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SPHERE COVERINGS, LATTICES, AND TILINGS
(in Low Dimensions)

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Contents

1. Introduction **1**
1.1. Dirichlet-Voronoi Polytopes . . . 1
1.2. Sphere Coverings and Sphere Packings . . . 2
1.3. Prerequisites . . . 4
1.4. Organisation of the Thesis . . . 4

2. Voronoï's Reduction Theory **11**
2.1. Delone Subdivisions . . . 12
2.2. Secondary Cones . . . 13
2.3. Voronoï's Principal Domain of the First Type . . . 15
2.3.1. Selling Parameters . . . 16
2.3.2. Computation of the Secondary Cone . . . 17
2.3.3. Selling's Reduction Algorithm . . . 18
2.4. Bistellar Neighbours . . . 21
2.5. Main Theorem of Voronoï's Reduction Theory . . . 23
2.6. Refinements and Sums . . . 25
2.7. Relations to the Theory of Secondary Polytopes . . . 27

3. Parallelohedra **29**
3.1. Definition and Basic Properties . . . 29
3.2. Voronoï's Conjecture . . . 30
3.3. Duality . . . 33
3.3.1. Definition . . . 33
3.3.2. Upper Bound Theorem . . . 34
3.3.3. Voronoi Vectors, Supporting Hyperplanes, and Facets . . . 34
3.3.4. Linearity and Rigidity . . . 35
3.4. Vonorms and Conorms . . . 36
3.5. Zonotopal Parallelohedra . . . 39
3.5.1. Definition and Basic Properties . . . 39
3.5.2. Seymour's Decomposition Theorem . . . 41
3.5.3. Delone Subdivisions, Dicings and Zonotopal Lattices . . . 44

4. Results in Low Dimensions **47**
4.1. Dimension 1 . . . 48
4.2. Dimension 2 . . . 48
4.3. Dimension 3 . . . 48
4.4. Dimension 4 . . . 50
4.4.1. Four-Dimensional Delone Triangulations . . . 50
4.4.2. Vonorms and Conorms in Dimension 4 . . . 53

4.4.3. Towards a Classification . . . 54
4.4.4. Diagrams for Zonotopal Cases . . . 54
4.4.5. Diagrams for Non-Zonotopal Cases . . . 55
4.4.6. More Data . . . 61
4.5. Dimension 5 . . . 62
4.5.1. The Missing Delone Triangulation . . . 62
4.5.2. Rigid Forms in Dimension 5 . . . 63
4.6. Dimension 6 and Higher . . . 64

5. Determinant Maximization **69**
5.1. The Determinant Maximization Problem . . . 69
5.2. Convexity of the Problem . . . 69
5.3. Relation to Semidefinite Programming . . . 70
5.4. Algorithms for the Determinant Maximization Problem . . . 71

6. Cayley-Menger Determinants **73**
6.1. Definition and Basic Properties . . . 73
6.2. The Radius of the Circumsphere of a Simplex . . . 74
6.3. A Linear Matrix Inequality . . . 75

7. Solving the Lattice Covering Problem **79**
7.1. A Restricted Lattice Covering Problem . . . 79
7.2. Interpretation of “Classical” Results . . . 82

8. Moments of Inertia **85**
8.1. The Moment of Inertia and the Circumradius of a Simplex . . . 85
8.2. Linear Optimization on Equidiscriminant Surfaces . . . 87
8.3. A Local Lower Bound . . . 88
8.4. Applications . . . 88
8.4.1. The Lattice Covering of A_d^* . . . 88
8.4.2. Four-dimensional Lattice Coverings . . . 93
8.4.3. Invariants and Symmetry . . . 95
8.4.4. Navigating in the Graph of Delone Triangulations . . . 95

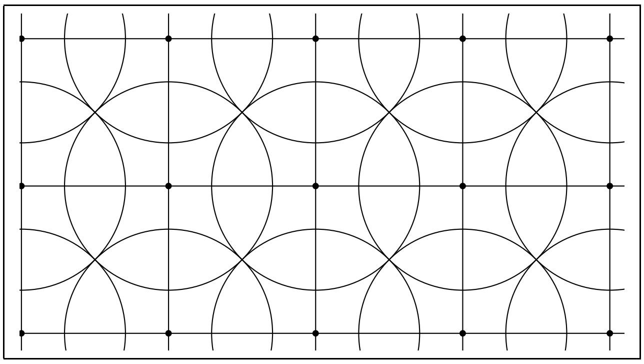
9. Results in Low Dimensions **99**
9.1. Dimension 1 . . . 99
9.2. Dimension 2 . . . 99
9.3. Dimension 3 . . . 100
9.4. Dimension 4 . . . 100
9.5. Dimension 5 . . . 104
9.6. Dimension 6 . . . 105
9.6.1. The best known 6-dimensional lattice covering . . . 105
9.6.2. The second best known 6-dimensional lattice covering . . . 107
9.7. Dimension 7 . . . 108
9.8. Dimension 8 and Higher . . . 109

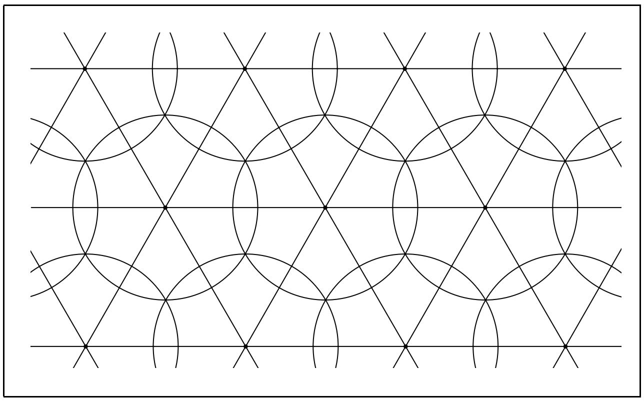
A. Glossary **111**
A.1. Geometry of Numbers . . . 111
A.2. Polyhedra and Polytopes . . . 113
A.3. (Realizable) Oriented Matroids . . . 113

Chapter 1.

Introduction

The main subject of this thesis is the geometry of low dimensional lattices. We concentrate on two central problems: the lattice covering problem and the classification of polytopes that tile space by lattice translates. The lattice covering problem asks for the most economical way to cover d-dimensional space by equal, overlapping spheres whose centers form a lattice. Let us look at the two most prominent plane lattices together with their coverings. It is obvious that the covering which belongs to the square lattice is less economical than the one that belongs to the hexagonal lattice.





The purpose of this introduction is two-fold. First, we define the rather intuitive notions we already used. We show how our future main actors, namely Dirichlet-Voronoi polytopes of lattices, come into the spotlight. Second, we give an outline of the thesis.

1.1. Dirichlet-Voronoi Polytopes

A lattice L is a discrete subgroup of a d-dimensional Euclidean vector space $(E, (\cdot, \cdot))$. For any lattice there are always linearly independent vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \in L$ so that they form a lattice basis, $L = \mathbb{Z}\boldsymbol{b}_1 \oplus \cdots \oplus \mathbb{Z}\boldsymbol{b}_n$. In the following we assume for simplicity and without loss of generality that n equals d. The geometry of a lattice is encoded in its Dirichlet-Voronoi polytope. This is the polytope which contains all those points lying closer to the origin than to all other lattice points:

$$\mathrm{DV}(L, (\cdot,\cdot)) = \{\boldsymbol{x} \in E : \text{for all } \boldsymbol{v} \in L \text{ we have } \mathrm{dist}(\boldsymbol{x}, \boldsymbol{0}) \leq \mathrm{dist}(\boldsymbol{x}, \boldsymbol{v})\}.$$

If it is clear which scalar product we are using, we simply write $\mathrm{DV}(L)$.

Dirichlet-Voronoi polytopes are very special polytopes. They tile E by translates of the form $\mathrm{DV}(L) + \boldsymbol{v}$, $\boldsymbol{v} \in L$, in a face-to-face manner. They and all their facets are centrally symmetric. If one knows the Dirichlet-Voronoi polytope of a lattice it is easy to determine geometrical information about the lattice. Its volume is the same as the lattice's volume, i.e. the volume of a fundamental parallelotope. A fundamental parallelotope is given by a lattice basis $\{\sum_i \alpha_i \boldsymbol{b}_i : \alpha_i \in [0,1]\}$. The Dirichlet-Voronoi polytope's circumsphere and the insphere give information about covering respectively about packing properties.

1.2. Sphere Coverings and Sphere Packings

A family of subsets $\mathcal{K} = (K_i)_{i\in I}$ of $\mathbb{R}^d$, I a set of indices, is called a *covering* of $\mathbb{R}^d$ if each point of $\mathbb{R}^d$ belongs to at least one of the sets K_i, i.e. $\mathbb{R}^d = \bigcup_{i\in I} K_i$. A covering of $\mathbb{R}^d$ is a *lattice covering* if it is of the form $(K+\boldsymbol{v})_{\boldsymbol{v}\in L}$ where L is a lattice. In summary, lattice coverings are those coverings which cover $\mathbb{R}^d$ by translated copies of a single body K and in addition the translates are the vectors of a lattice L.

Although the general notion of density is very intuitive, it is not easy to give a good definition which exhibits all pathological cases (see e.g. [Kup2000]). But since we are interested only in the case of lattice coverings, pathological cases do not exist and we are fine with the following quite classical definitions.

Let $C_d(\boldsymbol{p}, r) = \{\boldsymbol{x} \in \mathbb{R}^d : \max_i |x_i - p_i| \leq r\} = [-\frac{r}{2}, \frac{r}{2}]^d + \boldsymbol{p}$ be a cube with side length r centered at $\boldsymbol{p}$. We say that a collection $\mathcal{K} = (K_i)_{i\in I}$ of subsets of $\mathbb{R}^d$ has *density* $\rho(\mathcal{K})$ if the limit

$$\lim_{r\to\infty} \frac{\sum_{i\in I} \mathrm{vol}(C_d(\boldsymbol{0}, r) \cap K_i)}{\mathrm{vol}\, C_d(\boldsymbol{0}, r)}$$

exists and if it equals $\rho(\mathcal{K})$. Of course, there are cases where the limit does not exist and there are even cases where the formula does not make any sense. But for lattice coverings $(K+\boldsymbol{v})_{\boldsymbol{v}\in L}$ with measurable body K there is no such problem as demonstrated in the first chapter of ROGERS' little book [Rog1964]. There (Theorem 1.6) it is also shown that the density of a lattice covering $\mathcal{K} = (K+\boldsymbol{v})_{\boldsymbol{v}\in L}$ can be expressed as simple as $\rho(\mathcal{K}) = \frac{\mathrm{vol}\, K}{\mathrm{vol}\, L}$ where $\mathrm{vol}\, L$ is the volume of a fundamental parallelotope of the lattice L.

In the following we are only considering coverings consisting of solid spheres (balls). By $B_d(\boldsymbol{p}, r) = \{\boldsymbol{x} \in \mathbb{R}^d : \mathrm{dist}(\boldsymbol{x}, \boldsymbol{p}) \leq r\}$ we denote the d-dimensional closed ball with center $\boldsymbol{p}$ and radius r. A lattice L gives always a lattice covering of equal spheres $(B_d(\boldsymbol{v}, r))_{\boldsymbol{v}\in L}$ if the radius r is large enough. If we start with a lattice covering $(B_d(\boldsymbol{v}, r))_{\boldsymbol{v}\in L}$ and shrink the spheres until they finally do not cover the space any more, then the threshold value of the shrinking radius r defines the least dense covering of equal spheres with covering lattice L. The threshold value is called the *covering radius* of L

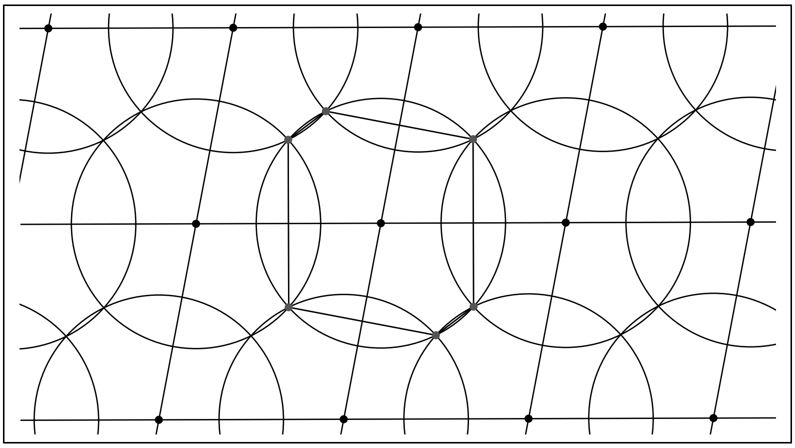
$$\mu(L) = \min\{r : (B_d(\boldsymbol{v}, r))_{\boldsymbol{v}\in L} \text{ is a lattice covering of } \mathbb{R}^d\}.$$

The *covering density* of the lattice covering given by the lattice L is therefore

$$\Theta(L) = \frac{\operatorname{vol} B_d(\mathbf{0}, \mu(L))}{\operatorname{vol} L} = \frac{\mu(L)^d}{\operatorname{vol} L} \operatorname{vol} B_d(\mathbf{0}, 1).$$

More precise: $\Theta(L)$ is the density of the least dense lattice covering of equal spheres with covering lattice L. Sometimes, the normalized covering density $\theta(L) = \frac{\mu(L)^d}{\operatorname{vol} L}$ gives nicer numbers than $\Theta(L)$.

The covering radius of a lattice can be determined geometrically as indicated in the following picture.



The covering radius is the radius of the circumsphere of the lattice's Dirichlet-Voronoi polytope, that is the largest distance between the midpoint and the vertices of a Dirichlet-Voronoi polytope. This requires a little proof. Let $r = \max\{\operatorname{dist}(\boldsymbol{x}, \mathbf{0}) : \boldsymbol{x} \in \operatorname{DV}(L)\}$ be the radius of the circumsphere of $\operatorname{DV}(L)$ and let $\mu(L)$ be the covering radius of L. We have $r \geq \mu(L)$ because every point $\boldsymbol{x} \in \mathbb{R}^d$ lies in a translate of the Dirichlet-Voronoi polytope $\boldsymbol{x} \in \operatorname{DV}(L) + \boldsymbol{v}$ for some $\boldsymbol{v} \in L$, therefore $B_d(\boldsymbol{v}, r)$ covers $\boldsymbol{x}$: $B_d(\boldsymbol{v}, r) \supseteq \operatorname{DV}(L) + \boldsymbol{v} \ni \{\boldsymbol{x}\}$. We have $r \leq \mu(L)$ because if we would have $r > \mu(L)$ then there would exist a vertex $\boldsymbol{x} \in \operatorname{DV}(L)$ with $\operatorname{dist}(\boldsymbol{x}, \mathbf{0}) = r$ and then $\boldsymbol{x}$ would not be covered by any of the balls $B_d(\boldsymbol{v}, \mu(L))$, $\boldsymbol{v} \in L$, since $\operatorname{dist}(\boldsymbol{x}, \boldsymbol{v}) \geq \operatorname{dist}(\boldsymbol{x}, \mathbf{0}) = r > \mu(L)$. Hence, $r = \mu(L)$.

Now we are ready to define the lattice covering problem.

Definition 1.2.1. (*Lattice Covering Problem*)

Given: Dimension d

Find: The value $\Theta_d = \min\{\Theta(L) : L \text{ is a } d\text{-dimensional lattice}\}$ together with a d-dimensional lattice L with $\Theta_d = \Theta(L)$.

Later we will prove that the function Θ is continuous so that we actually can talk about "$\min$" in the problem above. The lattice covering problem first appeared in a paper [Ker1939] by RICHARD KERSHNER in 1939. In contrast, the related lattice packing problem has been studied since centuries. The lattice packing problem asks for the most dense way to pack d-dimensional space by equal non-overlapping spheres whose centers form a lattice.

A family of subsets $\mathcal{K} = (K_i)_{i \in I}$ of $\mathbb{R}^d$, I a set of indices, is called a *packing* if no point in $\mathbb{R}^d$ belongs to the interiors of two different sets, i.e. $\operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset$ whenever $i \neq j$.

A packing of $\mathbb{R}^d$ is a *lattice packing* if it is of the form $(K + \boldsymbol{v})_{\boldsymbol{v}\in L}$ where L is a lattice. The value $\lambda(L) = \max\{r : (B(\boldsymbol{v}, r))_{\boldsymbol{v}\in L}$ is a lattice packing$\}$ is called the *packing radius* of L. Geometrically, $\lambda(L)$ can be interpreted as the radius of the insphere of the Dirichlet-Voronoi polytope of L. The *packing density* of the lattice packing given by L is $\Delta(L) = \frac{\operatorname{vol} B_d(\mathbf{0},\lambda(L))}{\operatorname{vol} L}$ and the *normalized packing density* is $\delta(L) = \frac{\lambda(L)^d}{\operatorname{vol} L}$.

Definition 1.2.2. (*Lattice Packing Problem*)

Given: Dimension d

Find: The value $\Delta_d = \max\{\Delta(L) : L$ is a d-dimensional lattice$\}$ together with a d-dimensional lattice L with $\Delta_d = \Delta(L)$.

One word about the relationship between the two problems. Currently, the lattice covering problem has been solved for dimensions $d = 1, 2, \ldots, 5$, whereas the lattice packing problem has been solved for dimensions $d = 1, 2, \ldots, 8$. The optimal lattice in dimension one and two are the same for both problems: $\mathbb{Z}^1$ and $\mathsf{A}_2 = \mathsf{A}_2^*$, the hexagonal lattice. In dimension three this is no longer the case. The body centered cubic lattice A_3^* whose Dirichlet-Voronoi polytope is a regular truncated octahedron gives the least dense lattice covering. The face centered cubic lattice A_3 whose Dirichlet-Voronoi polytope is a regular rhombic dodecahedron gives the most dense lattice packing. And in the dimension $4, \ldots, 8$ the lattice A_d^*, whose Dirichlet-Voronoi polytope is a permutahedron and which gives optimal lattice coverings in dimensions up to 5, gives a better lattice covering than the lattices $\mathsf{D}_4, \mathsf{D}_5, \mathsf{E}_6, \mathsf{E}_7, \mathsf{E}_8$ which give the optimal lattice packings in their dimensions. In dimension 24 the Leech lattice seems to be optimal for both problems. At the moment the relationship between the two problems is unclear and one might wonder if there exists any independent of the dimension.

1.3. Prerequisites

In our investigations we will use methods from geometry of numbers, polytope theory, combinatorics and optimization. We have collected some basic notation in the glossary at the end of the thesis. The equivalence of lattices and positive definite quadratic forms is a constant source of confusion and we did not make any effort to separate strictly between these two languages. The confused reader should first consult the section “Lattices vs. Quadratic Forms” in the glossary. Otherwise it might be helpful to have the books [Zie1995] (for polytopes and oriented matroids) and [GL1987] (for geometry of numbers) at hand. It should not be necessary to read the whole books. A quick look into the index will suffice in the most cases.

1.4. Organisation of the Thesis

Of course, the selection of the thesis’ title “Sphere Coverings, Lattices and Tilings (in Low Dimensions)” (and so is its contents) is influenced by the book “Sphere Packings, Lattices and Groups” by John H. Conway and Neil J.A. Sloane. There, the problem of packing spheres in Euclidean spaces of dimensions $1, 2, 3, 4, 5, \ldots$ is studied from many different angles. Arranging the spheres so that their centers form a lattice makes the problem far more accessible. It is an unwritten law (and an unproven statement) that lattices with many symmetries provide dense sphere packings; many exceptional groups pop up in this context. For sphere coverings this is not unconditionally true. The combinatorial structure of the underlying tiling by Dirichlet-Voronoi polytopes seems to be more important.

The suffix “(in Low Dimensions)” possesses two meanings. On the one hand, we are dealing in explicit calculations with lattices whose dimension seldom exceeds seven. On the other hand, the suffix also refers to the work of CONWAY and SLOANE. Starting from 1988 they published a series of papers named “Low-Dimensional Lattices I–VII”. One of their main goals is to simplify and systematize work of others. We will try hard to take this as a model although we frequently present some new material.

The thesis is divided into two parts:

Voronoi Reduction and Parallelohedra: One of our main tools is a reduction theory of positive definite quadratic forms that goes back to GEORGES F. VORONOÏ. We try to give a gentle introduction to VORONOÏ’s reduction theory that classifies positive definite quadratic forms according to their Delone subdivisions. Our approach to the reduction theory shows many similarities to the theory of regular subdivisions and secondary polytopes of finite point sets. This theory was recently developed by IZRAIL M. GEL’FAND, MIKHAIL M. KAPRANOV and ANDREI V. ZELEVINSKY. We show that VORONOÏ’s reduction theory is an analogue theory for infinite but periodic point sets.

Dual to the theory of Delone triangulations is the theory of primitive parallelohedra. A parallelohedron is a d-dimensional polytope P that tiles d-dimensional space in a face-to-face manner by translates of the form $P + \boldsymbol{v}$. With help of VORONOÏ’s reduction theory we classify all possible combinatorial types of parallelohedra up to dimension 4.

As we know since ancient times, the only plane parallelohedra are quadrangles and hexagons. The Russian crystallographer E.S. FEDOROV showed that there are five different types of parallelohedra in three dimensions: cubes, hexagonal prisms, truncated octahedra, rhombic dodecahedra and hexarhombic dodecahedra. BORIS N. DELONE tried to prove that there are 51 combinatorially non-equivalent four-dimensional parallelohedra. But he missed one type that later was discovered by MIKHAIL I. STOGRIN. As the main result of the first part we give a new and geometric classification working out a list of JOHN H. CONWAY. Form the 52 types there are 17 zonotopes and all the other 35 parallelohedra have the 24-cell as Minkowski summand. For the classification we use the vonorm/conorm method of JOHN H. CONWAY and NEIL J.A. SLOANE. We describe how their method fits into VORONOÏ’s reduction theory.

The complexity of parallelohedra grows enormously with their dimension. We do not give a complete classification in dimensions 5 and higher. We concentrate on characteristic phenomena and explore interesting effects.

The Lattice Covering Problem: In the second part we give an algorithm for the solution of the lattice covering problem. The existence of such an algorithm has only been anticipated by RYSHKOV and BARANOVSKII. For the design of our algorithm we combine classical methods in the geometry of numbers (going back to works of HERMANN MINKOWSKI, GEORGES F. VORONOÏ, BORIS N. DELONE, E.S. BARNES, SERGEI S. RYSHKOV, EVGENII P. BARANOVSKII, NIKOLAI P. DOLBILIN and MIKHAIL I. STOGRIN) with modern, numerical methods from convex optimization of the “semidefinite programming community”. Our algorithm is not only of theoretical interest. We implemented it and we found all locally optimal lattice coverings in dimensions up to 5. Thereby, we checked (and filled a gap in) a proof of RYSHKOV and BARANOVSKII. Furthermore, we found interesting lattice coverings in dimensions 6 which are less dense than the previous known ones. RYSHKOV asked for the lowest dimension where the lattice A_d^* does not give a globally optimal lattice covering. We show that $d = 6$ is the answer.

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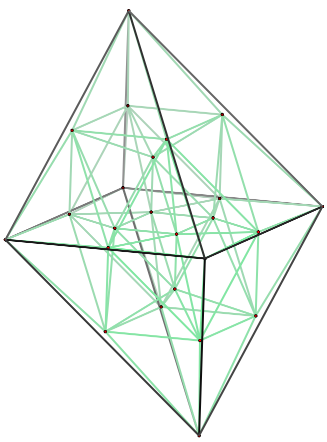
Voronoi Reduction & Parallelohedra



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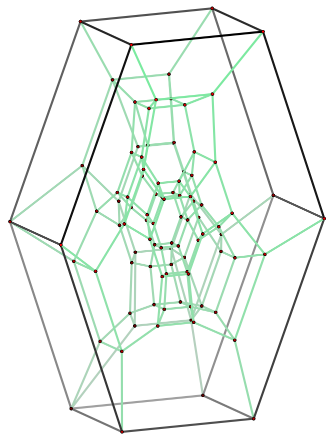

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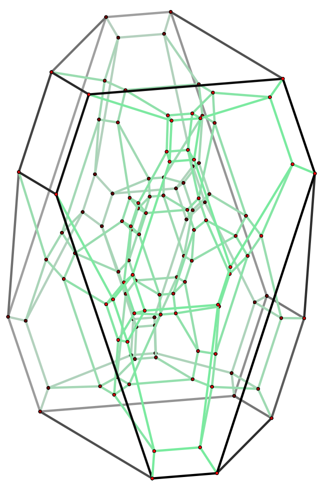

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Voronoi Reduction & Parallelohedra

In the first part, we present our main tools for investigating the geometry of lattices. These are Delone subdivisions, parallelohedra, and Dirichlet-Voronoi polytopes.

The Dirichlet-Voronoi polytope of a lattice is the set of all those points that are closer to the origin than to any other lattice point. Many geometric information is encoded in a Dirichlet-Voronoi polytope, e.g. the circumradius equals the lattice's covering radius, and the inradius equals the lattice's packing radius. A Dirichlet-Voronoi polytope of a lattice is a parallelohedron, i.e. it is a polytope which admits a face-to-face tiling of space by lattice translates. A conjecture of VORONOÏ states that every parallelohedron can be represented as a Dirichlet-Voronoi polytope. We report on the state-of-the-art of this conjecture and give a computational criterion to check whether a given parallelohedron can be represented as a Dirichlet-Voronoi polytope.

A central question is: How does the Dirichlet-Voronoi polytope change if we vary the underlying lattice? To formulate this question in mathematical terms we have to specify the parameter space in which we want to perform the variation. The cone of positive definite quadratic form turns out to be the right choice.

We describe VORONOÏ's reduction theory for positive definite quadratic forms. The discrete group $\mathsf{GL}_d(\mathbb{Z})$ acts on the cone of positive definite quadratic forms $\mathcal{S}^d_{>0}$. VORONOÏ's reduction theory gives a fundamental domain for $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$. This is a subset which behaves like $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$ up to boundary identifications, so that we have a parameter space for lattices where no two interior points represent the same lattice. VORONOÏ's reduction theory is based on Delone subdivisions which are tilings dual to tilings of Dirichlet-Voronoi polytopes. The main theorem of VORONOÏ's reduction theory gives us the possibility to enumerate all non-equivalent Delone subdivisions and so all non-equivalent Dirichlet-Voronoi polytopes of a given dimension. We give a contemporary view on this classical theory where we emphasize its relation to the theory of secondary polytopes.

We perform this classification for dimensions ≤ 4 and look at interesting effects and phenomena in higher dimensions. Instead of giving too technical descriptions involving zonotopal lattices, vonorms, conorms, etc. we gave pictures (generated with `polymake` and `javaview`) of Schlegel diagrams on the previous page which show how typical four-dimensional parallelohedra look like.

Chapter 2.

VORONOÏ's Reduction Theory

In this chapter we describe VORONOÏ's reduction theory for positive definite quadratic forms. The discrete group $\mathsf{GL}_d(\mathbb{Z})$ acts on the cone of positive definite quadratic forms $\mathcal{S}^d_{>0}$. Reduction means giving a fundamental domain for $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$. This is a subset which behaves like $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$ up to boundary identifications. More precisely, there are two fundamental tasks in the reduction theory of positive definite quadratic forms:

i) Define a reduction domain! A reduction domain is a subset $\mathcal{R} \subseteq \mathcal{S}^d_{>0}$ in which there is exactly one (up to boundary identifications) representative for each arithmetical equivalence class of positive quadratic forms.

ii) Describe an algorithm that for a positive definite quadratic form computes an arithmetically equivalent positive form lying in $\mathcal{R}$!

VORONOÏ's reduction theory provides a natural and geometric answer to the first task. The second task is much more difficult. No satisfying solution is known for any reduction theory. For VORONOÏ's reduction theory a solution is known for the dimensions $d = 2$ and $d = 3$. For dimension $d = 4$ a partial solution is known but here we are faced with some inherent difficulties.

VORONOÏ's reduction theory is based on secondary cones of Delone triangulations. The secondary cone of a fixed Delone triangulation is the set of all positive definite quadratic forms that have this fixed Delone triangulation. First, we determine the secondary cone of a Delone triangulation explicitly. It is always a full-dimensional open polyhedral cone in $\mathcal{S}^d_{>0}$. Then, we show that the cone $\mathcal{S}^d_{>0}$ can be partitioned face-to-face into secondary cones of Delone triangulations. The group $\mathsf{GL}_d(\mathbb{Z})$ is acting on this partition. Two secondary cones of Delone triangulation have a common facet whenever the corresponding Delone triangulations differ by a bistellar operation. Every subset of the topological closures of all non-equivalent secondary cones (after factoring out their symmetry) is a fundamental domain of $\mathcal{S}^d_{>0}/\mathsf{GL}_d(\mathbb{Z})$. There are only finitely many non-equivalent secondary cones. They can be enumerated completely by an algorithm. The boundaries of secondary cones of Delone triangulations correspond naturally to coarser Delone subdivisions. If Q lies on the boundary of the secondary cone of a Delone triangulation $\mathcal{D}$, then $\mathcal{D}$ is a refinement of Q's Delone subdivision.

The aim of this chapter is to give a contemporary synopsis of the second part of VORONOÏ's monograph [Vor1908]. In doing so, we emphasize relations to the theory of regular triangulations and secondary polytopes.

2.1. Delone Subdivisions

Definition 2.1.1. Let $Q \in \mathcal{S}^d_{>0}$ be a positive definite quadratic form. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ points in $\mathbb{Z}^d$. The polyhedron $L = \operatorname{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$ is called a *Delone polyhedron* of Q if there exists a point $\boldsymbol{c} \in \mathbb{R}^d$ and a real number $r \in \mathbb{R}$ with $Q[\boldsymbol{v}_i - \boldsymbol{c}] = \operatorname{dist}(\boldsymbol{v_i}, \boldsymbol{c})^2 = r^2$ for all $i = 1, 2, \ldots$ and for all other lattice points $\boldsymbol{v} \in \mathbb{Z}^d \backslash \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$ we have strict inequality $Q[\boldsymbol{v} - \boldsymbol{c}] > r^2$. The set of all Delone polyhedra

$$\operatorname{Del}(Q) = \{L : L \text{ is a Delone polyhedron of } Q\}$$

is called the *Delone subdivision* of Q. A *Delone triangulation* is a Delone subdivision that consists of simplices only.

In other words: We view $\mathbb{R}^d$ as Euclidean space with inner product $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^t)Q\boldsymbol{y}$. Then a Delone polytope L is defined by the ball $B_d(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^d : Q[\boldsymbol{x} - \boldsymbol{c}] \leq r^2\}$ as follows: The vertices of L are the only lattice points lying on the boundary of the ball while in the interior of the ball there are no lattice points. The polyhedron L is a lattice polyhedron. In Figure 2.1 we see how this construction works for the positive definite quadratic form $Q = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. In this Euclidean space spheres are given by ellipsoids.

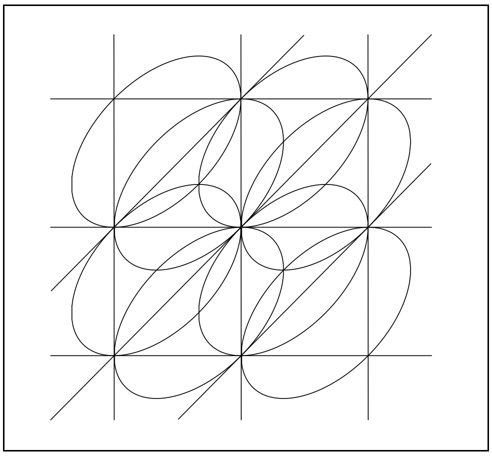


Figure 2.1. Empty Ellipsoids.

The Delone subdivision of a positive definite quadratic form is a periodic polyhedral subdivision of $\mathbb{R}^d$. We call two Delone polyhedra L and L' *equivalent* if there is a lattice vector $\boldsymbol{v} \in \mathbb{Z}^d$ with $L' = L + \boldsymbol{v}$.

In his work "Sur la sphère vide" ([Del1928]) DELONE describes this construction for arbitrary point sets. In [Vor1908] VORONOÏ already uses it for the special point set $\mathbb{Z}^d$. He calls them "l'ensemble (L) de simplexes charactérsant un type de paralléloèdres primitifs" and for this reason Delone subdivisions are sometimes called L-*partitions*; he defines Delone triangulations by dualizing tilings of so-called primitive Dirichlet-Voronoi polytopes which we treat in the next chapter.

Let Q be a semidefinite quadratic form that is arithmetically equivalent to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & Q' \end{smallmatrix}\right)$ where Q' is positive definite. In this case we can define a Delone subdivision for Q by taking literally Definition 2.1.1. Then the Delone subdivision contains unbounded polyhedra. For example, the positive semidefinite quadratic form given by the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ fulfills the requirements above. It has no zero-dimensional Delone polytopes. The one-dimensional Delone polyhedra are given by lines $\operatorname{conv}\{\boldsymbol{v} + \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) : x \in \mathbb{Z}\}$, $\boldsymbol{v} \in \mathbb{Z}^2$ (take $\boldsymbol{c} = \boldsymbol{v}$). The two-dimensional Delone polyhedra are given by horizontal strips $\operatorname{conv}\{\boldsymbol{v} + \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \boldsymbol{v} + \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) : x \in \mathbb{Z}\}$, $\boldsymbol{v} \in \mathbb{Z}^2$ (take $\boldsymbol{c} = \boldsymbol{v} + (0, \frac{1}{2})^t$).

Another construction of the Delone subdivision of a positive definite quadratic form is the *lifting construction* by BROWN ([Bro79]), and by EDELSBRUNNER and SEIDEL ([ES1986]). It is shown in Figure 2.2 for $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Consider the lifting map $l : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}$, $l(\boldsymbol{x}) = (\boldsymbol{x}, Q[\boldsymbol{x}])$, which lifts the points in $\mathbb{R}^d$ onto a paraboloid in $\mathbb{R}^d \times \mathbb{R}$. If we take the convex hull of the lifted lattice points $\operatorname{conv} l(\mathbb{Z}^d)$ and project its lower faces back down onto $\mathbb{R}^d$ we get the Delone subdivision of Q. The lower faces are those faces which can be seen from the "point" $(\mathbf{0}, -\infty)$. A set $\operatorname{conv} L$, $L \subseteq \mathbb{Z}^d$, is a Delone polytope of Q if and only if $\operatorname{conv} l(L)$ is a lower face of the set $\operatorname{conv} l(\mathbb{Z}^d)$.

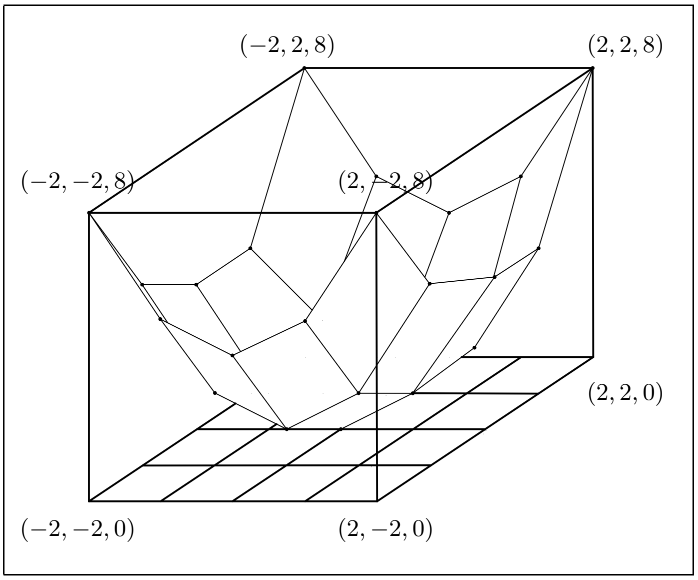


Figure 2.2. Lifting Construction.

The lifting construction provides an extremely useful criterion: Let L be a d-dimensional Delone polytope of Q and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$ be vertices of L which affinely span $\mathbb{R}^d$. Define the function $\chi : \mathbb{R}^d \to \mathbb{R}$ by

$$\chi(\boldsymbol{x}) = \begin{vmatrix} 1 & \ldots & 1 & 1 \\ \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_{d+1} & \boldsymbol{x} \\ Q[\boldsymbol{v}_1] & \ldots & Q[\boldsymbol{v}_{d+1}] & Q[\boldsymbol{x}] \end{vmatrix} \tag{2.1}$$

A lattice point $\boldsymbol{v} \in \mathbb{Z}^d$ is a vertex of the Delone polytope L if and only if $\chi(\boldsymbol{v}) = 0$. More generally, we can use this function to decide whether a point $\boldsymbol{x} \in \mathbb{R}^d$ lies inside, on, or outside the circumsphere of L depending on the sign of $\chi(\boldsymbol{x})$ and on the ordering of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$.

2.2. Secondary Cones

Let Q be a positive definite quadratic form whose Delone subdivision is a triangulation of $\mathbb{R}^d$. In this section we will determine all positive definite quadratic forms which have the same Delone subdivision as Q.

Definition 2.2.1. Let $\mathcal{D}$ be a subdivision of $\mathbb{R}^d$. The set

$$\boldsymbol{\Delta}(\mathcal{D}) := \{Q \in \mathcal{S}^d_{\geq 0} : \operatorname{Del}(Q) = \mathcal{D}\}$$

is called the *secondary cone** of the subdivision $\mathcal{D}$.

*In the Russian literature $\boldsymbol{\Delta}(\operatorname{Del}(Q))$, $Q \in \mathcal{S}^d_{\geq 0}$, is called the *$L$-type domain* of Q.

Our main insight of this section will be that the secondary cone of $\mathrm{Del}(Q)$ forms the interior of a polyhedral cone in $\mathcal{S}^d_{>0}$. We explicitly give a finite set of supporting hyperplanes which is determined by the non-equivalent $(d-1)$-dimensional cells of the Delone subdivision.

Let $\mathcal{D}$ be a Delone triangulation. We want to find the set of all positive definite quadratic forms Q with $\mathrm{Del}(Q) = \mathcal{D}$. Let $L = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d+1}\}$, $L' = \mathrm{conv}\{\boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{d+2}\}$ be two d-dimensional Delone simplices having a $(d-1)$-dimensional face $F = \mathrm{conv}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d+1}\}$ in common. We say (L, L') is a pair of *adjacent* simplices. If these two Delone simplices occur in the Delone triangulation of Q there has to be a ridge between $l(L)$ and $l(L')$ along $l(F)$. The situation is illustrated in Figure 2.3. The condition of "having a ridge" can be expressed as a linear inequality in the parameters q_{ij} of the matrix Q as we will see below.

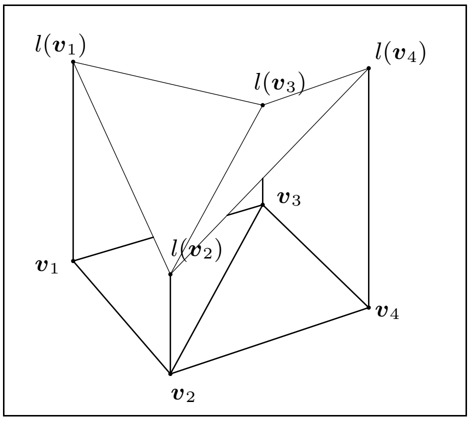


Figure 2.3. Ridge Between $l(\mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\})$ and $l(\mathrm{conv}\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\})$.

The points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+2}$ are affinely dependent. There exist real numbers $\alpha_1, \ldots, \alpha_{d+2}$ with $\sum_{i=1}^{d+2} \alpha_i = 0$ and $\sum_{i=1}^{d+2} \alpha_i \boldsymbol{v}_i = \boldsymbol{0}$. Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_{d+2}$ lie on different sides of the affine hyperplane $\mathrm{aff}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d+1}\}$ we can assume that α_1 and α_{d+2} are positive. Having a ridge along $l(F)$ means that in the lifting $Q[\boldsymbol{v}_{d+2}]$ lies above the affine hyperplane

$$\mathrm{aff}\, l(L) = \mathrm{aff}\{(\boldsymbol{v}_1, Q[\boldsymbol{v}_1]), \ldots, (\boldsymbol{v}_{d+1}, Q[\boldsymbol{v}_{d+1}])\}$$

and that $Q[\boldsymbol{v}_1]$ lies above the affine hyperplane

$$\mathrm{aff}\, l(L') = \mathrm{aff}\{(\boldsymbol{v}_2, Q[\boldsymbol{v}_2]), \ldots, (\boldsymbol{v}_{d+2}, Q[\boldsymbol{v}_{d+2}])\}.$$

This yields two inequalities

$$Q[\boldsymbol{v}_{d+2}] > -\frac{1}{\alpha_{d+2}} \sum_{i=1}^{d+1} \alpha_i Q[\boldsymbol{v}_i], \quad Q[\boldsymbol{v}_1] > -\frac{1}{\alpha_1} \sum_{i=2}^{d+2} \alpha_i Q[\boldsymbol{v}_i].$$

Since α_1 and α_{d+2} are both positive the two inequalities reduce to $\sum_{i=1}^{d+2} \alpha_i Q[\boldsymbol{v}_i] > 0$. This is a linear condition in the entries of the matrix Q since we fixed the lattice points $\boldsymbol{v}_i$, $i = 1, \ldots, d+2$.

Definition 2.2.2. Let $L = \mathrm{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}\}$ and $L' = \mathrm{conv}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d+2}\}$ be two d-dimensional simplices sharing the common facet $F = \mathrm{conv}\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d+1}\}$. Let $\alpha_1, \ldots, \alpha_{d+2}$ be real numbers with $\alpha_1 = 1$, $\sum_{i=1}^{d+2} \alpha_i = 0$ and $\sum_{i=1}^{d+2} \alpha_i \boldsymbol{v}_i = \boldsymbol{0}$. (We could fix α_1 to an arbitrary positive number. Then α_{d+2} is positive, too.) The *regulator* $\varrho_{(L,L')} \in (\mathcal{S}^d)^*$ of the pair of adjacent simplices (L, L') is the linear form $\varrho_{(L,L')}(Q) = \sum_{i=1}^{d+2} \alpha_i Q[\boldsymbol{v}_i]$.

From the above arguments it is clear that the secondary cone of $\mathcal{D}$ is bounded by the condition $\varrho_{(L,L')}(Q) > 0$ where (L, L') is a pair of adjacent simplices. This gives only finitely many conditions because we have for all $\boldsymbol{v} \in \mathbb{Z}^d$ the equality $\varrho_{(L+\boldsymbol{v},L'+\boldsymbol{v})} = \varrho_{(L,L')}$. Additionally, $\varrho_{(L,L')}$ is a positive multiple of $\varrho_{(L',L)}$. On the other hand, a quadratic form satisfying all these conditions is positive definite and its Delone subdivision coincides with $\mathcal{D}$. The positive definiteness follows from the conditions since they imply that the space is subdivided by *bounded* Delone polytopes. That the Delone subdivision of a quadratic form satisfying the conditions coincides with $\mathcal{D}$ follows by a reduction to the one-dimensional case where it is obvious. We summarize the main result of this section:

Theorem 2.2.3. Let Q be a positive definite quadratic form whose Delone subdivision is a triangulation. The secondary cone of the Delone triangulation $\mathrm{Del}(Q)$ is the full-dimensional open polyhedral cone

$$\boldsymbol{\Delta}(\mathrm{Del}(Q)) = \{Q' \in \mathcal{S}^d : \varrho_{(L,L')}(Q') > 0,\ (L, L') \text{ pair of adjacent simplices}\}.$$

2.3. VORONOÏ's Principal Domain of the First Type

As a first example and because of its importance in dimension 2 and 3 we derive the Delone subdivision of VORONOÏ*'s principal form of the first type* $Q[\boldsymbol{x}] = d\sum x_i^2 - \sum x_i x_j$ which is associated to the lattice A_d^*. The Delone subdivision of Q which is a triangulation can be described as follows: Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be the standard basis vectors of $\mathbb{Z}^d$. We set $\boldsymbol{e}_{d+1} = -\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_d$ so that we have $\boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{d+1} = \boldsymbol{0}$. For a permutation $\pi \in \mathsf{S}_{d+1}$ we define the d-dimensional simplex L_π by

$$L_\pi = \mathrm{conv}\{\boldsymbol{e}_{\pi(1)}, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)}, \ldots, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(d+1)}\}.$$

The set of simplices $\{L_\pi + \boldsymbol{v} : \boldsymbol{v} \in \mathbb{Z}^d, \pi \in \mathsf{S}_{d+1}\}$ defines a triangulation of $\mathbb{R}^d$ which we from now on denote by $\mathcal{D}_1$. The full-dimensional cells of the star of the origin are given by L_π, $\pi \in \mathsf{S}_{d+1}$. In the star two simplices L_π and $L_{\pi'}$ have a facet in common if and only if π and π' differ by a single transposition of two adjacent positions. On the left side of Figure 2.4 the star of the origin in dimension 2 is illustrated, on the right side we have a “fundamental domain” of the three-dimensional triangulation.

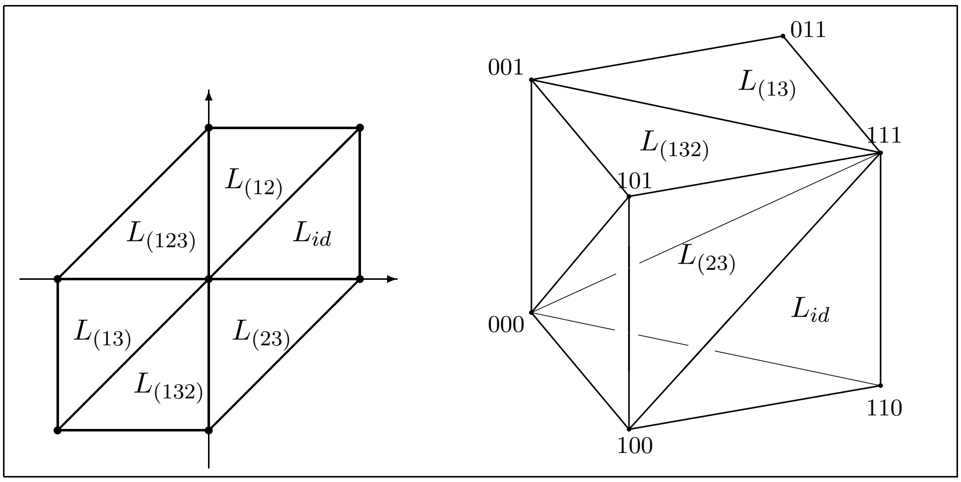


Figure 2.4. The Triangulation $\mathcal{D}_1$ in Dimension $d = 2$ and $d = 3$.

