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Shocks and Choices – an Analysis of Incomplete Market Models

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to my parents

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Contents

Abstract

The thesis is divided into two parts:
The first part deals with modelling long-range dependence in asset returns. Certain long-range dependence models, which have been suggested for financial modelling, fall outside the semimartingale set-up. We suggest Poisson shot noise processes as a skeleton of a long-range dependence model which provides an economic reasoning for long memory. We study weak convergence to a fractional Brownian motion. Whereas fractional Brownian motion allows for arbitrage, the shot noise processes themselves can be chosen arbitrage-free.

As complement we also investigate shot noise processes which consist of shots with finite limits. They converge to a Brownian motion, i.e. they have the same asymptotic behaviour as compound Poisson processes.

In the second part of the thesis we analyze American options and so-called “game options” in a general semimartingale setting. Game options naturally generalize American options by giving both counterparties the right to cancel the contract prematurely.

Whereas in recent years various suggestions have been made how to price European-type contingent claims in incomplete markets, up to now there is only little corresponding literature dealing with American options. Pricing the latter is conceptually more involved: in addition to the uncertainty caused by the underlyings, one has to take the seller’s ignorance of the buyer’s exercise strategy into account.

We generalize the “neutral derivative pricing” approach to American and game options which leads to *unique* “neutral” derivative price processes in incomplete markets.

An alternative approach is “utility-based indifference pricing” which was firstly suggested by Hodges and Neuberger (HN89) and which is by now a standard concept to valuate European style derivatives in incomplete markets. We generalize this concept to American options and so-called "chooser options". It leads to a quite surprising result concerning the exponential utility function.

Chapter 1

Introduction

1.1 Mathematical Finance - from modelling financial markets to hedging and pricing derivatives

Bachelier (Bac00) proposed to describe fluctuations in the price of a stock by a Brownian motion. Samuelson (Sam65) advocated a framework where the stock price is modeled by a *geometric* Brownian motion, which has the advantage that it does not generate negative prices. In this framework Black and Scholes (BS73) and Merton (Mer73) derived their celebrated formula for the price of a European call option. Their key discovery was that the option payoff can be replicated by a dynamic trading strategy holding positions in the underlying stock and the riskless bond. Consequently, all risk can be removed and the replication cost is the unique no-arbitrage price of the option, i.e. for every other option price either the seller or the buyer can make a riskless gain. Mathematically this is based on a representation theorem for Brownian martingales.

A serious disadvantage of this model is that it does not fit financial data very well. Nowadays it is well-known that the normal distribution is not a realistic model for the returns of most financial assets. One can often observe leptokurtic data, i.e. the increments of logarithmic prices have semi-heavy tails, such that the curtosis is higher than the curtosis of the normal distribution. In consequence of this, in recent years Lévy processes became important in modelling financial data. The distribution of the logarithmic prices generalizes from a normal distribution to an arbitrary infinitely divisible one. For example, Eberlein and Keller (EK95) proposed generalized hyperbolic Lévy processes resp. certain subclasses as a model for the logarithmic asset price processes and examined statistically their fit in a quite convincing way. These generalized hyperbolic distributions which model the increments of the logarithmic asset price, are a normal mean variance mixture. They were first introduced by Barndorff-Nielsen (BN77), who applied them to model grain size distributions of wind blown sands. Typical examples for these normal mixture models which play an increasing role also in the financial industry are the normal inverse Gaussian and the variance gamma model.

With the generalization from Brownian motion to general Lévy processes there arises the conceptual problem that financial models become *incomplete,* i.e. not every claim an be replicated. This has the consequence that simple no-arbitrage arguments alone are not sufficient to determine unique derivative prices. Consider a financial market consisting of two tradable securities: one risky asset modeled by a Lévy process and one riskless bond. Then, besides Brownian motion, Lévy processes with constant jump size, i.e. the sum of a multiple of a Poisson process and a linear drift, are the only examples where the market is complete, cf. Cox and Ross (CR76). This illustrates that complete markets are very special cases and incomplete markets is what we should expect.

There are, of course, various further generalizations, taking some dependency for the asset returns into account, as e.g. stochastic volatility models. The question arises what stochastic processes can be used to model asset prices. A limit is given in Delbaen and Schachermayer (DS94). It is shown that every adapted càdlàg locally bounded process that satisfies the *no free lunch with vanishing risk* (NFLVR) property for *simple integrands* is already a *semimartingale*. So, semimartingales are the most general "reasonable" stochastic processes to model asset prices. However, this arbitrage-argument is based on many idealized assumptions, as e.g. that investors are price-takers and there are no transaction costs. Dropping these assumptions one can of course think of more general stochastic processes than semimartingales.

1.2 European-type and American-type derivatives

In recent years various suggestions have been made how to price European-type contingent claims in incomplete markets (see Section 1.3). We think that by contrast there is only little corresponding literature dealing with American options. Pricing the latter is conceptually more involved: in addition to the uncertainty caused by the underlyings, one has to take the seller's ignorance of the buyer's exercise strategy into account.

Thus, one important issue of this thesis is the analysis of American options and their generalizations, so-called "game options" in a general semimartingale setting. Game options naturally generalize American options by giving both counterparties the right to cancel the contract prematurely.

Let us point out the main difference between European-type and American-type contingent claims. European-type claims can be characterized by a single random variable H being the (discounted) amount the seller has to pay to the buyer at maturity T. In this case we have a symmetric situation: one can interchange the roles of the seller and the buyer just by substituting H by $-H$. Examples are European call and put but also path-dependent options like Asian or lookback options. For a survey of exotic options we refer to Hull (Hul00), Chapter 18. Furthermore, in a frictionless financial market with a tradeable numeraire, expressing everything as multiples of this numeraire, it makes in principle no difference if the payoff takes place at time T or prematurely. So further examples are barrier options where the payoff takes place at a stopping time (which depends on the evolution of the underlying stock price and cannot be affected by the option buyer).

By contrast, for American style derivatives the buyer can *choose* her exercise time. There is an asymmetric situation. The seller does not know the exercise strategy of the buyer. Thus, American contingent claims are described by a stochastic process $L = (L_t)_{t\in[0,T]}$ instead of a single random variable H. If the buyer exercises the claim at time t the payoff is L_t. Of course, European claims can be interpreted as special cases of American claims by setting $L_t = -\infty$ for $t \in [0, T)$ and $L_T = H$.

Note however, in a *complete* market a rational buyer chooses her exercise strategy in such a way that she maximizes her expected (discounted) payoff under the unique equivalent martingale measure – independently of her utility function.

In *incomplete* markets the optimal exercise strategy of a rational buyer is not unique, a criterion may be provided by a utility function. Thus, one has also to take into consideration that the buyer can resell the option to another investor (with other preferences) on the market. In addition, one has to see that the "best" exercise strategy of the option buyer need not coincide with the "least favorable strategy" from the viewpoint of the option seller (as it is the case for complete markets). At this place we just want to outline that such considerations make the analysis of such contracts in the context of incomplete markets quite complicated.

In the context of utility-based indifference pricing we define a "still fair premium" for American claims by a *worst case* consideration, see Chapter 4. *Worst case analysis* means that the seller has to take all "possible" exercise strategies of the buyer into account. The advantage of this approach is that the seller need not know anything about the preferences of the buyer. The drawback is that it appears to be a quite pessimistic criterion and therefore it leads to quite high premiums.

Fortunately, in the context of "neutral derivative pricing" we can price American options (and even game options) without any worst case considerations, see Chapter 3.

1.3 On valuating and hedging of derivatives in incomplete markets

In the following we discuss several approaches for valuating and hedging in incomplete markets.

1.3.1 Super-replication

The (super-)hedging price is the smallest initial capital $\overline{V}$ that allows the *seller* of the claim to construct a portfolio which dominates the payoff process of the option. For a European style claim, which is given by a single payoff H paid out at maturity T, there exists an admissible strategy φ such that

$$\overline{V} + \int_0^T \varphi_u \, dS_u \geq H. \tag{3.2}$$

It turns out that

$$\overline{V} = \sup_{Q \in \mathscr{P}_e} E_Q(H), \tag{3.3}$$

where $\mathscr{P}_e$ is the set of all equivalent local martingale measures (cf. El Karoui and Quenez (EKQ95)). In Karatzas and Shreve (KS98) $\overline{V}$ is called the upper-hedging price of the claim H. Analogously one can define the lower-hedging price $\underline{V}$ as the highest price the *buyer* can afford to pay for the contingent claim and still be sure that by investment in the underlyings she can be guaranteed to have nonnegative wealth at T, once the payoff of the contingent claim has been received. We have

$$\underline{V} = \inf_{Q \in \mathscr{P}_e} E_Q(H). \tag{3.4}$$

For an American claim, given by a stochastic process L, the quantity $\overline{V}$ is the minimal amount such that

$$\overline{V} + \int_0^t \varphi_u \, dS_u \geq L_t, \quad t \in [0,T], \tag{3.5}$$

for an admissible strategy φ. Notice that in this definition of a superhedging price no stopping time is involved. So all exercise strategies of the option holder are allowed. Only the filtration of the option *seller* enters (3.5) (as φ has to be predictable with respect to the seller's filtration)

One obtains – as analogue to (3.3) – that

$$\overline{V} = \sup_{\tau \in \mathcal{S}} \sup_{Q \in \mathscr{P}_e} E_Q(L_\tau), \tag{3.6}$$

where $\mathcal{S}$ is the set of stopping times (cf. Karatzas and Kou (KK98), Kramkov (Kra96), Föllmer and Kabanov (FK98), and Föllmer and Kramkov (FK97)). An advantage of superhedging is that it is a quite intuitive concept: the superhedging-price is the smallest price leaving no risk to the hedger. It is at least an important reference point for the maximum price. Unfortunately, it yields only trivial upper bounds in many models of practical importance (cf. e.g. Eberlein and Jacod (EJ97), Frey and Sin (FS99), Cvitanić et al. (CPT99)). E.g., for a European call option with payoff $H = (S_T - K)^+$ one obtains in many models that $\overline{V} = S_0$, and the (static) superhedging strategy just consists in buying one stock at 0 and holding it until T.

1.3.2 Neutral derivative pricing

Davis (Dav97) defines the fair price at time 0 of a claim H paid out at maturity T as the price which makes investors indifferent between investing “a little of their funds” in the contract and not investing in the contract at all. A similar idea is followed up by Kallsen (Kal01), but in a more general setting allowing for intermediate trades also in the derivatives. Kallsen (Kal01) obtained a whole neutral price *process* for a European contingent claim. In this *dynamic* setting it is also possible to deal with American and game options – as we will see in Chapter 3 of this thesis. Let us explain the main idea of Kallsen (Kal01). As usual in *derivative pricing*, it is assumed that the stochastic processes describing the fluctuations in the underlying prices are *given* – by a semimartingale S^1 – whereas the derivative price process S^2 is to *determine*. A *neutral derivative price process* S^2 has essentially to satisfy two conditions. First the terminal condition $S^2_T = H$. Secondly, it is assumed that there is a representative agent maximizing her expected utility by choosing a dynamic portfolio (φ^1, φ^2) in the extended market (S^1, S^2), and in the optimum her demand for the derivative S^2 should vanish during the whole period $[0, T]$, i.e. $\varphi^2 = 0$. The idea behind the second condition is that derivative supply and demand has to be balanced as derivative securities have no counterpart in the world of commodities. In other words, if someone has a long position in a derivative somebody else has to be short. Obviously, this is in contrast to stocks. In the world of a representative agent (that means, all agents have the same utility function) this implies that the demand for derivatives of each agent has to be zero. More precisely, there has to exist an optimal strategy (φ^1, φ^2) with $\varphi^2 = 0$. It turns out that these two conditions uniquely determine a “neutral derivative price process” which can be represented as the conditional expectation of the final payoff H with respect to a special martingale measure $P^\star$, i.e. $S^2_t = E_{P^\star}(H|\mathcal{F}_t)$. $P^\star$ depends on the stock market, i.e. on the stochastic law of S^1, but for every claim H it is the same. So, one chooses one $P^\star$ from the set of equivalent martingale measures to price all derivatives. Summing up, in this model derivatives are tradeable assets like their underlyings with the only difference that the cumulative demand for it has to vanish.

1.3.3 Utility-based indifference pricing

At a first glance this concept seems to be quite similar to the previous one. Here, one takes the perspective of a particular counterparty and fixes the number of shares of the claim (say, 1 for an option buyer or -1 for an option seller). The *indifference premium* is a price such that the optimal expected utility among all portfolios containing the prespecified number of options coincides with the optimal expected utility among all portfolios without options. Put differently, the investor is indifferent to including the option into the portfolio, cf. equation (1.1) in Chapter 4 for an exact definition. This approach was firstly suggested by Hodges and Neuberger (HN89) and is by now a standard concept to valuate European style derivatives in incomplete markets.

Both, utility-based indifference pricing and neutral pricing rely on expected utility maximization and indifference to trading the option. Let us point out the differences between the two concepts. Indifference pricing takes an asymmetric point of view. Moreover, it depends decisively on the fixed number of claims under consideration. As far as options are concerned, intermediate trades are not allowed. Therefore, this approach is particularly well-suited for over-the-counter trades: Suppose that the buyer wants to purchase a specific contingent claim. Then she has to pay the seller at least the latter's indifference price in order to prompt her to enter the contract. This concept is also especially useful for insurance applications, see also the introduction of Chapter 4. It is a generalization of the classical zero-utility premium calculation principle in insurance mathematics where a random payoff is valuated without considering the possibility to hedge against it. For the classical approach see Gerber (Ger79) and for the financial generalization Schweizer (Sch01c).

An advantage of the indifference pricing approach is that one obtains not only prices but also hedging strategies. By contrast, neutral prices just lead to vanishing demand for derivatives and consequently there is nothing that can be interpreted as hedging strategy. On the other hand, the neutral prices are easier to compute as they are linear.

1.3.4 Quadratic approaches

This class of approaches consists of the so-called quadratic methods, see Schweizer (Sch01d) for a survey. We can divide this class of approaches into (local) risk-minimization approaches, proposed by Föllmer and Sondermann (FS86) for the case where the asset price process S is a martingale and generalized to semimartingales by Schweizer (Sch88; Sch91), and mean-variance hedging approaches, proposed by Bouleau and Lamberton (BL89) and Duffie and Richardson (DR91).

This approach has the big advantage that hedging strategies can be obtained quite explicitly. In addition, the quadratic approach can be embedded in the utility-indifference approach for an investor having mean/variance preferences, see Schweizer (Sch01c).

1.3.5 Quantile hedging and shortfall risk minimization

An undesirable feature of the quadratic approaches is the fact that they punish losses and gains equally. A way out of this is *quantile hedging* (see Föllmer and Leukert (FL99) and Spivak and Cvitanić (SC99)) or *efficient hedging* (see Föllmer and Leukert (FL00), Cvitanić and Karatzas (CK99), and Cvitanić (Cvi00)).

The seller minimizes the expected shortfall risk (or, more general, some lower partial moments) subjected to a given initial capital.

Let $l : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be her *loss function* which is an increasing convex function with $l(0) = 0$. She has to minimize

$$E\left[l(H - V_T)^+\right],$$

over all strategies φ, where $V_t = c + \int_0^t \varphi_u \, dS_u$, $t \in [0, T]$, is her wealth process and c her given initial capital. In this approach only losses are punished. However, gains are not rewarded.

1.4 Contribution of this thesis

The first part of this thesis, namely Chapter 2, deals with shot noise processes. Section 2.1 and 2.2, which coincide with Klüppelberg and Kühn (KK02b), are about modelling *long-range dependence* in asset returns. Certain long-range dependence models, which have been suggested for financial modelling, fall outside the semimartingale set-up. We suggest *Poisson shot noise processes* as a skeleton of a long-range dependence model which provides an economic reasoning for long memory. We study weak convergence to a fractional Brownian motion. Whereas fractional Brownian motion allows for arbitrage, the shot noise processes themselves can be chosen arbitrage-free.

In Section 2.3 we investigate – as complement to Section 2.1 and 2.2 – shot noise processes with finite shots, i.e. shots which possess finite and non-degenerated limits, when time tends to infinity. Such processes have already been analyzed in Klüppelberg and Mikosch (KM95b). We state limiting results for these kind of processes. It turns out that they converge to a Brownian motion, i.e. they have the same asymptotic behaviour as a compound Poisson process. So information with finite expansion cannot produce dependency in the limiting process.

Further in this thesis we analyze American options and so-called "game options" in a general semimartingale setting. Game options naturally generalize American options by giving both counterparties the right to cancel the contract prematurely.

In Chapter 3, which is submitted for publication in the form of Kallsen and Kühn (KK02a), we adapt the "neutral derivative pricing" approach as suggested by Kallsen (Kal01) to American and game options which leads to *unique* "neutral" derivative price processes in incomplete markets. This is also a generalization of the "marginal substitution value" approach for pricing European contingent claims as suggested in Davis (Dav97), cf. Subsection 1.3.2.

Chapter 4, which will appear in slightly different form as Kühn (Küh02), deals with an alternative approach called "utility-based indifference pricing" which was firstly suggested by Hodges and Neuberger (HN89) and which is by now a standard concept to valuate European style derivatives in incomplete markets. We generalize this concept to American and so-called "chooser options". It leads to a quite surprising result concerning the exponential utility function.

Chapter 5 coincides with Kühn (Küh01). Like Chapter 3 it deals with game options. But, the economic model is different. We assume that there is solely *one* option buyer/holder and *one* seller/writer and model the exercising of the option as a non-zero-sum stopping game taking trading possibilities *in the underlyings* explicitly into consideration. This corresponds to a game option which is not traded on a liquid market.

Chapter 2

Shot Noise Processes in Finance

Section 2.1 and 2.2 are an adapted version of Klüppelberg and Kühn (KK02b).

2.1 Introduction to shot noise processes

Whereas Lévy processes and stochastic volatility models are by now standard instruments to model stock prices, more recently long memory processes like fractional Brownian motion (FBM) have also attracted attention by stochastic analysts and mathematical finance researchers, cf. e.g., Hu and Øksendal (HØ99) and the references therein. For an introduction to FBM see Samorodnitsky and Taqqu (ST94). Certain financial time series show long memory properties as observed since the 1980s; see Granger (Gra80), resp. Granger and Joyeux (GJ80), and Mandelbrot (Man97). Such observation has led to an ongoing debate among econometricians and statisticians. It is obvious that any deterministic component like a small trend or business cycle can cause a *fictitious* long memory effect in a time series and it has been shown recently that also changepoints in a time series can exhibit such a long memory effect (Mikosch and Stărică (MS99)). More recently, Brody, Syroka, and Zervos (BSZ01) have investigated weather derivatives written on temperature-based indices, whose dynamics show long memory and can be modelled by fractional Ornstein-Uhlenbeck processes.

From the point of view of stochastic analysis FBM has the distinct disadvantage that it is not a semimartingale and allows for arbitrage; explicit arbitrage strategies have been found for FBM by Rogers (Rog97) and for geometric FBM by Cheridito (Che01a). But, as already mentioned there, the existence of an arbitrage possibility is no inherent property of long memory processes. It is rather a consequence of the *local* behaviour of FBM that is inconsistent with the properties of a semimartingale, whereas long-range dependence is a property of the long-run behaviour of a process.

In this chapter we answer the natural question for a possible economic explanation of logarithmic stock price processes to follow FBM. For instance, Brownian motion appears as Donsker limit of a random walk for relative price changes, as do Lévy processes in general. Stochastic volatility models have the obvious economic interpretation of a volatility changing in time depending on past prices, past volatilities and market conditions.

A first idea is to find a discrete skeleton, the most obvious one is a long memory linear model, more precisely an ARIMA(p, d, q) process with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0\,,\,0.5)$; for more details see Brockwell and Davis (BD87). Such models converge in a Donsker sense to FBM. Sottinen (Sot01) shows convergence of a special binary market model to FBM.

However, all this does not provide an economic reason, why to consider FBM or geometric FBM as a price model. Much more promising to us seems an idea by Stute (Stu00) who suggested geometric Brownian motion as price process, enriched by a geometric shot noise part. His model is given by

$$P(t) = \exp\{W(t) + S(t)\}, \quad t \geq 0, \tag{1.1}$$

where $(W(t))_{t\geq 0}$ is a Brownian motion and $(S(t))_{t\geq 0}$ is a shot noise model, which we define in a slight modification by

$$S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i) + \sum_{i=-1}^{-\infty} [X_i(t - T_i) - X_i(-T_i)]\,, \quad t \geq 0\,. \tag{1.2}$$

Here $X_i = (X_i(t))_{t\in\mathbb{R}}$, $i \in \mathbb{Z} \setminus \{0\}$, are i.i.d. stochastic processes on $\mathbb{R}$ such that $X_i(t) = 0$ for $t < 0$, independent of the two-sided homogeneous Poisson process N with rate $\alpha > 0$ and points $\ldots < T_{-2} < T_{-1} < T_1 < T_2 < \ldots$.

The shot noise model S is interpreted as a model for information provided by various sources which enters the price at random Poisson times. The arrival of information acts like a shock to the market which may change the price quite drastically and may also have some influence on the future price movements. The reason for this is that a new piece of information that is relevant for the stock price of a firm (e.g. a political decision or some rumor concerning a merger) needs some time to spread among the market participants. That means some traders have information earlier than others (think for example of insider-trading). Therefore it needs some time until the news reaches its maximum effect. Later on, some effects may fade away again, but it may as well happen that certain information has a long lasting influence on the price. In this way long memory is introduced into the economic model.

We obtain convergence to FBM. Moreover, we show that the model (1.2) itself can be chosen arbitrage-free (by the right choice of X_i near 0), only its limit model FBM allows for arbitrage.

Shot noise processes were used in various branches of stochastic modelling; references can be found in Klüppelberg and Mikosch (KM95b) and Klüppelberg, Mikosch, and Schärf (KTS01). Whereas in those papers limits for non-stationary shot noise models of the form $S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i), \;\; t \geq 0$, were investigated with a view towards applications in insurance, in Section 2.2 we work with a version of the process possessing stationary increments, which requires the introduction of the second sum in (1.2).

This chapter is organized as follows. First we investigate some properties of the restricted process $S\left|_{[0,t]}\right.$ which are important for applications in mathematical finance. In particular, we show how to construct an equivalent martingale measure. Hence, our model does not allow for arbitrage. In Subsection 2.2.2 we show weak convergence of a rescaled process to a FBM when the time horizon tends to infinity.

2.2 Long-range dependence

It is straightforward to see that S according to (1.2) has stationary increments. In Section 2.2, we restrict ourselves to the special case of multiplicative shots: for all $i \in \mathbb{Z} \setminus \{0\}$,

$$X_i(u) = g(u)Y_i, \quad u \geq 0, \tag{2.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a normalised regularly varying function in ∞ with index $\gamma \in (-1/2\,,\,1/2)$, i.e. g is continuously differentiable and $\lim_{u\to\infty} ug'(u)/g(u) = \gamma$, cf. Bingham, Goldie, and Teugels (BGT87). The Y_i are i.i.d. innovations with $EY_1 = 0$ and $EY_1^2 \in (0, \infty)$.

Example 2.2.1. Examples are $g(u) = (u+1)^\gamma$, $u \geq 0,\ \gamma \in (-1/2\,,\,1/2)$, as well as $g(u) = (u+1)^\gamma(1 \vee \ln u)$ or $g(u) = (u+1)^\gamma/(1 \vee \ln u)$.

Remark 2.2.2. Notice, that for $\gamma > 0$ the innovations that enter S are not absolutely summable. However, we show that for every $t \geq 0$ the limit in (1.2) exists almost surely and there exists a càdlàg version of S.

Lemma 2.2.3. *For $u, h \geq 0$, we have*

$$E\left[\sum_{i=-1}^{-\infty} \sup_{s\in[u-T_i,u+h-T_i]} \left[g'(s)\right]^2\right] < \infty.$$

Proof. As $\lim_{s\to\infty} sg'(s)/g(s) = \gamma$ it is straightforward to see that g' is regularly varying with index $\gamma - 1$. Therefore, for every $\varepsilon > 0$ there exists a $s_0 > 0$ s.t.

$$|g'(s)| \leq s^{\gamma-1+\varepsilon}, \quad \forall s \geq s_0. \tag{2.2}$$

For $u, h \geq 0$, we have

$$\begin{aligned}
&E\left[\sum_{i=-1}^{-\infty} \sup_{s\in[u-T_i,u+h-T_i]} \left[g'(s)\right]^2\right] \\
&\leq \sup_{0<s<\infty} \left[g'(s)\right]^2 \sum_{i=-1}^{-\infty} P\left(|T_i| < s_0 \vee \frac{|i|}{2\alpha}\right) + \sum_{i=-1}^{-\infty} \left(\frac{|i|}{2\alpha}\right)^{2(\gamma-1+\varepsilon)} =: A < \infty.
\end{aligned}$$

The finiteness of the first sum in A can be derived from general results for random walks with drift (see e.g. Theorem 3.3 in Chapter 3 of Gut (Gut88)) applied to the random walk $(|T_i| - |i|/(2\alpha))_{i=-1,-2,\ldots}$ (which has a positive drift and its increments are bounded from below). ☐

Proposition 2.2.4. *The process S as defined in (1.2) with (2.1) possesses a càdlàg version and has finite variation. Therefore, it is a semimartingale with respect to its natural filtration.*

Proof. For fixed t the existence of $S(t)$ can be derived from Theorem 1 in Westcott (Wes76). For the following, it is sufficient to consider the sum

$$\widetilde{S}(t) := \sum_{i=-1}^{-\infty} Y_i \left[g(t-T_i) - g(-T_i)\right], \quad t \geq 0. \tag{2.3}$$

For $u, h \geq 0$, we have

$$
\begin{aligned}
& E\left[\left(\widetilde{S}(u+h)-\widetilde{S}(u)\right)^2\right] \\
&= EY_1^2 E\left[\sum_{i=-1}^{-\infty}\left[g(u+h-T_i)-g(u-T_i)\right]^2\right] \\
&= EY_1^2 h^2 E\left[\sum_{i=-1}^{-\infty}\left[g'(\xi_i)\right]^2\right] \\
&\leq EY_1^2 h^2 A < \infty.
\end{aligned}
\tag{2.4}
$$

where $\xi_i \in (u-T_i, u+h-T_i)$. The inequalities hold due to Lemma 2.2.3.

Thus, Kolmogorov's continuity theorem ensures the existence of a continuous version of $\widetilde{S}$ (resp. a càdlàg version of S). We can approximate the variation of the process $\widetilde{S}$ by its variation on the dual grid $\{i2^{-n}t \mid i = 0, \ldots, 2^n\}$. Using Jensen's inequality and again (2.4) yields

$$
\begin{aligned}
& E\left[\sum_{k=0}^{2^n-1}\left|\widetilde{S}((k+1)2^{-n}t)-\widetilde{S}(k2^{-n}t)\right|\right] \\
&\leq 2^n\sqrt{E\left[\left(\widetilde{S}(2^{-n}t)-\widetilde{S}(0)\right)^2\right]} \\
&\leq t\sqrt{EY_1^2 A} < \infty.
\end{aligned}
\tag{2.5}
$$

Due to monotone convergence, we get the assertion by letting $n \to \infty$. □

Remark 2.2.5. The proof of Proposition 2.2.4 is based on the fact that $|T_i|$ grows with the same order as $|i|$, and the increments are nonnegative. So, it is also valid for renewal processes other than the Poisson process.

From now on, we work with the completed stochastic basis of $(\Omega, \mathcal{F}_t, (\mathcal{F}_s)_{0\leq s\leq t}, P)$, where

$$
\mathcal{F}_s = \sigma\left((Y_i)_{i\in\mathbb{Z}_-},\ (T_i)_{i\in\mathbb{Z}_-},\ (S(u))_{0\leq u\leq s}\right). \tag{2.6}
$$

Define f as *càglàd* modification of the process

$$
u \mapsto \sum_{i=N(u)}^{-\infty} Y_i g'(u-T_i). \tag{2.7}
$$

Similar to (2.4), by Kolmogorov's continuity theorem, there exists a continuous version of $\left(\sum_{i=-1}^{-\infty} Y_i g'(u-T_i)\right)_{u\geq 0}$, we call it $\widetilde{f}$. In addition, we define

$$
Z(t) := g(0)\sum_{i=1}^{N(t)} Y_i, \quad t \geq 0. \tag{2.8}
$$

Lemma 2.2.6. *With the quantities as defined above, S satisfies the stochastic differential equation*

$$dS(t) = dZ(t) + f(t)\,dt, \quad t \geq 0. \tag{2.9}$$

Proof. Step 1: First of all we have to show that we can interchange integration and summation, i.e.

$$\sum_{i=-1}^{-\infty} Y_i\,[g(t+h-T_i) - g(t-T_i)] = \int_t^{t+h} \widetilde{f}(u)\,du \quad P\text{-a.s.}, \quad \forall t,h \geq 0. \tag{2.10}$$

Remember that the addends entering the sums in (2.10) are not absolutely summable.

On the grid points $\{t + hj2^{-n} \mid n \in \mathbb{N},\ j = 1,\dots,2^n\}$ we have pointwise convergence by the martingale convergence theorem, i.e.

$$\sum_{i=-1}^{-m} Y_i g'(t + hj2^{-n} - T_i) \to \widetilde{f}(t + hj2^{-n}) \quad P\text{-a.s.}, \quad m \to \infty. \tag{2.11}$$

Then, we want to estimate the approximation error

$$G(m,n) := \sum_{i=-1}^{-m} Y_i a_i^{(n)} := \sum_{i=-1}^{-m} Y_i \left[h2^{-n} \sum_{j=1}^{2^n} g'(t + hj2^{-n} - T_i) - \{g(t+h-T_i) - g(t-T_i)\} \right].$$

As g' is continuous we have for all $i \in \mathbb{Z}_-$ that $a_i^{(n)} \to 0$ P-a.s., as $n \to \infty$. On the other hand

$$|a_i^{(n)}| \leq 2h \sup_{u \in (t-T_i, t+h-T_i)} |g'(u)| =: a_i, \tag{2.12}$$

and from Lemma 2.2.3 we obtain that $E\left[\sum_{i=-1}^{-\infty} a_i^2\right] < \infty$ and thus $E\left[\sum_{i=-1}^{-\infty} (a_i^{(n)})^2\right] \to 0$, as $n \to \infty$. Therefore, for fixed $n \in \mathbb{N}$ the sequence $(G(m,n))_{m \in \mathbb{N}}$ has a limit $G(\infty, n)$, both P-a.s. and in $L^2(P)$ (the former is due to martingale convergence and the latter by the Cauchy criterion). We have $EG(\infty,n)^2 = E\left[\sum_{i=-1}^{-\infty} (a_i^{(n)})^2\right] \to 0$, as $n \to \infty$. Thus, $G(\infty, n)$ tends to zero in probability for $n \to \infty$.

Now, we are ready to proof Equation (2.10). Let $\varepsilon > 0$ and $\omega \in \Omega$ be given (outside excluded null sets: in the whole verification, we use P-a.s. arguments only for countably many objects, due to completeness of the σ-algebra $\mathcal{F}_t$ we can throw away all P-null sets that make problems). As $\widetilde{f}$ is continuous the integral $\int_t^{t+h} \widetilde{f}(u)\,du$ can be approximated (ω-wise) by the Riemann sums $h2^{-n}\sum_{j=1}^{2^n} \widetilde{f}(t+hj2^{-n})$, $n \in \mathbb{N}$, i.e. there exists an n_1 s.t. for $n \geq n_1$

$$\left| h2^{-n}\sum_{j=1}^{2^n} \widetilde{f}(t+hj2^{-n}) - \int_t^{t+h} \widetilde{f}(u)\,du \right| \leq \frac{\varepsilon}{3}. \tag{2.13}$$

Since $G(\infty, n) \xrightarrow{P} 0$ as $n \to \infty$ we have

$$P\left\{G(\infty, n) \leq \varepsilon/6 \quad \text{infinitely often}\right\} = 1.$$

And as $G(m, n) \to G(\infty, n)$ when $m \to \infty$, we can find a $n_2 \geq n_1$ and $m_1 \in \mathbb{N}$ such that for all $m \geq m_1$

$$|G(m, n_2)| \leq \frac{\varepsilon}{3}. \tag{2.14}$$

For this n_2 we use (2.11), i.e. the convergence on the grid $\{t + hj2^{-n_2} \mid j = 1, \ldots, 2^{n_2}\}$, and we get for $m \geq m_2$

$$\begin{aligned} &\left| \sum_{i=-1}^{-m} Y_i h2^{-n_2} \sum_{j=1}^{2^{n_2}} g'(t+hj2^{-n_2} - T_i) - h2^{-n_2}\sum_{j=1}^{2^{n_2}} \widetilde{f}(t+hj2^{-n_2}) \right| \\ &= \left| h2^{-n_2}\sum_{j=1}^{2^{n_2}} \sum_{i=-1}^{-m} Y_i g'(t+hj2^{-n_2} - T_i) - h2^{-n_2}\sum_{j=1}^{2^{n_2}} \widetilde{f}(t+hj2^{-n_2}) \right| \leq \frac{\varepsilon}{3}. \end{aligned} \tag{2.15}$$

Putting (2.13), (2.14), and (2.15) together, we get that for $m \geq m_1 \vee m_2$

$$\left| \sum_{i=-1}^{-m} Y_i \left[g(t+h-T_i) - g(t-T_i)\right] - \int_t^{t+h} \widetilde{f}(u)\,du \right| \leq \varepsilon \tag{2.16}$$

and therefore (2.10) holds.

Step 2: Using (2.10) Equation (2.9) follows just from the calculation:

$$
\begin{aligned}
&S(t+h) - S(t) \\
&= \sum_{i=N(t)+1}^{N(t+h)} Y_i g(t+h-T_i) + \sum_{i=N(t)}^{-\infty} Y_i \left[g(t+h-T_i) - g(t-T_i)\right] \\
&= g(0) \sum_{i=N(t)+1}^{N(t+h)} Y_i + \sum_{i=N(t)+1}^{N(t+h)} Y_i \left[g(t+h-T_i) - g(0)\right] + \sum_{i=N(t)}^{-\infty} Y_i \left[g(t+h-T_i) - g(t-T_i)\right] \\
&= g(0) \sum_{i=N(t)+1}^{N(t+h)} Y_i + \sum_{i=N(t)+1}^{N(t+h)} Y_i \int_{T_i}^{t+h} g'(u-T_i)\, du + \int_t^{t+h} \sum_{i=N(t)}^{-\infty} Y_i g'(u-T_i)\, du \\
&= g(0) \sum_{i=N(t)+1}^{N(t+h)} Y_i + \int_t^{t+h} \sum_{i=N(t)+1}^{N(u)} Y_i g'(u-T_i)\, du + \int_t^{t+h} \sum_{i=N(t)}^{-\infty} Y_i g'(u-T_i)\, du \\
&= Z(t+h) - Z(t) + \int_t^{t+h} f(u)\, du. \qquad (2.17)
\end{aligned}
$$

□

2.2.1 Girsanov Theorem

Theorem 2.2.7. *If $g(0) > 0$, then there exists a probability measure $Q \sim P$ such that S is a local martingale with respect to Q.*

Proof. Step 1: Construction of a possible Q: under Q, the process Z should be a point process whose stochastic compensator has rate $-f$. Then by (2.9) S becomes a local martingale. This can be achieved by applying Girsanov's theorem for point processes, cf. Theorem T10 p. 241 in Brémaud (Bré81). Translated to our notation the theorem says: choose a $\widetilde{\mathcal{P}}$-measurable function $\phi > 0$ ($\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ whereas $\mathcal{P}$ is the $\mathcal{F}$-predictable σ-algebra on $\Omega \times \mathbb{R}_+$ and $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$) satisfying

$$\int_0^t \int_{\mathbb{R}} \phi(s,x) \alpha P(Y_1 \in dx)\, ds < \infty \quad P-\text{a.s.}, \qquad (2.18)$$

and define for $s \in [0,t]$

$$L_s := \exp\left\{ \int_0^s \int_{\mathbb{R}} \log \phi(u,x) \mu(du,dx) + \int_0^s \int_{\mathbb{R}} (1-\phi(u,x)) \alpha P(Y_1 \in dx)\, du \right\}, \quad (2.19)$$

or, equivalently,

$$dL_s = L_s \int_{\mathbb{R}} (\phi(s,x) - 1) \left\{ \mu(ds,dx) - \alpha P(Y_1 \in dx)\, ds \right\}, \quad L_0 = 1, \qquad (2.20)$$

where

$$\forall A \in \mathcal{B}(\mathbb{R}), \qquad \mu\left(\omega,(0,s]\times A\right) := \sum_{0<T_i\le s} I(Y_i \in A). \tag{2.21}$$

If $EL_t = 1$, then there is a $Q \sim P$ defined by

$$\frac{dQ}{dP} = L_t, \tag{2.22}$$

such that under Q the process Z is a point process with rate

$$\lambda_Q(s,dx) = \phi(s,x)\alpha P(Y_1 \in dx), \quad s \ge 0, x \in \mathbb{R}. \tag{2.23}$$

To make S a local martingale we need that

$$g(0)\int_{\mathbb{R}} x\lambda_Q(s,dx) = -f(s), \quad s \ge 0. \tag{2.24}$$

This can be achieved by setting

$$\phi(s,x) := \begin{cases} 1 & : \ f(s) < 0 \quad \text{and} \quad x < 0 \\ 1 - \dfrac{f(s)}{\alpha g(0) EY_1^+} & : \ f(s) < 0 \quad \text{and} \quad x \ge 0 \\ 1 + \dfrac{f(s)}{\alpha g(0) EY_1^-} & : \ f(s) \ge 0 \quad \text{and} \quad x < 0 \\ 1 & : \ f(s) \ge 0 \quad \text{and} \quad x \ge 0 \end{cases} \tag{2.25}$$

ϕ is $\widetilde{\mathcal{P}}$-measurable and strictly positive. $(L_s)_{0\le s\le t}$ is a local P-martingale and, due to positivity, a P-supermartingale, i.e. $EL_t \le 1$. To verify $EL_t = 1$ we make a localization: as f is càglàd we can define by $\tau_n := \inf\{s \ge 0 \mid |f(s+)| > n\}$ a sequence of stopping times with $|f^{\tau_n}| \le n$ and obtain due to $P\left(\sup_{s\in[0,t]} |f(s)| < \infty\right) = 1$ that $P(\tau_n \ge t) \to 1$ as $n \to \infty$. Define $\phi^n := \phi 1_{[0,\tau_n]} + 1_{(\tau_n,t]}$. For the corresponding density processes $(L_s^n)_{s\in[0,t]}$ and $dQ^n/dP = L_t^n$ we have indeed $EL_t^n = 1$ (cf. Theorem T11 p. 242 in Brémaud (Bré81)) and therefore

$$1 = E\left(L_t^n\right) = E\left(L_t 1_{\{\tau_n \ge t\}}\right) + E\left(L_t^n 1_{\{\tau_n < t\}}\right) = E\left(L_t 1_{\{\tau_n \ge t\}}\right) + Q^n(\tau_n < t). \tag{2.26}$$

Thus, it remains to show that $Q^n(\tau_n < t) \to 0$, as $n \to \infty$.

