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Probing Interacting Dark Sector Models with Large-Scale Structure

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Probing Interacting Dark Sector Models with Large-Scale Structure

DISSERTATION

von

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Abstract

This thesis investigates an interacting dark sector, where dark matter scatters off relativistic dark radiation, analogous to photon-electron interactions in the visible sector. This framework, described by the Effective Theory of Structure Formation (ETHOS), has the potential to address cosmological tensions, such as the S_8 discrepancy—a $2-3\sigma$ tension between high-redshift and low-redshift measurements of matter fluctuations. Using new datasets and advanced analyses, this work explores the impact of an interacting dark sector on structure formation. Key contributions include constraints from full-shape galaxy clustering data, state-of-the-art cluster abundance analyses with mass information from weak-lensing data, and forecasts for future galaxy cluster surveys. The thesis also refines the computation of the interaction rate in a non-Abelian $SU(N)$ dark sector, addressing the divergences and incorporating corrections. By combining cosmological probes and particle physics insights, this research advances our understanding of dark matter and its interactions, offering new perspectives on the unresolved mysteries of the Universe.

Zusammenfassung

Diese Dissertation untersucht einen interagierenden dunklen Sektor, in dem Dunkle Materie an relativistischer dunkler Strahlung streut, analog zu Photon-Elektron-Wechselwirkungen im sichtbaren Sektor. Dieses Szenario, beschrieben durch die Effektive Theorie der Strukturbildung (ETHOS), hat das Potenzial Diskrepanzen zwischen verschiedenen kosmologischen Beobachtungen aufzulösen, wie die S_8 -Diskrepanz – eine $2-3\sigma$ Abweichung zwischen hoch- und niedrigrotverschobenen Messungen der Materiedichtefluktuationen. Unter Verwendung neuer Datensätze und fortgeschrittener Analysen untersucht diese Arbeit die Auswirkungen eines interagierenden dunklen Sektors auf die Strukturbildung. Zu den wichtigsten Ergebnissen gehören Parameterbeschränkungen aus Daten zum Full-Shape Galaxien-Clustering, hochmodernste Analysen der Häufigkeit von Galaxienhaufen mit Massenbestimmungen beruhend auf dem schwachen Gravitationslinseneffekt, sowie Prognosen für zukünftige Untersuchungen von Galaxienhaufen. Darüber hinaus wird die Berechnung der Wechselwirkungsrate in einem nicht-Abelschen $SU(N)$ -dunklen Sektor präzisiert, wobei auftretende Divergenzen behandelt und Korrekturen berücksichtigt werden. Durch die Kombination von kosmologischen Beobachtungen und Erkenntnissen aus der Teilchenphysik erweitert diese Forschung unser Verständnis von Dunkler Materie und ihren Wechselwirkungen und bietet neue Perspektiven auf die ungelösten Rätsel des Universums.

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1 Introduction

Our understanding of the universe remains far from complete. From a physics perspective, we have a well-established theory that describes only about 5% of the universe’s composition—what we refer to as *visible matter*. The remaining 95% remains a profound mystery, with 27% attributed to *dark matter* (DM) and 68% to *dark energy* (DE) [4]. While dark energy drives the accelerated expansion of the universe, dark matter plays a crucial role in the formation and evolution of cosmic structures, providing the gravitational framework within which galaxies and galaxy clusters form.

Dark matter was first discovered through its gravitational influence on galaxies and galaxy clusters. Observations such as the rotation curves of spiral galaxies, gravitational lensing, and cosmic microwave background (CMB) anisotropies provide compelling evidence for its existence; see [4, 5] for a review. In physics, we have developed several experimental approaches to probe the properties of dark matter beyond its gravitational effects. These include *direct detection* experiments, which aim to observe rare interactions between dark matter particles and atomic nuclei; *indirect detection*, which searches for secondary particles produced by dark matter annihilation or decay; and *collider experiments*, where high-energy particle collisions, such as those at the Large Hadron Collider (LHC), attempt to produce dark matter particles in controlled environments, see [4, 5]. However, none of these methods was successful in finding DM.

Cosmology and astrophysics provide another powerful approach to studying dark matter by analyzing the structure of the universe—an approach that aligns with the observations that led to the evidence for DM’s existence. The standard cosmological model, known as Λ CDM, successfully describes the evolution of the universe with cold dark matter (CDM) and a cosmological constant (Λ) for dark energy, [6] provides an early review on this paradigm. However, Λ CDM remains an empirical model, and its underlying physics, particularly in relation to dark matter, is not yet fully understood. Cosmology and astrophysics provide unique opportunities to probe DM properties across different frontiers. At *the small-scale frontier*, galactic-scale observations test models like self-interacting DM [7, 8], which could explain anomalies like the core cusp problem [9]. *The intensity frontier* focuses on substructures, such as satellite galaxy populations, where models like warm DM or fuzzy DM (ultra-light scalar particles) are relevant [10]. Finally, *the precision frontier* leverages large-scale structure data to explore scenarios involving small deviations from Λ CDM like interacting dark sectors [11–13] or decaying DM to other types of particles; see e.g., [14–16].