In Section 2.3.2 we will compute the secondary cone of $\mathcal{D}_1$. It is

$$\boldsymbol{\Delta}(\mathcal{D}_1) = \{Q \in \mathcal{S}^d : q_{ij} < 0, i \neq j, \text{ and } \textstyle\sum_{i,j} q_{ij} > 0\}.$$

Hence, $\mathrm{Del}(Q) = \mathcal{D}_1$. Its topological closure $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$ is called VORONOÏ*'s principal domain of the first type*.

Before this computation we introduce in Section 2.3.1 the so-called Selling parameters for positive definite quadratic forms. Using these parameters it is possible to use symmetry properties of $\mathcal{D}_1$ in various computations. The automorphism group of $\mathcal{D}_1$ is isomorphic to the permutation group S_{d+1}.

In dimension 2 and 3 every positive definite quadratic form is arithmetically equivalent to one that lies in $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$. SELLING's reduction algorithm which we will present in Section 2.3.3 is a constructive proof of this fact. But in dimension 4 and higher there are positive definite quadratic forms which are not arithmetically equivalent to one in $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$.

This is not everything what has to be said about VORONOÏ's principal domain of the first type. In Chapter 3.5.2 we will continue our studies. Other sources for information on this domain are e.g. [CS1992] where the three-dimensional case is discussed in great detail and [Jan1998] where the four-dimensional case is described.

2.3.1. Selling Parameters

Usually we represent a positive definite quadratic form Q by a positive definite matrix $Q = (q_{ij})$. One disadvantage of this representation is that the coefficients are not of the "same type". The coefficients on the main diagonal are squared norms

$$q_{ii} = Q[\boldsymbol{e}_i] = s_Q(\boldsymbol{e}_i, \boldsymbol{e}_i) = (\boldsymbol{e}_i, \boldsymbol{e}_i)$$

and all other coefficients are inner products between different vectors

$$q_{ij} = \frac{1}{2}(Q[\boldsymbol{e}_i + \boldsymbol{e}_j] - Q[\boldsymbol{e}_i] - Q[\boldsymbol{e}_j]) = s_Q(\boldsymbol{e}_i, \boldsymbol{e}_j) = (\boldsymbol{e}_i, \boldsymbol{e}_j).$$

SELLING introduces in [Sel1874] parameters which are homogeneous: We simply forget the coefficients on the main diagonal q_{ii} but add inner products $q_{i,d+1} = q_{d+1,i} = (\boldsymbol{e}_i, \boldsymbol{e}_{d+1})$ with the additional vector $\boldsymbol{e}_{d+1} = -\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_d$. Then the parameters $q_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$, $i \neq j$, $i, j = 1, \ldots, d+1$ define the positive definite quadratic form Q completely since we get q_{ii} back by the relation $\sum_{j=1}^{d+1} q_{ij} = 0$. The parameters q_{ij}, $i \neq j$, $i, j = 1, \ldots, d+1$, are traditionally called *Selling parameters* of Q. For instance VORONOÏ's principal form of the first type

$$Q[\boldsymbol{x}] = 3x_1^2 + 3x_2^2 + 3x_3^3 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$$

is given

- by the positive definite matrix: $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$,
- and by the Selling parameters: $\begin{pmatrix} & -1 & -1 & -1 \\ -1 & & -1 & -1 \\ -1 & -1 & & -1 \\ -1 & -1 & -1 & \end{pmatrix}$.

By looking at the Selling parameters it is obvious that the form is invariant under the 24 permutations $\boldsymbol{e}_i \mapsto \boldsymbol{e}_{\pi(i)}$, $i = 1, \ldots, 4$, $\pi \in \mathsf{S}_4$.

The following formula makes use of Selling parameters and we will use it frequently in calculations where the triangulation $\mathcal{D}_1$ is involved.

Proposition 2.3.1. (SELLING*'s Formula*)
Let Q be a positive definite quadratic form and let (q_{ij}), $1 \leq i, j \leq d+1$, $i \neq j$, be its Selling parameters. The squared norm (with respect to Q) of a vector $\boldsymbol{x} = \sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_i$, $\alpha_i \in \mathbb{R}$, is given by

$$Q[\boldsymbol{x}] = -\sum_{1 \leq i < j \leq d+1} q_{ij}(\alpha_i - \alpha_j)^2.$$

Proof. We first consider the special case $\alpha_{d+1} = 0$ and reduce the general case to this special case afterwards. By applying the equations $q_{i,d+1} = -\sum_{j=1}^{d} q_{ij}$ and $q_{ij} = q_{ji}$ we have

$$\begin{aligned}
& -\sum_{1 \leq i < j \leq d+1} q_{ij}(\alpha_i - \alpha_j)^2 \\
= & -\sum_{1 \leq i < j \leq d} q_{ij}(\alpha_i - \alpha_j)^2 - \sum_{1 \leq i \leq d} q_{i,d+1}(\alpha_i - 0)^2 \\
= & -\sum_{1 \leq i < j \leq d} q_{ij}\alpha_i^2 + 2\sum_{1 \leq i < j \leq d} q_{ij}\alpha_i\alpha_j - \sum_{1 \leq i < j \leq d} q_{ij}\alpha_j^2 + \sum_{1 \leq i, j \leq d} q_{ij}\alpha_i^2 \\
= & \sum_{1 \leq i \leq d} q_{ii}\alpha_i^2 + 2\sum_{1 \leq i < j \leq d} q_{ij}\alpha_i\alpha_j \\
= & \; Q\left[\textstyle\sum_{1 \leq i \leq d+1} \alpha_i \boldsymbol{e}_i\right].
\end{aligned}$$

In the general case we could have $\alpha_{d+1} \neq 0$. Then we replace

$$\sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_i \quad \text{by} \quad \sum_{i=1}^{d+1} \alpha_i \boldsymbol{e}_i - \alpha_{d+1} \sum_{i=1}^{d+1} \boldsymbol{e}_i.$$

Now we can apply the formula we proved above to get the general formula

$$\begin{gathered}
Q\left[\textstyle\sum_{1 \leq i \leq d+1} \alpha_i \boldsymbol{e}_i\right] = Q\left[\textstyle\sum_{1 \leq i \leq d+1} (\alpha_i - \alpha_{d+1}) \boldsymbol{e}_i\right] \\
= -\sum_{1 \leq i < j \leq d+1} q_{ij}((\alpha_i - \alpha_{d+1}) - (\alpha_j - \alpha_{d+1}))^2 = -\sum_{1 \leq i < j \leq d+1} q_{ij}(\alpha_i - \alpha_j)^2.
\end{gathered}$$

$\diamond$

2.3.2. Computation of the Secondary Cone

Proposition 2.3.2. ([Vor1908], §102–§104)
The secondary cone of the triangulation $\mathcal{D}_1$ is given by the interior of VORONOÏ's principal domain of the first type

$$\begin{aligned}
\overline{\boldsymbol{\Delta}(\mathcal{D}_1)} &= \{Q \in \mathcal{S}^d : q_{ij} \leq 0,\ i \neq j, \text{ and } \textstyle\sum_{i,j} q_{ij} \geq 0\} \\
&= \{Q \in \mathcal{S}^d : q_{ij} \leq 0, 1 \leq i, j \leq d+1, i \neq j\}.
\end{aligned}$$

Proof. Let $\pi, \pi' \in \mathsf{S}_{d+1}$ be two permutations which differ only by a transposition of two adjacent symbols, $\pi' = \pi\ (i\ i{+}1)$. Then the simplices L_π and $L_{\pi'}$ of the triangulation $\mathcal{D}_1$ share a common facet. In the following we compute the regulator $\varrho_{(L_\pi, L_{\pi'})}$. We have

$$\begin{aligned} L_\pi &= \operatorname{conv}\{\boldsymbol{e}_{\pi(1)}, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)}, \ldots, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(d+1)}\} \\ L_{\pi'} &= \operatorname{conv}\{\boldsymbol{e}_{\pi(1)}, \ldots, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)}, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)} + \boldsymbol{e}_{\pi(i+1)}, \\ &\qquad \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i+1)}, \ldots, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(d+1)}\}, \end{aligned}$$

so that $\operatorname{conv}\{\boldsymbol{e}_{\pi(1)}, \ldots, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)}, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i+1)}, \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(d+1)}\}$ is the common facet of L_π and $L_{\pi'}$. The vertices that are not contained in the common facet are $\operatorname{vert} L_\pi \setminus \operatorname{vert} L_{\pi'} = \{\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i)}\}$, $\operatorname{vert} L_{\pi'} \setminus \operatorname{vert} L_\pi = \{\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)} + \boldsymbol{e}_{\pi(i+1)}\}$. An affine dependence among these vertices and the vertices of the common facet is

$$\begin{aligned} \boldsymbol{0} &= (\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i)}) + (\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)} + \boldsymbol{e}_{\pi(i+1)}) \\ &\quad - (\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)}) - (\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i+1)}). \end{aligned}$$

In the case $i = 1$ the sum $\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)}$ equals $\boldsymbol{0} = \boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(d+1)}$. Hence, the regulator $\varrho_{(L_\pi, L_{\pi'})}$ is given by

$$\begin{aligned} \varrho_{(L_\pi, L_{\pi'})}(Q) &= Q[\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i)}] + Q[\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)} + \boldsymbol{e}_{\pi(i+1)}] \\ &\quad - Q[\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i-1)}] - Q[\boldsymbol{e}_{\pi(1)} + \cdots + \boldsymbol{e}_{\pi(i+1)}] \\ &= -2(\boldsymbol{e}_{\pi(i)}, \boldsymbol{e}_{\pi(i+1)}) \\ &= -2q_{\pi(i),\pi(i+1)}. \end{aligned}$$

If we consider all pairs of adjacent simplices we see by Theorem 2.2.3 that the secondary cone of $\mathcal{D}_1$ is bounded by the hyperplanes $q_{ij} = 0$, $1 \leq i, j \leq d{+}1$, $i \neq j$. By the equations $q_{ij} = q_{ji}$, $q_{i,d+1} = -\sum_{j=1} q_{ij}$, we can transform the given inequalities into inequalities of the space of symmetric matrices $\mathcal{S}^d$. $\diamond$

2.3.3. SELLING's Reduction Algorithm

In dimension 2 and 3 every positive definite quadratic form is arithmetically equivalent to one that lies in $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$. Using Theorem 2.5.1 below one can give a systematic proof of this fact. Here we present a simpler ad-hoc proof. Suppose we are a given a positive definite quadratic form Q. Our goal is to find a transformation $A \in \mathsf{GL}_d(\mathbb{Z})$, $d = 2, 3$, so that the Selling parameters which define the form $A^t Q A$ are all non-positive.

Binary Case

We consider the two-dimensional case first. Assume that one Selling parameter of Q is positive, for instance $q_{12} > 0$. Then, by the unimodular transformation $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ we get

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^t \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} q_{11} & -q_{12} \\ -q_{12} & q_{22} \end{pmatrix} = \begin{pmatrix} q'_{11} & q'_{12} \\ q'_{12} & q'_{22} \end{pmatrix} = Q'.$$

The Selling parameters of Q' are

$$\begin{aligned} q'_{12} &= -q_{12} \\ q'_{13} &= -q'_{11} - q'_{12} = -q_{11} + q_{12} = q_{12} + q_{13} + q_{12} = 2q_{12} + q_{13} \\ q'_{23} &= -q'_{12} - q'_{22} = q_{12} - q_{22} = q_{12} + q_{12} + q_{23} = 2q_{12} + q_{23}. \end{aligned}$$

For the sum of negative Selling parameters we have the following relation between Q and Q':

$$-q'_{12} - q'_{13} - q'_{23} = -3q_{12} - q_{13} - q_{23} < -q_{12} - q_{13} - q_{23}.$$

For $q_{13} > 0$ the transformation $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and for $q_{23} > 0$ the transformation $\left(\begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix}\right)$ yield a similar effect on the sum of negative Selling parameters. So we get a sequence of decreasing sums of negative Selling parameters as long as one Selling parameter is positive. The sum of negative Selling parameters $-q_{12} - q_{13} - q_{23}$ equals $\frac{1}{2}(Q[\boldsymbol{e}_1] + Q[\boldsymbol{e}_2] + Q[\boldsymbol{e}_3])$. Since in a class of arithmetically equivalent positive definite quadratic forms the set $\{Q[\boldsymbol{v}] : \boldsymbol{v} \in \mathbb{Z}^d \backslash \{\boldsymbol{0}\}\}$ is bounded from below by the homogeneous minimum $\lambda(Q)$ and since the difference between two successive sums is at least $\frac{1}{2}\lambda(Q)$, the sequence is finite. So we have an algorithm which constructs the matrix A step-by-step. This algorithm is called SELLING's reduction algorithm. It reduces a binary positive definite quadratic form to one lying in $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$. Finally, by using permutations of the form $\boldsymbol{e}_i \mapsto \boldsymbol{e}_{\pi(i)}$, $\pi \in \mathsf{S}_3$ (which can be linearly extended to unimodular transformations) we can transform a positive definite quadratic form lying in $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$ to one that lies in the cone bounded by

$$q_{12} \leq 0, \quad q_{11} \leq q_{22}, \quad q_{11} \leq 2q_{12}. \tag{2.2}$$

This means that we can reduce a positive definite quadratic form to a canonical representative in the class of arithmetically equivalent forms. Supplementary, SELLING proved and it is not difficult to verify: The Selling parameters of a binary positive definite quadratic form are all non-positive if and only if the sum of negative Selling parameters is minimal in the class of arithmetically equivalent forms.

Ternary Case

Similar arguments can be used to define a reduction algorithm for ternary positive definite quadratic form. If $q_{12} > 0$, then the unimodular transformation $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ gives an arithmetically equivalent form whose sum of negative Selling parameters is smaller than the previous one. If we have $q_{ij} > 0$, then we can apply the permutation $\boldsymbol{e}_i \mapsto \boldsymbol{e}_1$ and $\boldsymbol{e}_j \mapsto \boldsymbol{e}_2$ that can be linearly extended to a unimodular transformation to have the situation $q_{12} > 0$.

Some Geometry

We want to close the discussion of the binary case with some geometric considerations and pictures. The set of two-dimensional positive semidefinite matrices $\mathcal{S}^2_{\geq 0}$ is bounded by HURWITZ's conditions $q_{11} \geq 0$ and $q_{11}q_{22} - q_{12}^2 \geq 0$. Hence, $\mathcal{S}^2_{\geq 0}$ is the upper half of a three-dimensional elliptic cone with fundamental axis $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

If we slice it by a hyperplane parallel to $q_{11} + q_{22} = 0$ (see Figure 2.5) we get the following (projective) picture which includes all geometric information because every ray beginning from the origin hits the hyperplane exactly once. The secondary cone of the two-dimensional triangulation $\mathcal{D}_1$ is given by the inequalities

$$-q_{12} > 0, \quad q_{11} + q_{12} > 0, \quad q_{12} + q_{22} > 0.$$

So its topological closure is a polyhedral cone with extreme rays

$$\overline{\boldsymbol{\Delta}(\mathcal{D}_1)} = \operatorname{cone}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}.$$

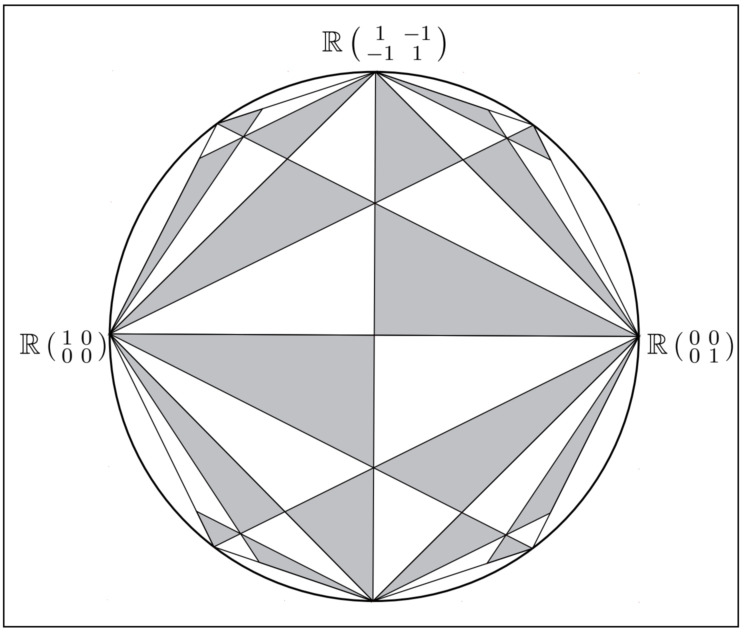


Figure 2.5. Cone of Positive Definite Matrices Sliced with $q_{11} + q_{22} = 1$.

The group

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\}$$

leaves the polyhedral cone $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$ fixed and only permutes the smaller triangular cones of the barycentric subdivision. This gives a geometric explanation of the polyhedral cone (2.2) in which we find exactly one representative of every arithmetical equivalence class.

Notice that this reduction domain is not the "standard one" given by the inequalities

$$-\tfrac{q_{11}}{2} < q_{12} \le \tfrac{q_{11}}{2},\ q_{11} \le q_{22}, \text{ and } 0 \le q_{12} \le \tfrac{q_{11}}{2} \text{ if } q_{11} = q_{22}$$

that goes back to LAGRANGE. For historical remarks on the reduction theory of binary quadratic forms we refer the interested reader to the book [SO1985] of SCHARLAU and OPOLKA.

The "Modulfigur" and Some History

Many mathematicians were and are interested in reduction domains of positive definite quadratic forms. One reason is that reduction domains connect different branches of mathematics very elegantly and unexpectedly. In [KF1890], page 242, KLEIN writes enthusiastically on the reduction domain of binary positive definite quadratic forms and explains how to draw a correct picture of it.

> "Ich habe diese Figur (an die sich eine Menge weiterer geometrischer Bemerkungen anknüpfen) in meinen Vorlesungen von 1877 wiederholt zur Sprache gebracht, weil dieselbe auch unter rein synthetischen Gesichtspunkten sehr bemerkenswert ist. Sie giebt uns nämlich das übersichtlichste Bild für die constructive Erledigung der in der synthetischen Geometrie fundamentalen Aufgabe, ein einförmiges Grundgebilde (hier unsere Ellipse) dadurch mit unendlich vielen Elementen zu überdecken, dass man zu drei willkürlich gegebenen Elementen desselben immer wieder das vierte harmonisch construiert.

> Im Herbst 1873 hatte ich mit dem verstorbenen CLIFFORD eine lebhafte Unterhaltung darüber, dass es als Aufgabe der modernen Mathematik betrachten müsse, die uns überkommenen, getrennt neben einander stehenden mathematischen Disciplinen in lebendige Wechselwirkung zu setzen; wir kamen überein, dass die für synthetische Geometrie und Zahlentheorie am schwierigsten sein möchte. *Die Figur (62) des Textes stellt diese Verbindung her.* Man wolle in dieser Hinsicht insbesondere die zahlentheoretischen Entwicklungen des folgenden Kapitels vergleichen."

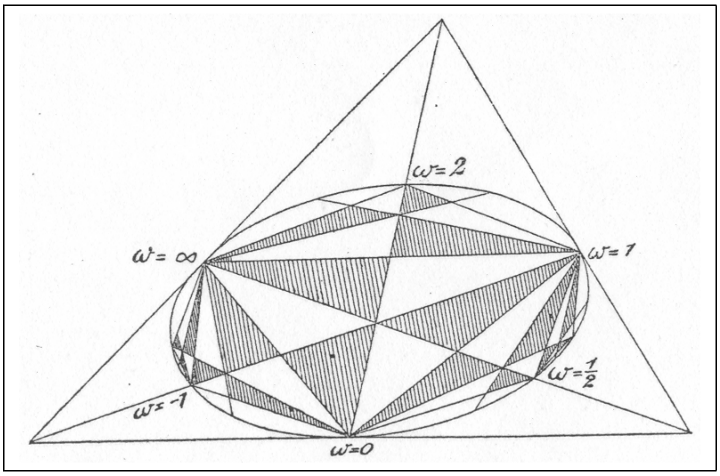


Figure 2.6. "Modulfigur" (from KLEIN and FRICKE's book [KF1890]).

In the book [Ter1988]† — following HILBERT's speech in memory of MINKOWSKI — TERRAS writes on page 113:

> "Much of this section is due to MINKOWSKI, who was the first to describe a fundamental domain for $\mathsf{GL}(n, \mathbb{Z})$. We will discuss another fundamental domain — that of GARNIER in Section 4.4.3. [...] There are indeed many unusual flowers in these higher dimensional gardens. The names of those who cultivated these flowers include: GAUSS, HERMITE, MINKOWSKI, VORONOÏ, SIEGEL, WEYL, WEIL, SATAKE, BAILY, BOREL, SERRE, HARISH-CHANDRA, MOSTOW, TAMAGAWA, MUMFORD, DELONE, RYSHKOV, ..."

2.4. Bistellar Neighbours

In Section 2.2 we saw that the secondary cone of all positive definite quadratic forms having the same Delone triangulation forms the interior of a full-dimensional polyhedral cone in $\mathcal{S}^d_{>0}$. Therefore, the cone of positive definite quadratic forms is tessellated by polyhedral secondary cones. In this section we will find out that the tessellation is a facet-to-facet tessellation. By a theorem of GRUBER and RYSHKOV we even have a face-to-face tessellation because "facet-to-facet implies face-to-face" ([GR1989]).

We will investigate what happens if we move a positive definite quadratic form continuously from the interior of one secondary cone to the interior of another one while crossing a facet of the first one. We will see that the polyhedral cones share the complete facet we crossed. It

†Its subtitle is "Revenge of the Higher Rank Symmetric Spaces and Their Fundamental Domains". We highly recommend it and its inspiring style!

can exactly be described how the two Delone triangulations belonging to the polyhedral cones differ. All $(d-1)$-dimensional cells in the Delone triangulation defining a regulator that gives the crossed facet perform a bistellar operation when the facet is transversally crossed.

Before we give the exact definitions and statements we illustrate in Figure 2.7 what happens in the two-dimensional case when we move from the positive definite quadratic form $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

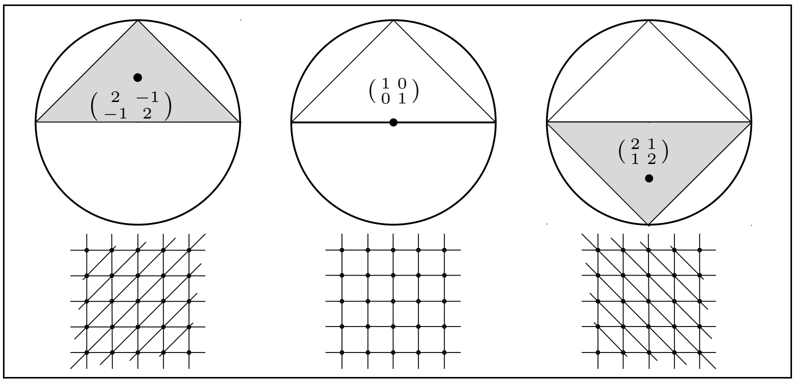


Figure 2.7. Construction of Bistellar Neighbours.

Bistellar operations are performed in so-called *repartitioning polytopes*. Repartitioning polytopes are d-dimensional Delone polytopes having $d+2$ vertices. A repartitioning polytope can be represented as convex hull of two Delone simplices having a common facet. Repartitioning polytopes were investigated by VORONOÏ in [Vor1908], §89, and he calls them simply "un polyèdre convexe K". The name "repartitioning polytope" was coined by RYSHKOV and BARANOVSKII in [RB1976], §9. Repartitioning polytopes are not only basic concepts in VORONOÏ's reduction theory. They also play an important role in the theory of hypermetric spaces where they correspond to facets in the hypermetric cone (see Chapter 15.2.2 in [DL1997] or originally in [AG1993]).

We summarize structural properties of repartitioning polytopes and more generally of d-dimensional polytopes with $d+2$ vertices in the following proposition. For the formulation it is convenient to use terminology from oriented matroid theory which we recall briefly. Let $V \subseteq \mathbb{R}^d$ be a finite set of points in $\mathbb{R}^d$. Every affine relation between these points $\sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} \boldsymbol{v} = \mathbf{0}$, $\sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}} = 0$, gives rise to a sign vector $X \in \{-1, 0, +1\}^V$, simply by $X_{\boldsymbol{v}} = \operatorname{sgn} \alpha_{\boldsymbol{v}}$. The *support* of the sign vector X is defined by $\underline{X} = \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} \neq 0\}$. We define the sets $X^+ = \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} = +1\}$, $X^- = \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} = -1\}$ and $X^0 = \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} = 0\}$. The set $\mathcal{V}(V)$ of all sign vectors is called the set of *vectors* of the *oriented matroid* $\mathcal{M}(V)$. A non-trivial vector of $\mathcal{M}(V)$ which has minimal support among all vectors is called a *circuit*.

Proposition 2.4.1. (*Repartitioning polytopes*)
Let V be a set of $d+2$ points which affinely spans $\mathbb{R}^d$. By C we denote one of the two circuits which are defined by the one-dimensional linear subspace of affine relations on V. The repartitioning polytope $\operatorname{conv} V$ has two different types of facets:

i) $|C^0|$ facets with $d+1$ vertices: $Q_{\boldsymbol{v}_0} = \operatorname{conv}(V \setminus \{\boldsymbol{v}_0\}), \boldsymbol{v}_0 \in C^0$.

ii) $|C^+| \cdot |C^-|$ facets with d vertices: $P_{\boldsymbol{v}_+,\boldsymbol{v}_-} = \operatorname{conv}(V \setminus \{\boldsymbol{v}_+, \boldsymbol{v}_-\})$, $\boldsymbol{v}_+ \in C^+$, $\boldsymbol{v}_- \in C^-$.

There exist exactly two triangulations of $\operatorname{conv} V$: $\mathcal{T}_+(V, C)$ with simplices $\operatorname{conv}(V \setminus \{\boldsymbol{v}_+\})$, $\boldsymbol{v}_+ \in C^+$, and $\mathcal{T}_-(V, C)$ with simplices $\operatorname{conv}(V \setminus \{\boldsymbol{v}_-\})$, $\boldsymbol{v}_- \in C^-$.

It is easy to prove this proposition by using the fact that the only circuits of the oriented matroid $\mathcal{M}(V)$ are C and $-C$. Figure 2.8 shows a two-dimensional repartitioning polytope together with its two triangulations.

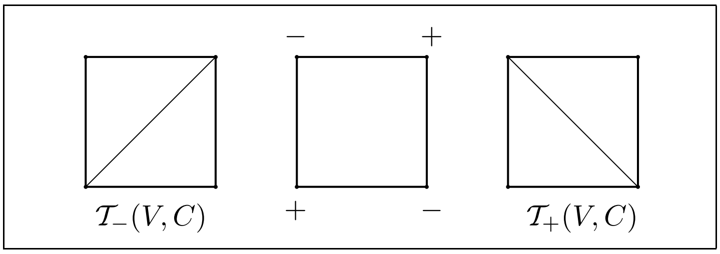


Figure 2.8. Two-Dimensional Repartitioning Polytope.

A bistellar operation replaces a given triangulation of a repartitioning polytope by the other possible one.

Definition 2.4.2. Let $\mathcal{T}$ be a triangulation of $\mathbb{R}^d$ and let F be a $(d-1)$-dimensional cell of $\mathcal{T}$. Then, F is contained in two simplices L and L' of $\mathcal{T}$. By V we denote the set of vertices of L and L', $V = \operatorname{vert} L \cup \operatorname{vert} L'$. By C we denote one of the two circuits of the oriented matroid $\mathcal{M}(V)$. The $(d-1)$-dimensional cell F is called a *flippable facet* of the triangulation $\mathcal{T}$ if one of the triangulations $\mathcal{T}_+(V,C)$ or $\mathcal{T}_-(V,C)$ is a subcomplex of $\mathcal{T}$. If F is a flippable facet of $\mathcal{T}$ and we replace the subcomplex $\mathcal{T}_+(V,C)$ by $\mathcal{T}_-(V,C)$ [respectively $\mathcal{T}_-(V,C)$ by $\mathcal{T}_+(V,C)$], then we get a new triangulation. This replacement is called *bistellar operation* or *flip*.

The facets of $\overline{\mathbf{\Delta}(\mathcal{D})}$ give the interesting bistellar operations of a Delone triangulation $\mathcal{D}$. A $(d-1)$-dimensional cell $L \cap L' \in \mathcal{D}$ is a flippable facet whenever the corresponding regulator $\varrho_{(L,L')}$ gives a facet-defining hyperplane of $\overline{\mathbf{\Delta}(\mathcal{D})}$ (see [Vor1908], §87–§88). This is clear since the repartitioning polytope $\operatorname{conv}(L \cup L')$ is a Delone polytope of the positive definite quadratic forms lying in the relative interior of the facet given by $\varrho_{(L,L')}$.

Let $\mathbf{F}$ be a facet of the polyhedral cone $\overline{\mathbf{\Delta}(\mathcal{D})}$. We describe how the Delone triangulation $\mathcal{D}$ changes if we move a positive definite quadratic form continuously. We start from the interior of $\overline{\mathbf{\Delta}(\mathcal{D})}$, then we move towards a relative interior point of $\mathbf{F}$ and finally we go infinitesimally further, leaving $\overline{\mathbf{\Delta}(\mathcal{D})}$. In every repartitioning polytope $V = \operatorname{conv}(L \cup L')$ where L, L' is a pair of adjacent simplices whose regulator defines $\mathbf{F}$, i.e. $\operatorname{lin} \mathbf{F} = \{Q \in \mathcal{S}^d : \varrho_{(L,L')}(Q) = 0\}$, we perform a bistellar operation. This gives a new triangulation $\mathcal{D}'$. It is a Delone triangulation again. The two secondary cones $\overline{\mathbf{\Delta}(\mathcal{D})}$ and $\overline{\mathbf{\Delta}(\mathcal{D}')}$ have the complete facet $\mathbf{F}$ in common. We say that $\mathcal{D}$ and $\mathcal{D}'$ are *bistellar neighbours*. In the chapter "Reconstruction de l'ensemble (L) de simplexes en un autre ensemble (L') de simplexes", [Vor1908] §91–§95, VORONOÏ computes the secondary cone of $\mathcal{D}'$ explicitly and in the next paragraph he shows that $\overline{\mathbf{\Delta}(\mathcal{D}')}$ has dimension $\frac{d(d+1)}{2}$.

2.5. Main Theorem of VORONOÏ's Reduction Theory

By constructing bistellar neighbours we could produce infinitely many Delone triangulations starting from the Delone triangulation $\mathcal{D}_1$ of VORONOÏ's principal form of the first type. But many of these will not be essentially new. A part of the infinite *flip graph* of two-dimensional Delone triangulation is given in Figure 2.9.

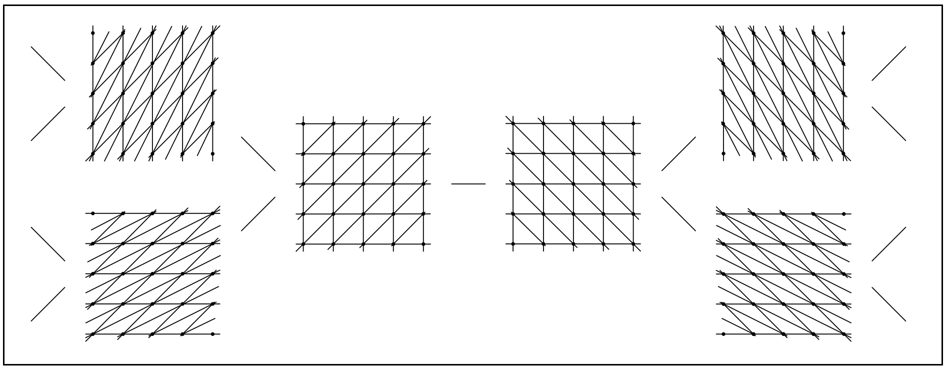


Figure 2.9. The Graph of Two-Dimensional Delone Triangulations.

If two positive definite quadratic forms Q and Q' are arithmetically equivalent, we have $Q' = A^t Q A$ for some $A \in \mathsf{GL}_d(\mathbb{Z})$, then their Delone subdivisions are related by the equation $\mathrm{Del}(Q') = A\,\mathrm{Del}(Q)$. The group $\mathsf{GL}_d(\mathbb{Z})$ is acting on the set of Delone subdivisions by $(A, \mathcal{D}) \mapsto A\mathcal{D}$ and it is acting on the set of secondary cones by $(A, \mathbf{\Delta}) \mapsto A^t \mathbf{\Delta} A$. We are only interested in the orbits of these group actions and there are only finitely many. VORONOÏ proved this by showing that there is a bound M depending only on the dimension so that the following holds: For every Delone triangulation there exists an equivalent one where the (integral) coordinates of the edges starting from the origin are bounded by M. Another proof was given by DEZA, GRISHUKHIN and LAURENT (see [DL1997], Chapter 13.3). They show that in a fixed dimension there are only finitely many Delone polytopes which are not arithmetically equivalent.

This yields the main theorem of VORONOÏ's reduction theory.

Theorem 2.5.1. (*Main Theorem of* VORONOÏ*'s Reduction Theory*)
The topological closures of secondary cones of Delone triangulations give a face-to-face tiling of the cone of positive semidefinite quadratic forms. Two secondary cones share a facet if and only if they are bistellar neighbours. The group $\mathsf{GL}_d(\mathbb{Z})$ acts on the tiling, and under this group action there are only finitely many non-equivalent secondary cones.

By Algorithm 1 we can enumerate all non-equivalent secondary cones of Delone triangulations (and thereby all non-equivalent Delone triangulations) in a given dimension.

Algorithm 1 Enumeration of all non-equivalent Delone triangulations.

Input: Dimension d.

Output: Set $\mathcal{R}$ of all non-equivalent d-dimensional Delone triangulations.

```
T ← {D_1}. R ← ∅.
while there is a D ∈ T do
    T ← T\{D}. R ← R ∪ {D}.
    compute the regulators of D.
    compute the facets F_1, ..., F_n of closure(Δ(D)).
    for i = 1, ..., n do
        compute the bistellar neighbour D_i of D which is defined by F_i.
        if D_i is not equivalent to a Delone triangulation in R ∪ {D_1, ..., D_{i-1}} then
            T ← T ∪ {D_i}.
        end if
    end for
end while
```

The secondary cones of Delone triangulations the algorithm produces can be used to define a reduction domain. Two positive definite quadratic forms lying in two non-equivalent secondary cones cannot be arithmetically equivalent. The only thing we have to consider is that secondary cones can have symmetry. Let $G \subseteq \mathsf{GL}_d(\mathbb{Z})$ be the group of symmetries of a secondary cone $\boldsymbol{\Delta}$. This is a finite group since $\boldsymbol{\Delta}$ is a polyhedral cone. Now choose a subset $\boldsymbol{\Delta}' \subseteq \boldsymbol{\Delta}$ such that $G\boldsymbol{\Delta}' = \boldsymbol{\Delta}$ and such that we have $A^t\boldsymbol{\Delta}'A = \boldsymbol{\Delta}'$ only if A is the identity. This can be done by using parts of the barycentric subdivision of $\mathcal{D}$.

2.6. Refinements and Sums

Until now we have only dealt with Delone *triangulations* and their secondary cones. Let us look at Delone *subdivisions* and find out how they fit into the theory we developed so far. Delone subdivisions are limiting cases of triangulations: Their secondary cones occur on the boundaries of full-dimensional secondary cones of Delone triangulations. Let $\mathcal{D}$ and $\mathcal{D}'$ be two Delone subdivisions. We say $\mathcal{D}$ is a *refinement* of $\mathcal{D}'$ if every Delone polytope of $\mathcal{D}$ is a subset of some Delone polytope of $\mathcal{D}'$. The following proposition shows that the relation between refinements, secondary cones and sums of positive semidefinite quadratic forms is very natural. It is not clear (at least not to the author) where this proposition was mentioned first. LOESCH gave it in his thesis [Loe1990]. Later, RYSHKOV who was not aware of LOESCH's thesis gives in [Rys1999] a statement equivalent to the following proposition. LOESCH's thesis is not easily available and RYSHKOV's paper does not contain a proof of the statement. We give the arguments in great detail here. Figure 2.10 visualizes what happens in the two-dimensional case.

Proposition 2.6.1. Let $\mathcal{D}$ be a Delone triangulation.

i) A positive semidefinite quadratic form Q lies in $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ if and only if $\mathcal{D}$ is a refinement of $\mathrm{Del}(Q)$.

ii) If two positive semidefinite quadratic forms Q and Q' both lie in $\overline{\boldsymbol{\Delta}(\mathcal{D})}$, then $\mathrm{Del}(Q+Q')$ is a common refinement of $\mathrm{Del}(Q)$ and $\mathrm{Del}(Q')$.

Proof. Throughout the proof we always assume that the vertices $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}$ of a Delone polytope L are ordered in such a way that for a point $\boldsymbol{x} \in \mathbb{R}^d$ we have

$$\begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{x} \\ Q[\boldsymbol{v}_1] & \dots & Q[\boldsymbol{v}_{d+1}] & Q[\boldsymbol{x}] \end{vmatrix} > 0$$

if and only if $\boldsymbol{x}$ lies outside the circumsphere of L (see (2.1)).

i) Suppose that $Q \in \overline{\boldsymbol{\Delta}(\mathcal{D})}$. We have to show that any d-dimensional Delone simplex $L = \mathrm{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}\}$ of $\mathcal{D}$ is contained in a Delone polytope of Q. Let Q_1 be a positive definite quadratic form lying in the interior of $\overline{\boldsymbol{\Delta}(\mathcal{D})}$. We consider the half-open segment $Q_t = (1-t)Q + tQ_1$, $t \in (0, 1]$, that is completely contained in the interior of $\overline{\boldsymbol{\Delta}(\mathcal{D})}$. Thus, all these positive definite quadratic forms have the Delone triangulation $\mathcal{D}$.

For every $\boldsymbol{v} \in \mathbb{Z}^d \backslash \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}\}$ and $\varepsilon > 0$ we have

$$\begin{aligned} 0 \quad &< \quad \begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q_\varepsilon[\boldsymbol{v}_1] & \dots & Q_\varepsilon[\boldsymbol{v}_d] & Q_\varepsilon[\boldsymbol{v}] \end{vmatrix} \\ &= \quad (1-\varepsilon) \begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q[\boldsymbol{v}_1] & \dots & Q[\boldsymbol{v}_d] & Q[\boldsymbol{v}] \end{vmatrix} + \varepsilon \begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q_1[\boldsymbol{v}_1] & \dots & Q_1[\boldsymbol{v}_d] & Q_1[\boldsymbol{v}] \end{vmatrix} \end{aligned}$$

Since the right side of this inequality is an affine function in ε we have

$$\begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q[\boldsymbol{v}_1] & \dots & Q[\boldsymbol{v}_d] & Q[\boldsymbol{v}] \end{vmatrix} \geq 0$$

if we take the limit $\varepsilon \searrow 0$. Hence, L is contained in a Delone polytope of Q.

Conversely, let $\mathcal{D}$ be a refinement of $\mathrm{Del}(Q)$. We define the number ε_0 by

$$\varepsilon_0 = \inf\{t \in [0,1] : Q_t \in \overline{\boldsymbol{\Delta}(\mathcal{D})}\}.$$

Assume that $\varepsilon_0 \neq 0$. Let K be a d-dimensional Delone polytope of Q_{ε_0} that is not a simplex. This does exist because Q_{ε_0} is a boundary point of $\overline{\boldsymbol{\Delta}(\mathcal{D})}$. Let L be a d-dimensional Delone simplex of $\mathcal{D}$ with vertices $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}$ and with $L \subseteq K$. For $\boldsymbol{v} \in \operatorname{vert} K \setminus \operatorname{vert} L$ we define the function

$$\chi(\varepsilon) = \begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q_\varepsilon[\boldsymbol{v}_1] & \dots & Q_\varepsilon[\boldsymbol{v}_d] & Q_\varepsilon[\boldsymbol{v}] \end{vmatrix}$$

We have the inequalities $\chi(1) > 0$, $\chi(\varepsilon_0) = 0$ and $\chi(0) \geq 0$ since $\mathcal{D}$ is a refinement of $\mathrm{Del}(Q)$. This cannot happen since χ is an affine function and $\chi(0)$ has to be negative. Hence, $\varepsilon_0 = 0$ and $Q \in \overline{\boldsymbol{\Delta}(\mathcal{D})}$.

ii) Let L be a Delone polytope of Q and let L' be one of Q'. We have to show that their intersection $L \cap L'$ is a Delone polytope of $Q + Q'$. We can assume that $L \cap L'$ is d-dimensional. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+2}$ be vertices of $L \cap L'$. Then,

$$\begin{aligned} &\begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v}_{d+2} \\ (Q+Q')[\boldsymbol{v}_1] & \dots & (Q+Q')[\boldsymbol{v}_{d+1}] & (Q+Q')[\boldsymbol{v}_{d+2}] \end{vmatrix} \\ = &\begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v}_{d+2} \\ Q[\boldsymbol{v}_1] & \dots & Q[\boldsymbol{v}_{d+1}] & Q[\boldsymbol{v}_{d+2}] \end{vmatrix} + \begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v}_{d+2} \\ Q'[\boldsymbol{v}_1] & \dots & Q'[\boldsymbol{v}_{d+1}] & Q'[\boldsymbol{v}_{d+2}] \end{vmatrix} \\ = &\ 0 + 0. \end{aligned}$$

Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d+1}$ be some vertices of $L \cap L'$ and let $\boldsymbol{v}$ be a lattice point that is not a vertex of $L \cap L'$. Suppose that $\boldsymbol{v} \notin \operatorname{vert} L$. Then,

$$\begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q[\boldsymbol{v}_1] & \dots & Q[\boldsymbol{v}_{d+1}] & Q[\boldsymbol{v}] \end{vmatrix} > 0, \quad \text{and} \quad \begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ Q'[\boldsymbol{v}_1] & \dots & Q'[\boldsymbol{v}_{d+1}] & Q'[\boldsymbol{v}] \end{vmatrix} > 0,$$

and so

$$\begin{vmatrix} 1 & \dots & 1 & 1 \\ \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{d+1} & \boldsymbol{v} \\ (Q+Q')[\boldsymbol{v}_1] & \dots & (Q+Q')[\boldsymbol{v}_{d+1}] & (Q+Q')[\boldsymbol{v}] \end{vmatrix} > 0.$$

◇

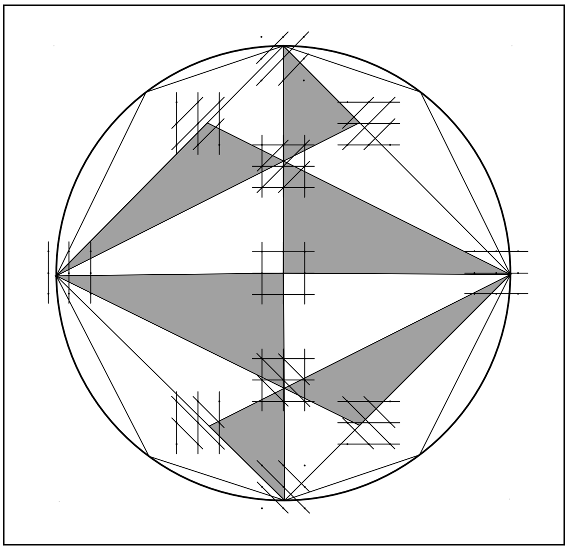


Figure 2.10. Refinements of Two-Dimensional Delone Subdivisions.

2.7. Relations to the Theory of Secondary Polytopes

Triangulations of discrete point sets have attracted many researchers in recent years. They have many applications, e.g. in computational geometry, optimization, algebraic geometry, topology, etc. One main tool to understand the structural behavior of triangulations of finite point sets is the theory of secondary polytopes invented by GEL'FAND, KAPRANOV and ZELEVINSKY ([GKZ1994]). We will describe how this theory is related to VORONOÏ'S reduction theory. We will find out that despite of different set-ups there are many similarities.

Let $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\} \subseteq \mathbb{R}^d$ be a finite set of points. Let $w : \mathcal{A} \to \mathbb{R}$ be a map that assigns to every point in $\mathcal{A}$ a weight. The set of weight maps forms a vector space over $\mathbb{R}$ which we denote by $\mathbb{R}^{\mathcal{A}}$. A lifting map $l : \mathcal{A} \to \mathbb{R}^d \times \mathbb{R}$, $l(\boldsymbol{a}_i) = (\boldsymbol{a}_i, w(\boldsymbol{a}_i))$ is defined by w which lifts each point $\boldsymbol{a}_i \in \mathcal{A}$ on its weight $w(\boldsymbol{a}_i)$. A subdivision of the convex polytope $\operatorname{conv} \mathcal{A}$ is induced by l: We take the convex hull of the lifted points $\operatorname{conv} l(\mathcal{A})$ and project its lower faces as seen from $(\mathbf{0}, -\infty)$ back down onto $\mathbb{R}^d$. A subdivision that can be obtained in this manner is called *regular subdivision*. Delone subdivisions (or more precisely Delone subdivisions of finitely many points) are regular subdivisions since the positive semidefinite quadratic form can be used as the weight function. We saw this already in Section 2.1.

