the same change in the investment strategy leads to about 24% increase of the VaR in the dangerous investment regime (Figure 6.4, left plot). Similar observations can be made for the VG model for the stock price. □

Chapter 7

Conclusion

The development of mathematical methods for integrated risk management is of theoretical as well as of practical interest. In this thesis we modeled the wealth of an insurance company which has the possibility to invest into a risky and a riskless asset under a constant mix strategy. From a theoretical point of view, we worked with a relatively complicated stochastic process (the IRP) based on underlying general Lévy processes with jumps. We investigated the distributional and the path-wise properties of the IRP. A transformation of the process was shown to have, under weak conditions, a stationary a.s. limit. We derived the right and the left tail behaviour of the resulting stationary distribution, and analysed the impact of different model subclasses (regimes) on it. Several quantile approximation methods based on these results were suggested, and their accuracy was investigated numerically. This enabled us to find explicit solutions to a special optimization problem of particular practical importance.

From a practical point of view, we tried to keep the model as simple as possible. Various relevant generalizations of the model were left for future research. These include, among the others:

- reinsurance or taxation issues;
- IBNR claims;
- dynamics of the premium rate;
- several insurance business lines;
- liquidity of the investments;
- stochastic interest rates;
- optimization within a risky portfolio with a large number of positions within;
- optimization with a dynamically changing investment strategy;
- dependence between the stock market and the total claim amount process;
- intertemporal dependence in the stock market.

Even without these features, the model poses a significant analytical challenge. Nevertheless, we derived a method to measure the risk in a stationary way by defining the risk measure Value-at-Risk (VaR) in our integrated framework. The VaR takes extreme risks into account during time intervals, where all parameters of the insurance model and the investment model are fixed. Only changes in these parameters would indicate a change in the risk as measured by the VaR.

The main focus was on computing the efficient frontier – the set of investment strategies which maximize the wealth of the insurance company subject to a risk constraint on the VaR. We suggested several methods to approximate the VaR and solved the optimizazion problem. In the case when the stock price follows a geometric Brownian motion, we derived quite explicit results. At first sight this is encouraging, as it allows a simple and straightforward calibration of the model, based e.g. on a moment matching procedure. However, we showed in a simple example that such a straightforward approach carries a significant model error risk. A particular advantage of this work is that the obtained results hold for the general class of Lévy processes with jumps, which will presumably describe the dynamics of the risky asset prices in a better way. Hence, for such models we are still able to compute the VaR and the optimal investment strategies by a slightly more complicated numerical methods than in the classical Brownian motion case.

Finally we investigated the impact of the different models for the insurance claims and for the investment process on the VaR and on the efficient frontier. We identified two different regimes: in the first the risk of the integrated model is driven mainly by the investment process and in the second – by dangerous large claims. For a fixed model for the stock price, the investment strategy has a greater impact on the VaR in the dangerous investment regime than in the dangerous claims regime. For a fixed model for the insurance claims, the more risky investment model we take, the greater is the impact of the investment strategy on the VaR. In practice, this could have important consequences in the risk management strategy of an insurance company.

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