What makes this endeavor a possibility is the wealth of cosmological and astrophysical data obtained from various surveys, spanning different times and scales. Examples include: gravitational weak-lensing measurements [17–20], galaxy clustering maps [21, 22], galaxy cluster abundance [23–25], and cosmic microwave background (CMB) anisotropies [26–

28]. Next-generation CMB and large-scale structure (LSS) observatories, such as CMB-S4 [29, 30], Euclid [31–33], the Vera C. Rubin Observatory [34], and Simons Observatory [35], will significantly enhance the quantity and quality of data, hence allow for precision tests of the properties of DM.

Another motivation to look into beyond- Λ CDM models is the existence of tensions in cosmology and astrophysics; like the H_0 tension [36, 37], the S_8 tension [38, 39] (relevant for this work), and small-scale problems [7, 8, 40]. The S_8 tension refers to a discrepancy between measurements of the amplitude of matter fluctuations in the universe at $8 h^{-1}\text{Mpc}$. Cosmic Microwave Background (CMB) data predict a higher value of $S_8 \approx 0.81 - 0.83$ [26, 27], while low-redshift probes like weak gravitational lensing and galaxy clustering favor a lower value around $0.76 - 0.79$ [17, 18, 25, 41], with a $2 - 3\sigma$ discrepancy. This could indicate unaccounted systematic effects or point to new physics beyond Λ CDM.

This work focuses on an *interacting dark sector*, where dark matter (or a fraction of it) scatters off relativistic species, called *dark radiation* (DR). This mimics the scattering between photons and electrons in the visible sector, which affects structure evolution. This possibility was introduced and studied in various works like [11–13, 42–48], and was shown to have potential in addressing the S_8 tension. A formalism to study the possible particle physics realization of an interacting dark sector and their impact on large scale structure was introduced in [13] called the Effective THeory Of Structure formation (ETHOS). It offers a simple, yet effective way to translate the microphysics details to macrophysical quantities.

In three works; [Paper I](#), [Paper II](#), and [Paper III](#), we went beyond what previously existed by performing analyses with new datasets in the interacting dark sector framework. In [Paper I](#) we explored constraints from full-shape (FS) galaxy clustering data, whereas in [Paper II](#) and [Paper III](#), we performed the state-of-the-art cluster abundance analysis, achieving constraints from current cluster samples and gaining insights on future samples.

Outline of the thesis

This thesis is structured as follows

Part I begins with an introduction to the two fundamental frameworks: the standard model of cosmology (Λ CDM) and the standard model of elementary particle physics. Chapter 2 provides an overview of modern cosmology, covering the concept of an expanding Universe and the fundamental equations governing its evolution. We then discuss the growth of structure, both in the linear regime and beyond, with a focus on spherical collapse and halo mass function. Additionally, we introduce the primary cosmological probes used in the subsequent analyses, which are either applied to this model for the first time or supplemented with complementary data for comparison. In Chapter 3, we review key aspects of the standard model of particle physics, with a particular focus on gauge theories, as they form the foundation of the interacting dark sector that will be

introduced later.

Part II in this work explores extensions beyond the CDM paradigm by introducing an *interacting dark sector*. In Chapter 4, we present the effective theory of structure formation (ETHOS), a framework designed to systematically describe this extension with effective parameters that impact structure formation. We will outline its formalism, discuss various particle physics realizations, and show its impact on structure formation—specifically, its effects on the matter power spectrum and the halo mass function. In Chapter 5, we focus on a non-Abelian $SU(N)$ dark sector, detailing its microscopic properties and addressing the divergences that arise in the interaction rate. We provide a refined computation of the interaction rate by incorporating corrections from all scattering channels (s , u , and t) and employing thermal field theory to regulate the t -channel divergence via the Debye mass of DR. This computation was a key result of [Paper I](#).

Finally, **Part III** is dedicated to detailing the research presented in the three papers on which this thesis is based. Chapter 6 focuses on the analysis of full-shape (FS) galaxy clustering data from BOSS DR12 within the framework of an interacting dark sector, as explored in [Paper I](#). In this work, we investigate three distinct particle physics models, deriving constraints on each and assessing their potential to alleviate the S_8 tension. Chapters 7 and 8 present analyses of galaxy cluster abundance, based on the work in [Paper II](#) and [Paper III](#), respectively. In Chapter 7, we analyze cluster abundance from SPT, incorporating mass information from DES Y3 and HST. Chapter 8 extends this analysis by forecasting constraints from ongoing and future surveys, including SPT-3G and CMB-S4, in combination with next-generation weak-lensing (ngWL) data. In both studies, we focus exclusively on the non-Abelian $SU(N)$ gauge theory.

We conclude this thesis in Chapter 9 and include three appendices to provide additional details. Appendix A provides an overview of conventions and useful formulae relevant to the Λ CDM framework discussed in Chapter 2. In Appendix B, we outline the derivation of the Boltzmann equations in the ETHOS parametrization, including additional steps to enhance clarity. Finally, Appendix C presents the complete set of posterior distributions for all parameters analyzed in each work.