Let $\mathcal{T}$ be a regular triangulation of $\operatorname{conv} \mathcal{A}$. We may ask what are the weight functions which define $\mathcal{T}$. What is the *secondary cone* of $\mathcal{T}$ in the parameter space $\mathbb{R}^{\mathcal{A}}$? Like in VORONOÏ's reduction theory it turns out that the secondary cone of $\mathcal{T}$ is a full-dimensional open polyhedral cone. The topological closures of the secondary cones of all regular triangulations tiles the space $\mathbb{R}^{\mathcal{A}}$ face-to-face. The tiling is called *secondary fan* of $\mathcal{A}$. If two secondary cones have a facet in common, then the corresponding regular triangulations differ by a bistellar operation in exactly one "repartitioning polytope" (in this context it is a polytope with $d + 2$ vertices without the condition of being a Delone polytope) that is defined by the facet. The faces in the secondary fan $\mathcal{A}$ are in a one-to-one correspondence to regular subdivisions in the essentially the same way we discussed in Section 2.6 for Delone subdivisions.

So far we have seen that the theory of regular subdivisions of finite point sets and the theory of Delone subdivisions of the lattice $\mathbb{Z}^d$ can be analogously developed. But there are differences. The parameter spaces are of completely different natures. For regular subdivisions it is the vector space $\mathbb{R}^{\mathcal{A}}$ and for Delone subdivisions we have the pointed cone $\mathcal{S}^d_{\geq 0}$. Groups play an important role for Delone subdivisions. The group $\mathbb{Z}^d$ is acting on Delone subdivisions by translations. On the set of secondary cones the group $\mathsf{GL}_d(\mathbb{Z})$ is acting.

If we order all regular subdivisions of $\operatorname{conv} \mathcal{A}$ by refinement we get a poset. This poset has a very nice combinatorially structure as proved by GEL'FAND, KAPRANOV and ZELEVINSKY: There exists a polytope — the *secondary polytope* $\Sigma(\mathcal{A})$ of $\mathcal{A}$ — whose normal fan equals the secondary fan of $\mathcal{A}$. So the refinement poset is anti-isomorphic to the face lattice of the secondary polytope. Regular triangulations are in one-to-one correspondence to the vertices, two regular triangulations differ by a bistellar operation if and only if their vertices are connected by an edge, etc. We do not know if there is a similar geometrical or combinatorial structure lurking behind the refinement poset of Delone subdivisions.

Question 2.7.1. Does there exist something similar to the secondary polytope for Delone subdivisions?

In Chapter 4 we will compute the complete refinement poset for the 2, 3 and 4-dimensional cases. Before attacking this challenging question the interested reader might find it helpful to consult some literature. There exists a vast amount of literature on triangulations and related topics. We can only provide some hopefully useful starting points: The construction of the secondary polytope can be best understood in the more general set-up of "fiber polytopes" by BILLERA and STURMFELS ([BS1992], see also Lecture 9 in [Zie1995]). In [San2002] SANTOS investigates the combinatorial structure of triangulations in a framework provided by oriented matroids. People with a background in algebraic geometry might benefit from the work of ASH, MUMFORD, RAPOPORT, TAI [AMRT1975], the more elementary accounts of NAMIKAWA [Nam1976], [Nam1980] where VORONOÏ's reduction theory is a central issue. Finally, the recent work of ALEXEEV [Ale2002] seems to be very relevant.

GEL'FAND, KAPRANOV and ZELEVINSKY show a possible direction of research in the introduction of their book [GKZ1994]: "A triangulation of a polytope Q can be viewed as a discrete analog of a Riemannian metric on Q. So $\Sigma(\mathcal{A})$ can be seen as a kind of combinatorial Teichmüller space parameterizing such metrics. This reminds us of the work of PENNER [Pen1993] who constructed a combinatorial model of the Teichmüller space of a Riemann surface in terms of curvilinear triangulations".

Chapter 3.

Parallelohedra

In the last chapter we studied Delone subdivisions. In this chapter we study tilings dual to Delone subdivisions. A parallelohedron is a polytope which admits a face-to-face tiling of the surrounding space by lattice translates. The maximal-dimensional cells of the dual tiling of a Delone subdivision are parallelohedra. They are called Dirichlet-Voronoi polytopes and can alternatively be defined as follows: a Dirichlet-Voronoi polytope contains all those points that are closer to the origin than to any other lattice point. It is a conjecture which goes back to VORONOÏ that the class of parallelohedra is exactly the class of Dirichlet-Voronoi polytopes. We give a computational criterion to check whether a given parallelohedron can be represented as a Dirichlet-Voronoi polytope.

We will explore the duality between Delone subdivisions and tilings by Dirichlet-Voronoi polytopes further. One central question is: How does the Dirichlet-Voronoi polytope vary if we vary the positive semidefinite quadratic form? We will see that this variation is linear in secondary cones of Delone triangulations and piecewise linear in the cone of positive semidefinite quadratic forms. CONWAY and SLOANE defined "vonorms" and "conorms" to parameterize this variation. Our results will give a clear picture of this parameterization.

Consider a Delone triangulation $\mathcal{D}$. A positive semidefinite quadratic form is called *rigid* if it defines an extreme ray of the secondary cone $\mathbf{\Delta}(\mathcal{D})$. Rigid forms are building blocks for Dirichlet-Voronoi polytopes: Every Dirichlet-Voronoi polytope is a Minkowski sum of rigid Dirichlet-Voronoi polytopes. Up to dimension 4 all these building blocks are well-known polytopes: only one-dimensional line segments and the four-dimensional 24-cell occur. Starting with dimension 5 the structure of rigid Dirichlet-Voronoi polytopes is getting more rich and more complicated.

If the secondary cone of a Delone subdivision is bounded by positive semidefinite quadratic forms of rank 1 only, then the corresponding Dirichlet-Voronoi polytopes are zonotopes. We will give a complete theory for zonotopal lattice tilings. The combinatorial theory of zonotopal lattice tilings is equivalent to the theory of regular oriented matroids.

3.1. Definition and Basic Properties

Let V be a d-dimensional real vector space. A parallelohedron $P \subseteq V$ is a polytope which admits a face-to-face tiling of the space V by translates. In this section we give another characterization of parallelohedra which can be used to decide whether a given polytope is a parallelohedron. This characterization was independently found by VENKOV ([Ven1954]) and by MCMULLEN ([McM1980]).

Definition 3.1.1. A d-dimensional polytope $P \subseteq V$ is called *parallelohedron* if it tiles V by translates, i.e. if there is a set $L \subseteq V$ such that

i) $V = \bigcup_{\boldsymbol{v} \in L}(P + \boldsymbol{v})$,

ii) for all $\boldsymbol{v}, \boldsymbol{w} \in L$ the intersection $(P+\boldsymbol{v}) \cap (P+\boldsymbol{w})$ is a common face of $P+\boldsymbol{v}$ and $P+\boldsymbol{w}$.

MINKOWSKI was the first who discovered some structural properties of parallelohedra. In [Min1897] he proves that every parallelohedron is centrally symmetric, has centrally symmetric facets and that every d-dimensional parallelohedron has not more than $2(2^d - 1)$ facets. In this context it is interesting that SHEPHARD showed (see [McM1976]) that a d-dimensional polytope ($d \geq 3$) is centrally symmetric whenever all its facets are centrally symmetric. Notice that not every face of a parallelohedron has to be centrally symmetric. For instance, the four-dimensional 24-cell is a parallelohedron, its 24 facets are octahedra, and the two-dimensional faces are triangles.

The only two-dimensional parallelohedra are centrally symmetric quadrangles and centrally symmetric hexagons. We can exploit this fact by projecting along the right $(d-2)$-dimensional faces to get more structural insights into higher-dimensional parallelohedra.

Definition 3.1.2. Let P be a polytope. A *belt* of P is a sequence of distinct facets $(F_0, \dots, F_{k-1})$ of P such that $F_i \cap F_{i+1}$ (we compute modulo k) is a $(d-2)$-dimensional face which is a translate of $F_0 \cap F_1$.

Each belt of a parallelohedron has length 4 or length 6: We project a parallelohedron along a $(d-2)$-dimensional face of a belt onto the two-dimensional subspace that is generated by the corresponding facet centers. This gives us a new parallelohedron in two dimensions. If it is a quadrangle we have a belt of length 4, if it is a hexagon we have a belt of length 6. It is astonishing and the proof is quite involved that the converse is also true.

Theorem 3.1.3. (VENKOV [Ven1954], MCMULLEN [McM1980])
A polytope is a parallelohedron if and only if it is a centrally symmetric polytope with centrally symmetric facets such that each belt contains either 4 or 6 facets. Furthermore, we can assume that the set of translates forms a lattice.

3.2. VORONOÏ's Conjecture

Let $L \subseteq V$ be a lattice and let $(\cdot,\cdot) : V \times V \to \mathbb{R}$ be an inner product. The *Dirichlet-Voronoi polytope* of the lattice $(L, (\cdot,\cdot))$ is given by

$$\mathrm{DV}(L, (\cdot,\cdot)) := \{\boldsymbol{x} \in V : \text{for all } \boldsymbol{v} \in L \text{ we have } \operatorname{dist}(\boldsymbol{x}, \boldsymbol{0}) \leq \operatorname{dist}(\boldsymbol{x}, \boldsymbol{v})\}.$$

Dirichlet-Voronoi polytopes of lattices are parallelohedra. The translates $\mathrm{DV}(L, (\cdot,\cdot)) + \boldsymbol{v}$, $\boldsymbol{v} \in L$, give a face-to-face tiling of V. VORONOÏ conjectured that all parallelohedra are Dirichlet-Voronoi polytopes of lattices.

Conjecture 3.2.1. (VORONOÏ's Conjecture)
For every parallelohedron $P \subseteq V$ there is a lattice $L \subseteq V$ and an inner product $(\cdot,\cdot) : V \times V \to \mathbb{R}$ such that the Dirichlet-Voronoi polytope of $(L, (\cdot,\cdot))$ is a translate of P.

In [Vor1908] he proves this conjecture for primitive parallelohedra (see below). He writes that he cannot prove it for imprimitive parallelohedra although he believes that the conjecture is true in these cases. Originally he writes in his second monograph *Nouvelles applications des* *paramètres continus à la théorie des formes quadratiques, Deuxième Mémoire, Recherches sur les parallélloèdres primitifs, Journal für die reine und angewandte Mathematik* **134** (1908), pages 210–211:

> On peut envisager le problème de partition uniforme de l'espace kanalytique à n dimension par de polyèdres convexes congruents indépendamment de la théorie des formes quadratiques.
>
> En appelent parallélloèdre chaque polyèdre convexe qui jouit de la propriété I, je démontre le remarquable théorème suivant.
>
> *En effectuant toutes les transformations linéaires possibles a l'aide du groupe continu de substitutions*
>
> $$x_i = a_i + \sum_{k=1}^{n} \alpha_i x'_k \quad (i = 1, 2, \ldots n)$$
>
> *à coefficients réels quelconques d'un parralléloèdre primitif, on obtient un esemble de parallélloèdres primtifs qui est parfaitement déterminé par une classe de formes quadratiques positives équivalentes, á condition qu'on ne considère pas comme différentes les formes quadratiques á coefficients proportionels.*
>
> En vertu de ce théorème, le problème de partition uniforme de l'espace à n dimensions par de parallélloèdres primitifs congruents se ramène toujours à l'étude des parallélloèdres primitifs correspondant aux formes quadratiques positives.
>
> **Je suis porté à croire, sans pouvoir le démontrer, que le théorème énoncé est aussi vrai pour les parallélloèdres imprimitifs.**

Currently the conjecture has only been proved in special cases.

Primitive parallelohedra: In [Vor1908] VORONOÏ, as he mentions above, proves the conjecture for primitive parallelohedra. These are d-dimensional parallelohedra where in each vertex of the tiling exactly the minimal number of $d+1$ parallelohedra meet. In this case the dual tiling consists only of simplices. This fact also characterizes primitive parallelohedra. In [Zhi1929] ZHITOMIRSKII relaxes the condition of primitivity. He shows that the conjecture is true for tilings of parallelohedra where in the dual tiling each two-dimensional face is a triangle (equivalently the considered parallelohedron has only belts of length 6).

Low dimensions: In [Del1929] DELONE shows that the conjecture is true in dimensions up to 4. STOGRIN indicates an alternative proof in [Sto1973]. In Chapter 4 we will classify all these parallelohedra.

Zonotopal parallelohedra: ERDAHL proves in [Erd1999] that every zonotope which tiles space by translates is a Dirichlet-Voronoi polytope. COXETER in [Cox1962], SHEPHARD in [She1974] and MCMULLEN in [McM1975] already anticipated a proof. In [Val2000] another proof is given where the connection to oriented matroids is emphasized. In Chapter 3.5 we present the theory of zonotopal parallelohedra.

In what follows we only deal with parallelohedra that are given by Dirichlet-Voronoi polytopes of lattices. But if one runs into a parallelohedron that is not apriori given as a Dirichlet-Voronoi polytope, then one might want to test if it can be represented as a Dirichlet-Voronoi polytope. By studying VORONOÏ's proof for primitive parallelohedra (one can benefit from the presentation in [GL1987]) we can extract the following proposition which yields a computational criterion afterwards.

Proposition 3.2.2. Let $P \subseteq V$ be a parallelohedron whose facet centers are $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{2n}$. The parallelohedron is the Dirichlet-Voronoi polytope of a lattice if and only if there exist linear functions $f_1, \ldots, f_{2n} \in V^*$ so that the following conditions hold:

i) $P = \bigcap_{i=1}^{2n} \{\boldsymbol{x} \in V : f_i(\boldsymbol{x}) \leq f_i(\boldsymbol{v}_i)\}$,

ii) $f_i = -f_{n+i}$, $i = 1, \ldots, n$,

iii) for every 6-belt $(F_i, -F_j, F_k, -F_i, F_j, -F_k)$ we have $f_i + f_j + f_k = 0$, i.e. all 6-belts are balanced.

Before proving this proposition we describe how it can be used as a computational criterion: Let $P = \bigcap_{i=1}^{2n} \{\boldsymbol{x} \in V : g_i(\boldsymbol{x}) \leq \alpha_i\}$ be a parallelohedron given by supporting hyperplanes. By scaling g_i and reordering we can assume that $\alpha_i = g_i(\boldsymbol{v}_i)$ and $g_i = -g_{n+i}$. The third condition is satisfied if the polyhedron

$$\{\boldsymbol{\beta} \in \mathbb{R}^{2n}_{\geq 0} : \beta_i g_i + \beta_j g_j + \beta_k g_k = 0 \text{ for every 6-belt } (F_i, -F_j, F_k, -F_i, F_j, -F_k)\}$$

has an interior point, which we can decide by linear programming.

Since we do not need Proposition 3.2.2 later we only sketch a proof. An extensive account to this and related themes is given by RYBNIKOV in [Ryb1999] where he works out the relationship between Dirichlet-Voronoi polytopes and the theory of stresses and liftings (extending work of CRAPO and WHITELEY on the so-called Maxwell-Cremona theory. RYBNIKOV writes: "The problem of determining whether a given tiling of the Euclidean space can be obtained as the projection of a convex surface has two origins – MAXWELL's correspondence in rigidity theory and VORONOÏ's generatrice in geometry of numbers and mathematical crystallography.").

Proof. (*Sketch*)
If P is the Dirichlet-Voronoi polytope of a lattice, we have $P = \mathrm{DV}(L, (\cdot,\cdot))$, then it is straightforward to show that P satisfies the three conditions. We simply use the linear functions $f_i(\cdot) = (\boldsymbol{v}_i, \cdot)$, $i = 1, \ldots, 2n$.

Let us look at the other implication. By Theorem 3.1.3 there exists a lattice L so that the family $(P + \boldsymbol{v})_{\boldsymbol{v} \in L}$ is a lattice tiling. On the set of lattice points we define an infinite graph whose vertices are the lattice points. Two vertices $\boldsymbol{v}$ and $\boldsymbol{w}$ are connected by an edge if the corresponding parallelohedra $P + \boldsymbol{v}$ and $P + \boldsymbol{w}$ share a common facet. This graph will turn out to be the 1-skeleton of the dual Delone subdivision and we will try to find a proper lifting of this graph.

We define a function $Q : L \to \mathbb{R}$ which simulates a positive definite quadratic form: Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{2n}$ be the lattice vectors so that $P \cap (P + \boldsymbol{w}_i)$ is a common facet of P and $P + \boldsymbol{w}_i$. Then we set $Q[\boldsymbol{0}] = 0$ and $Q[\boldsymbol{v} + \boldsymbol{w}_i] = Q[\boldsymbol{v}] + f_i(2\boldsymbol{v} + \boldsymbol{w}_i)$, $\boldsymbol{v} \in L$. Since we defined Q only according to neighbouring relations we have to check whether Q is well-defined, i.e. do different paths form $\boldsymbol{0}$ to $\boldsymbol{v}$ always lead to the same value $Q[\boldsymbol{v}]$? We prove this by using a technique from combinatorial topology ("elementary homotopies", see [Bjö1995]): Every two paths from $\boldsymbol{0}$ to $\boldsymbol{v}$ can be deformed into each other using only triangles and squares. Triangles belong to 6-belts which are balanced by condition (iii) and this implies that Q is well-defined on triangles. Squares belong to 4-belt which are balanced because of the central symmetry of P and P's facets and this implies that Q is well-defined on squares.

Then one has to show that Q is a positive definite quadratic form. This can be done by showing that Q has an associated inner product $s_Q(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^t)Q\boldsymbol{y}$. The bilinearity of s_Q follows directly from the definition of Q. The symmetry is a consequence of the central symmetry of P and P's facets. The positive definiteness is a consequence of the boundedness of P. The last step of the proof is to show that $P = \mathrm{DV}(L, s_Q)$ which at that point is straightforward. $\diamond$

3.3. Duality

Delone subdivisions are dual to tilings of Dirichlet-Voronoi polytopes. In this section we will explore this duality. After defining Dirichlet-Voronoi polytopes for quadratic forms we will give an upper bound theorem, a characterization of the facets and the important structural insight that Dirichlet-Voronoi polytopes behave linearly in secondary cones of Delone triangulations.

3.3.1. Definition

We already defined Dirichlet-Voronoi polytopes for lattices in the previous section. The following definition of Dirichlet-Voronoi polytopes for positive definite quadratic forms emphasizes their duality to the cells in the Delone subdivision; although it is not a direct translation of the definition for lattices. At the end of this section we list the main advantages of this definition.

Definition 3.3.1. Let $Q \in \mathcal{S}^d_{>0}$ be a positive definite quadratic form. Let $L = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots\}$ be a Delone polyhedron of Q. We define the *Dirichlet-Voronoi polytope* of Q corresponding to L by

$$\mathrm{DV}(Q, L) = \{\boldsymbol{x}^t Q \in (\mathbb{R}^d)^* : \text{for } \boldsymbol{v} \in \mathbb{Z}^d, i \in \{1, 2, \ldots\} \text{ we have } \mathrm{dist}(\boldsymbol{x}, \boldsymbol{v}_i) \leq \mathrm{dist}(\boldsymbol{x}, \boldsymbol{v})\}.$$

We also can use the definition for positive semidefinite quadratic forms which are arithmetically equivalent to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & Q' \end{smallmatrix}\right)$ where Q' is positive definite. For instance, let us consider again the positive semidefinite quadratic form of the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The Dirichlet-Voronoi polytope corresponding to the one-dimensional Delone polyhedron $\mathrm{conv}\{\boldsymbol{v} + \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) : x \in \mathbb{Z}\}$ is the line segment $\mathrm{conv}\{(0, v_2 + 1/2), (0, v_2 - 1/2)\}$, and the one corresponding to the two-dimensional Delone polyhedron $\mathrm{conv}\{\boldsymbol{v} + \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \boldsymbol{v} + \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right) : x \in \mathbb{Z}\}$ is the point $\{(0, v_2 + 1/2)\}$.

The following proposition states the duality relationship between the two tilings. Its proof is straightforward.

Proposition 3.3.2. Let $Q \in \mathcal{S}^d_{\geq 0}$ be a positive semidefinite quadratic form with Delone subdivision $\mathrm{Del}(Q)$.

i) Let $L, L' \in \mathrm{Del}(Q)$ Delone polyhedra of Q. L is a face of L' if and only if $\mathrm{DV}(Q, L')$ is a face of $\mathrm{DV}(Q, L)$.

ii) For every Delone polyhedron $L \in \mathrm{Del}(Q)$ we have $\dim L + \dim \mathrm{DV}(Q, L) = d$.

Definition 3.3.1 is due to NAMIKAWA ([Nam1976]). It has three important features:

i) The facet normals only depend on the Delone triangulation $\mathcal{D}$ and not on the positive definite quadratic form.

ii) Dirichlet-Voronoi polytopes are always bounded polytopes, even in the semidefinite case.

iii) The Dirichlet-Voronoi polytope of a positive definite quadratic form Q corresponding to the cell $\{\boldsymbol{0}\}$ and the one of an associated lattice $(\mathbb{Z}^d, s_Q)$, $s_Q(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^t Q \boldsymbol{y}$, are affinely isomorphic because

$$\mathrm{DV}(Q, \{\boldsymbol{0}\}) = (Q\,\mathrm{DV}(\mathbb{Z}^d, s_Q(\cdot, \cdot))^t.$$

3.3.2. Upper Bound Theorem

Using duality we easily get an upper bound for the number of vertices of d-dimensional Dirichlet-Voronoi polytopes: Let Q be a positive definite quadratic form whose Delone subdivision is a triangulation. By duality every vertex of $\mathrm{DV}(Q, \{\mathbf{0}\})$ is given by a d-dimensional Delone simplex. The vertices of a Delone polytope belong to the lattice $\mathbb{Z}^d$, so the volume of a d-dimensional Delone polytope is at least $1/d!$. Because every fundamental domain of $\mathbb{Z}^d$ has unit volume, there are at most $d!$ vertices of $\mathrm{DV}(Q, \{\mathbf{0}\})$ that are not translates of each other. If $\boldsymbol{v}$ is a vertex of $\mathrm{DV}(Q, \{\mathbf{0}\})$, then there exists a Delone simplex $L = \mathrm{conv}\{\mathbf{0}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ defining $\boldsymbol{v}$. The d Delone simplices $L - \boldsymbol{v}_i$, $i = 1, \ldots, d$, define vertices of $\mathrm{DV}(Q, \{\mathbf{0}\})$ that are translates of $\boldsymbol{v}$. Thus, $\mathrm{DV}(Q, \{\mathbf{0}\})$ has at most $(d + 1)!$ vertices. This upper bound holds also for positive definite quadratic forms whose Delone subdivision is no triangulation because they are limits of triangulations.

In [Vor1908], §63–§68, §101, VORONOÏ refines this observation to get an upper bound theorem for Dirichlet-Voronoi polytopes. The Dirichlet-Voronoi polytope of VORONOÏ's principal form of the first type is a permutahedron. The number of k-dimensional faces of a d-dimensional permutahedron is $(d - k + 1)! \left\{ {d+1 \atop d-k+1} \right\}$ where $\left\{ {d \atop k} \right\}$ are the Stirling numbers of the second kind (the number of k-element partitions of a d-element set). The permutahedron is an extreme Dirichlet-Voronoi polytope, no d-dimensional Dirichlet-Voronoi polytope has more k-dimensional faces.

$\mathbf{d}$	$\mathbf{f_0}$	$\mathbf{f_1}$	$\mathbf{f_2}$	$\mathbf{f_3}$	$\mathbf{f_4}$	$\mathbf{f_5}$	$\mathbf{f_6}$	$\mathbf{f_7}$
1	2	1						
2	6	6	1					
3	24	36	14	1				
4	120	240	150	30	1			
5	720	1,800	1,560	540	62	1		
6	5,040	15,120	16,800	8,400	1,806	126	1	
7	40,320	141,120	191,520	126,000	40,824	5,796	254	1

Table 3.1. Extreme f-Vectors of d-Dimensional Dirichlet-Voronoi Polytopes.

3.3.3. Voronoi Vectors, Supporting Hyperplanes, and Facets

In this short section we want to characterize the facets of a Dirichlet-Voronoi polytope.

Definition 3.3.3. Let $Q \in \mathcal{S}^d_{\geq 0}$ be a positive semidefinite quadratic form with Delone subdivision $\mathrm{Del}(Q)$. A vector $\boldsymbol{v} \in \mathbb{Z}^d$ is called *Voronoi vector* if the affine hyperplane

$$H_{Q,\boldsymbol{v}} = \{\boldsymbol{y} \in (\mathbb{R}^d)^* : \boldsymbol{y}\boldsymbol{v} = \frac{1}{2}\boldsymbol{v}^t Q \boldsymbol{v}\} = \{\boldsymbol{x}^t Q : \mathrm{dist}(\boldsymbol{x}, \mathbf{0}) = \mathrm{dist}(\boldsymbol{x}, \boldsymbol{v})\}$$

is a supporting hyperplane of $\mathrm{DV}(Q, \{\mathbf{0}\})$.

Voronoi vectors are the shortest vectors in the cosets $\mathbb{Z}^d/2\mathbb{Z}^d$, more precisely:

Proposition 3.3.4. (*Characterization of Voronoi Vectors*)
Let $Q \in \mathcal{S}^d_{\geq 0}$ be a positive semidefinite quadratic form with Delone subdivision $\mathrm{Del}(Q)$. A vector $\boldsymbol{v} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ is a Voronoi vector if and only if it is a shortest vector (we use the seminorm defined by Q) in the coset $\boldsymbol{v} + 2\mathbb{Z}^d$. The hyperplane $H_{Q,\boldsymbol{v}}$ defines a facet of $\mathrm{DV}(Q, \{\mathbf{0}\})$ if and only if $\pm\boldsymbol{v}$ are the only shortest vectors in the coset $\boldsymbol{v} + 2\mathbb{Z}^d$.

A proof of this characterization is not difficult and one can find it e.g. in [Vor1908], or in [CS1992].

We illustrate this proposition in Figure 3.1 for the positive definite quadratic form $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Here the Voronoi vectors are $(\pm 1, 0)^t$, $(0, \pm 1)^t$, and $(\pm 1, \pm 1)^t$ and only the four vectors $(\pm 1, 0)$, $(0, \pm 1)$ define facet defining hyperplane.

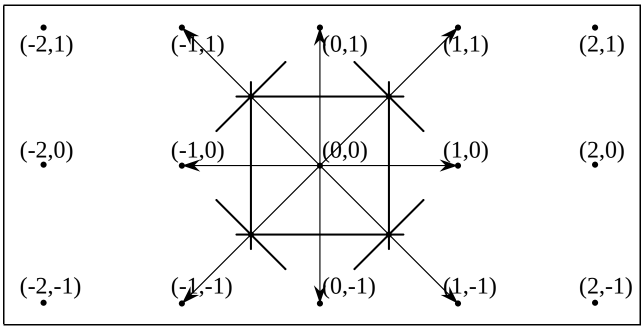


Figure 3.1. Voronoi Vectors of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

3.3.4. Linearity and Rigidity

Now we show that Dirichlet-Voronoi polytopes behave linearly in the topological closure of the secondary cone of a Delone triangulation. This is a major structural insight with many consequences and applications.

Proposition 3.3.5. Let $\mathcal{D}$ be a Delone triangulation. For positive semidefinite quadratic forms $Q_1, \ldots, Q_n \in \overline{\boldsymbol{\Delta}(\mathcal{D})}$ and non-negative numbers $\alpha_1, \ldots, \alpha_n$ we have

$$\mathrm{DV}(\sum_{i=1}^{n} \alpha_i Q_i, \{\mathbf{0}\}) = \sum_{i=1}^{n} \alpha_i \, \mathrm{DV}(Q_i, \{\mathbf{0}\}).$$

Proof. It suffices to consider only two summands Q_1 and Q_2. From Proposition 2.6.1 we conclude that all positive semidefinite quadratic forms in $\overline{\boldsymbol{\Delta}(\mathcal{D})}$ have the same set of Voronoi vectors. We defined Dirichlet-Voronoi polytopes in such a way that the facet normals are Voronoi vectors, and so that they only depend on $\mathcal{D}$. The support function of polytopes respects Minkowski sums. For $Q_1, Q_2 \in \overline{\boldsymbol{\Delta}(\mathcal{D})}$ and a Voronoi vector $\boldsymbol{v}$ of Q_1 and Q_2 we have

$$H_{Q_1+Q_2,\boldsymbol{v}} = H_{Q_1,\boldsymbol{v}} + H_{Q_2,\boldsymbol{v}}.$$

This shows that every facet of $\mathrm{DV}(Q_1 + Q_2, \{\mathbf{0}\})$ is the Minkowski sum of faces of $\mathrm{DV}(Q_1, \{\mathbf{0}\})$ and $\mathrm{DV}(Q_2, \{\mathbf{0}\})$. ◇

This proposition is extremely useful. It shows that and how we get all Dirichlet-Voronoi polytopes by summing up Dirichlet-Voronoi polytopes of semidefinite quadratic forms belonging to extreme rays of secondary cones. We consider all those positive semidefinite quadratic forms $Q_1, \ldots, Q_n$ that belong to the extreme rays of the secondary cone of the Delone triangulation $\mathcal{D}$. Proposition 3.3.5 says that $\mathrm{DV}(Q_i, \{\mathbf{0}\})$, $i = 1, \ldots, n$, are *the* building blocks of all Dirichlet-Voronoi polytopes belonging to quadratic forms lying in $\overline{\boldsymbol{\Delta}(\mathcal{D})}$: Every $Q \in \overline{\boldsymbol{\Delta}(\mathcal{D})}$ can be written as a non-negative linear combination of the Q_i's (we have $Q = \sum_{i=1}^{n} \alpha_i Q_i$, $\alpha_i \geq 0$) and by Proposition 3.3.5 the Dirichlet-Voronoi polytope can be written in exactly the same way by taking weighted Minkowski sums (we have $\mathrm{DV}(\sum_{i=1}^{n} \alpha_i Q_i, \{\mathbf{0}\}) = \sum_{i=1}^{n} \alpha_i \, \mathrm{DV}(Q_i, \{\mathbf{0}\})$).

In case that we know all non-equivalent extreme rays of a given dimension, we can generate all Dirichlet-Voronoi polytopes by taking Minkowski sums. We will use this fact to classify Dirichlet-Voronoi polytopes in dimensions up to 4 in the next chapter.

All these considerations lead to the definition of rigid forms respectively to rigid lattices.

Definition 3.3.6. Let Q be a positive definite quadratic form. The dimension of the secondary cone of its Delone subdivision is called *non-rigidity degree* of Q. If the non-rigidity degree of Q equals 1, we say Q is *rigid*.

Since the definition above does not depend on the choice of Q in the class of arithmetically equivalent positive definite quadratic forms this defines “non-rigidity degree” and “rigidity” also for lattices. In the next chapter we will see that the only rigid lattices in dimensions $d \leq 4$ are the one-dimensional lattice $\mathbb{Z}^1$ whose Dirichlet-Voronoi polytope is a line segment and the four-dimensional lattice D_4 whose Dirichlet-Voronoi polytope is the 24-cell. A lattice is rigid if and only if its Dirichlet-Voronoi polytope cannot be written as a non-trivial Minkowski sum of two Dirichlet-Voronoi polytopes.

BARANOVSKII and GRISHUKHIN were the first who studied rigid positive definite quadratic forms. The main result of their article [BG2001] is a formula for the computation of the non-rigidity degree. The moral of its proof is that every affine dependency between the vertices of a Delone polytope gives a linear dependency between the entries of the matrix Q.

Proposition 3.3.7. Let Q be a positive definite quadratic form. The non-rigidity degree of Q is given by $\frac{d(d+1)}{2} - \dim S(Q)$, where $S(Q)$ is the subspace of $\mathcal{S}^d$ that is defined by

$$S(Q) := \left\langle \left\{ \begin{array}{lcl} Q' \in \mathcal{S}^d & : & \sum_{i=2}^n \alpha_i Q'[\boldsymbol{v}_1 - \boldsymbol{v}_i] = 0, \\ & & \text{where } L = \operatorname{conv}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\} \text{ is a Delone polytope of } Q, \\ & & \sum_{i=1}^n \alpha_i \boldsymbol{v}_i = \mathbf{0} \text{ is a minimal affine dependency} \end{array} \right\} \right\rangle_{\mathbb{R}} .$$

Furthermore, the linear span of $\boldsymbol{\Delta}(\mathrm{Del}(Q))$ equals $S(Q)$.

We think that it is important to find a good, i.e. structural, characterization of rigid positive definite quadratic forms. This proposition is only a first step.

Similar questions have been studied in other settings. If we formulate the question of being rigid for Delone subdivisions, then a Delone subdivision $\mathcal{D}$ belongs to a rigid positive definite quadratic form if there is no Delone subdivision coarser than $\mathcal{D}$. In the theory of regular triangulations and secondary polytopes these coarsest subdivisions are in 1-to-1 correspondence to the facets of the secondary polytope. But also in this setting no structural characterization is known although there has been some progress (see [BGS1993], and the more recent [San2001]).

3.4. Vonorms and Conorms

CONWAY and SLOANE introduce “vonorms” and “conorms” in [CS1992] for lattices. We extend their definitions to positive semidefinite quadratic forms. This gives us the opportunity to understand the basic properties of vonorms and conorms better. The main result of this section is that vonorms and conorms are piecewise linear functions in the cone of positive semidefinite quadratic forms. CONWAY and SLOANE conjecture that the vonorms and conorms of a lattice characterize the lattice uniquely. Our main result enables us to show that this is locally true and it enables us to check this conjecture in every given dimension algorithmically.

The vonorm map of a positive semidefinite quadratic form Q assigns to a coset $\boldsymbol{v} + 2\mathbb{Z}^d$ the squared norm of the shortest lattice vector in this coset. The definition is motivated by the characterization of Voronoi vectors in Proposition 3.3.4 where we stated that every shortest lattice vector in a coset $\boldsymbol{v} + 2\mathbb{Z}^d$ defines a supporting hyperplane of $\mathrm{DV}(Q, \{\mathbf{0}\})$.

Definition 3.4.1. Let $Q \in \mathcal{S}^d_{\geq 0}$ be a positive semidefinite quadratic form. We define the *vonorm map* $\mathrm{vo}_Q : \mathbb{Z}^d/2\mathbb{Z}^d \to \mathbb{R}$ as follows: For a coset $\boldsymbol{v} + 2\mathbb{Z}^d \in \mathbb{Z}^d/2\mathbb{Z}^d$ we set

$$\mathrm{vo}_Q(\boldsymbol{v} + 2\mathbb{Z}^d) = \min\{Q[\boldsymbol{w}] : \boldsymbol{w} \in \boldsymbol{v} + 2\mathbb{Z}^d\}.$$

Let $\chi : \mathbb{Z}^d/2\mathbb{Z}^d \to \{\pm 1\}$ be a group homomorphism (a character of the group $\mathbb{Z}^d/2\mathbb{Z}^d$). We define the *conorm map* of Q by

$$\mathrm{co}_Q(\chi) = -\frac{1}{2^{d-1}} \sum_{\boldsymbol{v}+2\mathbb{Z}^d \in \mathbb{Z}^d/2\mathbb{Z}^d} \chi(\boldsymbol{v} + 2\mathbb{Z}^d)\, \mathrm{vo}_Q(\boldsymbol{v} + 2\mathbb{Z}^d).$$

The conorm map is, apart from the scale factor $-\frac{1}{2^{d-1}}$, the discrete Fourier transform of the vonorm map. The vonorm map can be reconstructed from the conorm map by

$$\mathrm{vo}_Q(\boldsymbol{v} + 2\mathbb{Z}^d) = \sum_{\chi : \chi(\boldsymbol{v}+2\mathbb{Z}^d) = -1} \mathrm{co}_Q(\chi).$$

This is an immediate corollary of the inversion formula for discrete Fourier transforms. For all $\boldsymbol{v}+2\mathbb{Z}^d$ which are subsets of the totally isotropic subspace $Q^{-1}[\{\mathbf{0}\}]$ we have $\mathrm{vo}_Q(\boldsymbol{v}+2\mathbb{Z}^d) = 0$.

In explicit calculations it is convenient to identify the d-dimensional vector space of group homomorphisms $\{\chi : \mathbb{Z}^d/2\mathbb{Z}^d \to \{\pm 1\}\}$ with the space of binary row vectors $\mathbb{F}_2^d$ using the canonical isomorphism. For convenience and future reference we explicitly define this canonical isomorphism.

Let $\chi : \mathbb{Z}^d/2\mathbb{Z}^d \to \{\pm 1\}$ be a group homomorphism and let $\boldsymbol{e}_i$ be the i-th unit vector. Then we identify χ with $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{F}_2^d$ componentwise by

$$x_i = \begin{cases} 0, & \text{if } \chi(\boldsymbol{e}_i + 2\mathbb{Z}^d) = +1 \\ 1, & \text{if } \chi(\boldsymbol{e}_i + 2\mathbb{Z}^d) = -1 \end{cases}$$

Conversely, let $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{F}_2^d$ be a binary vector. Then we identify $\boldsymbol{x}$ with the group homomorphism $\chi : \mathbb{Z}^d/2\mathbb{Z}^d \to \{\pm 1\}$ by

$$\chi((v_1, \ldots, v_d)+2\mathbb{Z}^d) = f(v_1)\cdot f(v_2) \cdots f(v_d), \quad \text{where } f(v_i) = \begin{cases} -1, & \text{if } v_i \text{ is odd and } x_i = 1 \\ 1, & \text{otherwise} \end{cases}$$

For instance, the vector $(1, 1) \in \mathbb{F}_2^2$ gives the group homomorphism $\chi : \mathbb{Z}^2/2\mathbb{Z}^2 \to \{\pm 1\}$

$$\begin{array}{rclcl rclcl}
\chi((0,0)^t + 2\mathbb{Z}^2) & = & 1 \cdot 1 & = & 1 & \chi((1,0)^t + 2\mathbb{Z}^2) & = & (-1) \cdot 1 & = & -1 \\
\chi((0,1)^t + 2\mathbb{Z}^2) & = & 1 \cdot (-1) & = & -1 & \chi((1,1)^t + 2\mathbb{Z}^2) & = & (-1) \cdot (-1) & = & 1.
\end{array}$$

Example 3.4.2. Let us compute the vonorm map and the conorm map of $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ whose geometry is illustrated in Figure 3.1. Its vonorm map is given by

$$\begin{array}{rcl rcl}
\mathrm{vo}_Q((0,0)^t + 2\mathbb{Z}^2) & = & 0 & \mathrm{vo}_Q((1,0)^t + 2\mathbb{Z}^2) & = & 1 \\
\mathrm{vo}_Q((0,1)^t + 2\mathbb{Z}^2) & = & 1 & \mathrm{vo}_Q((1,1)^t + 2\mathbb{Z}^2) & = & 2,
\end{array}$$

and its conorm map is given by the formula

$$\begin{aligned}\mathrm{co}_Q(\chi) &= -\frac{1}{2}\big(\chi((0,0)^t + 2\mathbb{Z}^2)\,\mathrm{vo}_Q((0,0)^t + 2\mathbb{Z}^2) \\ &\quad + \chi((0,1)^t + 2\mathbb{Z}^2)\,\mathrm{vo}_Q((0,1)^t + 2\mathbb{Z}^2) \\ &\quad + \chi((1,0))^t + 2\mathbb{Z}^2)\,\mathrm{vo}_Q((1,0)^t + 2\mathbb{Z}^2) \\ &\quad + \chi((1,1)^t + 2\mathbb{Z}^2)\,\mathrm{vo}_Q((1,1)^t + 2\mathbb{Z}^2)\big)\,,\end{aligned}$$

and by the values (here we used the identification we defined above)

$$\begin{array}{rclcl}\mathrm{co}_Q(0,0) &=& -\frac{1}{2}(0+1+1+2) &=& -2\\ \mathrm{co}_Q(0,1) &=& -\frac{1}{2}(0-1+1-2) &=& 1\\ \mathrm{co}_Q(1,0) &=& -\frac{1}{2}(0+1-1-2) &=& 1\\ \mathrm{co}_Q(1,1) &=& -\frac{1}{2}(0-1-1+2) &=& 0.\end{array}$$

In [CS1992] Conway and Sloane state that the conorm map varies continuously with the lattice and that this is one of its most useful properties (other useful properties are: the conorm map is an invariant of the lattice, and all symmetries of the lattice arise from symmetries of the conorm map). Here, we will turn this qualitative statement into a quantitative one.

Proposition 3.4.3. The vonorm map and the conorm map are piecewise linear maps if we view them as maps from $\mathcal{S}^d_{\geq 0}$ to $\mathbb{R}^{2^d}$. Let $\mathcal{D}$ be a Delone triangulation, $Q_1, \ldots, Q_n \in \overline{\mathbf{\Delta}(\mathcal{D})}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{\geq 0}$. Then we have for all $\boldsymbol{v} \in \mathbb{Z}^d$ and for all group homomorphisms $\chi : \mathbb{Z}^d/2\mathbb{Z}^d \to \{\pm 1\}$

$$\begin{aligned}\mathrm{vo}_{\sum_{i=1}^n \alpha_i Q_i}(\boldsymbol{v} + 2\mathbb{Z}^d) &= \sum_{i=1}^n \alpha_i\,\mathrm{vo}_{Q_i}(\boldsymbol{v} + 2\mathbb{Z}^d)\\ \mathrm{co}_{\sum_{i=1}^n \alpha_i Q_i}(\chi) &= \sum_{i=1}^n \alpha_i\,\mathrm{co}_{Q_i}(\chi).\end{aligned}$$

Proof. Let $\mathcal{D}$ be a Delone triangulation, and $Q, Q' \in \overline{\mathbf{\Delta}(\mathcal{D})}$. Then, the Voronoi vectors of Q and Q' coincide. For a Voronoi vector $\boldsymbol{v}$ and non-negative numbers α, α' we have

$$(\alpha Q + \alpha' Q')[\boldsymbol{v}] = \alpha Q[\boldsymbol{v}] + \alpha' Q'[\boldsymbol{v}].$$

Hence,

$$\mathrm{vo}_{\alpha Q + \alpha' Q'}(\boldsymbol{v} + 2\mathbb{Z}^d) = \alpha\,\mathrm{vo}_Q(\boldsymbol{v} + 2\mathbb{Z}^d) + \alpha'\,\mathrm{vo}_{Q'}(\boldsymbol{v} + 2\mathbb{Z}^d),$$

and the conorm map inherits this linearity. ◇

Conway and Sloane conjecture that the vonorm map characterizes a lattice.

Conjecture 3.4.4. (Conway & Sloane, [CS1992])
Let Q, Q' be positive semidefinite quadratic forms. If their vonorm maps coincide, i.e. if we have $\mathrm{vo}_Q = \mathrm{vo}_{Q'}$, then Q and Q' are arithmetically equivalent.

Now we show by an easy argument that this conjecture is locally true. Let $\mathcal{D}$ be a Delone triangulation. Let Q, Q', $Q \neq' Q$, be two positive semidefinite quadratic forms lying in $\overline{\mathbf{\Delta}(\mathcal{D})}$. Assume that they have the same vonorm map. As a result of Proposition 3.4.3 the vonorm maps of all forms in the segment $[Q, Q']$ coincide. This cannot happen because infinitesimally changes of a positive semidefinite quadratic form changes the vonorm map.

Together with VORONOÏ's reduction theory the preceding considerations provide an algorithm which proves or disproves the conjecture of CONWAY and SLOANE in every given dimension d. For two non-equivalent secondary cones of d-dimensional Delone triangulations $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}'$ we compute the extreme rays $\overline{\boldsymbol{\Delta}} = \operatorname{cone}\{Q_1, \ldots, Q_n\}$, $\overline{\boldsymbol{\Delta}'} = \operatorname{cone}\{Q'_1, \ldots, Q'_{n'}\}$ and their vonorms. Then we check whether solutions of the system of the $2^d + n + n'$ linear (in-)equalities

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i \operatorname{vo}_{Q_i}(\boldsymbol{v}) &= \sum_{j=1}^{n'} \alpha'_j \operatorname{vo}_{Q'_j}(\boldsymbol{v}), \quad \boldsymbol{v} \in \{0,1\}^d, \\ \alpha_i &\geq 0, \quad i = 1, \ldots, n, \\ \alpha'_j &\geq 0, \quad j = 1, \ldots, n', \end{aligned}$$

that depends on the parameters α_i, α'_j defines only arithmetically equivalent boundaries of the secondary cones $\overline{\boldsymbol{\Delta}}$ and $\overline{\boldsymbol{\Delta}'}$. The conjecture is true if and only if for all pairs of non-equivalent secondary cones this is the case. In Chapter 4.4 we will use this approach to give a proof of the conjecture for $d = 4$ (in [CS1992] CONWAY and SLOANE state that the conjecture is true in dimensions $d \leq 4$ without providing a proof). Trivially (since there is only one non-equivalent secondary cone), the conjecture is true in the two-dimensional and in the three-dimensional case.

3.5. Zonotopal Parallelohedra

In this section we study zonotopal parallelohedra. For zonotopal parallelohedra VORONOÏ's conjecture holds (see [Erd1999]). So we concentrate our attention on lattices whose Dirichlet-Voronoi polytopes are zonotopes. The methods we present here can also be used to give a proof of VORONOÏ's conjecture for zonotopes (see [Val2000]). We provide a link between the theory of (regular) oriented matroids and the theory of lattices whose Dirichlet-Voronoi polytope is a zonotope. The main advantage of this approach is the strict separation between combinatorial and metrical data. This approach was initiated by GERRITZEN [Ger1982] and LOESCH [Loe1990].

We suggest that readers who are familiar with the theory of oriented matroids and the transitions between oriented matroids, zonotopes, graphs and hyperplane arrangements should only browse through this section. We use the usual connections and we only interlace the definition of zonotopal lattices into the framework of oriented matroid theory. It will suffice to understand that we can get Dirichlet-Voronoi polytopes of cographical lattices from the permutahedron by deleting edges and that there is exactly one four-dimensional Dirichlet-Voronoi zonotope, namely the one of the graphical lattice $L_{\mathsf{K}_{3,3}}$, that does not originate from the permutahedron.

And we suggest that all other readers should just read happily ahead.