Step 2: In this step we show the non-explosiveness of a special point process $\widetilde{N}$. In step 3 we derive from this that $Q^n(\tau_n < t) \to 0$, as $n \to \infty$.

First of all we have to redefine our model on a new probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, where $\widetilde{\Omega} = \widetilde{\Omega}_1 \times \widetilde{\Omega}_2 \times \widetilde{\Omega}_3$, $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_1 \otimes \widetilde{\mathcal{F}}_2 \otimes \widetilde{\mathcal{F}}_3$, and $\widetilde{P} = \widetilde{P}_1 \otimes \widetilde{P}_2 \otimes \widetilde{P}_3$.

On $\widetilde{\Omega}_1$ we define three i.i.d. sequences $(U_i^1)_{i\in\mathbb{N}}$, $(U_i^2)_{i\in\mathbb{N}}$, and $(I(A_i))_{i\in\mathbb{N}}$. They are mutually independent. U_i^1, U_i^2 are uniformly distributed on $(0,1)$ and the events A_i have probabilty $\widetilde{P}_1(A_i) = p := F_Y(0)$, where F_Y is the given distribution function of the jumps under the objective probability measure. We define Y_i by

$$Y_i := I(A_i)F_Y^{\leftarrow}(p + (1-p)U_i^1) + I(A_i^C)F_Y^{\leftarrow}(pU_i^2), \tag{2.27}$$

where $F_Y^{\leftarrow}$ is the generalized inverse of F_Y. Obviously, Y_1 has under $\widetilde{P}_1$ the distribution function F_Y. The aim of this construction is the following: under the new measure $\widetilde{Q}^n$, which we will obtain, as in the previous step, from $\widetilde{P}$ by the density L_t^n, the random variables $U_i^1, U_i^2,\ i = 1, 2, \ldots$ remain independent and uniformly distributed on $(0,1)$. Only the distribution of $I(A_i)$ changes.

In addition, $\widetilde{\Omega}_1$ covers the independent random variable C_1, which has under $\widetilde{P}_1$ the same distribution as $\sup_{s\in[0,t]} \left|\sum_{i=-1}^{-\infty} Y_i g'(s - T_i)\right|$ under P. Since the paths $s \mapsto \sum_{i=-1}^{-\infty} Y_i g'(s - T_i)$ are bounded on $[0,t]$ P-a.s., we have $\widetilde{P}_1(C_1 < \infty) = 1$.

$\widetilde{\Omega}_2$ covers an increasing sequence $(T_i)_{i\in\mathbb{N}}$. The increments of $(T_i)_{i\in\mathbb{N}}$ are independent and exponentially-distributed with parameter α. They correspond to the jump times in the original model.

$\widetilde{\Omega}_3$ covers an increasing sequence $(\widehat{T}_i)_{i\in\mathbb{N}}$. The increments of $(\widehat{T}_i)_{i\in\mathbb{N}}$ are independent and exponentially-distributed with parameter 1.

Now we construct further quantities on these spaces.

We define on $\widetilde{\Omega}_1$ a sequence $(\widetilde{Y}_i)_{i\in\mathbb{N}}$ of nonnegative random variables by

$$\widetilde{Y}_i := F_Y^{\leftarrow}(p + (1-p)U_i^1) \vee (-F_Y^{\leftarrow}(pU_i^2)).$$

Notice that $\widetilde{Y}_i \geq |Y_i|$.

Let $c_2 := \sup_{s\in[0,t]} |g'(s)|$, and $c_3 := [g(0)(E_P Y_1^+ \wedge E_P Y_1^-)]^{-1}$. On $\widetilde{\Omega}_3$ we define a new increasing sequence $(\widetilde{Y}_i)_{i\in\mathbb{N}}$ by

$$\widetilde{T}_1 := \frac{\widehat{T}_1}{\alpha + c_3 C_1},$$

$$\widetilde{T}_{n+1} := \widetilde{T}_n + \frac{\widehat{T}_{n+1} - \widehat{T}_n}{\alpha + c_3(C_1 + c_2 \sum_{i=1}^n \widetilde{Y}_i)}, \quad n = 1, 2, \dots .$$

$\widetilde{N}$ denotes the counting process of $(\widetilde{Y}_i)_{i\in\mathbb{N}}$. Let

$$\widetilde{f}(s) = C_1 + c_2 \sum_{i=1}^{\widetilde{N}(s)} \widetilde{Y}_i, \quad s \in [0, \widetilde{T}_\infty). \tag{2.28}$$

By construction, $\widetilde{N}(s)$ possesses the rate $\alpha + c_3\widetilde{f}(s-)$. We want to show that $\widetilde{N}$ is nonexplosive on $[0,t]$, i.e. $\widetilde{P}\left(\widetilde{T}_n \geq t\right) \to 1$ for $n \to \infty$. That implies

$$\sup_{s\in[0,t]} \widetilde{f}(s) = \widetilde{f}(t) < \infty, \quad \widetilde{P}\text{-a.s.} \tag{2.29}$$

Therefore, define $\widetilde{\mathcal{F}}_i = \sigma\left(\widetilde{T}_j,\ j = 0, \dots, i\right) \vee \sigma\left(C_1,\ (\widetilde{Y}_j)_{j\in\mathbb{N}}\right)$, $i = 0, 1, \dots$, with the convention $\widetilde{T}_0 := 0$.

As an easy consequence from the law of large numbers, we have

$$\sum_{i=1}^n \widetilde{Y}_i \leq C_2 n, \quad \forall n \in \mathbb{N}, \quad \widetilde{P}_1\text{-a.s.}, \tag{2.30}$$

where C_2 is an $\widetilde{\mathcal{F}}_0$-measurable real valued random variable. (2.30) yield that $(\widetilde{f})^{\widetilde{T}_{i+1}-} \leq C_1 + c_2 C_2 i$, $\widetilde{P}$-a.s., for all $i \in \mathbb{N}$, and consequently the rate of $\widetilde{N}$ is bounded on $[0, \widetilde{T}_{i+1}]$ by $\alpha + c_3(C_1 + c_2 C_2 i) =: C_1' + C_2' i$. Hence, for nonnegative $\widetilde{\mathcal{F}}_0$-measurable real valued random variables a_i we have that

$$\widetilde{P}\left(\widetilde{T}_{i+1} - \widetilde{T}_i > a_i \mid \widetilde{\mathcal{F}}_i\right) \geq \exp\{-a_i(C_1' + C_2' i)\}, \quad \widetilde{P}\text{-a.s.} \tag{2.31}$$

Notice that $\widetilde{T}_{i+1} - \widetilde{T}_i$ is independent of $\widetilde{Y}_{i+1}, \widetilde{Y}_{i+2}, \dots$. Choosing $a_i = \ln 2/(C_1' + C_2' i)$ we obtain

$$\widetilde{P}\left(\widetilde{T}_{i+1} - \widetilde{T}_i > \frac{\ln 2}{C_1' + C_2' i} \mid \widetilde{\mathcal{F}}_i\right) \geq 1/2, \quad \widetilde{P}\text{-a.s.} \tag{2.32}$$

Furthermore, we have

$$\widetilde{T}_n = \sum_{i=0}^{n-1} (\widetilde{T}_{i+1} - \widetilde{T}_i) \geq \sum_{i=0}^{n-1} I\left(\widetilde{T}_{i+1} - \widetilde{T}_i > \frac{\ln 2}{C_1' + C_2' i}\right) \frac{\ln 2}{C_1' + C_2' i}, \tag{2.33}$$

and

$$\widetilde{P}\left(\sum_{i=0}^{\infty} \frac{\ln 2}{C_1' + C_2' i} = \infty \mid \widetilde{\mathcal{F}}_0\right) = 1, \quad \widetilde{P}\text{-a.s.} \tag{2.34}$$

Putting (2.33), (2.34), and (2.32) together we arrive in view of Lemma A1.2 at

$$\begin{aligned}&\widetilde{P}\left(\sum_{i=0}^{\infty}(\widetilde{T}_{i+1}-\widetilde{T}_i)=\infty \mid \widetilde{\mathcal{F}}_0\right)\\ &\geq \widetilde{P}\left(\sum_{i=0}^{\infty} I\left(\widetilde{T}_{i+1}-\widetilde{T}_i > \frac{\ln 2}{C_1'+C_2'i}\right)\frac{\ln 2}{C_1'+C_2'i}=\infty \mid \widetilde{\mathcal{F}}_0\right)=1, \quad \widetilde{P}\text{-a.s.,}\end{aligned} \tag{2.35}$$

i.e., by Fubini's theorem, $\widetilde{T}_n \nearrow \infty$ as $n \to \infty$, $\widetilde{P}$-a.s., and hence we obtain (2.29).

Step 3: On the new space define f analogously to (2.7), and let again $\tau_n := \inf\{s \geq 0 \mid |f(s+)| > n\}$. By some "monotonicity arguments" we want to show that (2.29) implies that $Q^n(\tau_n < t) \to 0$, as $n \to \infty$.

$\widetilde{Q}^n$ is defined with respect to $\widetilde{P}$ by the density L_t^n, as in the previous step. By comparing the $\widetilde{P}$-rate of $\widetilde{N}$ with the $\widetilde{Q}^n$-rate of N on the stochastic intervals $(\widetilde{T}_m, \widetilde{T}_{m+1}]\cap[0,t]$ and $(T_m, T_{m+1}]\cap [0,t]$, resp., we obtain for all *positive* $\sigma(U_i^1, U_i^2,\ i\in\mathbb{N}, C_1)$-measurable random variables a and for all $B_1,\ldots,B_m \in \mathcal{B}(\mathbb{R}_+)$

$$\begin{aligned}&\widetilde{P}(\widetilde{T}_{m+1} > (\widetilde{T}_m + a)\wedge t \mid \widetilde{T}_1 \in B_1,\ldots,\widetilde{T}_m-\widetilde{T}_{m-1}\in B_m; \sigma(U_i^1,U_i^2,\ i\in\mathbb{N},\ C_1))\\ &\leq \widetilde{Q}^n(T_{m+1} > (T_m+a)\wedge t \mid T_1\in B_1,\ldots,T_m-T_{m-1}\in B_m; \sigma(U_i^1,U_i^2,\ i\in\mathbb{N},\ C_1)),\end{aligned} \tag{2.36}$$

$\widetilde{P}$-a.s. This holds as $\widetilde{T}_{m+1}-\widetilde{T}_m$ is $\widetilde{P}$-independent of $\sigma(U_i^1,U_i^2,\ i=m+1,m+2,\ldots)$ and $T_{m+1}-T_m$ is $\widetilde{Q}^n$-independent of $\sigma(U_i^1,U_i^2,\ i=m+1,m+2,\ldots)$. Whereas the former independence is trivial, the latter independence is due to the special form of ϕ in (2.25) and the construction of Y_i in (2.27). In view of Lemma A1.1, (2.36) implies that

$$\widetilde{P}\left(\widetilde{T}_m > t \mid \sigma(U_i^1,U_i^2,\ i\in\mathbb{N},C_1)\right) \leq \widetilde{Q}^n\left(T_m > t \mid \sigma(U_i^1,U_i^2,\ i\in\mathbb{N},C_1)\right),\ \widetilde{P}\text{-a.s.} \tag{2.37}$$

Let us show that

$$
\begin{aligned}
&\widetilde{Q}^n(\tau_n < t)\\
&\leq \widetilde{Q}^n\left(C_1 + \sup_{s\in[0,t]}\left|\sum_{i=1}^{N(s)} g'(s-T_i)Y_i\right| > n\right)\\
&\leq \widetilde{Q}^n\left(C_1 + c_2\sum_{i=1}^{N(t)}|Y_i| > n\right)\\
&= \widetilde{Q}^n\left(\widetilde{Q}^n\left(C_1 + c_2\sum_{i=1}^{N(t)}|Y_i| > n \;\middle|\; \sigma(U_i^1, U_i^2,\ i\in\mathbb{N}, C_1)\right)\right)\\
&= \widetilde{P}\left(\widetilde{Q}^n\left(C_1 + c_2\sum_{i=1}^{N(t)}|Y_i| > n \;\middle|\; \sigma(U_i^1, U_i^2,\ i\in\mathbb{N}, C_1)\right)\right)\\
&\leq \widetilde{P}\left(\widetilde{Q}^n\left(C_1 + c_2\sum_{i=1}^{N(t)}\widetilde{Y}_i > n \;\middle|\; \sigma(U_i^1, U_i^2,\ i\in\mathbb{N}, C_1)\right)\right)\\
&\leq \widetilde{P}\left(\widetilde{P}\left(C_1 + c_2\sum_{i=1}^{\widetilde{N}(t)}\widetilde{Y}_i > n \;\middle|\; \sigma(U_i^1, U_i^2,\ i\in\mathbb{N}, C_1)\right)\right)\\
&= \widetilde{P}(\widetilde{f}(t) > n).
\end{aligned}
\tag{2.38}
$$

The first two inequalities are trivial estimations. The second equality holds as (U_i^1), (U_i^2), and C_1 have under $\widetilde{Q}^n$ the same distribution as under $\widetilde{P}$. For the third inequality we use that $|Y_i| \leq \widetilde{Y}_i$. The fourth inequality holds by (2.37). As by (2.29) $\widetilde{P}(\widetilde{f}(t) > n) \to 0$ for $n \to \infty$, (2.38) yields the assertion. □

Remark 2.2.8. An alternative way to proof this theorem is to use Theorem 3.6 in Jacod (Jac75) which guarantees the existence of a measure Q such that $(T_n, Y_n)_{n\in\mathbb{N}}$ is under Q a (possibly exploding) marked point process with rate

$$
\lambda_Q(s,dx) = \begin{cases}
\alpha P(Y_1 \in dx) & : \ f(s) < 0 \ \text{ and } \ x < 0,\\
\alpha P(Y_1 \in dx)\left(1 - \dfrac{f(s)}{\alpha g(0) E Y_1^+}\right) & : \ f(s) < 0 \ \text{ and } \ x \geq 0,\\
\alpha P(Y_1 \in dx)\left(1 + \dfrac{f(s)}{\alpha g(0) E Y_1^-}\right) & : \ f(s) \geq 0 \ \text{ and } \ x < 0,\\
\alpha P(Y_1 \in dx) & : \ f(s) \geq 0 \ \text{ and } \ x \geq 0,
\end{cases}
$$

on the whole interval $[0,t] \cap [0, T_\infty)$. In contrast to step 1 in our proof it is not necessary to make a localization. To obtain $Q \sim P$ it is (also) to verify that $(T_n)_{n\in\mathbb{N}}$ is not exploding under Q.

Remark 2.2.9. A heuristic explanation why S is arbitrage-free is that though an investor could profit by the stochastic drift $f(s)\,ds$, there remains the risk dZ_s that consists of jumps. Therefore, it cannot be controlled as effective as for FBM which has continuous sample paths. In contrast, the fractional binary market model in Sottinen (Sot01) obviously allows for arbitrage as - given the history of the process - it can happen that the discounted stock price increases with probability one. The same phenomena occurs in our model when setting $g(0) = 0$.

Remark 2.2.10. Q is obviously not unique. The unities in (2.25) can be replaced by every other element of $\mathbb{R}_+ \setminus \{0\}$.

We add to S an independent Brownian motion σB, $\sigma \geq 0$. To transfer an additive to a geometric model, there are two common approaches in mathematical finance: the Doléans Dade exponential and the ordinary exponential of the process S. In the first case the price process of the asset satisfies the following SDE

$$dP(t) = P(t-)\left(dS(t) + \sigma\, dB(t)\right), \quad t \geq 0, \qquad P(0) = p_0 > 0. \tag{2.39}$$

$(S(s) + \sigma B(s))_{s\in[0,t]}$ is a local Q-martingale and hence $(P(s))_{s\in[0,t]}$, cf. e.g. Theorem 17 of Chapter III in Protter (Pro92). If $\Delta S > -1$ $(P(s))_{s\in[0,t]}$ is positive.

In the second case, i.e. setting

$$\widetilde{P}(t) = p_0 \exp\left\{S(t) + \sigma B(t) - \frac{\sigma^2}{2}t\right\}, \quad t \geq 0, \tag{2.40}$$

we have by Itô's formula

$$d\widetilde{P}(t) = \widetilde{P}(t-)\left(dS(t) + e^{\Delta S(t)} - 1 - \Delta S(t) + \sigma\, dB(t)\right), \quad t \geq 0, \tag{2.41}$$

and by (2.9)

$$d\widetilde{P}(t) = \widetilde{P}(t-)\left(e^{\Delta Z(t)} - 1 + df(t) + \sigma\, dB(t)\right), \quad t \geq 0.. \tag{2.42}$$

Thus, condition (2.24) has to be replaced by

$$\int_{\mathbb{R}} \left(e^{g(0)x} - 1\right) \lambda_Q(s, dx) = -f(s), \quad s \geq 0. \tag{2.43}$$

This can be achieved by setting

$$\widetilde{\phi}(s,x) := \begin{cases} 1 & : \quad f(s) < 0 \quad \text{and} \quad x < 0 \\ \dfrac{1 - Ee^{-g(0)Y_1^-}}{Ee^{g(0)Y_1^+} - 1} - \dfrac{f(s)}{\alpha\left(Ee^{g(0)Y_1^+} - 1\right)} & : \quad f(s) < 0 \quad \text{and} \quad x \geq 0 \\ 1 + \dfrac{f(s)}{\alpha\left(1 - Ee^{-g(0)Y_1^-}\right)} & : \quad f(s) \geq 0 \quad \text{and} \quad x < 0 \\ \dfrac{1 - Ee^{-g(0)Y_1^-}}{Ee^{g(0)Y_1^+} - 1} & : \quad f(s) \geq 0 \quad \text{and} \quad x \geq 0 \end{cases} \tag{2.44}$$

With the same arguments as in the proof of Theorem 2.2.7 one verifies that by plugging $\widetilde{\phi}$ into (2.19) one obtains a measure $\widetilde{Q}$, equivalent to P, under which $\widetilde{P}$ becomes a local martingale.

2.2.2 Asymptotic theory

Now, we introduce for $t > 0$ the rescaled process

$$S_x(t) = \frac{S(xt)}{\sigma(t)}, \quad x \in [0, \infty), \tag{2.45}$$

where $\sigma^2(t) = \text{var}\,(S(t))$ and show weak convergence to a FBM.

Theorem 2.2.11. *Let B^H be a FBM with Hurst parameter $H = \gamma + 1/2$ for $\gamma \in (0\,,\,1/2)$. Then*

$$S_{\cdot}(t) \xrightarrow{d} B^H, \quad t \to \infty, \tag{2.46}$$

where the convergence holds in $D[0,\infty)$ equipped with the metric of uniform convergence on compacta and the projection σ-algebra.

Remark 2.2.12. *If $\gamma \in (-1/2\,,\,0]$ we still have the convergence of the finite-dimensional distributions. This follows from Steps 1-2 in the proof of Theorem 2.2.11, which go through for $\gamma \in (-1/2\,,\,1/2)$.*

Proof. Since the limit process has continuous sample paths we can equivalently consider weak convergence with respect to the Skorohod J_1-metric on $D[0,\infty)$, see e.g. (16.4) in Billingsley (Bil99). By Billingsley (Bil99), Theorem 16.7 and Theorem 13.1, we have to show weak convergence of the finite-dimensional distributions and tightness of $\left(S_{\cdot}(t)|_{[0,M]}\right)_{t\in\mathbb{R}_+}$ for each $M \in \mathbb{R}_+$.

Step 1: By Campbell's theorem (cf. Daley and Vere-Jones (DVJ88)) we have for $0 \le s \le t$

$$\begin{aligned} \mathrm{Cov}(S(s), S(t)) &= \alpha \int_0^s E X_1(u) X_1(u+t-s)\, du \\ &\quad + \alpha \int_0^\infty E\left[X_1(s+u) - X_1(u)\right]\left[X_1(t+u) - X_1(u)\right]\, du, \end{aligned} \tag{2.47}$$

and obtain for $0 \le x \le y$

$$\frac{\mathrm{Cov}(S(xt), S(yt))}{\sigma^2(t)} = \frac{\int_0^x \frac{g(ut) g((u+y-x)t)}{g(t)^2}\, du + \int_0^\infty \frac{[g((x+u)t) - g(ut)][g((y+u)t) - g(ut)]}{g(t)^2}\, du}{\int_0^1 \frac{g(ut)^2}{g(t)^2}\, du + \int_0^\infty \frac{[g((1+u)t) - g(ut)]^2}{g(t)^2}\, du}. \tag{2.48}$$

We show that the r.h.s. of (2.48) converges when t tends to infinity to

$$\frac{\int_0^x u^\gamma (u+y-x)^\gamma\, du + \int_0^\infty \left[(x+u)^\gamma - u^\gamma\right]\left[(y+u)^\gamma - u^\gamma\right]\, du}{\frac{1}{2\gamma+1} + \int_0^\infty \left[(1+u)^\gamma - u^\gamma\right]^2\, du}, \tag{2.49}$$

which is the covariance function of $B^{\gamma+1/2}$ (it is sufficient to verify it for $x = y$ as the increments of the processes are stationary). For fixed $u \in \mathbb{R}_+ \setminus \{0\}$ convergence is obvious. By Potter bounds the integral on compacta converges. But, for the integral on $(0, \infty)$ we need an integrable dominating function for

$$\left| \frac{\left[g((x+u)t) - g(ut)\right]\left[g((y+u)t) - g(ut)\right]}{g^2(t)} \right|. \tag{2.50}$$

We have for a $\xi \in (u, x+u)$

$$\left| \frac{g((x+u)t) - g(ut)}{g(t)} \right| = \left| xt \frac{g'(\xi t)}{g(t)} \right| = \left| x \frac{t g'(t)}{g(t)} \frac{g'(\xi t)}{g'(t)} \right| \le 2x|\gamma| u^{\gamma-1}, \quad u \ge 1, t \ge t_0,$$

where the last inequality holds as g is *normalised* regularly varying with index γ and $g'(\lambda t)/g'(t)$ converges to $\lambda^{\gamma-1}$ *uniformly* in $\lambda \in [1, \infty)$, when $t \to \infty$, cf. Bingham, Goldie, and Teugels (BGT87), Theorem 1.5.2. As $\gamma < 1/2$ we have the required integrable dominating function. Hence we have shown that

$$\frac{\mathrm{Cov}(S(xt), S(yt))}{\sigma^2(t)} \to \mathrm{Cov}(B^{\gamma+1/2}(x), B^{\gamma+1/2}(y)), \quad t \to \infty. \tag{2.51}$$

Let $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, d \in \mathbb{N}$, $0 \le x_1 < \ldots < x_d < \infty$, and consider

$$Z(t) := \sum_{i=1}^d \lambda_i S_{x_i}(t), \quad t \ge 0, \tag{2.52}$$

By (2.51) the variance of $Z(t)$ converges to those of $\sum_{i=1}^d \lambda_i B^{\gamma+1/2}(x_i)$.

Step 2: Now we verify a condition (which is very similar to the Lindeberg condition) for $Z(t)$ to ensure that it converges to a normal limit. $Z(t)$ has zero mean and if not all λ_i vanish, $\mathrm{var}(Z(t)) \to c$ as $t \to \infty$, for a $c > 0$. Hence by Theorem 3 in Lane (Lan84) we have to show that for every $\varepsilon > 0$

$$\sigma(t)^{-2} \int_{\varepsilon\sigma(t)}^{\infty} y \quad \left[\int_0^{x_d t} P\left(\left| Y_1 \sum_{i=1}^{d} \lambda_i g(x_i t - u) \right| > y \right) du \right. \\ \left. + \int_0^{\infty} P\left(\left| Y_1 \sum_{i=1}^{d} \lambda_i \left(g(u + x_i t) - g(u) \right) \right| > y \right) du \right] dy \to 0, \quad t \to \infty.$$

It is sufficient to verify that for every $\lambda, \varepsilon > 0$

$$\sigma(t)^{-2} \int_{\varepsilon\sigma(t)}^{\infty} y \int_0^{\infty} P\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| > y\right) du\, dy \to 0, \quad t \to \infty, \quad \text{and} \tag{2.53}$$

$$\sigma(t)^{-2} \int_{\varepsilon\sigma(t)}^{\infty} y \int_0^{t} P\left(\lambda \left| Y_1 g(u) \right| > y\right) du\, dy \to 0, \quad t \to \infty. \tag{2.54}$$

Ad (2.53): we have

$$\begin{aligned} &\sigma(t)^{-2} \int_{\varepsilon\sigma(t)}^{\infty} y \int_0^{\infty} P\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| > y\right) du\, dy \\ &= \sigma(t)^{-2} \int_0^{\infty} \int_{\varepsilon\sigma(t)}^{\infty} y P\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| > y\right) dy\, du \\ &= \frac{1}{2} \sigma(t)^{-2} \int_0^{\infty} E\left(\left(\lambda \left| Y_1 \left(g(u+t) - g(u) \right) \right| - \varepsilon\sigma(t) \right)^{+} \right)^2 du \\ &= \frac{1}{2} \sigma(t)^{-2} t g(t)^2 \int_0^{\infty} E\left(\left(\lambda \left| Y_1 \frac{g((u+1)t) - g(ut)}{g(t)} \right| - \varepsilon \frac{\sigma(t)}{g(t)} \right)^{+} \right)^2 du. \end{aligned} \tag{2.55}$$

Since

$$\int_0^{\infty} E\left(\lambda Y_1 \frac{g((u+1)t) - g(ut)}{g(t)} \right)^2 du = \lambda^2 E Y_1^2 \int_0^{\infty} \left(\frac{g((u+1)t) - g(ut)}{g(t)} \right)^2 du < \infty,$$

we have an integrable function that dominates the integrand in the last line of (2.55). With (2.47) in mind it is easy to see that $\sigma^2(t) = c g(t)^2 t \left(1 + o(1)\right)$, $c > 0$, $t \to \infty$. Therefore, dominated convergence implies that the last line of (2.55) converges to zero as $t \to \infty$. (2.54) can be proven in a similar way. Thus $Z(t) \xrightarrow{d} \sum_{i=1}^{d} \lambda_i B^{\gamma+1/2}(x_i)$, $t \to \infty$, and the Cramér-Wold device yields the convergence of the finite-dimensional distributions.

Step 3: Finally, we check tightness. The family of processes $\left((\sigma(t)^{-1}g(0)\sum_{i=1}^{N(\cdot t)} Y_i) \mid_{[0,M]}\right)_{t\in\mathbb{R}_+}$ is obviously tight. Thus we can replace g by $\widetilde{g} = g - g(0)$. Since the increments of S are stationary we have

$$E\left(S_y(t) - S_x(t)\right)^2 = \frac{\sigma^2\left((y-x)t\right)}{\sigma^2(t)}. \tag{2.56}$$

Due to (2.51) σ^2 is regular varying with index $1+2\gamma$. Therefore, $h(s) := \sigma^2(s)/s^{1+\gamma}$ is regular varying with index $\gamma > 0$, and due to

$$\begin{aligned}
\sigma^2(s) &= EY_1^2\left\{\int_0^s \widetilde{g}^2(u)\,du + \int_0^\infty \left[\widetilde{g}(s+u) - \widetilde{g}(u)\right]^2\,du\right\}\\
&\le EY_1^2\left\{\int_0^s \left(\int_0^u \widetilde{g}'(v)\,dv\right)^2 du + s^2\int_0^\infty \sup_{\xi\ge u}\left[\widetilde{g}'(\xi)\right]^2\,du\right\}\\
&\le EY_1^2\left\{\frac{1}{3}s^3\sup_{0<u<\infty}\left[\widetilde{g}'(u)\right]^2 + s^2\int_0^\infty \sup_{\xi\ge u}\left[\widetilde{g}'(\xi)\right]^2\,du\right\},
\end{aligned} \tag{2.57}$$

h is bounded near zero. Therefore, $h((y-x)t)/h(t)$ converges to $(y-x)^\gamma$ for $t\to\infty$, *uniformly* in $x<y$ on compact subsets of $\mathbb{R}_+$, cf. Bingham, Goldie, and Teugels (BGT87), Theorem 1.5.2. This implies that for each $M>0$ and $t\ge t_M$

$$\frac{\sigma^2\left((y-x)t\right)}{\sigma^2(t)} \le 2(y-x)^{1+\gamma}, \quad \forall\, 0\le x<y\le M. \tag{2.58}$$

This (together with Cauchy-Schwarz's inequality) ensures the tightness condition (13.14) in Billingsley (Bil99) (recall that $\gamma>0$). $\square$

2.2.3 Conclusion

We have constructed Poisson shot noise processes whose finite-dimensional distributions are close to those of FBM, but which lead to arbitrage-free models for stock prices. By way of contrast, if the shots $(X_i)_{i\in\mathbb{Z}\setminus\{0\}}$ have no jumps at zero and there is no additional Brownian noise σW in (2.39) and (2.40) respectively, our model obviously allows for arbitrage, even with so-called "simple" trading strategies.

These results can also be considered as supplements to recent work of Cheridito (Che00; Che01b). He has excluded arbitrage from FBM by changing slightly the convolution kernel in the

Mandelbrot-Van Ness representation of FBM or, alternatively, he considered, for $H\in(3/4\,,\,1]$, the process $B^H + \varepsilon B^{1/2}$ ($\varepsilon > 0$ arbitrary small) instead of just B^H. This leads to complete models whereas our models are incomplete.

2.3 Finite shots

In this section we want to investigate shot noise processes with finite, non-degenerated shots, i.e. $X_i(u)$, $i \in \mathbb{N}$, possess finite limits $X_i(\infty) \not\equiv 0$ for $u \to \infty$. Define

$$S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i)\,, \quad t \geq 0\,, \tag{3.1}$$

where $(X_i(u))_{u\in\mathbb{R}}$, $i \in \mathbb{N}$ are i.i.d. copies of a stochastic process $(X(u))_{u\in\mathbb{R}}$ which are independent of the homogeneous Poisson process N. Let $X(u) = 0$ for $u < 0$. In general the increments of S are neither independent nor stationary.

An important application of (3.1) is the modelling of delay in claim settlement in insurance portfolios. In that case $S(t)$ is interpreted as the total claim amount in an insurance portfolio and the process $X_i(\cdot - T_i)$ describes the pay-off procedure of the ith individual claim. If $t > T_i$ claim i is already occured and $X_i(\infty) - X_i(t - T_i)$ can be interpreted as its amount which is not yet reported to the insurance company. For more details see Klüppelberg and Mikosch (KM95b),(KM95a) and Klüppelberg and Severin (KS02b).

Consider for $t > 0$ the rescaled processes

$$S_x(t) := \frac{S(xt) - \mu(xt)}{\sigma(t)}, \quad x \in [0,\infty), \tag{3.2}$$

where $\mu(t) = ES(t)$ and $\sigma^2(t) = \mathrm{var}\,(S(t))$.

In this section we state some limiting results for (3.2).

Proposition 2.3.1. *Suppose that $X(u) \to X(\infty)$ P-a.s. as $u \to \infty$ with $EX^2(\infty) > 0$ and $E\left(\sup_{0\leq u<\infty} X^2(u)\right) < \infty$, then the finite-dimensional distributions of the process $S_\cdot(t)$ converge to those of B, where B is a standard Brownian motion on $[0,\infty)$.*

Remark 2.3.2. Proposition 2.3.1 states that $S_\cdot(t)$ has the same asymptotic behaviour as

$$\frac{\sum_{i=1}^{N(xt)} X_i(\infty) - \alpha xtEX(\infty)}{\sqrt{\mathrm{var} X(\infty)t}}$$

It is a result similar to Proposition 3.4 in Klüppelberg and Mikosch (KM95b). But it requires no monotonicity of the sample paths $X(u)$ and – more interesting – $X(u)$ may approach $X(\infty)$ arbitrary slowly. On the other hand, for *functional* convergence we need additional assumptions to ensure tightness. This is formulated in Theorem 2.3.4. Proposition 2.3.1 in itself can also be interpreted as a negative result: if $X(u)$ has a finite limit, then $S_{\cdot}(t)$ can converge (weakly) to nothing else than Brownian motion.

Proof of Proposition 2.3.1. Step 1: First, we notice that due to Campbell's theorem (cf. Daley and Vere-Jones (DVJ88)) we have $\sigma^2(t) = \alpha \int_0^t EX^2(u)\,du$, and as a simple consequence of the assumptions

$$\sigma^2(t) = \alpha t EX^2(\infty)\,(1+o(1))\,, \quad t \to \infty.$$

Let $x \in (0,\infty)$. We decompose $S_x(t)$ in the following way:

$$\begin{aligned}
S_x(t) &= \sigma^{-1}(t)\left\{\sum_{i=1}^{N(xt)} X_i(xt-T_i) - \mu(xt)\right\} \\
&= \left[\alpha t EX^2(\infty)\right]^{-1/2}(1+o(1)) \\
&\times \left[\sum_{i=1}^{N(xt)}\left\{X_i(xt-T_i) - X_i(\infty) - E\left(X(xt-T_i)\mid T_i\right) + EX(\infty)\right\}\right. \\
&+ \sum_{i=1}^{N(xt)}\left\{E\left(X(xt-T_i)\mid T_i\right) - EX(\infty)\right\} - \alpha\int_0^{xt}\left\{EX(u)-EX(\infty)\right\}du \\
&+ \left.\sum_{i=1}^{N(xt)} X_i(\infty) - \alpha xt EX(\infty)\right] \\
&= I_1(x) + I_2(x) + I_3(x).
\end{aligned} \tag{3.3}$$

In Step 2 and Step 3 resp. we show by L^2-arguments that for a fixed $x \in (0,\infty)$ $I_1(x) \stackrel{d}{\to} 0$ and $I_2(x) \stackrel{d}{\to} 0$ resp., for $t \to \infty$. This and an application of the continuous mapping theorem results in the assertion, as the finite-dimensional distributions of the standard compound Poisson process $I_3(x)$ converges to B, cf. Klüppelberg and Mikosch (KM95b) or Gut (Gut88).

Step 2: Ad $I_1(x)$: We introduce the normalized shots

$$\widetilde{X}_i(u) := X_i(u) - EX(u). \tag{3.4}$$

Let $\varepsilon > 0$ be given. Due to the assumptions there exists a $K_\varepsilon \in \mathbb{R}_+$ s.t.

$$E\left(\sup_{K_\varepsilon \le u < \infty}\left(\widetilde{X}(u) - \widetilde{X}(\infty)\right)^2\right) \le \frac{EX^2(\infty)\varepsilon}{2x}. \tag{3.5}$$

As we consider for fixed $\varepsilon > 0$ the limit for $t \to \infty$ we can assume w.l.o.g. that $xt - K_\varepsilon > 0$. We have

$$\begin{aligned}
(1 + o(1))\, EI_1^2 &= \frac{1}{\alpha t EX^2(\infty)} E\left(\sum_{i=1}^{N(xt)}\left\{\widetilde{X}_i(xt - T_i) - \widetilde{X}_i(\infty)\right\}\right)^2 \\
&= \frac{1}{\alpha t EX^2(\infty)} E\left(\sum_{i=1}^{N(xt-K_\varepsilon)}\left\{\widetilde{X}_i(xt - T_i) - \widetilde{X}_i(\infty)\right\}\right)^2 \\
&+ \frac{1}{\alpha t EX^2(\infty)} E\left(\sum_{i=N(xt-K_\varepsilon)+1}^{N(xt)}\left\{\widetilde{X}_i(xt - T_i) - \widetilde{X}_i(\infty)\right\}\right)^2 \\
&= J_1 + J_2.
\end{aligned}$$

$$\begin{aligned}
J_1 &= \frac{1}{\alpha t EX^2(\infty)} \sum_{k=0}^{\infty} P(N(xt - K_\varepsilon) = k) \\
&\quad \times \sum_{i=1}^{k} E\left[E\left(\left(\widetilde{X}_i(K_\varepsilon + (xt - K_\varepsilon)U_i) - \widetilde{X}_i(\infty)\right)^2 \mid U_i\right)\right],
\end{aligned}$$

where the U_i are uniformly distributed on $(0, 1)$ and stochastically independent of $\widetilde{X}_i$. This is due to the well-known "order-statistic property" of Poisson arrival times: given $N(u) = k$ the random vector $(T_1, \ldots T_k)$ has the same distribution as the order statistics of a sample of k i.i.d. random variables with uniform distribution on $[0, u]$.

Together with (3.5) it implies

$$J_1 \le \frac{1}{\alpha t EX^2(\infty)} \sum_{k=0}^{\infty} P(N(xt - K_\varepsilon) = k) k \frac{EX^2(\infty)\varepsilon}{2x} = \frac{\alpha(xt - K_\varepsilon)}{\alpha t}\frac{\varepsilon}{2x} \le \frac{\varepsilon}{2}.$$

Now, as $J_2 = O(1/t)$, $t \to \infty$ (for fixed K_ε), for t big enough it is smaller than $\varepsilon/2$, thus $J_1 + J_2 \le \varepsilon$. Therefore $I_1(x) \xrightarrow{d} 0$, $t \to \infty$.