All equations in this work are in natural units; meaning the speed of light c , the Planck constant $\hbar$ and the Boltzmann constant k_B are set to unity ($c = \hbar = k_B = 1$).

Part I

Standard Models

2 Standard Model of Cosmology (Λ CDM)

The concordance cosmological model, known as the Λ CDM model, has been shaped by several pivotal astrophysical discoveries and theoretical advancements. It is built upon key contributions from *general relativity*, *particle physics*, and *statistical physics*, each playing a crucial role in shaping our understanding of the cosmos. *General relativity* (GR) provides the theoretical foundation for cosmic dynamics through Einstein's field equations, leading to the *Friedmann equations*, which describe the expansion of the Universe in a homogeneous and isotropic space. *Particle physics* contributes by explaining the early Universe's thermal history, particularly *Big Bang Nucleosynthesis* (BBN), which determines the primordial abundances of light elements. *Statistical physics* plays a central role in describing the evolution of primordial density fluctuations, using *Boltzmann equations* to study the evolution of particle distributions in the early Universe. Additionally, statistical methods underpin large-scale structure formation, with *perturbation theory* explaining the growth of cosmic structures from initial quantum fluctuations seeded during *inflation*.

Moreover, the foundation of the Λ CDM model was laid by several early key observations. The first crucial discovery was the *expansion of the Universe*, revealed by Edwin Hubble in 1929 through the redshift-distance relation, confirming predictions from GR and the Friedmann equations. This was followed by the detection of the CMB radiation in 1965 by Penzias and Wilson, providing strong evidence for the *Big Bang theory* and ruling out steady-state models. Another major milestone was the successful prediction and measurement of BBN abundances, particularly those of hydrogen, helium, and deuterium, which matched theoretical expectations and reinforced the hot, dense early Universe scenario. In the following decades, observations of *galaxy clustering* and *large-scale structure* suggested the presence of an unseen mass component, leading to the development of the *cold dark matter* (CDM) hypothesis. The combination of these discoveries provided the first strong empirical foundation for a Universe dominated by dark matter and governed by general relativity, setting the stage for the later refinement of the Λ CDM model.

These are the key components that define the cosmological model; further details can be found in [49–52]. In the following sections, we introduce the main concepts relevant to this work. We begin by outlining the fundamental quantities that describe an expanding Universe in Sec. 2.1. Section 2.2 focuses on the fundamental equations governing cosmology, including the Friedmann equations, Boltzmann equations, and Einstein equations. In Section 2.3, we discuss the growth of structure, covering both the linear regime—characterized by the matter power spectrum—and the nonlinear regime, emphasizing spherical collapse and the halo mass function. Finally, in Section 2.4, we introduce the cosmological probes used in the subsequent analyses.

2.1 Expanding Universe

As in any theory, we start by making general assumptions. In the case of Cosmology, we have what is called THE COSMOLOGICAL PRINCIPLE, which states that the Universe on large scales is spatially *homogeneous* and *isotropic*. In this context, *homogeneity* refers to the Universe appearing uniform at all points in space, while *isotropy* means it looks the same in every direction. These assumptions imply that the large-scale properties of the universe do not depend on position, and they ensure that the fundamental laws of physics apply universally. Of course, this is not absolutely true; yet, it is a powerful starting point. And we will learn later about how we can deviate from it. The simplest geometry that upholds the cosmological principle is a flat Euclidean space. Considering the abundant evidence we have that our Universe is expanding and including the time component, we work in what is known as the Friedmann-Lemaître-Robertson-Walker metric. In physical time t and comoving space coordinates $\boldsymbol{x}$, the FLRW metric is given by

$$ds^2 = -dt^2 + a(t) d\boldsymbol{x}^2, \quad (2.1)$$

$a(t)$ is the scaling factor quantifying the expansion of the Universe. Conventionally, the scale factor at present time is set to unity, $a(t_0) = a_0 = 1$. The subscript 0 denotes the value of a quantity evaluated today.

To study the history of the Universe, we need to determine how the scale factor a evolves over cosmic time t . Therefore, it is useful to define the *Hubble rate*

$$H(t) \equiv \frac{1}{a} \frac{da}{dt}, \quad (2.2)$$

which quantifies the rate of expansion of the Universe. Its value today is denoted as H_0 , and known as the *Hubble constant*

$$H_0 = (67.27 \pm 0.60) \text{ km s}^{-1} \text{ Mpc}^{-1}, \quad (2.3)$$

as measured by CMB anisotropies from Planck 2018 (at 68% CI) [53]. This value is lower than the one inferred from local measurements, such as the value from the SH0ES collaboration with $H_0 = 73.5 \pm 1.4$ [37]. The two measurements are in $\sim 4 \sigma$ tension, leading to one of the most pressing problems in cosmology today. Possible explanations include systematic errors or new physics beyond- Λ CDM. Models that might explain this discrepancy deal with changes in the early Universe; these include early dark energy model and early modified gravity, see [54] for a review.