3.5.1. Definition and Basic Properties

Let E be a finite set. Let $(\mathbb{R}^E, (\cdot,\cdot))$ be a Euclidean space where the standard basis vectors form an orthogonal basis (but not necessarily an orthonormal basis). Let $L \subseteq \mathbb{Z}^E$ be a lattice and let $\boldsymbol{v} \in L$ be a lattice vector. The *support* of $\boldsymbol{v}$ is $\underline{\boldsymbol{v}} := \{e \in E : v_e \neq 0\}$. A lattice vector $\boldsymbol{v}$ is called *elementary* if $\boldsymbol{v} \in \{-1, 0, +1\}^E \backslash \{\mathbf{0}\}$ and if $\boldsymbol{v}$ has minimal support. Two lattice vectors $\boldsymbol{v}, \boldsymbol{w}$ are called *conformal* if $v_e \cdot w_e \geq 0$ for all $e \in E$. A lattice L is called *zonotopal* if every lattice vector of L can be written as a sum of pairwise conformal elementary lattice vectors. The definition is highly motivated by TUTTE's theory of regular chain groups. In [Tut1971] one finds proofs of all our statements not involving the Euclidean structure. Statements that depend on the Euclidean structure are proved e.g. in [Val2000].

When $L \subseteq \mathbb{Z}^E$ is a zonotopal lattice, the set of elementary vectors are cocircuits of an oriented matroid. We denote this oriented matroid by $\mathcal{M}(L)$. It is a regular oriented matroid. Conversely, for every regular oriented matroid $\mathcal{M}$ there is a zonotopal lattice L with $\mathcal{M} = \mathcal{M}(L)$. Let us translate the oriented matroid operations "dualization", "contraction" and "deletion" into the language of zonotopal lattices. Let A be a subset of E. For a lattice vector $\boldsymbol{v} \in \mathbb{Z}^E$ we define the restriction $\boldsymbol{v}_{|A} \in \mathbb{Z}^A$ by $(\boldsymbol{v}_{|A})_e = v_e$ for $e \in A$. Let S be a subset of E. We define the *zonotopal dual* of L, the *contraction* L/S, and the *deletion* $L\backslash S$ by

$$
\begin{aligned}
L^{\perp} &= \{\boldsymbol{v} \in \mathbb{Z}^E : \text{we have } \textstyle\sum_{e\in E} v_e w_e = 0 \text{ for all } \boldsymbol{w} \in L\}\\
L/S &= \{\boldsymbol{v}_{|E\backslash S} : \boldsymbol{v} \in L \text{ and } \underline{\boldsymbol{v}} \cap S = \emptyset\}\\
L\backslash S &= \{\boldsymbol{v}_{|E\backslash S} : \boldsymbol{v} \in L\}.
\end{aligned}
$$

The class of zonotopal lattices is closed under these operations. For the corresponding regular oriented matroids we have

$$\mathcal{M}(L^{\perp}) = \mathcal{M}^*(L), \;\; \mathcal{M}(L/S) = \mathcal{M}(L)/S, \;\; \mathcal{M}(L\backslash S) = \mathcal{M}(L)\backslash S.$$

Let L be a zonotopal lattice. A lattice obtained from L by a sequence of deletions and contractions is called *minor* of L.

The Dirichlet-Voronoi polytope of L is a zonotope because we have

$$\mathrm{DV}(L, (\cdot,\cdot)) = \pi(\mathrm{DV}(\mathbb{Z}^E, (\cdot,\cdot))) = \pi\left([-1/2, 1/2]^E\right)$$

where π is the orthogonal projection of $\mathbb{R}^E$ onto the linear subspace spanned by L. In Figure 3.2 the case $L = \mathbb{Z}(1,1,0)^t + \mathbb{Z}(0,1,1)^t$, $(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + x_2y_2 + x_3y_3$, is demonstrated. The elementary vectors of L are those Voronoi vectors that define a facet of $\mathrm{DV}(L, (\cdot,\cdot))$. The combinatorial structure of the Dirichlet-Voronoi polytope is completely determined by the oriented matroid $\mathcal{M}(L)$: the face lattices of the polytope $\mathrm{DV}(L, (\cdot,\cdot))$ and the one of the oriented matroid $\mathcal{M}(L)$ coincide.

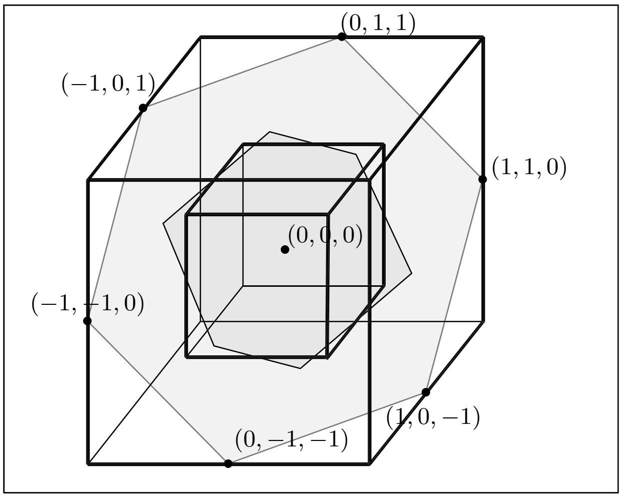


Figure 3.2. Dirichlet-Voronoi Polytope of a Zonotopal Lattice.

The geometrical realization of $\mathrm{DV}(L, (\cdot,\cdot))$ depends on the norms of the standard basis vectors. We denote the standard basis vectors of $\mathbb{R}^E$ by $\boldsymbol{e}_i$, $i \in E$. The edges of the zonotope $\mathrm{DV}(L, (\cdot,\cdot))$ are translates of $\pi(\boldsymbol{e}_i)$ if i is contained in the support of some lattice vector. The edge lengths are given by

$$\|\pi(\boldsymbol{e}_i)\|^2 = \frac{(\boldsymbol{e}_i, \boldsymbol{e}_i)\det(L/\{i\})}{\det L}.$$

The determinant of L is given by

$$\sum_{B\in\mathcal{B}}\prod_{i\in B}(\boldsymbol{e}_i, \boldsymbol{e}_i) = (\boldsymbol{e}_j, \boldsymbol{e}_j)\det(L/\{j\}) + \det(L\backslash\{j\})$$

where $\mathcal{B}$ is the set of basis of $\mathcal{M}(L)$ and $j \in E$ is contained in the support of some lattice vector.

3.5.2. SEYMOUR's Decomposition Theorem

The combinatorial structure of zonotopal lattices and their Dirichlet-Voronoi polytopes is well-understood by SEYMOUR's decomposition theorem for regular matroids. We formulate it in terms of zonotopal lattices.

Theorem 3.5.1. (SEYMOUR*'s Decomposition Theorem*, [Sey80])
Every zonotopal lattice can be decomposed into 1-sums, 2-sums, and 3-sums of cographical lattices, graphical lattices and lattices of type R_{10}.

In the following sections we will introduce the main examples of zonotopal lattices: cographical lattices, graphical lattices and lattices of type R_{10}. For the definitions of 1-sums, 2-sums, and 3-sums the interested reader is referred to SEYMOUR's original paper, to TRÜMPER's book [Trü1992] that also deals with algorithmic aspects or to [Val2000] where 1-sums, 2-sums, and 3-sums are defined in the context of zonotopal lattices.

With help of this theorem one can classify zonotopal lattices according to their combinatorial structure. A d-dimensional zonotopal lattice is called *maximal* if it is not a minor of another d-dimensional zonotopal lattice. The classification of maximal zonotopal lattices has been carried out up to dimension 5 by ERDAHL and RYSHKOV ([ER1994]). In [DG1999] DANILOV and GRISHUKHIN work out the case $d = 6$ where they explicitly make use of SEYMOUR's decomposition theorem.

Cographical Lattices

Let $G = (V, E)$ be a connected graph with directed edges. We denote the set of all oriented minimal cuts (cocircuits) of G by $\mathcal{C}^*(G)$. For every cocircuit C^* we define a lattice vector $\boldsymbol{v}(C^*) \in \{-1, 0, 1\}^E$ by

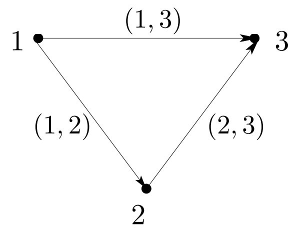
$$\boldsymbol{v}(C^*)_e = \begin{cases} +1, & \text{if } e \text{ is an outgoing edge of the minimal cut,} \\ -1, & \text{if } e \text{ is an ingoing edge of the minimal cut,} \\ 0, & \text{if } e \text{ is not an edge of the minimal cut.} \end{cases}$$

The lattice $L^*_G = \sum_{C^*\in\mathcal{C}^*(G)} \mathbb{Z}\boldsymbol{v}(C^*) \subseteq \mathbb{Z}^E$ is called *cographical*. Actually, L^*_G describes a set of lattices with a fixed combinatorial structure. Together with a corresponding inner product L^*_G is a zonotopal lattice of dimension $|V| - 1$.

The Dirichlet-Voronoi polytope of the cographical lattice $L^*_{\mathsf{K}_d}$ is a $(d-1)$-dimensional permutahedron. We get all Dirichlet-Voronoi polytopes of $(d-1)$-dimensional cographical lattices by deleting edges of the permutahedron which correspond to edges we deleted from the complete graph K_d. Each edge of the graph corresponds to a rigid rank-1-form whose Dirichlet-Voronoi polytope is a one-dimensional line segment.

Exactly the lattices which are associated to the positive definite quadratic forms lying in the topological closure of VORONOÏ's principal domain of the first type are cographical. We prove this using Delone graphs*. Let Q be a positive definite quadratic form which lies in the topological closure of VORONOÏ's principal domain of the first type. The Selling parameters q_{ij}, $1 \leq i,j \leq d+1$, $i \neq j$, of Q are all non-positive. We define the *Delone graph* of Q by $G_Q = (\{1,\ldots,d+1\}, E)$ with $(i,j) \in E$ whenever $i < j$ and $q_{ij} < 0$. Then, the form Q is associated to the lattice $L^*_{G_Q}$. Hence, we have a convenient description of the face lattices for Dirichlet-Voronoi polytopes of lattices associated to positive definite quadratic forms lying in $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$.

Example 3.5.2. Let us look at the two-dimensional case and the graph K_3.



$$\text{cocircuits:}\quad \begin{array}{cccccc} & (1,2) & (1,3) & (2,3) & \\ \pm(& 1 & 1 & 0 &) \\ \pm(& 0 & 1 & 1 &) \\ \pm(& 1 & 0 & -1 &) \end{array}$$

The two vectors $\boldsymbol{b}_1 = (1,1,0)^t$, $\boldsymbol{b}_2 = (0,-1,-1)^t$ form a lattice basis of the two-dimensional cographical lattice $L^*_{K_3} \subseteq \mathbb{Z}^3$. The Dirichlet-Voronoi polytope of $L^*_{K_3}$ is a two-dimensional permutahedron aka a hexagon. In Figure 3.2 one finds an illustration of our construction.

The inner product of $\mathbb{R}^3$ is given by $(\boldsymbol{e}_i, \boldsymbol{e}_i) = \lambda_i$, $\lambda_i > 0$. The Gram matrix of the basis $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ is $G_{(\boldsymbol{b}_1,\boldsymbol{b}_2)} = \begin{pmatrix} \lambda_1+\lambda_2 & -\lambda_2 \\ -\lambda_2 & \lambda_2+\lambda_3 \end{pmatrix}$. This quadratic form lies in VORONOÏ's principal domain of the first type. Thus we have a 1-to-1 correspondence between the edges of the graph and the rigid forms of $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$:

edge of K_3		**rigid form**
$(1,2)$	$\leftrightarrow$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$(1,3)$	$\leftrightarrow$	$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
$(2,3)$	$\leftrightarrow$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Hence, Figure 3.3 below is a dual picture of Figure 2.10. The extreme rays of the secondary cone $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$ correspond to graphs with one edge and to one-dimensional cographical lattices whose Dirichlet-Voronoi polytopes are line segments. The two-dimensional faces correspond to graphs with two edges and to two-dimensional cographical lattices whose Dirichlet-Voronoi polytopes are quadrangles. Finally the three-dimensional face corresponds to the complete graph K_3 and to two-dimensional cographical lattices whose Dirichlet-Voronoi polytopes are hexagons.

* DELONE introduced these graphs when he studied 3-dimensional Bravais lattices in the context of positive definite quadratic forms ([Del1932])

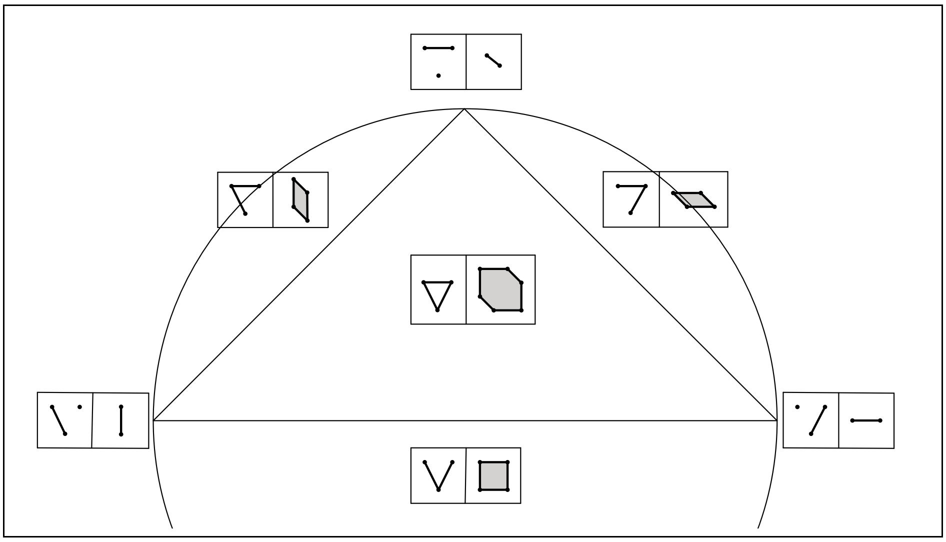


Figure 3.3. Dirichlet-Voronoi Polytopes of Cographical Lattices.

Graphical Lattices

Not all zonotopal lattices are cographical. The minimal zonotopal lattice which is not cographical is the graphical lattice $L_{\mathsf{K}_{3,3}}$ where $\mathsf{K}_{3,3}$ denotes the complete bipartite graph on 3 and 3 vertices.

Let $G = (V, E)$ be a connected graph with directed edges. We denote the set of all circuits of G by $\mathcal{C}(G)$. For every circuit C of G we define a lattice vector $\boldsymbol{v}(C) \in \{-1, 0, 1\}^E$ by

$$\boldsymbol{v}(C)_e = \begin{cases} +1, & \text{if } e \text{ is a positive edge of the circuit,} \\ -1, & \text{if } e \text{ is a negative edge of the circuit,} \\ 0, & \text{if } e \text{ is not an edge of the circuit.} \end{cases}$$

The lattice $L_G = \sum_{C \in \mathcal{C}(G)} \mathbb{Z}\boldsymbol{v}(C) \subseteq \mathbb{Z}^E$ is called *graphical*. Actually, L_G describes a set of lattices with a fixed combinatorial structure. With a corresponding inner product L_G is a zonotopal lattice of dimension $|E| - |V| + 1$.

A theorem of TUTTE ([Tut1958], [Tut1959]) characterizes graphical and cographical lattices. A zonotopal lattice is graphical if and only if it has no minor that is combinatorially isomorphic to the cographical lattice $L^*_{\mathsf{K}_5}$ or $L^*_{\mathsf{K}_{3,3}}$. Conversely, a zonotopal lattice is cographical if and only if it has no minor combinatorially isomorphic to the graphical lattices L_{K_5} or $L_{\mathsf{K}_{3,3}}$. TUTTE's theorem is a generalization of KURATOWSKI's prominent characterization for planar graphs.

Dirichlet-Voronoi polytopes of graphical lattices are combinatorially equivalent if the corresponding graphs do have the same circuits up to ordering and sign changes. WHITNEY's 2-isomorphism theorem ([Whi1933]) says when two graphs have the same circuits (up to ordering and sign changes). That is if one can transform the first graph by *Whitney flips* into the second graph. There are two types of *Whitney flips*: A 1-flip either glues two components by identifying two vertices or it decomposes a component by removing an edge which corresponds to a cocircuit whose cardinality is 1. In a 2-flip a graph is decomposed along a minimal cut whose support has cardinality 2. Let (v_1, w_1), (v_2, w_2) be the edges belonging to the cut. Then, the two parts are glued together again by the edges (v_1, w_2), (v_2, w_1).

Lattices of Type R_{10}

Lattices of type R_{10} are zonotopal lattices whose oriented matroid is isomorphic to the 5-dimensional lattice $\mathsf{R}_{10} \subseteq \mathbb{Z}^{10}$ given by the rows of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This lattice (together with the standard inner product in $\mathbb{R}^{10}$) is zonotopal. It is the smallest zonotopal lattice that is neither cographical nor graphical.

3.5.3. Delone Subdivisions, Dicings and Zonotopal Lattices

Finally, we determine Delone subdivisions of positive semidefinite quadratic forms associated to zonotopal lattices. One goal is to give a dictionary that translates our language of zonotopal lattices into the language of lattice dicings by ERDAHL ([ER1994], [Erd1999]).

We already saw that face lattices of zonotopal Dirichlet-Voronoi polytopes are face lattices of regular oriented matroids. From oriented matroid theory (see [BVSWZ1993]) we know that face lattices of realizable oriented matroids (and regular oriented matroids are realizable by definition) are face lattices of central hyperplane arrangements. A central hyperplane arrangement is a finite collection of real hyperplanes in $\mathbb{R}^d$ having the origin as common point. A central hyperplane arrangement in $\mathbb{R}^d$ gives a face lattice of a regular oriented matroid if the intersection of $d-2$ hyperplanes is contained in either two or three intersections of $d-1$ hyperplanes.

Let $L \subseteq \mathbb{Z}^n$ be a d-dimensional zonotopal lattice. Let $\mathcal{A} = (H_1, \ldots, H_n)$ be a central hyperplane arrangement which belongs to the regular oriented matroid $\mathcal{M}(L)$. We turn $\mathcal{A}$ into a periodic arrangement of hyperplanes: $(H_1 + \boldsymbol{v}_1, \ldots, H_n + \boldsymbol{v}_n)_{\boldsymbol{v}_i \in \mathbb{Z}^d}$. This gives us n families of parallel equispaced hyperplanes. By a theorem of BRYLAWSKI and LUCAS [BL1976] there exists a linear map that transforms the periodic arrangement into a *lattice dicing*, i.e. through each vertex of the periodic arrangement there goes exactly one hyperplane of each family. This subdivision of $\mathbb{R}^d$ is up to an affine isomorphism the Delone subdivision of a positive semidefinite quadratic form associated to the zonotopal lattice L we started with. In Figure 3.4 (an affine image of) this construction is illustrated.

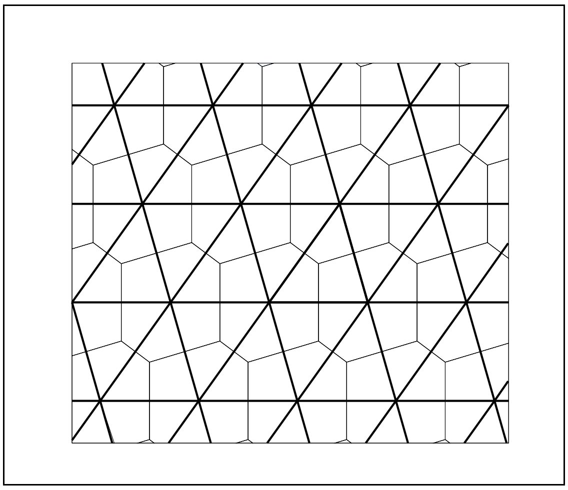


Figure 3.4. A Lattice Dicing.

Let $\mathcal{D}$ be a Delone subdivision given by a lattice dicing. Every family of parallel hyperplanes of $\mathcal{D}$ is itself a Delone subdivision of a positive semidefinite quadratic form of rank 1. Hence, the secondary cone of $\mathcal{D}$ is bounded by extreme rays of positive semidefinite quadratic forms of rank 1. Conversely, every Delone subdivision whose secondary cone is bounded by extreme rays of rank 1 is a lattice dicing. The corresponding Dirichlet-Voronoi polytope is a zonotope. A theorem of KORKINE and ZOLOTAREV (in the literature it is often referred to HELLER) implies that if the secondary cone of a Delone subdivision belonging to a zonotopal lattice has full dimension $\frac{d(d+1)}{2}$, then it belongs to the cographical lattice $L^*_{\mathsf{K}_d}$. All other secondary cones belonging to zonotopal lattices do not have full dimension. For specialists: This gives a proof of DICKSON's Theorem ([Dic1972]).

Chapter 4.

Results in Low Dimensions

In this chapter we classify Dirichlet-Voronoi polytopes of positive definite quadratic forms using the methods we described in the two previous chapters. The classification, which is equivalent to the classification of Delone subdivision of positive definite quadratic forms, is performed in two steps. First we classify all non-equivalent Delone triangulations. Then we compute the extreme rays of every secondary cone. Let $\overline{\mathbf{\Delta}(\mathcal{D})}$ be the secondary cone of a Delone triangulation $\mathcal{D}$. After we computed its extreme rays $\overline{\mathbf{\Delta}(\mathcal{D})} = \mathrm{cone}\{R_1, \ldots, R_n\}$ we find all combinatorial types of Dirichlet-Voronoi polytope of quadratic forms lying in $\overline{\mathbf{\Delta}(\mathcal{D})}$ among

$$\alpha_1 \,\mathrm{DV}(R_1, \{\mathbf{0}\}) + \cdots + \alpha_n \,\mathrm{DV}(R_n, \{\mathbf{0}\}), \quad \alpha_i \in \{0, 1\}.$$

We discuss the dimensions one, two, and three only very briefly. In each of these dimensions there only exists one non-equivalent Delone triangulation. Hence, we only have to classify co-graphical lattices. The one-dimensional case is trivial. The two-dimensional case is known since ancient times: only quadrangles and hexagons tile the plane by translates. The three-dimensional case was solved by FEDEROV in 1885 who showed that there are 5 three-dimensional polytopes that tile space by translates.

We focus on the four-dimensional case. DELONE (later corrected by STOGRIN) was the first who tried to give a classification. Here, the number of non-equivalent Delone triangulations equals 3. Using the vonorm/conorm method we succeed to give a classification of combinatorially distinct four-dimensional Dirichlet-Voronoi polytopes that can be done by hand calculations. There are 52 combinatorially distinct four-dimensional Dirichlet-Voronoi polytopes. Our approach was suggested by CONWAY who gave a complete list of classification symbols without showing its completeness and even without providing further explanations. We succeeded in the challenge of giving a combinatorial/geometrical interpretation of CONWAY's list.

In the five dimensional case we only report on a computation of all non-equivalent Delone triangulations. BARANOVSKII and RYSHKOV were the first who tried this. They found 221 non-equivalent Delone triangulations. But they missed one type which was observed by ENGEL. ENGEL and GRISHUKHIN identified the missed type. Our computations confirm their result. ENGEL reports that there are $179,372$ combinatorially distinct five-dimensional Dirichlet-Voronoi polytopes. In dimension 6 the number of non-equivalent Delone triangulations explodes. Up to now we found more than $250,000$ non-equivalent Delone triangulations.

As a first summary we have the following table:

Dimension	1	2	3	4	5	6
# Delone triangulations	1	1	1	3	222	$> 250,000$
# Dirichlet-Voronoi polytopes	1	2	5	52	$179,372$	$\gg 250,000$

4.1. Dimension 1

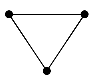
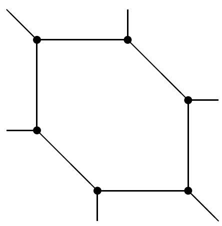
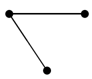
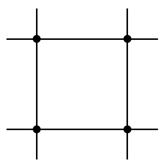
We include the trivial one-dimensional case only for the sake of completeness. Let $Q = (q_{11})$ be a unary positive definite quadratic form. The one-dimensional Delone simplices $[0,1] + \boldsymbol{v}$, $\boldsymbol{v} \in \mathbb{Z}^1$, define the Delone triangulation of Q. The Dirichlet-Voronoi polytope of an associated lattice is a line segment.

4.2. Dimension 2

In Chapter 2.3.3 we saw that every binary positive definite quadratic form is arithmetically equivalent to a from that lies in the topological closure of VORONOÏ's principal domain of the first type

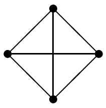
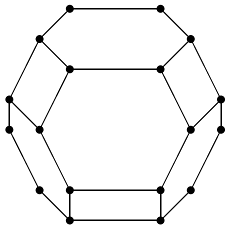
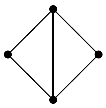
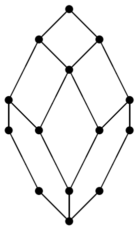
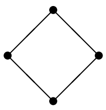
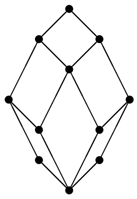
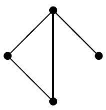
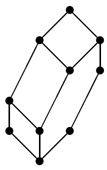
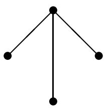
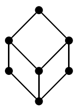
$$\begin{aligned}\overline{\boldsymbol{\Delta}(\mathcal{D}_1)} &= \{Q \in \mathcal{S}^2_{\geq 0} : q_{11} + q_{12} \geq 0, q_{12} \leq 0, q_{12} + q_{22} \geq 0\} \\ &= \operatorname{cone}\left\{\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & -1\\ -1 & 1\end{smallmatrix}\right)\right\}.\end{aligned}$$

In Chapter 3.5.2 we proved that these forms are associated to cographical lattices and that we can describe the combinatorial structure of the corresponding Dirichlet-Voronoi polytopes by connected graphs with three vertices. Hence, we have the following classification. The number d on the left side gives the dimension of the face of $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$ where the dual Delone subdivision is being realized.

d	Delone Graph	Dirichlet-Voronoi Polytope	Quadratic Form	Name
3	 K_3		$\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$	**HEXAGON**
2	 $1+1$		$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	**SQUARE**

4.3. Dimension 3

From dimension 2 to dimension 3 nothing spectacular happens. One reason for this is that by Theorem 3.1.3 three-dimensional parallelohedra are zonotopes. Like in the binary case, every ternary positive definite quadratic form is arithmetically equivalent to a form lying in VORONOÏ's principal domain of the first type. Hence, we only have to deal with cographical lattices and we can classify all Dirichlet-Voronoi polytopes of three-dimensional lattice by connected graphs with four vertices.

d	Delone Graph	Polytope	Form	Name
6	 K_4		$\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$	**TRUNCATED OCTAHEDRON**
5	 $\mathsf{K}_4 - 1$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	**HEXA-RHOMBIC DODECAHEDRON**
4	 C_4		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	**RHOMBIC DODECAHEDRON**
4	 $\mathsf{K}_3 + 1$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**HEXAGONAL PRISM**
3	 $1+1+1$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	**CUBE**

We adopted the way of drawing the five parallelohedra from [CS1992]. The procedure of edge deletion becomes visible. First, the Russian crystallographer FEDEROV gave a complete classification of three-dimensional parallelohedra. He and his work was extremely influential for VORONOÏ's memoir [Vor1908].

4.4. Dimension 4

In [Del1929] DELONE tried to prove that there are 51 different combinatorial types of Dirichlet-Voronoi polytopes of four-dimensional lattices. But he missed one type which was found by STOGRIN later in [Sto1973]*. There are 52 different combinatorial types. From these 17 types are zonotopes and in the other 35 types the 24-cell appears as Minkowski summand.

First of all, DELONE proves in [Del1929] VORONOÏ's conjecture for four-dimensional parallelohedra by a skillful use of Schlegel diagrams. Then he tries to enumerate all Dirichlet-Voronoi polytopes in dimension 4 by investigating the faces of the secondary cones of the three non-equivalent four-dimensional Delone triangulation.

DELONE writes in [Del1929] on page 161:

> G. VORONOJ a démontré dans le remarquable mémoire mentionné plus haut (v. l'introduction) qu'il n'existe que 3 domaines de Dirichlet primitifs dans l'espace à 4 dimensions ; les paralléloèdres 1, 2 et 3 [...] sont donc ces domaines. Il est facile de démontrer que chaque domaine de Dirichlet qui n'est pas primitifs peut être obtenu d'un domaine primitifs, si l'on fait disparaître dans ce domaine certaines arêtes.
> [...]
> On obtiendrait ainsi 3072 paralléloèdres. Mais parmi ces paralléloèdres il peut se trouver des paralléloèdres identiques. Cette identité ne peut pas toujour être tout de suite remarquée au moyen de la comparaison des figures correspondantes, parce que ces figures ne représentent que de projections et celles-ci peuvent être des projections différentes d'un même paralléloèdre. En me servant de quelques symétries particulières et d'autres méthodes particulières, dans le détail desquelles je ne veux pas entrer ici, j'ai trouvé le résultat final suivant :
>
> **THÉORÈME III : «Il existe 51 et seulement 51 partitions différentes de l'espace à 4 dimensions ».**

The statement above points out that it will remain unclear why DELONE missed one type. Here, we will use DELONE's approach but we will give many details and natural representatives of the relevant positive definite quadratic forms. At the same time we give a geometrical and combinatorial interpretation of the classification symbols CONWAY used in [Con1997]. There he showed how to exploit the symmetry of the conorms of four-dimensional lattices to give a complete list of four-dimensional Dirichlet-Voronoi polytopes. But CONWAY only gave a list without proving its completeness.

By using a method called "zone reduction" and with help of computer programs ENGEL gives another classification of four-dimensional Dirichlet-Voronoi polytopes in [Eng1992]. At the end of our four-dimensional journey we will compare his classification with DELONE's and the one we obtained.

4.4.1. Four-Dimensional Delone Triangulations

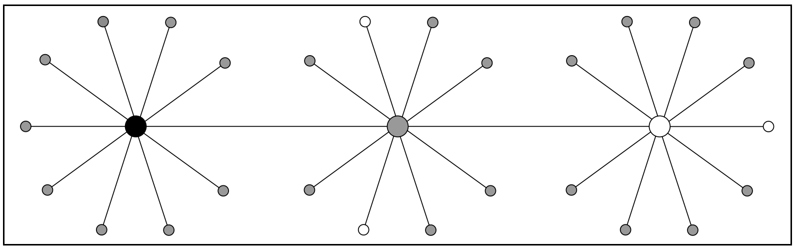
Using VORONOÏ's algorithm (Algorithm 1 in Chapter 2.5) we can list† all three non-equivalent Delone triangulations $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ in dimension 4. We start with the Delone triangulation $\mathcal{D}_1$ of

*Actually, the article provides much more than just the omitted combinatorial type. It shows how one can use and extend the theory of Dirichlet-Voronoi polytopes to understand three-dimensional crystallographic groups.

†VORONOÏ performed this computation at the end of his memoir [Vor1908].

VORONOÏ's principal form of the first type. The triangulation $\mathcal{D}_1$ has ten bistellar neighbours, all equivalent to $\mathcal{D}_2$. Among the bistellar neighbours of $\mathcal{D}_2$ we find $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. Whereas we find among the bistellar neighbours of $\mathcal{D}_3$ only $\mathcal{D}_2$ and $\mathcal{D}_3$ and no triangulation equivalent to $\mathcal{D}_1$. The simplices of the Delone triangulations $\mathcal{D}_2$ and $\mathcal{D}_3$ are listed in Chapter 8.4.2, and the simplices of $\mathcal{D}_1$ in Chapter 2.3.

With VORONOÏ's algorithm we also get an infinite tree of secondary cones with three non-equivalent nodes. The tree shows the combinatorial structure of the tiling of $\mathcal{S}^4_{>0}$ by secondary cones. It is represented by the following diagram where we factored out the $\mathsf{GL}_4(\mathbb{Z})$ action.



In the diagram the black node corresponds to secondary cones equivalent to $\mathbf{\Delta}(\mathcal{D}_1)$, the grey node corresponds to secondary cones equivalent to $\mathbf{\Delta}(\mathcal{D}_2)$ and the white node to secondary cones equivalent to $\mathbf{\Delta}(\mathcal{D}_3)$. Two nodes are connected by an edge if and only if the secondary cones have a facet in common (if and only if the Delone triangulations are bistellar neighbours). In the infinite tree every black node is surrounded by ten grey nodes, every grey node is surrounded by one black, six grey and three white nodes, and every white node is surrounded by nine grey nodes and one white node. The group $\mathsf{GL}_4(\mathbb{Z})$ is acting on the tree.

We give an explicit description of representatives for the three non-equivalent secondary cones. We specify these three polyhedral cones by their facet-defining hyperplanes and by their extreme rays.

Extreme Rays

By R_i, $i = 1, \ldots, 12$, we denote the following positive semidefinite quadratic forms that are the extreme rays of the secondary cones $\mathbf{\Delta}(\mathcal{D}_i)$, $i = 1, 2, 3$. Notice that our list of extreme rays implies that up to isomorphism there is only one rigid positive definite quadratic form in dimension 4. It is associated to the root lattice D_4 whose Dirichlet-Voronoi polytope is the 24-cell. Since we only consider symmetric matrices it suffices to give the lower triangular entries. We do this purely because of aesthetical reasons.

$$R_1 = \begin{pmatrix} 1 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_3 = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_5 = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_6 = \begin{pmatrix} 1 & & & \\ 0 & 0 & & \\ -1 & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_7 = \begin{pmatrix} 1 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ -1 & 0 & 0 & 1 \end{pmatrix}, R_8 = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & -1 & 1 & \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_9 = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

$$R_{10} = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{pmatrix}, R_{11} = \begin{pmatrix} 4 & & & \\ 2 & 4 & & \\ -2 & -2 & 4 & \\ -2 & -2 & 0 & 4 \end{pmatrix}, \quad R_{12} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & -1 & 1 & \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

The Black Triangulation $\mathcal{D}_1$

The secondary cones which is represented by the big black node is given by the inequalities

$$\begin{aligned} q_{21} &< 0 \\ q_{31} &< 0 \\ q_{41} &< 0 \\ q_{42} &< 0 \\ q_{43} &< 0 \\ q_{11}+q_{21}+q_{31}+q_{41} &> 0 \\ q_{21}+q_{22}+q_{32}+q_{42} &> 0 \\ q_{31}+q_{32}+q_{33}+q_{43} &> 0 \\ q_{41}+q_{42}+q_{43}+q_{44} &> 0 \end{aligned}$$

or equivalently by the extreme rays $\boldsymbol{\Delta}(\mathcal{D}_1) = \operatorname{int}(\operatorname{cone}\{R_1, \dots, R_{10}\})$.

The Grey Delone Triangulation $\mathcal{D}_2$

The one which is represented by the big grey node is given by the inequalities

$$\begin{aligned} q_{21} &> 0 \\ q_{21}+q_{31} &< 0 \\ q_{21}+q_{32} &< 0 \\ q_{21}+q_{41} &< 0 \\ q_{21}+q_{42} &< 0 \\ q_{43} &< 0 \\ q_{11}+q_{31}+q_{41} &> 0 \\ q_{22}+q_{32}+q_{42} &> 0 \\ q_{31}+q_{32}+q_{33}+q_{43} &> 0 \\ q_{41}+q_{42}+q_{43}+q_{44} &> 0 \end{aligned}$$

or equivalently by the extreme rays $\boldsymbol{\Delta}(\mathcal{D}_2) = \operatorname{int}(\operatorname{cone}\{R_1, \dots, R_4, R_6, \dots, R_{11}\})$.

The White Delone Triangulation $\mathcal{D}_3$

The one which belongs to the big white node is given by the inequalities

$$\begin{aligned} q_{21}-q_{43} &> 0 \\ q_{21}+q_{31} &< 0 \\ q_{21}+q_{32} &< 0 \\ q_{21}+q_{41} &< 0 \\ q_{21}+q_{42} &< 0 \\ q_{43} &> 0 \\ q_{11}+q_{31}+q_{41}+q_{43} &> 0 \\ q_{22}+q_{32}+q_{42}+q_{43} &> 0 \\ q_{31}+q_{32}+q_{33}+q_{43} &> 0 \end{aligned}$$

$$q_{41} + q_{42} + q_{43} + q_{44} \quad > \quad 0$$

or equivalently by the extreme rays $\mathbf{\Delta}(\mathcal{D}_3) = \operatorname{int}(\operatorname{cone}\{R_1, \ldots, R_4, R_6, \ldots, R_9, R_{11}, R_{12}\})$.

The Reduction Theory of CHARVE and HOFREITER

In the last section we gave a fundamental domain of $\mathcal{S}^4_{>0}/\mathsf{GL}_4(\mathbb{Z})$ that is divided into three polyhedral cones. So it is natural to distinguish between three different types of positive definite quadratic forms in four variables. Now the following question arises: Given a positive definite quadratic form $Q \in \mathcal{S}^4_{>0}$, to which of the three types does Q belong? Using a reduction theory of CHARVE and HOFREITER that generalizes SELLING's reduction theory we can answer this question algorithmically. We refer the interested reader to the original papers [Cha1882], [Hof1933], and to §117 of [Vor1908].

4.4.2. Vonorms and Conorms in Dimension 4

In this section we show that the conjecture of CONWAY and SLOANE (see Chapter 3.4) is true for quaternary positive definite quadratic forms. We show that the vonorms, respectively the conorms, characterize the arithmetical equivalence classes of quaternary positive definite quadratic forms.

It is more convenient to use the conorms because the conorms of the forms $R_1, \ldots, R_{10}, R_{12}$ differ only in one non-trivial character from zero where the character 0000 is the trivial character. We have

$$\begin{array}{llllllllll} & \operatorname{co}_{R_1}(1000) & = & \operatorname{co}_{R_2}(0100) & = & \operatorname{co}_{R_3}(0010) & = & \operatorname{co}_{R_4}(0001) & = & \operatorname{co}_{R_5}(1100) \\ = & \operatorname{co}_{R_6}(1010) & = & \operatorname{co}_{R_7}(1001) & = & \operatorname{co}_{R_8}(0110) & = & \operatorname{co}_{R_9}(0101) & = & \operatorname{co}_{R_{10}}(0011) \\ = & \operatorname{co}_{R_{12}}(1111) & = & 1. & & & & & & \end{array}$$

Here we only listed the non-trivial non-zero conorms. For the positive definite quadratic form R_{11} which is associated to a scaled version of the root lattice D_4 we have

$$\begin{array}{lll} \operatorname{co}_{R_{11}}(\chi) & = & -1, \quad \text{if } \chi \in \{1100, 1011, 0111\}, \\ \operatorname{co}_{R_{11}}(\chi) & = & 1, \qquad \text{otherwise.} \end{array}$$

To proof the conjecture of CONWAY and SLOANE in dimension 4 we have to check the following. Let $Q_i \in \overline{\mathbf{\Delta}(\mathcal{D}_i)}$, $Q_j \in \overline{\mathbf{\Delta}(\mathcal{D}_j)}$, $i \neq j$ be two positive definite quadratic forms whose conorm maps coincide, then the forms coincide, too. In particular they lie on the common boundary of $\overline{\mathbf{\Delta}(\mathcal{D}_i)}$ and $\overline{\mathbf{\Delta}(\mathcal{D}_j)}$. We can consider conorms instead of vonorms since the vonorm map can be reconstructed from the vonorm map.

We have to distinguish between three cases. We give the complete arguments for the case $i = 1$, $j = 2$ only. The other two cases work out in the same manner. Suppose for $\alpha_1, \ldots, \alpha_{10} \in \mathbb{R}_{\geq 0}$, $\beta_1, \ldots, \beta_4, \beta_6, \ldots, \beta_{11} \in \mathbb{R}_{\geq 0}$ we have

$$\alpha_1 \operatorname{co}_{R_1} + \cdots + \alpha_{10} \operatorname{co}_{R_{10}} = \beta_1 \operatorname{co}_{R_1} + \cdots + \beta_4 \operatorname{co}_{R_4} + \beta_6 \operatorname{co}_{R_6} + \cdots + \beta_{11} \operatorname{co}_{R_{11}} .$$

When we plug in $\chi = 1011$, the left hand side vanishes, the right hand side equals $-\beta_{11}$. Hence, $\beta_{11} = 0$. When we plug in $\chi = 1100$, the left hand side equals α_5 and the right hand side vanishes. Hence, $\alpha_5 = 0$. This shows that every pair of positive definite quadratic forms satisfying the equation above lies on the common boundary of $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$ and $\overline{\mathbf{\Delta}(\mathcal{D}_2)}$.

4.4.3. Towards a Classification

Now we are nearly ready to give a classification of all combinatorial types of four-dimensional Dirichlet-Voronoi polytopes. The idea is very simple: For every face of the three secondary cones given above the Dirichlet-Voronoi polytope of the associated lattices can be computed and afterwards all of them can be put into equivalence classes according to their combinatorial structure.

In principle we only have to consider the following (≤ 3072) quadratic forms

$$\alpha_1 R_1 + \cdots + \alpha_{10} R_{10} \in \overline{\mathbf{\Delta}(\mathcal{D}_1)}, \quad \alpha_i \in \{0,1\},$$
$$\beta_1 R_1 + \cdots + \beta_4 R_4 + \beta_6 R_6 + \cdots + \beta_{11} R_{11} \in \overline{\mathbf{\Delta}(\mathcal{D}_2)}, \quad \beta_i \in \{0,1\},$$
$$\gamma_1 R_1 + \cdots + \gamma_4 R_4 + \gamma_6 R_6 + \cdots + \gamma_9 R_9 + \gamma_{11} R_{11} + \gamma_{12} R_{12} \in \overline{\mathbf{\Delta}(\mathcal{D}_3)}, \quad \gamma_i \in \{0,1\}.$$

But among these forms many give Dirichlet-Voronoi polytopes of the same combinatorial structure. In the following we look at invariants to simplify the isomorphism tests. Then, we only need computations which we can do by hand without using a computer. First, we are distinguishing between zonotopal and non-zonotopal Dirichlet-Voronoi polytopes. The forms $\sum_{i=1}^{10} \alpha_i R_i$, $\alpha_i \in \{0,1\}$, and the form $R_1 + \cdots + R_4 + R_6 + \cdots + R_9 + R_{12}$ give zonotopal Dirichlet-Voronoi polytopes which can be classified with method from the theory of zonotopal parallelohedra we introduced in Chapter 3.5. The Dirichlet-Voronoi polytopes of the forms $R_{11} + \delta_1 R_1 + \cdots \delta_4 R_4 + \delta_6 R_6 + \cdots + \delta_{10} R_{10} + \delta_{12} R_{12}$, $\delta_i \in \{0,1\}$ have the 24-cell as a Minkowski summand and so they cannot be zonotopes. The four-dimensional non-zonotopal Dirichlet-Voronoi polytopes can be classified by special diagrams of CONWAY. In [Con1997] CONWAY gives a list of all four-dimensional Dirichlet-Voronoi polytopes without proving its completeness. Our computations are inspired by his (after-)thoughts and we are using his classification symbols. In some sense we give the geometric justification for his classification symbols. So, how does CONWAY feel the form of a four-dimensional lattice?

4.4.4. Diagrams for Zonotopal Cases

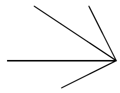
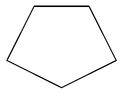
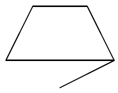
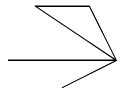
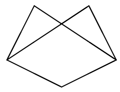
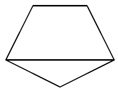
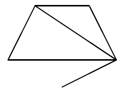
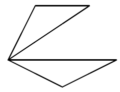
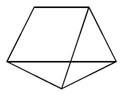
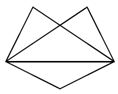
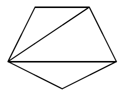
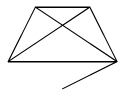
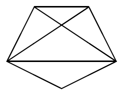
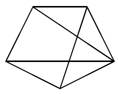
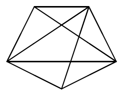
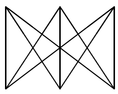
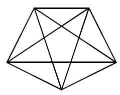
We have seen in the last chapter that all but one four-dimensional zonotopal Dirichlet-Voronoi polytopes are cographical and the exceptional one is the Dirichlet-Voronoi polytope of the graphical lattice $L_{\mathsf{K}_{3,3}}$. This lattice is associated to the positive definite quadratic forms that lie in the relative interior of the nine-dimensional $\operatorname{cone}\{R_1, \ldots, R_4, R_6, \ldots, R_9, R_{12}\}$.

By listing all connected subgraphs of the complete graph on five vertices K_5 we find all cographical lattices. Sequence A001349 in SLOANE's On-Line Encyclopedia of Integer Sequences tells us that there are 21 non-isomorphic connected graphs with five vertices. Two graphs determine the same combinatorial type of Dirichlet-Voronoi polytope if and only if they have the same set of cocircuits. The 2-isomorphism theorem of WHITNEY we described in Section 3.5.2 gives a necessary and sufficient criterion for two graphs on five vertices which define the same set of cocircuits.

So we get the following classification of the zonotopal Dirichlet-Voronoi polytopes in four dimensions. The number on the left side is the dimension of the reduction domain's face where the corresponding combinatorial type is being realized.

The list of CONWAY contains an error[‡]: His type K_4 should be replaced by $\mathsf{C}_{221} + 1$.