Step 3: Ad $I_2(x)$: Let $\varepsilon > 0$ be given. Analogue to (3.5) there exists a real number, again called K_ε, s.t.

$$E\left(\sup_{K_\varepsilon \le u < \infty}(X(u) - X(\infty))^2\right) \le \frac{EX^2(\infty)\varepsilon}{2x}, \tag{3.6}$$

and $xt - K_\varepsilon > 0$. Again, separate the jumps "shortly" before xt from the others.

$$
\begin{aligned}
&(1+o(1))\, I_2 \\
&= \frac{1}{\sqrt{\alpha t E X^2(\infty)}} \\
&\times \left[\sum_{i=1}^{N(xt-K_\varepsilon)} \left\{ E\left(X(xt-T_i) \mid T_i\right) - EX(\infty) \right\} - \alpha \int_{K_\varepsilon}^{xt} \left\{ EX(u) - EX(\infty) \right\}\, du \right. \\
&+ \left. \sum_{i=N(xt-K_\varepsilon)+1}^{N(xt)} E\left(X(xt-T_i) \mid T_i\right) - \alpha \int_0^{K_\varepsilon} \left\{ EX(u) - EX(\infty) \right\}\, du \right] \\
&= \widetilde{J}_1 + \widetilde{J}_2.
\end{aligned}
$$

$\widetilde{J}_1$ and $\widetilde{J}_2$ are stochastically independent and have expectation equal to zero. Furthermore, again by the "order-statistic property" of Poisson arrival times, we obtain for $\widetilde{J}_1$ that

$$
\begin{aligned}
E\widetilde{J}_1^2 \;&=\; \frac{1}{\alpha t E X^2(\infty)} \left[E\left(\sum_{i=1}^{N(xt-K_\varepsilon)} \left\{ E\left(X(xt-T_i) \mid T_i\right) - EX(\infty) \right\} \right)^2 \right. \\
&\quad \left. - \alpha^2 \left[\int_{K_\varepsilon}^{xt} \left\{ EX(u) - EX(\infty) \right\}\, du \right]^2 \right] \\
&=\; \frac{1}{\alpha t E X^2(\infty)} \\
&\quad \times \left[\sum_{k=0}^{\infty} P(N(xt-K_\varepsilon)=k) E\left(\sum_{i=1}^{k} \left\{ E\left(X(K_\varepsilon + (xt-K_\varepsilon)U_i) \mid U_i\right) - EX(\infty) \right\} \right)^2 \right. \\
&\quad \left. - \alpha^2 \left[\int_{K_\varepsilon}^{xt} \left\{ EX(u) - EX(\infty) \right\}\, du \right]^2 \right] \\
&=\; \frac{1}{\alpha t E X^2(\infty)} \\
&\quad \times \left[\sum_{k=0}^{\infty} P(N(xt-K_\varepsilon)=k) \left\{ kE\left(E\left(X(K_\varepsilon + (xt-K_\varepsilon)U_1) \mid U_1\right) - EX(\infty)\right)^2 \right. \right. \\
&\quad \left. + k(k-1)\left[E\left(X(K_\varepsilon + (xt-K_\varepsilon)U_1)\right) - EX(\infty)\right]^2 \right\} \\
&\quad \left. - \alpha^2 \left[\int_{K_\varepsilon}^{xt} \left\{ EX(u) - EX(\infty) \right\}\, du \right]^2 \right] \\
&=\; \frac{1}{\alpha t E X^2(\infty)} \\
&\quad \times \left[\sum_{k=0}^{\infty} P(N(xt-K_\varepsilon)=k) kE\left(E\left(X(K_\varepsilon + (xt-K_\varepsilon)U_1) \mid U_1\right) - EX(\infty)\right)^2 \right], \qquad (3.7)
\end{aligned}
$$

where the U_i are i.i.d., uniformly distributed on $(0,1)$ and stochastically independent of X_i. For the last equality we have used that

$$\sum_{k=0}^{\infty} P(N(xt-K_\varepsilon)=k)k(k-1)=\alpha^2\left(xt-K_\varepsilon\right)^2$$

and

$$E\left(X(K_\varepsilon+(xt-K_\varepsilon)U_1)\right)=\frac{1}{xt-K_\varepsilon}\int_{K_\varepsilon}^{xt} EX(u)\,du.$$

Putting (3.7) and (3.6) together we arrive at

$$E\widetilde{J}_1^2\leq\frac{1}{\alpha tEX^2(\infty)}\sum_{k=0}^{\infty}P(N(xt-K_\varepsilon)=k)k\frac{EX^2(\infty)\varepsilon}{2x}=\frac{\alpha(xt-K_\varepsilon)}{\alpha t}\frac{\varepsilon}{2x}\leq\frac{\varepsilon}{2}.$$

Now, as $E\widetilde{J}_2^2=O(1/t)$, $t\to\infty$ (for fixed K_ε), for t big enough it is smaller than $\varepsilon/2$, thus $(1+o(1))\,EI_2^2=E\widetilde{Y}_1^2+E\widetilde{Y}_2^2\leq\varepsilon$. Therefore $I_2(x)\xrightarrow{d}0$, $t\to\infty$. □

Remark 2.3.3. In comparison to the proof of Proposition 3.4 in Klüppelberg and Mikosch (KM95b) we make in (3.3) a sharper estimation of the difference

$$\frac{S(xt)-\mu(xt)}{\sigma(t)}-\frac{\sum_{i=1}^{N(xt)}X_i(\infty)-\alpha xtEX(\infty)}{\sqrt{\mathrm{var}X(\infty)t}}.$$

By this we need no additional restrictions concerning the rate with which $X(u)$ converges to $X(\infty)$, as $u\to\infty$. But, this comes at the price that our arguments only work for fixed $x>0$. Thus we have only proved convergence of the finite-dimensional distributions. To obtain weak convergence in a functional sense we additionally need that $\left(S_{\cdot}(t)|_{[0,M]}\right)_{t\in\mathbb{R}_+}$ is tight for each $M\in\mathbb{R}_+$. In Theorem 2.3.4 we give some extra conditions that ensure this.

Since the limit process has continuous sample paths we can equivalently consider weak convergence with respect to the Skorohod J_1-metric on $D[0,\infty)$, see e.g. (16.4) in Billingsley (Bil99). By Billingsley (Bil99), Theorem 16.7 and Theorem 13.1, we have to show weak convergence of the finite-dimensional distributions and tightness of $\left(S_{\cdot}(t)|_{[0,M]}\right)_{t\in\mathbb{R}_+}$ for each $M\in\mathbb{R}_+$.

In the following, we already start with shots having expectations equal to zero, i.e. $EX_i(u)=0$, for all $u\in\mathbb{R}$, $i\in\mathbb{N}$. Thus I_2 defined in (3.3) vanishes.

Theorem 2.3.4. *Let the assumptions of Proposition 2.3.1 be satisfied. If in addition the total variation of X on $(0,\infty)$ has finite fourth moment and there exists a $\delta > 0$ s.t.*

$$E\left(\sup_{s\in\mathbb{R}}\left(\widetilde{X}(s+h)-\widetilde{X}(s)\right)^4\right)=\mathcal{O}\left(h^\delta\right), \quad h\to 0, \tag{3.8}$$

then $S_{\cdot}(t)$ converges weakly to B, as $t\to\infty$, where the convergence holds in $D[0,\infty)$ equipped with the metric of uniform convergence on compacta and the projection σ-algebra.

Remark 2.3.5. Condition (3.8) is satisfied if $\widetilde{X}$ is Hölder continuous with exponent $\delta > 0$, not necessarily uniformly in $\omega\in\Omega$, but the bound must possess a fourth moment.

Remark 2.3.6. As the limit has continuous paths it is equivalent to convergence with respect to the Skorohod J_1-topology.

Proof. In consideration of Proposition 2.3.1, it remains to show that I_1 in (3.3) is tight. For this it is sufficient to verify tightness of

$$\widetilde{I}_1(x) := \frac{1}{\sqrt{t}}\sum_{i=1}^{N(xt)}\widetilde{X}_i(xt-T_i),$$

as for $t^{-1/2}\sum_{i=1}^{N(xt)}\widetilde{X}_i(\infty)$ tightness is obvious due to independent increments.

We apply an appropriate condition on the moments of the increments of $S_{\cdot}(t)$, cf. Billingsley

(Bil99) (13.14) with $\beta = 1$. Let $0 \le x_1 \le x_2 \le x_3 \le M$. We have

$$
\begin{aligned}
& E\left(\widetilde{I}_1(x_3) - \widetilde{I}_1(x_2)\right)^2 \left(\widetilde{I}_1(x_2) - \widetilde{I}_1(x_1)\right)^2 \\
&= \frac{1}{t^2} E\left(\sum_{i=1}^{N(x_3 t)} \widetilde{X}_i(x_3 t - T_i) - \sum_{i=1}^{N(x_2 t)} \widetilde{X}_i(x_2 t - T_i)\right)^2 \\
&\times \left(\sum_{i=1}^{N(x_2 t)} \widetilde{X}_i(x_2 t - T_i) - \sum_{i=1}^{N(x_1 t)} \widetilde{X}_i(x_1 t - T_i)\right)^2 \\
&= \frac{1}{t^2} E\left[\sum_{i_1,i_2,j_1,j_2=1}^{N(x_3 t)} \left(\widetilde{X}_{i_1}(x_3 t - T_{i_1}) - \widetilde{X}_{i_1}(x_2 t - T_{i_1})\right)\left(\widetilde{X}_{i_2}(x_3 t - T_{i_2}) - \widetilde{X}_{i_2}(x_2 t - T_{i_2})\right)\right. \\
&\left. \times \left(\widetilde{X}_{j_1}(x_2 t - T_{j_1}) - \widetilde{X}_{j_1}(x_1 t - T_{j_1})\right)\left(\widetilde{X}_{j_2}(x_2 t - T_{j_2}) - \widetilde{X}_{j_2}(x_1 t - T_{j_2})\right)\right] \\
&= \frac{1}{t^2} E\left[\sum_{i \ne j}^{N(x_3 t)} \left(\widetilde{X}_i(x_3 t - T_i) - \widetilde{X}_i(x_2 t - T_i)\right)^2 \left(\widetilde{X}_j(x_2 t - T_j) - \widetilde{X}_j(x_1 t - T_j)\right)^2\right] \\
&+ \frac{2}{t^2} E\left[\sum_{i_1 \ne i_2}^{N(x_3 t)} \left(\widetilde{X}_{i_1}(x_3 t - T_{i_1}) - \widetilde{X}_{i_1}(x_2 t - T_{i_1})\right)\left(\widetilde{X}_{i_1}(x_2 t - T_{i_1}) - \widetilde{X}_{i_1}(x_1 t - T_{i_1})\right)\right. \\
&\left. \times \left(\widetilde{X}_{i_2}(x_3 t - T_{i_2}) - \widetilde{X}_{i_2}(x_2 t - T_{i_2})\right)\left(\widetilde{X}_{i_2}(x_2 t - T_{i_2}) - \widetilde{X}_{i_2}(x_1 t - T_{i_2})\right)\right] \\
&+ \frac{1}{t^2} E\left[\sum_{i=1}^{N(x_3 t)} \left(\widetilde{X}_i(x_3 t - T_i) - \widetilde{X}_i(x_2 t - T_i)\right)^2 \left(\widetilde{X}_i(x_2 t - T_i) - \widetilde{X}_i(x_1 t - T_i)\right)^2\right] \\
&=: J_1 + J_2 + J_3
\end{aligned}
\tag{3.9}
$$

The second equality holds as the $\widetilde{X}_i$ are equal to zero on the negative half-line. The third equality is due to the fact that every addend containing a single index drops out as $E\widetilde{X}_i(u) = 0$ for all $u \in \mathbb{R}$. We get

$$
\begin{aligned}
& J_1 \\
&= \tfrac{1}{t^2} \textstyle\sum_{k=0}^{N(x_3 t)} P\left(N(x_3 t) = k\right) k(k-1) \\
&\quad \times E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1)\right)^2 E\left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1)\right)^2 \\
&= \alpha^2 x_3^2 E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1)\right)^2 E\left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1)\right)^2
\end{aligned}
\tag{3.10}
$$

where U_1 is uniformly distributed on $(0, x_3 t)$ and independent of $\widetilde{X}_1$.

Consequently there is a constant K s.t.

$$J_1 \le x_3^2 K. \tag{3.11}$$

But, for the case that x_3 is not so small but rather the difference $x_3 - x_1$ we need a tougher estimation: for $0 \le x_2 < x_3$ we have

$$\begin{aligned}
&E\left(\widetilde{X}_1(x_3t-U_1)-\widetilde{X}_1(x_2t-U_1)\right)^2\\
&=\frac{1}{\left[\frac{x_3}{x_3-x_2}\right]}\sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} E\left[\left(\widetilde{X}_1(x_3t-U_1)-\widetilde{X}_1(x_2t-U_1)\right)^2\middle|U_1\in\left((i-1)\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]},i\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right]\\
&=\frac{1}{\left[\frac{x_3}{x_3-x_2}\right]}\sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} E\left[\left(\widetilde{X}_1(U_1)-\widetilde{X}_1(U_1-(x_3-x_2)t)\right)^2\middle|U_1\in\left((i-1)\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]},i\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right]\\
&\le\frac{1}{\left[\frac{x_3}{x_3-x_2}\right]}\sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} EV\left(\widetilde{X}_1;\left((i-2)\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]},i\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right)^2,
\end{aligned}$$

where $V(f;I)$ denotes the total variation of the function f on the interval I. By using the fact that $V(f;\sum_k I_k)=\sum_k V(f;I_k)$ for disjunctive intervals (I_k), we go on with

$$\begin{aligned}
&E\left(\widetilde{X}_1(x_3t-U_1)-\widetilde{X}_1(x_2t-U_1)\right)^2\\
&\le\frac{1}{\left[\frac{x_3}{x_3-x_2}\right]}\left\{\sum_{i=1\ 2\nmid i}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} EV\left(\widetilde{X}_1;\left((i-2)\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]},i\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right)^2\right.\\
&\left.+\sum_{i=2\ 2\mid i}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} EV\left(\widetilde{X}_1;\left((i-2)\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]},i\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right)^2\right\}\\
&\le\frac{2}{\left[\frac{x_3}{x_3-x_2}\right]}EV\left(\widetilde{X}_1;(0,x_3t]\right)^2\\
&\le\frac{2}{\left[\frac{x_3}{x_3-x_2}\right]}EV\left(\widetilde{X}_1;(0,\infty)\right)^2\\
&\le\frac{2(x_3-x_2)}{x_2}EV\left(\widetilde{X}_1;(0,\infty)\right)^2. \qquad (3.12)
\end{aligned}$$

In the same way, we get

$$\begin{aligned}
&E\left(\widetilde{X}_1(x_2t-U_1)-\widetilde{X}_1(x_1t-U_1)\right)^2\\
&\le\frac{2(x_2-x_1)}{x_1}EV\left(\widetilde{X}_1;(0,\infty)\right)^2,
\end{aligned}$$

and thus there is another constant $\widetilde{K} \in \mathbb{R}$ s.t.

$$J_1 \leq x_3^2 \frac{(x_3 - x_1)^2}{x_1^2} \widetilde{K}. \tag{3.13}$$

Putting (3.11) and (3.13) together implies that for all $0 \leq x_1 \leq x_3 \leq M$

$$J_1 \leq \min \left\{ x_3^2 K, x_3^2 \frac{(x_3 - x_1)^2}{x_1^2} \widetilde{K} \right\} \leq 4 \max\{K, \widetilde{K}\} \left(x_3 - x_1\right)^2, \tag{3.14}$$

where the second inequality in (3.14) can easily be verified by a case differentiation $x_1 \leq x_3/2$ and $x_1 > x_3/2$.

J_2 can be treated in an analogue way, as in each addent enter two different (and therefore independent) shots. As an analogue to (3.10) we obtain

$$\begin{aligned} & J_2 \\ & = \frac{2}{t^2} \sum_{k=0}^{N(x_3 t)} P\left(N(x_3 t) = k\right) k(k-1) \\ & \quad \times \left[E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1) \right) \left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1) \right) \right]^2 \\ & = 2\alpha^2 x_3^2 \left[E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1) \right) \left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1) \right) \right]^2 \\ & = 2\alpha^2 x_3^2 E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1) \right)^2 E\left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1) \right)^2 \end{aligned} \tag{3.15}$$

where U_1 is again uniformly distributed on $(0, x_3 t)$ and independent of $\widetilde{X}_1$. Thus by the arguments leading to (3.14) we arrive at

$$J_2 \leq K'(x_3 - x_1)^2. \tag{3.16}$$

By contrast, for J_3 we need the additional assumption (3.8) as in each addent only a single shot enters and we have in general no product of independent random variables. The argumentation is quite similar but although not the same as for J_1 and J_2. We obtain

$$\begin{aligned} & J_3 \\ & = \frac{1}{t^2} \sum_{k=0}^{N(x_3 t)} P\left(N(x_3 t) = k\right) k \\ & \quad \times E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1) \right)^2 \left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1) \right)^2 \\ & = \frac{\alpha x_3}{t} E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1) \right)^2 \left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1) \right)^2 \\ & \leq \frac{\alpha x_3}{t} \sqrt{E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1) \right)^4 E\left(\widetilde{X}_1(x_2 t - U_1) - \widetilde{X}_1(x_1 t - U_1) \right)^4}, \end{aligned} \tag{3.17}$$

where U_1 is uniformly distributed on $(0, x_3 t)$ and independent of $\widetilde{X}_1$. We have

$$\begin{aligned}
&E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1)\right)^4 \\
&\leq E\left(\sup_{s\in\mathbb{R}}\left(\widetilde{X}_1(s + (x_3 - x_2)t) - \widetilde{X}_1(s)\right)^4\right) \\
&\leq C((x_3 - x_2)t)^\delta,
\end{aligned}$$

where the last inequality holds for a $C \in \mathbb{R}_+$, due to (3.8) and the finiteness of $EV\left(\widetilde{X}_1; (0,\infty)\right)^4$. Thus, we obtain for t big enough (as w.l.o.g. $\delta < 1$)

$$J_3 \leq \frac{\alpha x_3}{t} C(x_3 t)^\delta \leq x_3^{1+\delta}. \tag{3.18}$$

But again, for the case that x_3 is not so small but rather the difference $x_3 - x_1$ we need a tougher estimation. For $0 \leq x_2 < x_3$ we have

$$\begin{aligned}
&E\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1)\right)^4 \\
&= \frac{1}{\left[\frac{x_3}{x_3 - x_2}\right]} \sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} E\left[\left(\widetilde{X}_1(x_3 t - U_1) - \widetilde{X}_1(x_2 t - U_1)\right)^4 \middle| U_1 \in \left((i-1)\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}, i\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right] \\
&= \frac{1}{\left[\frac{x_3}{x_3 - x_2}\right]} \sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} E\left[\left(\widetilde{X}_1(U_1) - \widetilde{X}_1(U_1 - (x_3 - x_2)t)\right)^4 \middle| U_1 \in \left((i-1)\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}, i\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right] \\
&\leq \frac{1}{\left[\frac{x_3}{x_3 - x_2}\right]} \sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} E\left[\left(\widetilde{X}_1(U_1) - \widetilde{X}_1(U_1 - (x_3 - x_2)t)\right)^2\right. \\
&\left.\times \sup_{s\in\mathbb{R}}\left(\widetilde{X}_1(s + (x_3 - x_2)t) - \widetilde{X}_1(s)\right)^2 \middle| U_1 \in \left((i-1)\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}, i\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right] \\
&\leq \frac{1}{\left[\frac{x_3}{x_3 - x_2}\right]} \sum_{i=1}^{\left[\frac{1}{\frac{x_3}{x_3-x_2}}\right]} E\left[V\left(\widetilde{X}_1; \left((i-2)\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}, i\frac{x_3 t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right)^2\right. \\
&\left.\times \sup_{s\in\mathbb{R}}\left(\widetilde{X}_1(s + (x_3 - x_2)t) - \widetilde{X}_1(s)\right)^2\right],
\end{aligned}$$

where $V(f; I)$ is the total variation of the function f on the interval I. We go on by applying

Cauchy-Schwarz's inequality

$$
\begin{aligned}
&E\left(\widetilde{X}_1(x_3t-U_1)-\widetilde{X}_1(x_2t-U_1)\right)^4\\
&\le \frac{1}{\left[\frac{x_3}{x_3-x_2}\right]}\left\{E\left(\sup_{s\in\mathbb{R}}\left(\widetilde{X}_1(s+(x_3-x_2)t)-\widetilde{X}_1(s)\right)^4\right)\right\}^{1/2}\\
&\times\left\{E\left[\sum_{i=1}^{\frac{1}{\left[\frac{x_3}{x_3-x_2}\right]}} V\left(\widetilde{X}_1;\left((i-2)\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]},i\frac{x_3t}{\left[\frac{x_3}{x_3-x_2}\right]}\right]\right)^2\right]^2\right\}^{1/2}\\
&\le \frac{2}{\left[\frac{x_3}{x_3-x_2}\right]}\sqrt{E\left(\sup_{s\in\mathbb{R}}\left(\widetilde{X}_1(s+(x_3-x_2)t)-\widetilde{X}_1(s)\right)^4\right)EV\left(\widetilde{X}_1;(0,x_3t]\right)^4}\\
&\le \frac{2}{\left[\frac{x_3}{x_3-x_2}\right]}\sqrt{C((x_3-x_2)t)^\delta EV\left(\widetilde{X}_1;(0,\infty)\right)^4}\\
&\le \frac{2\sqrt{C}(x_3-x_2)^{1+\delta/2}t^{\delta/2}}{x_2}\sqrt{EV\left(\widetilde{X}_1;(0,\infty)\right)^4}
\end{aligned}
\tag{3.19}
$$

The second inequality in (3.19) is by the same argument leading to (3.12). In the same manner, we get

$$
\begin{aligned}
&E\left(\widetilde{X}_1(x_2t-U_1)-\widetilde{X}_1(x_1t-U_1)\right)^4\\
&\le \frac{2\sqrt{C}(x_2-x_1)^{1+\delta/2}t^{\delta/2}}{x_1}\sqrt{EV\left(\widetilde{X}_1;(0,\infty)\right)^4}.
\end{aligned}
\tag{3.20}
$$

Putting (3.17), (3.19), and (3.20) together, we obtain that there is a constant $\widetilde{K}\in\mathbb{R}$ s.t.

$$
J_3 \le \frac{x_3}{t}\frac{(x_3-x_1)^{1+\delta/2}t^{\delta/2}}{x_1}\widetilde{K}\le x_3\frac{(x_3-x_1)^{1+\delta/2}}{x_1},
\tag{3.21}
$$

for t big enough. (3.18), (3.21) imply that for all $0\le x_1\le x_3\le 1$

$$
J_3\le \min\left\{x_3^{1+\delta}, x_3\frac{(x_3-x_1)^{1+\delta/2}}{x_1}\right\}\le 2^{1+\delta/2}\left(x_3-x_1\right)^{1+\delta/2},
\tag{3.22}
$$

where the second inequality in (3.14) can easily be verified by a case differentiation $x_1\le x_3/2$ and $x_1>x_3/2$.

Putting (3.14), (3.16), and (3.22) together yields in view of (3.9)

$$
E\left(\widetilde{I}_1(x_3)-\widetilde{I}_1(x_2)\right)^2\left(\widetilde{I}_1(x_2)-\widetilde{I}_1(x_1)\right)^2\le \bar{K}(x_3-x_1)^{1+\delta/2}
$$

for t big enough and therfore the assertion. □

Chapter 3

Neutral Pricing of American and Game Type Derivatives

This chapter is an adapted version of Kallsen and Kühn (KK02a).

3.1 Introduction

In recent years various suggestions have been made how to price European-type contingent claims in incomplete markets. By contrast, there is only little corresponding literature dealing with American options. Pricing the latter is conceptually more involved: In addition to the uncertainty caused by the underlyings, one has to take the seller's ignorance of the buyer's exercise strategy into account. If we fix a stopping time as exercise time, then the American option reduces to a European claim. It is obvious that the American option should be worth at least as much as the most valuable of these implied European claims. In the financial literature the price of an American option is often just *defined* as the supremum of all European style claims corresponding to arbitrary stopping times of the buyer. Consequently, the problem of pricing American options is reduced to the simpler problem of pricing European contingent claims. However, this concept already implies by definition that an American option is not worth more than the highest priced of its implied European-style derivatives, i.e. the *right* to choose the exercise time has no value in itself. To us, this is not entirely obvious because in the American case the seller faces the disadvantage not to know the preferred stopping time of the buyer.

In complete markets, arbitrage arguments suffice to derive unique prices for American contingent claims. Here, it turns out that the fair price is indeed the supremum of the implied European option values (cf. Bensoussan (Ben84) and Karatzas (Kar88)). Analogous results are shown in varying degrees of generality for the superhedging price in incomplete markets (cf. Karatzas and Kou (KK98), Kramkov (Kra96), Föllmer and Kabanov (FK98), and Föllmer and Kramkov (FK97)). This price denotes the smallest initial capital that allows to construct a portfolio which dominates the payoff process of the option. Although superhedging is an interesting concept from a theoretical point of view, it yields only trivial upper bounds in many models of practical importance (cf. e.g. Eberlein and Jacod (EJ97), Frey and Sin (FS99), Cvitanic et al. (CPT99)). This is somewhat unsatisfactory.

Utility-based indifference pricing is a concept which has been applied explicitly to American options. Here, one takes the perspective of a particular counterparty and fixes the number of shares of the claim (say, 1 for an option buyer or -1 for an option seller). The *indifference premium* is a price such that the optimal expected utility among all portfolios containing the prespecified number of options coincides with the optimal expected utility among all portfolios without option. Put differently, the investor is indifferent to including the option into the portfolio. Taking the perspective of the option buyer, it turns out that the indifference price is indeed the supremum of the indifference prices of the implied European claims (cf. Davis and Zariphopoulou (DZ95)). Surprisingly, this is not true for the option seller: Unless exponential utility is chosen, it may happen that a reasonable indifference premium for an American option exceeds the indifference price of all implied European claims (cf. Kühn (Küh02) and Proposition 4.2.11 in this thesis, resp.).

In this chapter we show that the concept of *neutral derivative pricing*, as suggested in Kallsen (Kal01), can be adapted quite naturally to American options. Neutral prices occur if traders maximize their expected utility and if derivative supply and demand are balanced. More precisely, a derivative price process is called *neutral* if the optimal portfolio contains no contingent claim. We will see that the neutral price of an American option coincides as in the complete case with the supremum of the neutral prices of all implied European claims.

Both utility-based indifference pricing and neutral pricing rely on expected utility maximization and indifference to trading the option. Let us point out the differences between the two concepts. Indifference pricing takes an asymmetric point of view. Moreover, it depends decisively on the fixed number of claims under consideration. As far as options are concerned, intermediate trades are not allowed. Therfore, this approach is particularly well suited for over-the-counter trades: Suppose that the buyer wants to purchase a specific contingent claim. Then he has to pay the seller at least her indifference price in order to prompt her to enter the contract.

The concept of neutral pricing, on the other hand, takes a symmetric point of view. It assumes that options are traded in arbitrary positive and negative amounts. It tries to mimic the economic reasoning in complete markets by substituting utility maximizers for arbitrage traders. Neutral prices are the unique prices such that neither buyer nor seller takes advantage from trading the claim. For motivation of neutral derivative pricing, references, and connections to other approaches in the literature we refer the reader to Kallsen (Kal01).

As mentioned above, neutral pricing relies on utility maximization for portfolios containing derivatives. This is a non-trivial issue in the presence of American-type contingent claims. The point is that short positions in the claim may suddenly be terminated if the buyer exercises the option. Therefore, investment in American claims corresponds to investment under specific short-selling constraints (cf. Section 3.3).

In the present chapter, American options are treated as special cases of game contingent claims. The latter naturally generalize American contingent claims by giving both counterparties the right to cancel the contract prematurely. This generalization requires some mathematical but no additional conceptual efforts. By contrast, it makes the neutral pricing approach even more transparent.

A *game contingent claim (GCC)*, as introduced in Kifer (Kif00), is a contract between a seller A and a buyer B which can be terminated by A and exercised by B at any time $t \in [0,T]$ up to a maturity date T when the contract is terminated anyway. More precisely, the contract may be specified in terms of stochastic processes $(L_t)_{t\in[0,T]}$, $(U_t)_{t\in[0,T]}$ with $L_t \leq U_t$ for $t \in [0,T]$ and $L_T = U_T$. If A terminates the contract at time t before it is exercised by B, she has to pay B the amount U_t. If B exercises the option before it is terminated by A, he is paid L_t. For motivation and examples for this kind of derivatives we refer the reader to (Kif00).

With American options the right to terminate the contract is restricted to the buyer B. Formally, they can be interpreted as game contingent claims by setting $U_t := m$ for $t \in [0,T)$, where $m \in \mathbb{R} \cup \{\infty\}$ exceeds the maximal payoff of the American option, e.g. $m = \infty$ in the unbounded case. This allows us to consider both kinds of options in a common framework.

Similarly as American options correspond to optimal stopping problems, GCC's incorporate a Dynkin game: If seller A selects stopping time τ^U as cancellation time and buyer B chooses stopping time τ^L as exercise time, then A pledges to pay B at time $\tau^L \wedge \tau^U$ the amount

$$R(\tau^L, \tau^U) = L_{\tau^L} 1_{\{\tau^L \leq \tau^U\}} + U_{\tau^U} 1_{\{\tau^U < \tau^L\}}.$$

In complete markets with a unique equivalent martingale measure $P^\star$, the random payoff $R(\tau^L, \tau^U)$ has the unique fair value $E_{P^\star}(R(\tau^L, \tau^U))$ at time 0. In analogy to American options, the buyer may want to choose his stopping time so as to maximize $E_{P^\star}(R(\tau^L, \tau^U))$ whereas the seller tries to minimize the same value. This is precisely the situation of a zero-sum *Dynkin stopping game*. It is well-known that such a game has a unique value in the sense that

$$\inf\nolimits_{\tau^U} \sup\nolimits_{\tau^L} E_{P^\star}(R(\tau^L, \tau^U)) = \sup\nolimits_{\tau^L} \inf\nolimits_{\tau^U} E_{P^\star}(R(\tau^L, \tau^U)) \tag{1.1}$$

(cf. Lepeltier and Maingueneau (LM84)). Kifer (Kif00) shows by hedging arguments that this value is in fact the unique no-arbitrage price of the GCC.

In incomplete markets these arguments fail because perfect replication is usually impossible. But it turns out that the price process of a GCC corresponds again to the value of a Dynkin game if we apply the neutral pricing approach. The unique equivalent martingale measure in Equation (1.1) is replaced with a properly chosen *neutral pricing measure*.

The chapter is organized as follows. Section 3.2 summarizes and states some facts on utility maximization. These are needed in the subsequent section to address the derivative pricing problem for game contingent claims. The appendix contains some auxiliary results from stochastic calculus.

Throughout, we use the notation of Jacod and Shiryaev (JS87) (henceforth JS) and Jacod (Jac79), (Jac80). The components of a vector x are denoted by superscripts. Increasing processes are identified with their corresponding Lebesgue-Stieltjes measure. Stochastic integrals are written in dot notation, i.e. $\varphi \cdot S_t$ means $\int_0^t \varphi_s dS_s$.

3.2 Utility maximization

The derivative pricing approach in Section 3.3 relies on assumptions concerning investors who maximize their expected utility. Therefore, we discuss two kinds of portfolio optimization problems in this section, based on the classical *utility of terminal wealth* and on *local utility* as in Kallsen (Kal99), respectively.

Our mathematical framework for a frictionless market model is as follows: Fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, P)$ in the sense of JS, I.1.2. In this section we consider traded *securities* $1, \ldots, d$ whose price processes are expressed in terms of multiples of a *numeraire* security 0. Put differently, these securities are modelled by their discounted *price process* $S := (S^1, \ldots, S^d)$. We assume that S is a $\mathbb{R}^d$-valued semimartingale.

3.2.1 Utility of terminal wealth

In this subsection we consider an investor who tries to maximize utility from terminal wealth. Her *initial endowment* is denoted by $\varepsilon \in (0, \infty)$. *Trading strategies* are modelled by $\mathbb{R}^d$-valued, predictable stochastic processes $\varphi = (\varphi^1, \ldots, \varphi^d) \in L(S)$, where φ^i_t denotes the number of shares of security i in the investor's portfolio at time t. A strategy φ belongs to the set $\mathfrak{S}$ of all *admissible* strategies if its discounted *wealth process* $V(\varphi) := \varepsilon + \varphi \cdot S$ is nonnegative (no debts allowed).

Trading constraints are expressed in terms of subsets of the set of all trading strategies. More specifically, we consider a process Γ whose values are convex cones in $\mathbb{R}^d$. The *constrained set* of trading strategies $\mathfrak{S}(\Gamma)$ is the subset of admissible strategies φ which satisfy $(\varphi^1, \ldots, \varphi^d)_t \in \Gamma_t$ pointwise on $\Omega \times [0, T]$. Important examples are $\Gamma := \mathbb{R}^d$ (no constraints) and $\Gamma := (\mathbb{R}_+)^d$ (no short sales).

The investor's preferences are modelled by a strictly concave *utility* function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ which is continuously differentiable on $(0, \infty)$ and satisfies $\lim_{x\to 0} u'(x) = \infty$, $\lim_{x\to\infty} u'(x) = 0$, and $\limsup_{x\to\infty} xu'(x)/u(x) < 1$ (i.e. it is of reasonable asymptotic elasticity in the sense of Kramkov and Schachermayer (KS99), Definition 2.2). Her aim is to make the best out of her money in the following sense:

Definition 3.2.1. *We say that* $\varphi \in \mathfrak{S}(\Gamma)$ *is an* optimal strategy for terminal wealth under the constraints Γ *if it maximizes* $\widetilde{\varphi} \mapsto E(u(V_T(\widetilde{\varphi})))$ *over all* $\widetilde{\varphi} \in \mathfrak{S}(\Gamma)$. *(By convention, we set* $E(u(V_T(\widetilde{\varphi}))) := -\infty$ *if* $E(-u(V_T(\widetilde{\varphi})) \vee 0) = \infty$.*)*

Optimal portfolios are characterized by the following result. Many references to related statements in the literature can be found in Kallsen (Kal01), Section 2.2 and Schachermayer (Sch01a).

Lemma 3.2.2. *Let* $\varphi \in \mathfrak{S}(\Gamma)$ *with finite expected utility. Then we have equivalence between:*

(i) φ *is optimal for terminal wealth under the constraints* Γ.

(ii) $u'(V_T(\varphi))((\psi-\varphi)\cdot S_T)$ *is integrable and has non-positive expectation for any* $\psi \in \mathfrak{S}(\Gamma)$ *with* $E(u(V_T(\psi))) > -\infty$.

PROOF. 2⇒1: Let $\psi \in \mathfrak{S}(\Gamma)$ with $E(u(V_T(\psi))) > -\infty$. Since u is concave, we have

$$\begin{aligned} E(u(\varepsilon + \psi \cdot S_T)) &\leq E(u(\varepsilon + \varphi \cdot S_T)) + E(u'(\varepsilon + \varphi \cdot S_T)((\psi - \varphi) \cdot S_T)) \\ &\leq E(u(\varepsilon + \varphi \cdot S_T)), \end{aligned}$$

which yields the assertion.

1⇒2: Let $\psi \in \mathfrak{S}(\Gamma)$ with $E(u(V_T(\psi))) > -\infty$. Define $\widetilde{\psi} := \varphi + \frac{1}{2}(\psi - \varphi)$ and $\psi^{(\lambda)} := \varphi + \lambda(\psi - \varphi)$ for $\lambda \in [0,1]$. Since $\mathfrak{S}(\Gamma)$ is convex and u is concave, we have that $\widetilde{\psi} \in \mathfrak{S}(\Gamma)$ and $E(u(V_T(\widetilde{\psi}))) > -\infty$. From

$$\begin{aligned} -\infty &< E(u(\varepsilon + \psi \cdot S_T)) \\ &\leq E(u(\varepsilon + \varphi \cdot S_T)) + E(u'(\varepsilon + \varphi \cdot S_T)((\psi - \varphi) \cdot S_T)) \end{aligned}$$

and $E(u(V_T(\varphi))) < \infty$ it follows that $E((u'(\varepsilon + \varphi \cdot S_T)((\psi - \varphi) \cdot S_T))^-) < \infty$. Similarly,

$$\begin{aligned} -\infty &< E(u(\varepsilon + \psi \cdot S_T)) \\ &\leq E(u(\varepsilon + \widetilde{\psi} \cdot S_T)) + \frac{1}{2} E(u'(\varepsilon + \widetilde{\psi} \cdot S_T)((\psi - \varphi) \cdot S_T)) \end{aligned}$$

implies that $E((u'(\varepsilon + \widetilde{\psi} \cdot S_T)((\psi - \varphi) \cdot S_T))^-) < \infty$.

Let $\lambda \in (0, \frac{1}{2}]$. By optimality of φ, we have

$$0 \geq E(u(\varepsilon + \psi^{(\lambda)} \cdot S_T)) - E(u(\varepsilon + \varphi \cdot S_T)),$$

which equals $\lambda E(\xi^{(\lambda)}((\psi - \varphi) \cdot S_T))$ for some random variable $\xi^{(\lambda)}$ with values in $[u'(\varepsilon + \varphi \cdot S_T), u'(\varepsilon + \widetilde{\psi} \cdot S_T)]$ or $[u'(\varepsilon + \widetilde{\psi} \cdot S_T), u'(\varepsilon + \varphi \cdot S_T)]$, respectively. Note that

$$\begin{aligned}
&(\xi^{(\lambda)}((\psi - \varphi) \cdot S_T))^- \\
&\leq (u'(\varepsilon + \varphi \cdot S_T)((\psi - \varphi) \cdot S_T))^- + (u'(\varepsilon + \widetilde{\psi} \cdot S_T)((\psi - \varphi) \cdot S_T))^-,
\end{aligned}$$

where the latter sum is in $L^1(P)$.

Since $\psi^{(\lambda),} \cdot S_T \to \varphi \cdot S_T$, we have that $\xi^{(\lambda)} \to u'(\varepsilon + \varphi \cdot S_T)$ a.s. for $\lambda \to 0$. Fatou's lemma yields $E(u'(\varepsilon + \varphi \cdot S_T)((\psi - \varphi) \cdot S_T)) \leq \liminf_{\lambda \to 0} E(\xi^{(\lambda)}((\psi - \varphi) \cdot S_T))$. It follows that $E(u'(\varepsilon + \varphi \cdot S_T)((\psi - \varphi) \cdot S_T)) \leq 0$ as claimed. $\square$

Suppose that φ is an optimal strategy for terminal wealth without constraints (i.e. for $\Gamma = \mathbb{R}^d$). If the probability space is finite, then

$$\frac{u'(V_T(\varphi))}{E(u'(V_T(\varphi)))}$$

is the density of some equivalent martingale measure (EMM) $P^\star$ (cf. (Kal01), Corollary 2.7). In addition, this measure solves some dual minimization problem (cf. Schachermayer (Sch01a), Theorem 2.3). In general markets, the density process of $P^\star$ is replaced with a supermartingale which may not be the density process of a probability measure, let alone an EMM (cf. Kramkov and Schachermayer (KS99), Section 5). Nevertheless, in many models of practical importance the *dual measure* $P^\star$ exists and it is at least a *σ-martingale measure*, i.e. $S^1, \ldots, S^d$ are σ-martingales relative to $P^\star$. Since it plays a key role in the neutral pricing approach, we call $P^\star$ *neutral pricing measure for terminal wealth*.

Definition 3.2.3. *Suppose that φ is an optimal strategy for terminal wealth without constraints (i.e. for $\Gamma = \mathbb{R}^d$) and, moreover, has finite expected utility. If $u'(V_T(\varphi))/E(u'(V_T(\varphi)))$ is the density of some σ-martingale measure $P^\star$, we call $P^\star$* dual measure *or* neutral pricing measure for terminal wealth.