H_0 is commonly parameterized by a dimensionless parameter h defined as

$$H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}. \quad (2.4)$$

As was just discussed, there is a discrepancy in the measurement of the precise value of H_0 from different probes. To make other measurements insensitive to its precise value, it is common to add h to other units, for e.g., $h^{-1} \text{ Mpc}$ as a length unit in cosmology.

Another important parameter stemming from the expansion of the Universe is the

cosmological redshift, denoted with parameter z . It describes how a physical wavelength from a distant object is stretched by the expansion of space as it travels through it. So that eventually, its observed wavelength is larger than its emitted one. This stretching factor is defined in terms of redshift z as

$$1 + z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a_{\text{obs}}}{a_{\text{emit}}} = \frac{1}{a_{\text{emit}}} . \quad (2.5)$$

This is an important and useful quantity because most of our observational information about the Universe comes to us through light waves.

The total comoving distance that light could have traveled since $t = 0$ to now (in the absence of interactions) is defined as

$$\tau(t) \equiv \int_0^\infty \frac{dt'}{a(t')} . \quad (2.6)$$

This is also referred to as *conformal time*, which is relevant for writing equations and interpreting them. Equations describing the propagation of light in the conformal time have the same form as those in flat, non-expanding space. The metric in conformal time

$$ds^2 = a^2(\tau)(-d\tau^2 + d\mathbf{x}^2) , \quad (2.7)$$

and the conformal Hubble rate is defined as

$$\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\tau} = aH . \quad (2.8)$$

2.2 Fundamental equations

General relativity coupled with statistical mechanics forms the basis of most needed equations in cosmology. General relativity describes the only relevant long-range force, which is gravity. It also provides a description of the background spacetime through which matter moves. The Einstein equation is relevant for this, which after evaluating it in an FLRW metric gives rise to what is known as the *Friedmann equations*. On the other hand, statistical mechanics provides a framework for describing the collective, average behavior of matter. In this context, we use the *Boltzmann equation* which describes the evolution of the phase space distribution functions of various particle species in an expanding Universe. Then we go back to the Einstein equation, but this time in a perturbed Universe.

2.2.1 Friedmann equations

The starting point is the Einstein equation given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} , \quad (2.9)$$

where $G_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant, $g_{\mu\nu}$ is the FLRW metric as defined in Eq. (2.1), and G is Newton's constant. $T_{\mu\nu}$ represents the stress-energy tensor,

$$T_{\mu\nu} = \text{diag} [\rho, \mathcal{P}, \mathcal{P}, \mathcal{P}] , \quad (2.10)$$

here ρ is the energy density, and $\mathcal{P}$ is the pressure. Evaluating the Einstein equation in the FLRW metric

$$H^2(t) = \frac{8\pi G}{3}\rho = H_0^2 \left(\Omega_r a^{-4}(t) + \Omega_m a^{-3}(t) + \Omega_\Lambda \right) . \quad (2.11)$$

This equation describes the evolution of different components in accordance with how their energy density scales. $\Omega_i \equiv \rho_i(t_0)/\rho_{\text{cr}}$ refers to the density of radiation ($i = r$) or matter ($i = m$) evaluated in the present time (t_0) with respect to the critical density $\rho_{\text{cr}} = 3H_0^2/8\pi G$. While the dark energy component is considered to be constant at all times with $\Omega_\Lambda = \rho_\Lambda/\rho_{\text{cr}} = \Lambda/8\pi G\rho_{\text{cr}}$.

The Friedmann equation as given in Eq. (2.11) describes a Λ CDM model with two main assumptions: the density of dark energy is always constant and the Universe is flat. These two assumptions can be loosened to have a more general form of the equation as follows

$$H^2(t) = \frac{8\pi G}{3}\rho = H_0^2 \left(\Omega_r a^{-4}(t) + \Omega_m a^{-3}(t) + \Omega_{\text{de}} a^{-3(1+w)}(t) + \Omega_k a^{-2}(t) \right) . \quad (2.12)$$

The first modification concerns dark energy, which evolves over time depending on the equation of state parameter w . Defined as the ratio of dark energy pressure to its energy density, $w = P_{\text{de}}/\rho_{\text{de}}$, this parameter determines the nature of cosmic acceleration. This extension is commonly referred to as the w CDM model. Three distinct scenarios arise based on its value. First, *Cosmological Constant* ($w = -1$): Dark energy behaves as a constant vacuum energy density, corresponding to the cosmological constant Λ , as described in Eq. (2.11). Second, *Phantom Energy* ($w < -1$): the energy density of dark energy increases over time, leading to an acceleration stronger than in the Λ CDM model. Third, *Dynamical Dark Energy* ($-1 < w < -1/3$): In this case, dark energy is represented by a dynamical field (e.g., a rolling scalar field), with an energy density that evolves as the universe expands. A more general framework allows w to evolve with time, following the parametrization $w(a) = w_0 + w_a(1 - a)$, where w_0 is the present-day value and w_a describes its evolution with the scale factor a . This extended model is known as the $w_0 w_a$ CDM model [55].