[‡]This fact was pointed out to me by DEZA and GRISHUKHIN

4	 $1+1+1+1$			
5	 C_5	 C_4+1	 C_3+1+1	
6	 C_{222}	 C_{321}	 $\mathsf{C}_{221}+1$	 $\mathsf{C}_3+\mathsf{C}_3$
7	 K_5-2-1	 C_{2221}	 K_5-3	 K_4+1
8	 K_5-2	 K_5-1-1		
9	 K_5-1	 $\mathsf{K}_{3,3}$		
10	 K_5			

4.4.5. Diagrams for Non-Zonotopal Cases

For the non-zonotopal cases we have to study the faces of the polyhedral cones $\overline{\mathbf{\Delta}(\mathcal{D}_2)}$ and $\overline{\mathbf{\Delta}(\mathcal{D}_3)}$ which contain R_{11}. Assume that there is a unimodular transformation $A \in \mathsf{GL}_4(\mathbb{Z})$ that transforms one face

$$\mathbf{F}_1 = \operatorname{cone}\{\alpha_1 R_1, \dots, \alpha_4 R_4, \alpha_6 R_6, \dots, \alpha_{10} R_{10}, R_{11}, \alpha_{12} R_{12}\},\ \alpha_i \in \{0,1\},\ \alpha_{10} \neq \alpha_{12},$$

into another face

$$\mathbf{F}_2 = \operatorname{cone}\{\beta_1 R_1, \dots, \beta_4 R_4, \beta_6 R_6, \dots, \beta_{10} R_{10}, R_{11}, \beta_{12} R_{12}\},\ \beta_i \in \{0,1\},\ \beta_{10} \neq \beta_{12},$$

i.e. the faces $\mathbf{F}_1$ and $\mathbf{F}_2$ are arithmetically equivalent. Then the corresponding forms

$$Q_1 = \alpha_1 R_1 + \cdots + \alpha_4 R_4 + \alpha_6 R_6 + \cdots + \alpha_{10} R_{10} + R_{11} + \alpha_{12} R_{12}$$
$$Q_2 = \beta_1 R_1 + \cdots + \beta_4 R_4 + \beta_6 R_6 + \cdots + \beta_{10} R_{10} + R_{11} + \beta_{12} R_{12}$$

are arithmetically equivalent because the conorm functions characterize arithmetical equivalence classes of quaternary positive definite quadratic forms. Furthermore, the Dirichlet-Voronoi polytopes of Q_1 and Q_2 are affinely equivalent, and the Dirichlet-Voronoi polytopes of forms lying in $\mathbf{F}_1$ and $\mathbf{F}_2$ are combinatorially equivalent. Conversely, suppose that the forms Q_1, Q_2 are arithmetically equivalent. Then the corresponding faces $\mathbf{F}_1$, $\mathbf{F}_2$ are arithmetically equivalent, too.

Furthermore, the Dirichlet-Voronoi polytopes of quadratic forms lying in $\mathbf{F}_1$ and $\mathbf{F}_2$ are combinatorially equivalent. We will classify the faces of $\overline{\mathbf{\Delta}(\mathcal{D}_2)}$ and $\overline{\mathbf{\Delta}(\mathcal{D}_3)}$ containing R_{11} according to their arithmetical equivalence classes. A priori this gives a finer classification as if we would classify them according to the combinatorial structure of the corresponding Dirichlet-Voronoi polytopes. Later, we will see that both relations actually coincide.

Now we are going to introduce the diagrams for the non-zonotopal cases. We arrange all characters in the following two-dimensional array. Here we are using the representation of the characters $\chi : \mathbb{Z}^4/2\mathbb{Z}^4 \to \{\pm 1\}$ by elements of $\mathbb{F}_2^4$ we introduced in Section 3.4.

0000	1100	1011	0111
1110	0010	0101	1001
1101	0001	0110	1010
0011	1111	1000	0100

In the first row we have the elements of the subspace $U = \{0000, 1100, 1011, 0111\}$. In the other rows we find the other cosets of $\mathbb{F}_2^4/U$. Later we will see the reason for using this particular arrangement: The automorphism group of the rigid form R_{11} acts on the characters and leaves the subspace U fixed.

For the positive definite quadratic form

$$Q_1 = \alpha_1 R_1 + \cdots + \alpha_4 R_4 + \alpha_6 R_6 + \cdots + \alpha_{10} R_{10} + R_{11}, \quad \alpha_i \in \{0, 1\},$$

which lies in $\overline{\mathbf{\Delta}(\mathcal{D}_2)}$, we determine the conorms and arrange them in the same way as the characters. In Section 4.4.2 we already determined the conorms of $R_1, \ldots, R_{12}$ and by Proposition 3.4.3 we only have to add them up in the right fashion. We do not care about the value of co_{Q_1} of the trivial character 0000.

$$\mathrm{co}_{Q_1}\left(\begin{array}{|c|c|c|c|}\hline 0000 & 1100 & 1011 & 0111 \\ \hline 1110 & 0010 & 0101 & 1001 \\ \hline 1101 & 0001 & 0110 & 1010 \\ \hline 0011 & 1111 & 1000 & 0100 \\ \hline \end{array}\right) = \left(\begin{array}{|c|c|c|c|}\hline * & -1 & -1 & -1 \\ \hline 1 & \alpha_3+1 & \alpha_9+1 & \alpha_7+1 \\ \hline 1 & \alpha_4+1 & \alpha_8+1 & \alpha_6+1 \\ \hline \alpha_{10}+1 & 1 & \alpha_1+1 & \alpha_2+1 \\ \hline \end{array}\right)$$

For the quadratic form

$$Q_2 = \beta_1 R_1 + \cdots + \beta_4 R_4 + \beta_6 R_6 + \cdots + \beta_9 R_9 + R_{11} + \beta_{12} R_{12}, \quad \beta_i \in \{0, 1\},$$

lying in $\overline{\mathbf{\Delta}(\mathcal{D}_3)}$ we do the same:

$$\mathrm{co}_{Q_2}\left(\begin{array}{|c|c|c|c|}\hline 0000 & 1100 & 1011 & 0111 \\ \hline 1110 & 0010 & 0101 & 1001 \\ \hline 1101 & 0001 & 0110 & 1010 \\ \hline 0011 & 1111 & 1000 & 0100 \\ \hline \end{array}\right) = \left(\begin{array}{|c|c|c|c|}\hline * & -1 & -1 & -1 \\ \hline 1 & \beta_3+1 & \beta_9+1 & \beta_7+1 \\ \hline 1 & \beta_4+1 & \beta_8+1 & \beta_6+1 \\ \hline 1 & \beta_{12}+1 & \beta_1+1 & \beta_2+1 \\ \hline \end{array}\right)$$

We see that in these cases the conorms of 1100, 1011 0111 are always -1 and the conorms of 1110, 1101 are always 1. The other non-trivial conorms are either 1 or 2 according to the following structure of rigid rank-1-forms.

	R_3	R_9	R_7
	R_4	R_8	R_6
R_{10}	R_{12}	R_1	R_2

To represent e.g. the positive definite quadratic form $Q = R_1 + R_2 + R_4 + R_6 + R_7 + R_{11}$ we will use the *diagram*

	○	○	○
○			●
○	●		●
		●	●

The white dots indicate that the conorms of 1100, 1011, 0111 are always -1 and that the conorms of 1110, 1101 are always 1. The black dots indicate that the conorms of 1001, 0001, 1010, 1000, 0100 equal 2 (respectively that we have the rank-1-summands R_7, R_4, R_6, R_1, R_2). All non-trivial conorms we have not mentioned so far are 1.

When do two diagrams describe arithmetically equivalent positive definite quadratic forms? A positive definite quadratic form that is represented by a diagram has always R_{11} as a summand. If two diagrams represent arithmetically equivalent forms, then the form differ by a modular transformation that leaves R_{11} fixed. The automorphism group of R_{11} is the Weyl group of the root system F_4 (see e.g. [Bou1968]). It has order 1152 and is generated by the matrices

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ G_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 \end{pmatrix},\ G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For example, for $Q' = G_1^t Q G_1$ we have

$$Q' = \begin{pmatrix} 6 & & & \\ 2 & 5 & & \\ -2 & -3 & 6 & \\ -3 & -2 & -1 & 7 \end{pmatrix}$$

and the corresponding diagram

	○	○	○
○			●
○	●	●	
●		●	

Exercise 4.4.1. Write Q' in the form $Q' = \sum_{i=1}^{12} \alpha_i R_i$ and determine the conorms of Q'.

Now we see how the diagram changes when we perform the transformation G_1. We project G_1 into $\mathsf{GL}_4(\mathbb{F}_2)$ by reducing the matrix entries modulo 2. Then, we interpret the new matrix as a linear map operating from the right on the row space of characters. We get the following transformations of the black dots

$$1001 \mapsto 1001,\quad 0001 \mapsto 1000,\quad 1010 \mapsto 0011,\quad 1000 \mapsto 0001.$$

We have to change from column space to row space because we are dealing with the conorm map that essentially is the Fourier transform of the vonorm map. By projecting the whole automorphism group of R_{11} into $\mathsf{GL}_4(\mathbb{F}_2)$ by reducing the matrix entries modulo 2 we get the group of all linear transformations that leave the subspace $U = \mathbb{F}_2(1100) + \mathbb{F}_2(1011) + \mathbb{F}_2(0111)$ fixed. It is easy to count that this group has order $576 = 1152/2$.

Lemma 4.4.2. Two diagrams describe arithmetically equivalent positive definite quadratic forms if and only if there is a linear map $\varphi : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ which leaves the subspace $U = \mathbb{F}_2(1100) + \mathbb{F}_2(1011) + \mathbb{F}_2(0111)$ fixed and which transforms one diagram into the other one.

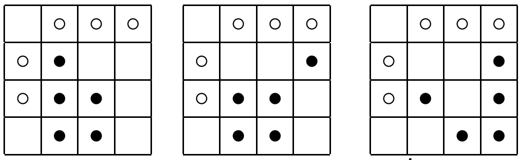
Whenever the numbers of black dots differ in two diagrams, these diagrams cannot specify arithmetically equivalent forms. We define two linear transformations $\varphi_i : \mathbb{F}_2^4 \to \mathbb{F}_2^4$, $i = 1, 2$, by

$$\varphi_1 : \begin{cases} 1100 & \mapsto & 1100 \\ 0111 & \mapsto & 0111 \\ 0010 & \mapsto & 0001 \\ 0001 & \mapsto & 0010 \end{cases}, \quad \varphi_2 : \begin{cases} 1100 & \mapsto & 1100 \\ 0111 & \mapsto & 0111 \\ 0011 & \mapsto & 0001 \\ 0001 & \mapsto & 0011 \end{cases}.$$

The first map leaves the first and the last row of a diagram fixed and interchanges the second with the third row. The second map leaves the first and second row fixed and interchanges the third with the fourth row. In this way we can arrange the black dots so that their number does not decrease from row to row. Let p be the number of squares without a black dot in the second row, q be the number of squares without a black dot in the third row, and r be the number of squares without a black dot in the last row. Then we have $p \geq q \geq r$. For (p, q, r) we have twenty possibilities:

$$(4,4,4), (4,4,3), (4,4,2), (4,4,1), (4,3,3), (4,3,2), (4,3,1), (4,2,2), (4,2,1), (4,1,1),$$
$$(3,3,3), (3,3,2), (3,3,1), (3,2,2), (3,2,1), (3,1,1), (2,2,2), (2,2,1), (2,1,1), (1,1,1).$$

But these triples do not characterize arithmetical equivalence classes. We will decorate the triples with $+$, $-$ or primes to get symbols which characterize the classes. We introduce the decorations by an example which is the same CONWAY used in [Con1997]. We show that the following three diagrams of type $(3, 2, 2)$ give three pairwise non-equivalent positive definite quadratic forms.



$322+$ $\qquad$ $322-$ $\qquad$ $322'- = 322'$

We denote the characters which belong to the squares without black dots by δ_1, δ_2, δ_3 (for the second row), ε_1, ε_2 (for the third row) and ζ_1, ζ_2 (for the last row). There are 12 different sums of the form $\delta_i + \varepsilon_j + \xi_k$. In the first diagram these 12 sums are

$$\begin{array}{rcccl}
\delta_1+\varepsilon_1+\zeta_1 &=& 1110+1101+0011 &=& 0000\\
\delta_1+\varepsilon_1+\zeta_2 &=& 1110+1101+0100 &=& 0111\\
\delta_1+\varepsilon_2+\zeta_1 &=& 1110+1010+0011 &=& 0111\\
\delta_1+\varepsilon_2+\zeta_2 &=& 1110+1010+0100 &=& 0000\\
\delta_2+\varepsilon_1+\zeta_1 &=& 0101+1101+0011 &=& 1011\\
\delta_2+\varepsilon_1+\zeta_2 &=& 0101+1101+0100 &=& 1100\\
\delta_2+\varepsilon_2+\zeta_1 &=& 0101+1010+0011 &=& 1100\\
\delta_2+\varepsilon_2+\zeta_2 &=& 0101+1010+0100 &=& 1011\\
\delta_3+\varepsilon_1+\zeta_1 &=& 1001+1101+0011 &=& 0111\\
\delta_3+\varepsilon_1+\zeta_2 &=& 1001+1101+0100 &=& 0000\\
\delta_3+\varepsilon_2+\zeta_1 &=& 1001+1010+0011 &=& 0000\\
\delta_3+\varepsilon_2+\zeta_2 &=& 1001+1010+0100 &=& 0111.
\end{array}$$

These sums yield 4 times 0000, whereas in the second diagram only 2 of these sums yield 0000, namely

$$\begin{array}{rcccl}
\delta_1+\varepsilon_1+\zeta_1 &=& 1110+1101+0011 &=& 0000\\
\delta_1+\varepsilon_2+\zeta_2 &=& 1110+1010+0100 &=& 0000.
\end{array}$$

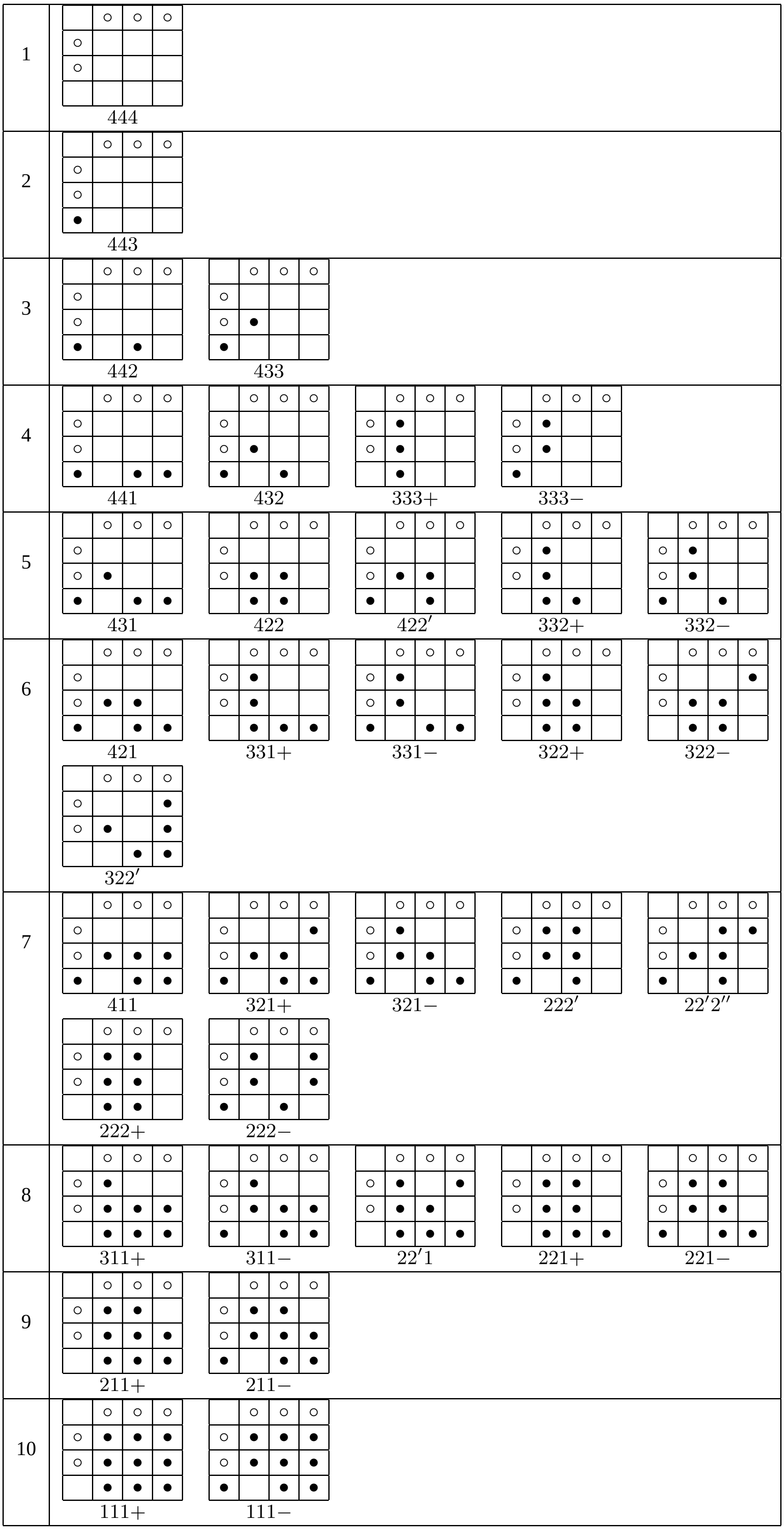
So the first and the second diagram cannot be isomorphic. In general, we append $+$ or $-$ if $\delta_i + \varepsilon_j + \zeta_k$ gives more or less than the "expected" number $pqr/4$ times 0000. But the third diagram is also of type $322-$ although it is not isomorphic to the second. Notice that the rows of the diagram are the cosets of $\mathbb{F}_2^4/(\mathbb{F}_2(1100) + \mathbb{F}_2(1011))$. Hence, every $A \in \mathsf{GL}_4(\mathbb{F}_2)$ which transforms the second diagram into the third diagram also transforms the sets

$$\{\varepsilon_1, \varepsilon_2\} = \{1101, 1010\}, \{\zeta_1, \zeta_2\} = \{0011, 0100\}$$

of the second diagram into the sets

$$\{\varepsilon_1, \varepsilon_2\} = \{1101, 0110\}, \{\zeta_1, \zeta_2\} = \{0011, 1111\}$$

of the third diagram. This is not possible because in the second diagram we have $\varepsilon_1 + \varepsilon_2 = \zeta_1 + \zeta_2 = 0111$ and in the third one we have $1011 = \varepsilon_1 + \varepsilon_2 \neq \zeta_1 + \zeta_2 = 1100$. In general, we append a prime to a 2 in the symbol if in the symbol there are at least two 2's and if the corresponding sums of characters are different. In the last case we could reduce the symbol $322'-$ to $322'$ without loosing the uniqueness of the symbol. In the following we will use this reduction whenever possible. Finally we can classify all non-zonotopal Dirichlet-Voronoi polytopes in four dimensions. We give the complete classification on the next page.



1
444
2
443
3
442
433
4
441
432
333+
333−
5
431
422
422′
332+
332−
6
421
331+
331−
322+
322−
322′
7
411
321+
321−
222′
22′2″
222+
222−
8
311+
311−
22′1
221+
221−
9
211+
211−
10
111+
111−

4.4.6. More Data

CONWAY	DELONE	ENGEL	f_0	f_1	f_2	f_3	group order
K_5	1.	$30-2$	120	240	150	30	240
K_5-1	4.	$28-4$	96	198	130	28	24
$K_{3,3}$	19.	$30-1$	102	216	144	30	144
K_5-2	6.	$24-16$	72	150	102	24	16
K_5-1-1	5.	$26-8$	78	168	116	26	16
K_5-2-	7.	$24-12$	60	134	98	24	16
C_{2221}	11.	$22-2$	54	116	84	22	96
K_5-3	10.	$20-3$	54	114	80	20	16
K_4+1	8.	$16-1$	48	96	64	16	96
C_{222}	9.	$22-1$	46	108	84	22	96
C_{321}	12.	$20-2$	24	94	72	20	24
K_4	13.	$14-2$	36	74	52	14	32
C_3+C_3	16.	$12-1$	36	72	48	12	288
C_5	14.	$20-1$	30	70	60	20	240
C_4+1	15.	$14-1$	28	62	48	14	96
C_3+1+1	17.	$10-1$	24	48	34	10	96
$C_{1+1+1+1}$	18.	$8-1$	16	32	24	8	384
$111+$	3.	$30-4$	120	240	150	30	72
$111-$	2.	$30-3$	120	240	150	30	24
$211+$	21.	$28-6$	104	212	136	28	8
$211-$	20.	$28-5$	104	212	136	28	8
$311+$	24.	$28-3$	94	198	132	28	8
$311-$	23.	$28-2$	94	198	132	28	24
$22'1$	22.	$26-10$	88	184	122	26	4
$221+$	26.	$26-9$	88	184	122	26	16
$221-$	25.	$26-11$	88	184	122	26	16
411	29.	$28-1$	88	192	132	28	24
$321+$	28.	$26-6$	78	170	118	26	4
$321-$	27.	$26-7$	78	170	118	26	4
$222'$	30.	$24-18$	72	156	108	24	8
$22'2''$	32.	$24-19$	72	156	108	24	24
$222+$	31.	$24-17$	72	156	108	24	96
$222-$	33.	$24-20$	72	156	108	24	96
421	36.	$26-5$	72	164	118	26	8
$331+$	45.	$26-3$	68	156	114	26	24
$331-$	34.	$26-4$	68	156	114	26	8
$322+$	44.	$24-14$	62	142	104	24	16
$322-$	39.	$24-15$	62	142	104	24	16
$322'$	43.	$24-13$	62	142	104	24	4
431	35.	$26-2$	62	150	114	26	12
422	37.	$24-11$	56	136	104	24	32
$422'$	38.	$24-10$	56	136	104	24	16
$332+$	41.	$24-8$	52	128	100	24	8
$332-$	46.	$24-9$	52	128	100	24	8
441	40.	$26-1$	56	144	114	26	96
432	42.	$24-7$	46	122	100	24	8
$333+$		$24-6$	42	114	96	24	24
$333-$	47.	$24-5$	42	114	96	24	72
442	48.	$24-4$	40	116	100	24	64
433	49.	$24-3$	36	108	96	24	24
443	50.	$24-2$	30	102	96	24	96
444	51.	$24-1$	24	96	96	24	1152

In the previous table we compared our classification of four-dimensional Dirichlet-Voronoi polytopes to the already existing classifications of DELONE and ENGEL. We summarized CONWAY's, DELONE's and ENGEL's classification symbol together with f-vector and order of automorphism group of every type.

By the automorphism group of a type, say $311+$, we mean all linear transformations that leave the corresponding cone, here $\mathrm{cone}\{R_1, R_2, R_3, R_4, R_6, R_8, R_{11}, R_{12}\}$, pointwise fixed.

Another problem arose: How can we distinguish e.g. the Dirichlet-Voronoi polytopes of type $111+$ and $111-$ combinatorially? This cannot be done by the f-vector alone. We have to use finer combinatorial invariants. We looked at two-dimensional faces. Dirichlet-Voronoi polytopes of type $111+$ have 6 triangles, 54 quadrangles, 54 pentagons and 36 hexagons on the boundary. Dirichlet-Voronoi polytopes of type $111-$ have 72 quadrangles, 36 pentagons and 42 hexagons on the boundary.

In DELONE's and in ENGEL's classification we find even more data on the four-dimensional Dirichlet-Voronoi polytope. This also can be used to show that the types are pairwise combinatorially distinct.

4.5. Dimension 5

RYSHKOV achieved the first step towards a classification of all non-equivalent five-dimensional DELONE triangulations. In [Rys1973] he determines all 76 non-equivalent 1-skeletons of Delone triangulations (so-called "C-types") in dimension 5. Together with BARANOVSKII he refined this result to find 221 non-equivalent Delone triangulations. They documented the calculations in [BR1973] and in greater detail in [RB1976]. They claimed to give a complete list but it was only almost complete. With help of a computer ENGEL showed that there are 222 non-equivalent types (see [Eng1998]). But the list of BARANOVSKII and RYSHKOV and the one of ENGEL are not directly comparable. In [EG2002] ENGEL and GRISHUKHIN undertake the non-trivial task to identify the missing Delone triangulation. In this article they also correct several errors in both lists.

By an implementation in `C++` of VORONOÏ's algorithm we confirm the number of 222 non-equivalent five-dimensional Delone triangulations. On a standard Intel Pentium computer the complete classification takes about 15 minutes. However, the computation can be sped up considerably because we test two Delone triangulations for being equivalent in a rather naive way. In Chapter 8.4.3 we will show an alternative and more efficient isomorphism test.

We do not want to print our complete data here. In the near future we will make it available on the world wide web. We only give the data of the missed Delone triangulation. As a byproduct of our classification we confirm the list of seven rigid five-dimensional positive definite quadratic forms given by BARANOVSKII and GRISHUKHIN in [BG2001].

In principle we could use our classification of the five-dimensional Delone triangulation to give a classification of all combinatorial types of five-dimensional Dirichlet-Voronoi polytopes. We could use similar methods as we did in the four-dimensional case. ENGEL ([Eng2000]) reports that there are $179,372$ combinatorially distinct five-dimensional Dirichlet-Voronoi polytopes.

4.5.1. The Missing Delone Triangulation

In our list the missing Delone triangulation is $\#164$. The automorphism group of the missing Delone triangulation is nearly trivial (it has order 4), so we do not give all simplices of the triangulation. We give the facet-defining inequalities of the secondary cone instead. To accustom the reader to the data that we will make available on the world wide web, we give them in the Polyhedra H-Format (Version 1997) that was defined by AVIS and FUKUDA and that is implemented in their programs `lrs` and `cdd`. Let $A \in \mathbb{Z}^{m \times d}$ be an integral matrix, and let $\mathbf{b} \in \mathbb{Z}^m$ be a vector. The Polyhedra H-Format of the system $A\boldsymbol{x} \leq \boldsymbol{b}$ of m inequalities in d variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)^t$ is

H-representation
begin
$m \quad d+1$ **integer**
$\boldsymbol{b} \quad -A$
end

In our situation we have $\boldsymbol{x} = (q_{11}, q_{21}, q_{22}, q_{31}, q_{32}, q_{33}, \ldots, q_{55})^t$.

```
* secondary cone #164
H-representation
begin
18 16 integer
0  0 -1  0  0  0  0  0 -1  0  0 -1 -1 -1 -1 -1
0  0 -1  0  0  0  0  0  0  0  0 -1  0  0  0  0
0  0 -1  0  0  0  0  0  0  0  0  0 -1  0  0  0
0  0  0  0 -1  0  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0 -1 -1 -1 -1  0  0  0 -1  0
0  0  0  0  0  0  0 -1  1  0  0  0  0  0  0  0
0  0  0  0  0  0  0  0  0 -1  0  0  0  0 -1  0
0  0  0  0  0  0  0  0  0  0  0  0  0 -1 -1  0
0  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0
0  0  0  0  1  1  1  1  1  2  1  0  0  1  1  0
0  0  0  1  0  1  0  0  1  0  0  0  1  0  0  0
0  0  1  0  0  0  0  0  1  0  0  0  0  0  0  0
0  0  1  0  1  0  0  1  1  1  1  1  1  1  1  1
0  0  0  0  1 -1  0  0  0  0  0  0  0  0  0  0
0  0  0  0  0  0  0  1  1  0  1  0  0  0  1  0
0  0  1  0  1  1  1  0  1  1  0  1  1  2  1  1
0  0  1  0  0  0  0  0  1  0  0  1  1  0  1  1
0  1  0  0  1  0  0  1  0  0  0  1  0  0  0  0
end
```

4.5.2. Rigid Forms in Dimension 5

Using a corrected list of ENGEL's classification BARANOVSKII and GRISHUKHIN found seven rigid five-dimensional positive definite quadratic forms. To confirm their list we computed the extreme rays of the 222 non-equivalent secondary cones. We found the same rigid forms. For future reference we only give the Gram matrices here and refer the interested reader to the article [BG2001] for more details. Our numbering coincides with the one of BARANOVSKII and GRISHUKHIN.

$$R_1 = \begin{pmatrix} 2 & & & & \\ 0 & 2 & & & \\ 1 & -1 & 2 & & \\ -1 & 0 & -1 & 2 & \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 3 & & & & \\ 1 & 2 & & & \\ 1 & 0 & 2 & & \\ -2 & -1 & -1 & 3 & \\ -2 & -1 & -1 & 1 & 3 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 3 & & & & \\ 1 & 3 & & & \\ 1 & 1 & 3 & & \\ -1 & -1 & -1 & 3 & \\ -2 & -2 & -2 & 0 & 5 \end{pmatrix} \quad R_4 = \begin{pmatrix} 3 & & & & \\ 1 & 3 & & & \\ -1 & -1 & 3 & & \\ -1 & -1 & 1 & 3 & \\ -1 & -1 & -1 & -1 & 3 \end{pmatrix}$$

$$R_5 = \begin{pmatrix} 5 & & & & \\ 1 & 3 & & & \\ -2 & -1 & 5 & & \\ -2 & -1 & 2 & 5 & \\ -1 & -1 & -2 & -2 & 5 \end{pmatrix} \quad R_6 = \begin{pmatrix} 5 & & & & \\ 1 & 5 & & & \\ -1 & -1 & 5 & & \\ -1 & -1 & 1 & 5 & \\ -2 & -2 & -2 & -2 & 6 \end{pmatrix}$$

$$R_7 = \begin{pmatrix} 4 & & & & \\ 2 & 6 & & & \\ -1 & -2 & 4 & & \\ -1 & -2 & 1 & 4 & \\ -2 & -2 & -1 & -1 & 6 \end{pmatrix}$$

4.6. Dimension 6 and Higher

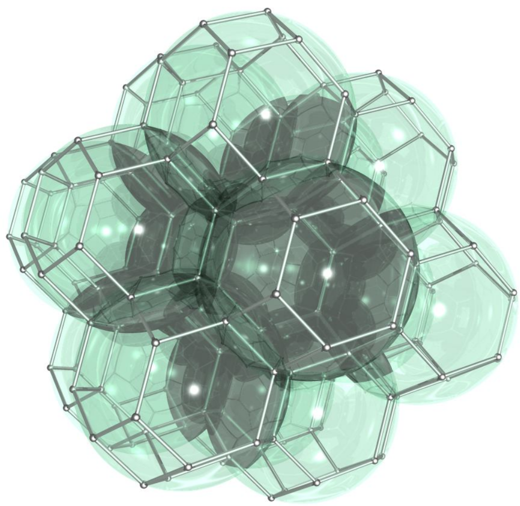
Not much is known about Delone triangulations in dimensions higher than 5. Currently, there are no realistic bounds known for the number of non-equivalent Delone triangulations in a given dimension. With help of our computer program we found more than $250,000$ non-equivalent Delone triangulations in dimension 6. Mainly due to memory limitations — we have to deal with 360 six-dimensional simplices per triangulation — we were not able to push the classification further. Nevertheless, we think that a complete classification might be possible with up-to-date computers. But for this a careful reexamination and reimplementation of our program would be necessary. On the other hand, we have no hope that a classification of all non-equivalent seven-dimensional Delone triangulations is feasible in the near future.

Even about rigid positive definite quadratic forms we do not know much. With one exception: For the class of positive definite quadratic forms associated to root lattices (or to their duals) DEZA and GRISHUKHIN give a complete answer in [DG2002]:

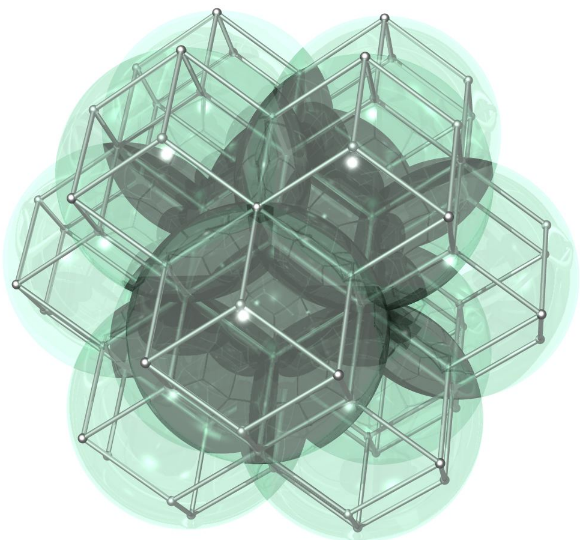
Lattice	$\mathsf{A}_1 = \mathsf{A}_1^*$	A_d	A_d^*	D_d	D_{2d+1}^*	D_{2d}^*	E_d	E_d^*
Dimension d	1	≥ 2	≥ 1	≥ 4	≥ 2	≥ 2	$6,7,8$	$6,7,8$
Non-rigidity deg.	1	$(d+1)$	$\frac{d(d+1)}{2}$	1	$2d+1$	1	1	1

— II —

The Lattice Covering Problem



$$\mathbf{\Theta}(\mathsf{A}_3^*) < \mathbf{\Theta}(\mathsf{A}_3)$$



— II —

The Lattice Covering Problem

The lattice covering problem asks for the most economical way to cover d-dimensional space by equal overlapping spheres whose centers form a lattice.
In this part, we present an algorithm which solves the lattice covering problem in any given dimension. The proposed algorithm has two phases. The first phase generates all non-equivalent Delone triangulations. Here, techniques form Part I are applied. In the second phase we have to solve an optimization problem for each generated Delone triangulation.
The optimization problem looks as follows: the set of all positive definite quadratic forms with fixed Delone triangulation is identified with a set of lattices whose Dirichlet-Voronoi polytopes are all combinatorially equivalent and can be transformed into each other continuously. The covering density of a lattice covering is the volume of the Dirichlet-Voronoi polytope's circumsphere divided by the volume of the Dirichlet-Voronoi polytope. Since this value is invariant under scaling of the lattice, the radius of the circumsphere can be fixed to 1. Then we maximize the volume of the Dirichlet-Voronoi polytope in order to get an economical lattice covering.

Chapter 5 "Determinant Maximization" introduces a general determinant maximization problem which is suitable for the above approach. It is a convex programming problem and can be transformed into a semidefinite programming problem. These kind of problems can be solved efficiently, e.g. by interior-point algorithms for which implementations are available. To assemble the original optimization problem into the framework of determinant maximization problems the constraint "the radius of the circumsphere of a Dirichlet-Voronoi polytope equals one" has to be formulated as a so-called linear matrix (in-)equality. We do this in Chapter 6 by using Carley-Monger determinants. Then, in Chapter 7 "Solving the Lattice Covering Problem" the methods of the preceding chapters are fit together into an algorithm that solves the lattice covering problem. In this context we interpret different classical results of the theory of lattice coverings.
The number of non-equivalent Delone triangulations grows enormously with the dimension. Solving a single optimization problem belonging to a Delone triangulation is time-consuming. So it is desirable to have an apriori lower bound for the covering density for those lattice coverings associated to a given Delone triangulation. For this purpose we introduce the method of the moments of inertia in Chapter 8.
In Chapter 9 "Results in Low Dimensions" we demonstrate that the algorithm is not only of theoretical interest. We have implemented the algorithm. For the dimensions up to 5 we could reproduce (check, extend, and rarely correct) all previous known results. For dimensions 6 and 7 many new interesting lattice coverings were found. For the dimensions 8 to 24 we give a report on the state-of-the-art.

Following the tradition of Part I we included motivating and somehow chaotic pictures on the previous page. We created the picture with `povray`. They show sphere coverings belonging to the three-dimensional lattices A_3^* and A_3.

Chapter 5.

Determinant Maximization

In this chapter we introduce determinant maximization problems. Determinant maximization problems are convex programming problems and are, in a sense, equivalent to the more popular semidefinite programming problems. In the last years, semidefinite programming problems and determinant maximization problems became standard problem classes in the theory of convex optimization. For both classes efficient algorithms and implementations are available.

Later on we will see how the lattice covering problem can be formulated naturally as a finite number of determinant maximization problems.

5.1. The Determinant Maximization Problem

Following VANDENBERGHE, BOYD, and WU ([VBW1998]) we say that a *determinant maximization problem* is an optimization problem of the form

minimize	$\boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x})$
subject to	$G(\boldsymbol{x})$ is a positive definite matrix,
	$F(\boldsymbol{x})$ is a positive semidefinite matrix.

The optimization vector is $\boldsymbol{x} \in \mathbb{R}^d$, the row vector $\boldsymbol{c}^t \in (\mathbb{R}^d)^*$ defines a linear form, the maps $G : \mathbb{R}^d \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^d \to \mathbb{R}^{n\times n}$ are affine:

$$\begin{aligned} G(\boldsymbol{x}) &= G_0 + x_1G_1 + \cdots + x_dG_d, \\ F(\boldsymbol{x}) &= F_0 + x_1F_1 + \cdots + x_dF_d, \end{aligned}$$

and $G_i \in \mathbb{R}^{m\times m}$, $F_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \dots, d$, are symmetric matrices. We will also write the *linear matrix inequalities* which define the constraints of the determinant maximization problem by $G(\boldsymbol{x}) \succ 0$ and $F(\boldsymbol{x}) \succeq 0$.

A point $\boldsymbol{x} \in \mathbb{R}^d$ is a *feasible solution* of the determinant maximization problem if it satisfies the two constraints: (i) the matrix $G(\boldsymbol{x})$ is positive definite, (ii) the matrix $F(\boldsymbol{x})$ is positive semidefinite. It is called a *strictly feasible solution* if the matrix $F(\boldsymbol{x})$ is positive definite.

5.2. Convexity of the Problem

In the definition of the determinant maximization problem we used $\min \boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x})$ instead of the more intuitive $\max \boldsymbol{c}^t\boldsymbol{x} + \det G(\boldsymbol{x})$ because in the first formulation the determinant maximization problem is a convex programming problem. In the following we will show the convexity of the problem. We have to show that the objective function $\boldsymbol{x} \mapsto \boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x})$

is convex on the set of feasible solutions and the set of feasible solutions is also convex. This is evident from the following arguments due to MINKOWSKI [Min1905] (see also [GL1987], §39).

The objective function is convex since it is the sum of two convex functions: the linear function $\boldsymbol{x} \mapsto \boldsymbol{c}^t\boldsymbol{x}$ and the composed function $\boldsymbol{x} \mapsto -\log G(\boldsymbol{x})$ where $x \mapsto G(\boldsymbol{x})$ is an affine transformation. Now $X \mapsto -\log\det X$ is a strictly convex function on the set of positive definite matrices. To prove this it suffices to show that the function $X \mapsto -\log\det X$ is strictly convex on any line segment $[X, Y] = \{tX + (1-t)Y : t \in [0,1], X \neq Y\}$ in $\mathcal{S}^m_{>0}$. Therefore, we compute the second derivative of the one-dimensional function $f(t) = -\log\det(tX + (1-t)Y)$ and see that it is always positive: There is a matrix A with determinant 1 whose inverse simultaneously diagonalizes X and Y. Hence, $X = A^t \operatorname{diag}(x_1, \ldots, x_m)A$, $Y = A^t \operatorname{diag}(y_1, \ldots, y_m)A$ and

$$\begin{aligned}
f(t) &= -\log(y_1 + t(x_1 - y_1)) - \cdots - \log(y_m + t(x_m - y_m)),\\
\frac{\partial f}{\partial t}(t) &= -\frac{x_1 - y_1}{y_1 + t(x_1 - y_1)} - \cdots - \frac{x_m - y_m}{y_m + t(x_m - y_m)},\\
\frac{\partial^2 f}{\partial t^2}(t) &= \left(\frac{x_1 - y_1}{y_1 + t(x_1 - y_1)}\right)^2 + \cdots + \left(\frac{x_m - y_m}{y_m + t(x_m - y_m)}\right)^2 > 0.
\end{aligned}$$

The set of feasible solutions is convex since it is the intersection of the two convex sets $\{\boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x})$ is positive semidefinite$\}$ and $\{\boldsymbol{x} \in \mathbb{R}^d : G(\boldsymbol{x})$ is positive definite$\}$. The first set equals $F^{-1}(\mathcal{S}^n_{\geq 0})$ and is convex because preimages of convex sets are again convex. Additionally, a straight forward computation shows that $\mathcal{S}^n_{\geq 0}$ is indeed convex. The same argument works for the second set $G^{-1}(\mathcal{S}^m_{>0})$.

5.3. Relation to Semidefinite Programming

If for all $x \in \mathbb{R}^d$ the matrix $G(\boldsymbol{x})$ is the identity matrix, then a determinant maximization problem reduces to a semidefinite programming problem. A *semidefinite programming problem* is an optimization problem of the form

minimize	$\boldsymbol{c}^t\boldsymbol{x}$
subject to	$F(\boldsymbol{x})$ is a positive semidefinite matrix.

In the last twenty years numerous people worked in the field of semidefinite programming problems. It unifies standard problems in convex optimization, e.g. linear and quadratic programming. Many problems in combinatorial optimization and engineering can be formulated as semidefinite programming problems. Furthermore, semidefinite programming problems are convex programming problems, they have a rich duality theory and can be solved efficiently.

Currently there exist two different types of algorithms which efficiently solve semidefinite programming problems. These are ellipsoid and interior-point methods. Both have many variants and the exact technical descriptions are quite complicated. They can approximate the solution of a semidefinite programming problem within any specified accuracy and run in polynomial time if the instances are “well-behaving”. But these theoretical results are definitely not an issue for us. We do not want to go further into details since for our application it suffices to use the methods more or less as a black box as long as they perform well in our instances. Instead we only want to understand the underlying principles of the specific interior-point algorithms. Nowadays, they are much more efficient in practice than ellipsoid methods. For more information on the exciting topic of semidefinite programming the interested reader is referred to the vast amount of literature which to a great extend is available on the World Wide Web. Good starting points which contain various points of views are [VB1996], [GLS1988], [NN1994], [Goe1997], [WSV2000], and the web site* of CHRISTOPH HELMBERG.

In [NN1994] NESTEROV and NEMIROVSKY developed a framework for the design of efficient interior-point algorithms for general and for specific convex programming problems. There (§6.4.3), they also showed that the determinant maximization problem can be cast into a semidefinite programming problem by a transformation which can be computed in polynomial time. Since their transformation uses more than linear time, their result is mainly of theoretical interest. Nevertheless, there exists a polynomial time algorithm which solves the determinant maximization problem.

5.4. Algorithms for the Determinant Maximization Problem

It is faster to solve the determinant maximization problem directly than to use the transformation of NESTEROV and NEMIROVSKY. VANDENBERGHE, BOYD, WU and independently TOH give in [VBW1998] and in [Toh1999] interior-point algorithms for the determinant maximization problem. Both algorithms fit into the general framework of NESTEROV and NEMIROVSKY.

The key fact is that the function

$$\varphi(\boldsymbol{x}) = \begin{cases} -\log\det F(\boldsymbol{x}) & \text{if } F(\boldsymbol{x}) \succ 0, \\ +\infty & \text{otherwise,} \end{cases}$$

is a *barrier function* for the feasible domain $\{\boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x}) \succeq 0\}$. A barrier function for a domain C is a smooth and convex function with $\lim\limits_{\boldsymbol{x}\to\partial C} \varphi(\boldsymbol{x}) = +\infty$. Then, it is intuitively clear that the minimum $\boldsymbol{x}^*(\alpha)$ of the function

$$\varphi_\alpha(\boldsymbol{x}) = \alpha(\boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x})) + \varphi(\boldsymbol{x})$$

gives the minimum of the original problem as $\alpha \to \infty$. The minimization of the function φ_α is an *unconstrained* optimization problem to which NEWTON's method can be applied. Altogether we can use a interior penalty scheme to solve our original problem. Now it is clear where the name "interior-point method" comes from: all intermediate solutions lie in the interior of the set of feasible solutions.

input	strictly feasible $\boldsymbol{x} \in \mathbb{R}^d$, positive number α
repeat	compute approximate minimum $\boldsymbol{x}^*$ of φ_α by NEWTON's method with 1. iterate $\boldsymbol{x}$.
	$\boldsymbol{x} \leftarrow \boldsymbol{x}^*$.
	increase α.
until	$\boldsymbol{x}$ is an approximate solution of the problem.

Two problems of the "algorithm" above are apparent: how do we increase the penalty parameter α and how do we decide whether $\boldsymbol{x}$ is an approximate solution? Both problems are highly non-trivial and one has to work very carefully through the technicalities to get a polynomial time algorithm. We just glimpse at the ideas.

It is a well-known fact that NEWTON's method converges very fast if the first iterate lies near to the minimum. In [NN1994] one can find a very detailed analysis of NEWTON's method. The increment of the penalty parameter α is adjusted in such a way that NEWTON's method finds the next minimum in a constant number of iterations.

*http://www-user.tu-chemnitz.de/~helmberg/semidef.html

To certify that we have found an optimum of a determinant maximization problem we make use of duality theory. To a determinant maximization problem we associate the *dual problem*

maximize	$\log\det W - \operatorname{trace}(G_0 W) - \operatorname{trace}(F_0 Z) + m$
subject to	$\operatorname{trace}(G_i W) + \operatorname{trace}(F_i Z) = c_i, i = 1, \ldots, d,$
	W is a positive definite matrix,
	Z is a positive semidefinite matrix.

The optimization variables are $W \in \mathbb{R}^{m\times m}$ and $Z \in \mathbb{R}^{n\times n}$. The primal determinant maximization problem and its dual problem are connected as follows (for proofs see §3 of [VBW1998]).

Theorem 5.4.1. Let p^* be the optimal value of the primal problem and let d^* be the optimal value of the dual problem. Then, we always have the inequality $p^* \geq d^*$. If the primal problem is strictly feasible, the optimal solution of the dual problem is achieved and vice versa. In both cases we have equality $p^* = d^*$.

Suppose that the primal problem has a strictly feasible solution. Then, a primal feasible solution $\boldsymbol{x}$ is optimal if and only if there exists a positive semidefinite matrix $Z \in \mathbb{R}^{n\times n}$ such that $F(\boldsymbol{x})Z = 0$ and

$$\operatorname{trace}(G_i G(\boldsymbol{x})^{-1}) + \operatorname{trace}(F_i Z) = c_i,\ i = 1, \ldots, d.$$

Chapter 6.

Cayley-Menger Determinants

Let L be a lattice and let $Q = (q_{ij})$ be a positive definite quadratic form associated to L. To express the lattice covering problem in a finite number of determinant maximization problems we formulate the fact that the covering radius of L is bounded by a constant, say 1, in terms of linear matrix inequalities in the parameters q_{ij}. The covering radius of L, which is the circumsphere of the lattice's Dirichlet-Voronoi polytopes, is bounded by 1 if and only if the circumradius of every Delone polytope is bounded by 1. So the first goal is to give a linear matrix inequality for a simplex having a circumsphere of radius at most 1. In this chapter we will achieve this by using Cayley-Menger determinants.

6.1. Definition and Basic Properties

We define the *Cayley-Menger determinant* of n points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, where the pairwise distances $\operatorname{dist}(\boldsymbol{x}_i, \boldsymbol{x}_j)$ are given, by

$$\operatorname{CM}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 1 & \operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_1)^2 & \ldots & \operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_n)^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \operatorname{dist}(\boldsymbol{x}_n, \boldsymbol{x}_1)^2 & \ldots & \operatorname{dist}(\boldsymbol{x}_n, \boldsymbol{x}_n)^2 \end{vmatrix}. \tag{6.1}$$

Cayley-Menger determinants give universal relations between the distances of points in affine Euclidean spaces.