In some cases the neutral pricing measure for terminal wealth can be computed explicitly:

Example 3.2.4. *Suppose that $S^1, \ldots, S^d$ are positive processes of the form $S^i = S^i_0 \mathscr{E}(L^i)$ for $i = 1, \ldots, d$, where L is a $\mathbb{R}^d$-valued Lévy process with characteristic triplet (b, c, F) relative* *to some truncation function* $h : \mathbb{R}^d \to \mathbb{R}^d$ *(i.e. a PIIS in the sense of JS, II.4.1). In the last couple of years, processes of this type have become popular for securities models, since they are mathematically tractable and provide a good fit to real data (cf. Eberlein and Keller (EK95), Eberlein et al. (EKP98), Madan and Seneta (MS90), Barndorff-Nielsen (BN98)). Suppose that* u *is of power or logarithmic type, i.e.* $u(x) = x^{1-p}/(1-p)$ *for some* $p \in \mathbb{R}_+ \setminus \{0,1\}$ *or* $u(x) = \log x$*, which corresponds to the case* $p = 1$*. Assume that there exists some* $\gamma \in \mathbb{R}^d$ *such that*

$$F(\{x \in \mathbb{R}^d : 1 + \gamma x \leq 0\}) = 0,$$

$$\int \left|\frac{x}{(1+\gamma x)^p} - h(x)\right| F(dx) < \infty,$$

and

$$b - pc\gamma + \int \left(\frac{x}{(1+\gamma x)^p} - h(x)\right) F(dx) = 0.$$

Let $Z := \mathscr{E}(-p\gamma L^c + ((1+\gamma x)^{-p} - 1) * (\mu^L - \nu^L))$*, where* L^c *denotes the continuous martingale part of* L *and* μ^L, ν^L *the random measure of jumps of* L *and its compensator. In the proof of Kallsen (Kal00), Theorem 3.2 it is shown that* Z *is the density process of the dual measure* $P^\star$*, which is even an equivalent martingale measure in this case. Relative to* $P^\star$*,* L *is a Lévy process with characteristic triplet* $(b^\star, c, F^\star)$*, where* $(dF^\star/dF)(x) = (1+\gamma x)^{-p}$ *and* $b^\star = -\int (x - h(x)) F^\star(dx)$*.*

Example 3.2.5. *In the case of logarithmic utility* $u(x) = \log x$*, the neutral pricing measure for terminal wealth can be calculated explicitly for a large number of semimartingale models (cf. Goll and Kallsen (GK01), Section 6).*

3.2.2 Local utility

Secondly, we turn to portfolio optimization based on local utility. We assume that S is a $\mathbb{R}^d$-valued special semimartingale. Denote by (b, c, F, A) differential characteristics of S in the sense of Definition A2.1, but relative to the truncation function $h(x) = x$. This choice of truncation function is possible because S is special. It is typically straightforward to obtain the differential characteristics from other local descriptions of S e.g. in terms of stochastic differential equations or one-step transition densities in the discrete-time case.

In this subsection, the family of trading strategies under consideration is the set $\mathfrak{S}'$ of all predictable $\mathbb{R}^d$-valued processes $\varphi = (\varphi^1, \dots, \varphi^d)$ satisfying the integrability condition

$$E\left(\left(|\varphi b| + \varphi c \varphi + \int ((\varphi x)^2 \wedge |\varphi x|) F(dx)\right) \cdot A_T\right) < \infty.$$

Similarly to above, we denote by $\mathfrak{S}'(\Gamma)$ the set of all trading strategies in $\mathfrak{S}'$ meeting the cone constraints Γ. In order to avoid technical proofs, we assume that there exist polyhedral cones $K_1, \dots, K_n \subset \mathbb{R}^d$ and predictable sets $D_1, \dots, D_n$ such that $\Gamma_t(\omega) = \cap_{\{i\in\{1,\dots,n\}:(\omega,t)\in D_i\}} K_i$ for $(\omega, t) \in \Omega \times [0, T]$. The *utility function* $u : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following conditions: u is twice continuously differentiable, the derivatives u', u'' are bounded with $\lim_{x\to\infty} u'(x) = 0$, moreover $u(0) = 0$, $u'(0) = 1$, $u'(x) > 0$ and $u''(x) < 0$ for any $x \in \mathbb{R}$. For any $\psi \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ the random variable

$$\gamma_t(\psi) := \psi b_t + \frac{u''(0)}{2} \psi c_t \psi + \int (u(\psi x) - \psi x) F_t(dx)$$

is termed *local utility* of ψ in t.

Definition 3.2.6. *We call a strategy* $\varphi \in \mathfrak{S}'(\Gamma)$ locally optimal under the constraints Γ *if*

$$E(\gamma(\varphi) \cdot A_T) \geq E(\gamma(\psi) \cdot A_T)$$

for any $\psi \in \mathfrak{S}'(\Gamma)$.

For motivation of local optimality we refer the reader to Kallsen (Kal99). Intuitively, a locally optimal strategy maximizes the expected utility of the gains over infinitesimal time intervals, or put differently, the expected utility of consumption among all strategies whose financial gains are immediately consumed.

Locally optimal portfolios can be determined by pointwise solution of equations in $\mathbb{R}^d$:

Theorem 3.2.7. *A trading strategy* $\varphi \in \mathfrak{S}'(\Gamma)$ *is locally optimal under the constraints* Γ *if and only if*

$$b_t + u''(0) c_t \varphi_t + \int x(u'(\varphi_t x) - 1) F_t(dx) \in \Gamma_t^\circ \tag{2.2}$$

$P \otimes A$*-a.e., where* $\Gamma_t^\circ := \{y \in \mathbb{R}^d : xy \leq 0$ *for any* $x \in \Gamma_t\}$ *denotes the polar cone of* Γ_t*.*

Proof. In view of Farkas' lemma (cf. Rockafellar and Wets (RW98), Lemma 6.45), Theorem 3.2.7 follows from Kallsen (Kal99), Theorem 3.5. Strictly speaking, Kallsen (Kal99) considers a narrower set-up where A and Γ are deterministic. As is pointed out in Kallsen (Kal02), the statements in Kallsen (Kal99) remain valid for $A \in \mathscr{A}^+_{\text{loc}}$. Moreover, a careful inspection of the proofs of Proposition 3.10 and Theorem 3.5 in that paper reveals that these results hold for random constraints of the above type as well. $\square$

Neutral pricing of European contingent claims is discussed in Kallsen (Kal02) in the context of local utility. A key role is played by the corresponding neutral pricing measure, which is defined as follows:

Definition 3.2.8. *Suppose that there exists a locally optimal strategy $\varphi \in \mathfrak{S}'$ without constraints (i.e. for $\Gamma = \mathbb{R}^d$). Moreover, assume that the local martingale*

$$Z := \mathscr{E}\left(u''(0)\varphi \cdot S^c + \frac{u'(\varphi x) - 1}{1 + V} * (\mu^S - \nu^S)\right)$$

is a martingale, where μ^S, ν^S are the random measure of jumps of S and its compensator, $V_t := \int (u'(\varphi_t x) - 1)\nu^S(\{t\} \times dx)$ for $t \in [0, T]$, and S^c denotes the continuous local martingale part of S. Then the probability measure $P^\star \sim P$ defined by $dP^\star/dP = Z_T$ is called neutral pricing measure for local utility.

Since the determination of the optimal strategy φ reduces to solving Equation (2.2) with $\Gamma^\circ_t = \{0\}$, the neutral pricing measure for local utility is often easier to obtain than the neutral pricing measure for terminal wealth. For concrete examples cf. Kallsen (Kal02), Section 5.

3.3 Neutral pricing

In this section we turn to the valuation of game contingent claims. Let us briefly review the idea of neutral pricing. For references and connections to similar approaches in the literature we refer the reader to Kallsen (Kal01).

In complete models there exist unique arbitrage-free derivative values. The assertion that real market prices have to coincide with these values can be easily justified. It suffices to assume the existence of traders (from now on called *derivative speculators*) who exploit favourable market conditions once they detect them. The existence of derivative speculators explains why the market price cannot deviate too strongly from the right value: If it did, the huge demand for (resp. supply of) the mispriced security would push its price immediately closer to the rational value. The only assumption on the preferences of the speculators is that they do not reject riskless profits – which most people may agree on. The elegance of this approach comes at a price. It only works in complete models, or more exactly, for attainable claims.

We extend this reasoning to incomplete markets by imposing stronger assumptions on the preferences of derivative speculators. We suppose that they trade by maximizing a specific kind of utility. The role of the unique arbitrage-free price will now be played by the *neutral* derivative value. This is the unique price such that the speculators' optimal portfolio contains no contingent claim. Similarly as in the complete case we argue that the speculators' presence should prevent the market price from deviating too strongly from the neutral value.

The general setting is as in the previous section. We distinguish two kinds of securities: *underlyings* $1,\dots,m$ and *derivatives* $m+1,\dots,m+n$. We assume that the derivatives are game contingent claims with discounted *exercise process* L^i and discounted *cancellation process* U^i, where L^i and U^i are semimartingales with $L^i < U^i$ as well as $L^i_- < U^i_-$ on $[0,T[$ and $L^i_T = U^i_T$ for $i = m+1,\dots,m+n$. European and American options are treated as special cases of game contingent claims as it is explained in Remark 2 below. We call semimartingales $S^{m+1},\dots,S^{m+n}$ *derivative price processes* if $L^i \leq S^i \leq U^i$ for $i = m+1,\dots,m+n$. As noted above, we are interested in derivative price processes that have a neutral effect on the market in the sense that they do not cause supply of or demand for contingent claims by derivative speculators.

Speculators may not be able to hold arbitrary amounts of game contingent claims because these contracts can be cancelled. If the market price approaches the upper cancellation value U^i, it may happen that all options vanish from the market because they are terminated by the sellers. So a long position in the option is no longer feasible. Conversely, all derivative contracts may be exercised by the claim holders if the market price coincides with the exercise value L^i. This terminates short positions in the claim. However, as long as the derivative price stays above the exercise value, nobody will exercise the option because selling it on the market yields a higher reward. Similarly, there is no danger that the seller of a GCC cancels the contract as long as the cancellation value exceeds the market price. Summing up, the derivative speculators are facing trading constraints Γ given by

$$\begin{aligned}\Gamma_t \quad := \quad & \{x \in \mathbb{R}^{m+n} : \text{ for } i = m+1, \dots, m+n \text{ we have } x^i \geq 0 \text{ if } S^i_{t-} = L^i_{t-} \\ & \text{and } x^i \leq 0 \text{ if } S^i_{t-} = U^i_{t-}\}. \end{aligned} \tag{3.1}$$

In the following subsections, we treat neutral pricing separately for utility of terminal wealth and for local utility, respectively.

3.3.1 Terminal wealth

We start by assuming that derivative speculators are identical investors trying to maximize expected utility from terminal wealth. Moreover, we suppose that the neutral pricing measure for terminal wealth $P^\star$ in the sense of Definition 3.2.3 exists for the underlyings' market $S^1, \dots, S^m$. As explained above, we look for neutral derivative prices in the following sense:

Definition 3.3.1. *We call derivative price processes $S^{m+1}, \dots, S^{m+n}$* neutral for terminal wealth *if there exists a strategy $\overline{\varphi}$ in the extended market $S^1, \dots, S^{m+n}$ which is optimal for terminal wealth under the constraints Γ and satisfies $\overline{\varphi}^{m+1} = \dots = \overline{\varphi}^{m+n} = 0$.*

The following main result of this chapter treats existence and uniqueness of neutral derivative price processes. Moreover, it shows that they are recovered as the value of a Dynkin game relative to the neutral pricing measure $P^\star$.

Theorem 3.3.2. *Suppose that $L^{m+1}, \dots, L^{m+n}$ and $U^{m+1}, \dots, U^{m+n}$ are bounded. Then there exist unique neutral derivative price processes. These are given by*

$$\begin{aligned} S^i_t \quad &= \quad \operatorname{ess\,inf}_{\tau^U \in \mathscr{T}_t} \operatorname{ess\,sup}_{\tau^L \in \mathscr{T}_t} E^\star(R^i(\tau^L, \tau^U)|\mathscr{F}_t) \\ &= \quad \operatorname{ess\,sup}_{\tau^L \in \mathscr{T}_t} \operatorname{ess\,inf}_{\tau^U \in \mathscr{T}_t} E^\star(R^i(\tau^L, \tau^U)|\mathscr{F}_t) \end{aligned} \tag{3.2}$$

for $t \in [0, T]$, $i = m+1, \dots, m+n$, where $\mathscr{T}_t$ denotes the set of $[t, T]$-valued stopping times and

$$R^i(\tau^L, \tau^U) := \begin{cases} L^i_{\tau^L} & \textit{if } \tau^L \leq \tau^U \\ U^i_{\tau^U} & \textit{otherwise.} \end{cases}$$

Moreover, the extended market $S^1, \dots, S^{m+n}$ satisfies condition NFLVR in the sense of

Definition 3.3.3. *We say that the market $S = (S^1, \dots, S^{m+n})$ satisfies the condition* no free lunch with vanishing risk (NFLVR) *if 0 is the only non-negative element of the $L^\infty(P)$-closure of the set $C := \{f \in L^\infty(P) : f \leq \psi \cdot S_T$ for some $\psi \in \mathfrak{S}(\Gamma)\}$. (Note that this is a straightforward extension of the usual NFLVR condition in Delbaen and Schachermayer (DS94), Definition 2.8 to markets containing game contingent claims.)*

Proof of Theorem 3.3.2. Step 1: By Lepeltier and Maingueneau (LM84), Théorème 9 and Corollaire 12, there exist right-continuous adapted processes $S^{m+1}, \dots, S^{m+n}$ satisfying Equation (3.2). Fix $i \in \{m+1, \dots, m+n\}$. Define stopping times $T_1^k := \inf\{t \in \mathbb{R}_+ : S_t^i \geq U_t^i - 1/k\}$ for any $k \in \mathbb{N}$ and $T_1 := \sup_{k \in \mathbb{N}} T_1^k$. By Lepeltier and Maingueneau (LM84), Théorème 11 and Dellacherie and Meyer (DM82), Theorem VI.3, $(S^i)^{T_1^k}$ is a $P^\star$-supermartingale for any $k \in \mathbb{N}$. Obviously, $(S^i_{T_1^k})_{k \in \mathbb{N}}$ converges for $k \to \infty$ $P^\star$-a.s. to

$$R := U_{T_1} 1_{\cup_{k \in \mathbb{N}}\{T_1^k = T_1\}} + U_{T_1-} 1_{\cap_{k \in \mathbb{N}}\{T_1^k < T_1\}}.$$

Define an adpated right-continuous process $\overline{S}^i$ by

$$\overline{S}^i_t := \begin{cases} S_t^i, & \text{if } t < T_1 \text{ or } t = 0, \\ U_{T_1-}, & \text{if } 0 \neq t \geq T_1 \text{ and } T_1^k < T_1 \text{ for any } k \in \mathbb{N}, \\ U_{T_1}, & \text{if } 0 \neq t \geq T_1 \text{ and } T_1^k = T_1 \text{ for some } k \in \mathbb{N}, \end{cases}$$

i.e. $\overline{S}^i = \sum_{k \in \mathbb{N}} (S^i)^{T_1^k} 1_{]T_1^{k-1}, T_1^k]} + R 1_{(\cup_{k \in \mathbb{N}}[0, T_1^k])^C}$ (with the convention $]T_1^{-1}, T_1^0] := [T_1^0]$).

Let $s, t \in [0, T]$ with $s \leq t$. If $s \in (\cup_{k \in \mathbb{N}}[0, T_1^k])^C$, then $\overline{S}^i_s = R = \overline{S}^i_t$ and hence $E^\star(\overline{S}^i_t | \mathscr{F}_s) = \overline{S}^i_s$. Now, let $s \in]T_1^{k-1}, T_1^k]$ for some $k \in \mathbb{N}$. Then

$$\overline{S}^i_s = (S^i)_s^{T_1^l} \geq E^\star((S^i)_t^{T_1^l} | \mathscr{F}_s) = E^\star(\overline{S}^i_{T_1^l \wedge t} | \mathscr{F}_s)$$

for $l \geq k$. Moreover, dominated convergence yields that $E^\star(\overline{S}^i_{T_1^l \wedge t} | \mathscr{F}_s) \to E^\star(\overline{S}^i_t | \mathscr{F}_s)$ in measure for $l \to \infty$. Hence $\overline{S}^i_s \geq E^\star(\overline{S}^i_t | \mathscr{F}_s)$. Altogether, it follows that $\overline{S}^i$ is a $P^\star$-supermartingale. Hence, $(S^i)^{T_1}$ is a semimartingale.

For $l \in \mathbb{N} \setminus \{0, 1\}$ define $T_l := \sup_{k \in \mathbb{N}} T_l^k$ where $T_l^k := \inf\{t \geq T_{l-1} : S_t^i \leq L_t^i + 1/k\}$ for $l = 2, 4, 6, \dots$ and $T_l^k := \inf\{t \geq T_{l-1} : S_t^i \geq U_t^i - 1/k\}$ for $l = 3, 5, 7, \dots$ Similarly to above, one shows by induction that $(S^i)^{T_l}$ is a semimartingale for any $l \in \mathbb{N}$.

Step 2: We keep the notation from the previous step. Fix $l \in \mathbb{N}$. For $t_0 \in [0, T]$ and $k \in \mathbb{N}$ define stopping times $\tau_{t_0,k} := \inf\{t \geq t_0 : (S^i)_t^{T_l} \leq (L^i)_t^{T_l} + 1/k\} \wedge T$. From Lepeltier and

Mainguenau (LM84), Théorème 11 it follows that $1_{]t_0,\tau_{t_0,k}]} \cdot (S^i)^{T_l}$ is a $P^\star$-submartingale for any $t_0 \in [0,T]$, $k \in \mathbb{N}$. In particular, we have

$$b^\star + \int (x - h(x))F^\star(dx) \geq 0 \tag{3.3}$$

$P \otimes A$-a.e. on $]t_0, \tau_{t_0,k}]$ (cf. Lemma A2.3), where $(b^\star, c^\star, F^\star, A)$ denote $P^\star$-differential characteristics of the semimartingale $(S^i)^{T_l}$ in the sense of Definition A2.1. Since

$$\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\} \cap]0,T] = \cup_{t_0 \in \mathbb{Q} \cap [0,T]} \cup_{k \in \mathbb{N}}]t_0, \tau_{t_0,k}],$$

it follows that Equation (3.3) holds $P \otimes A$-a.e. on $\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\}$. Therefore, $1_{\{(L^i)^{T_l}_- < (S^i)^{T_l}_-\}} \cdot (S^i)^{T_l}$ is a $P^\star$-σ-submartingale (cf. Kallsen and Shiryaev (KS01), Lemma 2.5 and Lemma A2.3). Analogously, it follows that $1_{\{(S^i)^{T_l}_- < (U^i)^{T_l}_-\}} \cdot (S^i)^{T_l}$ is a $P^\star$-σ-supermartingale, and hence $1_{\{(L^i)^{T_l}_- < (S^i)^{T_l}_- < (U^i)^{T_l}_-\}} \cdot (S^i)^{T_l}$ is a $P^\star$-σ-martingale.

Step 3: We keep the notation from the previous steps. Let $T_\infty := \lim_{l \to \infty} T_l$. Since L^i, U^i are $P^\star$-special semimartingales with integrable L^i_0, U^i_0, they are locally in class $\mathscr{H}^1$ in the sense of Definition A2.4 and relative to $P^\star$ (cf. Dellacherie and Meyer (DM82), VII.99). Denote by $(\sigma_k)_{k \in \mathbb{N}}$ a corresponding localizing sequence. Fix $k \in \mathbb{N}$. By Proposition A2.6, applied to $(L^i)^{T_l \wedge \sigma_k}$, $(S^i)^{T_l \wedge \sigma_k}$, and $(U^i)^{T_l \wedge \sigma_k}$, it follows that $\sup_{l \in \mathbb{N}} \|(S^i)^{T_l \wedge \sigma_k}\|_{\mathscr{H}^1} < \infty$, which in turn implies that $(S^i)^{T_\infty \wedge \sigma_k}$ is a semimartingale (cf. Proposition A2.7). Therefore, $(S^i)^{T_\infty}$ is a local semimartingale and hence a semimartingale. In particular, it has left-hand limits at T_∞. Since $L^i_{t-} < U^i_{t-}$ for $t < T$, this is only possible if $T_\infty = T$. Consequently, S^i is a semimartingale.

Step 4: Let Z denote the density process of $P^\star$ and φ an optimal strategy for terminal wealth in the market $S^1, \dots, S^m$. We want to show that the $\mathbb{R}^{m+n}$-valued process $\overline{\varphi} := (\varphi, 0) \in \mathfrak{S}(\Gamma)$ is an optimal strategy for terminal wealth under the constraints Γ, now referring to the extended market $S := (S^1, \dots, S^{m+n})$. Since $ZE(u'(V_T(\varphi)))$ coincides with the optimal solution $\widehat{Y}(y)$ to the dual problem in Kramkov and Schachermayer (KS99), Theorem 2.2, we have that $(\varphi \cdot (S^1, \dots, S^m))Z$ is a martingale. This implies that $\overline{\varphi} \cdot S = \varphi \cdot (S^1, \dots, S^m)$ is a $P^\star$-martingale (cf. JS, III.3.8).

Consider a strategy $\psi \in \mathfrak{S}(\Gamma)$ in the extended market. Let $(b^\star, c^\star, F^\star, A)$ be $P^\star$-differential characteristics of S in the sense of Definition A2.1. The same argument as in Step 2 shows that

$$b^{\star,i} + \int (x^i - h^i(x))F^\star(dx) \geq 0$$

$P \otimes A$-a.e. on $\{L^i_- < S^i_-\}$ and ≤ 0 on $\{S^i_- < U^i_-\}$ for $i = m+1, \dots, m+n$. Since $S^1, \dots, S^m$ are $P^\star$-σ-martingales, we have

$$b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx) = 0$$

for $i = 1, \dots, m$. From the form of the constraints Γ it follows that

$$\psi^i(b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx)) \leq 0$$

for $i = m+1, \dots, m+n$, which yields that

$$\psi(b^\star + \int (x - h(x)) F^\star(dx)) \leq 0 \quad P \otimes A - \text{a.e.}$$

In view of Kallsen and Shiryaev (KS01), Lemma 2.5 and Lemma A2.3, this implies that $\psi \cdot S$ is a $P^\star$-σ-supermartingale. By Goll and Kallsen (GK01), Proposition 7.9, this process and hence also $(\psi - \overline{\varphi}) \cdot S$ is even a $P^\star$-supermartingale. In particular, we have

$$E(u'(V_T(\overline{\varphi}))((\psi - \overline{\varphi}) \cdot S)) = E(u'(V_T(\overline{\varphi})))E^\star((\psi - \overline{\varphi}) \cdot S) \leq 0.$$

Due to Lemma 3.2.2, $\overline{\varphi}$ is an optimal strategy for terminal wealth under the constraints Γ. Hence, $S^{m+1}, \dots, S^{m+n}$ are neutral price processes for terminal wealth.

Step 5: For the uniqueness part assume that $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative price processes corresponding to some optimal strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m, 0, \dots, 0)$ in the extended market $\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$. Since $\widetilde{\varphi}$ does not contain any derivative, we have that $(\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m)$ is an optimal strategy for the small market $S^1, \dots, S^m$ with the same expected utility. Similarly, the expected utility of φ in the small market and of $\overline{\varphi} = (\varphi, 0)$ in the extended market $\widetilde{S}$ tally. Since φ is optimal in the small market $S^1, \dots, S^m$, it follows that $\overline{\varphi} \in \mathfrak{S}'(\Gamma)$ is optimal in the extended market $\widetilde{S}$ under the constraints Γ. Hence we may w.l.o.g. assume that $\widetilde{\varphi} = \overline{\varphi}$.

Fix $i \in \{m+1, \dots, m+n\}$. Firstly, we show that $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-submartingale for any predictable subset D of $\{L^i_- < \widetilde{S}^i_-\}$. Since $\widetilde{S}^i$ is bounded, we have that $1_D \cdot \widetilde{S}^i$ is locally bounded. Hence, there exists an increasing sequence $(T_k)_{k \in \mathbb{N}}$ of stopping times with $P^\star(T_k = T) \to 1$ and $\sup_{t \in [0,T]} |(1_D \cdot \widetilde{S}^i)^{T_k}_t| \leq k$. Fix $k \in \mathbb{N}$, $s, t \in [0, T]$ with $s \leq t$, and $F \in \mathscr{F}_s$. Define an admissible strategy $\psi \in \mathfrak{S}(\Gamma)$ in the market

$\widetilde{S} := (S^1, \dots, S^m, \widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n})$ by $\psi^j := 0$ for $j \neq i$ and

$$\psi^i = -\frac{\varepsilon}{4k} 1_{D \cap [0,T_k] \cap (F \times]s,t])}.$$

Lemma 3.2.2 and the fact that $\overline{\varphi} \cdot \widetilde{S} = \varphi \cdot (S^1, \dots, S^m)$ is a $P^\star$-martingale yield that

$$\begin{aligned} & -\frac{\varepsilon}{4k} E^\star(((1_D \cdot \widetilde{S}^i)_t^{T_k} - (1_D \cdot \widetilde{S}^i)_s^{T_k}) 1_F) \\ &= E^\star((\psi - \overline{\varphi}) \cdot \widetilde{S}_T) + E^\star(\overline{\varphi} \cdot \widetilde{S}_T) \\ &= (E(u'(V_T(\overline{\varphi}))))^{-1} E(u'(V_T(\overline{\varphi}))((\psi - \overline{\varphi}) \cdot \widetilde{S}_T)) \\ &\leq 0. \end{aligned}$$

Therefore, $(1_D \cdot \widetilde{S}^i)^{T_k}$ is a $P^\star$-submartingale, which implies that $1_D \cdot \widetilde{S}^i$ is a local $P^\star$-submartingale. Similarly, it follows that $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-supermartingale for any predictable subset D of $\{\widetilde{S}^i_- < U^i_-\}$.

Define stopping times $\tau_{t_0,k} := \inf\{t \geq t_0 : S^i_t \leq \widetilde{S}^i_t + 1/k\}$ for any $t_0 \in [0,T]$, $k \in \mathbb{N}$. Note that $\{S^i_- > \widetilde{S}^i_-\} \cap]0,T] = \cup_{t_0 \in \mathbb{Q} \cap [0,T]} \cup_{k \in \mathbb{N}}]t_0, \tau_{t_0,k}]$. Fix $t_0 \in [0,T]$, $k \in \mathbb{N}$. Since

$$\{L^i_- < S^i_-\} \cap \{\widetilde{S}^i_- < U^i_-\} \supset \{S^i_- > \widetilde{S}^i_-\},$$

we have that $1_{]t_0,\tau_{t_0,k}]} \cdot S^i$ and hence also $((S^i)_t^{\tau_{t_0,k}})_{t \in [t_0,T]}$ is a $P^\star$-σ-submartingale. By Goll and Kallsen (GK01), Proposition 7.9, this process is even a $P^\star$-submartingale. Similarly, it follows that $((\widetilde{S}^i)_t^{\tau_{t_0,k}})_{t \in [t_0,T]}$ is a $P^\star$-supermartingale. Since $(S^i)_T^{\tau_{t_0,k}} \leq (\widetilde{S}^i)_T^{\tau_{t_0,k}} + 1/k$, we have that $(S^i)_{t_0}^{\tau_{t_0,k}} \leq (\widetilde{S}^i)_{t_0}^{\tau_{t_0,k}} + 1/k$ P-a.s. for any $k \in \mathbb{N}$. Consequently, $S^i_{t_0} \leq \widetilde{S}^i_{t_0}$ P-a.s.. Since this holds for any $t_0 \in \mathbb{Q} \cap [0,T]$, we have that $S^i \leq \widetilde{S}^i$ by right-continuity. Similarly, it is shown that $\{S^i < \widetilde{S}^i\}$ is evanescent, which yields the uniqueness of neutral price processes for terminal wealth.

Step 6: The NFLVR property of the price process S is shown in the usual way: Let $\psi \in \mathfrak{S}(\Gamma)$. In Step 4 it is shown that $\psi \cdot S$ is a $P^\star$-supermartingale and hence $E^\star(f) \leq 0$ for any $f \in C$. Since $P^\star \sim P$, this is also true for any f in the $L^\infty(P)$-closure of C. Therefore $f = 0$ P-a.s. for any such f with $f \geq 0$. $\square$

Remarks.

(i) If $\sup_{t\in[0,T]}|L_t^i|$ and $\sup_{t\in[0,T]}|U_t^i|$ are $P^\star$-integrable instead of bounded for $i=m+1,\ldots,m+n$, we still have the existence of neutral derivative prices for terminal wealth. As Kifer (Kif00) points out, the results of Lepeltier and Maingueneau (LM84) hold also if L^i, U^i satisfy the above integrability conditions. The existence follows now from Steps 1–4 in the proof of Theorem 3.3.2.

(ii) European options with bounded discounted terminal payoff R^i at time T may be considered as special cases of game contingent claims by letting

$$L_t^i := \begin{cases} \text{ess inf } R^i - 1 & \text{if } t < T \\ R^i & \text{if } t = T \end{cases}$$

and

$$U_t^i := \begin{cases} \text{ess sup } R^i + 1 & \text{if } t < T \\ R^i & \text{if } t = T. \end{cases}$$

If we assume the absence of arbitrage, the price of the European claim will never leave the interval $[\text{ess inf } R^i, \text{ess sup } R^i]$. Therefore, the additional right to cancel the contract prematurely is worthless. Equation (3.2) reduces to

$$S_t^i = E^\star(R^i|\mathscr{F}_t)$$

for European options.

American options with bounded exercise process L^i and final payoff L_T^i are treated similarly by defining

$$U_t^i := \begin{cases} \text{ess sup } (\sup_{t\in[0,T]} L_t^i) + 1 & \text{if } t < T \\ L_T^i & \text{if } t = T. \end{cases}$$

The neutral price process S^i in Equation (3.2) now has the form of a Snell envelope:

$$S_t^i = \text{ess sup}_{\tau\in\mathscr{T}_t} E^\star(L_\tau^i|\mathscr{F}_t).$$

Moreover, an inspection of the proof reveals that we can slightly weaken the conditions on L^i in the American option case. It is enough to assume that L^i is a càdlàg, adapted process instead of a semimartingale.

(iii) In general, neutral derivative prices for terminal wealth depend on the utility function u, the time horizon T, the initial endowment ε, and the numeraire. In the setting of Example 3.2.4, the density process of $P^\star$ does not depend on T and ε. Therefore, neutral prices do not depend on the time horizon and the initial endowment of derivative speculators in this case.

Logarithmic utility is even more agreeable in this respect: As it is discussed in Goll and Kallsen (GK01), Section 6, the neutral prices relative to $P^\star$ depend neither on T, ε, nor on the chosen numeraire. Moreover, the density process of $P^\star$ can be calculated explicitly even in very complex models.

3.3.2 Local utility

In this subsection, we suppose that derivative speculators maximize their local utility. Similarly to above, we assume that the neutral pricing measure for local utility $P^\star$ exists for the underlyings' market $S^1, \dots S^m$ (cf. Definition 3.2.8).

Definition 3.3.4. *We call derivative price processes* $S^{m+1}, \dots, S^{m+n}$ neutral for local utility *if there exists a strategy* $\overline{\varphi}$ *in the extended market* $S^1, \dots, S^{m+n}$ *which is locally optimal under the constraints* Γ *and satisfies* $\overline{\varphi}^{m+1} = \dots = \overline{\varphi}^{m+n} = 0$.

The following result corresponds to Theorem 3.3.2 in the local utility setting.

Theorem 3.3.5. *Suppose that* L^i, U^i *are special semimartingales and that* $\sup_{t\in[0,T]} |L^i_t|$ *and* $\sup_{t\in[0,T]} |U^i_t|$ *are* $P^\star$*-integrable for* $i = m+1, \dots, m+n$. *Then there exist unique neutral derivative price processes. These are given by*

$$\begin{aligned} S^i_t &= \operatorname{ess\,inf}_{\tau^U\in\mathscr{T}_t} \operatorname{ess\,sup}_{\tau^L\in\mathscr{T}_t} E^\star(R^i(\tau^L,\tau^U)|\mathscr{F}_t) \\ &= \operatorname{ess\,sup}_{\tau^L\in\mathscr{T}_t} \operatorname{ess\,inf}_{\tau^U\in\mathscr{T}_t} E^\star(R^i(\tau^L,\tau^U)|\mathscr{F}_t) \end{aligned}$$

for $t \in \mathbb{R}_+$, $i = m+1, \dots, m+n$, *where* $\mathscr{T}_t$ *and* $R^i(\tau^L, \tau^U)$ *are defined as in Theorem 3.3.2. Moreover, the extended market* $S^1, \dots, S^{m+n}$ *satisfies condition NFLVR in the sense of Definition 3.3.3.*

Proof. Steps 1–3 and 6 are shown literally as in the proof of Theorem 3.3.2. Only Steps 4 and 5 have to be modified slightly.

Step 4: Since $L^i \leq S^i \leq U^i$, we have that S^i is a special semimartingale for $i = m+1, \ldots, m+n$ (cf. Kallsen (Kal02), Proposition 3.7). Similarly as in Step 4 of the proof of Theorem 3.3.2 we want to show that $\overline{\varphi} := (\varphi, 0) \in \mathfrak{S}'(\Gamma)$ is a locally optimal strategy for $S = (S^1, \ldots, S^{m+n})$, where φ denotes an optimal strategy in the small market $S^1, \ldots, S^m$. Denote by (b, c, F, A) the P-differential characteristics of S relative to $h(x) = x$. In view of Theorem 3.2.7 we have to show that

$$b + u''(0)c\overline{\varphi} + \int x(u'(\overline{\varphi}x) - 1)F(dx) \in \Gamma^\circ. \tag{3.4}$$

Note that $\Gamma^\circ_t = \{y \in \{0\}^m \times \mathbb{R}^n :$ For $i = m+1, \ldots, m+n$ we have $y^i \geq 0$ if $L^i_{t-} < S^i_{t-}$ and $y^i \leq 0$ if $S^i_{t-} < U^i_{t-}\}$. From the Girsanov-Jacod-Mémin theorem it follows that the $P^\star$-differential characteristics $(b^\star, c^\star, F^\star, A)$ of S relative to some truncation function $h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ satisfy the equation

$$\begin{aligned} b^{\star,i}_t + \int (x^i - h^i(x))F^\star_t(dx) &= b^i_t + u''(0)c^{i\cdot}_t\overline{\varphi}_t + \int x^i\left(\frac{u'(\overline{\varphi}_t x)}{1+V_t} - 1\right)F_t(dx) \\ &= \frac{1}{1+V_t}\left(b^i_t + u''(0)c^{i\cdot}_t\overline{\varphi}_t + \int x^i(u'(\overline{\varphi}_t x) - 1)F_t(dx)\right) \end{aligned} \tag{3.5}$$

for $i = 1, \ldots, m+n$, where V_t is defined as in Definition 3.2.8 (cf. Kallsen (Kal02), Steps 3 and 4 on page 122 for the arguments in detail). Since φ is optimal in the small market, Theorem 3.2.7 yields that expression (3.5) equals 0 for $i = 1, \ldots, m$. The same argument as in Step 2 of the proof of Theorem 3.3.2 shows that the left-hand side of Equation (3.5) is non-negative on $\{L^i_{t-} < S^i_{t-}\}$ (resp. non-positive on $\{S^i_{t-} < U^i_{t-}\}$) for $i = 1, \ldots, m+n$. Together, it follows that Condition (3.4) holds. Therefore, $S^{m+1}, \ldots, S^{m+n}$ are neutral price processes for local utility.

Step 5: For the uniqueness part assume that $\widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n}$ are neutral derivative price processes corresponding to some locally optimal strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \ldots, \widetilde{\varphi}^m, 0, \ldots, 0)$ in the extended market $\widetilde{S} := (S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$. As in Step 5 of the proof of Theorem 3.3.2 we may w.l.o.g. assume that $\widetilde{\varphi} = \overline{\varphi}$.

In this step, we denote by (b, c, F, A) the P-differential characteristics of $\widetilde{S}$ relative to $h(x) = x$. Since $\overline{\varphi}$ is an optimal strategy, Theorem 3.2.7 yields that Condition (3.4) holds $P \otimes A$-a.e. As in the previous step, we express this condition in terms of the $P^\star$-differential characteristics $(b^\star, c^\star, F^\star, A)$ of $\widetilde{S}$ relative to some truncation function $h : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$. Fix $i \in \{m+1, \dots, m+n\}$. Then the $P^\star$-drift $b^{\star,i} + \int (x^i - h^i(x))F^\star(dx)$ of $\widetilde{S}^i$ it is non-negative on $\{L^i_{t-} < \widetilde{S}^i_{t-}\}$ resp. non-positive on $\{\widetilde{S}^i_{t-} < U^i_{t-}\}$. Due to Kallsen and Shiryaev (KS01), Lemma 2.5 and Lemma A2.3, this means that $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-submartingale for any predictable subset D of $\{L^i_{t-} < \widetilde{S}^i_{t-}\}$ and $1_D \cdot \widetilde{S}^i$ is a $P^\star$-σ-supermartingale for any predictable subset D of $\{\widetilde{S}^i_{t-} < U^i_{t-}\}$. The uniqueness of neutral price processes follows now as in the second half of Step 5 in the proof of Theorem 3.3.2. $\square$

Remark 2 following Theorem 3.3.2 holds accordingly in this setting.

3.4 Some supplementing considerations

In section 3.3 we have argued that by choosing their trading strategies under the constraints Γ as defined in (3.1) the derivative speculators are not disturbed by possible terminations of their counterparties. Thus the set Γ is small enough for the optimisation problem that is behind the definition of neutral derivative prices (Def. 3.3.1). The other way round, in this section we give reasons why Γ is actually big enough.

Therefore, let D^i_L and D^i_U resp. be predictable subsets of $\{S^i_- = L^i_-\}$ and $\{S^i_- = U^i_-\}$ resp., $i = m+1, \dots, m+n$. Assume derivative speculators are facing trading constraints $\widetilde{\Gamma}$ (instead of Γ) given by

$$\begin{aligned} \widetilde{\Gamma}_{\omega,t} \;\; := \;\; & \{x \in \mathbb{R}^{m+n} : \text{ for } i = m+1, \dots, m+n \text{ we have } x^i \geq 0 \text{ if } (\omega,t) \in D^i_L \\ & \text{and } x^i \leq 0 \text{ if } (\omega,t) \in D^i_U\}, \quad (\omega,t) \in \Omega \times [0,T], \end{aligned}$$

with the corresponding constrained set of trading strategies $\mathfrak{S}(\widetilde{\Gamma})$ defined in subsection 3.2.1.