The second change in Eq. (2.12) is introducing the curvature density Ω_k . The value of Ω_k determines the geometry and expansion of the Universe. Specifically: $\Omega_k = 0$ describes a flat Universe, $\Omega_k > 0$ for an open Universe, and $\Omega_k < 0$ for a closed Universe.

w CMD, $w_0 w_a$ CDM, and including Ω_k are standard extensions of the Λ CDM model. They have been investigated in cosmological inference from several data such as [17, 18, 25–27, 41]. So far, most data point to a flat Universe with a constant density of dark energy. Nevertheless, recent work, such as in [56], has stimulated a discussion about an evolving dark energy equation of state.

2.2.2 Boltzmann equation

The Boltzmann equation describes the evolution of the distribution function of a particle species due to the spatial expansion and the momentum exchanges through interactions or possible decays. In its very compact version, it can be written as

$$\frac{df}{dt} = C[f], \quad (2.13)$$

where the left-hand side (LHS) is a total time derivative of the distribution function $f(\mathbf{x}, \mathbf{p}, t)$, where $\mathbf{p} \equiv m\dot{\mathbf{x}}$ is the particle momentum vector. When the total derivative is set to zero, the number of particles in a given element of phase space is conserved. The right-hand side (RHS) is the collision term, which contains any possible interaction, particle decay, annihilation, and pair production. In the presence of a collision term, the number density of a particle species is not conserved anymore.

As we know, our Universe has structures (galaxies, galaxy clusters, and voids), meaning it is inhomogeneous. Yet, CMB data provide clear evidence that the early Universe was much smoother, with an almost uniform temperature exhibiting small fluctuations on the order $\delta T/T \sim 10^{-5}$. Therefore, we can approximate the Universe to be homogeneous and isotropic with small perturbations that grew over time. Linear Perturbation theory provides a powerful framework to study the evolution of these tiny perturbations.

We start by introducing a perturbed spacetime

$$ds^2 = -[1 + 2\Psi(\mathbf{x}, t)]dt^2 + a^2(t)\delta_{ij}[1 + 2\Phi(\mathbf{x}, t)]dx^i dx^j, \quad (2.14)$$

where Ψ and Φ are the scalar potentials associated with the perturbation in the time-time and space-space components, respectively¹, in a conformal-Newtonian-gauge. In principle there could be other perturbations; vector and tensor perturbations, yet, we are interested here only in the scalar ones.

The total derivative in Eq. (2.14) can be written as a function of partial derivatives of the distribution function. We first need to know how particles move within the perturbed spacetime. For that, we evaluate the geodesic equation to compute dx^i/dt , dp^i/dt . The total derivative becomes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{p}{E} \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i} - \left[H + \dot{\Phi} + \frac{E}{ap} \hat{p}^i \frac{\partial \Psi}{\partial x^i} \right] p \frac{\partial f}{\partial p}, \quad (2.15)$$

$E(p) = \sqrt{p^2 + m^2}$ is the particle energy, and the dot represents the time derivative. Here we decompose the vector of the momentum into an amplitude $p = |\mathbf{p}|$ and a unit vector $\hat{p}^i$, as $p^i \equiv p\hat{p}^i$.

For a general interaction process $(1)_{\mathbf{p}_1} + (2)_{\mathbf{p}_2} \longleftrightarrow (3)_{\mathbf{p}_3} + (4)_{\mathbf{p}_4}$, the collision term is

¹In the literature, Ψ and Φ can be interchanged in describing space or time component, and might have a different sign convention.

defined as

$$\begin{aligned} \mathcal{C}[f_1(\mathbf{p}_1)] = & \frac{1}{2E_1(p)} \int d\Pi_2 d\Pi_3 d\Pi_4 \left(|\mathcal{M}^2| \right) (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4) \\ & \times \left\{ f_3(\mathbf{p}_3) f_4(\mathbf{p}_4) [1 \pm f_1(\mathbf{p}_1)] [1 \pm f_2(\mathbf{p}_2)] \right. \\ & \left. - f_1(\mathbf{p}_1) f_2(\mathbf{p}_2) [1 \pm f_3(\mathbf{p}_3)] [1 \pm f_4(\mathbf{p}_4)] \right\}, \end{aligned} \quad (2.16)$$

where $d\Pi_s \equiv d^3p_s/(2\pi)^3 2E_s$, and $\delta^4(\dots)$ is the Dirac delta in four dimensions, it ensures energy-momentum conservation. $|\mathcal{M}^2|$ is the squared scattering amplitude, it encodes the particle physics details of the interaction and can be computed with Feynman diagrams. The first term represents the forward reaction and the second one refers to the reverse reaction. Depending on the case, the reaction might go mostly in one direction. To account for quantum effects, $[1 \pm f_s(\mathbf{p}_s)]$ factors are included for Pauli blocking (-) or Bose enhancement (+). They can be neglected in the non-relativistic limit.