Let us look at the simplest case: the case of three points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ on the line $\mathbb{R}^1$. The Cayley-Menger determinant of these three points vanishes: By computing the determinant we see that $\operatorname{CM}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ factors

$$\begin{aligned} \operatorname{CM}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \;=\; & (\operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_3) + \operatorname{dist}(\boldsymbol{x}_2, \boldsymbol{x}_3)) \\ & \cdot (-\operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_3) + \operatorname{dist}(\boldsymbol{x}_2, \boldsymbol{x}_3)) \\ & \cdot (\operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_2) - \operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_3) + \operatorname{dist}(\boldsymbol{x}_2, \boldsymbol{x}_2)) \\ & \cdot (\operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \operatorname{dist}(\boldsymbol{x}_1, \boldsymbol{x}_3) - \operatorname{dist}(\boldsymbol{x}_2, \boldsymbol{x}_3)). \end{aligned}$$

Now it is easy to see that $\operatorname{CM}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = 0$ because one of the last three factors vanishes since one point has to lie between the other two.

Cayley-Menger determinants and Euclidean spaces are linked concepts. Cayley-Menger determinants are the “main” syzygies of the Euclidean invariants and with their help it is easy to decide whether a distance space can be embedded into a Euclidean space (for more information on these topics see e.g. [Blu1970], [Hav1991], [Dal1995], [DL1997]). In our account we will follow mainly BERGER's book [Ber1987].

The only property of Cayley-Menger determinants we need in the following is:

Lemma 6.1.1. Given $d+2$ points in d-dimensional Euclidean space $(E, (\cdot,\cdot))$, then their Cayley-Menger determinant vanishes.

Proof. The points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+2}$ are affinely dependent. So there exist numbers $\lambda_1, \ldots, \lambda_{d+2}$ with $\lambda_1 + \cdots + \lambda_{d+2} = 0$ and $\sum_{i=1}^{d+2} \lambda_i \boldsymbol{x}_i = \boldsymbol{0}$. Then, the function $\boldsymbol{y} \mapsto \sum_{i=1}^{d+2} \lambda_i \operatorname{dist}(\boldsymbol{x}_i, \boldsymbol{y})^2$ is a constant function because

$$\sum_{i=1}^{d+2} \lambda_i \operatorname{dist}(\boldsymbol{x}_i, \boldsymbol{y})^2 = \sum_{i=1}^{d+2} \lambda_i \left((\boldsymbol{x}_i, \boldsymbol{x}_i) - 2(\boldsymbol{x}_i, \boldsymbol{y}) + (\boldsymbol{y}, \boldsymbol{y})\right) = \sum_{i=1}^{d+2} \lambda_i (\boldsymbol{x}_i, \boldsymbol{x}_i).$$

Using this for $\boldsymbol{y} = \boldsymbol{x}_i$, $i = 1, \ldots, d+2$, we see that the vector

$$(-\sum_{i=1}^{d+2} \lambda_i (\boldsymbol{x}_i, \boldsymbol{x}_i), \lambda_1, \ldots, \lambda_{d+2})^t$$

lies in the kernel of the matrix which is used in (6.1) to define the Cayley-Menger determinant, hence $\operatorname{CM}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{d+2}) = 0$. ◇

6.2. The Radius of the Circumsphere of a Simplex

It is obvious that a d-dimensional simplex in d-dimensional Euclidean space is defined up to Euclidean isometries by its edge lengths. Then (without knowing anything about invariant theory) it is clear that the radius of the simplex' circumsphere can somehow expressed by its edge lengths. By using Cayley-Menger determinants one can find a simple formula for the circumradius.

Lemma 6.2.1. Let $L = \operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_d\}$ be a d-dimensional simplex in d-dimensional Euclidean space. Then the uniquely determined circumsphere of L has the squared radius

$$R^2 = -\frac{1}{2} \cdot \frac{\det\left(\operatorname{dist}(\boldsymbol{v}_i, \boldsymbol{v}_j)^2\right)_{0 \le i,j \le d}}{\operatorname{CM}(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_d)}.$$

Proof. Let $\boldsymbol{c}$ be the center of the circumsphere of L and let $R = \operatorname{dist}(\boldsymbol{c}, \boldsymbol{v}_i)$ be the circumradius. Due to Lemma 6.1.1 we have $\operatorname{CM}(\boldsymbol{c}, \boldsymbol{v}_0, \ldots, \boldsymbol{v}_d) = 0$. Then, looking at the matrix which defines the Cayley-Menger determinant yields the desired result:

$$\begin{aligned}
\operatorname{CM}(\boldsymbol{c}, \boldsymbol{v}_0, \ldots, \boldsymbol{v}_d) &= \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & R^2 & \ldots & R^2 \\ 1 & R^2 & \operatorname{dist}(\boldsymbol{v}_0, \boldsymbol{v}_0)^2 & \ldots & \operatorname{dist}(\boldsymbol{v}_0, \boldsymbol{v}_d)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & R^2 & \operatorname{dist}(\boldsymbol{v}_d, \boldsymbol{v}_0)^2 & \ldots & \operatorname{dist}(\boldsymbol{v}_d, \boldsymbol{v}_d)^2 \end{vmatrix} \\
&= - \begin{vmatrix} 1 & 0 & R^2 & \ldots & R^2 \\ 0 & 1 & 1 & \ldots & 1 \\ 1 & R^2 & \operatorname{dist}(\boldsymbol{v}_0, \boldsymbol{v}_0)^2 & \ldots & \operatorname{dist}(\boldsymbol{v}_0, \boldsymbol{v}_d)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & R^2 & \operatorname{dist}(\boldsymbol{v}_d, \boldsymbol{v}_0)^2 & \ldots & \operatorname{dist}(\boldsymbol{v}_d, \boldsymbol{v}_d)^2 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & R^2 & \ldots & R^2 \\ 1 & 0 & 1 & \ldots & 1 \\ R^2 & 1 & \operatorname{dist}(\boldsymbol{v}_0, \boldsymbol{v}_0)^2 & \ldots & \operatorname{dist}(\boldsymbol{v}_0, \boldsymbol{v}_d)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^2 & 1 & \operatorname{dist}(\boldsymbol{v}_d, \boldsymbol{v}_0)^2 & \ldots & \operatorname{dist}(\boldsymbol{v}_d, \boldsymbol{v}_d)^2 \end{vmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{vmatrix} -R^2 & 1 & R^2 & \dots & R^2 \\ 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & \operatorname{dist}(\boldsymbol{v}_0,\boldsymbol{v}_0)^2 & \dots & \operatorname{dist}(\boldsymbol{v}_0,\boldsymbol{v}_d)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \operatorname{dist}(\boldsymbol{v}_d,\boldsymbol{v}_0)^2 & \dots & \operatorname{dist}(\boldsymbol{v}_d,\boldsymbol{v}_d)^2 \end{vmatrix} \\
&= \begin{vmatrix} -2R^2 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & \operatorname{dist}(\boldsymbol{v}_0,\boldsymbol{v}_0)^2 & \dots & \operatorname{dist}(\boldsymbol{v}_0,\boldsymbol{v}_d)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \operatorname{dist}(\boldsymbol{v}_d,\boldsymbol{v}_0)^2 & \dots & \operatorname{dist}(\boldsymbol{v}_d,\boldsymbol{v}_d)^2 \end{vmatrix} \\
&= -2R^2\,\mathrm{CM}(\boldsymbol{v}_0,\dots,\boldsymbol{v}_d) - \det\left(\operatorname{dist}(\boldsymbol{v}_i,\boldsymbol{v}_j)^2\right)_{0\le i,j\le d}.
\end{aligned}
$$

◇

6.3. A Linear Matrix Inequality

By the previous lemma our first goal is within reach. To find a linear matrix inequality for the fact that the circumradius of a simplex is bounded by 1 we only have to transform the formula stated in the lemma to the right form. The idea of using Cayley-Menger determinants to find the linear matrix inequality is highly inspired by the paper [DDRS1970] of DELONE, DOLBILIN, RYSHKOV and STOGRIN.

Proposition 6.3.1. Let $L = \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_1,\dots,\boldsymbol{v}_d\} \subseteq \mathbb{R}^d$ with $\boldsymbol{v}_0 = \boldsymbol{0}$ be a d-dimensional simplex. A positive definite quadratic form $Q = (q_{ij})$ gives the scalar product of a Euclidean space $(\mathbb{R}^d, (\cdot,\cdot))$, $(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{x}^t Q \boldsymbol{y}$, in which the radius of the circumsphere of L is at most 1 if and only if the following linear matrix inequality (in the parameters q_{ij}) is satisfied:

$$
\mathrm{BR}_L(Q) = \begin{pmatrix} 4 & (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_2,\boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \\ (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1,\boldsymbol{v}_d) \\ (\boldsymbol{v}_2,\boldsymbol{v}_2) & (\boldsymbol{v}_2,\boldsymbol{v}_1) & (\boldsymbol{v}_2,\boldsymbol{v}_2) & \dots & (\boldsymbol{v}_2,\boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d,\boldsymbol{v}_d) & (\boldsymbol{v}_d,\boldsymbol{v}_1) & (\boldsymbol{v}_d,\boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \end{pmatrix} \succeq 0.
$$

Proof. As a first step transform the nominator of the formula given in the previous lemma using the so-called covariance map, i.e. replace $\operatorname{dist}(\boldsymbol{x},\boldsymbol{y})^2$ by $(\boldsymbol{x},\boldsymbol{x}) - 2(\boldsymbol{x},\boldsymbol{y}) + (\boldsymbol{y},\boldsymbol{y})$. This gives

$$
\begin{aligned}
&\det\left(\operatorname{dist}(\boldsymbol{v}_i,\boldsymbol{v}_j)^2\right)_{0\le i,j\le d} \\
&= \det\left((\boldsymbol{v}_i,\boldsymbol{v}_i) - 2(\boldsymbol{v}_i,\boldsymbol{v}_j) + (\boldsymbol{v}_j,\boldsymbol{v}_j)\right)_{0\le i,j\le d} \\
&= \begin{vmatrix} 0 & (\boldsymbol{v}_1,\boldsymbol{v}_1) & \dots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \\ (\boldsymbol{v}_1,\boldsymbol{v}_1) & (\boldsymbol{v}_1,\boldsymbol{v}_1) - 2(\boldsymbol{v}_1,\boldsymbol{v}_1) + (\boldsymbol{v}_1,\boldsymbol{v}_1) & \dots & (\boldsymbol{v}_1,\boldsymbol{v}_1) - 2(\boldsymbol{v}_1,\boldsymbol{v}_d) + (\boldsymbol{v}_d,\boldsymbol{v}_d) \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d,\boldsymbol{v}_d) & (\boldsymbol{v}_d,\boldsymbol{v}_d) - 2(\boldsymbol{v}_d,\boldsymbol{v}_1) + (\boldsymbol{v}_1,\boldsymbol{v}_1) & \dots & (\boldsymbol{v}_d,\boldsymbol{v}_d) - 2(\boldsymbol{v}_d,\boldsymbol{v}_d) + (\boldsymbol{v}_d,\boldsymbol{v}_d) \end{vmatrix} \\
&= \begin{vmatrix} 0 & (\boldsymbol{v}_1,\boldsymbol{v}_1) & \dots & (\boldsymbol{v}_d,\boldsymbol{v}_d) \\ (\boldsymbol{v}_1,\boldsymbol{v}_1) & -2(\boldsymbol{v}_1,\boldsymbol{v}_1) & \dots & -2(\boldsymbol{v}_1,\boldsymbol{v}_d) \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d,\boldsymbol{v}_d) & -2(\boldsymbol{v}_d,\boldsymbol{v}_1) & \dots & -2(\boldsymbol{v}_d,\boldsymbol{v}_d) \end{vmatrix}
\end{aligned}
$$

$$= \frac{1}{4} \cdot \begin{vmatrix} 0 & -2(\boldsymbol{v}_1, \boldsymbol{v}_1) & \dots & -2(\boldsymbol{v}_d, \boldsymbol{v}_d) \\ -2(\boldsymbol{v}_1, \boldsymbol{v}_1) & -2(\boldsymbol{v}_1, \boldsymbol{v}_1) & \dots & -2(\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \ddots & \vdots \\ -2(\boldsymbol{v}_d, \boldsymbol{v}_d) & -2(\boldsymbol{v}_d, \boldsymbol{v}_1) & \dots & -2(\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix}$$

$$= \frac{1}{4} \cdot (-2)^{d+1} \cdot \begin{vmatrix} 0 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix}.$$

As a second step the denominator of the formula given in the previous lemma is being transformed similarly which gives

$$\begin{aligned} &\mathrm{CM}(\boldsymbol{v}_0, \dots, \boldsymbol{v}_d) \\ &= \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & (\boldsymbol{v}_0, \boldsymbol{v}_0) - 2(\boldsymbol{v}_0, \boldsymbol{v}_0) + (\boldsymbol{v}_0, \boldsymbol{v}_0) & \dots & (\boldsymbol{v}_0, \boldsymbol{v}_0) - 2(\boldsymbol{v}_0, \boldsymbol{v}_d) + (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\boldsymbol{v}_d, \boldsymbol{v}_d) - 2(\boldsymbol{v}_d, \boldsymbol{v}_0) + (\boldsymbol{v}_0, \boldsymbol{v}_0) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) - 2(\boldsymbol{v}_d, \boldsymbol{v}_d) + (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \\ &= \frac{1}{4} \cdot (-2)^{d+2} \cdot \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & (\boldsymbol{v}_0, \boldsymbol{v}_0) & \dots & (\boldsymbol{v}_0, \boldsymbol{v}_d) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\boldsymbol{v}_d, \boldsymbol{v}_0) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \\ &= \frac{1}{4} \cdot (-2)^{d+2} \cdot \begin{vmatrix} 0 & 1 & \mathbf{0}^t \\ 1 & 0 & \boldsymbol{v}_0^t \\ \vdots & \vdots & \vdots \\ 1 & 0 & \boldsymbol{v}_d^t \end{vmatrix} \begin{vmatrix} 1 & 0 & \mathbf{0}^t \\ 0 & 1 & \mathbf{0}^t \\ \mathbf{0} & 0 & Q \end{vmatrix} \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 \\ \mathbf{0} & \boldsymbol{v}_0 & \dots & \boldsymbol{v}_d \end{vmatrix} \\ &= \frac{1}{4} \cdot (-2)^{d+2} \cdot (-1) \cdot \begin{vmatrix} (\boldsymbol{v}_1, \boldsymbol{v}_1) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_1) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix}. \end{aligned}$$

Hence,

$$R^2 = -\frac{1}{4} \cdot \frac{\begin{vmatrix} 0 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix}}{\det\left((\boldsymbol{v}_i, \boldsymbol{v}_j)\right)_{1 \le i,j \le d}}. \tag{6.2}$$

If $R \le 1$, then

$$4 \cdot \det\left((\boldsymbol{v}_i, \boldsymbol{v}_j)\right)_{1 \le i,j \le d} + \begin{vmatrix} 0 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \ge 0,$$

which is equivalent to

$$\begin{vmatrix} 4 & 0 & 0 & \dots & 0 \\ 0 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} + \begin{vmatrix} 0 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \ge 0,$$

therefore

$$\begin{vmatrix} 4 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_1, \boldsymbol{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{v}_d, \boldsymbol{v}_d) & (\boldsymbol{v}_d, \boldsymbol{v}_1) & (\boldsymbol{v}_d, \boldsymbol{v}_2) & \dots & (\boldsymbol{v}_d, \boldsymbol{v}_d) \end{vmatrix} \geq 0.$$

After reordering the columns of the above matrix ($1 \leftrightarrow d+1$, $2 \leftrightarrow d$, $\dots$), we see that all main minors are non-negative. Then, by the criterion of HURWITZ the matrix has to be positive semidefinite. $\diamond$

Chapter 7.

Solving the Lattice Covering Problem

In this chapter we present an algorithm which solves the lattice covering problem in any dimension d.

Our algorithm computes all locally optimal lattice coverings. These are only finitely many because we will see that for every fixed Delone triangulation $\mathcal{D}$ there exists at most one positive definite quadratic form which lies in the topological closure of the secondary cone of $\mathcal{D}$ giving a locally optimal covering density. So, we fix a Delone triangulation and try to find the positive definite quadratic form which minimizes the density function in the topological closure of the secondary cone of the fixed Delone triangulation. We will formulate this restricted lattice covering problem as a determinant maximization problem. RYSHKOV and BARANOVSKII anticipated that an algorithm for the lattice covering problem does exist. In [RB1976] (page 115) they write "Lemma 20.5 can be used for developing "machine" methods for finding the minima of the functions $\varphi_i(f)$; that is, these minima can be found to within a defined accuracy using a computer".

Algorithms which solve the dual lattice packing problem have a long history in the geometry of numbers. The first algorithms were already proposed by MINKOWSKI in [Min1905] and by VORONOÏ in [Vor1907]. These algorithms have been successfully applied in dimensions up to 7 (see [Bar1957] for $d = 6$, [Jaq1993] for $d = 7$; currently the group around MARTINET is working on the case $d = 8$). For more information on this topic see [CS1988b], [Mar2003], the catalogue of lattices* by NEBE and SLOANE, and the catalogue of perfect lattices† by MARTINET and BATUT.)

7.1. A Restricted Lattice Covering Problem

Recall that the covering density of a positive definite quadratic form Q in d variables is given by $\Theta(Q) = \sqrt{\frac{\operatorname{vol} B_d(\mathbf{0},\mu(Q))}{\det Q}}$ where $\mu(Q) = \max_{\boldsymbol{x}\in\mathbb{R}^d} \min_{\boldsymbol{v}\in\mathbb{Z}^d} Q[\boldsymbol{x}-\boldsymbol{v}]$ is the inhomogeneous minimum of Q. Scaling of Q by a positive real number α leaves the covering density function invariant:

$$\Theta(\alpha Q) = \sqrt{\frac{\operatorname{vol} B_d(\mathbf{0},\mu(\alpha Q))}{\det(\alpha Q)}} = \sqrt{\frac{\operatorname{vol} B_d(\mathbf{0},\alpha\mu(Q))}{\alpha^d \det Q}} = \sqrt{\frac{\alpha^d \operatorname{vol} B_d(\mathbf{0},\mu(Q))}{\alpha^d \det Q}} = \Theta(Q).$$

Consequently we can restrict our attention to those positive definite quadratic forms Q with

*http://www.research.att.com/~njas/lattices/index.html

†http://www.math.u-bordeaux.fr/~martinet

$\mu(Q) = 1$. Hence, we solve the lattice covering problem if we solve the optimization problem

maximize	$\det(Q)$
subject to	Q is a positive definite matrix,
	$\mu(Q) = 1$,

where the optimization variables are q_{ij}, the entries of the symmetric matrix Q. The major disadvantage of this optimization problem is that the second constraint is not expressible as a convex condition in the optimization variables q_{ij} and that the problem has many local maxima. A locally optimal solution is also called *locally optimal lattice covering*.

We will circumvent this by splitting the original problem into a finite number of determinant maximization problems. For every Delone triangulation $\mathcal{D}$ we solve the optimization problem

maximize	$\det(Q)$
subject to	Q is a positive definite matrix,
	$Q \in \overline{\boldsymbol{\Delta}(\mathcal{D})}$,
	$\mu(Q) \leq 1$.

The relaxation of no longer requiring $\mu(Q) = 1$ in the third constraint does not give more optimal solutions because with Q also $\frac{1}{\mu(Q)}Q$ satisfies the constraints. Now, we have to show that this is indeed a determinant maximization problem. We have seen in Theorem 2.5.1 that the second constraint can be expressed with inequalities linear in q_{ij}. The constraint $\mu(Q) \leq 1$ is equivalent to the fact that the radius of the circumsphere of any full-dimensional Delone simplex $L \in \mathcal{D}$ is at most one. For a Delone simplex L this can be expressed by a linear matrix inequality $\mathrm{BR}_L(Q) \succeq 0$ in the variables q_{ij} as stated in Proposition 6.3.1.

A determinant maximization problem is of the form

minimize	$\boldsymbol{c}^t\boldsymbol{x} - \log\det G(\boldsymbol{x})$
subject to	$G(\boldsymbol{x})$ is a positive definite matrix,
	$F(\boldsymbol{x})$ is a positive semidefinite matrix.

In our case the optimization vector $\boldsymbol{x}$ is given by the vector of coefficients of Q

$$\boldsymbol{x} = (q_{11}, q_{21}, q_{22}, q_{31}, \dots, q_{dd})^t \in \mathbb{R}^{d(d+1)/2},$$

and the linear matrix inequality $G(\boldsymbol{x}) \succ 0$ is given by

$$G(\boldsymbol{x}) = q_{11}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \end{pmatrix} + q_{22}\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \end{pmatrix} + \dots + q_{dd}\begin{pmatrix} 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$
$$+ q_{21}\begin{pmatrix} 0 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} + q_{31}\begin{pmatrix} 0 & 0 & \frac{1}{2} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} + \dots + q_{d,d-1}\begin{pmatrix} 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \frac{1}{2} \\ 0 & \dots & \frac{1}{2} & 0 \end{pmatrix},$$

such that $G(\boldsymbol{x}) = Q$. The two other constraints $Q \in \overline{\boldsymbol{\Delta}(Q)}$ and $\mu(Q) \leq 1$ can be encoded by two block matrices in the linear matrix inequality $F(\boldsymbol{x}) \succeq 0$. Instead of struggling with indices and notation we demonstrate the encoding in a simple two-dimensional example which differs from the general case only by the number of subblock matrices involved.

Let $\mathcal{D}_1$ be the Delone triangulation of VORONOÏ's principal form of the first type $Q[\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)] = 2x_1^2 + 2x_2^2 - 2x_1x_2$. From Section 2.3 we know that the topological closure of the secondary cone of $\mathcal{D}_1$ is given by the linear inequalities

$$\begin{array}{ccccccc} q_{11} & + & q_{21} & & & \geq & 0 \\ & - & q_{21} & & & \geq & 0 \\ & & q_{21} & + & q_{22} & \geq & 0 \end{array}$$

Every matrix Q whose coefficients satisfy the above inequalities belongs to $\overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$.

As we saw in Section 2.3, $\mathcal{D}_1$ is given by the set of simplices $\{\boldsymbol{v} + L_\pi : \boldsymbol{v} \in \mathbb{Z}^d, \pi \in \mathsf{S}_3\}$ where $L_\pi = \operatorname{conv}\{\boldsymbol{e}_{\pi(1)}, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)}, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)} + \boldsymbol{e}_{\pi(3)}\}$, and $\boldsymbol{e}_1 = (1,0)^t$, $\boldsymbol{e}_2 = (0,1)^t$, $\boldsymbol{e}_3 = (-1,-1)^t$. Every full-dimensional simplex of $\mathcal{D}_1$ is either a translate of L_{id} or of $L_{(13)}$. Furthermore $L_{(13)}$ is transformed into L_{id} by the map $\boldsymbol{x} \to -\boldsymbol{x}$. In Figure 7.1 we show the simplices of $\mathcal{D}_1$ containing the origin.

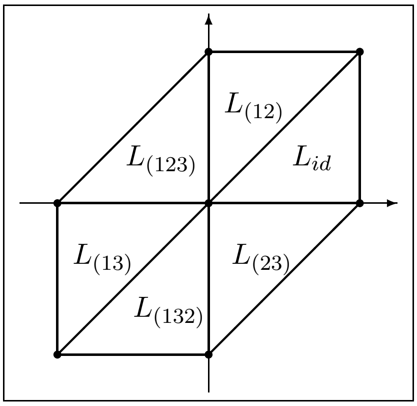


Figure 7.1. Delone Triangulation $\mathcal{D}_1$.

Thus, we have $\mu(Q) \leq 1$ for a $Q \in \overline{\boldsymbol{\Delta}(\mathcal{D}_1)}$ if and only if the radius of the circumsphere of $L_{id} = \operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ with $\boldsymbol{v}_0 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\boldsymbol{v}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\boldsymbol{v}_2 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, is at most 1. This translates into the linear matrix inequality

$$\begin{aligned} \mathrm{BR}_{L_{id}}(Q) &= \begin{pmatrix} 4 & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) \\ (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_1) & (\boldsymbol{v}_1, \boldsymbol{v}_2) \\ (\boldsymbol{v}_2, \boldsymbol{v}_2) & (\boldsymbol{v}_2, \boldsymbol{v}_1) & (\boldsymbol{v}_2, \boldsymbol{v}_2) \end{pmatrix} \\ &= \begin{pmatrix} 4 & q_{11} & q_{11} + 2q_{21} + q_{22} \\ q_{11} & q_{11} & q_{11} + q_{21} \\ q_{11} + 2q_{21} + q_{22} & q_{11} + q_{21} & q_{11} + 2q_{21} + q_{22} \end{pmatrix} \succeq 0. \end{aligned}$$

A block matrix is positive semidefinite if and only if each of its blocks is positive semidefinite. Finally, the linear matrix inequality $F(\boldsymbol{x}) \succeq 0$ looks as follows

$$\begin{aligned} & F((q_{11}, q_{21}, q_{22})^t) \\ &= \begin{pmatrix} \boxed{\begin{matrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{0} \end{matrix}} & \\ & \boxed{\begin{matrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} \end{pmatrix} + q_{11} \begin{pmatrix} \boxed{\begin{matrix} \boxed{1} & & \\ & \boxed{0} & \\ & & \boxed{0} \end{matrix}} & \\ & \boxed{\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}} \end{pmatrix} + \end{aligned}$$

$$+q_{21}\begin{pmatrix} \begin{array}{|ccc|}\hline \boxed{1} & & \\ & \boxed{-1} & \\ & & \boxed{1} \\ \hline\end{array} & \\ & \begin{array}{|ccc|}\hline 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \\ \hline\end{array}\end{pmatrix} + q_{22}\begin{pmatrix} \begin{array}{|ccc|}\hline \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{1} \\ \hline\end{array} & \\ & \begin{array}{|ccc|}\hline 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline\end{array}\end{pmatrix}.$$

In the general case we have for any linear inequality which is needed to describe the secondary cone one 1×1 block matrix. For any non-equivalent d-dimensional simplex $L \in \mathcal{D}$ we have the $(d+1) \times (d+1)$ block matrix $\mathrm{BR}_L(Q)$.

7.2. Interpretation of "Classical" Results

We have seen above that the lattice covering problem can be split into a finite number of restricted lattice covering problems. Then every restricted problem is expressible as a determinant maximization problem. In the "classical" works on the lattice covering problem by BARNES, DICKSON and by DELONE, DOLBILIN, RYSHKOV, and STOGRIN the restricted lattice covering problem was also considered. There, ad hoc methods were used to find locally optimal solutions. Many calculations were performed by hand. In the process of simplifying these a number of interesting structural properties of locally optimal lattice coverings were found. Here, we want to interpret these properties in the context of the determinant maximization problem.

Let $\mathcal{D}$ be a Delone triangulation. In [BD1967] BARNES and DICKSON show that there is at most one positive definite quadratic form Q (together with all positive multiples) with $\mathrm{Del}(Q) = \mathcal{D}$ and giving a locally optimal lattice covering. They use analytical methods for the proof. Another proof of this fact which is based on convexity arguments and which gives a clear geometric picture is given in [DDRS1970] by DELONE, DOLBILIN, RYSHKOV, and STOGRIN. We want to find a positive definite quadratic form Q with $\mu(Q) \leq 1$ and with maximum determinant. Let $L_1, L_2, \ldots, L_n$ be representatives of all non-equivalent full-dimensional Delone simplices of $\mathcal{D}$. Then we have $\mu(Q) \leq 1$ if and only if for every L_i the linear matrix inequality $\mathrm{BR}_{L_i}(Q) \succeq 0$ holds. The set of all $Q \in \mathcal{S}^d_{>0}$ for which the linear matrix inequality $\mathrm{BR}_{L_i}(Q) \succeq 0$ holds is convex. Therefore, the set

$$\{Q \in \mathcal{S}^d_{>0} : \mu(Q) \leq 1\} = \bigcap_{i=1}^{n} \{Q \in \mathcal{S}^d_{>0} : \mathrm{BR}_{L_i}(Q) \succeq 0\}$$

is convex, too. For every positive real number D the determinant-D-surface

$$\mathcal{S}^d_{>0}(D) = \{Q \in \mathcal{S}^d_{>0} : \det Q = D\}$$

is *strictly* convex, i.e. the interior of the segment $[Q_1, Q_2]$ with $Q_1 \in \mathcal{S}^d_{>0}(D)$, $Q_2 \in \mathcal{S}^d_{>0}(D)$ lies above the surface $\mathcal{S}^d_{>0}(D)$: for every $\alpha \in (0,1)$ we have $\det(\alpha Q_1 + (1-\alpha)Q_2) > D$ (see Section 5.2). Thus, there is exactly one Q with $\mu(Q) \leq 1$ and maximum determinant.

Note that the Q above does not necessarily give a locally optimal lattice covering when Q lies on the boundary of $\overline{\boldsymbol{\Delta}(\mathcal{D})}$. In [Dic1968] DICKSON reports that for dimensions $d \geq 14$ there is no positive definite quadratic form giving a locally optimal lattice covering which lies in the interior of the secondary cone $\overline{\boldsymbol{\Delta}(\mathcal{D}_2)}$. Here the Delone triangulation $\mathcal{D}_2$ is the (up to equivalence) unique bistellar neighbour of the Delone triangulation $\mathcal{D}_1$ belonging to VORONOÏ's principal form of the first type. The positive definite quadratic form Q which gives the best lattice covering in $\overline{\mathbf{\Delta}(\mathcal{D}_2)}$ lies on the boundary of $\overline{\mathbf{\Delta}(\mathcal{D}_2)}$ with $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$. It does not give a locally optimal lattice covering because VORONOÏ's principal form of the first type which lies in the interior of $\overline{\mathbf{\Delta}(\mathcal{D}_1)}$ gives a locally optimal lattice covering in any dimension (see Theorem 8.4.1). A boundary form Q gives a locally optimal lattice covering if and only if for every Delone triangulation $\mathcal{D}'$ with $Q \in \overline{\mathbf{\Delta}(\mathcal{D}')}$ we have for all $Q' \in \overline{\mathbf{\Delta}(\mathcal{D}')}$ the inequality $\Theta(Q) \leq \Theta(Q')$. At the moment there is no positive definite quadratic form known which lies on the boundary giving a locally optimal lattice covering. But we strongly believe that they do exist (e.g. the positive definite quadratic form associated to the Leech lattice).

As an immediate consequence of the uniqueness we have the following invariant property first discovered by BARNES and DICKSON ([BD1967]): Let $A \in \mathsf{GL}_d(\mathbb{Z})$ be a unimodular transformation which leaves the Delone triangulation $\mathcal{D}$ fixed: $A\mathcal{D} = \mathcal{D}$. Then A also leaves the secondary cone of $\mathcal{D}$ fixed: $A^t\mathbf{\Delta}(\mathcal{D})A = \mathbf{\Delta}(\mathcal{D})$. If $Q \in \mathbf{\Delta}(\mathcal{D})$ gives a locally optimal lattice covering, then Q has to be invariant under A, otherwise $A^tQA \in \mathbf{\Delta}(\mathcal{D})$ would give another locally optimal lattice covering contradicting the uniqueness of Q. Hence the automorphism group of $\mathcal{D}$ is a subgroup of the automorphism group of Q. In [Dic1968] DICKSON shows that if $Q' \in \mathbf{\Delta}(Q)$ gives an optimal lattice covering among all $Q' \in \mathbf{\Delta}(Q)$ with $\mathrm{Aut}(\mathcal{D}) \subseteq \mathrm{Aut}(Q')$, then Q gives a locally optimal lattice covering.

In [BD1967] BARNES and DICKSON give a criterion which can be used to decide whether a given positive definite quadratic form Q whose Delone subdivision is a triangulation gives a locally optimal lattice covering. Let Q^{-1} be the positive definite quadratic form inverse to Q. Then Q gives a locally optimal lattice covering if and only if there exist real numbers $\lambda_L \geq 0$ such that Q^{-1} can be expressed as

$$Q^{-1}[\boldsymbol{x}] = \sum_L \lambda_L \left(\sum_{i=0}^{d} \alpha_i (\boldsymbol{v}_i^t \boldsymbol{x})^2 - (\boldsymbol{c}_L^t \boldsymbol{x})^2 \right),$$

where $L = \mathrm{conv}\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_d\}$ runs over all d-dimensional Delone simplices with $\boldsymbol{v}_0 = \mathbf{0}$, and $\boldsymbol{c}_L = \sum_i \alpha_i \boldsymbol{v}_i$, $\sum_i \alpha_i = 1$, is the center of the circumsphere of the simplex L. This criterion can be viewed as a variant of the optimality criterion for the determinant maximization problem given by duality theory which was stated in Theorem 5.4.1. The criterion of BARNES and DICKSON asks for the feasibility of a linear optimization problem. So from the computational point of view it is much simpler than the optimality criterion for determinant maximization problems. Moreover, it is an analogue to the so-called “eutactic” criterion for locally optimal lattice packings (see [Mar2003]). A positive definite quadratic form Q is said to be *eutactic* if there exist real numbers $\lambda_{\boldsymbol{v}} > 0$ such that Q^{-1} can be expressed as

$$Q^{-1}[\boldsymbol{x}] = \sum_{\boldsymbol{v}} \lambda_{\boldsymbol{v}} (\boldsymbol{v}^t \boldsymbol{x})^2$$

where $\boldsymbol{v}$ runs over all shortest vectors $\boldsymbol{v} \in \mathbb{Z}^d$ of Q with $Q[\boldsymbol{v}] = \min_{\boldsymbol{w} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} Q[\boldsymbol{w}]$. Shortest vectors appear always in pairs and we take for every pair of shortest vectors $(\boldsymbol{v}, -\boldsymbol{v})$ only one. Every positive definite quadratic form which gives a locally optimal lattice packing has to be eutactic. VORONOÏ proved this in [Vor1907].

All the investigations above have the disadvantage that they only work if the considered positive definite quadratic form lives in the interior of the secondary cone of a Delone triangulation, i.e. that its Delone subdivision is a triangulation. In [DDRS1970] DELONE, DOLBILIN, RYSHKOV and STOGRIN explored what can happen if the Delone subdivision of a positive definite quadratic form is not a triangulation. Let $\mathcal{D}$ be a Delone triangulation and let Q be the optimal solution of the restricted lattice covering problem as given in the beginning of Section 7.1. Now relax this problem by omitting the second constrained "Q lies in the topological closure of the secondary cone of $\mathcal{D}$". This relaxed problem is again a determinant maximization problem. Let Q' be an optimal solution. Then we can have the following three cases.

i) If Q' lies in the interior of $\overline{\mathbf{\Delta}(\mathcal{D})}$, then $Q' = Q$ and Q gives a locally optimal lattice covering.

ii) If Q' lies outside $\overline{\mathbf{\Delta}(\mathcal{D})}$, then obviously $Q' \neq Q$, and Q lies on the boundary of $\overline{\mathbf{\Delta}(\mathcal{D})}$. Furthermore, Q does not give a locally optimal lattice covering.

iii) If Q' lies on the boundary of $\overline{\mathbf{\Delta}(\mathcal{D})}$, then $Q = Q'$ and Q gives a locally optimal lattice covering if and only if Q gives in this way a local minimum for all neighbouring secondary cones.

Chapter 8.

Moments of Inertia

In this section we give a simple and efficiently computable local lower bound of the covering density function. The local lower bound does only apply to those positive definite quadratic forms lying in the topological closure of the secondary cone of a given Delone triangulation. For the computation of the lower bound we only need to know the coordinates of the simplices of the considered Delone triangulation.

The method goes back to RYSHKOV and DELONE. It is called the method of the moments of inertia because the central idea in its proof is analogous to the Parallel Axis Theorem in classical mechanics. Let $I_{\boldsymbol{m}}$ be the moment of inertia of a body about a fixed axis passing through the body's center of gravity $\boldsymbol{m}$. Then the moment of inertia $I_{\boldsymbol{x}}$ about another fixed axis $\boldsymbol{x}$ parallel to the former one can be determined by $I_{\boldsymbol{c}} = Mr^2 + I_{\boldsymbol{m}}$ where M is the mass of the body and r is the distance between the two axis.

The method of the moments of inertia can be applied in many different situations:

- We will use it to prove that VORONOÏ's principal form $Q[\boldsymbol{x}] = d\sum x_i^2 - \sum_{i\neq j} x_i x_j$ of the first type provides the optimal lattice covering among all positive definite quadratic forms with the same Delone triangulation as Q.
- We will apply the method to solve the lattice covering problem in dimensions $d = 2, 3, 4$.
- We will use the local lower bounds as a heuristic measure to find good lattice coverings in dimension $d = 6, 7$: In these dimensions the number of non-equivalent Delone triangulations starts to explode, there are more than $250,000$ non-equivalent Delone triangulations in dimension 6, and it is not clear which Delone triangulation admits a good lattice covering. We encode the set of Delone triangulations as an undirected labeled graph. Every triangulation represents a node. We connect two triangulations by an edge if they are bistellar neighbours. The nodes are labeled by the local lower bound of the triangulation. Then we try to find the nodes that give globally optimal labeling by a randomized greedy approach.
- Finally, we will show that the local lower bounds of two equivalent Delone triangulations coincide. This give a strong invariant.

In our account and especially in the first three sections of this chapter we mainly follow [RB1976], §23. On a single bound for the covering density for each of the L-type n-dimensional lattices.

8.1. The Moment of Inertia and the Circumradius of a Simplex

Let $P \subseteq \mathbb{R}^d$ be a finite set of points in d-dimensional Euclidean space $(\mathbb{R}^d, (\cdot,\cdot))$. We interpret the points of P as masses with unit weight. The *moment of inertia* of the points about a point $\boldsymbol{x} \in \mathbb{R}^d$

is defined as $I_{\boldsymbol{x}}(P) = \sum_{\boldsymbol{v}\in P} \operatorname{dist}(\boldsymbol{x},\boldsymbol{v})^2$. The *centroid* of P (center of gravity) is given by $\boldsymbol{m} = \frac{1}{|P|}\sum_{\boldsymbol{v}\in P}\boldsymbol{v}$. From the equations

$$\begin{aligned}\operatorname{dist}(\boldsymbol{x},\boldsymbol{v})^2 &= (\boldsymbol{x}-\boldsymbol{v},\boldsymbol{x}-\boldsymbol{v})\\ &= (\boldsymbol{x}-\boldsymbol{m},\boldsymbol{x}-\boldsymbol{m}) + (\boldsymbol{m}-\boldsymbol{v},\boldsymbol{m}-\boldsymbol{v}) + 2(\boldsymbol{x}-\boldsymbol{m},\boldsymbol{m}-\boldsymbol{v})\\ &= \operatorname{dist}(\boldsymbol{x},\boldsymbol{m})^2 + \operatorname{dist}(\boldsymbol{m},\boldsymbol{v})^2 + 2(\boldsymbol{x}-\boldsymbol{m},\boldsymbol{m}-\boldsymbol{v}),\end{aligned}$$

and $\sum_{\boldsymbol{v}\in P}(\boldsymbol{m}-\boldsymbol{v}) = \boldsymbol{0}$, we derive the following formula (APOLLONIUS' formula which relates to the Parallel Axis Theorem in classical mechanics, see [Ber1987] §9.7.6)

$$I_{\boldsymbol{x}}(P) = |P| \operatorname{dist}(\boldsymbol{x},\boldsymbol{m})^2 + I_{\boldsymbol{m}}(P). \tag{8.1}$$

Hence, the moment of inertia about the centroid $\boldsymbol{m}$ is minimal.

If the points of P form the vertices of a d-dimensional simplex, then (8.1) gives a relationship between the radius of the circumsphere R, the center of the circumsphere $\boldsymbol{c}$, and the moment of inertia about the centroid $\boldsymbol{m}$ of P:

$$R^2 = \frac{I_{\boldsymbol{c}}(P)}{d+1} = \operatorname{dist}(\boldsymbol{c},\boldsymbol{m})^2 + \frac{I_{\boldsymbol{m}}(P)}{d+1}.$$

We can compute $I_{\boldsymbol{m}}(P)$ using only the edge lengths of the simplex P which gives nicer formulas and makes some computations less laboriously. For every vertex $\boldsymbol{w} \in \operatorname{vert} P$ we have by definition $I_{\boldsymbol{w}}(P) = \sum_{\boldsymbol{v}\in P} \operatorname{dist}(\boldsymbol{w},\boldsymbol{v})^2$. Summing up and using (8.1) gives

$$\sum_{\boldsymbol{w}\in P} I_{\boldsymbol{w}}(P) = \sum_{\boldsymbol{w}\in P}((d+1)\operatorname{dist}(\boldsymbol{w},\boldsymbol{m}) + I_{\boldsymbol{m}}(P)) = 2(d+1)I_{\boldsymbol{m}}(P).$$

So, we get

$$I_{\boldsymbol{m}}(P) = \frac{1}{d+1}\sum_{\{\boldsymbol{v},\boldsymbol{w}\}\subseteq P} \operatorname{dist}(\boldsymbol{v},\boldsymbol{w})^2. \tag{8.2}$$

Let $\mathcal{D}$ be a Delone triangulation of $\mathbb{R}^d$, let $L_1,\dots,L_n$ be the d-dimensional simplices of the star of a lattice point (say e.g. the origin), and let $\boldsymbol{m}_i$ be the centroid of L_i, $i = 1,\dots,n$. The arithmetical mean of the moments of inertia about the centroids of L_i with respect to a positive definite quadratic form Q is defined as

$$I_{\mathcal{D}}(Q) = \frac{1}{n}\sum_{i=1}^{n} I_{\boldsymbol{m}_i}(L_i),$$

and is called the *central moment of inertia* of $\mathcal{D}$ with respect to Q (note that we are now dealing with the scalar product given by Q: $\operatorname{dist}(\boldsymbol{x},\boldsymbol{y})^2 = Q[\boldsymbol{x}-\boldsymbol{y}]$).

Proposition 8.1.1. The central moment of inertia of $\mathcal{D}$ with respect to Q yields a lower bound of the inhomogeneous minimum of Q if $\mathcal{D}$ is a refinement of $\operatorname{Del}(Q)$. In this case we have $\mu(Q) \geq \frac{1}{d+1} I_{\mathcal{D}}(Q)$.

Proof. Let R_i be the radius and $\boldsymbol{c}_i$ be the center of the circumsphere of the simplex L_i, then

$$\begin{aligned}\mu(Q) &= \max_{i=1,\dots,n} R_i^2 &&= \max_{i=1,\dots,n}\left(\operatorname{dist}(\boldsymbol{c}_i,\boldsymbol{m}_i)^2 + \tfrac{I_{\boldsymbol{m}_i}(L_i)}{d+1}\right)\\ &\geq \max_{i=1,\dots,n} \tfrac{I_{\boldsymbol{m}_i}(L_i)}{d+1} &&\geq \tfrac{1}{(d+1)n}\sum_{i=1}^{n} I_{\boldsymbol{m}_i}(L_i)\\ &= \tfrac{1}{d+1} I_{\mathcal{D}}(Q).\end{aligned}$$

◇

If $\mathcal{D}$ is a Delone triangulation, the function $I_{\mathcal{D}}$ can be used to find a lower bound of the covering density for all positive definite quadratic forms Q with $\operatorname{Del}(Q) \subseteq \mathcal{D}$. For this we minimize the linear function $I_{\mathcal{D}}$ over all positive definite quadratic forms with a fixed determinant.

8.2. Linear Optimization on Equidiscriminant Surfaces

Minimizing the linear function $I_{\mathcal{D}}$ over all positive definite quadratic forms with a fixed determinant is computationally easy but requires some preceding mathematical work.

Definition 8.2.1. (*Determinant-D-Surface/Equidiscriminant Surface*)
Let D be a positive real number. The $\left(\frac{d(d+1)}{2} - 1\right)$-dimensional submanifold of $\mathcal{S}^d_{>0}$

$$\mathcal{S}^d_{>0}(D) = \{Q \in \mathcal{S}^d_{>0} : \det Q = D\}$$

is called the *determinant-D-surface*. Obviously, we have the partition $\mathcal{S}^d_{>0} = \bigcup_{D \in \mathbb{R}_{>0}} \mathcal{S}^d_{>0}(D)$.

The vector space $\mathcal{S}^d$ of symmetric $(d \times d)$-matrices is equipped with the scalar product $\langle F, G\rangle = \mathrm{Trace}(FG)$. Let f be a linear function f on $\mathcal{S}^d$. Then, by identifying $(\mathcal{S}^d)^*$ with $\mathcal{S}^d$ using the scalar product, we can write $f(\cdot) = \langle F, \cdot\rangle$ for a symmetric matrix $F \in \mathcal{S}^d$. Suppose F is positive definite, then f has a unique local minimum on the determinant-D-surface which we now explicitly compute.

Proposition 8.2.2. Let $f(\cdot) = \langle F, \cdot\rangle \in (\mathcal{S}^d)^*$ be a linear function on $\mathcal{S}^d$, and $F = (f_{ij})_{1 \le i,j,\le d}$ a positive definite matrix. Then f has a unique local minimum on the determinant-D-surface. Its value is $d\sqrt[d]{D \det F}$ and the minimum is attained at the positive definite matrix $\sqrt[d]{D \det F}F^{-1}$.

Proof. We will make use of Lagrange multipliers. Consider the Lagrangian

$$L(Q; \lambda) = f(Q) + \lambda(D - \det Q).$$

The partial derivatives of L are

$$\frac{\partial L}{\partial q_{ii}}(Q; \lambda) = f_{ii} - \lambda Q_{ii}, \quad \frac{\partial L}{\partial q_{ij}}(Q; \lambda) = 2f_{ij} - 2\lambda Q_{ij},$$

where the Q_{ij}'s are the cofactors of the matrix Q (the $(d-1) \times (d-1)$-matrix Q_{ij} is obtained from Q by eliminating row i and column j and by multiplication with $(-1)^{i+j}$).