Corollary 3.4.1. *Suppose that $L^{m+1}, \dots, L^{m+n}$ and $U^{m+1}, \dots, U^{m+n}$ are bounded and $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative price processes in the sense of Definition 3.3.1 but with respect to $\mathfrak{S}(\widetilde{\Gamma})$ instead of $\mathfrak{S}(\Gamma)$. Then $\widetilde{S}^i = S^i$ up to evanescence, for $i = m+1, \dots, m+n$, where S^i are defined in (3.2).*

Proof. Assume that $\widetilde{S}^{m+1}, \dots, \widetilde{S}^{m+n}$ are neutral derivative price processes with respect to $\mathfrak{S}(\widetilde{\Gamma})$, i.e. there is some strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \dots, \widetilde{\varphi}^m, 0, \dots, 0)$, which is optimal in the extended market

$\widetilde{S} := (S^1, \ldots, S^m, \widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n})$ under the constraint $\widetilde{\Gamma}$. As $\widetilde{\varphi} \in \mathfrak{S}(\Gamma)$ and $\mathfrak{S}(\Gamma) \subset \mathfrak{S}(\widetilde{\Gamma})$ $\widetilde{\varphi}$ is also an optimal strategy under $\mathfrak{S}(\Gamma)$, i.e. $\widetilde{S}^{m+1}, \ldots, \widetilde{S}^{m+n}$ are neutral price processes in the sense of Definition 3.3.1 (i.e. with $\mathfrak{S}(\Gamma)$). Therefore the assertions follows from step 5 in the proof of Theorem 3.3.2. Notice that in this step we do not make use of the fact that $S^{m+1}, \ldots, S^{m+n}$ are neutral price processes in that context. □

Remark 3.4.2. However, in general the *existence* of neutral prices gets lost: just take $D_L^1 = \varnothing$ (i.e. short positions in the derivative are always allowed) and $L_t^1 = 1 - t$, for $t \in [0, T]$. Then, we have for the ess inf-ess sup-process $S^1 = L^1$, which is obviously no neutral price process: it is decreasing with probability one and therefore short positions in it are worthwhile for every agent with increasing utility function.

Nethertheless, S^i, $i \in \{m+1, \ldots, m+n\}$, as defined in (3.2), are the only *candidates* for neutral price processes. So, it makes sense to define D_L^i and D_U^i depending on S^i and show under which conditions the existence in Theorem 3.3.2 remains valid.

Let $(b^\star, c^\star, F^\star, A)$ be $P^\star$-differential characteristics of S in the sense of Definition A2.1. Take e.g. the lower bound: on $\{S_-^i = L_-^i\}$ we have that

$$b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx) \leq 0$$

$P \otimes A$-a.e. (see step 4 of Theorem 3.3.2). It turns out that on $\{b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx) = 0\}$ we can allow for short positions in asset i, but on $\{b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx) < 0\}$ we have to rule it out. Analogously the upper bound can be devided in two. We state this in the following

Corollary 3.4.3. *Suppose that* $L^{m+1}, \ldots, L^{m+n}$ *and* $U^{m+1}, \ldots, U^{m+n}$ *are bounded.* S^i *is the neutral derivative price process with respect to* $\mathfrak{S}(\widetilde{\Gamma})$ *iff*

$$\{b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx) < 0\} \subset D_L^i \quad P \otimes A - a.e.$$

and

$$\{b^{\star,i} + \int (x^i - h^i(x)) F^\star(dx) > 0\} \subset D_U^i \quad P \otimes A - a.e.$$

Remark 3.4.4. Again, for the "if-part" we only need that for $i = m+1, \ldots, m+n$ $\sup_{t \in [0,T]} |L_t^i|$ and $\sup_{t \in [0,T]} |U_t^i|$ are $P^\star$-integrable instead of bounded.

Remark 3.4.5. Remember from step 4 in the proof of Theorem 3.3.2 that we have that

$$b^{\star,i} + \int (x^i - h^i(x))F^\star(dx) \geq 0$$

$P \otimes A$-a.e. on $\{L^i_- < S^i_-\}$ and ≤ 0 on $\{S^i_- < U^i_-\}$ for $i = m+1, \ldots, m+n$.

Proof. Sufficiency: Consider a strategy $\psi \in \mathfrak{S}(\widetilde{\Gamma})$ from the extended market. From the assumption for the constraints $\widetilde{\Gamma}$ it follows that

$$\psi^i(b^{\star,i} + \int (x^i - h^i(x))F^\star(dx)) \leq 0$$

for $i = m+1, \ldots, m+n$, $P \otimes A$-a.e. and we can proceed as in step 4 of Theorem 3.3.2 to show that $\overline{\varphi} = (\varphi, 0)$ is an optimal strategy for terminal wealth.

Necessarity: Assume that S^i, $i = m+1, \ldots, m+n$ is a neutral derivative price process. By the same arguments as in step 5 in the proof of Theorem 3.3.2 we have that $1_{(D^i_L)^C} \cdot S^i$ is a $P^\star$-σ-submartingale. With Lemma A.2 this implies that $b^{\star,i} + \int (x^i - h^i(x))F^\star(dx) \geq 0$ $P \otimes A$-a.e. on $(D^i_L)^C$. As the analogous assertion holds for D^i_U we are ready. □

Remark 3.4.6. *All the previous considerations hold also in the context of* local utility maximization.

After this formal treatment let us interprete the results economically.

Assume that the speculator has no trading constraints. Then, roughly speaking, the postulation that under a derivative price process S^i her optimal portfolio contains no derivatives ensures that the option price is neither too high nor too low, in the sense that there is neither a negative nor a positive demand. Making *a priori* the restriction of no short sales, as e.g. at the lower bound $S^i_- = L^i_-$, the speculator may invest in the option, but does not do it at the optimum. From this one can (only) conclude that the price S^i_- is at least not too low. On the other hand, at $S^i_- = L^i_-$ it is merely the exercise value. So, a lower price $S^i_- < L^i_-$ is definitively too low. It allows for arbitrage, resp. the optimal demand is $\varphi^i = +\infty$ (buying an option and exercising it immediately yields the riskless gain $L^i_- - S^i_- > 0$). By contrast, for $S^i_- = L^i_-$ we cannot exclude that S^i is too high in the sense that an optimal φ^i from an unconstrained strategy set would be strictly negative. Summing up, at $S^i_- = L^i_-$ we can say that a lower price would be too low, but we cannot exclude that $S^i_- = L^i_-$ is too high. At $S^i_- = U^i_-$ we can argue the opposite way round. This interpretation corresponds somehow to the nonexistence of a neutral price process, whereas in the case of existence the price process is unique, cf. Corrolary 3.4.1 and the following remark.

Corrolary 3.4.3 tells us that the bounds $S^i_- = L^i_-$ and $S^i_- = U^i_-$, resp., can be devided in two, resp. Whereas on $\{b^{\star,i} + \int(x^i - h^i(x))F^\star(dx) = 0$ there is an *indifference situation* in our economy at $\{b^{\star,i} + \int(x^i - h^i(x))F^\star(dx) < 0$ and $\{b^{\star,i} + \int(x^i - h^i(x))F^\star(dx) > 0$, resp., the option has to be exercised and bought back, resp.

Chapter 4

Valuation of Contingent Claims with Embedded Options in Incomplete Markets

This chapter is an adapted version of Kühn (Küh02).

4.1 Introduction to utility based valuation

We are interested in options where the holder purchases the right to choose (in a predefined way) among several random payoffs offered by the seller. Such options could be a chooser option having the feature that, after a specified period of time, the holder can choose whether the option is a call or a put (cf. for example Hull (Hul00)), or an American option that can be exercised at any time up to the expiration date (and so the discounted payoff depends on the stopping time). Another example is an installment option, i.e. an European option in which the premium is paid in a series of installments and the holder has the right to terminate payments at any payment date, but then the option matures automatically (cf. Karsenty and Sikorav (KS96)). In an insurance context, it could be a pension scheme where the policy-holder reaching a special age can swap his right to a pension for a single payment. Such choices offered by an insurance contract are called "embedded options". Many examples for such "options" are given in Held (Hel99).

In all these cases the insurer (writer of the option) has the problem that he does not know at time 0 which random payoff the insured (holder of the option) is going to choose.

In the spirit of Schweizer (Sch01c), we consider a general model combining financial market risk and traditional actuarial risk. **Hereafter, we just call the insurer/writer "she" and the insured/holder "he".**

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let the $\mathbb{R}^d$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ model the *discounted* price processes of the d risky assets available for trade. Θ is a suitable space of admissible trading strategies to be specified later.

Definition 4.1.1. *Let u be her utility function. It is a mapping from the set of random variables into $\mathbb{R}$ that is monotone in the sense that $X \leq Y$ P-a.s. implies $u(X) \leq u(Y)$.*

The classical actuarial variance principle would correspond to $u(X) = E_P(X) - \lambda \mathrm{Var}(X)$, $\lambda > 0$, but it is known *not* to be monotone. The monotonicity is necessary for our valuation principle to be consistent with no-arbitrage.

For pricing random payoffs in incomplete markets Schweizer (Sch01c) introduces – in the most general form – an indifference principle in the framework of financial markets. The idea is as follows: she can decide whether she insures a risk B, an $\mathcal{F}_T$-measurable random variable, for a premium h or not. The utility-indifference premium is defined as the premium which makes her indifferent with regard to this decision. She also takes into consideration that she can perhaps (partly) hedge the risk.

Definition 4.1.2. *h is called a* "utility-indifference premium" *if it satisfies*

$$\sup_{\vartheta\in\Theta} u\left(c + h - B + \int_0^T \vartheta_t \, dS_t\right) = \sup_{\vartheta\in\Theta} u\left(c + \int_0^T \vartheta_t \, dS_t\right), \tag{1.1}$$

where c is her initial capital.

For the variance and the standard derivation principle, closed-form valuations for many practically relevant products combining financial and actuarial risk, as for example unit-linked life insurance contracts or so-called financial stop-loss reinsurance contracts, are given by Møller (Møl00), using general results of Schweizer (Sch01c). For the exponential utility function, more recently, Becherer (Bec01), see Theorem 2.4.1, derives a recursive computation formula for the premium considering a model which consists of a complete financial market and additional independent actuarial risk observed at discrete points of time.

The aim of this chapter is to generalize this concept to situations where the random payment is not fixed at the beginning, but during the policy term the holder can choose in a contractually predefined way between several scenarios.

In the first instance, we consider a model with only one predefined decision time at which the holder can choose among a finite number of payoffs (Section 4.2). In Section 4.3, we deal with American style contingent claims where the holder can stop the contract before maturity T. More technical lemmas are left to the appendix.

4.2 Choice among a finite number of payoffs

Let $\mathcal{B} = \{B_1, \dots, B_k\}$ be a set of contingent claims, i.e. each B_i is an $\mathcal{F}_T$ measurable positive random variable. *He* can choose among these k different payoffs at the predefined stopping time τ (using the information $\mathcal{F}_\tau$). This means that there is a set of permissible decision rules

$$\mathcal{D} = \{\delta : \Omega \to \{1, \dots, k\},\ \ \mathcal{F}_\tau - \text{measurable}\}.$$

The final payment depending on δ is then

$$B^\delta = \sum_{i=1}^{k} \mathbf{1}(\delta = i) B_i\,. \tag{2.2}$$

We call $(B^\delta)_{\delta \in \mathcal{D}}$ a *general claim*.

Example 4.2.1 (Chooser option). *At a fixed time $T^* \in (0, T)$ the holder of a chooser option can decide whether the option is a call or a put (here, with the same strike price K), i.e. $B_1 = (S_T^{(1)} - K)^+$, $B_2 = (K - S_T^{(1)})^+$, and $\tau \equiv T^* \in (0, T)$.*

Our generalization of the utility-indifference principle is as follows:

We suggest to determine the premium as the minimal amount she must receive at time 0 such that for **all** *possible decision functions δ (he could hypothetically have) her utility is at least as big as the utility she would have if she did not offer this contingent claim.*

This means, she will be on the safe side even if she knows nothing about his preferences/decision function. Such a premium is reasonable in the following sense: if the premium was smaller she would offer him a decision possibility that decreases her utility. For a single decision function δ

the random payoff B^δ is uniquely determined. Therefore, in the case of *no* financial market the premium described above would simply be the supremum over all utility-indifference premiums – according to (1.1) – related to all possible B^δ (Theorem 4.2.9 case (i)).

But the existence of a financial market makes things more complicated: she can (partially) hedge the risk carried by the claim. The crucial point about this is that she does not know his decision function δ, and therefore she must choose her trading strategy independently of it. Only from time $\tau(\omega)$ on she can choose a strategy depending on her information $\delta(\omega)$. We formalize this in the following way:

Definition 4.2.2. *We call h a* "still fair premium" *if*

$$\sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta) \in \Theta^{k+1}} \inf_{\delta \in \mathcal{D}} u\left(c + h - B^\delta + \int_0^T \vartheta_t^\delta \, dS_t\right) = \sup_{\vartheta \in \Theta} u\left(c + \int_0^T \vartheta_t \, dS_t\right), \tag{2.3}$$

where

$$\vartheta_t^\delta(\omega) := \begin{cases} \vartheta_t(\omega) & : \quad t \leq \tau(\omega), \\ {}^i\vartheta_t(\omega) & : \quad t > \tau(\omega) \quad \textit{and} \quad \delta(\omega) = i. \end{cases} \tag{2.4}$$

As τ is a stopping time, it can be shown by standard arguments that ϑ^δ is predictable if ϑ and ${}^i\vartheta$ are predictable.

Remark 4.2.3. Definition 4.2.2 takes into account that from time $\tau(\omega)$ on she knows his *effective* decision $\delta(\omega)$. This is a priori not the same as if she received at time $\tau(\omega)$ his whole *decision function* $\delta : \Omega \to \{1, \dots, k\}$. In the latter case one would replace the lhs of (2.3) by

$$\sup_{\vartheta \in \Theta} \inf_{\delta \in \mathcal{D}} \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - B^\delta + \int_0^\tau \vartheta_t \, dS_t + \int_\tau^T \widetilde{\vartheta}_t \, dS_t\right),$$

at which she could choose an optimal $\widetilde{\vartheta}$ depending on δ. But, it turns out that these two concepts are equivalent under all expected utility functions $u(\cdot) = E_P(U(\cdot))$, see Lemma A3.2.

The strategy space Θ has to satisfy the

Assumption 4.2.4. *All elements of Θ are $(\mathcal{F}_t)_{t \in [0,T]}$-predictable and S-integrable, i.e. $\Theta \subset L(S)$. Θ is linear, and for all $\delta \in \mathcal{D}$ the following implication is valid:*

If $\vartheta, {}^i\vartheta \in \Theta$, $i = 1, \dots, k$, then the compound strategy ϑ^δ is also an element of Θ.

The latter is essential: a strategy ϑ is admissible if and only if its restriction to $(0, \tau]$ and its restriction to $(\tau, T]$ are both admissible. Therefore, it allows us to verify the admissibility of a strategy seperately on $(0, \tau]$ and $(\tau, T]$. So, it is a quite natural assumption. But unfortunately, it is not as harmless as it looks like. For example, the set of all predictable trading strategies such that the discounted gain process $\int_0^t \vartheta_u \, dS_u$ is bounded from below (but not necessarily from above) does obviously not satisfy Assumption 4.2.4, as the insurer's credit limit for the second period is determined through her trading gains in the first period.

As $L(S)$ is linear an example satisfying Assumption 4.2.4 is

$$\Theta_1 = \left\{ \vartheta \in L(S) \,\middle|\, \int_0^t \vartheta_u \, dS_u \text{ is bounded uniformly in } t \text{ and } \omega \right\}. \tag{2.5}$$

Θ_1 is rather small, but in Delbaen et al. (DGR$^+$02) resp. Kabanov and Stricker (KS02a) it is shown that under some restrictions (in particular the standing assumption that S is locally bounded) for exponential utility the maximization problem (1.1) with $\Theta = \Theta_1$ has the same value as for much bigger Θ. Another choice satisfying Assumption 4.2.4 is

$$\Theta_2 = \left\{ \vartheta \in L(S) \,\middle|\, \int_0^t \vartheta_u \, dS_u \text{ is a martingale resp. a special set of martingale measures} \right\}.$$

Remark 4.2.5. A complementary approach would consist in defining the premium – similarly to Davis and Zariphopoulou (DZ95) – from *his* point of view: the maximum premium he is prepared to pay for the claim. As *he* determines the decision, for him δ is not uncertain but he can maximize over it. The "utility-indifference premium" h' would then be determined by

$$\sup_{\vartheta \in \Theta} \sup_{\delta \in \mathcal{D}} u \left(c' - h' + B^\delta + \int_0^T \vartheta_t \, dS_t \right) = \sup_{\vartheta \in \Theta} u \left(c' + \int_0^T \vartheta_t \, dS_t \right),$$

where c' is *his* initial capital. But, in actuarial mathematics the zero utility principle is traditionally considered from the insurer's viewpoint. That makes economically more sense as "utility-indifference" assumes *perfect competition* (and therefore homogeneous preferences) that is more likely between insurance companies than between the insureds. Moreover, we are interested in hedging strategies for the *insurer* against the risk carried by the claim. With (2.3) one can obtain a minimax strategy.

First we show that h solving (2.3) is consistent with no-arbitrage. We take over the concept of Definition 4.2 in Karatzas and Kou (KK98):

Definition 4.2.6. *Suppose that h' is the price of the general claim $(B^\delta)_{\delta\in\mathcal{D}}$ defined in (2.2). We say that there is an* arbitrage opportunity, *if there exists either*

(i) some compound strategy $\widehat{\vartheta}^\bullet$ - according to (2.4) - that satisfies

$$x + \int_0^T \widehat{\vartheta}_t^\delta \, dS_t \geq B^\delta \quad P\text{-a.s.} \qquad \forall \delta \in \mathcal{D}$$

for some $x < h'$, or

(ii) some $\widehat{\delta} \in \mathcal{D}$ and some $\widehat{\vartheta} \in \Theta$ such that

$$-x + \int_0^T \widehat{\vartheta}_t \, dS_t + B^{\widehat{\delta}} \geq 0 \quad P\text{-a.s.}$$

for some $x > h'$.

Theorem 4.2.7. *Let (2.3) have a unique solution h. Then, for $h' = h$ neither she (case (i)) nor he (case (ii)) has an arbitrage opportunity.*

Proof. (i) Suppose that (i) is satisfied for $h' = h$. Due to the linearity of Θ, the mapping $\vartheta^\delta \mapsto \vartheta^\delta + \widehat{\vartheta}^\delta$ is a bijection of the set of permissible compound trading strategies into itself. This (first equality) and the monotonicity of u (first inequality) yield:

$$\begin{aligned}
&\sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c + x - B^\delta + \int^T \vartheta_t^\delta \, dS_t\right) \\
&= \sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c + x - B^\delta + \int_0^T \widehat{\vartheta}_t^\delta \, dS_t + \int_0^T \vartheta_t^\delta \, dS_t\right) \\
&\geq \sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c + \int_0^T \vartheta_t^\delta \, dS_t\right) \\
&= \sup_{\vartheta\in\Theta} u\left(c + \int_0^T \vartheta_t \, dS_t\right).
\end{aligned} \tag{2.6}$$

The last equality holds due to Assumption 4.2.4. On the other hand, we have by monotonicity of u and the uniqueness of h:

$$\sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c + x - B^\delta + \int_0^T \vartheta_t^\delta \, dS_t\right) < \sup_{\vartheta\in\Theta} u\left(c + \int_0^T \vartheta_t \, dS_t\right), \tag{2.7}$$

i.e. a contradiction to (2.6).

(ii) For similar reasons (ii) cannot be satisfied for $h' = h$. □

Theorem 4.2.8. *Assume that for every $\delta \in \mathcal{D}$ there exists a unique utility-indifference premium h_δ for the claim B^δ, i.e. h_δ solves equation (1.1) with $B = B^\delta$. If h solves (2.3), then $h \geq \sup_{\delta\in\mathcal{D}} h_\delta$.*

Proof. Let us first show that for all $\delta_0 \in \mathcal{D}$:

$$\begin{aligned}
&\sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c+h-B^\delta+\int_0^T \vartheta_t^\delta\, dS_t\right)\\
&\leq \sup_{\vartheta\in\Theta} \inf_{\delta\in\mathcal{D}} \sup_{\widetilde{\vartheta}\in\Theta} u\left(c+h-B^\delta+\int_0^\tau \vartheta_t\, dS_t+\int_\tau^T \widetilde{\vartheta}_t\, dS_t\right)\\
&\leq \inf_{\delta\in\mathcal{D}} \sup_{\vartheta\in\Theta} u\left(c+h-B^\delta+\int_0^T \vartheta_t\, dS_t\right)\\
&\leq \sup_{\vartheta\in\Theta} u\left(c+h-B^{\delta_0}+\int_0^T \vartheta_t\, dS_t\right). \qquad (2.8)
\end{aligned}$$

The first inequality is valid as in the second term, for every $\delta \in \mathcal{D}$, we can choose $\widetilde{\vartheta} = \sum_{i=1}^k \mathbf{1}(\delta = i, t > \tau)\, {}^i\vartheta_t \in \Theta$, by Assumption 4.2.4. The second inequality holds since, again by Assumption 4.2.4,

$$\vartheta, \widetilde{\vartheta} \in \Theta \Longrightarrow \mathbf{1}(t \leq \tau)\vartheta + \mathbf{1}(t > \tau)\widetilde{\vartheta} \in \Theta.$$

On the other hand, we have by the definitions of h and h_{δ_0}

$$\begin{aligned}
&\sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c+h-B^\delta+\int_0^T \vartheta_t^\delta\, dS_t\right)\\
&= \sup_{\vartheta\in\Theta} u\left(c+\int_0^T \vartheta_t\, dS_t\right)\\
&= \sup_{\vartheta\in\Theta} u\left(c+h_{\delta_0}-B^{\delta_0}+\int_0^T \vartheta_t\, dS_t\right) \qquad (2.9)
\end{aligned}$$

for all $\delta_0 \in \mathcal{D}$. Putting (2.8) and (2.9) together, we get

$$\begin{aligned}
&\sup_{\vartheta\in\Theta} u\left(c+h-B^{\delta_0}+\int_0^T \vartheta_t\, dS_t\right)\\
&\geq \sup_{\vartheta\in\Theta} u\left(c+h_{\delta_0}-B^{\delta_0}+\int_0^T \vartheta_t\, dS_t\right) \quad \forall\delta_0 \in \mathcal{D},
\end{aligned}$$

and, due to monotonicity of u and uniqueness of h_{δ_0}, this implies $h \geq h_{\delta_0}$ for all $\delta_0 \in \mathcal{D}$ and therefore the assertion. $\square$

Theorem 4.2.9. *Assume that for every $\delta \in \mathcal{D}$ there exists a unique utility-indifference premium h_δ for the claim B^δ, i.e. h_δ solves equation (1.1) with $B = B^\delta$, and let h be a unique solution of (2.3). If one of the following conditions holds:*

(i) there is no financial market, i.e. $\Theta = \{0\}$,

(ii) the financial market is complete, i.e. there is a unique equivalent martingale measure Q, $\Theta = \Theta_2$, and every $B_i \in L^1(\Omega, \mathcal{F}, Q)$, or

(iii) u is the expected exponential utility function, i.e.

$$u(X) = E_P\left[-\exp\left(-\alpha X\right)\right], \tag{2.10}$$

for some risk aversion parameter $\alpha > 0$, and $|B_i - B_j| \in L^\infty(\Omega, \mathcal{F}, P)$,

then we have $h = \sup_{\delta \in \mathcal{D}} h_\delta$.

Proof. (i) obvious.

(ii) Define

$$\delta_{\max} = \arg\max_{i=1,\dots,k} \left\{E_Q\left[B_i \mid \mathcal{F}_\tau\right]\right\}, \tag{2.11}$$

using arbitrary versions of the conditional expectations. It is evident that $\delta_{\max}$ is $\mathcal{F}_\tau$-measurable. Theorem 4.2.7 implies that $h_{\delta_{\max}}$ according to (1.1) is the unique no-arbitrage price for the attainable claim $B^{\delta_{\max}}$, i.e. $h_{\delta_{\max}} = E_Q\left(B^{\delta_{\max}}\right)$. Due to completeness (cf. e.g. Jacka (Jac92)) and the optional stopping theorem there exists a permissible strategy $\widehat{\vartheta}$ such that

$$E_Q\left[B^{\delta_{\max}} \mid \mathcal{F}_\tau\right] = h_{\delta_{\max}} + \int_0^\tau \widehat{\vartheta}_t \, dS_t \quad P\text{-a.s.}$$

Furthermore, there are permissible strategies ${}^i\widehat{\vartheta}$ such that

$$B_i = E_Q\left[B_i \mid \mathcal{F}_\tau\right] + \int_\tau^T {}^i\widehat{\vartheta}_t \, dS_t \quad P\text{-a.s.,} \qquad i = 1, \dots, k.$$

Therefore, starting with initial capital $h_{\delta_{\max}}$ and choosing the compound strategy $\widehat{\vartheta}^\delta$, according to (2.4), we can superhedge all claims $(B^\delta)_{\delta \in \mathcal{D}}$:

$$h_{\delta_{\max}} + \int_0^T \widehat{\vartheta}_t^\delta \, dS_t = B^\delta + E_Q\left[B^{\delta_{\max}} \mid \mathcal{F}_\tau\right] - E_Q\left[B^\delta \mid \mathcal{F}_\tau\right] \geq B^\delta \quad P\text{-a.s.}$$

So, $h_{\delta_{\max}}$ is the unique no-arbitrage price of the general claim $(B^\delta)_{\delta\in\mathcal{D}}$. Due to Theorem 4.2.7, this implies $h = h_{\delta_{\max}}$.

(iii) We set

$$\delta_{\max} = \arg \max_{i=1,\dots,k} \left\{ \operatorname{ess\,inf}_{\vartheta\in\Theta} E_P \left[\exp\left(\alpha \left(B_i - \int_\tau^T \vartheta_t \, dS_t \right) \right) |\mathcal{F}_\tau \right] \right\}, \tag{2.12}$$

using arbitrary versions of the essential infima. By Assumption 4.2.4, the first supremum in (2.13) below can be split into two parts. Then, as $h_{\delta_{\max}}$ exists and $|B_i - B_j| \in L^\infty(\Omega, \mathcal{F}, P)$, Lemma A3.1 can be applied. The last equality is the assertion of Lemma A3.2:

$$\begin{aligned}
&\sup_{\vartheta\in\Theta} u\left(c + h - B^{\delta_{\max}} + \int_0^T \vartheta_t \, dS_t \right) \\
&= \sup_{\vartheta\in\Theta} \sup_{\widetilde{\vartheta}\in\Theta} u\left(c + h - B^{\delta_{\max}} + \int_0^\tau \vartheta_t \, dS_t + \int_\tau^T \widetilde{\vartheta}_t \, dS_t \right) \\
&= \sup_{\vartheta\in\Theta} \inf_{\delta\in\mathcal{D}} \sup_{\widetilde{\vartheta}\in\Theta} u\left(c + h - B^{\delta} + \int_0^\tau \vartheta_t \, dS_t + \int_\tau^T \widetilde{\vartheta}_t \, dS_t \right) \\
&= \sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c + h - B^{\delta} + \int_0^T \vartheta_t^\delta \, dS_t \right).
\end{aligned} \tag{2.13}$$

Altogether, we obtain $h = h_{\delta_{\max}}$. $\square$

Remark 4.2.10. We have some kind of minimax-principle:

$$\begin{aligned}
&\sup_{\vartheta\in\Theta} \inf_{\delta\in\mathcal{D}} u\left(c + h - B^{\delta} + \int_0^T \vartheta_t \, dS_t \right) \\
\leq\; &\sup_{(\vartheta, {}^1\vartheta, \dots, {}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c + h - B^{\delta} + \int_0^T \vartheta_t^\delta \, dS_t \right) \\
\leq\; &\sup_{\vartheta\in\Theta} \inf_{\delta\in\mathcal{D}} \sup_{\widetilde{\vartheta}\in\Theta} u\left(c + h - B^{\delta} + \int_0^\tau \vartheta_t \, dS_t + \int_\tau^T \widetilde{\vartheta}_t \, dS_t \right) \\
\leq\; &\inf_{\delta\in\mathcal{D}} \sup_{\vartheta\in\Theta} u\left(c + h - B^{\delta} + \int_0^T \vartheta_t \, dS_t \right).
\end{aligned}$$

By Theorem 4.2.9, in the case of exponential utility, the last two inequalities are equalities. The first inequality, however, can still be strict.

Proposition 4.2.11. *Let $U \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ be a utility function with $U' > 0$, $U'' < 0$, and $u(\cdot) = E_P[U(\cdot)]$. If U is not of the form $U(c) = a_0 - a_1 \exp\{-\alpha c\}$, $a_0 \in \mathbb{R}$, $a_1, \alpha > 0$, then there exists a general claim $(B^\delta)_{\delta\in\mathcal{D}}$ such that $h > \sup_{\delta\in\mathcal{D}} h_\delta$.*

Remark 4.2.12. (a) Proposition 4.2.11 has a nice economical interpretation: for each utility function U with varying risk aversion $r(c) := -U''(c)/U'(c)$, one can construct a counterexample with $h > \sup_{\delta\in\mathcal{D}}$. In this case there need not exist a least favorable decision function $\delta_{\max}$, as in (2.12), which does not depend on her wealth $c + h + \int_0^\tau \vartheta_t \, dS_t$ at time τ and therefore on her strategy ϑ **until** time τ. We will illustrate this in Example 4.2.13.

(b) This is an interesting analogy to the assertion in Gerber (Ger79), p. 77, concerning premium calculation principles which is also caused by the (non)constancy of the risk aversion of the utility function: "A principle of zero utility is iterative, if and only if it is an exponential principle or the net premium principle."

Proof. To construct a counterexample it is sufficient to look at a simple discrete two-period binomial model. She has initial capital c. There are a riskless bond identical 1, a tradeable risky asset with $S_0 = 1$ and for some $s^u \in \mathbb{R}$

$$S_2 = S_1 = \begin{cases} s^u & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

(so trading in the second period can be ignored), and another random variable Y, *stochastically independent* of S, with

$$Y = \begin{cases} y_0 & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

for some $y_0 \in \mathbb{R}$. At time $\tau \equiv 1$, having the information S_1, the holder can choose between *two* payoffs at time 2, a constant payoff $B_1 \equiv x_0 \in \mathbb{R}$ and $B_2 = Y$. As S_1 can take two different values, there are four possible decision functions $\mathcal{D} = \{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\}$ (where δ^{ij} means that he decides for B_i if $S_1 = 3$ and for B_j if $S_1 = 0$). We have four free parameters, namely c, s^u, y_0, and x_0 to construct a counterexample.

For each $(x, c) \in \mathbb{R}_+ \times \mathbb{R}$ let $y = y(x, c)$ be the unique solution of

$$U(c - x) = \frac{1}{2}\left[U(c) + U(c - y)\right], \quad y \in [x, 2x]. \tag{2.14}$$

It is given by

$$y(x, c) = c - U^{-1}(2U(c - x) - U(c)),$$

and for the partial derivative with respect to c we have

$$y_c(x,c) = \frac{U'(c) + U'(c - y(x,c)) - 2U'(c-x)}{U'(c - y(x,c))}. \tag{2.15}$$

Taking in equation (2.14) the first and second partial derivative with respect to x, respectively, and then setting $x = 0$, one obtains (note $y(0,c) = 0$)

$$y_x(0,c) = 2 \quad \text{and} \quad y_{xx}(0,c) = \frac{2U''(c)}{U'(c)} = -2r(c), \quad \forall c \in \mathbb{R}.$$

By the Taylor expansion

$$y(x,c) = 2x - r(c)x^2 + x^2 \int_0^1 \lambda[y_{xx}((1-\lambda)x, c) - y_{xx}(0,c)]\, d\lambda,$$

and due to $U \in \mathcal{C}^3(\mathbb{R},\mathbb{R})$, we obtain

$$y_c(x,c) = (-r'(c) + o(1))x^2, \quad x \to 0, \tag{2.16}$$

where the convergence holds uniformly on compacta in c. As U is neither linear nor of exponential type, we have $r' \not\equiv 0$. Thus, there exist some $c_0 \in \mathbb{R}$ and $\varepsilon > 0$ s.t. w.l.o.g.

$$r'(c) < 0, \quad \forall c \in [c_0 - \varepsilon, c_0 + \varepsilon] \tag{2.17}$$

and therefore, due to (2.16) and the continuity of r', there exists $x_0 > 0$ arbitrary small s.t.

$$y_c(x_0, c) > 0 \quad \forall c \in [c_0 - \varepsilon, c_0 + \varepsilon]. \tag{2.18}$$

We want to lead this to a contradiction to $h = \sup_{\delta \in \mathcal{D}} h_\delta$ or, equivalently, to

$$\sup_{\vartheta \in \mathbb{R}} \inf_{\delta^{ij}} E_P \left[U\left(c_0 + \vartheta(S_1 - S_0) - B^{\delta^{ij}} \right) \right] = \inf_{\delta^{ij}} \sup_{\vartheta \in \mathbb{R}} E_P \left[U\left(c_0 + \vartheta(S_1 - S_0) - B^{\delta^{ij}} \right) \right] \tag{2.19}$$

First of all, choose $c = c_0$ and $s^u > 2$, but close to 2 (i.e. $E_P[S_1 - S_0] \gtrapprox 0$) such that for fixed $\delta \in \{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\}$ the maximization problem possesses a maximizer ϑ^{ij} (which is then, due to the strict concavity of U, unique). ϑ^{ij} should be positive but not too big, more precisely,

$$c_0 - \varepsilon < c_0 - \vartheta^{ij} < c_0 + \vartheta^{ij}(s^u - 1) < c_0 + \varepsilon \quad i,j = 1,2.$$

This is possible as $U''(c_0) < 0$, $E_P[S_1 - S_0] > 0$, and x_0 resp. $y_0 \in (x_0, 2x_0)$ are arbitrarily small. Then, choose $y_0 \in (x_0, 2x_0)$ such that

$$E_P[U(c_0 + \vartheta^{12}(S_1 - S_0) - B^{12})] = E_P[U(c_0 + \vartheta^{11}(S_1 - S_0) - B^{11})] =: V.$$

This implies that $y_0 \geq y(x_0, c_0 - \vartheta^{11})$ but $y_0 \leq y(x_0, c_0 - \vartheta^{12})$. As $y(x_0, \cdot)$ is increasing we obtain $(0 <)\vartheta^{12} \leq \vartheta^{11}$ and $y_0 < \min\{y(x_0, c_0 + \vartheta^{11}(s^u - 1)), y(x_0, c_0 + \vartheta^{12}(s^u - 1))\}$. We arrive at

$$\begin{aligned} V &< \min\left\{E_P[U(c_0 + \vartheta^{12}(S_1 - S_0) - B^{22})], E_P[U(c_0 + \vartheta^{11}(S_1 - S_0) - B^{21})]\right\} \\ &\leq \min\left\{E_P[U(c_0 + \vartheta^{22}(S_1 - S_0) - B^{22})], E_P[U(c_0 + \vartheta^{21}(S_1 - S_0) - B^{21})]\right\}, \end{aligned}$$

i.e. δ^{11} and δ^{12} are indeed the least favorable decision functions.

To disprove (2.19) it is enough to show that $\vartheta^{12} \neq \vartheta^{11}$ (as maximizers are unique). Assume that $\vartheta^{12} = \vartheta^{11} =: \vartheta^{\mathrm{opt}}$. This implies $y_0 = y(x_0, c_0 - \vartheta^{\mathrm{opt}})$. On the one hand we have

$$E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{12})] - E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{11})] = o(\vartheta - \vartheta^{\mathrm{opt}}), \quad \vartheta \to \vartheta^{\mathrm{opt}},$$

as both expectations take their maximum in ϑ^{opt}. On the other hand we have by (2.15) and (2.18)

$$\begin{aligned} &E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{12})] - E_P[U(c_0 + \vartheta(S_1 - S_0) - B^{11})] \\ &= \frac{1}{4}U(c_0 - \vartheta) + \frac{1}{4}U(c_0 - \vartheta - y(x_0, c_0 - \vartheta^{\mathrm{opt}})) - \frac{1}{2}U(c_0 - \vartheta - x_0) \\ &= \Bigg[\frac{1}{4}U'(c_0 - \vartheta^{\mathrm{opt}}) + \frac{1}{4}U'(c_0 - \vartheta^{\mathrm{opt}} - y(x_0, c_0 - \vartheta^{\mathrm{opt}})) \\ &\quad - \frac{1}{2}U'(c_0 - \vartheta^{\mathrm{opt}} - x_0) + o(1)\Bigg] (\vartheta^{\mathrm{opt}} - \vartheta) \\ &= \frac{1}{4}\left[\underbrace{y_c(x_0, c_0 - \vartheta^{\mathrm{opt}})U'(c_0 - \vartheta^{\mathrm{opt}} - y(x_0, c_0 - \vartheta^{\mathrm{opt}}))}_{>0} + o(1)\right] (\vartheta^{\mathrm{opt}} - \vartheta), \quad \vartheta \to \vartheta^{\mathrm{opt}}. \end{aligned}$$

But this is a contradiction. □

Example 4.2.13. We want to illustrate the different situations for exponential utility in contrast to other utility functions. Therefore we take the example above with $U = \log$. She has initial capital $c_0 = 4$ and $s^u = 3$, $x_0 = 1$. The optimal amount ϑ of assets she has to buy at time 0 depends on δ but she must find a joint strategy ϑ. By choosing y_0 in such a way that the optimal utilities for δ^{11} and δ^{12} are the same and smaller than the utilities for δ^{21}, δ^{22}. That is the case for $y \approx 1.7518$ (cf. Figure 4.1). Then $h_{\delta^{11}} = h_{\delta^{12}} = 1$. But as $\vartheta^{11} \neq \vartheta^{12}$ there exists no **joint** ϑ which brings in at least that utility. Therefore $h > \sup_{\delta \in \mathcal{D}} h_\delta = 1$. In Figure 4.2 the same situation is plotted, but with exponential utility. Here, there is always a least favorable decison function $\delta_{\max}$ (as defined in (2.12), in the example it is δ^{22}). It brings in the smallest expected utility for all strategies ϑ.

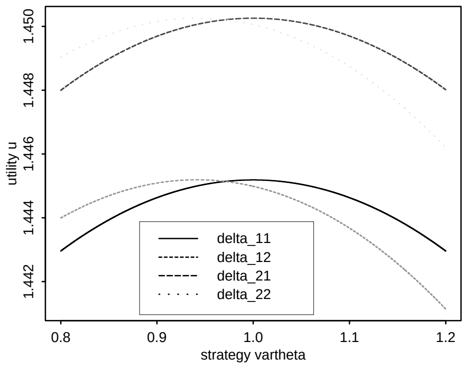

Figure 4.1: The writer's expected logarithmic utility as a function of her strategy, plotted for the four different random payoffs B^{δ}.