Photons

For photons, we consider a perturbed distribution function, where at zero-order it is described by a Bose-Einstein distribution ($f^{(0)} \equiv [\exp\{\frac{p}{T}\} - 1]^{-1}$). We include perturbations at linear order through $\Theta = \frac{\delta T}{T}$. In general, it takes the form

$$\begin{aligned} f(\mathbf{x}, \mathbf{p}, t) &= \left[\exp \left\{ \frac{p}{T(t)[1 + \Theta(\mathbf{x}, \hat{\mathbf{p}}, t)]} \right\} - 1 \right]^{-1} \\ &= f^{(0)}(p, t) - p \frac{\partial f^{(0)}(p, t)}{\partial p} \Theta(\mathbf{x}, \hat{\mathbf{p}}, t), \end{aligned} \quad (2.17)$$

The total derivative in Eq. (2.15) can be further simplified for massless particles where $E = p$. Keeping only terms at first order, it becomes

$$\left. \frac{df}{dt} \right|_{\text{first order}} = -p \frac{\partial f^{(0)}}{\partial p} \left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right]. \quad (2.18)$$

At the time before recombination, photons interact with electrons via Compton scattering. The temperature of the Universe at that time is $T \sim 0.3 \text{ eV}$, which is much smaller than the electron's mass ($m_e/T \gg 1$). This means that the electron's momentum is much higher than the photon's momentum and allows us to take the non-relativistic limit of the interaction, $E_\gamma = p \sim T$, $E_e - m_e = q^2/(2m_e) \sim T \rightarrow q \sim \sqrt{2m_e T}$

The squared amplitude, averaged over initial states and summed over final states, is given by

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 8\pi\sigma_T m_e^2, \quad (2.19)$$

where σ_T is the Thomson cross-section. Note that we are averaging here over the angle

for simplicity; this has a subdominant effect on the final result. For completeness, one can keep the angular dependence, which leads to a small polarization effect in the CMB.

Evaluating the collision term gives

$$\mathcal{C}[f(\mathbf{p})] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \mathbf{v}_b], \quad (2.20)$$

where $\mathbf{v}_b$ is baryons' velocity (same as electrons velocity as will be described below when discussing the Boltzmann equations for baryons). It is useful to work in Fourier space and expand temperature fluctuations in terms of Legendre polynomials as

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^1 \frac{d\mu}{2} \mathcal{P}_l(\mu) \Theta(\mu), \quad (2.21)$$

where $\mu = (\mathbf{k} \cdot \hat{\mathbf{p}})/k$ is the cosine of the angle between the wavenumber $\mathbf{k}$ and the photon direction $\hat{\mathbf{p}}$. Θ_0 defines the *monopole*, which is the integral over all angles and is related to the density contrast via $\delta_\gamma = 4\Theta_0$, and Θ_1 is the dipole, which defines the direction of movement of photons and is related to its velocity via $\mathbf{v}_\gamma = -3i\Theta_1$.

Combining both sides of the Boltzmann equation from the results in Eq. (2.18) and Eq. (2.20), we get

$$\dot{\Theta} + ik\mu\Theta + \dot{\Phi} + ik\mu\Psi = -\dot{\kappa} [\Theta_0 - \Theta + \mu v_b] \quad (2.22)$$

here $\dot{\kappa} = -an_e\sigma_T$ is the optical depth.

Dark matter

The main steps of obtaining the Boltzmann equation for dark matter is very similar to photons with a few differences. First, in Λ CDM paradigm, DM is assumed to be collisionless, i.e., we do not have to deal with any collision term. Second, DM is non-relativistic and massive; its energy can be approximated to $E \approx m(1 + \frac{1}{2} \frac{p^2}{m^2})$. This leads to a velocity term $p/E \sim p/m$ in the total derivative in Eq. (2.15), which suppresses any free-streaming effect. Therefore, DM can be treated as an effective fluid that can be described by only the first two moments (density and velocity dispersion). Unlike photons, for DM we do not need to know the specific distribution function at zero-order. First, we multiply the total derivative in Eq. (2.15) by $d^3p/(2\pi)^3$ and integrate. By recalling the definition of the number density given in Eq. (A.2) and velocity, we get an equation for the number density as

$$\frac{\partial n_c}{\partial t} + \frac{1}{a} \frac{\partial(n_c v^i)}{\partial x^i} + 3 \left[H + \frac{\partial \Phi}{\partial t} \right] n_c = 0 \quad (2.23)$$

We expand n_c to a zero-order plus some perturbation δ_c .