If f has a local minimum, then λ has to fulfill the equations

$$f_{ii} - \lambda Q_{ii} = 0, \quad 2(f_{ij} - \lambda Q_{ij}) = 0, \tag{8.3}$$

so we get $\lambda = \frac{f_{ij}}{Q_{ij}}$ whenever $Q_{ij} \neq 0$. Using this relation we are able to compute λ in terms of the coefficients f_{ij}. The determinant of the matrix Q^{-1} is $\det Q^{-1} = \frac{1}{\det Q} = \frac{1}{D}$. By Laplacian expansion and (8.3) we also have

$$\det Q^{-1} = \det\left(\frac{1}{\det Q}(Q_{ij})\right) = \frac{1}{D^d}\det\left(\frac{f_{ij}}{\lambda}\right) = \frac{1}{\lambda^d D^d}\det F.$$

Thus, $\frac{1}{D} = \frac{1}{\lambda^d D^d}\det F$ and $\lambda = \frac{1}{D}\sqrt[d]{D \det F}$. Let $Q = (q_{ij})$ be a critical point, then again by Laplacian expansion

$$\begin{aligned} L(Q; \lambda) &= \sum_{1 \le i,j \le d} f_{ij} q_{ij} + \lambda(D - \det(q_{ij})) \\ &= \sum_{1 \le i,j \le d} \lambda Q_{ij} q_{ij} + 0 \\ &= \lambda \sum_{i=1}^{d}\sum_{j=1}^{d} q_{ij} Q_{ij} \\ &= \lambda d D. \end{aligned}$$

Since f is continuous and bounded from below by 0, f has a minimum. So $\lambda dD = d\sqrt[d]{D \det F}$ is f's unique minimum. It is attained at the positive definite matrix $\sqrt[d]{D \det F}F^{-1}$ because

$$f\left(\sqrt[d]{D \det F}F^{-1}\right) = \left\langle F, \sqrt[d]{D \det F}F^{-1}\right\rangle = d\sqrt[d]{D \det F},$$

and $\det\left(\sqrt[d]{D \det F}F^{-1}\right) = D$. $\diamond$

8.3. A Local Lower Bound

Now we can plug Propositions 8.1.1 and Proposition 8.2.2 together. This yields a local lower bound for the covering density of a positive definite quadratic form. "Local" means that we first have to fix a Delone triangulation and then the lower bound applies only for the positive definite quadratic forms lying in the topological closure of the secondary cone of the fixed Delone triangulation.

Proposition 8.3.1. Let $\mathcal{D}$ be a Delone triangulation. Let Q be a positive definite quadratic form for which $\mathcal{D}$ is a refinement of $\mathrm{Del}(Q)$. Then we have a lower bound for the normalized covering density of Q:

$$\theta(Q) \geq \theta_*(\mathcal{D}) = \sqrt{\left(\frac{d}{d+1}\right)^d \det F},$$

where F is the positive definite matrix given by the equation $I_{\mathcal{D}}(\cdot) = \langle F, \cdot\rangle$. We denote the local lower bound for the Delone triangulation $\mathcal{D}$ by $\theta_*(\mathcal{D})$.

Proof. Since $I_{\mathcal{D}}$ is a linear function there is a symmetric matrix F with $I_{\mathcal{D}}(\cdot) = \langle F, \cdot\rangle$. For every positive definite matrix Q we have $\langle F, Q\rangle = I_{\mathcal{D}}(Q) > 0$. Since $\mathcal{S}^d_{>0}$ is a self-dual cone, F is positive definite. Now we can apply Proposition 8.2.2: On the determinant-D-surface $I_{\mathcal{D}}$ has the unique minimum $d\sqrt[d]{D \det F}$. Using this with $D = \det Q$ and using Proposition 8.1.1 we get a lower bound for $\theta(Q)$: The normalized covering density of Q is at least

$$\theta(Q) = \sqrt{\frac{\mu(Q)^d}{\det Q}} \geq \sqrt{\left(\frac{I_{\mathcal{D}}(Q)}{d+1}\right)^d / \det Q} \geq \sqrt{\frac{d^d \det Q \det F}{(d+1)^d \det Q}} = \sqrt{\left(\frac{d}{d+1}\right)^d \det F}$$

$\diamond$

8.4. Applications

8.4.1. The Lattice Covering of A_d^*

As a first application of the method we show that VORONOÏ's principal form of the first type $Q[\boldsymbol{x}] = d\sum x_i^2 - \sum_{i\neq j} x_i x_j$, which is associated to the lattice A_d^* that is the dual of the root lattice A_d, gives a locally optimal lattice covering. This was independently discovered by GAMECKII ([Gam1962], [Gam1963]) and by BLEICHER ([Ble1962]).

Let $\mathcal{D}_1 = \mathrm{Del}(Q)$ be the Delone triangulation of Q. The determinant of Q is $(d+1)^{d-1}$. Then, there exists a unique local minimum of the covering density function in the intersection of the topological closure of $\mathcal{D}_1$'s secondary cone and the determinant-$(d+1)^{d-1}$-surface. The minimum is attained at Q. Every binary and every ternary positive definite quadratic form can be transformed by a unimodular integral transformation to a form which lies in $\mathcal{D}_1$'s secondary cone. Hence, the following theorem solves the lattice covering problem in two and three dimensions.

The possibility of applying the method to VORONOÏ's principal form of the first type is indicated in [RB1976] §23.5 where it is worked out only for dimension 5*.

Theorem 8.4.1. (GAMECKII, BLEICHER)
Let $\mathcal{D}_1$ be the Delone triangulation of VORONOÏ's principal form of the first type. Then for every $Q \in \overline{\mathbf{\Delta}(\mathcal{D}_1)}$ the following inequality holds

$$\theta(Q) \geq \sqrt{\left(\frac{d(d+2)}{12(d+1)}\right)^d (d+1)}.$$

This inequality is tight if and only if Q is VORONOÏ's principal form of the first type. In other words the lattice A_d^* gives a locally optimal lattice covering in every dimension.

Proof. We have described the Delone triangulation of Q already in Chapter 2.3. Its set of d-dimensional simplices consists of $\{\boldsymbol{v} + L_\pi : \boldsymbol{v} \in \mathbb{Z}^d, \pi \in \mathsf{S}_{d+1}\}$, where L_π is the simplex

$$L_\pi = \operatorname{conv}\{\boldsymbol{e}_{\pi(1)}, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)}, \ldots, \boldsymbol{e}_{\pi(1)} + \boldsymbol{e}_{\pi(2)} + \cdots + \boldsymbol{e}_{\pi(d+1)}\},$$

$\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ are the standard basis vectors of $\mathbb{Z}^d$ complemented by $\boldsymbol{e}_{d+1} = -\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_d$. With SELLING's formula (Proposition 2.3.1)

$$\left\|\sum\nolimits_{i=1}^{d+1} \alpha_i \boldsymbol{e}_i\right\|^2 = \sum_{k<l} -q_{kl}(\alpha_k - \alpha_l)^2, \text{ where } q_{kl} = (\boldsymbol{e}_k, \boldsymbol{e}_l),\ k, l = 1, \ldots, d+1, \tag{8.4}$$

it is possible to use the symmetry of the Delone triangulation in the following calculations.

First, we show that the positive definite matrix F with $I_{\mathcal{D}_1}(\cdot) = \langle F, \cdot\rangle$ is given by the quadratic form $F[\boldsymbol{x}] = \alpha_d \sum_{1\leq i\leq j\leq d} x_i x_j$ where α_d is a positive real number. That means F is a scaled version of the "first perfect" quadratic form which is associated to the root lattice A_d. Next, we compute the positive scaling factor α_d, so that we can apply Proposition 8.3.1.

We have

$$\begin{aligned} I_{\mathcal{D}_1}(Q) &= \tfrac{1}{(d+1)!} \sum_{\pi\in \mathsf{S}_{d+1}} I_{\boldsymbol{m}_\pi}(L_\pi) \\ &= \tfrac{1}{(d+1)!} \sum_{\pi\in \mathsf{S}_{d+1}} \tfrac{1}{d+1} \sum_{1\leq i<j\leq d+1} \|\boldsymbol{e}_{\pi(i+1)} + \boldsymbol{e}_{\pi(i+2)} + \cdots + \boldsymbol{e}_{\pi(j)}\|^2 \quad \text{(by (8.2))} \\ &= \tfrac{1}{(d+1)!(d+1)} \sum_{\pi\in \mathsf{S}_{d+1}} \sum_{1\leq i<j\leq d+1} \sum_{1\leq k,l\leq d+1} -q_{kl}\delta_{\pi,ij,kl} \quad \text{(by (8.4))}, \end{aligned}$$

where

$$\delta_{\pi,ij,kl} = \begin{cases} 1, & \text{if } |\{k,l\} \cap \{\pi(i+1), \pi(i+2), \ldots, \pi(j)\}| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If we count the number of ones in $\delta_{\pi,ij,kl}$, $\pi \in \mathsf{S}_{d+1}$, we find

$$\sum_{\pi\in \mathsf{S}_{d+1}} \delta_{\pi,ij,kl} = 2(d-1)!(j-i)((d+1)-(j-i))$$

because

- ▶ fixing $\pi^{-1}(k) \in \{i+1, \ldots, j\}$ and $\pi^{-1}(l) \in \{1, \ldots, d+1\}\backslash\{i+1, \ldots, j\}$ gives a factor of $(j-i)((d+1)-(j-i))$,

* "For convenience we consider the case $n = 5$. In the context of arbitrary n it is not extremely difficult to obtain the estimate [...] of the covering density of type I lattices ..."

- after fixation of $\pi^{-1}(k)$ and $\pi^{-1}(l)$ all other images of the permutation π^{-1} can be chosen arbitrarily which gives a factor of $(d-1)!$,
- interchanging the roles of k and l gives another factor of 2.

So, we obtain

$$I_{\mathcal{D}_1}(Q) \quad = \quad \frac{1}{(d+1)!(d+1)} \sum_{1\le k,l\le d+1} -q_{kl} \sum_{1\le i<j\le d+1} 2(d-1)!(d+1-(j-i))(j-i).$$

We simplify the sum $\sum_{1\le i<j\le d+1}(d+1-(j-i))(j-i)$. Instead of summing over i and j we sum over $m=j-i$. For fixed m a pair (i,j) with $m=j-i$ appears exactly $d+1-m$ times in $\{(i,j) : 1\le i<j\le d+1\}$. So,

$$\sum_{1\le i<j\le d+1}(d+1-(j-i))(j-i) = \sum_{m=1}^{d+1}(d+1-m)m(d+1-m)$$

Continuing the computation gives

$$\begin{aligned}
I_{\mathcal{D}_1}(Q) &= \frac{2(d-1)!}{(d+1)!(d+1)} \sum_{1\le k,l\le d+1} -q_{kl} \sum_{m=1}^{d+1}(d+1-m)^2 m \\
&= \alpha_d \sum_{1\le k<l\le d+1} -q_{kl} \quad \text{with } \alpha_d = \tfrac{2(d-1)!}{(d+1)!(d+1)} \sum_{m=1}^{d+1}(d+1-m)^2 m \\
&= \alpha_d \sum_{1\le k<l\le d} -q_{kl} + \alpha_d \sum_{1\le k\le d} -q_{k,d+1} \\
&= \alpha_d \sum_{1\le k<l\le d} -q_{kl} + \alpha_d \sum_{1\le k,l\le d} q_{kl} \\
&= \alpha_d \sum_{1\le l\le k\le d} q_{kl}
\end{aligned}$$

The computation for α_d is an exercise in the arithmetics of sums of powers (see e.g. [GKP1994], Chapter 6.5 “Bernoulli numbers”).

$$\begin{aligned}
\alpha_d &= \frac{2(d-1)!}{(d+1)!(d+1)} \sum_{m=1}^{d+1}(d+1-m)^2 m \\
&= \frac{2(d-1)!}{(d+1)!(d+1)} \sum_{m=1}^{d}(m^3-2(d+1)m^2+(d+1)^2 m) \\
&= \frac{2(d-1)!}{(d+1)!(d+1)} \Bigg(\frac{1}{4}(d+1)^4 - \frac{1}{2}(d+1)^3 + \frac{1}{4}(d+1)^2 \\
&\qquad - 2(d+1)\left(\frac{1}{3}(d+1)^3 - \frac{1}{2}(d+1)^2 + \frac{1}{6}(d+1)\right) \\
&\qquad + (d+1)^2\left(\frac{1}{2}(d+1)^2 - \frac{1}{2}(d+1)\right)\Bigg) \\
&= \frac{2(d-1)!}{(d+1)!(d+1)} \left(\frac{1}{12}(d+1)^4 - \frac{1}{12}(d+1)^2\right) \\
&= \frac{d+2}{6}.
\end{aligned}$$

Now we are ready to apply Proposition 8.3.1. The symmetric matrix which corresponds to the first perfect form $\boldsymbol{x} \mapsto \sum x_i + \sum_{i<j} x_i x_j$ has determinant $(d+1)/2^d$. Hence,

$$\det F = \left(\frac{\alpha_d}{2}\right)^d (d+1) = \left(\frac{d+2}{12}\right)^d (d+1)$$

showing the first statement of the theorem.

To accomplish the proof we have to show that the inequality is tight for VORONOÏ's principal form of the first type $Q[\boldsymbol{x}] = d \sum x_i^2 - \sum_{i \neq j} x_i x_j$. The centroid of the Delone simplex $L_{id} = \operatorname{conv}\{\boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_2, \ldots, \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{d+1}\}$ is given by $\boldsymbol{m}_{id} = \frac{1}{d+1}(d, d-1, \ldots, 1)$. We will show that it is at the same time the center of the circumsphere of L_{id}. We have to verify that the distances between $\boldsymbol{m}_{id}$ and the vertices of L_{id} are all equal:

$$\operatorname{dist}(\boldsymbol{m}_{id}, \boldsymbol{e}_1) = \operatorname{dist}(\boldsymbol{m}_{id}, \boldsymbol{e}_1 + \boldsymbol{e}_2) = \ldots = \operatorname{dist}(\boldsymbol{m}_{id}, \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{d+1}).$$

Below, it will turn out that the distances are all equal to $\sqrt{\frac{d(d+2)}{12}}$. The norm belonging to Q is invariant under permutation of coordinates. As a consequence we see that the circumradius of each simplex equals $\sqrt{d(d+2)/12}$. So, the inhomogeneous minimum of Q is $\mu(Q) = d(d+2)/12$. The determinant of Q is $(d+1)^{d-1}$. This gives the desired value of the normalized covering density of Q:

$$\theta(Q) = \sqrt{\left(\frac{d(d+2)}{12(d+1)}\right)^d (d+1)}.$$

For $n \in \{1, \ldots, d\}$ (the case $n = d+1$ equals the case $n = d$ up to changing of signs and permutation of coordinates) we have

$$\begin{aligned} \operatorname{dist}(\boldsymbol{m}_{id}, \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n)^2 &= \frac{1}{(d+1)^2} \|(-1, -2, \ldots, -n, d-n, d-n-1, \ldots, 1)\|^2 \\ &= \frac{1}{(d+1)^2} \|(-1, \ldots, -n, 1, \ldots, d-n)\|^2 \end{aligned}$$

where we again used the fact that the norm belonging to Q is invariant under permutation of coordinates. We have to compute the following product

$$(-1, \ldots, -n, 1, \ldots, d-n) \begin{pmatrix} d & -1 & -1 & \ldots & -1 & -1 \\ -1 & d & -1 & \ldots & -1 & -1 \\ \ldots \\ -1 & -1 & -1 & \ldots & -1 & d \end{pmatrix} \begin{pmatrix} -1 \\ \vdots \\ -n \\ 1 \\ \vdots \\ d-n \end{pmatrix}.$$

This is a rather cumbersome computation. It did not appear in the literature but it is completely elementary. We start with the following expression where $i \in \{1, \ldots, n\}$ respectively

$j \in \{1, \dots, d-n\}$ denote an i-th column respectively an $(n+j)$-th column

$$
\begin{aligned}
& (-1,-2,\dots,-n,1,\dots,d-n)\begin{pmatrix} \vdots \\ -di + \sum\limits_{k=1,k\neq i}^{n} k - \sum\limits_{k=1}^{d-n} k \\ \vdots \\ \sum\limits_{k=1}^{n} k + dj - \sum\limits_{k=1,k\neq j}^{d-n} k \\ \vdots \end{pmatrix} \\
= \;& \sum_{i=1}^{n}(-i)\left(-di + \sum_{k=1,k\neq i}^{n} k - \sum_{k=1}^{d-n} k\right) + \sum_{j=1}^{d-n} j \left(\sum_{k=1}^{n} k + dj - \sum_{k=1,k\neq j}^{d-n} k\right) \\
= \;& \sum_{i=1}^{n} i \left(di - \frac{n(n+1)}{2} + i + \frac{(d-n)(d-n+1)}{2}\right) \\
& + \sum_{j=1}^{d-n} j \left(\frac{n(n+1)}{2} + dj - \frac{(d-n)(d-n+1)}{2} + j\right) \\
= \;& (d+1)\sum_{i=1}^{n} i^2 + \left(\frac{(d-n)(d-n+1)}{2} - \frac{n(n+1)}{2}\right)\sum_{i=1}^{n} i \\
& + (d+1)\sum_{j=1}^{d-n} j^2 + \left(\frac{n(n+1)}{2} - \frac{(d-n)(d-n+1)}{2}\right)\sum_{j=1}^{d-n} j \\
= \;& (d+1)\left(\frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1)\right) \\
& + \left(\frac{(d-n)(d-n+1)}{2} - \frac{n(n+1)}{2}\right)\left(\frac{1}{2}(n+1)^2 - \frac{1}{2}(n+1)\right) \\
& + (d+1)\left(\frac{1}{3}(d-n+1)^3 - \frac{1}{2}(d-n+1)^2 + \frac{1}{6}(d-n+1)\right) \\
& + \left(\frac{n(n+1)}{2} - \frac{(d-n)(d-n+1)}{2}\right)\left(\frac{1}{2}(d-n+1)^2 - \frac{1}{2}(d-n+1)\right) \\
= \;& \frac{d(d+1)(d+1)(d+2)}{12}.
\end{aligned}
$$

Then we get the desired result $\operatorname{dist}(\boldsymbol{m}_{id}, \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n)^2 = \frac{1}{(d+1)^2} \cdot \frac{d(d+1)^2(d+2)}{12} = \frac{d(d+2)}{12}$. $\diamond$

One last comment on the last part of the proof: In [Ble1962] BLEICHER went a different and less elementary path and got some more information. Let Q be a positive definite quadratic form which lies in the secondary cone $\boldsymbol{\Delta}(\mathcal{D}_1)$ where $\mathcal{D}_1$ is the Delone triangulation of VORONOÏ's principal form of the first type. BLEICHER determined all radii of the circumspheres of the simplices in the Delone triangulations $\mathcal{D}_1$. For the Delone simplex L_π, $\pi \in \mathsf{S}_{d+1}$, he computed

$$R(L_\pi)^2 = -\frac{1}{4\det Q}\det\begin{pmatrix} 0 & Y_\pi \\ Y_\pi^t & Q \end{pmatrix},$$

where $Y_\pi = (y_1, \dots, y_d) \in \mathbb{R}^{1\times d}$, $q_{k,d+1} = q_{kk} - q_{1k} - \dots - q_{k-1,k} - q_{k+1,k} - \dots - q_{dk}$, and

$$\begin{aligned}
y_1 &= q_{1,\pi(2)} + q_{1,\pi(3)} + \dots + q_{1,\pi(d)} + q_{1,\pi(d+1)}\\
y_2 &= -q_{2,\pi(1)} + q_{2,\pi(3)} + \dots + q_{2,\pi(d)} + q_{2,\pi(d+1)}\\
&\vdots\\
y_k &= -q_{k,\pi(1)} - \dots - q_{k,\pi(k-1)} + q_{k,\pi(k+1)} + \dots + q_{k,\pi(d)} + q_{k,\pi(d+1)}\\
&\vdots\\
y_d &= -q_{d,\pi(1)} - \dots - q_{d,\pi(d-1)} + q_{d,\pi(d+1)}.
\end{aligned}$$

The next table shows numerical values of $\theta(\mathsf{A}_d^*) = \sqrt{\left(\frac{d(d+2)}{12(d+1)}\right)^d (d+1)}$ up to dimension $d = 24$.

dimension d	normalized covering density $\theta(\mathsf{A}_\mathbf{d}^*)$
2	0.384900
3	0.349386
4	0.357771
5	0.403566
6	0.493668
7	0.647571
8	0.903205
9	1.330585
10	2.059363
11	3.333843
12	5.624446
13	9.857770
14	17.900873
15	33.600994
16	65.061343
17	129.718168
18	265.880009
19	559.436387
20	1206.788059
21	2665.722767
22	6023.337013
23	13908.241579
24	32789.139836

Table 8.1. Numerical Values of the Normalized Covering Density of A_d^*.

8.4.2. Four-dimensional Lattice Coverings

We can apply the method to solve the lattice covering problem in dimension 4. In Chapter 4.4 we saw that there are three non-equivalent Delone triangulations in dimension 4: $\mathcal{D}_1$ ("the black node") is the Delone triangulation of VORONOÏ's principal form of the first type, $\mathcal{D}_2$ ("the grey node") is the only bistellar neighbour of $\mathcal{D}_1$, and $\mathcal{D}_3$ ("the white node") is a bistellar neighbour of $\mathcal{D}_2$. Using Proposition 8.3.1 DELONE and RYSHKOV determine in [DR1963] the local lower bounds $\theta_*(\mathcal{D}_i)$ for the normalized covering densities of lattice coverings which belong to $\mathcal{D}_i$, $i = 1, 2, 3$. They got the relation $\theta_*(\mathcal{D}_2) > \theta_*(\mathcal{D}_3) > \theta_*(\mathcal{D}_1)$. Thus the second part of Theorem 8.4.1

shows that *the lattice* A_4^* *gives the optimal four-dimensional lattice covering*. Since the solution only needs calculations which can easily done by hand, we do this illustrative computation here.

The lower bound for $\mathcal{D}_1$:

From Theorem 8.4.1 we know $\theta_*(\mathcal{D}_1) \approx 0.357771$ and $\theta_*(\mathcal{D}_1) = \theta(\mathsf{A}_4^*)$.

The lower bound for $\mathcal{D}_2$:

The Delone triangulation $\mathcal{D}_2$ is encoded by the following twelve simplices which we write down in RYSHKOV's "snake" notation

$$\begin{array}{lcllcl}
\langle \boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_5\rangle & = & L_{\mathrm{XI}} & \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_5\rangle & = & L_{\mathrm{II}} \\
\langle \boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_5, \boldsymbol{e}_4\rangle & = & L_{\mathrm{III}} & \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_3, \boldsymbol{e}_5, \boldsymbol{e}_4\rangle & = & L_{\mathrm{X}} \\
\langle \boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_5\rangle & = & L_{\mathrm{XII}} & \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_5\rangle & = & L_{\mathrm{VI}} \\
\langle \boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_2, \boldsymbol{e}_5, \boldsymbol{e}_3\rangle & = & L_{\mathrm{VII}} & \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_4, \boldsymbol{e}_5, \boldsymbol{e}_3\rangle & = & L_{\mathrm{IX}} \\
\langle \boldsymbol{e}_1, \boldsymbol{e}_5, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\rangle & = & L_{\mathrm{IV}} & \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_5, \boldsymbol{e}_3, \boldsymbol{e}_4\rangle & = & L_{\mathrm{V}} \\
\langle \boldsymbol{e}_1, \boldsymbol{e}_5, \boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_3\rangle & = & L_{\mathrm{VIIII}} & \langle \boldsymbol{e}_1, \boldsymbol{e}_2 - \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_5, \boldsymbol{e}_4, \boldsymbol{e}_3\rangle & = & L_{\mathrm{I}}
\end{array}$$

where the snake notation is

$$\langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_5\rangle = \mathrm{conv}\{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \dots, \boldsymbol{v}_1 + \dots + \boldsymbol{v}_5\},$$

and $\boldsymbol{e}_5 = -\boldsymbol{e}_1 - \dots - \boldsymbol{e}_4$. The Roman numbers refers to VORONOÏ's original numbering ([Vor1908], page 169). From the twelve simplices above we get all four-dimensional simplices of the triangulation by $\pm\langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_5\rangle + \boldsymbol{w}$, $\boldsymbol{w} \in \mathbb{Z}^4$. The snake notation has the advantage that the computation of $I_{\mathcal{D}_2}(Q)$ is very easy. Suppose $\boldsymbol{m}$ is the centroid of the simplex $L = \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_5\rangle$, then by (8.2) we have

$$\begin{aligned}
I_{\boldsymbol{m}}(L) \;=\; & \frac{1}{5}(Q[\boldsymbol{v}_2] + Q[\boldsymbol{v}_2 + \boldsymbol{v}_3] + Q[\boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4] + Q[\boldsymbol{v}_2 + \boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5] \\
& + Q[\boldsymbol{v}_3] + Q[\boldsymbol{v}_3 + \boldsymbol{v}_4] + Q[\boldsymbol{v}_3 + \boldsymbol{v}_4 + \boldsymbol{v}_5] \\
& + Q[\boldsymbol{v}_4] + Q[\boldsymbol{v}_4 + \boldsymbol{v}_5] \\
& + Q[\boldsymbol{v}_5]).
\end{aligned}$$

Now, we get $I_{\mathcal{D}_2}(\cdot) = \langle F, \cdot\rangle$ with

$$F = \frac{10}{5! \cdot 5}\begin{pmatrix} 60 & 12 & 30 & 30 \\ 12 & 60 & 30 & 30 \\ 30 & 30 & 60 & 30 \\ 30 & 30 & 30 & 60 \end{pmatrix} \quad \text{and} \quad \det F = \frac{8}{25} = \frac{2^3}{5^2},$$

and by Proposition 8.3.1 every lattice covering which belongs to $\mathcal{D}_2$ has a normalized covering density of at least

$$\theta_*(\mathcal{D}_2) = \sqrt{\frac{4^4}{5^4}\det F} = \sqrt{\frac{2^{11}}{5^6}} = \frac{32}{125}\sqrt{2} \approx 0.3620386719$$

The lower bound for $\mathcal{D}_3$:

As in the case of $\mathcal{D}_2$, the Delone triangulation $\mathcal{D}_3$ is encoded by the following twelve simplices which are written in snake notation. Again do the Roman numbers refer to VORONOÏ's original numbering ([Vor1908], page 173).

$$\begin{array}{lcllcl}
L_{\mathrm{I}} & = & \langle \boldsymbol{e}_4, \boldsymbol{e}_3 - \boldsymbol{e}_4, \boldsymbol{e}_2 + \boldsymbol{e}_4, \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_5\rangle & L_{\mathrm{VII}} & = & \langle \boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_2, \boldsymbol{e}_5\rangle \\
L_{\mathrm{II}} & = & \langle \boldsymbol{e}_4, \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2, \boldsymbol{e}_5\rangle & L_{\mathrm{VIII}} & = & \langle \boldsymbol{e}_2 + \boldsymbol{e}_4, \boldsymbol{e}_3 - \boldsymbol{e}_4, \boldsymbol{e}_4, \boldsymbol{e}_1, \boldsymbol{e}_5\rangle \\
L_{\mathrm{III}} & = & \langle \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_5\rangle & L_{\mathrm{IX}} & = & \langle \boldsymbol{e}_2 + \boldsymbol{e}_4, \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_5\rangle \\
L_{\mathrm{IV}} & = & \langle \boldsymbol{e}_1 + \boldsymbol{e}_4, \boldsymbol{e}_3 - \boldsymbol{e}_4, \boldsymbol{e}_4, \boldsymbol{e}_2, \boldsymbol{e}_5\rangle & L_{\mathrm{X}} & = & \langle \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_5\rangle \\
L_{\mathrm{V}} & = & \langle \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_3 - \boldsymbol{e}_4, \boldsymbol{e}_2 + \boldsymbol{e}_4 + \boldsymbol{e}_5\rangle & L_{\mathrm{XI}} & = & \langle \boldsymbol{e}_1, \boldsymbol{e}_3, \boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_5\rangle \\
L_{\mathrm{VI}} & = & \langle \boldsymbol{e}_2, \boldsymbol{e}_1 - \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_5\rangle & L_{\mathrm{XII}} & = & \langle \boldsymbol{e}_1, \boldsymbol{e}_4, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_5\rangle
\end{array}$$

Now, we get $I_{\mathcal{D}_3}(\cdot) = \langle F, \cdot \rangle$ with

$$F = \frac{10}{5! \cdot 5} \begin{pmatrix} 60 & 12 & 30 & 30 \\ 12 & 60 & 30 & 30 \\ 30 & 30 & 60 & 18 \\ 30 & 30 & 18 & 60 \end{pmatrix} \quad \text{and} \quad \det F = \frac{196}{625} = \frac{2^2 7^2}{5^4},$$

and by Proposition 8.3.1 every lattice covering belonging to $\mathcal{D}_3$ has a normalized covering density of at least

$$\theta_*(\mathcal{D}_3) = \sqrt{\frac{4^4}{5^4} \det F} = \sqrt{\frac{2^{10} 7^2}{5^8}} = \frac{224}{625} = 0.3584$$

8.4.3. Invariants and Symmetry

The method of the moments of inertia is very useful in the context of symmetry detection.

Let $\mathcal{D}$, $\mathcal{D}'$ be two equivalent Delone triangulations. Then there exists a unimodular transformation $A \in \mathsf{GL}_d(\mathbb{Z})$ with $A\mathcal{D} = \mathcal{D}'$. As in Proposition 8.3.1, let F, F' be the positive definite matrices with $I_{\mathcal{D}}(\cdot) = \langle F, \cdot \rangle$, $I_{\mathcal{D}'}(\cdot) = \langle F', \cdot \rangle$. From the definition of the functions $I_{\mathcal{D}}$, $I_{\mathcal{D}'}$ the equality $I_{\mathcal{D}}(A^t Q A) = I_{A\mathcal{D}}(Q) = I_{\mathcal{D}'}(Q)$ that holds for all $Q \in \mathcal{S}^d_{>0}$ is obvious. Hence,

$$\langle F, A^t Q A \rangle = \operatorname{trace}(F A^t Q A) = \operatorname{trace}(A F A^t Q) = \langle A F A^t, Q \rangle = \langle F', Q \rangle.$$

and $F' = AFA^t$ since $\langle \cdot, \cdot \rangle$ is a non-degenerate scalar product. This implies the following lemma which is extremely useful in practical computations.

Lemma 8.4.2. Let $\mathcal{D}$, $\mathcal{D}'$ be two equivalent Delone triangulations. As in Proposition 8.3.1 let F, F' be the positive definite matrices with $I_{\mathcal{D}}(\cdot) = \langle F, \cdot \rangle$ and $I_{\mathcal{D}'} = \langle F', \cdot \rangle$. Then the local lower bounds for both Delone triangulation coincide:

$$\theta_*(\mathcal{D}) = \sqrt{\left(\frac{d}{d+1}\right)^d \det F} = \sqrt{\left(\frac{d}{d+1}\right)^d \det F'} = \theta_*(\mathcal{D}').$$

The automorphism group of the Delone triangulation $\mathcal{D}$ is a subgroup of the automorphism group of F, i.e.

$$\operatorname{Aut}(\mathcal{D}) = \{A \in \mathsf{GL}_d(\mathbb{Z}) : A\mathcal{D} = \mathcal{D}\} \subseteq \{A \in \mathsf{GL}_d(\mathbb{Z}) : A^t F A = F\} = \operatorname{Aut}(F).$$

As an indication that the local lower bound is indeed a very strong invariant we describe what happens for the dimensions $d = 4$ and $d = 5$. In the four-dimensional case there are 3 non-equivalent Delone triangulations and they are completely separated by the invariant. In the five-dimensional case there are 222 non-equivalent Delone triangulations and we get the following statistic in this case: There are 212 different values for the local lower bounds. They range from ≈ 0.396911 to ≈ 0.421017. In eight cases two non-equivalent Delone triangulations yield the same local lower bound and in one case three non-equivalent Delone triangulations yield the same local lower bound.

8.4.4. Navigating in the Graph of Delone Triangulations

We view the set of Delone triangulations as an undirected labeled graph. A node represents a Delone triangulation and two nodes are adjacent if their Delone triangulations are bistellar neighbours. Let $\mathcal{D}$ be a Delone triangulation. We label its node by the local lower bound $\theta_*(\mathcal{D})$. We can use the labeling in two different ways.

► On the one hand it is clear that if the labeling of a node is large, then the considered Delone triangulation does not admit a good lattice covering. This was used in the last section to solve the lattice covering problem in dimension 4.

In this way the five-dimensional lattice covering problem cannot be solved. From the 222 non-equivalent Delone triangulations there are 20 whose local lower bound is smaller than $\theta(\mathsf{A}_5^*)$.

In [Rys1965]† RYSHKOV shows that in the neighbourhood of the Delone triangulation $\mathcal{D}_1$ of VORONOÏ's principal form of the first type the local lower bounds are higher than $\theta_*(\mathcal{D}_1) = \theta(\mathsf{A}_d^*)$ in any dimension $d \geq 5$. He shows that for the Delone triangulations $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ we have the inequalities $\theta_*(\mathcal{D}_2), \theta_*(\mathcal{D}_3), \theta_*(\mathcal{D}_4) > \theta_*(\mathcal{D}_1)$. The Delone triangulations adjacent to $\mathcal{D}_1$ are all equivalent to $\mathcal{D}_2$. The Delone triangulations adjacent to $\mathcal{D}_2$ are all equivalent either to $\mathcal{D}_3$ or to $\mathcal{D}_4$.

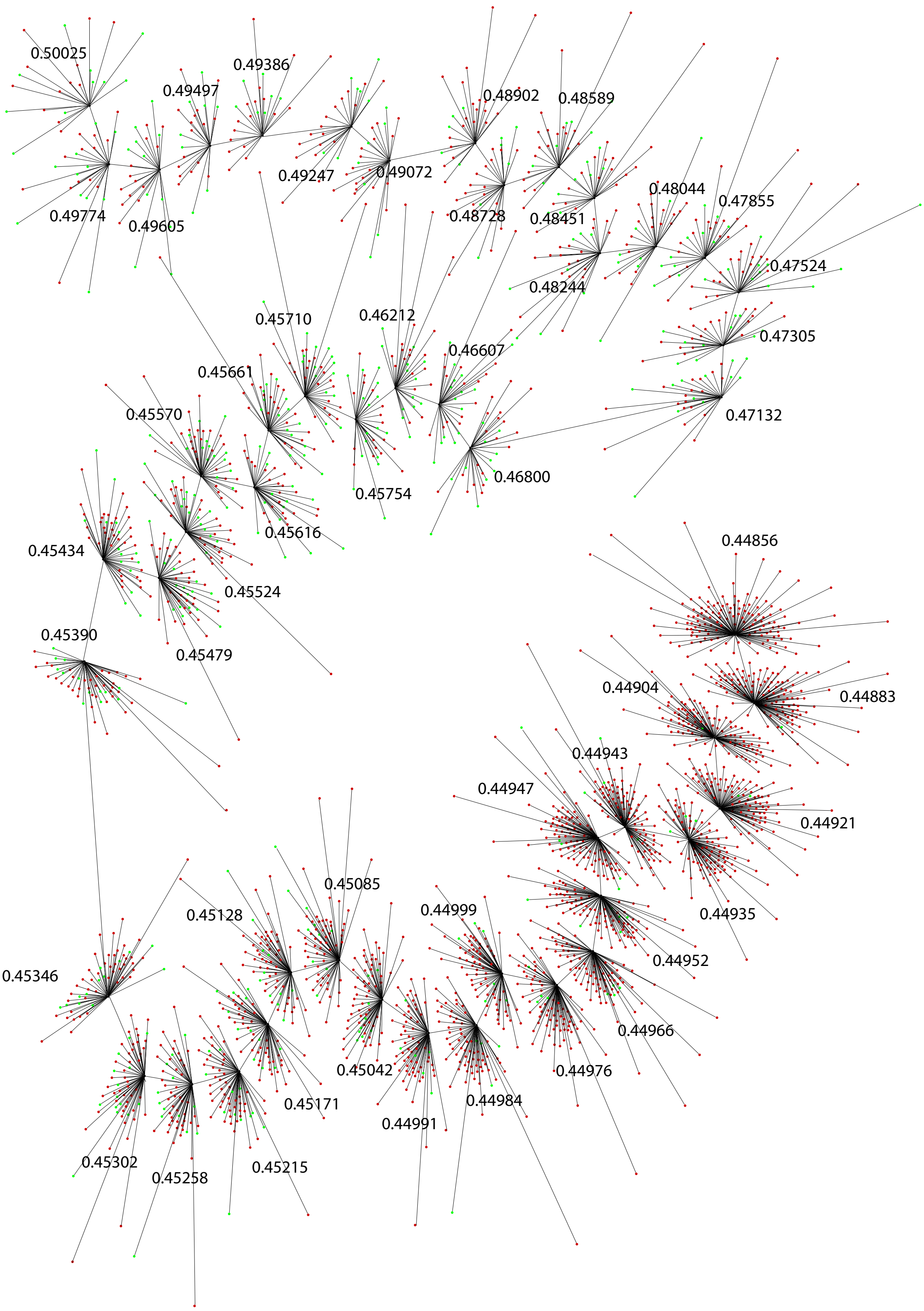
In [BT1972] BARNES and TRENERRY investigate a Delone triangulation $\mathcal{D}_{\mathrm{BT}}$ that does have the same automorphism group as $\mathcal{D}_1$. Their triangulation does exist only in every odd dimension starting from five. By a direct computation of the locally optimal lattice covering they show that the best lattice covering belonging to this Delone triangulation is denser than the lattice covering given by A_d^*. Our computation show that in dimension 5 it gives the second best locally optimal lattice covering. It is also possible to show $\theta_*(\mathcal{D}_{\mathrm{BT}}) = \theta_*(\mathcal{D}_1)$.

► On the other hand we can hope that $\mathcal{D}$ admits a good lattice covering if the local lower bound is small.

Beginning with dimension $d = 6$ the number of non-equivalent Delone triangulations starts to explode. Up to now we produced more than $250,000$ of them and we think that there are several millions.

In dimension 6 the hope that good local lower bounds yield good lattice coverings is partially fulfilled. We demonstrate this in a typical example (see the figure on the next page). We start from the Delone triangulation of VORONOÏ's principal form of the first type. From the discussion above we know that its local lower bound gives a local minimum in the set of node labels. We take a random walk of length 50. Then, we find a node labeled by ≈ 0.50025. From this we proceed by taking a neighbouring node having the smallest local lower bound (In the figure, nodes which have smaller labels than the current one are marked with green circles, the other are marked with red circles). By repeating this greedy strategy we result in a node labeled by ≈ 0.44856. At the moment this node is interesting for several "extremeness" properties. It yields the smallest known local lower bound and it has the largest known number of neighbours, namely 130. As we will see in the next chapter there exists a locally optimal lattice covering which belongs to this node with normalized covering density ≈ 0.477217. At present this is the second best known 6-dimensional lattice covering.

†Caution! In this paper there are several misprints. See [RB1976] for corrections.


0.50025
0.49386
0.49497
0.48902
0.48589
0.49247
0.49072
0.49774
0.49605
0.48728
0.48451
0.48044
0.47855
0.47524
0.48244
0.45710
0.46212
0.47305
0.46607
0.45661
0.45570
0.47132
0.46800
0.45754
0.45616
0.45434
0.44856
0.45524
0.45390
0.45479
0.44904
0.44883
0.44943
0.44947
0.44921
0.45085
0.45128
0.44999
0.44935
0.44952
0.45346
0.44966
0.45042
0.44976
0.45171
0.44984
0.44991
0.45302
0.45215
0.45258

Chapter 9.

Results in Low Dimensions

In the last chapters we developed an algorithm for the solution of the lattice covering problem in any given dimension. Now we want to demonstrate that this algorithm is not purely of theoretical interest. We implemented the algorithm in C++. We used the package MAXDET* of WU, VANDENBERGHE and BOYD and the package lrs† of AVIS as subroutines. The implemented algorithm is able to solve the lattice covering problem in the dimensions $d = 1, \ldots, 5$, and it produces interesting lattice coverings in the dimensions $d = 6, 7$ on a usual Intel Pentium based computer. In higher dimensions the implementation does not perform very well mainly due to memory limitations.

Another purpose of this chapter is to present the state-of-the-art of the lattice covering problem in low dimensions together with the history of the results. Low dimension means that we restrict our attention to lattices up to dimension 24 where the miraculous Leech lattice enters the scene. So it is a partial update of Chapter 2.1 in CONWAY and SLOANE's book [CS1988a].

9.1. Dimension 1

The one-dimensional case is entirely trivial. The lattice covering given by the lattice $\mathbb{Z}^1$ provide a sphere covering $[-\frac{1}{2}, \frac{1}{2}] + \boldsymbol{v}$, $\boldsymbol{v} \in \mathbb{Z}^1$, which is at the same time a sphere packing. Hence, the covering density equals one which cannot be improved.

9.2. Dimension 2

In the introduction we already saw that the optimal lattice covering of the plane is provided by the hexagonal lattice. This is also the optimal sphere covering of the plane which was first proved by KERSHNER in [Ker1939]. FEJES TÓTH gives in his book [Fej1953] different proofs for this fact. Since for general sphere coverings new aspects come into play and since the two-dimensional case is the only non-trivial case where the optimality of a sphere covering is proven at the moment, it is a must for us to give at least the main arguments here.

From EULER's formula for planar graphs it follows that the number of vertices per polygon in a polygonal subdivision of the plane is at most six. The area of a polygon with n vertices which can be inscribed into the unit circle is at most $\frac{n}{2} \sin \frac{2\pi}{n}$. The maximum is only attained for regular n-gons.

Given a covering of the plane by unit circles $(B_2(\boldsymbol{v}_i, 1))_{i \in \mathbb{N}}$. The Voronoi subdivision of the discrete set of circle centers is a polygonal subdivision of the plane. Let $(P_i)_{i \in \mathbb{N}}$ be the

*http://www.stanford.edu/~boyd/MAXDET.html

†http://cgm.cs.mcgill.ca/~avis/C/lrs.html

family of polygons of the Voronoi subdivision. Each P_i is inscribed into the corresponding unit circle. Hence, the area of P_i is at most $\frac{n_i}{2}\sin\frac{2\pi}{n_i}$ where n_i is the number of vertices of P_i. Since $f(x) = \frac{x}{2}\sin\frac{2\pi}{x}$ is a concave function we can apply JENSEN's inequality, so that for every $n \in \mathbb{N}$

$$\frac{\sum_{i=1}^{n} \text{area}(P_i)}{n} \leq \frac{\sum_{i=1}^{n} f(n_i)}{n} \leq f\left(\frac{\sum_{i=1}^{n} n_i}{n}\right).$$

From this we see that the area of an average polygon in the Voronoi subdivision is at most $f(6) = \frac{3\sqrt{3}}{2}$. The covering density of the sphere covering $(B_2(\boldsymbol{v}_i, 1))_{i\in\mathbb{N}}$ is the area of a unit circle divided by the area of an average polygon of the Voronoi subdivision. Thus,

$$\Theta((B_2(\boldsymbol{v}_i, 1))_{i\in\mathbb{N}}) \geq \frac{2\pi}{\sqrt{27}}.$$

Furthermore this lower bound is tight and the optimal sphere covering are almost unique: FEJES TÓTH shows that the lower bound is attained only by hexagonal-like sphere coverings.

9.3. Dimension 3

VORONOÏ's principal form of the first type $Q[\boldsymbol{x}] = 3x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$ provides the thinnest lattice covering of three-dimensional Euclidean space. We gave a proof in Section 8.4.1. The lattice which is associated to Q is the body centered cubic lattice A_3^*. The Dirichlet-Voronoi polytope of A_3^* is the truncated octahedron which is an Archimedian solid.

The optimality of Q was first proven by BAMBAH in [Bam1954a]. This paper is remarkable because there techniques like Dirichlet-Voronoi polytopes and reduction theory are used. Although BAMBAH uses the reduction theory due to SEEBER (see [Gau1840]) which is not well-suited for the lattice covering problem. Later, BARNES substantially simplifies BAMBAH's proof in [Bar1956] where he uses VORONOÏ's reduction theory. He also anticipates that this is the right setup for solving the lattice covering problem in dimensions higher than three. A third proof of the optimality of Q was given by FEW in [Few1956]. He demonstrates that the three-dimensional case can be solved without using reduction theory mainly by elementary means.

At the moment no attempt is known to the author to show that the optimal lattice covering also gives the optimal sphere covering. It is probably very hard to prove this covering type "Kepler conjecture".

9.4. Dimension 4

VORONOÏ's principal form of the first type

$$Q[\boldsymbol{x}] = (x_1, x_2, x_3, x_4)\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

gives the least dense four-dimensional lattice covering with covering density $\Theta(Q) \approx 1.765528$ ($\theta(Q) \approx 0.357771$). In [DR1963] DELONE and RYSHKOV prove this conjecture of BAMBAH [Bam1954b] by using the method of the moments of inertia which we described in Chapter 8.

In [BR1966] BARANOVSKII and RYSHKOV show that the positive definite quadratic form

$$Q[\boldsymbol{x}] = (x_1, x_2, x_3, x_4)\begin{pmatrix} 3-\gamma & \gamma & -1 & -1 \\ \gamma & 3-\gamma & -1 & -1 \\ -1 & -1 & 2+2\beta & -\beta \\ -1 & -1 & -\beta & 2+2\beta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where $\beta \approx 0.544$, $\gamma \approx 0.499$ gives a locally optimal lattice covering. The numbers β and γ are roots of the polynomials

$$81\beta^5 + 234\beta^4 - 84\beta^3 - 601\beta^2 - 156\beta + 252 = 0,$$
$$\gamma = \tfrac{(18\beta^2+39\beta+10)\beta}{(\beta+2)(3\beta+14)}.$$

Later we will demonstrate how to find polynomials which can be used to specify locally optimal lattice coverings exactly. The Delone subdivision of the positive definite quadratic form Q corresponds with the triangulation $\mathcal{D}_2$ ("the grey node" in Chapter 4.4). The covering density of the lattice covering provided by Q is $\Theta(Q) \approx 1.883855$ ($\theta(Q) \approx 0.381749$).