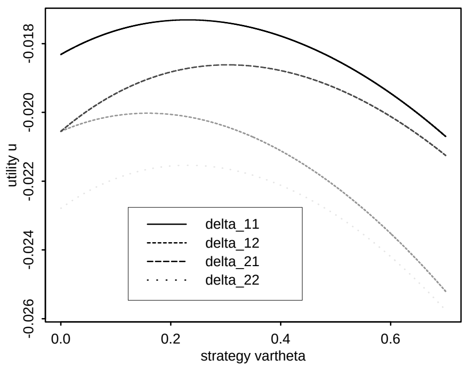

Figure 4.2: Same situation as in Figure 4.1, but with expected exponential utility ($\alpha = 1$). $B^{\delta^{22}}$ is least favorable for *every* strategy $\vartheta \in \mathbb{R}$ (cf. (A.1)).

4.3 American style contingent claims

An American contingent claim is a financial instrument modeled by an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted process $(f_t)_{t\in[0,T]}$. If he exercises the claim at time t he gets a payoff with *discounted* value f_t. We assume a *deterministic* riskless interest rate $(r_t)_{t\in[0,T]}$. Then, it makes no difference whether he is paid off at time t or at time T with interest. So, in our model we can assume that the paying off takes place at time T (but of course the amount is known at the exercise time t). Here, the claim can only be exercised at $0 = t_0 < t_1 < \ldots < t_k = T$.

Definition 4.3.1. *Let $\mathcal{S}$ be the set of stopping times resp. $(\mathcal{F}_t)_{t\in[0,T]}$ with values in $\{t_0 \ldots , t_k\}$.*

In the framework of *complete financial markets* American contingent claims have - due to a superhedging opportunity (for a proof in continuous time cf. Karatzas (Kar88)) - a unique no-arbitrage price. In the presence of constraints on portfolios, Karatzas and Kou (KK98) give *intervals* of no-arbitrage prices.

Example 4.3.2 (Surrender option in unit-linked life insurance contract). Consider a pure endowment life insurance contract that is linked to an equity index $(S_t^{(1)})_{t\in[0,T]}$. At time T the amount $\max\{S_T^{(1)}, K\}$ is paid contingent on survival of the policy-holder. Let T_1 be the remaining lifetime of the insured at time 0. Then $f_T = \mathbf{1}(T_1 > T)\max\{S_T^{(1)}, K\} \times e^{-\int_0^T r_s\,ds}$. But, the policy holder has the right to terminate the contract at $t_1, t_2, \ldots , t_{k-1}$. Then he gets a payoff depending on $S_{t_i}^{(1)}$ and t_i that is predefined in the contract. So, $f_t = \mathbf{1}(T_1 > t)g(S_t^{(1)}, t)e^{-\int_0^t r_s\,ds}$. Notice that, if the policy-holder dies before the payoff time τ, his following decision would be irrelevant as the payoff is then always 0.

Grosen and Jørgensen (GJ97) describe the practical importance of this example and price such contracts in the context of complete financial markets, not considering mortality risk (as in Example 4.3.2).

Analogous to Definition 2.3 we define :

Definition 4.3.3. *We call h a "still fair premium" if*

$$\sup_{(\vartheta,{}^0\vartheta,\ldots,{}^{k-1}\vartheta)\in\Theta^{k+1}} \inf_{\tau\in\mathcal{S}} u\left(c + h - f_\tau + \int_0^T \vartheta_t^\tau\, dS_t\right) = \sup_{\vartheta\in\Theta} u\left(c + \int_0^T \vartheta_t\, dS_t\right), \qquad (3.1)$$

where

$$\vartheta_t^\tau(\omega) := \begin{cases} \vartheta_t(\omega) & : \quad t \le \tau(\omega), \\ {}^i\vartheta_t(\omega) & : \quad t > \tau(\omega) \quad \textit{and} \quad \tau(\omega) = t_i, \end{cases} \tag{3.2}$$

and $\tau \in \mathcal{S}$ is his stopping time.

The interpretation is as follows : until the exercise time $\tau(\omega)$ she does only know that $\tau(\omega) > t$ and therefore she has to choose a strategy ϑ independently of τ that comes into effect till $\tau(\omega)$. From $\tau(\omega)$ on she can choose a strategy depending on $\tau(\omega)$ (the information she has). Define for $0 = t_0 < t_1 < \cdots < t_k = T$ recursively :

$$\tau_{\max}(t_k) = t_k, \tag{3.3}$$

$$\tau_{\max}(t_{i-1}) := \begin{cases} t_{i-1} & : \quad \omega \in A_{i-1}, \\ \tau_{\max}(t_i) & : \quad \text{otherwise}, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} A_{i-1} \quad = \quad & \Big\{ e^{\alpha f_{t_{i-1}}} \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp\left(-\alpha \int_{t_{i-1}}^T \vartheta_t \, dS_t \right) | \mathcal{F}_{t_{i-1}} \right] \\ & \ge \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left[\exp\left(\alpha \left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t \, dS_t \right) \right) | \mathcal{F}_{t_{i-1}} \right] \Big\} . \end{aligned}$$

Theorem 4.3.4. *Assume that for every $\tau \in \mathcal{S}$ there exists a unique utility-indifference premium h^τ for the claim f_τ, i.e. h^τ solves equation (1.1) with $B = f_\tau$, and let h solve (3.1). Then we have*

(a) $h \ge \sup_{\tau \in \mathcal{S}} h^\tau$ and

(b) if in addition one of the conditions (i), (ii), (iii) of Theorem 4.2.9 with f_{t_i} instead of B_i is satisfied and f is bounded, we have $h = \sup_{\tau \in \mathcal{S}} h^\tau$.

Proof. Part (a) is analog to the proof of Theorem 4.2.8. In part (b), case (i) is evident and (ii) is standard (cf. e.g. Elliott and Kopp (EK99)). So, we restrict ourselves to case (iii).

Recall the proof of Theorem 4.2.9(iii): there was only one decision time τ and – roughly speaking – it was based on the fact that *at time* τ the claim $B^{\delta_{\max}}$ is less favorable for her than each other B^δ, in the sense of (A.1). Now, the payoff $f_{\tau_{\max}(t_0)}$ is least favorable, but necessarily only at t_0. But as until τ a joint strategy ϑ comes into effect, we cannot argue as simple as in the proof of Theorem 4.2.9. But we know that at t_1 $f_{\tau_{\max}(t_1)}$ is less favorable than all f_τ with $\tau \geq t_1$, etc. (cf. Lemma A3.3). So, we can argue successively till $t_k = T$:

For every $\varepsilon > 0$, we take strategies $\vartheta^{(1)}, \ldots, \vartheta^{(k)}$ that are "ε-optimal" for $f_{\tau_{\max}(t_1)}, \ldots,$ $f_{\tau_{\max}(t_k)}$ and let them come into effect (depending on $\tau \in \mathcal{S}$) on the stochastic intervals

$$(0, \tau \wedge t_1], (\tau \wedge t_1, \tau \wedge t_2], \ldots (\tau \wedge t_{k-1}, \tau],$$

where $(a, a] := \varnothing$. Each time the approximation error is smaller than ε, uniformly in $\tau \in \mathcal{S}$ (cf. Lemma A3.6). With that and by applying Lemma A3.3, we obtain

$$\begin{aligned}
& \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - f_{\tau_{\max}(t_0)} + \int_0^T \widetilde{\vartheta}_t \, dS_t\right) \\
\stackrel{L.\ A3.3}{\leq} & \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - \left[\mathbf{1}(\tau = t_0) f_\tau + \mathbf{1}(\tau > t_0) f_{\tau_{\max}(t_1)}\right] + \int_0^T \widetilde{\vartheta}_t \, dS_t\right) \\
\stackrel{L.\ A3.6}{\leq} & \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - \left[\mathbf{1}(\tau = t_0) f_\tau + \mathbf{1}(\tau > t_0) f_{\tau_{\max}(t_1)}\right] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} \, dS_t \right. \\
& \left. + \int_{\tau \wedge t_1}^T \widetilde{\vartheta}_t \, dS_t\right) + \varepsilon \\
\stackrel{L.\ A3.3}{\leq} & \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - \left[\mathbf{1}(\tau \leq t_1) f_\tau + \mathbf{1}(\tau > t_1) f_{\tau_{\max}(t_2)}\right] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} \, dS_t \right. \\
& \left. + \int_{\tau \wedge t_1}^T \widetilde{\vartheta}_t \, dS_t\right) + \varepsilon \\
\stackrel{L.\ A3.6}{\leq} & \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - \left[\mathbf{1}(\tau \leq t_1) f_\tau + \mathbf{1}(\tau > t_1) f_{\tau_{\max}(t_2)}\right] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} \, dS_t \right. \\
& \left. + \int_{\tau \wedge t_1}^{\tau \wedge t_2} \vartheta_t^{(2)} \, dS_t + \int_{\tau \wedge t_2}^T \widetilde{\vartheta}_t \, dS_t\right) + 2\varepsilon \\
& \vdots \\
\leq & \sup_{\widetilde{\vartheta} \in \Theta} u\left(c + h - f_\tau + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)} \, dS_t + \int_{\tau \wedge t_1}^{\tau \wedge t_2} \vartheta_t^{(2)} \, dS_t \right. \\
& \left. + \ldots + \int_{\tau \wedge t_{k-1}}^{\tau} \vartheta_t^{(k)} \, dS_t + \int_\tau^T \widetilde{\vartheta}_t \, dS_t\right) + k\varepsilon.
\end{aligned}$$

By setting

$$\widehat{\vartheta}_t = \vartheta_t^{(i)}, \qquad \text{if } t_{i-1} < t \leq t_i,$$

and taking the infimum over all $\tau \in \mathcal{S}$, we get

$$\begin{aligned}
&\sup_{\vartheta\in\Theta} u\left(c+h-f_{\tau_{\max}(t_0)}+\int_0^T \vartheta_t\,dS_t\right)\\
&\le \inf_{\tau\in\mathcal{S}}\sup_{\widetilde{\vartheta}\in\Theta} u\left(c+h-f_\tau+\int_0^\tau \widehat{\vartheta}_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)+k\varepsilon\\
&\le \sup_{\vartheta\in\Theta}\inf_{\tau\in\mathcal{S}}\sup_{\widetilde{\vartheta}\in\Theta} u\left(c+h-f_\tau+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)+k\varepsilon.
\end{aligned}$$

As ε can be choosen arbitrary small, this implies

$$\begin{aligned}
&\sup_{\vartheta\in\Theta} u\left(c+h-f_{\tau_{\max}(t_0)}+\int_0^T \vartheta_t\,dS_t\right)\\
&\le \sup_{\vartheta\in\Theta}\inf_{\tau\in\mathcal{S}}\sup_{\widetilde{\vartheta}\in\Theta} u\left(c+h-f_\tau+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right).
\end{aligned} \tag{3.5}$$

Putting (3.5) and Lemma A3.4 together yields

$$\begin{aligned}
&\sup_{\vartheta\in\Theta} u\left(c+h-f_{\tau_{\max}(t_0)}+\int_0^T \vartheta_t\,dS_t\right)\\
&\le \sup_{\vartheta\in\Theta}\sup_{\widetilde{\vartheta}\in\Theta}\inf_{\tau\in\mathcal{S}} u\left(c+h-f_\tau+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\\
&\le \sup_{(\vartheta,{}^0\vartheta,\dots,{}^{k-1}\vartheta)\in\Theta^{k+1}}\inf_{\tau\in\mathcal{S}} u\left(c+h-f_\tau+\int_0^T \vartheta_t^\tau\,dS_t\right),
\end{aligned}$$

and therefore, due to monotonicity of u and uniqueness of $h^{\tau_{\max}(t_0)}$, $h \le h^{\tau_{\max}(t_0)}$. This completes the proof of Theorem 4.3.4. □

Remark 4.3.5. (a) If the denominator in (3.6) below does not vanish with positive probability, we can recursively define the "still fair conditional time t_i premium" X_{t_i} of the *American contingent claim* :

$$X_{t_k} = f_{t_k},$$

$$X_{t_{i-1}} = \max\left\{ f_{t_{i-1}}, \frac{1}{\alpha}\ln\frac{\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(X_{t_i}-\int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]}{\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(-\alpha\int_{t_{i-1}}^T \vartheta_t\,dS_t\right)|\mathcal{F}_{t_{i-1}}\right]}\right\}, \tag{3.6}$$

for $i = 1,\dots,k$. If the financial market is complete $(X_{t_i})_{i=0,\dots,k}$ coincides with the Snell envelope of the discrete process $(f_{t_i})_{i=0,\dots,k}$.

(b) The denominator in (3.6) can vanish even if we assume the existence of an equivalent (local) martingale measure, let S be locally bounded, and take the rather small set of strategies Θ_1, defined in (2.5). This shows the example in Lemma 3.8 of Schachermayer (Sch01b). But, if we have in addition an equivalent local martingal measure with finite relative entropy it follows from Theorem 1 in Delbaen et al. (DGR$^+$02) that the denominator in (3.6) is P-a.s. positive.

4.4 Conclusions

In this chapter, we have studied the following questions: can the possibility for the holder to choose have a value in itself and how can the writer of a claim hedge simultaneously against different risks related to different decision functions of the holder.

It turns out that in the case of exponential utility the utility-indifference premium which covers the claim related to the least favorable decision function is sufficient for all other decision functions the holder could hypothetically take.

An important application is a unit-linked life insurance contract that can be terminated by the policy-holder (cf. Example 4.3.2). Working with exponential utility, it would be reasonable to define the payoff, the holder gets if he terminates the contract, as the current conditional premium for the final payoff. Then, the optimal hedging strategies for all possible stopping times coincide until the termination time.

Chapter 5

Game Contingent Claims in Complete and Incomplete Markets

This chapter is an adapted version of Kühn (Küh01).

5.1 Introduction to game contingent claims

A game contingent claim (GCC) is a contract between a seller A and a buyer B which enables A to terminate it and B to exercise it at any time $t \in \{t_0, \ldots, t_k\}$ up to a maturity date $T = t_k$ when the contract is terminated anyway.

More precisely, let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness, and let $(U_{t_i})_{i=0,\ldots,k}$, $(L_{t_i})_{i=0,\ldots,k}$, $(M_{t_i})_{i=0,\ldots,k}$ be sequences of real-valued random variables adapted to $(\mathcal{F}_{t_i})_{i=0,\ldots,k}$ with $L_{t_i} \le M_{t_i} \le U_{t_i}$ for $i = 0, \ldots, k-1$ and $L_{t_k} = M_{t_k} = U_{t_k}$. If A terminates the contract at time t_i before B exercises then A should pay B the amount U_{t_i}. The other way around, A should pay B only L_{t_i}. If A terminates and B exercises at the same time, then A pays B the amount M_{t_i}.

Definition 5.1.1. *Let $\mathcal{S}_i$, $i = 0, \ldots, k$, be the sets of all stopping times resp. $(\mathcal{F}_t)_{t\in[0,T]}$ with values in $\{t_i, \ldots, t_k\}$.*

The above contract can be formulated as follows. If A selects a cancellation time $\sigma \in \mathcal{S}_0$

and B selects an exercise time $\tau \in \mathcal{S}_0$, then A pledges to pay B at time $\sigma \wedge \tau$ the amount

$$R(\sigma, \tau) = U_\sigma I(\sigma < \tau) + L_\tau I(\tau < \sigma) + M_\tau I(\tau = \sigma).$$

The frictionless financial market consists of d risky assets whose discounted price processes are modeled by the $\mathbb{R}^d$-valued semimartingale S and one riskless asset with discounted price process equal to 1. We denote by Θ a suitable space of admissible trading strategies to be specified later.

Example 5.1.2 (Israeli call option). *An American style call option with strike price K where also the seller can terminate the contract, but at the expense of a penalty $\delta_{t_i} \geq 0$, i.e. $L_{t_i} = (S^{(1)}_{t_i} - K)^+$, $U_{t_i} = (S^{(1)}_{t_i} - K)^+ + \delta_{t_i}$, and $M_{t_i} = (S^{(1)}_{t_i} - K)^+ + \delta_{t_i}/2$.*

Such a game version of an American option is safer for an investment company which issues it, and so it can be sold cheaper than the corresponding American option. As pointed out in Kifer (Kif00), essentially any contract in modern life presumes explicitly or implicitly a cancellation option by each side which then has to pay some penalty, and so it is natural to introduce a buyback option to contingent claims, as well. An example which has already been traded on real markets is a Liquid Yield Option Note (LYON). It is discussed in McConnell and Schwarz (MS86) - on a rather heuristical level without indicating a connection to a Dynkin game.

In a complete market (i.e. L, M, U are replicable by trading in S) one can solve our problem without letting enter the agents' preferences: A just wants to minimize $E_Q\left(R(\sigma, \tau)\right)$ whereas B wants to maximize the same expression (Q is the unique equivalent martingale measure). Thus, we have a zero-sum Dynkin stopping game. It is well-known that such a game has a *unique* value, cf. Ohtsubo (Oht86). Kifer (Kif00) shows by hedging-arguments that this value is also the *unique* no-arbitrage price of the GCC. In other words, the expectation of the (discounted) payoff under the unique equivalent martingale measure is the variable to be maximized resp. minimized, and this ensures consistency with the principle of no-arbitrage. Consequently, one has to solve a classical Dynkin game.

In incomplete markets this argument fails because there is more than one equivalent martingale measure. It is possible to superhedge the claim and get an interval of no-arbitrage prices, but then the feature of a stochastic game gets lost.

We suggest a utility maximization approach that takes trading possibilities **explicitly** into consideration. This approach is very popular for valuating European style contingent claims in the context of incomplete markets; see e.g. Hodges and Neuberger (HN89), Delbaen et al. (DGR$^+$02), or Davis (Dav97). For American style contingent claims see Davis and Zariphopoulou (DZ95).

Let $u_1, u_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be nondecreasing and concave; they are the utility functions of the seller resp. the buyer. Each "player" chooses a stopping time $\sigma \in \mathcal{S}_0$ (resp. $\tau \in \mathcal{S}_0$) and a trading strategy $\vartheta \in \Theta$, whose i-th component ϑ_t^i, $i = 1, \dots, d$, represents the number of shares of asset i held in the portfolio at time $t \in [0, T]$. The seller wants to maximize

$$E_P \left(u_1 \left(C_1 - R(\sigma, \tau) + \int_0^T \vartheta_t \, dS_t \right) \right), \tag{1.1}$$

while the buyer wants to maximize

$$E_P \left(u_2 \left(C_2 + R(\sigma, \tau) + \int_0^T \vartheta_t \, dS_t \right) \right). \tag{1.2}$$

So, the agents are solely interested in *terminal* wealth. The rv $C_i \in \mathcal{F}_T$ ($i = 1, 2$) is the exogenous endowment of the i-th player. This randomness especially makes sense for the buyer, who perhaps buys the claim to hedge against another risk in his portfolio.

In the whole chapter, the space Θ of admissible trading strategies has to satisfy the

Assumption 5.1.3. *All elements of Θ are $(\mathcal{F}_t)$-predictable and S-integrable. Θ is linear, and for all $t_i \in \{t_1, \dots, t_{k-1}\}$, $A \in \mathcal{F}_{t_i}$ the following implication is valid:*

If $\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(3)} \in \Theta$, then the compound strategy

$$\vartheta_t := \begin{cases} \vartheta_t^{(1)} & : \quad t \le t_i, \\ \vartheta_t^{(2)} & : \quad t > t_i \quad \textit{and} \quad \omega \in A, \\ \vartheta_t^{(3)} & : \quad t > t_i \quad \textit{and} \quad \omega \notin A. \end{cases} \tag{1.3}$$

is also an element of Θ.

The latter is essential as it allows a successive optimization, first over all stategies $(\vartheta_t)_{t \in (t_i, T]}$ (fixing one strategy $(\vartheta_t)_{t \in (0, t_i]}$), and then over all $(\vartheta_t)_{t \in (0, t_i]}$. So, it is a quite natural assumption. But unfortunately, it is not as harmless as it looks like. For example, the set of all predictable trading strategies such that the discounted gain process $\int_0^t \vartheta_u \, dS_u$ is bounded from below (but not necessarily from above) does obviously not satisfy Assumption 5.1.3.

A permissible choice of Θ is for example

$$\Theta_1 = \left\{ \vartheta \in L(S) \,\middle|\, \int_0^t \vartheta_u \, dS_u \text{ is bounded uniformly in } t \text{ and } \omega \right\}. \tag{1.4}$$

Θ_1 is rather small, but in Delbaen et al. (DGR$^+$02) resp. Kabanov and Stricker (KS02a) it is shown for exponential utility that under the assumption that S is locally bounded and admits an equivalent local martingale measure with finite entropy the maximization problems (1.1) and (1.2) with $\Theta = \Theta_1$ have the same values as for much bigger Θ. Another permissible choice is

$$\Theta_2 = \Bigg\{ \vartheta \in L(S) \mid \quad \int_0^t \vartheta_u \, dS_u \text{ is a martingale w.r.t. a special set } \mathcal{P} \text{ of absolutely} \\ \text{continuous local martingale measures} \Bigg\}.$$

Remark 5.1.4. Analogously to Kühn (Küh02), one can define from the seller's point of view a "still fair premium" for the GCC which coincides with the unique no-arbitrage price if the market is complete. But the main aim of this chapter is not to determine a "premium" or "price" for the claim, but rather to describe the "game", defined above, that takes place after the premium has been paid till maturity - and compare the situations of complete and incomplete markets.

Definition 5.1.5. *We say that a pair* $(\sigma^*, \tau^*) \in \mathcal{S}_0 \times \mathcal{S}_0$ *is a Nash (or a non-cooperative) equilibrium point, if for all* $(\sigma, \tau) \in \mathcal{S}_0 \times \mathcal{S}_0$

$$\sup_{\vartheta \in \Theta} E_P \left(u_1 \left(C_1 - R(\sigma^*, \tau^*) + \int_0^T \vartheta_t \, dS_t \right) \right) \geq \sup_{\vartheta \in \Theta} E_P \left(u_1 \left(C_1 - R(\sigma, \tau^*) + \int_0^T \vartheta_t \, dS_t \right) \right),$$

and

$$\sup_{\vartheta \in \Theta} E_P \left(u_2 \left(C_2 + R(\sigma^*, \tau^*) + \int_0^T \vartheta_t \, dS_t \right) \right) \geq \sup_{\vartheta \in \Theta} E_P \left(u_2 \left(C_2 + R(\sigma^*, \tau) + \int_0^T \vartheta_t \, dS_t \right) \right).$$

Remark 5.1.6. To simplify the notation and to stress the point that the interdependence between the agents' decisions only takes place through the stopping times and not through the trading strategies, we have not explicitly taken the chosen trading strategies into the definition of a Nash equilibrium. But of course, the outcome would be the same.

Without a financial market, i.e. $\Theta = \{0\}$, we have a nonzero-sum extension of a Dynkin game. This has been thoroughly investigated by many authors, firstly and independently of each other by Ohtsubo (Oht87) and Morimoto (Mor86) for a discrete time space. Their results can be directly transfered to our model (1.1)/(1.2), when $\Theta = \{0\}$, and ensure the existence of equilibrium points. Nevertheless, the existence of a financial market makes things more complicated.

5.2 The case of exponential utility

In this section, we assume that both seller and buyer have an *exponential utility function*, i.e.

$$u_1(x) = 1 - e^{-\alpha_1 x}, \tag{2.5}$$

$$u_2(x) = 1 - e^{-\alpha_2 x}, \tag{2.6}$$

for some *risk aversion parameters* $\alpha_1, \alpha_2 > 0$. Now, we define stopping times $(\sigma_0, \tau_0) \in \mathcal{S}_0 \times \mathcal{S}_0$ that will turn out to be equilibrium points.

Define, for $0 = t_0 < t_1 < \cdots < t_k = T$ recursively (in reverse order of time):

$$\sigma_k = t_k, \ \ \tau_k = t_k, \tag{2.7}$$

$$\sigma_{i-1} := \begin{cases} t_{i-1} & : \ \omega \in A_{i-1}, \\ \sigma_i & : \ \text{otherwise}, \end{cases} \tag{2.8}$$

$$\tau_{i-1} := \begin{cases} t_{i-1} & : \ \omega \in B_{i-1}, \\ \tau_i & : \ \text{otherwise}, \end{cases} \tag{2.9}$$

where A_{i-1} and B_{i-1} have to satisfy

$$\begin{aligned} A_{i-1} = \Bigg\{ & e^{\alpha_1 U_{t_{i-1}}} \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \Big| \mathcal{F}_{t_{i-1}} \right) \\ & \leq \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_1 \left(C_1 - R(\sigma_i, \tau_i) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \Big| \mathcal{F}_{t_{i-1}} \right) \Bigg\} \setminus B_{i-1}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} B_{i-1} = \Bigg\{ & e^{-\alpha_2 L_{t_{i-1}}} \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_2 \left(C_2 + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \Big| \mathcal{F}_{t_{i-1}} \right) \\ & \leq \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P \left(e^{-\alpha_2 \left(C_2 + R(\sigma_i, \tau_i) + \int_{(t_{i-1},T]} \vartheta_t \, dS_t \right)} \Big| \mathcal{F}_{t_{i-1}} \right) \Bigg\} \setminus A_{i-1}. \end{aligned} \tag{2.11}$$

Remark 5.2.1. We have $A_{i-1} \cap B_{i-1} = \varnothing$ (i.e. the players never stop at the same time) and the system (2.10)/(2.11) has at least one solution.

Remark 5.2.2. Due to $L_{t_{i-1}} \leq U_{t_{i-1}}$, for the seller it would be better that the buyer would stop the game as if he did it himself (and vice versa). This tends to result in a negative attitude towards stopping.

Theorem 5.2.3. *Let u_1, u_2 be the exponential utility functions (2.5) resp. (2.6), $L_{t_i}, U_{t_i} \in L^\infty(\Omega, \mathcal{F}, P)$, $i = 0, \dots, k$, and*

$$E_P\left(u_1\left(C_1 + \int_0^T \vartheta_t^{(1)}\, dS_t\right)\right) > -\infty, \tag{2.12}$$

resp.

$$E_P\left(u_2\left(C_2 + \int_0^T \vartheta_t^{(2)}\, dS_t\right)\right) > -\infty, \tag{2.13}$$

for some strategies $\vartheta^{(1)}, \vartheta^{(2)} \in \Theta$. Then, each pair $(\sigma_0, \tau_0) \in \mathcal{S}_0 \times \mathcal{S}_0$ satisfying (2.7)-(2.11) is a Nash equilibrium in the sense of Definition 5.1.5.

Proof. Let $(\sigma_i)_{i=0,\dots,k}$ and $(\tau_i)_{i=0,\dots,k}$ satisfy (2.7)-(2.11). To proof the optimality of σ_0 (for τ_0 the argumentation is analogous and therefore omitted) it is sufficient to show that for all $i = 0, \dots, k$ and $\sigma \in \mathcal{S}_i$ P-a.s.

$$\operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma_i,\tau_i) + \int_{(t_i,T]} \vartheta_t\, dS_t\right)} |\mathcal{F}_{t_i}\right) \leq \operatorname*{ess\,inf}_{\vartheta \in \Theta} E_P\left(e^{-\alpha_1\left(C_1 - R(\sigma,\tau_i) + \int_{(t_i,T]} \vartheta_t\, dS_t\right)} |\mathcal{F}_{t_i}\right).$$

This is done by backward induction: for $i = k$ we have $\sigma = t_k = \sigma_k$. $i \rightsquigarrow i-1$: for all $A \in \mathcal{F}_{t_{i-1}}$

we have by definition of τ_{i-1} and σ_{i-1}

$$\begin{aligned}
&\int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_{i-1},\tau_{i-1})+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP\\
&= \int_{A\cap\{\tau_{i-1}=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-L_{i-1}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP\\
&+ \int_{A\cap\{\tau_{i-1}>t_{i-1}\}} \min\left\{\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-U_{t_{i-1}}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right),\right.\\
&\left.\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\right\}\,dP\\
&\leq \int_{A\cap\{\tau_{i-1}=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-L_{t_{i-1}}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP \qquad (2.14)\\
&+ \int_{A\cap\{\tau_{i-1}>t_{i-1}\}\cap\{\sigma=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-U_{i-1}+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP\\
&+ \int_{A\cap\{\tau_{i-1}>t_{i-1}\}\cap\{\sigma>t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP.
\end{aligned}$$

Furthermore, due to (2.12) and as $R(\sigma_i,\tau_i)$ is bounded, we can apply Theorem A4.2 and obtain

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) \qquad (2.15)\\
&= \operatorname*{ess\,inf}_{\vartheta\in\Theta'} E_P\left[e^{-\alpha_1\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t}\operatorname*{ess\,inf}_{\tilde\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_i,\tau_i)+\int_{(t_i,T]}\tilde\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_i}\right)\,\middle|\mathcal{F}_{t_{i-1}}\right]
\end{aligned}$$

P-a.s., where

$$\Theta' = \left\{\vartheta\in\Theta\,\middle|\,E_P\left(e^{-\alpha_1\left(C_1+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)<\infty\quad P\text{-a.s.}\right\}.$$

We can now apply the induction assumption for $\sigma'=\sigma\vee t_i\in\mathcal{S}_i$ to the last expression in (2.14). Then, we again make use of (2.15) for σ' instead of σ_i. Finally, we obtain as $L_{t_{i-1}}\leq M_{t_{i-1}}$ that

$$\begin{aligned}
&\int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma_{i-1},\tau_{i-1})+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP\\
&\leq \int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha_1\left(C_1-R(\sigma,\tau_{i-1})+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right)\,dP.
\end{aligned}$$

□

Remark 5.2.4. We want to construct an example for which no Nash equilibrium exists. We take *logarithmic* utility functions, i.e. $u_i = \log$ (i=1,2), and a discrete two-period binomial model. There are a riskless bond with value identical to 1, a tradeable risky asset with $S_0 = 1$ and

$$S_2 = S_1 = \begin{cases} 3 & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

(so trading in the second period can be ignored and the trading strategy consists of the number $\vartheta \in \mathbb{R}$ of risky assets held in the first period), and another random source H, *stochastically independent* of S, with

$$H = \begin{cases} 1.7522 & : \text{ with probability } 1/2 \\ 0 & : \text{ with probability } 1/2 \end{cases}$$

$U_2 = L_2 = H$ is the final payoff. If A cancels at time 1 before B he has to pay a constant amount $U_1 = 1$ and vice versa B gets the smaller constant payoff $L_1 = 0.9$ (stopping at time 0 is excluded by prohibitive disadvantageous payoffs). A has initial capital $c_1 = 5$ whereas B has the random endowment $c_2 = 10.692 - H$.

At time 1, having the information S_1, both players can decide whether to stop or not. As S_1 can take two different values, each player can choose between four possible stopping times, symbolized by $\{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\}$ resp. $\{\varepsilon^{11}, \varepsilon^{12}, \varepsilon^{21}, \varepsilon^{22}\}$ (where "ij" means: stopping at time i if $S_1 = 3$ and at time j if $S_1 = 0$).

The example is constructed in such a way that no stopping-strategy δ^{ij}, $i, j = 1, 2$ can be part of an equilibrium: given δ^{ij}, there are uniquely determined optimal responses $\varepsilon^{i'j'}$ and $\delta^{i''j''}$. And, we always have $\delta^{ij} \neq \delta^{i''j''}$, indeed:

$$\delta^{11} \rightsquigarrow \varepsilon^{22} \rightsquigarrow \delta^{21}, \quad \delta^{12} \rightsquigarrow \varepsilon^{22} \rightsquigarrow \delta^{21}, \quad \delta^{21} \rightsquigarrow \varepsilon^{12} \rightsquigarrow \delta^{22}, \quad \delta^{22} \rightsquigarrow \varepsilon^{22} \rightsquigarrow \delta^{21}.$$

Remark 5.2.5. *Why does Theorem 5.2.3 fail in Remark 5.2.4 ?*

The exponential utility function has for every initial capital $x \in \mathbb{R}$ the same *risk aversion* $\alpha = -u''(x)/u'(x)$. Therefore, for each player there exists - given the "state of the world" at time 1 (here: $S_1 = 3$ resp. $S_1 = 0$) and the chosen stopping decision of the other player - an optimal stopping decision that is *independent* of the capital $\vartheta(S_1 - S_0)$ gained until 1, and thus independent of his trading strategy $\vartheta \in \mathbb{R}$. As a consequence, the optimal stopping decision for one “state of the world” does not depend on things that happen on other “states of the world”. That is in contrast to other utility functions: due to the varying risk aversion the interdependence arises through the choice of ϑ.

To construct a Nash equilibrium for exponential utility let (for example) the seller determine his optimal cancellation strategy assuming that the buyer never stops. Then, on the set A_1 where the seller cancels the optimal responding buyer does not terminate (as $M_1 \leq U_1$). Here the seller’s hypothesis is self-fulfilling. On the set $\Omega \setminus A_1$ where the seller does not cancel the optimal responding buyer can terminate (cross the seller’s hypothesis), but as $M_1 \geq L_1$ this does not motivate the seller to change his initial strategy and to stop on this set, as well. As for the exponential utility the optimal decision for one “state of the world” does not depend on things that happen on other “states of the world”, this “state-wise” argumentation is valid. Therefore, the seller need not change his stopping strategy *at all* and we have an equilibrium. For over utility function this “state-wise” argumentation fails and the seller could change his stopping-stategy on another state where his hypothesis was actually right. This is visible in Remark 5.2.4:

$$\varepsilon^{22} \rightsquigarrow \delta^{21} \rightsquigarrow \varepsilon^{12} \rightsquigarrow \delta^{22}.$$

5.3 The case of a complete market

If the financial market is complete, i.e. there exists a unique equivalent martingale measure Q, we get for general utility functions a result similar to Theorem 5.2.3. In addition, the values of the game for seller and buyer are unique. So, we have a similar property as in a zero-sum stopping game.

We can define a corresponding zero-sum stopping game which has the unique value V_0

$$V_0 = \inf_{\sigma \in \mathcal{S}_0} \sup_{\tau \in \mathcal{S}_0} E_Q \left(R(\sigma, \tau) \right) = \sup_{\tau \in \mathcal{S}_0} \inf_{\sigma \in \mathcal{S}_0} E_Q \left(R(\sigma, \tau) \right). \tag{3.1}$$

Analogously to Kifer (Kif00), it turns out that $(\sigma_0, \tau_0) \in \mathcal{S}_0 \times \mathcal{S}_0$, defined as in (2.7)-(2.11), but taking

$$A_{i-1} = \left\{ U_{t_{i-1}} \leq E_Q \left(R(\sigma_i, \tau_i) \, \middle| \mathcal{F}_{t_{i-1}} \right) \right\}, \tag{3.2}$$

and

$$B_{i-1} = \left\{ L_{t_{i-1}} \geq E_Q \left(R(\sigma_i, \tau_i) \left| \mathcal{F}_{t_{i-1}} \right. \right) \right\}, \tag{3.3}$$

is a saddlepoint of (3.1).

Lemma 5.3.1. *Let $\Theta = \Theta_2$ with $\mathcal{P} = \{Q\}$, let u be a utility function, $H \in L^1(\Omega, \mathcal{F}, Q)$, and $C \in \mathcal{F}_T$, then we have*

$$\sup_{\vartheta \in \Theta} E_P \left(u \left(C + H + \int_0^T \vartheta_t \, dS_t \right) \right) = \sup_{\vartheta \in \Theta} E_P \left(u \left(C + E_Q \left(H \right) + \int_0^T \vartheta_t \, dS_t \right) \right).$$

Proof. Due to the completeness (cf. e.g. Jacka (Jac92)), H can be represented by a constant plus a stochastic integral, i.e. there exists a $\widehat{\vartheta} \in \Theta$ such that P-a.s.

$$H = E_Q \left(H \right) + \int_{(0,T]} \widehat{\vartheta}_t \, dS_t,$$

and due to the linearity of Θ, the mapping $\vartheta \mapsto \vartheta + \widehat{\vartheta}$ is a bijection of Θ into itself. □

Theorem 5.3.2. *Let $L_{t_i}, U_{t_i} \in L^1(\Omega, \mathcal{F}, Q)$, i=0, ... ,k, and $\Theta = \Theta_2$ with $\mathcal{P} = \{Q\}$. Then*

(i) the pair (σ_0, τ_0) according to (3.2)/(3.3) is a Nash equilibrium in the sense of Definition 5.1.5, and

(ii) if in addition

$$-\infty < \sup_{\vartheta \in \Theta} E_P \left(u_1 \left(C_1 - V_0 + \int_0^T \vartheta_t \, dS_t \right) \right) < u_1 \left(\infty \right), \tag{3.4}$$

and

$$-\infty < \sup_{\vartheta \in \Theta} E_P \left(u_2 \left(C_2 + V_0 + \int_0^T \vartheta_t \, dS_t \right) \right) < u_2 \left(\infty \right), \tag{3.5}$$

then all other Nash equilibria (σ^*, τ^*) have the same pair of values, i.e.

$$\sup_{\vartheta \in \Theta} E_P \left(u_1 \left(C_1 - R(\sigma^*, \tau^*) + \int_0^T \vartheta_t \, dS_t \right) \right) = \sup_{\vartheta \in \Theta} E_P \left(u_1 \left(C_1 - R(\sigma_0, \tau_0) + \int_0^T \vartheta_t \, dS_t \right) \right),$$

and

$$\sup_{\vartheta \in \Theta} E_P \left(u_2 \left(C_2 + R(\sigma^*, \tau^*) + \int_0^T \vartheta_t \, dS_t \right) \right) = \sup_{\vartheta \in \Theta} E_P \left(u_2 \left(C_2 + R(\sigma_0, \tau_0) + \int_0^T \vartheta_t \, dS_t \right) \right).$$

Proof. (i) follows immediately from the respective assertions for the zero-sum game (3.1) and Lemma 5.3.1. For (ii) one needs in addition the fact that the mappings

$$\widetilde{u}_i : \quad \mathbb{R} \longrightarrow \mathbb{R} \cup \{\pm\infty\}, \quad x \mapsto \sup_{\vartheta \in \Theta} E_P \left(u_i \left(C_i + x + \int_0^T \vartheta_t \, dS_t \right) \right), \quad i = 1, 2,$$

satisfy $\widetilde{u}_1(x) < \widetilde{u}_1(-V_0)$, for $x < -V_0$, resp. $\widetilde{u}_2(x) < \widetilde{u}_2(V_0)$, for $x < V_0$. So (σ^*, τ^*) is an equilibrium for (1.1)/(1.2) if and *only if* it is an equilibrium for (3.1).