$$n_c(\mathbf{x}, t) = n_c^{(0)}(t) [1 + \delta_c(\mathbf{x}, t)] \quad (2.24)$$

Substituting this equation in Eq. (2.23) and keeping only terms at first order, we get the first Boltzmann equation for the density of DM

$$\dot{\delta}_c + \theta_c + 3\dot{\Phi} = 0. \quad (2.25)$$

In the same way, we compute the second moment by multiplying Eq. (2.15) with $d^3p \mathbf{p}/(2\pi)^3$, integrating over momentum and keeping only terms at first order. We get the second Boltzmann equation for DM velocity

$$\dot{\theta}_c + \mathcal{H}\theta_c - k^2\Psi = 0. \quad (2.26)$$

Baryons

In cosmology the term *baryons* refers to both electrons and nucleons (protons and neutrons). Because electrons are coupled to protons via the Coulomb scattering, they are treated as one fluid with a common overdensity δ_b

$$\frac{\rho_e - \bar{\rho}_e}{\bar{\rho}_e} = \frac{\rho_p - \bar{\rho}_p}{\bar{\rho}_p} \equiv \delta_b, \quad (2.27)$$

and move with the same velocity

$$\mathbf{v}_e = \mathbf{v}_p \equiv \mathbf{v}_b. \quad (2.28)$$

Now, we can proceed in the same way as in the case of DM to get Boltzmann equations. The main difference is that baryons have a collision term, mainly due to the Compton scattering between electrons and photons, as discussed previously for photons. The Coulomb scattering mainly influences small-scale interactions and local thermalization rather than the overall evolution of baryons on cosmological scales.

The first equation is for the density contrast. The collision term is set here to zero because Compton scattering is elastic and preserves the number density.

$$\dot{\delta}_b + \theta_b + 3\dot{\Phi} = 0. \quad (2.29)$$

Then, we have the second equation for velocity

$$\dot{\theta}_b + \mathcal{H}\theta_b - c_s^2 k^2 \delta_b - k^2\Psi = \frac{\dot{\kappa}}{R} [\theta_b - \theta_\gamma], \quad (2.30)$$

with $R = (3\rho_b)/(4\rho_\gamma)$, c_s is the sound speed. This term was previously neglected for DM because it is non-relativistic and massive. It is important to include for baryons because of the momentum gained from the interaction with photons. On the LHS, the term accounts for the drag force due to the tight coupling between baryons and photons before recombination. This term ensures that baryons are dragged along by photons forming one fluid. In this fluid, photons provide pressure and baryons contribute to the mass. This makes two forces fighting against each other, leading to the propagation of

sound waves in the fluid. the sound speed is given by

$$c_s = \frac{1}{\sqrt{3(1+R)}}. \quad (2.31)$$

This effect leads to what is known as Baryon Acoustic Oscillations (BAO), which can be measured by different methods and used as a cosmological probe, as will be discussed in Sec. 2.4.3. This feature is also important in understanding the CMB, and we will discuss this in Sec. 2.4.1

Neutrinos

Neutrinos are considered relativistic and massless in the early Universe and follow an equilibrium distribution with a temperature $T_\nu(a)$ at zeroth order, which is given by a Fermi-Dirac distribution. The procedure of obtaining Boltzmann equations for neutrinos is analogous to photons. We start by introducing a perturbed distribution function at linear order with

$$f_\nu(\mathbf{x}, \mathbf{p}, t) = \left[1 - \mathcal{N}(\mathbf{x}, \mathbf{p}, t) p \frac{d}{dp} \right] f_\nu^{(0)}(p) \quad (2.32)$$

where $\mathcal{N}(\mathbf{x}, \mathbf{p}, t)$ represents the temperature fluctuation. Neutrinos are not known to have an interaction with any other species, except gravity. Therefore, there is no need to deal with a collision term. The Boltzmann equation for neutrinos at first order reads

$$\dot{\mathcal{N}} + ik\mu \frac{p}{E_\nu(p)} \mathcal{N} + \dot{\Phi} + ik\mu \frac{E_\nu(p)}{p} \Psi = 0. \quad (2.33)$$

Studies of measuring neutrinos' oscillations have provided strong evidence for non-zero neutrino masses with minimal values of $\sum m_\nu = 0.06 \text{ eV}$, see [57] for a recent review. If neutrinos are massive, the effect of their mass becomes important at late time and has to be considered. For that, we will need to keep terms with the prefactor p/E_ν , which slows down the effect of free-streaming.

Constraining the sum of neutrino masses is still under investigation from different probes. The upper limit from the CMB data is $\sum m_\nu < 0.26 \text{ eV}$ from Planck 2018 data, and it tightens when including BAO measurements $\sum m_\nu < 0.13 \text{ eV}$, see [26].

2.2.3 Einstein equations

Another important aspect in cosmology is the relation between matter perturbations and gravity potentials. Starting again from the Einstein equation (Eq. (2.9)), which we previously used to obtain the Friedman equation in Sec. 2.2.1. We use it now in a perturbed Universe with the metric in Eq. (2.14). Considering the energy-momentum tensor

$$T_\nu^\mu = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{P^\mu P^\nu}{E_p} f_i(t, \mathbf{k}, \mathbf{p}), \quad (2.34)$$

where i refers to different components (radiation, matter, neutrinos, $\dots$). The time (00) component reads

$$k^2\Psi + 3\mathcal{H}(\Psi' - \Phi\mathcal{H}) = 4\pi G a^2 \sum_i \rho_i^{(0)} \delta_i \quad (2.35)$$

The space (ij) component is

$$k^2(\Psi + \Phi) = -32\pi G a^2 \left(\rho_\gamma^{(0)} \Theta_2 + \rho_\nu^{(0)} \mathcal{N}_2 \right) \quad (2.36)$$

Today the density of photons and neutrinos is subdominant compared to the matter component or dark energy. Then, the radiation's quadrupole (Θ_2 and $\mathcal{N}_2$) can be neglected, therefore, Eq. (2.36) gives $\Phi \approx -\Psi$. Plugging this in Eq. 2.34 leave us with

$$\delta_m = \frac{2k^2 a}{3\Omega_m H_0^2} \Psi = -\frac{2k^2 a}{3\Omega_m H_0^2} \Phi. \quad (2.37)$$

This is the Newtonian limit, which leads to the Poisson equation in matter-dominating Universe.