In [Bar1965] and [Bar1966] BARANOVSKII finds a third locally optimal lattice covering which is provided by

$$Q[\boldsymbol{x}] = (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & \alpha & -1 & -1 \\ \alpha & 2 & -1 & -1 \\ -1 & -1 & 2 & 1-\alpha \\ -1 & -1 & 1-\alpha & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

where $\alpha = (5 - \sqrt{13})/2$. The Delone subdivision of the positive definite quadratic form Q corresponds with the triangulation $\mathcal{D}_3$ ("the white node" in Chapter 4.4). The covering density is $\Theta(Q) = \frac{16(1-\alpha)^2}{(2-\alpha)^3(1+\alpha)} \cdot \operatorname{vol} B_4(\mathbf{0}, 1) \approx 1.928782$ ($\theta(Q) \approx 0.390853$).

Independently does DICKSON give the locally optimal lattice covering of VORONOÏ's third domain in [Dic1966] and in [Dic1967] he gives a complete list of all locally optimal four-dimensional lattice coverings. Especially the second paper is interesting because there the methods developed by BARNES and DICKSON which we described in Section 7.2 are used to give an almost algorithmic proof.

Our implementation takes less than a second of CPU-time to yield the same result. As a "proof" of this and to give at least one "picture" of the implementation we provide a screen shot. There the phases of the implemented algorithm are visible:

i) estimate the covering density by the method of the moments of inertia,

ii) compute the covering density with the help of the package `MAXDET`,

iii) compute the bistellar neighbours,

iv) test if the new neighbours are isomorphic to the already known ones.

```
[geometry16:~/src/LatticeCovering] vallenti% time ./classify
Looking at #0 (normalized covering density >= 0.357771)
* Computing exact normalized covering density...
 iters        obj           gap
    5      -1.35e+00       7.00e+01
    7      -1.80e+00       2.25e+00
    9      -2.04e+00       7.26e-02
   11      -2.06e+00       2.34e-03
   13      -2.06e+00       7.51e-05
   16      -2.06e+00       4.60e-06
   19      -2.06e+00       8.25e-07
  2.0000
 -0.5000  2.0000
 -0.5000 -0.5000  2.0000
 -0.5000 -0.5000 -0.5000  2.0000
 * normalized thickness = 0.357771
```

```
 * There are 10 neighbours.
 * (0->1) (1=1) (2=1) (3=1) (4=1) (5=1) (6=1) (7=1) (8=1) (9=1)

Looking at #1 (normalized covering density >= 0.362039)
* Computing exact normalized covering density...
 iters        obj            gap
    7      -1.27e+00       4.71e+01
   11      -1.76e+00       1.52e+00
   17      -1.92e+00       5.39e-02
   21      -1.93e+00       1.77e-03
   26      -1.93e+00       8.25e-05
   31      -1.93e+00       3.85e-06
   35      -1.93e+00       6.91e-07
  1.7852
  0.3571  1.7852
 -0.7141 -0.7141  2.2054
 -0.7141 -0.7141 -0.3886  2.2054
 * normalized covering density = 0.381749
 * There are 10 neighbours.
 * (0=1) (1=1) (2=1) (3=1) (4->2) (5=2) (6=2) (7=1) (8=0) (9=1)

Looking at #2 (normalized covering density >= 0.3584)
* Computing exact normalized covering density...
 iters        obj            gap
    6      -1.28e+00       4.73e+01
   11      -1.75e+00       1.52e+00
   19      -1.88e+00       6.72e-02
   24      -1.88e+00       2.24e-03
   30      -1.88e+00       8.84e-05
   35      -1.88e+00       3.48e-06
   39      -1.88e+00       6.24e-07
  2.1514
  0.7500  2.1514
 -1.0757 -1.0757  2.1514
 -1.0757 -1.0757  0.3257  2.1514
 * normalized covering density = 0.390853
 * There are 10 neighbours.
 * (0=1) (1=1) (2=1) (3=1) (4=1) (5=1) (6=1) (7=1) (8=2) (9=1)
--
Classification completed!
There are 3 non-equivalent Delone triangulations in dimension 4.
0.730u 0.010s 0:01.49 49.6%     0+0k 0+3io 0pf+0w
[geometry16:~/src/LatticeCovering] vallenti%
```

The positive definite quadratic forms found by the computer are only an approximation of the desired local minima. But this is not a problem since the algorithm can at least in principle approximate the solution to every given precision and since we can "beautify" the nasty numbers to find the polynomials whose roots the "real" numbers are. We will demonstrate the "beautification" in the case of the local optimal solution with Delone triangulation $\mathcal{D}_2$ where we make use of the computational algebra system MAGMA[‡].

The automorphism group of the Delone triangulation $\mathcal{D}_2$ has order 24. The group is generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[‡] http://magma.maths.usyd.edu.au/magma

From Section 7.2 we know that if the locally optimal solution Q lies in the interior of $\Delta(\mathcal{D}_2)$, then Q has to be invariant under $\mathrm{Aut}(\mathcal{D}_2)$, therefore Q lies in the subspace with basis

$$A = \begin{pmatrix} 3 & 0 & -1 & -1 \\ 0 & 3 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 3 & -1 & -1 \\ 3 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

The set of all non-equivalent (under the group of translations) simplices is partitioned by the action of the group $\mathrm{Aut}(\mathcal{D}_2)$ into two orbits with representatives

$$\begin{aligned} L_{\mathrm{II}} &= \mathrm{conv}\{(0,0,0,0)^t, (1,-1,0,0)^t, (1,0,0,0)^t, (1,0,1,0)^t, (1,0,1,1)^t\}, \\ L_{\mathrm{XI}} &= \mathrm{conv}\{(0,0,0,0)^t, (1,0,0,0)^t, (1,0,1,0)^t, (1,1,1,0)^t, (1,1,1,1)^t\}, \end{aligned}$$

where to the orbit of L_{II} the simplices $L_{\mathrm{II}}, L_{\mathrm{X}}, L_{\mathrm{VI}}, L_{\mathrm{IX}}, L_{\mathrm{V}}, L_{\mathrm{I}}$ belong, and to the orbit of L_{XI} the simplices $L_{\mathrm{XI}}, L_{\mathrm{III}}, L_{\mathrm{XII}}, L_{\mathrm{VII}}, L_{\mathrm{IV}}, L_{\mathrm{VIII}}$.

Given the positive definite quadratic form

$$\tilde{Q} = \begin{pmatrix} 1.7852 & 0.3571 & -0.7141 & -0.7141 \\ 0.3571 & 1.7852 & -0.7141 & -0.7141 \\ -0.7141 & -0.7141 & 2.2054 & -0.3886 \\ -0.7141 & -0.7141 & -0.3886 & 2.2054 \end{pmatrix}$$

found by the algorithm, we compute $\mathrm{BR}_{L_1}(Q) \approx 0.4905 > 0$, $\mathrm{BR}_{L_2}(Q) \approx -0.0002 \approx 0$. So it is reasonable to assume that the inhomogeneous minimum of 1 is only attained at the centers of the circumspheres of the simplices which lie in the orbit of L_2.

Since the density function is invariant under scaling we can normalize the quadratic form $xA+yB+zC$, $x, y, z \in \mathbb{R}$, which lies in the invariant subspace by setting $z = 1$. The determinant of $Q(x,y) = xA + yB + C$ is given by the polynomial

$$\det Q(x,y) = f(x,y) = 3x^4+6x^3y-18x^3-18x^2y+27x^2-6xy^3+18xy^2-3y^4+18y^3-27y^2$$

The radius of the circumsphere around the simplex L_2 is given by the rational expression (see (6.2))

$$R(x,y)^2 = \frac{g(x,y)}{h(x,y)} = \frac{\begin{array}{c} 6x^5 - 18x^4y - 12x^4 - 6x^3y^2 + 48x^3y + 6x^3 + 42x^2y^3 - 36x^2y^2 \\ +114x^2y - 72x^2 - 48xy^3 - 168xy^2 - 24y^5 + 48y^4 + 48y^3 + 72y^2 \end{array}}{\begin{array}{c} 3x^4 + 6x^3y - 18x^3 - 18x^2y + 27x^2 - 6xy^3 + 18xy^2 - 3y^4 \\ +18y^3 - 27y^2 \end{array}}$$

Now we try to find the minima of the function $\theta(Q(x,y))^2 = R(x,y)^4/\det Q(x,y)$ by setting the partial derivatives to zero. The set of critical points is then given by the affine variety of the ideal

$$\begin{aligned} I \;=\; \langle &-\frac{\partial f(x,y)}{\partial x} g(x,y)h(x,y) + 4f(x,y)\frac{\partial g(x,y)}{\partial x} h(x,y) - 4f(x,y)g(x,y)\frac{\partial h(x,y)}{\partial x}, \\ &-\frac{\partial f(x,y)}{\partial y} g(x,y)h(x,y) + 4f(x,y)\frac{\partial g(x,y)}{\partial y} h(x,y) - 4f(x,y)g(x,y)\frac{\partial h(x,y)}{\partial y} \rangle. \end{aligned}$$

Using Gröbner basis techniques — decompose the radical ideal $\sqrt{I}$ into prime ideals — we decompose the variety $\mathbf{V}(I)$ into four irreducible varieties

$$\begin{aligned} \mathbf{V}(I) \;=\; & \mathbf{V}(x+y-3) \,\cup\, \mathbf{V}(x+y) \,\cup\, \mathbf{V}(x-y) \\ & \cup\, \mathbf{V}(473x - 22512y^4 - 70584y^3 - 26351y^2 + 23757y - 2421, \\ & \quad 84y^5 + 243y^4 + 35y^3 - 113y^2 + 30y - 2). \end{aligned}$$

In this decomposition the first three subvarieties are lines and since we know that the locally optimal solution Q is unique they cannot give Q. So we have to look at the last subvariety which consists out of five points which are all real:

$$\begin{aligned}(x_1, y_1) &\approx (0.0942, 0.3188),\\ (x_2, y_2) &\approx (-1.389, 0.2552),\\ (x_3, y_3) &\approx (0.5396, 0.1080),\\ (x_4, y_4) &\approx (-0.2341, -1.0919),\\ (x_5, y_5) &\approx (-2.0951, -2.4829).\end{aligned}$$

Then the point (x_3, y_3) gives the desired solution so that $x_3A + y_3B + C$ provides a locally optimal lattice covering which coincides with the one computed by the algorithm after a suitable scaling.

What is the general pattern behind this beautification process? We use the symmetry of $\mathcal{D}$ to find the subspace in which Q lies. The simplices of the Delone triangulation which have circumradius 1 give equality constraints. Then, we maximize the determinant of the quadratic forms lying in the subspace subject to the equality constraints. For this optimization problem, which involves only algebraic equations, we can use Gröbner basis techniques.

9.5. Dimension 5

In a series of papers RYSHKOV and BARANOVSKII solved the five-dimensional lattice covering problem. In [Rys1973] RYSHKOV introduces the concept of C-types. Two Delone triangulations are of the same *C-type* if their 1-skeletons (the graph consisting of vertices and edges of the triangulation) coincide. He gives an algorithm to find all non-equivalent C-types in any given dimension. He computes that there are 3 non-equivalent C-types in dimension 4 and that there are 76 non-equivalent C-types in dimension 5. Using this list BARANOVSKII and RYSHKOV enumerate 221 (of 222) non-equivalent 5-dimensional Delone triangulations in [BR1973]. They describe the triangulations in more detail in [BR1975]. In the last paper of the series [RB1975] they show that the lattice A_5^* provides the least dense 5-dimensional lattice covering. In their proof they do not find all locally optimal lattice coverings. By using estimations (like the moments of inertia) they merely show that all local minima exceed the covering density of A_5^*.

Since the papers in the series are very dense and not easy to read RYSHKOV and BARANOVSKII prepared a 140-pages long monograph [RB1976] based on their investigations. There they comment on using an algorithmic approach to the lattice covering problem like ours:

- “Attempts to apply VORONOÏ’s algorithm for $d > 4$ have run into colossal computational difficulties.”
- “Such an approach is extremely difficult for $d = 5$ (and is therefore uninteresting).”

Using our algorithm we produced a complete table of all non-equivalent locally optimal lattice coverings in dimension 5. The computation takes about 20 minutes on a standard Intel Pentium computer. As mentioned in Section 4.5 RYSHKOV and BARANOVSKII missed the Delone triangulation $\#164$ which fortunately does not give a thinner lattice covering than the lattice A_5^*.

In the near future we will make our computations available on the world wide web. There, one will see e.g. that there exist 222 non-equivalent minima of the covering density function ranging from ≈ 0.403566 to ≈ 0.535956. This means that all locally optimal solutions are attained in the interior of the secondary cones. Our list also implies that the positive definite quadratic form of BARNES and TRENERRY ([BT1972]) yields the second best locally optimal lattice covering. It has normalized covering density ≈ 0.423671. Here we only show what happened with $\#164$:

```
Looking at #164 (normalized covering density >= 0.406124)
 * Computing exact normalized covering density...
 iters        obj           gap
   12      -7.61e-01       1.20e+02
   17      -1.24e+00       3.03e+00
   28      -1.41e+00       8.12e-02
   35      -1.41e+00       2.06e-03
   42      -1.41e+00       5.47e-05
   47      -1.41e+00       1.45e-06
   51      -1.41e+00       3.85e-08
 1.9341845097
 0.7956950345 1.8678576859
 -0.5256981646-0.7187301788 1.6137547961
 -0.6460313338-0.5359019448-0.3586190581 1.6830691309
 -0.9121187120-0.8730186521 0.3479116634-0.2439723760 1.9251704528
 * There are 18 neighbours.
 * (0=144) (1->206) (2=204) (3=162) (4=163) (5=86) (6=204) (7=85) (8=156)
   (9=161) (10=165) (11=163) (12=185) (13=86) (14=144) (15=162) (16=165)
   (17=156)
```

9.6. Dimension 6

Up to now, we found 65 non-equivalent 6-dimensional lattice coverings which are locally optimal and which are better lattice coverings than the one given by the lattice A_6^*. These lattice coverings were found by the heuristic method we explained in Section 8.4.4. We do not claim that the list is complete in any sense. In [Rys1967] RYSHKOV raises the question of finding the first dimension d for which there is a better lattice covering than the one given by A_d^*. Hence, $d = 6$ is the answer.

We have $\theta(\mathsf{A}_6^*) \approx 0.493668$. We found two lattice coverings with normalized covering density of about 0.477. All other good lattice coverings found have a normalized covering density of at least 0.485. We give a detailed report on the best two lattice coverings found.

9.6.1. The best known 6-dimensional lattice covering

The best known 6-dimensional lattice covering has normalized covering density of ≈ 0.476962 which is some percentage less than the former best known one with ≈ 0.493668. In this section we describe some data for the new lattice covering. But at the moment we lack a good interpretation of this result.

The Delone triangulation which belongs to the best known 6-dimensional lattice covering has 100 bistellar neighbours. Its local lower bound is ≈ 0.449368 which is less than its neighbours' values. The linear automorphism group of the Delone triangulation is the group $G_1 = \langle g_1, g_2 \rangle$ generated by the two matrices

$$g_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is a subgroup of $\mathsf{GL}_6(\mathbb{Z})$ and has order 240. The set of all non-equivalent (under the group of translations) simplices is partitioned by the action of the group G_1 into eight orbits with representatives

$$\begin{array}{lcllcl}
L_1 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_5,\boldsymbol{v}_7,\boldsymbol{v}_9,\boldsymbol{v}_{10},\boldsymbol{v}_{11},\boldsymbol{v}_{13}\} & L_2 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_5,\boldsymbol{v}_7,\boldsymbol{v}_9,\boldsymbol{v}_{10},\boldsymbol{v}_{13},\boldsymbol{v}_{15}\}\\
L_3 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4,\boldsymbol{v}_{13},\boldsymbol{v}_{14}\} & L_4 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_7,\boldsymbol{v}_9,\boldsymbol{v}_{10},\boldsymbol{v}_{11},\boldsymbol{v}_{13},\boldsymbol{v}_{15}\}\\
L_5 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_5,\boldsymbol{v}_6,\boldsymbol{v}_8,\boldsymbol{v}_9,\boldsymbol{v}_{12},\boldsymbol{v}_{14}\} & L_6 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_5,\boldsymbol{v}_8,\boldsymbol{v}_9,\boldsymbol{v}_{11},\boldsymbol{v}_{12},\boldsymbol{v}_{14}\}\\
L_7 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_5,\boldsymbol{v}_9,\boldsymbol{v}_{11},\boldsymbol{v}_{12},\boldsymbol{v}_{13},\boldsymbol{v}_{14}\} & L_8 &=& \operatorname{conv}\{\boldsymbol{v}_0,\boldsymbol{v}_5,\boldsymbol{v}_8,\boldsymbol{v}_{11},\boldsymbol{v}_{12},\boldsymbol{v}_{13},\boldsymbol{v}_{14}\}
\end{array}$$

where

$$\begin{array}{lcllcl}
\boldsymbol{v}_0 &=& (0,0,0,0,0,0)^t & \boldsymbol{v}_1 &=& (0,1,0,0,0,0)^t\\
\boldsymbol{v}_2 &=& (0,1,0,0,-1,0)^t & \boldsymbol{v}_3 &=& (0,1,1,0,0,1)^t\\
\boldsymbol{v}_4 &=& (0,1,1,1,0,0)^t & \boldsymbol{v}_5 &=& (1,0,0,0,0,0)^t\\
\boldsymbol{v}_6 &=& (1,0,0,0,0,1)^t & \boldsymbol{v}_7 &=& (1,0,0,0,1,0)^t\\
\boldsymbol{v}_8 &=& (1,0,0,-1,0,1)^t & \boldsymbol{v}_9 &=& (1,0,-1,0,0,0)^t\\
\boldsymbol{v}_{10} &=& (1,0,-1,0,0,-1)^t & \boldsymbol{v}_{11} &=& (1,0,-1,-1,0,0)^t\\
\boldsymbol{v}_{12} &=& (1,0,-1,-1,-1,0)^t & \boldsymbol{v}_{13} &=& (1,1,0,0,0,0)^t\\
\boldsymbol{v}_{14} &=& (1,1,0,0,0,1)^t & \boldsymbol{v}_{15} &=& (1,1,0,1,1,0)^t
\end{array}$$

The first four orbits have at each case length 60 and the four last orbits have at each case length 120. The set of all quadratic forms which are invariant under the group G_1 is a four-dimensional subspace with basis

$$\begin{pmatrix}
1 &&&&&\\
0 & 0 &&&&\\
1 & 0 & 1 &&&\\
-2 & 0 & -2 & 4 &&\\
0 & 0 & 0 & 0 & 0 &\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 &&&&&\\
1 & 0 &&&&\\
1 & 0 & 2 &&&\\
-2 & 0 & -2 & 4 &&\\
1 & -1 & 0 & 0 & 0 &\\
-2 & 1 & -1 & 0 & 1 & 0
\end{pmatrix},$$

$$\begin{pmatrix}
0 &&&&&\\
0 & 1 &&&&\\
0 & 0 & 1 &&&\\
1 & -1 & 0 & 1 &&\\
0 & 0 & 0 & -1 & 1 &\\
-1 & 0 & -1 & 1 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 &&&&&\\
0 & 0 &&&&\\
0 & 1 & 0 &&&\\
-1 & -2 & -3 & 8 &&\\
0 & 1 & 1 & -2 & 0 &\\
1 & -1 & 0 & 2 & -1 & 0
\end{pmatrix}$$

The positive definite quadratic form lying in the secondary cone of the described Delone triangulation and giving a locally optimal lattice covering is

$$Q_6^1 \approx \begin{pmatrix}
2.0550 &&&&&\\
-0.9424 & 1.9227 &&&&\\
1.1126 & -0.5773 & 2.0930 &&&\\
0.2747 & -0.7681 & -0.4934 & 1.7550 &&\\
-0.9424 & 0.3651 & -0.5773 & -0.7681 & 1.9227 &\\
-0.6153 & -0.3651 & -0.9804 & 0.7681 & -0.3651 & 1.9227
\end{pmatrix}.$$

Its normalized covering density is $\theta(Q_6^1) \approx 0.476962$.

At least in principle we could use the techniques we have demonstrated in the four-dimensional case to beautify the numbers. Actually we do not expect that it will give any insight to the "real" nature and origin of this lattice covering. Unfortunately we do not have an interpretation of this lattice covering.

Question 9.6.1.

▶ Is there any known or nice structure related to this lattice covering?

▶ Is there a 6-dimensional lattice covering which is better than the given one?

9.6.2. The second best known 6-dimensional lattice covering

The second best known 6-dimensional lattice covering we already met in Section 8.4.4 seems to be easier to understand than the best known one. Its Delone triangulation has 130 bistellar neighbours which is extreme at the moment. Its local lower bound is ≈ 0.448561 which is also extreme at the moment. The linear automorphism group of the Delone triangulation has order 3840 and it is

$$G_2 = \left\langle \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix} \right\rangle.$$

The set of all non-equivalent (under the group of translations) simplices is partitioned by the action of the group into three orbits with representatives

$$\begin{aligned} L_1 &= \operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_8, \boldsymbol{v}_{10}\} \\ L_2 &= \operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_5, \boldsymbol{v}_7, \boldsymbol{v}_{11}\} \\ L_3 &= \operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_4, \boldsymbol{v}_6, \boldsymbol{v}_8, \boldsymbol{v}_9, \boldsymbol{v}_{10}, \boldsymbol{v}_{11}\} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{v}_0 &= (0,0,0,0,0,0)^t & \boldsymbol{v}_1 &= (0,0,0,0,1,-1)^t \\ \boldsymbol{v}_2 &= (0,0,0,0,1,0)^t & \boldsymbol{v}_3 &= (0,0,1,0,1,-1)^t \\ \boldsymbol{v}_4 &= (0,1,-1,0,0,1)^t & \boldsymbol{v}_5 &= (0,1,0,0,1,0)^t \\ \boldsymbol{v}_6 &= (1,-1,-1,-1,-1,0)^t & \boldsymbol{v}_7 &= (1,-1,0,-1,0,-1)^t \\ \boldsymbol{v}_8 &= (1,0,-1,-1,-1,0)^t & \boldsymbol{v}_9 &= (1,0,-1,0,-1,1)^t \\ \boldsymbol{v}_{10} &= (1,0,-1,0,0,0)^t & \boldsymbol{v}_{11} &= (1,0,0,0,0,0)^t \end{aligned}$$

In the first two orbits there are at each case 320 simplices and in the last one there are 80 simplices. The lattice covering found by our algorithm is given by

$$Q_6^2 \approx \begin{pmatrix} 1.9982 & & & & & \\ 0.5270 & 1.9982 & & & & \\ 0.5270 & 0.5270 & 1.9982 & & & \\ -0.5270 & -0.5270 & -0.5270 & 1.9982 & & \\ 0.9440 & -0.5270 & -0.5270 & -0.9440 & 1.9982 & \\ 0.5270 & -0.9440 & 0.5270 & -0.5270 & 0.9440 & 1.9982 \end{pmatrix}.$$

Its normalized covering density is $\theta(Q_6^2) \approx 0.477217$. In the following we want to beautify the numbers. We demonstrated the technique already in Section 9.4. We report the results in a rather telegraphic style. The positive definite quadratic form Q_6^2 has to be invariant under the group G_2. Thus it lies in the two-dimensional subspace spanned by

$$A = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ 1 & 0 & 0 & -1 & 1 & \\ 0 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 1 & 1 & 0 & & & \\ -1 & -1 & -1 & 0 & & \\ -2 & -1 & -1 & 2 & 0 & \\ 1 & 2 & 1 & -1 & -2 & 0 \end{pmatrix}.$$

Using the approximation of the positive definite quadratic form Q_6^2 given above we conclude that $\mathrm{BR}_{L_1}(Q_6^2) = 0$ and $\mathrm{BR}_{L_2}(Q_6^2), \mathrm{BR}_{L_3}(Q_6^2) > 0$. We have to minimize the function $x \mapsto \frac{R_1^6}{\det A+xB}$ where R_1 denotes the circumradius of the simplex L_1 and find out that x has to be a root of the $x^2 - \frac{169}{382}x + \frac{9}{191}$. Hence $x = \frac{1}{764}(169+\sqrt{1057})$. This also enables us to give a nice integral approximation of the lattice covering. The integral positive definite quadratic form

$$\begin{pmatrix} 19 & & & & & \\ 5 & 19 & & & & \\ 5 & 5 & 19 & & & \\ -5 & -5 & -5 & 19 & & \\ 9 & -5 & -5 & -9 & 19 & \\ 5 & -9 & 5 & -5 & 9 & 19 \end{pmatrix}$$

has normalized covering density $\frac{47045881}{206524416} \approx 0.477282$. Some more data: The corresponding lattice has minimum 9, covering radius $9/2$, kissing number 32 and determinant $3226944 = 2^6 \cdot 3 \cdot 7^5$. But again we lack an interpretation of this!

9.7. Dimension 7

Dimension 7 is the largest dimension which our implementation can reasonably handle. Unlike in dimension 6 we had no success so far in finding a lattice covering that is thinner than the one given by A_7^*. We want to report on those lattice coverings which we found and which have "extreme" properties at the moment.

Maximum number of neighbours

The Delone triangulation $\mathcal{D}$ of the positive definite quadratic form

$$Q = \begin{pmatrix} 1.904925 & & & & & & \\ 0.500997 & 1.904925 & & & & & \\ 0.500997 & 0.500997 & 1.904925 & & & & \\ -0.523372 & -0.523372 & -0.523372 & 1.711906 & & & \\ 1.072379 & -0.331548 & -0.331548 & -0.973369 & 2.141536 & & \\ 0.500997 & -0.902931 & 0.500997 & -0.523372 & 1.072379 & 1.904925 & \\ -0.385543 & -0.385543 & -0.385543 & 0.241082 & -0.721332 & -0.385543 & 1.439959 \end{pmatrix}$$

has 211 bistellar neighbours ($\theta_*(\mathcal{D}) \approx 0.56761$, $\theta(Q) \approx 0.70161$).

Minimum local lower bound

The Delone triangulation $\mathcal{D}$ of the positive definite quadratic form

$$Q = \begin{pmatrix} 1.878537 & & & & & & \\ -0.788448 & 2.018228 & & & & & \\ 0.246709 & -1.220731 & 2.305654 & & & & \\ 0.537635 & -0.771141 & 0.235302 & 1.892539 & & & \\ -0.531172 & 0.862986 & -0.340367 & -0.515488 & 1.898709 & & \\ -0.297546 & -0.456190 & 0.735788 & -0.319872 & 0.181170 & 1.093520 & \\ -0.256090 & -0.242451 & -0.822380 & -0.271715 & -1.035561 & -0.574974 & 2.125296 \end{pmatrix}$$

has local lower bound $\theta_*(\mathcal{D}) \approx 0.56582$ (130 bistellar neighbours, $\theta(Q) \approx 0.65292$).

Second best known locally optimal lattice covering

The covering density of the positive definite quadratic form

$$Q = \begin{pmatrix} 1.863443 & & & & & & \\ -0.777709 & 2.009567 & & & & & \\ 0.242848 & -1.217673 & 2.311683 & & & & \\ 0.524683 & -0.764348 & 0.242660 & 1.891535 & & & \\ -0.522409 & 0.861928 & -0.348180 & -0.508457 & 1.894410 & & \\ -0.299146 & -0.460471 & 0.735739 & -0.320881 & 0.170282 & 1.098942 & \\ -0.252218 & -0.243976 & -0.825929 & -0.278885 & -1.032467 & -0.563856 & 2.122792 \end{pmatrix}$$

is $\theta(Q) \approx 0.651192$. The corresponding Delone triangulation has 143 bistellar neighbours and its local lower bound is ≈ 0.565825.

We do not claim and do not even dare to conjecture that any of these lattice coverings are extreme in any sense. Moreover, we leave it as a challenge to the reader to find improvements.

9.8. Dimension 8 and Higher

In our last section on the lattice covering problem we give a table of the least dense known lattice coverings in dimensions up to 24. At the same time this list gives the least dense known sphere coverings in dimensions up to 24 since there is no covering of equal spheres known which is better than the best known lattice covering. Our Table 9.1 is an update of Table 2.1 in [CS1988a]. We first give the table and comment it afterwards.

d	lattice covering	density Θ	normalized density θ	bound
1	$\mathbb{Z}^1$	1	0.5	0.5
2	A_2^*	1.209199	0.384900	0.3849
3	A_3^*	1.463505	0.349386	0.3419
4	A_4^*	1.765529	0.357771	0.3360
5	A_5^*	2.124286	0.403566	0.3581
6	Q_6^1	2.464803	0.476962	0.4087
7	A_7^*	3.059621	0.647571	0.4949
8	A_8^*	3.665949	0.903205	0.6319
9	A_9^5	4.340185	1.315802	0.8460
10	A_{10}^*	5.251713	2.059363	1.183
11	A_{11}^4	5.598338	2.971353	1.721
12	A_{12}^*	7.510113	5.624446	2.597
13	A_{13}^*	8.976769	9.857770	4.055
14	A_{14}^5	6.368635	10.627419	6.537
15	A_{15}^*	12.816873	33.600994	10.86
16	A_{16}^*	15.310927	65.061343	18.56
17	A_{17}^*	18.287811	129.718168	32.57
18	A_{18}^*	21.840949	265.880009	58.63
19	A_{19}^*	26.081820	559.436387	108.1
20	A_{20}^*	31.143448	1206.788059	204.0
21	A_{21}^*	37.184568	2665.722767	393.5
22	Λ_{22}^*	≤ 27.8839	3783.2116	775.2
23	Λ_{23}^*	≤ 15.3218	4020.7771	1558
24	Λ_{24}	7.903536	4096	3193

Table 9.1. Best known lattice coverings up to dimension 24.

In [Cox1951] COXETER gives a list of locally optimal lattice packings which are related to Lie groups. One of his infinite series of locally optimal lattice packings is given by A_d^r which is defined by the positive definite quadratic form

$$\sum_{i=1}^{d} x_i^2 + q(1 - \frac{1}{r})x_d^2 - \sum_{1<i<j<d-1} x_i x_j - x_q x_d,$$

where $d = qr - 1 > 1$ and $r > 1$. In [Bar1994] BARANOVSKII determines the covering density of the lattice covering given by A_9^5. The covering radius of the lattice A_9^5 is $\mu(\mathsf{A}_9^5) = \sqrt{3/5}$ and its determinant is $5^8/2^{26}$. By $\theta(\mathsf{A}_9^5) \approx 1.315802$ it is slightly better than the one given by A_9^* ($\theta(\mathsf{A}_9^*) \approx 1.330585$). Recently, ANZIN extended BARANOVSKII's work. In [Anz2002] he computes the covering densities of A_{11}^4 and A_{14}^5: The covering radii are $\mu(\mathsf{A}_{11}^4) = \sqrt{19/32}$, $\mu(\mathsf{A}_{14}^5) = \sqrt{71/100}$ and the determinants are $\det(\mathsf{A}_{11}^4) = 3/2^{13}$, $\det(\mathsf{A}_{14}^5) = 3/(5 \cdot 2^{14})$. In a private communication ANZIN stated that he computed the covering densities of lattice coverings given by A_d^r in other dimensions. They also give better coverings than the corresponding A_d^*.

It is not surprising that the Leech lattice Λ_{24} yields the best known lattice covering and it is not too brave to conjecture that it does give the optimal 24-dimensional sphere covering. The covering density of the Leech lattice was computed by CONWAY, PARKER and SLOANE (Chapter 23 of [CS1988a]). Expanding this work BORCHERDS, CONWAY and QUEEN computed the Dirichlet-Voronoi polytope of Λ_{24} (Chapter 25 of [CS1988a]). It seems that as a "corollary" of the existence of the Leech lattice the duals of the laminated lattices Λ_{22} and Λ_{23} give good lattice coverings. Their covering densities were estimated by SMITH [Smi1988]. For the definitions of these exceptional lattices and much more we refer to [CS1988a].

In [CFR1959] COXETER, FEW, ROGERS give a lower bound for the covering density of general sphere coverings. This bound is sharp for the two-dimensional case only. The values of their lower bound were copied one-to-one from [CS1988a].

There was not much activity in the last years on finding good lattice coverings in high dimensions. The book of ROGERS [Rog1964] contains the most recent results on the asymptotic behavior of lattice coverings.

Finally, we list our most tantalizing questions in the theory of lattice coverings.

- Solve the lattice covering problem in dimension 6!
- Find an interpretation of Q_6^1 or a better lattice covering in dimension 6!
- Try to understand the lattice covering given by A_d^r!
- Find construction methods for good lattice coverings!
- Find good sphere coverings that are not lattice coverings!
- Improve Table 9.1! Especially, improve the lower bounds!

Chapter A.

Glossary

A.1. Geometry of Numbers

Euclidean Spaces

A d-dimensional *Euclidean space* is a pair $(E, (\cdot,\cdot))$ consisting of a d-dimensional real vector space and an inner product $(\cdot,\cdot) : E \times E \to \mathbb{R}$. By an inner product we mean a positive definite symmetric bilinear form. A Euclidean space is a normed space. Its norm function is $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. In the case the "inner product" is only positive semidefinite, $\|\cdot\|$ is called *seminorm*. A Euclidean space is a metric space. Its distance function is $\operatorname{dist}(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$. By $B_d(\boldsymbol{c}, r) = \{\boldsymbol{x} \in E : \operatorname{dist}(\boldsymbol{x}, \boldsymbol{c}) \leq r\}$ we denote the d-dimensional closed ball with center $\boldsymbol{c} \in E$ and radius $r \in \mathbb{R}_{\geq 0}$. Let $V \subseteq E$ be a subset of E. By $\operatorname{lin} V = \{\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i : n \in \mathbb{N}, \alpha_i \in \mathbb{R}, \boldsymbol{v}_i \in V\}$ we denote the *linear span* of V, by $\operatorname{aff} V = \{\sum_{i=1}^{n} \alpha_i \boldsymbol{v}_i : n \in \mathbb{N}, \alpha_i \in \mathbb{R}, \sum_{i=1}^{n} \alpha_i = 1, \boldsymbol{v}_i \in V\}$ we denote the *affine span* of V.

Lattices

Let $(E, (\cdot,\cdot))$ be a d-dimensional Euclidean space. A subset $L \subseteq E$ is called an n-dimensional *lattice* in E if there exist linearly independent vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ so that $L = \{\sum_{i=1}^{n} \alpha_i \boldsymbol{b}_i : \alpha_i \in \mathbb{Z}\}$. The vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$ are called a *lattice basis* of L. To emphasize that our lattice live in a Euclidean space we sometimes even write $(L, (\cdot,\cdot))$ instead of L. From now on we assume that $n = d$. Another family of lattice vectors $(\boldsymbol{b}'_1, \dots, \boldsymbol{b}'_n)$ forms a lattice basis if and only if there exists an *integral unimodular transformation* $A \in \mathsf{GL}_n(\mathbb{Z})$ so that $A(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n) = (\boldsymbol{b}'_1, \dots, \boldsymbol{b}'_n)$. A lattice basis gives a *Gram matrix* of L: $G(L, (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)) = ((\boldsymbol{b}_i, \boldsymbol{b}_j))_{1 \leq i,j \leq n}$. This is a positive definite matrix. The determinant of a Gram matrix is the *determinant* of the lattice L: $\det L = \det((\boldsymbol{b}_i, \boldsymbol{b}_j))$. The *volume* of the lattice L is $\operatorname{vol} L = \sqrt{\det L}$. The volume of L equals the volume of a fundamental parallelotope $\mathcal{P}(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n) = \{\sum_{i=1}^{n} \alpha_i \boldsymbol{b}_i : \alpha_i \in [0,1]\}$. Two d-dimensional lattices $L, L' \subseteq E$ are called *isometric* if there exists an *isometry* between L and L'. This is a group homomorphism $\Phi : L \to L'$ so that $(\Phi(\boldsymbol{v}), \Phi(\boldsymbol{w})) = (\boldsymbol{v}, \boldsymbol{w})$ for all $\boldsymbol{v}, \boldsymbol{w} \in L$. The set of all isometries $\Phi : L \to L$ is called the *automorphism group* of L.

Dual Lattices

Every d-dimensional lattice $L \subseteq E$ in d-dimensional Euclidean space has a *dual lattice* L^*. It is given by $L^* = \{\boldsymbol{x} \in E : (\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{Z} \text{ for all } \boldsymbol{v} \in L\}$.

Special Lattices

In our investigations we will sometimes meet prominent lattices, e.g. root lattices and the Leech lattice. We do not uncover the 24-dimensional mystery Leech lattice here. The root lattices are defined as follows:

$$
\begin{aligned}
\mathsf{A}_n &= \{(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} : x_0 + \cdots + x_n = 0\}\\
\mathsf{D}_n &= \{(x_1, \ldots, x_n) \in \mathbb{Z}^{n} : x_1 + \cdots + x_n \text{ is even}\}\\
\mathsf{E}_8 &= \{(x_1, \ldots, x_8) : \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \tfrac{1}{2} + \mathbb{Z}, x_1 + \cdots + x_8 \text{ is even}\}\\
\mathsf{E}_7 &= \{(x_1, \ldots, x_8) \in \mathsf{E}_8 : x_1 + \cdots + x_8 = 0\}\\
\mathsf{E}_6 &= \{(x_1, \ldots, x_8) \in \mathsf{E}_8 : x_1 + x_8 = x_2 + \cdots + x_7 = 0)\}
\end{aligned}
$$

For more information on these lattices, on the Leech lattice, and on other important lattices consult Chapter 4 in [CS1988a].

Quadratic Forms

By $\mathcal{S}^d$ we denote the $\frac{d(d+1)}{2}$-dimensional space of real symmetric $(d \times d)$-matrices. The quadratic form which corresponds to the symmetric matrix $Q \in \mathcal{S}^d$ is given by $Q[\boldsymbol{x}] = (\boldsymbol{x}^t)Q\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^d$. *We do not distinguish between quadratic forms and symmetric matrices.* We denote the open cone of all positive definite quadratic forms by $\mathcal{S}^d_{>0} = \{Q \in \mathcal{S}^d : Q[\boldsymbol{x}] > 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^d \backslash \{\boldsymbol{0}\}\}$, and the closed cone of all positive semidefinite quadratic forms by $\mathcal{S}^d_{\geq 0}$. In the context of optimization problems we sometimes write $Q \succ 0$ instead of $Q \in \mathcal{S}^d_{>0}$ and $Q \succeq 0$ instead of $Q \in \mathcal{S}^d_{\geq 0}$. Two quadratic forms $Q, Q' \in \mathcal{S}^d$ are called *arithmetically equivalent* if there exists an integral unimodular transformation $A \in \mathsf{GL}_d(\mathbb{Z})$ with $Q[A\boldsymbol{x}] = Q'[\boldsymbol{x}]$.

Lattices vs. Quadratic Forms

There is a canonical bijection between the isometry classes of lattices and the arithmetical equivalence classes of positive definite quadratic forms.

A class of arithmetically equivalent positive definite quadratic forms defines an isometry class of lattices. Let Q be a positive definite quadratic form. This is mapped to the lattice $\mathbb{Z}^d$ together with the following scalar product: For $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{Z}^d$ we have $s_Q(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^t)Q\boldsymbol{y}$. If we don't expect any confusions we write simply $(\cdot, \cdot)$ instead of $s_Q(\cdot, \cdot)$. All lattices lying in the isometry class of $(\mathbb{Z}^d, s_Q)$ are called *associated* to Q.

A class of isometric lattices defines an arithmetical equivalence class of positive definite quadratic forms. Let L be a d-dimensional lattice. Let $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d)$ be a basis of L. Then the Gram matrix $G(L, (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d))$ gives a positive definite quadratic form. All positive definite quadratic forms lying in the arithmetical equivalence class of $G(L, (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d))$ are called *associated* to L.

It turns out that the language of lattices is appropriate for discussing geometry whereas the language of positive definite quadratic forms is appropriate for computing.

MORAL: Think in lattices, compute with quadratic forms!

Lattice Constants

	lattice covering problem	**lattice packing problem**
lattice	covering radius $\mu(L) = \min\{r : (B_d(\boldsymbol{v}, r))_{\boldsymbol{v}} \text{ is covering}\}$ covering density $\Theta(L) = \frac{\operatorname{vol} B_d(\boldsymbol{0}, \mu(L))}{\operatorname{vol} L}$ normalized covering density $\theta(L) = \frac{\mu(L)^d}{\operatorname{vol} L}$	packing radius $\lambda(L) = \max\{r : (B_d(\boldsymbol{v}, r))_{\boldsymbol{v}} \text{ is packing}\}$ packing density $\Delta(L) = \frac{\operatorname{vol} B_d(\boldsymbol{0}, \lambda(L))}{\operatorname{vol} L}$ normalized packing density $\delta(L) = \frac{\lambda(L)^d}{\operatorname{vol} L}$

	lattice covering problem	**lattice packing problem**
quadratic form	inhomogeneous minimum $\mu(Q) = \max_{\boldsymbol{x}\in\mathbb{R}^d} \min_{\boldsymbol{v}\in\mathbb{Z}^d} Q[\boldsymbol{x}-\boldsymbol{v}]$	homogeneous minimum $\lambda(Q) = \min_{\boldsymbol{v}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} Q[\boldsymbol{v}]$
	covering density $\Theta(Q) = \sqrt{\frac{\operatorname{vol} B_d(\boldsymbol{0},\mu(Q))}{\det Q}}$	packing density $\Delta(Q) = \sqrt{\frac{\operatorname{vol} B_d(\boldsymbol{0},\lambda(Q))}{\det Q}}$
	normalized covering density $\theta(Q) = \sqrt{\frac{\mu(Q)^d}{\det Q}}$	normalized packing density $\delta(Q) = \sqrt{\frac{\lambda(Q)^d}{\det Q}}$

A.2. Polyhedra and Polytopes

Polyhedra and polytopes are special convex subsets of affine space. A *polyhedron* is the intersection of finitely many halfspaces. A *polytope* is a bounded polyhedron. Conversely a polytope is the convex hull of finitely many points and a polyhedron is a sum of a polytope and a polyhedral cone.

Faces, Facets, Face Lattices

A *face* of a polytope P is the intersection of P and the boundary hyperplane of a halfspace containing P. A *facet* is a face of codimension 1. A *ridge* is a face of codimension 2. An *edge* is a face of dimension 1. A *vertex* is a face of dimension 0. Caution! Face lattices are not lattices! Face lattices are partially ordered sets. The face lattice of a polytope is the set of all its faces ordered by set theoretic inclusion.

Complexes, Subdivisions, Triangulations

A *polyhedral complex* $\mathcal{P}$ is a set of polyhedra which satisfies the following two conditions: (i) If a polyhedron P belongs to $\mathcal{P}$, then all faces of P are again in $\mathcal{P}$. (ii) The intersection of two polyhedra $P, Q \in \mathcal{P}$ is a face of P and Q. Sometimes, we call the elements of $\mathcal{P}$ *cells*. We say that a polyhedral complex is a *subdivision* or a *tiling* of the set $\bigcup_{P\in\mathcal{P}} P$. A *polytopal complex* is a polyhedral complex that only contains polytopes. If a polytopal complex contains simplices only we say that we have a *triangulation* of $\bigcup_{P\in\mathcal{P}} P$.

Zonotopes

A *zonotope* is a polytope whose faces are all centrally symmetric. Equivalently, zonotopes are those polytopes one gets by projecting higher-dimensional cubes $[-1,1]^n$.

A.3. (Realizable) Oriented Matroids

Basic Notation

Let $V \subseteq \mathbb{R}^d$ be a finite set of points. Every affine relation between these points (they are of the form $\sum_{\boldsymbol{v}\in V} \alpha_{\boldsymbol{v}}\boldsymbol{v} = 0$) gives rise to a sign vector $X \in \{-1,0,+1\}^V$ simply by $X_{\boldsymbol{v}} = \operatorname{sgn}\alpha_{\boldsymbol{v}}$, $\boldsymbol{v} \in V$. The *support* of the sign vector X is defined by $\underline{X} := \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} \neq 0\}$. Further useful notations are $X^+ := \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} = +1\}$, $X^- := \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} = -1\}$, and $X^0 := \{\boldsymbol{v} \in V : X_{\boldsymbol{v}} = 0\}$.

Circuits, Vectors

The set $\mathcal{V}(V)$ of all sign vectors which come from affine relations of V is called the set of *vectors* of the *oriented matroid* $\mathcal{M}(V)$. A non-trivial vector of $\mathcal{M}(V)$ which has minimal support among all these vectors is called a *circuit*.

Cocircuits, Covectors

For the set of points V we can define another set of sign vectors by the values of affine functions $\{(\operatorname{sgn} f(\boldsymbol{v}))_{\boldsymbol{v}\in V} : f$ affine function on $\mathbb{R}^d\}$. These are the *covectors* $\mathcal{V}^*(V)$ of $\mathcal{M}(V)$ which are dual to $\mathcal{V}(V)$ by the combinatorial relation

$$X \perp Y :\Longleftrightarrow (\underline{X} \cap \underline{Y} = \emptyset) \vee (\exists \boldsymbol{v}, \boldsymbol{w} \in V : X_{\boldsymbol{v}} Y_{\boldsymbol{v}} = +1, X_{\boldsymbol{w}} Y_{\boldsymbol{w}} = -1).$$

A non-trivial covector of $\mathcal{M}(V)$ having minimal support among all these covectors is called a *cocircuit*. We partially order the set of sign vectors by "$0 < +$" and "$0 < -$" so that the partial ordering on the set of sign vectors is understood componentwise. The partially ordered set of covectors is called the *face lattice* of an oriented matroid.

Directed Graphs

Let $G = (V, E)$, $E \subseteq V \times V$, be a directed graph. A *circuit* C of G is a special subset of E: C is a cycle and C is minimal with respect to inclusion having this property. A *minimal cut* C^* of G is a special subset of E: After removing all the edges belonging to C^* the number of connected components of G increases by one and C^* is minimal with respect to inclusion having this property.

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