This strict monotonicity can be derived as follows: the monotonicity and concavity of u_i imply the respective properties of $\widetilde{u}_i$ (for the latter implication one makes use of the fact that a convex combination of admissible strategies is again an admissible strategy). By $\widetilde{u}_1(-V_0) > -\infty$ resp. $\widetilde{u}_2(V_0) > -\infty$ and dominated convergence we conclude that $\widetilde{u}_i(\infty) = u_i(\infty)$. Therefore, (3.4) resp. (3.5) implies the required strict monotonicity. □

Remark 5.3.3. The uniqueness of the values is due to the fact that in the *complete* market there is never an incentive for *both* players to stop. Only if both A and B are indifferent, i.e. on $\{U_{t_{i-1}} = E_Q \left(R(\sigma_i, \tau_i) | \mathcal{F}_{t_{i-1}} \right) = L_{t_{i-1}}\}$ the behaviour can be different for different Nash equilibria, but that has no influence on the expected utility.

So, we have a characteristic of a zero-sum game. In a certain sense, this gives a different argument for Kifer's approach in (Kif00).

Appendix

A1 Some auxiliary results

Lemma A1.1. *Let $g : \mathbb{R}^n \to \mathbb{R}_+$ be semi-continuous from below, (component-wise) monotone, and bounded, and let F, H be distribution functions on $\mathbb{R}^n$. Denote by F_i and H_i, resp., the ith marginal distribution function of F and H, resp. Suppose that*

$$F_1(x_1) \geq H_1(x_1), \quad \forall x_1 \in \mathbb{R},$$

and for $i = 2, \dots, n$

$$F_i(x_i | x_1, \dots, x_{i-1}) \geq H_i(x_i | x_1, \dots, x_{i-1}), \quad \forall x_1, \dots, x_i \in \mathbb{R}.$$

Then

$$\int_{\mathbb{R}^n} g(x_1, \dots, x_n) F(dx_1, \dots, dx_n) \leq \int_{\mathbb{R}^n} g(x_1, \dots, x_n) H(dx_1, \dots, dx_n).$$

Proof. *Step 1:* For $n = 1$ we have due to the semi-continuity of g from below that $g(x_1) > t \Leftrightarrow x_1 > g^{\leftarrow}(t)$, where $g^{\leftarrow}$ is the *generalized inverse* of g defined as $g^{\leftarrow}(t) := \inf\{x \in \mathbb{R} \mid g(x) > t\}$. Therefore we obtain

$$\int_{\mathbb{R}} g(x_1) F(dx_1) = \int_{\mathbb{R}_+} (1 - F(g^{\leftarrow}(t)) dt,$$

and the assertion holds for $n = 1$.

Step 2: Rewrite the intregral over $\mathbb{R}^n$ as iterated integrals over $\mathbb{R}$ with respect to the conditional marginal distribution functions. Then, by using step 1, we can estimate iteratively:

$$
\begin{aligned}
&\int_{\mathbb{R}^n} g(x_1,\dots,x_n)F(dx_1,\dots,dx_n)\\
&= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} g(x_1,\dots,x_n)F_n(dx_n|x_1,\dots,x_{n-1})\dots F_1(dx_1)\\
&\overset{\text{step 1}}{\leq} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} g(x_1,\dots,x_n)H_n(dx_n|x_1,\dots,x_{n-1})\dots F_1(dx_1)\\
&\ \vdots\\
&\overset{\text{step 1}}{\leq} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} g(x_1,\dots,x_n)H_n(dx_n|x_1,\dots,x_{n-1})\dots H_1(dx_1)\\
&= \int_{\mathbb{R}^n} g(x_1,\dots,x_n)H(dx_1,\dots,dx_n).
\end{aligned}
$$

□

Lemma A1.2. *Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers with $\sum_{k=1}^{\infty} a_k = \infty$, $\varepsilon > 0$, and $(A_k)_{k\in\mathbb{N}}$ a sequence of events with $P(A_{k+1} \mid \sigma(A_1,\dots,A_k)) \geq \varepsilon$, P-a.s. for all $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k I(A_k) = \infty$, P-a.s.*

Proof. Define $X_n := \sum_{k=2}^{n} a_k I(A_k)$ and $\widetilde{X}_n := \sum_{k=2}^{n} a_k I(\widetilde{A}_k)$, where $(\widetilde{A}_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence with $P(\widetilde{A}_1) = \varepsilon$. By Kolmogorov's "three-series theorem", cf. e.g. Theorem IX.9.3 in Feller (Fel71), $\widetilde{X}_n \to \infty$ for $n \to \infty$, P-a.s.. Forthermore, we obtain for each $n \in \mathbb{N}$ and $s \in \mathbb{R}_+$ by Lemma A1.1, applied to $g(x_1,\dots,x_n) = I(\sum_{k=1}^{n} a_k x_k > s)$, that $P(\widetilde{X}_n > s) \leq P(X_n > s)$. Thus $X_n \to \infty$ in probability and due to monotonicity also almost-surely. □

A2 Some results from stochastic calculus

In this part of the appendix we state some auxiliary results from stochastic calculus. Firstly, we consider the σ-supermartingale property in terms of semimartingale characteristics. Secondly, we turn to the $\mathscr{H}^1$-norm of semimartingales.

Definition A2.1. *Let X be a $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν) relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. By JS, II.2.9 there exists some predictable process $A \in \mathscr{A}^+_{\text{loc}}$, some predictable $\mathbb{R}^{d\times d}$-valued process c, whose values are non-negative, symmetric matrices, and some transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that*

$$dB_t = b_t\, dA_t,\ \ dC_t = c_t\, dA_t,\ \ \nu(dt, dx) = F(t, dx)\, dA_t.$$

We call (b, c, F, A) differential characteristics *of* X.

Remark A2.2. Usually, in the stochastic analysis, a semimartingale is associated with a triplet of predictable characteristics (B, C, ν). However, for various purposes it seems reasonable to desintegrate the chosen characteristics, cf. Kabanov (Kab97).

One should observe that the differential characteristics are not unique: E.g. $(2b, 2c, 2F, \frac{1}{2}A)$ yields another version. Typical choices for A are $A_t := t$ (e.g. for Lévy processes, diffusions, Itô processes etc.) and $A_t := \sum_{s\leq t} 1_{\mathbb{N}\setminus\{0\}}(s)$ (discrete-time processes). Especially for $A_t = t$, one can interpret b_t or rather $b_t + \int(x - h(x))F_t(dx)$ as a drift rate, c_t as a diffusion coefficient, and F_t as a local jump measure. As the following result shows, a non-positive or vanishing drift corresponds to a σ-supermartingale or σ-martingale, respectively. These processes play an important role in the context of fundamental theorems of asset pricing (cf. Delbaen and Schachermayer (DS98), Kabanov (Kab97), Cherny and Shiryaev (CS01)). For background on σ-localization and the related classes of processes we refer the reader to Goll and Kallsen (GK01).

Lemma A2.3. *Let X be a semimartingale in $\mathbb{R}^d$ with differential characteristics (b, c, F, A). Fix $i \in \{1, \dots, d\}$. Then X^i is a σ-supermartingale iff $\int |x^i - h^i(x)|F(dx) < \infty$ and*

$$b^i + \int (x^i - h^i(x))F(dx) \leq 0 \quad (P \otimes A) - \textit{a.e.}$$

If we replace ≤ 0 with $= 0$ or ≥ 0, we obtain corresponding statements for σ-martingales and σ-submartingales, respectively.

Proof. We use the notation of Goll and Kallsen (GK01), Section 7 (henceforth GK).

$\Rightarrow$: This is shown in the first part of the proof of GK, Proposition 7.9.

$\Leftarrow$: From JS, II.2.29, II.2.13, I.3.10 it follows that X is a local supermartingale if we have, in addition, $\int |x^i - h^i(x)|F(dx) \in L(A)$, i.e. if $X \in \mathscr{D}_{\mathrm{loc}}$. Since $X \in \mathscr{D}_\sigma$ (cf. GK, Lemma 7.6), X belongs to the σ-localized class of the set of local supermartingales, which coincides with the set of σ-supermartingales (cf. GK, Lemma 7.4). $\square$

In the proof of Theorem 3.3.2 we make use of the $\mathscr{H}^1$-norm in the sense of Emery (Eme78), Protter (Pro77), (Pro78), (Pro92), Dellacherie and Meyer (DM82). Note that we treat the value

X_0 differently from e.g. Dellacherie and Meyer (DM82) because we use the conventions of JS as far as starting values of $[X, X]$, ΔX etc. are concerned.

Definition A2.4. *For any real-valued semimartingale X we define*

$$\begin{aligned} \|X\|_{\mathscr{H}^1} &:= \inf\Bigg\{E\Bigg(|X_0| + \mathrm{Var}(A)_\infty + \sqrt{[M,M]_\infty}\Bigg) : \\ &\qquad X = X_0 + M + A \text{ with } M \in \mathscr{M}_{\mathrm{loc}}, A \in \mathscr{V}\Bigg\}, \end{aligned}$$

where $\mathrm{Var}(A)$ denotes the variation process of A. By $\mathscr{H}^1$ we denote the set of all real-valued semimartingales X with $\|X\|_{\mathscr{H}^1} < \infty$.

Proposition A2.5. *Let X be a non-negative semimartingale. Then $1_{\{X_-=0\}} \cdot X \in \mathscr{V}^+$.*

Proof. This is shown by applying the Itô-Meyer formula to $X^- = -(X \wedge 0)$. Indeed, since $X^- = 0$, Jacod (Jac79), (5.49) yields that

$$0 = -\frac{1}{2}1_{\{X_-=0\}} \cdot X + \frac{1}{2}L^0 + \sum_{t \leq \cdot} \frac{1}{2}1_{\{X_-=0\}}\Delta X_t,$$

where L^0 denotes the local time of X in 0 in the sense of Jacod (Jac79), (5.47). Since L^0 is increasing and $\Delta X \geq 0$ on $\{X_- = 0\}$, it follows that $1_{\{X_-=0\}} \cdot X$ is increasing as well. □

Proposition A2.6. *Let L, X, U be real-valued semimartingales with $L \leq X \leq U$ and such that $1_{\{L_- < X_-\}} \cdot X$ is a σ-submartingale and $1_{\{X_- < U_-\}} \cdot X$ is a σ-supermartingale. Then*

$$\|X\|_{\mathscr{H}^1} \leq c(\|L\|_{\mathscr{H}^1} + \|U\|_{\mathscr{H}^1})$$

for some $c \in \mathbb{R}_+$ which is independent of L, X, U.

Proof. In this proof, we write $Y^*_\infty := \sup_{t \in \mathbb{R}_+} |Y_t|$ for any semimartingale Y and $\mathrm{Var}(Y)$ for the variation process of any $Y \in \mathscr{V}$.

Step 1: W.l.o.g. L, U are special because otherwise $\|L\|_{\mathscr{H}^1} = \infty$ or $\|U\|_{\mathscr{H}^1} = \infty$ (cf. JS, I.4.23). By Kallsen (Kal02), Proposition 3.7, X is special as well. Denote by $X = X_0 + M^X + A^X$, $U = U_0 + M^U + A^U$, $L = L_0 + M^L + A^L$ the canonical decompositions of the special semimartingales X, L, U into a local martingale and a process of finite variation, respectively.

Step 2: By JS, I.3.13, there exist predictable processes H^X, H^L, H^U such that $A^X = H^X \cdot A$, $A^L = H^L \cdot A$, $A^U = H^U \cdot A$, where $A := \mathrm{Var}(A^X) + \mathrm{Var}(A^L) + \mathrm{Var}(A^U) \in \mathscr{V}^+$ is predictable.

Since $1_{\{L_-<X_-\}} \cdot X = 1_{\{L_-<X_-\}} \cdot M^X + (1_{\{L_-<X_-\}} H^X) \cdot A$ is a σ-submartingale, we have that $H^X \geq 0$ $(P \otimes A)$-almost everywhere on $\{L_- < X_-\}$. Similarly, it follows that $H^X \leq 0$ $(P \otimes A)$-almost everywhere on $\{X_- < U_-\}$. Proposition A2.5 yields that

$$1_{\{L_-=X_-\}} \cdot (M^X - M^L) + (1_{\{L_-=X_-\}}(H^X - H^L)) \cdot A = 1_{\{L_-=X_-\}} \cdot (X - L) \in \mathscr{V}^+.$$

From JS, I.3.17 and the uniqueness of the special semimartingale decomposition it follows that $(1_{\{L_-=X_-\}}(H^X - H^L)) \cdot A \in \mathscr{V}^+$, which implies that $H^X \geq H^L$ $(P \otimes A)$-almost everywhere on $\{L_- = X_-\}$. Similarly, we have $H^X \leq H^U$ $(P \otimes A)$-almost everywhere on $\{X_- = U_-\}$. Altogether, it follows that $|H^X| \leq |H^L| + |H^U|$ $(P \otimes A)$-almost everywhere. Consequently, we have

$$\mathrm{Var}(A^X) = |H^X| \cdot A \leq |H^L| \cdot A + |H^U| \cdot A = \mathrm{Var}(A^L) + \mathrm{Var}(A^U).$$

Step 3: Since

$$M^X = X - A^X - X_0 \geq L - A^X - X_0 = L_0 - X_0 + A^L - A^X + M^L$$

and

$$M^X \leq U_0 - X_0 + A^U - A^X + M^U,$$

we have that

$$|M^X| \leq |L_0| + |U_0| + |A^L| + |A^U| + |A^X| + |M^L| + |M^U|$$

and hence by Step 2

$$M^{X,*}_\infty \leq |L_0| + |U_0| + 2\mathrm{Var}(A^L)_\infty + 2\mathrm{Var}(A^U)_\infty + M^{L,*}_\infty + M^{U,*}_\infty.$$

By the Burkhölder-Davis-Gundy inequality (cf. Jacod (Jac79), (2.34)), it follows that there exists some constant $c_1 \geq 2$ such that $E(M^{L,*}_\infty) \leq c_1 E(\sqrt{[M^L, M^L]_\infty})$ and likewise for U. By (DM82), VII.98, there exists some constant $c_2 \geq 1$ such that

$$E(|L_0| + \mathrm{Var}(A^L)_\infty + \sqrt{[M^L, M^L]_\infty}) \leq c_2 \|L\|_{\mathscr{H}^1}$$

and likewise for U. Together, it follows that $E(M^{X,*}_\infty) \leq c_1 c_2 (\|L\|_{\mathscr{H}^1} + \|U\|_{\mathscr{H}^1})$.

Step 4: From Step 2 we conclude that

$$|X_0| + \mathrm{Var}(A^X) \leq |L_0| + \mathrm{Var}(A^L) + |U_0| + \mathrm{Var}(A^U),$$

which implies that

$$E(|X_0| + \mathrm{Var}(A^X)_\infty) \leq c_2(\|L\|_{\mathscr{H}^1} + \|U\|_{\mathscr{H}^1}).$$

By the Burkhölder-Davis-Gundy inequality (cf. Jacod (Jac79), (2.34)), there exists some constant $c_3 \geq 1$ such that

$$E(\sqrt{[M^X, M^X]_\infty}) \leq c_3 E(M_\infty^{X,*}).$$

Altogether, it follows that

$$\|X\|_{\mathscr{H}^1} \leq E(|X_0| + \mathrm{Var}(A^X)_\infty + \sqrt{[M^X, M^X]_\infty}) \leq (c_2 + c_1 c_2 c_3)(\|L\|_{\mathscr{H}^1} + \|U\|_{\mathscr{H}^1}).$$

□

Proposition A2.7. *Let X be an adapted real-valued process and $(T_n)_{n\in\mathbb{N}}$ an increasing sequence of stopping times such that X^{T_n} is a semimartingale for any $n \in \mathbb{N}$. If we have $\sup_{n\in\mathbb{N}} \|X^{T_n}\|_{\mathscr{H}^1} < \infty$, then X^{T_∞} is a semimartingale, where $T_\infty := \sup_{n\in\mathbb{N}} T_n$.*

Proof. It is easy to see that $(X^{T_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{H}^1$. Due to completeness (cf. Dellacherie and Meyer (DM82), VII.98) there is a limit in $\mathscr{H}^1$ which coincides with X on the set $[0, T_\infty[$. □

A3 Auxiliary results for the proof of Theorems 4.2.9 and 4.3.4

Lemma A3.1. *Let Z be an $\mathcal{F}_\tau$-measurable random variable and u the expected exponential utility function, defined in Chapter 4 Equation (2.10). If $\sup_{\vartheta\in\Theta} u\left(Z - B_i + \int_\tau^T \vartheta_t\, dS_t\right) > -\infty$ for $i = 1, \ldots, k$ then we have for all $\delta \in \mathcal{D}$*

$$\sup_{\vartheta\in\Theta} u\left(Z - B^{\delta_{\max}} + \int_\tau^T \vartheta_t\, dS_t\right) \leq \sup_{\vartheta\in\Theta} u\left(Z - B^{\delta} + \int_\tau^T \vartheta_t\, dS_t\right), \tag{A.1}$$

where $\delta_{\max}$ is defined in Chapter 4 Equation (2.12).

Proof. Let us first show that infimum and integral can be interchanged, i.e.

$$\begin{aligned} &\inf_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(-Z + B^\delta - \int_\tau^T \vartheta_t\, dS_t\right)\right)\right] \\ &= E_P\left[e^{-\alpha Z} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left(\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t\, dS_t\right)\right) |\mathcal{F}_\tau\right)\right]. \end{aligned} \tag{A.2}$$

General properties of the essential infimum (cf. e.g. Karatzas and Shreve (KS98)) guarantee that there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of admissible strategies such that

$$\inf_{n\in\mathbb{N}} E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t^n\, dS_t\right)\right)|\mathcal{F}_\tau\right] = \operatorname{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t\, dS_t\right)\right)|\mathcal{F}_\tau\right].$$

For two strategies $\vartheta^{(1)}, \vartheta^{(2)} \in \Theta$ define

$$\vartheta_t^{(3)} = \begin{cases} \mathbf{1}(t>\tau)\vartheta_t^{(1)} & : \quad E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_s^{(1)}\, dS_s\right)\right)|\mathcal{F}_\tau\right] \\ & \quad\le E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_s^{(2)}\, dS_s\right)\right)|\mathcal{F}_\tau\right], \\ \mathbf{1}(t>\tau)\vartheta_t^{(2)} & : \quad \text{otherwise.} \end{cases}$$

We have $\vartheta^{(3)} \in \Theta$,

$$\begin{aligned} &E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t^{(3)}\, dS_t\right)\right)|\mathcal{F}_\tau\right] \\ &= \min\left\{E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t^{(1)}\, dS_t\right)\right)|\mathcal{F}_\tau\right], E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t^{(2)}\, dS_t\right)\right)|\mathcal{F}_\tau\right]\right\} \end{aligned}$$

and therefore inf-stability. Hence, there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}} \subset \Theta$ such that

$$\begin{aligned} &e^{-\alpha Z} E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t^n\, dS_t\right)\right)|\mathcal{F}_\tau\right] \\ &\searrow e^{-\alpha Z} \operatorname{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t\, dS_t\right)\right)|\mathcal{F}_\tau\right], \quad n \nearrow \infty, \end{aligned} \tag{A.3}$$

and the left-hand side is dominated by the integrable random variable $e^{-\alpha Z} E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t^1\, dS_t\right)\right)|\mathcal{F}_\tau\right]$. So, (A.2) holds due to the dominated convergence theorem. The result follows immediately from the definition of u and the fact that for all $\delta \in \mathcal{D}$

$$\operatorname{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(B^{\delta_{\max}} - \int_\tau^T \vartheta_t\, dS_t\right)\right)|\mathcal{F}_\tau\right] \ge \operatorname{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(B^\delta - \int_\tau^T \vartheta_t\, dS_t\right)\right)|\mathcal{F}_\tau\right].$$

$\square$

Lemma A3.2. *Let $U : \mathbb{R} \to \mathbb{R}$ be a monotone utility function and $u(\cdot) = E_P[U(\cdot)]$. If $\sup_{(\vartheta,{}^1\vartheta,\ldots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c+h-B^\delta+\int_0^T \vartheta_t^\delta\, dS_t\right) > -\infty$ then*

$$\begin{aligned} &\sup_{\vartheta\in\Theta} \inf_{\delta\in\mathcal{D}} \sup_{\widetilde{\vartheta}\in\Theta} u\left(c+h-B^\delta+\int_0^\tau \vartheta_t\, dS_t + \int_\tau^T \widetilde{\vartheta}_t\, dS_t\right) \\ &= \sup_{(\vartheta,{}^1\vartheta,\ldots,{}^k\vartheta)\in\Theta^{k+1}} \inf_{\delta\in\mathcal{D}} u\left(c+h-B^\delta+\int_0^T \vartheta_t^\delta\, dS_t\right), \end{aligned}$$

where ϑ^δ is defined in Chapter 4 Equation (2.4).

Proof. Let us show that

$$\begin{aligned}
&\sup_{\vartheta\in\Theta}\inf_{\delta\in\mathcal{D}}\sup_{\widetilde{\vartheta}\in\Theta} E_P\left[U\left(c+h-B^\delta+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right]\\
=\;&\sup_{\vartheta\in\Theta}\inf_{\delta\in\mathcal{D}} E_P\left[\operatorname*{ess\,sup}_{\widetilde{\vartheta}\in\Theta} E_P\left[U\left(c+h-B^\delta+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_\tau\right]\right]\\
=\;&\sup_{\vartheta\in\Theta}\inf_{\delta\in\mathcal{D}} E_P\left[\operatorname*{ess\,sup}_{\widetilde{\vartheta}\in\Theta}\sum_{i=1}^k \mathbf{1}(\delta=i)E_P\left[U\left(c+h-B_i+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_\tau\right]\right]\\
=\;&\sup_{\vartheta\in\Theta}\inf_{\delta\in\mathcal{D}} E_P\left[\sum_{i=1}^k \mathbf{1}(\delta=i)\operatorname*{ess\,sup}_{\widetilde{\vartheta}\in\Theta} E_P\left[U\left(c+h-B_i+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_\tau\right]\right]\\
=\;&\sup_{\vartheta\in\Theta} E_P\left[\min_{i=1,\dots,k}\operatorname*{ess\,sup}_{{}^i\vartheta\in\Theta} E_P\left[U\left(c+h-B_i+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_\tau\right]\right]\\
=\;&\sup_{\vartheta\in\Theta}\sup_{({}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^k} E_P\left[\min_{i=1,\dots,k} E_P\left[U\left(c+h-B_i+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T {}^i\vartheta_t\;dS_t\right)|\mathcal{F}_\tau\right]\right]\\
=\;&\sup_{(\vartheta,{}^1\vartheta,\dots,{}^k\vartheta)\in\Theta^{k+1}}\inf_{\delta\in\mathcal{D}} E_P\left[\sum_{i=1}^k \mathbf{1}(\delta=i)U\left(c+h-B_i+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T {}^i\vartheta_t\;dS_t\right)\right]
\end{aligned}\tag{A.4}$$

The first equality holds by a similar argument leading to (A.2), the third by Assumption 4.2.4. For the fourth equality, we use the fact that the infimum is attained in

$$\delta=\arg\min_{i=1,\dots,k}\left\{\operatorname*{ess\,sup}_{\widetilde{\vartheta}\in\Theta} E_P\left[U\left(c+h-B_i+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_\tau\right]\right\}.$$

The crucial fifth equality in (A.4) holds by the dominated convergence theorem and the argument leading to (A.3). In addition, we use the obvious fact that for sequences of random variables $(Z^i_n)_{n\in\mathbb{N}}$ if $Z^i_n\uparrow Z^i$ as $n\uparrow\infty$ for $i=1,\dots,k$ then also $\min\{Z^1_n,\dots,Z^k_n\}\uparrow\min\{Z^1,\dots,Z^k\}$ as $n\uparrow\infty$. □

Lemma A3.3. *Let $\widetilde{\tau}\in\mathcal{S}$, $A\in\mathcal{F}_{\widetilde{\tau}}$, Z be an $\mathcal{F}_{\widetilde{\tau}}$-measurable random variable and u the expected exponential utility function (2.10). If $\sup_{\vartheta\in\Theta} u\left(Z-f_i+\int_{\widetilde{\tau}}^T\vartheta_t\,dS_t\right)>-\infty$ for $i=0,\dots,k$, then we have for all $\tau\in\mathcal{S}$ with $\tau\geq\widetilde{\tau}$:*

$$\sup_{\vartheta\in\Theta} u\left(Z-f_{\tau_{\max}(\widetilde{\tau})}\mathbf{1}_A-f_\tau\mathbf{1}_{\Omega\setminus A}+\int_{\widetilde{\tau}}^T\vartheta_t\,dS_t\right)\leq\sup_{\vartheta\in\Theta} u\left(Z-f_\tau+\int_{\widetilde{\tau}}^T\vartheta_t\,dS_t\right),$$

where $\tau_{\max}$ is defined in Chapter 4, (3.3)/(3.4).

Proof. Using (A.2) it remains to show that for all $\tau \geq \widetilde{\tau}$

$$\begin{aligned}&\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(\widetilde{\tau})} - \int_{\widetilde{\tau}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{\widetilde{\tau}}\right]\\ &\geq \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau} - \int_{\widetilde{\tau}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{\widetilde{\tau}}\right]. \qquad (A.5)\end{aligned}$$

In addition, it is sufficient to proof (A.5) for deterministic stopping times $\widetilde{\tau} = t_i$. This is done by induction (in reverse order of time): for $i = k$ we have $\tau = t_k = \tau_{\max}(t_k)$. $i \rightsquigarrow i-1$: for all $A \in \mathcal{F}_{t_{i-1}}$ we have per definition of $\tau_{\max}(t_{i-1})$

$$\begin{aligned}&\int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(t_{i-1})} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]dP\\ &= \int_A \max\Bigg\{\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{t_{i-1}} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right],\\ &\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]\Bigg\}\,dP\\ &\geq \int_{A\cap\{\tau=t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{t_{i-1}} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]dP\\ &+ \int_{A\cap\{\tau>t_{i-1}\}} \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]dP. \qquad (A.6)\end{aligned}$$

Furthermore, as f is bounded, we can apply Theorem A4.2 and obtain

$$\begin{aligned}&\operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]\\ &= \operatorname*{ess\,inf}_{\vartheta\in\Theta'} E_P\Bigg[\exp\left(-\alpha\int_{t_{i-1}}^{t_i} \vartheta_t\,dS_t\right)\\ &\quad\times \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_i}^T \widetilde{\vartheta}_t\,dS_t\right)\right)|\mathcal{F}_{t_i}\right]|\mathcal{F}_{t_{i-1}}\Bigg] \quad P\text{-a.s.}, \qquad (A.7)\end{aligned}$$

where

$$\Theta' = \left\{\vartheta\in\Theta \,\middle|\, E_P\left[\exp\left(-\alpha\int_{t_{i-1}}^T \vartheta_t\,dS_t\right)|\mathcal{F}_{t_{i-1}}\right] < \infty \quad P\text{-a.s.}\right\}.$$

We can now apply the induction assumption for $\tau' = \tau \vee t_i$ to the last expression in (A.6). Then, we again make use of (A.7) for τ' instead of $\tau_{\max}(t_i)$ and obtain

$$\begin{aligned}&\int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau_{\max}(t_{i-1})} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]dP\\ &\geq \int_A \operatorname*{ess\,inf}_{\vartheta\in\Theta} E_P\left[\exp\left(\alpha\left(f_{\tau} - \int_{t_{i-1}}^T \vartheta_t\,dS_t\right)\right)|\mathcal{F}_{t_{i-1}}\right]dP.\end{aligned}$$

□

Lemma A3.4. *Let u be the expected exponential utility function defined in Chapter 4 Equation (2.10). If*

$$\sup_{\vartheta\in\Theta}\sup_{\widetilde{\vartheta}\in\Theta}\inf_{\tau\in\mathcal{S}} u\left(c+h-f_\tau+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)>-\infty,$$

then

$$\begin{aligned}&\sup_{\vartheta\in\Theta}\inf_{\tau\in\mathcal{S}}\sup_{\widetilde{\vartheta}\in\Theta} u\left(c+h-f_\tau+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\\ &=\sup_{\vartheta\in\Theta}\sup_{\widetilde{\vartheta}\in\Theta}\inf_{\tau\in\mathcal{S}} u\left(c+h-f_\tau+\int_0^\tau \vartheta_t\,dS_t+\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right).\end{aligned}$$

Proof. We have to show that

$$\begin{aligned}&\inf_{\vartheta\in\Theta}\sup_{\tau\in\mathcal{S}}\inf_{\widetilde{\vartheta}\in\Theta} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right)\right]\\ &=\inf_{\vartheta\in\Theta}\inf_{\widetilde{\vartheta}\in\Theta}\sup_{\tau\in\mathcal{S}} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right)\right].\end{aligned}$$

Therefore, it suffices to show that for every fixed $\vartheta\in\Theta$

$$\begin{aligned}&\sup_{\tau\in\mathcal{S}}\inf_{\widetilde{\vartheta}\in\Theta} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right)\right]\\ &=\inf_{\widetilde{\vartheta}\in\Theta}\sup_{\tau\in\mathcal{S}} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right)\right].\end{aligned}\tag{A.8}$$

Of course, the right-hand side is at least as big as the left-hand side. For the converse, we use the fact (cf. Lemma A3.5) that there exists a sequence of strategies $(\widetilde{\vartheta}^n)_{n\in\mathbb{N}}\subset\Theta$ such that *for all* $i=0,\dots,k-1$

$$E_P\left[\exp\left(-\alpha\int_{t_i}^T \widetilde{\vartheta}^n_t\,dS_t\right)|\mathcal{F}_{t_i}\right]\searrow \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left[\exp\left(-\alpha\int_{t_i}^T \widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_{t_i}\right].\tag{A.9}$$

With this special sequence we want to approximate the left-hand side of (A.8) from above:

$$\begin{aligned}
&\sup_{\tau\in\mathcal{S}} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}^n_t\,dS_t\right)\right)\right]\\
&-\sup_{\tau\in\mathcal{S}}\inf_{\widetilde{\vartheta}\in\Theta} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right)\right]\\
&\le \sup_{\tau\in\mathcal{S}}\left\{E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}^n_t\,dS_t\right)\right)\right]\right.\\
&\left.-\inf_{\widetilde{\vartheta}\in\Theta} E_P\left[\exp\left(\alpha\left(f_\tau-\int_0^\tau \vartheta_t\,dS_t-\int_\tau^T \widetilde{\vartheta}_t\,dS_t\right)\right)\right]\right\}\\
&=\sup_{\tau\in\mathcal{S}}\sum_{i=0}^k\left\{E_P\left[\mathbf{1}(\tau=t_i)\exp\left(\alpha\left(f_{t_i}-\int_0^{t_i}\vartheta_t\,dS_t-\int_{t_i}^T\widetilde{\vartheta}^n_t\,dS_t\right)\right)\right]\right.\\
&\left.-E_P\left[\mathbf{1}(\tau=t_i)\exp\left(\alpha\left(f_{t_i}-\int_0^{t_i}\vartheta_t\,dS_t\right)\right)\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta}E_P\left(\exp\left(-\alpha\int_{t_i}^T\widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_{t_i}\right)\right]\right\}\\
&=\sup_{\tau\in\mathcal{S}}\sum_{i=0}^k E_P\left[\mathbf{1}(\tau=t_i)\exp\left(\alpha\left(f_{t_i}-\int_0^{t_i}\vartheta_t\,dS_t\right)\right)\right.\\
&\left.\times\underbrace{\left\{E_P\left[\exp\left(-\alpha\int_{t_i}^T\widetilde{\vartheta}^n_t\,dS_t\right)|\mathcal{F}_{t_i}\right]-\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta}E_P\left[\exp\left(-\alpha\int_{t_i}^T\widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_{t_i}\right]\right\}}_{\ge 0\ P\text{-a.s.}}\right]\\
&\le\sum_{i=0}^k E_P\left[\exp\left(\alpha\left(f_{t_i}-\int_0^{t_i}\vartheta_t\,dS_t\right)\right)\right.\\
&\qquad\left.\times\left\{E_P\left[\exp\left(-\alpha\int_{t_i}^T\widetilde{\vartheta}^n_t\,dS_t\right)|\mathcal{F}_{t_i}\right]-\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta}E_P\left[\exp\left(-\alpha\int_{t_i}^T\widetilde{\vartheta}_t\,dS_t\right)|\mathcal{F}_{t_i}\right]\right\}\right].
\end{aligned}$$

Due to (A.9) and the dominated convergence theorem, the last term tends to zero as n tends to infinity. □

Lemma A3.5. *There exists a sequence of strategies* $(\vartheta^n)_{n\in\mathbb{N}}\subset\Theta$ *such that* for all $i=0,\dots,k-1$

$$E_P\left[\exp\left(-\alpha\int_{t_i}^T\vartheta^n_t\,dS_t\right)|\mathcal{F}_{t_i}\right]\searrow\operatorname*{ess\,inf}_{\vartheta\in\Theta}E_P\left[\exp\left(-\alpha\int_{t_i}^T\vartheta_t\,dS_t\right)|\mathcal{F}_{t_i}\right]\quad P\text{-a.s.}\tag{A.10}$$

Proof. General properties of the essential infimum guarantee that for each $i=0,\dots,k-1$ there exists a sequence $(\vartheta^{n,i})_{n\in\mathbb{N}}\subset\Theta$ satisfying (A.10). It remains to show that there is a *joint* sequence. We define such a joint sequence $(\widehat{\vartheta}^n)_{n\in\mathbb{N}}\subset\Theta$ recursively on the intervals

$(t_{k-1}, T], (t_{k-2}, t_{k-1}], \dots, (0, t_1]$: For $t \in (t_{k-1}, T]$ we set :

$$\widehat{\vartheta}_t^n = \vartheta_t^{n,\widehat{j}}, \quad \text{where } \widehat{j} = \arg \min_{j=0,\dots,k-1} \left\{ E_P \left[\exp\left(-\alpha \int_{t_{k-1}}^T \vartheta_t^{n,j}\, dS_t \right) |\mathcal{F}_{t_{k-1}} \right] \right\},$$

and for $t \in (t_{i-1}, t_i]$

$$\widehat{\vartheta}_t^n = \vartheta_t^{n,\widehat{j}}, \quad \text{where } \widehat{j} = \arg \min_{j=0,\dots,k-1} \left\{ E_P \left[\exp\left(-\alpha \left(\int_{t_{i-1}}^{t_i} \vartheta_t^{n,j}\, dS_t + \int_{t_i}^T \widehat{\vartheta}_t^n\, dS_t \right) \right) |\mathcal{F}_{t_{i-1}} \right] \right\}.$$

It is obvious that for all $n \in \mathbb{N}$, $i = 0, \dots, k-1$

$$E_P \left[\exp\left(-\alpha \int_{t_i}^T \widehat{\vartheta}_t^n\, dS_t \right) |\mathcal{F}_{t_i} \right] \le E_P \left[\exp\left(-\alpha \int_{t_i}^T \vartheta_t^{n,i}\, dS_t \right) |\mathcal{F}_{t_i} \right].$$

That implies the assertion. $\square$

Lemma A3.6. *Under the conditions of Theorem 4.3.4(b) case (iii), for every $\varepsilon > 0$ there exist $\vartheta^{(i)} \in \Theta$, $i = 1, \dots, k$ such that* for all $\tau \in \mathcal{S}$, $i = 1, \dots, k$

$$\begin{aligned}
& \sup_{\widetilde{\vartheta} \in \Theta} u\Bigg(c + h - \left[\mathbf{1}(\tau \le t_{i-1}) f_\tau + \mathbf{1}(\tau > t_{i-1}) f_{\tau_{\max}(t_i)} \right] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)}\, dS_t + \dots \\
& \quad + \int_{\tau \wedge t_{i-2}}^{\tau \wedge t_{i-1}} \vartheta_t^{(i-1)}\, dS_t + \int_{\tau \wedge t_{i-1}}^T \widetilde{\vartheta}_t\, dS_t \Bigg) \\
\le\ & \sup_{\widetilde{\vartheta} \in \Theta} u\Bigg(c + h - \left[\mathbf{1}(\tau \le t_{i-1}) f_\tau + \mathbf{1}(\tau > t_{i-1}) f_{\tau_{\max}(t_i)} \right] + \int_0^{\tau \wedge t_1} \vartheta_t^{(1)}\, dS_t + \dots \\
& \quad + \int_{\tau \wedge t_{i-2}}^{\tau \wedge t_{i-1}} \vartheta_t^{(i-1)}\, dS_t + \int_{\tau \wedge t_{i-1}}^{\tau \wedge t_i} \vartheta_t^{(i)}\, dS_t + \int_{\tau \wedge t_i}^T \widetilde{\vartheta}_t\, dS_t \Bigg) + \varepsilon. \qquad \text{(A.11)}
\end{aligned}$$

Proof. In the first expression, the supremum over all strategies $(\widetilde{\vartheta})_{t \in (\tau \wedge t_{i-1}, T]}$ can be split into two suprema : one over all $(\widetilde{\vartheta})_{t \in (\tau \wedge t_{i-1}, \tau \wedge t_i]}$ and the other over all $(\widetilde{\vartheta})_{t \in (\tau \wedge t_i, T]}$. So, it remains to show that $\vartheta^{(i)}$ in (A.11) can be chosen independent of $\tau \in \mathcal{S}$. By putting in the definition of u and interchanging infimum and expectation, one can see that the set $\{\tau \le t_{i-1}\}$ does not have any influence on the difference between the two suprema in (A.11) as $(\tau \wedge t_{i-1}, \tau \wedge t_i] = \varnothing$ on the set $\{\tau \le t_{i-1}\}$. Therefore, we only consider the set $\{\tau > t_{i-1}\}$:

$$\begin{aligned}
& \inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp\left\{\alpha\left(f_{\tau_{\max}(t_i)} - \ldots - \int_{t_{i-1}}^{\tau\wedge t_i} \vartheta_t^{(i)}\, dS_t - \int_{\tau\wedge t_i}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right\}\right] \\
& - \inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \ldots - \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)}\, dS_t - \int_{t_{i-1}}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right)\right] \\
\le \;& \inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \ldots - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)}\, dS_t - \int_{t_i}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right)\right] \\
& - \inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \ldots - \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)}\, dS_t - \int_{t_{i-1}}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right)\right] \\
= \;& E_P \left[\mathbf{1}(\tau > t_{i-1}) \exp\left(-\alpha\left(\int_0^{t_1} \vartheta_t^{(1)}\, dS_t + \ldots + \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)}\, dS_t\right)\right)\right. \\
& \times \left\{\operatorname{ess}\inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)}\, dS_t - \int_{t_i}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right) |\mathcal{F}_{t_{i-1}}\right]\right. \\
& \left.\left. - \operatorname{ess}\inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right) |\mathcal{F}_{t_{i-1}}\right]\right\}\right] \\
\le \;& E_P \left[\exp\left(-\alpha\left(\int_0^{t_1} \vartheta_t^{(1)}\, dS_t + \ldots + \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)}\, dS_t\right)\right)\right. \\
& \times \left\{\operatorname{ess}\inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)}\, dS_t - \int_{t_i}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right) |\mathcal{F}_{t_{i-1}}\right]\right. \\
& \left.\left. - \operatorname{ess}\inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_{t_{i-1}}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right) |\mathcal{F}_{t_{i-1}}\right]\right\}\right] \\
= \;& \inf_{\widetilde{\vartheta}\in\Theta} E_P \left[e^{\alpha\left(f_{\tau_{\max}(t_i)} - \int_0^{t_1} \vartheta_t^{(1)}\, dS_t - \ldots - \int_{t_{i-1}}^{t_i} \vartheta_t^{(i)}\, dS_t - \int_{t_i}^{T} \widetilde{\vartheta}_t\, dS_t\right)}\right] \\
& - \inf_{\vartheta\in\Theta} \inf_{\widetilde{\vartheta}\in\Theta} E_P \left[\exp\left(\alpha\left(f_{\tau_{\max}(t_i)} - \int_0^{t_1} \vartheta_t^{(1)}\, dS_t\right.\right.\right. \\
& \qquad\qquad \left.\left.\left. - \ldots - \int_{t_{i-2}}^{t_{i-1}} \vartheta_t^{(i-1)}\, dS_t - \int_{t_{i-1}}^{t_i} \vartheta_t\, dS_t - \int_{t_i}^{T} \widetilde{\vartheta}_t\, dS_t\right)\right)\right].
\end{aligned}$$

The last term does not depend on τ any more, and by suitable choice of $\vartheta^{(i)}$ it can be made arbitrary small. □

A4 Iterative application of the essential infimum

We want to give full details about the iterative application of the essential infimum in Equation (2.15) of Chapter 5 and in Equation (A.7) in the Appendix A3, resp.

Definition A4.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a nonempty family of ran-*

dom variables defined on $(\Omega, \mathcal{F}, P)$. *The* essential infimum *of* $\mathcal{X}$, *denoted by* $\operatorname{ess\,inf} \mathcal{X}$, *is a random variable* X^* *satisfying*

(i) $\forall X \in \mathcal{X}$, $X^* \leq X$ *P-a.s., and*

(ii) if Y *is a random variable satisfying* $Y \leq X$ *P-a.s. for all* $X \in \mathcal{X}$, *then* $Y \leq X^*$ *P-a.s.*

The essential infimum exists (for a proof see Gihman and Skorohod (GS79)) and is obviously unique P-a.s.

Theorem A4.2. *Let* C *be an* $\mathcal{F}_T$*-measurable random variable and assume that*

$$\Theta' := \left\{ \vartheta \in \Theta \,\middle|\, E_P\left(e^{-\alpha\left(C + \int_{(t_{i-1},T]} \vartheta_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_{i-1}} \right) < \infty \quad P\text{-a.s.} \right\} \neq \varnothing.$$

Then we have

$$\begin{aligned} &\operatorname{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_{i-1},T]} \vartheta_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_{i-1}} \right) && \text{(A.1)} \\ &= \operatorname{ess\,inf}_{\vartheta\in\Theta'} E_P\left[e^{-\alpha \int_{(t_{i-1},t_i]} \vartheta_t\, dS_t} \operatorname{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_i,T]} \widetilde{\vartheta}_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_i} \right) \,\middle|\mathcal{F}_{t_{i-1}} \right], \quad P\text{-a.s.} \end{aligned}$$

Proof. Due to Assumption 5.1.3 one can rewrite Θ as a product space consisting of strategies $\vartheta \in \Theta$ coming into effect on $(t_{i-1}, t_i]$ and strategies $\widetilde{\vartheta} \in \Theta$ coming into effect on $(t_i, T]$, i.e.

$$\begin{aligned} &\operatorname{ess\,inf}_{\vartheta\in\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_{i-1},T]} \vartheta_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_{i-1}} \right) && \text{(A.2)} \\ &= \operatorname{ess} \inf_{(\vartheta,\widetilde{\vartheta})\in\Theta\times\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_{i-1},t_i]} \vartheta_t\, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_{i-1}} \right) \quad P\text{-a.s.} \end{aligned}$$

Then, one can split the essential infimum over the product space into two essential infima (using the same arguments as for the infimum in $\mathbb{R}$):

$$\begin{aligned} &\operatorname{ess} \inf_{(\vartheta,\widetilde{\vartheta})\in\Theta\times\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_{i-1},t_i]} \vartheta_t\, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_{i-1}} \right) && \text{(A.3)} \\ &= \operatorname{ess\,inf}_{\vartheta\in\Theta} \operatorname{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_{i-1},t_i]} \vartheta_t\, dS_t + \int_{(t_i,T]} \widetilde{\vartheta}_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_{i-1}} \right) \quad P\text{-a.s.} \end{aligned}$$

For every fixed strategy $\widehat{\vartheta} \in \Theta$ we have of course that

$$E_P\left(e^{-\alpha\left(C + \int_{(t_i,T]} \widehat{\vartheta}_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_i} \right) \geq \operatorname{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C + \int_{(t_i,T]} \widetilde{\vartheta}_t\, dS_t\right)} \,\middle|\mathcal{F}_{t_i} \right)$$

P-a.s., and general properties of the essential infimum (cf. e.g. Gihman and Skorohod (GS79)) guarantee that the essential infimum can be approximated by a countable set of elements of Θ, i.e. there exists a sequence $(\widetilde{\vartheta}^{(n)})_{n\in\mathbb{N}} \subset \Theta$ s.t.

$$\begin{aligned}
&\inf_{n\in\mathbb{N}} E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(n)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right)\\
&= \operatorname{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right) \quad P\text{-a.s.},
\end{aligned}$$

where the $\inf$ is understood pointwise. For two strategies $\widetilde{\vartheta}^{(1)}, \widetilde{\vartheta}^{(2)} \in \Theta$ define

$$\widetilde{\vartheta}^{(3)}_t = \begin{cases} \mathbf{1}(t>t_i)\widetilde{\vartheta}^{(1)}_t & : \quad E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(1)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right) \\ & \qquad \le E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(2)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right), \\ \mathbf{1}(t>t_i)\widetilde{\vartheta}^{(2)}_t & : \quad \text{otherwise.} \end{cases}$$

Due to Assumption 5.1.3 we have $\widetilde{\vartheta}^{(3)} \in \Theta$, and in addition

$$\begin{aligned}
&E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(3)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right)\\
&= \min\left\{E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(1)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right), E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(2)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right)\right\},
\end{aligned}$$

and therefore inf-stability. Hence, there exists a sequence $(\widetilde{\vartheta}^n)_{n\in\mathbb{N}} \in \Theta$ such that

$$\begin{aligned}
&E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(n)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right) \qquad \text{(A.4)}\\
&\searrow \operatorname{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right) \quad P\text{-a.s.}, n\to\infty.
\end{aligned}$$

Take a $\vartheta \in \Theta'$. (A.4) implies

$$\begin{aligned}
&e^{-\alpha\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t}\min\left\{E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}^{(n)}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right),\right.\\
&\left.E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\vartheta_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right)\right\}\\
&\searrow e^{-\alpha\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t}\operatorname{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right) \quad P\text{-a.s.}, \qquad \text{(A.5)}
\end{aligned}$$

as $n\to\infty$. The sequence in (A.5) is dominated by the random variable

$$E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},T]}\vartheta_t\,dS_t\right)}\,|\mathcal{F}_{t_i}\right),$$

which has $P-a.s.$ finite $P\left(\bullet\,\middle|\mathcal{F}_{t_{i-1}}\right)$-expectation. So, we can apply the dominated convergence theorem for conditional expectations to (A.5). Then, we take the essential infimum over all $\vartheta \in \Theta'$ on both sides:

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta\in\Theta'} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) \\
&= \operatorname*{ess\,inf}_{\vartheta\in\Theta'} E_P\left[e^{-\alpha\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_i}\right)\,\middle|\mathcal{F}_{t_{i-1}}\right]
\end{aligned} \tag{A.6}$$

P-a.s. It remains to show that it makes no difference whether the essential infimum in the *first* expression of (A.6) is taken over all $\vartheta \in \Theta$ or only over all $\vartheta \in \Theta'$. Take at first an arbitrary $\vartheta \in \Theta$ and define

$$A = \left\{\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) < \infty\right\} \tag{A.7}$$

The essential infimum in (A.7) can be monotonously approximated by a sequence $(\widetilde{\vartheta}^{(n)})_{n\in\mathbb{N}} \subset \Theta$. That implies

$$A^{(n)} := \left\{E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t^{(n)}\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) < \infty\right\} \nearrow A \quad P\text{-a.s.},$$

as $n \to \infty$. Let $\widehat{\vartheta} \in \Theta' \neq \varnothing$ and define

$$\vartheta_t^{(n)} := \begin{cases} \vartheta_t & : \quad t \le t_i \quad \text{and} \quad \omega \in A^{(n)}, \\ \widetilde{\vartheta}_t^{(n)} & : \quad t > t_i \quad \text{and} \quad \omega \in A^{(n)}, \\ \widehat{\vartheta}_t & : \quad \text{otherwise.} \end{cases}$$

$\vartheta^{(n)}$ are by construction elements of Θ'. Furthermore, $A^{(n)} \cup (\Omega \setminus A) \nearrow \Omega$, P-a.s., as $n \to \infty$, and on $A^{(n)} \cup (\Omega \setminus A)$ we have

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t^{(n)}\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) \\
&\le \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\vartheta\in\Theta'} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) \\
&= \operatorname*{ess\,inf}_{\vartheta\in\Theta} \operatorname*{ess\,inf}_{\widetilde{\vartheta}\in\Theta} E_P\left(e^{-\alpha\left(C+\int_{(t_{i-1},t_i]}\vartheta_t\,dS_t+\int_{(t_i,T]}\widetilde{\vartheta}_t\,dS_t\right)}\,\middle|\mathcal{F}_{t_{i-1}}\right) \quad P\text{-a.s.}
\end{aligned} \tag{A.8}$$

Putting (A.2), (A.3), (A.6), and (A.8) together, this implies the assertion. $\square$

Bibliography

[Bac00] Bachelier. Théorie de la spéculation. *Annales Scientifiques de l'École Normale Supérieure*, 17:21–86, 1900.

[BD87] P.J. Brockwell and R.A. Davis. *Time Series: Theory and Methods*. Springer-Verlag, New York, 1987.

[Bec01] D. Becherer. *Rational Hedging and Valuation with Utility-based Preferences*. PhD thesis, TU Berlin, 2001.

[Ben84] A. Bensoussan. On the theory of option pricing. *Acta Applicandae Mathematicae*, 2:139–158, 1984.

[BGT87] N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation*. Cambridge University Press, 1987.

[Bil99] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, second edition, 1999.

[BL89] N. Bouleau and D. Lamberton. Residual Risks and Hedging Strategies in Markovian Markets. *Stochastic Processes and their Applications*, 33:131–150, 1989.

[BN77] O.E. Barndorff-Nielsen. Exponential decreasing distributions for the logarithm of particle size. *Proceeding of the Royal Society London A*, 353:401–419, 1977.

[BN98] O.E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance & Stochastics*, 2:41–68, 1998.

[Bré81] P. Brémaud. *Point Processes and Queues, Martingale Dynamics*. Springer-Verlag, New York, 1981.

[BS73] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–659, 1973.

[BSZ01] D.C. Brody, J. Syroka, and M. Zervos. Dynamical pricing of weather derivatives. *Journal of Quantitative Finance*, 1:1–10, 2001.

[Che00] P. Cheridito. Regularized fractional Brownian motion and option pricing. *Preprint ETH Zürich, Available at* `http://www.math.ethz.ch/ dito/`, 2000.

[Che01a] P. Cheridito. Arbitrage in fractional Brownian motion models. *Preprint ETH Zürich, Available at* `http://www.math.ethz.ch/ dito/`, 2001.

[Che01b] P. Cheridito. Mixed fractional Brownian motion. *Bernoulli*, 7:913–934, 2001.

[CK99] J. Cvitanić and I. Karatzas. On dynamic measures of risk. *Finance & Stochastics*, 3:451–482, 1999.

[CPT99] J. Cvitanić, H. Pham, and N. Touzi. Super-replication in stochastic volatility models under portfolio constraints. *Journal of Applied Probability*, 36:523–545, 1999.

[CR76] J. Cox and S. Ross. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166, 1976.

[CS01] A. Cherny and A. Shiryaev. Vector stochastic integrals and the fundamental theorems of asset pricing. Preprint, 2001.

[Cvi00] J. Cvitanić. Minimizing expected loss of hedging in incomplete and constrained markets. *SIAM J. Contr. & Optim.*, 38:1050–1066, 2000.

[Dav97] M.H.A. Davis. Opting pricing in incomplete markets. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*, pages 216–226. Cambridge University Press, 1997.

[DGR+02] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12:99–123, 2002.

[DM82] C. Dellacherie and P. Meyer. *Probabilities and Potential B*. North-Holland, Amsterdam, 1982.

[DR91] D. Duffie and H.R. Richardson. Mean-variance hedging in continuous time. *The Annals of Applied Probability*, 1:1–15, 1991.

[DS94] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[DS98] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen*, 312:215–250, 1998.

[DVJ88] D. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer-Verlag, Berlin and New York, 1988.

[DZ95] M. Davis and T. Zariphopoulou. American options and transaction fees. In M. Davis, editor, *Mathematical Finance*, volume 65 of *The IMA Volumes in Mathematics and its Applications*, pages 47–61. Springer, New York, 1995.

[EJ97] E. Eberlein and J. Jacod. On the range of option prices. *Finance & Stochastics*, 1:131–140, 1997.

[EK95] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1:281–299, 1995.

[EK99] R.J. Elliott and P.E. Kopp. *Mathematics of Financial Markets*. Springer-Verlag, 1999.

[EKP98] E. Eberlein, U. Keller, and K. Prause. New insights into smile, mispricing and value at risk: The hyperbolic model. *Journal of Business*, 71:371–406, 1998.

[EKQ95] N. El Karoui and M. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization*, 33:29–66, 1995.

[Eme78] M.. Emery. Stabilité des solutions des équations différentielles stochastiques. Application aux intégrales multiplicatives stochastiques. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 41:241–262, 1978.

[Fel71] W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 2. Wiley, New York, 2 edition, 1971.

[FK97] H. Föllmer and D. Kramkov. Optional decompositions under constraints. *Probability Theory and Related Fields*, 109:1–25, 1997.

[FK98] H. Föllmer and Y. Kabanov. Optional decomposition and Lagrange multipliers. *Finance & Stochastics*, 2:69–81, 1998.

[FL99] H. Föllmer and P. Leukert. Quantile hedging. *Finance & Stochastics*, 3:251–273, 1999.

[FL00] H. Föllmer and P. Leukert. Efficient hedging: Cost versus shortfall risk. *Finance & Stochastics*, 4:117–146, 2000.

[FS86] H. Föllmer and D. Sondermann. Hedging of non-redundant contingent claims. In W. Hildenbrand and A. Mas-Colell, editors, *Contributions to Mathematical Economics*, pages 205–223. North-Holland, 1986.

[FS99] R. Frey and C. Sin. Bounds on European option prices under stochastic volatility. *Mathematical Finance*, 9(2):97–116, 1999.

[Ger79] H. Gerber. *An Introduction to Mathematical Risk Theory*. Number 8 in Huebner Foundation Monograph Series. 1979.

[GJ80] C.W.J. Granger and R. Joyeux. An introduction to long-memory time series models and fractional differencing. *J. Time Series Anal.*, 1:15–29, 1980.

[GJ97] A. Grosen and P.L. Jørgensen. Valuation of early exercisable interest rate guarantees. *Journal of Risk and Insurance*, 64(3):481–503, 1997.

[GK01] T. Goll and J. Kallsen. A complete explicit solution to the log-optimal portfolio problem. Technical Report 31/2001, Mathematische Fakultät Universität Freiburg i. Br., 2001.

[Gra80] C.W.J. Granger. Long memory relationships and the aggregation of dynamic models. *J. Econometrics*, 14:227–238, 1980.

[GS79] I.I. Gihman and A.V. Skorohod. *Controlled Stochastic Processes*. Springer-Verlag, 1979.

[Gut88] A. Gut. *Stopped Random Walks*. Springer-Verlag, New York, 1988.

[Hel99] W.C. Held. *Optionen in Lebensversicherungsverträgen*. ifa-Schriftenreihe, Ulm, 1999.

[HN89] S.D. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, 8:222–239, 1989.

[HØ99] Y. Hu and B. Øksendal. Fractional White Noise Calculus and Applications to Finance. *Preprint University of Oslo, Available at* `http://www.math.uio.no/eprint/pure_math/1999/10-99.html`, 1999.

[Hul00] J.C. Hull. *Options, Futures and other Derivatives*. Prentice-Hall, Inc., fourth edition, 2000.

[Jac75] J. Jacod. Multivariate Point Processes: Predictable Projection, Radon-Nikodym Derivatives, Representation of Martingales. *Probability Theory and Related Fields*, 31:235–253, 1975.

[Jac79] J. Jacod. *Calcul Stochastique et Problèmes de Martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[Jac80] J. Jacod. Intégrales stochastiques par rapport à une semi-martingale vectorielle et changements de filtration. In *Séminaire de Probabilités XIV, 1978/79*, volume 784 of *Lecture Notes in Mathematics*, pages 161–172. Springer, Berlin, 1980.

[Jac92] S.D. Jacka. A martingale representation result and an application to incomplete financial markets. *Mathematical Finance*, 2(4):239–250, 1992.

[JS87] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Berlin, 1987.

[Kab97] Yu. Kabanov. On the FTAP of Kreps-Delbaen-Schachermayer. In *Statistics and control of stochastic processes (Moscow, 1995/1996)*, pages 191–203. World Scientific, River Edge, NJ, 1997.

[Kal99] J. Kallsen. A utility maximization approach to hedging in incomplete markets. *Mathematical Methods of Operations Research*, 50:321–338, 1999.

[Kal00] J. Kallsen. Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research*, 51:357–374, 2000.

[Kal01] J. Kallsen. Utility-based derivative pricing in incomplete markets. In H. Geman, D. Madan, S. Pliska, and T. Vorst, editors, *Mathematical Finance – Bachelier Congress 2000*, pages 313–338, Berlin, 2001. Springer.

[Kal02] J. Kallsen. Derivative pricing based on local utility maximization. *Finance & Stochastics*, 6:115–140, 2002.

[Kar88] I. Karatzas. On the pricing of American options. *Applied Mathematics and Optimization*, 17:37–60, 1988.

[Kif00] Y. Kifer. Game options. *Finance & Stochastics*, 4:443–463, 2000.

[KK98] I. Karatzas and S. Kou. Hedging American contingent claims with constrained portfolios. *Finance & Stochastics*, 2:215–258, 1998.

[KK02a] J. Kallsen and C. Kühn. Pricing Derivatives of American and Game Type in Incomplete Markets. *Submitted for publication.*, 2002.

[KK02b] C. Klüppelberg and C. Kühn. Fractional Brownian motion as weak limit of Poisson shot noise processes – with applications to finance. *Submitted for publication*, 2002.

[KM95a] C. Klüppelberg and T. Mikosch. Delay in claim settlement and ruin probability approximations. *Scand. Act. J.*, 2:154–168, 1995.

[KM95b] C. Klüppelberg and T. Mikosch. Explosive Poisson shot noise processes with applications to risk reserves. *Bernoulli*, 1:125–147, 1995.

[Kra96] D. Kramkov. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields*, 105:459–479, 1996.

[KS96] F. Karsenty and J. Sikorav. Installment plain, over the rainbow. *Risk publication*, pages 203–206, 1996.

[KS98] I. Karatzas and S. Shreve. *Methods of Mathematical Finance*. Springer, Berlin, 1998.

[KS99] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9:904–950, 1999.

[KS01] J. Kallsen and A. Shiryaev. Time change representations of stochastic integrals. *Theory of Probability and its Applications,*, forthcoming, 2001.

[KS02a] Y. Kabanov and C. Stricker. On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. *Mathematical Finance*, 12:125–134, 2002.

[KS02b] C. Klüppelberg and M. Severin. Prediction of outstanding insurance claims. *Submitted for publication. Available at* `http:` `www-m4.mathematik.tu-muenchen.de/m4/Papers/`, 2002.

[KTS01] C. Klüppelberg, Mikosch T., and A. Schärf. Regular variation in the mean and stable limits for Poisson shot noise. *Bernoulli, to appear. Available at* `http:www-m4.mathematik.tu-muenchen.de/m4/Papers/`, 2001.

[Küh01] C. Kühn. Game contingent claims in complete and incomplete markets. *Submitted for publication*, 2001.

[Küh02] C. Kühn. Pricing contingent claims in incomplete markets when the holder can choose among different payoffs. *To appear in Insurance: Mathematics & Economics*, 2002.

[Lan84] J.A. Lane. The central limit theorem for the Poisson shot-noise process. *J. Appl. Prob.*, 21:287–301, 1984.

[LM84] J. Lepeltier and M. Maingueneau. Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics*, 13:25–44, 1984.

[Man97] B. Mandelbrot. *Fractals and Scaling in Finance, Discontinuity, Concentration Risk*. Springer-Verlag, New York, 1997.

[Mer73] R. Merton. Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[Møl00] T. Møller. *Quadratic Hedging Approaches and Indifference Pricing in Insurance*. PhD thesis, University of Copenhagen, 2000.

[Mor86] H. Morimoto. Non-zero-sum discrete parameter stochastic games with stopping times. *Probab. Th. Rel. Fields*, 72:155–160, 1986.

[MS86] J.J. McConnell and E.S. Schwartz. LYON Taming. *Journal of Finance*, 16(3):561–576, 1986.

[MS90] D. Madan and E. Seneta. The VG model for share market returns. *Journal of Business*, 63:511–524, 1990.

[MS99] T. Mikosch and C. Stărică. Change of structure in financial time series, long range dependence and the GARCH model. *Preprint, Available at* `http://www.math.ku.dk/ mikosch/preprint.html`, 1999.

[Oht86] Y. Ohtsubo. Optimal stopping in sequential games with or without a constraint of always terminating. *Mathematics of Operations Research*, 11:591–607, 1986.

[Oht87] Y. Ohtsubo. A nonzero-sum extension of Dynkin's stopping problem. *Mathematics of Operations Research*, 12(2):277–296, 1987.

[Pro77] P. Protter. On the existence, uniqueness, convergence and explosions of solutions of systems of stochastic integral equations. *The Annals of Probability*, 5:243–261, 1977.

[Pro78] P. Protter. H^p stability of solutions of stochastic differential equations. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 44:337–352, 1978.

[Pro92] P. Protter. *Stochastic Integration and Differential Equations*. Springer, Berlin, second edition, 1992.

[Rog97] L.C.G. Rogers. Arbitrage with fractional Brownian motion. *Mathematical Finance*, 7:95–105, 1997.

[RW98] T. Rockafellar and R. Wets. *Variational Analysis*. Springer, Berlin, 1998.

[Sam65] P. Samuelson. Rational theory of warrant pricing. *Industrial Management Review*, 6:13–31, 1965.

[SC99] G. Spivak and J. Cvitanić. Maximizing the probability of a perfect hedge. *The Annals of Applied Probability*, 9:1303–1328, 1999.

[Sch88] M. Schweizer. *Hedging of Options in a General Semimartingale Model*. PhD thesis, ETH Zürich, 1988.

[Sch91] M. Schweizer. Option hedging for semimartingales. *Stochastic Processes and their Applications*, 37:339–363, 1991.

[Sch01a] W. Schachermayer. Optimal investment in incomplete financial markets. In H. Geman, D. Madan, S. Pliska, and T. Vorst, editors, *Mathematical Finance – Bachelier Congress 2000*, pages 427–462, Berlin, 2001. Springer.

[Sch01b] W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability*, 11(3):694–734, 2001.

[Sch01c] M. Schweizer. From actuarial to financial valuation principles. *Insurance: Mathematics & Economics*, 28(1):31–47, 2001.

[Sch01d] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, 2001.

[Sot01] T. Sottinen. Fractional Brownian motion, random walks and binary market models. *Finance & Stochastics*, 5:343–355, 2001.

[ST94] G. Samorodnitsky and M.S. Taqqu. *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall, New York, 1994.

[Stu00] W. Stute. Jump diffusion processes with shot noise effects and their applications to finance. *Preprint University of Gießen*, 2000.

[Wes76] M. Westcott. On the existence of a generalized shot-noise process. In E.J. Williams, editor, *Studies in Probability and Statistics: Papers in Honour of Edwin J.G. Pitman*, pages 73–88. North-Holland, Amsterdam, 1976.

List of Figures

List of Abbreviations and Symbols

$\mathscr{A}$	set of processes with integrable variation	
$\mathscr{A}^+$	set of integrable increasing processes	
A^C	complement of the set A: $A^C = \Omega \setminus A$	
a.e.	almost everywhere	
a.s.	almost surely	
$\arg\max$	$\arg\max_{i=1,\dots,k} x_i = \min\{i \in \{1,\dots,k\} \mid x_i = x_1 \vee \dots \vee x_k\}$	
$\arg\min$	$\arg\min_{i=1,\dots,k} x_i = \min\{i \in \{1,\dots,k\} \mid x_i = x_1 \wedge \dots \wedge x_k\}$	
B	standard Brownian motion	
B^H	fractional Brownian motion with Hurst parameter H	
(B, C, ν)	semimartingale characteristic	96
(b, c, F, A)	differential characteristic of a semimartingale	97
$\mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d_+)$	Borel-σ-algebra on $\mathbb{R}^d$ and $\mathbb{R}^d_+$	
$\mathscr{C}$	a class of semimartingales	
$\mathscr{C}_{\mathrm{loc}}$	localized class of $\mathscr{C}$: $X \in \mathscr{C}_{\mathrm{loc}} \Leftrightarrow \exists$ sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ s.t. $T_n \uparrow \infty$ a.s. and $X^{T_n} \in \mathscr{C}$, for each n, where $X^{T_n}_t = X_{t \wedge T_n}$	
$\mathscr{C}_\sigma$	σ-localized class of $\mathscr{C}$: $X \in \mathscr{C}_\sigma \Leftrightarrow \exists$ sequence of predictable sets $(D_n)_{n\in\mathbb{N}}$ s.t. $D_n \uparrow \Omega \times \mathbb{R}_+$ up to an evanescent set and $X^{D_n} \in \mathscr{C}$, for each n, where $X^{D_n}_t(\omega) = X_0 I((\omega, 0) \in D_n) + (I(D_n) \cdot X_t)(\omega)$	
$\mathcal{C}^n(\mathbb{R}, \mathbb{R})$	set of all functions $\mathbb{R} \to \mathbb{R}$ which are n times continuously differentiable	
cadlag	continu a droite avec des limites a gauche	
caglad	continu a gauche avec des limites a droite	
$D[0, \infty)$	set of all function $\mathbb{R}_+ \to \mathbb{R}$ which are cadlag	
(D)	a stochastic process X is of class (D) iff the set of random variables	

	$\{X_T \mid T$ finite-valued stopping time$\}$ is uniformly integrable	
$\mathscr{D}$	set of all semimartingales s.t. for every $t \in \mathbb{R}_+$ the stopped process X^t is of class (D)	
$d(\cdot,\cdot)$	metric	
ΔX	$\Delta X_t = X_t - X_{t-}$	
E	expectation with respect to P	
$E^\star$	expectation with respect to $P^\star$	
E_Q	expectation with respect to Q	
$\mathscr{E}(X)$	stochastic exponential of the semimartingale X	
ess sup$X$	essential supremum of $\{X\}$, for the random variable X, with respect to the trivial σ-algebra $\{\varnothing, \Omega\}$	
ess inf$X$	essential infimum of $\{X\}$, for the random variable X, with respect to the trivial σ-algebra $\{\varnothing, \Omega\}$	
ess $\sup_{X\in\mathcal{X}} E(X\|\mathcal{F})$	essential supremum of the set $\{E(X\|\mathcal{F}) \mid X \in \mathcal{X}\}$ with respect to the σ-algebra $\mathcal{F}$	
ess $\inf_{X\in\mathcal{X}} E(X\|\mathcal{F})$	essential infimum of the set $\{E(X\|\mathcal{F}) \mid X \in \mathcal{X}\}$ with respect to the σ-algebra $\mathcal{F}$	
evanescence	a random set $A \subset \Omega \times \mathbb{R}_+$ is evanescent iff $P(\{\omega \in \Omega \mid \exists t \in \mathbb{R}_+ \text{ s.t. } (\omega, t) \in A\}) = 0$	
FBM	fractional Brownian motion	
F_i	ith marginal distribution function of distribution function F: $F_i(x_i) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} F(dx_1, \ldots, dx_{i-1}, x_i, dx_{i+1}, \ldots, dx_n)$	
$F_i(\cdot\|\cdot)$	ith marginal conditional distribution function given $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$	
$g^{\leftarrow}$	generalized inverse of g: $g^{\leftarrow}(t) = \inf\{x \in \mathbb{R} \mid g(x) > t\}$	
Γ	stochastic process whose values are convex cones in $\mathbb{R}^d$	45
$\mathfrak{S}(\Gamma)$	constrained set of trading strategies for terminal wealth	45
$\mathfrak{S}'(\Gamma)$	constrained set of trading strategies for local utility	49
Γ_t°	the polar cone of Γ_t: $\Gamma_t^\circ = \{y \in \mathbb{R}^d : x^\top y \leq 0 \text{ for any } x \in \Gamma_t\}$	
$\\|X\\|_{\mathscr{H}^1}$	$\mathscr{H}^1$-norm of a semimartingale X	98

$I(A)$	indicator function of the set A	
iff	if and only if	
i.i.d.	independent, identically distributed	
$\lim$	limes	
$\liminf$	limes inferior	
$\limsup$	limes superior	
indistinguishability	two stochastic processes X, Y are indistinguishable iff $\{X = Y\}$ is evanescent	
inf	infimum	
$L(S)$	set of predictable and S-integrable processes	
$L^\infty(\Omega, \mathcal{F}, P)$ $\equiv L^\infty(P)$	space of all real-valued random variables X s.t. $\operatorname{ess\,sup}\lvert X\rvert < \infty$	
lhs	left-hand side	
$\ln x$	logarithm with basis e	
$l(x)$	slowly varying function	
L^a	local time of the semimartingale X in state a in the sense of Jacod (Jac79), Definition 5.47	98
$\mathbb{N}$	set of positive integers	
$N, N(t), N_t$	point process	
$o(1)$	$a(x) = o(b(x))$ as $x \to x_0$ means that $\lim_{x\to x_0} a(x)/b(x) = 0$	
$O(1)$	$a(x) = O(b(x))$ as $x \to x_0$ means that $\limsup_{x\to x_0} \lvert a(x)/b(x)\rvert < \infty$	
$(\Omega, \mathcal{F}, P)$	underlying probability space	
ω	$\omega \in \Omega$ random outcome	
P	objective probability measure	
$P(A \mid B\,;\,\mathcal{F})$	conditional P-probability of the event A under the σ-algebra $\mathcal{F}$ given the event B: $P(A \mid B\,;\,\mathcal{F}) = P(A \cap B \mid \mathcal{F})/P(B \mid \mathcal{F})$	
$P_1 \otimes P_2$	product measure of the measures P_1 and P_2	
$\mathscr{P}_a$	$Q \in \mathscr{P}_a$ iff $Q \ll P$ and the (discounted) price process S is a	

	local Q-martingale	
$\mathscr{P}_e$	$Q \in \mathscr{P}_e$ iff $Q \sim P$ and the (discounted) price process S is a local Q-martingale	
$P^\star$	the neutral pricing measure for terminal wealth resp. local utility	47, 50
$\mathcal{P}$	predictable σ-algebra on $\Omega \times \mathbb{R}_+$	
$\mathcal{P}_1 \otimes \mathcal{P}_2$	product σ-algebra of $\mathcal{P}_1$ and $\mathcal{P}_2$	
$\mathbb{Q}$	set of rational numbers	
Q	(martingale) probability measure	
$Q \ll P$	Q is absolutely continuous with respect to P: $P(A) = 0 \Rightarrow Q(A) = 0 \;\; \forall A \in \mathcal{F}$	
$Q \sim P$	Q is equivalent to P: $Q(A) = 0 \Leftrightarrow P(A) = 0 \;\; \forall A \in \mathcal{F}$	
$\mathbb{R}$, $\mathbb{R}_+$	real line, nonnegative half-line	
$\mathbb{R}^d$	set of d-dimensional real vectors	
rhs	right-hand side	
semi-continuity	a function $g : M \to \mathbb{R}$ is semi-continuous from *below* in $x_0 \in M$ iff $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $d(x, x_0) \leq \delta \Rightarrow g(x) \geq g(x_0) - \varepsilon$ g is semi-continuous from *above* iff $-g$ is semi-continuous from below	
$\sigma(X_s, \; s \in A)$	the σ-algebra generated by $\{X_s, \; s \in A\}$	
sup	supremum	
T	maturity $T \in \mathbb{R}_+$	
$(T_n)_{n \in \mathbb{N}}$	localizing sequence: increasing sequence of stopping times s.t. $T_n \uparrow \infty$ P-a.s., i.e. $P(T_n \geq T) \to 1$	
Θ	subset of $L(S)$: set of admissible portfolio strategies	
ϑ	portfolio strategy	
ϑ^δ	compound portfolio strategy in the sense of equation (2.4)	68
ϑ^τ	compound portfolio strategy in the sense of equation (3.2)	79
$\int \vartheta \, dS \equiv \vartheta \cdot S$	stochastic integral: gains from trade	
$\int_{\tau_1}^{\tau_2} \vartheta \, dS \equiv \int_{(\tau_1, \tau_2]} \vartheta \, dS$	stochastic integral: gains from trade in the period between τ_1 and τ_2	
$V(f; I)$	total variation of the function f on the interval I	

$\mathrm{Var}(A)$	variation process of process A
$\mathrm{var}(X)$	variance of the random variable X
$\mathscr{V}$	set of processes with finite variation
$\mathscr{V}^+$	set of finite-valued increasing processes
w.l.o.g.	without loss of generality
X^τ	stopped process $X^\tau_t = X_{t\wedge\tau}$
$X^{\tau-}$	stopped process $X^{\tau-}_t = X_t I(0 \le t < \tau) + X_{\tau-} I(t \ge \tau)$
$X(\infty)$	$\lim_{u\to\infty} X(u)$
$X(t-)$	$\lim_{u\uparrow t} X(u)$
$\{X = Y\}$	$\{X = Y\} = \{(\omega, t) \in \Omega \times [0, T] \mid X_t(\omega) = Y_t(\omega)\}$,
	for the stochastic processes X, Y
$[X, Y]$	quadratic co-variation of the two semimartingales X and Y,
	$[X, Y] = XY - X_0 Y_0 - X_- \cdot Y - Y_- \cdot X$
Y^*	$Y^* = \sup_{t\in\mathbb{R}_+} \lvert Y_t\rvert$ for the stochastic process Y
Y^+, Y^-	positive and negative part of the random variable Y:
	$Y^+ = Y \vee 0$, $Y^- = (-Y) \vee 0$
$y_x,\ y_{xx}$	first and second derivatives of the function y with respect to x
$\mathbb{Z}$	set of integers
$\lvert\cdot\rvert$	absolute value
$\xrightarrow{d}$	convergence in distribution

Curriculum Vitae

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German citizen, date of birth $12/25/1973$

Education

$1999-2002$	Ph.D. student and scholarship holder at the Graduiertenkolleg "Applied Algorithmic Mathematics" at Munich University of Technology Supervisor of Ph.D. thesis: Prof. C. Klüppelberg Title: "Shocks an Choices – an Analysis of Incomplete Market Models"
$1993-1999$	Study of mathematics, economics, and computer science at Universität Marburg, diploma in mathematical economics Title of diploma thesis (in German) supervised by Prof. J. Steinebach: "Statistical Analysis of Change Parameters based on Invariance Principles"
1993	German Abitur, Altkönigschule Kronberg

Awards

2000	One of four first prizes for the best diploma thesis at the annual conference of the "DMV" (the association of German mathematicians)

Research Interests

Mathematical finance and stochastic analysis; in particular pricing and hedging derivatives of American type (and more general of "game" typ) in incomplete financial markets, hedging under transaction costs, stochastic control theory, alternative stock price models, functional limit theorems

Teaching Experience

$10/01/97 - 03/31/98$:	Tutor in Linear Programming, Philipps-University Marburg
$10/01/99 - 03/31/2000$:	Tutor in Statistics, Munich University of Technology

Talks

- Seminar on “Integrated Risk Management”, Swiss Re, Zurich, Switzerland, April 2000
- EURANDOM-Workshop on “Risk, Insurance, and Extremes”, Eindhoven, The Netherlands, August 2000
- “Students’ conference” of the association of German mathematicans, Dresden, Germany, September 2000
- Workshop on “Stochastic Approaches in Finance, Insurance, and Physics”, Munich, Germany, September 2000
- Summer School on “Stochastics and Finance”, Barcelona, Spain, September 2001
- Colloquium on “Actuarial Mathematics and Mathematical Finance”, Chair Prof. Schweizer, Munich, Germany, January 2002
- “German Open Conference on Probability and Statistics”, Magdeburg, Germany, March 2002
- “Munich Spring School on Mathematical Finance”, Munich, Germany, April 2002

Organization of Conferences

- Workshop on “Extreme Value Theory and Financial Risk” (together with Claudia Klüppelberg), Munich, Germany, December 1999
- Workshop on “Stochastic Approaches in Finance, Insurance, and Physics”(together with Andreas Kunz), Munich, Germany, September 2000
- “Munich Spring School on Mathematical Finance”(together with Claudia Klüppelberg and Andreas Kunz), Munich, Germany, April 2002

Publications / Preprints

- Kühn, C., Steinebach, J. (2002) On the estimation of change parameters based on weak invariance principles. Proceedings of the “Fourth Hungarian Colloquium on Limit Theorems in Probability and Statistics” (1999), to appear in Bolyai Society Mathematical Studies (in print).

- Kühn, C. (2001) An estimator of the number of change points based on a weak invariance principle. Statistics and Probability Letters, 51(2), 189-196.

- Kühn, C. (2001) Pricing contingent claims in incomplete markets when the holder can choose between different payoffs. To appear in Insurance: Mathematics and Economics.

- Kühn, C. (2001) Game contingent claims in complete and incomplete markets. Submitted for publication.

- Kallsen, J., Kühn, C. (2002) Pricing Derivatives of American and Game Type in Incomplete Markets. Submitted for publication.

- Klüppelberg, C., Kühn, C. (2002) Fractional Brownian motion as a weak limit of Poisson shot noise processes – with applications to finance. Submitted for publication.