2.3 Growth of structure

In studying the growth of structure, we want to know how tiny perturbations in the early Universe evolve in time to form the large-scale structure we observe today from galaxies to clusters. This growth is driven primarily by gravitational instability, where overdense regions become more massive as they pull in surrounding matter. First, we will deal with the linear growth, governed by the equations we introduced previously in Sec. 2.2. This is enough to describe the structure at large scales. But, when we reach small scales, the evolution of overdensities becomes more complicated and requires further tools to describe it. First we need to have initial conditions. The theory of *inflation*, introduced in [58] in 1980 by Alan Guth, provides a mechanism for generating primordial perturbations around the smooth Universe. Along the way, it also solves other problems, like: the flatness problem, and the horizon problem. The flatness problem refers to the fact that the Universe appears to be very close to geometrically flat today; this means that the total density of the Universe is very close to the critical density. Inflation solves this problem by exponentially expanding the Universe by an enormous factor in a very short period of time. The horizon problem refers to the observation that different regions in the Universe, which are separated by large distances, have nearly identical physical properties; for example, CMB photons have a nearly uniform temperature of $T_0 = 2.728 \pm 0.002$ K. Because the Universe expands so rapidly, due to inflation, regions that were originally very close to each other and in causal contact were stretched to vast distances, far beyond the observable horizon today. As a result, these regions share the same physical properties, such as temperature, because they were in thermal equilibrium before inflation. More information on inflation and its theories can be found in recent reviews as [59, 60].

2.3.1 Linear evolution

To describe the distribution of matter in the Universe, we begin with the total matter density contrast δ_m

$$\delta_m = \frac{\rho_c}{\rho_m} \delta_c + \frac{\rho_b}{\rho_m} \delta_b, \quad (2.38)$$

where $\rho_m = \rho_c + \rho_b$ represents the total matter density, consisting of cold dark matter (CDM) and baryonic matter. The statistical properties of the matter distribution are typically characterized by the matter power spectrum, which is defined through the Fourier-space correlation function of the density contrast

$$\langle \delta_m(\eta, \mathbf{k}) \delta_m(\eta, \mathbf{k}') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_L(k, a). \quad (2.39)$$

The matter power spectrum, $P_L(k, a)$, encodes information about the growth and distribution of cosmic structures. To determine its exact form, one must solve the coupled Boltzmann and Einstein equations, introduced in previous sections, while incorporating initial conditions from inflation. Since these equations are highly complex, numerical computations using cosmological Boltzmann solvers such as **CAMB** [61, 62] and **CLASS** [63, 64] are required. However, an approximate analytical expression for $P_L(k, a)$ is given by [65]

$$P_L(k, a) = \frac{8\pi^2}{25} \frac{A_s}{\Omega_m} \frac{k^{n_s}}{H_0^4 k_p^{n_s-1}} T^2(k) D_+^2(a). \quad (2.40)$$

Here, $k_p = 0.05 \text{ Mpc}^{-1}$ is the pivot scale, A_s is the amplitude of primordial scalar fluctuations, and n_s is the spectral index of the primordial power spectrum. The transfer function $T(k)$ encodes the evolution of perturbations across different scales due to effects such as radiation pressure, baryonic acoustic oscillations, and free-streaming of relativistic particles. The growth factor $D_+(a)$ quantifies how density fluctuations evolve over time.

For large scales ($k \ll k_{\text{eq}}$), where k_{eq} is the wavenumber corresponding to matter-radiation equality, the transfer function asymptotically approaches unity, meaning that these modes evolve linearly from the early Universe. However, at small scales ($k \gg k_{\text{eq}}$), the transfer function suppresses power due to various effects, including radiation pressure and collisionless damping mechanisms like free-streaming. While analytical approximations, such as the BBKS (Bardeen-Bond-Kaiser-Szalay) method, exist for estimating $T(k)$, precise calculations require numerical methods.

The large- and small-scale behavior of the power spectrum follows a well-known scaling relation

$$P_L(k) \propto \begin{cases} k & k \ll k_{\text{eq}}, \\ k^{-3} \ln(k) & k \gg k_{\text{eq}}. \end{cases} \quad (2.41)$$

Observationally, the matter power spectrum can be inferred from large-scale structure surveys, including galaxy clustering measurements, weak gravitational lensing (cosmic shear), and Lyman- α forest data, as well as from cosmic microwave background (CMB) anisotropies and baryon acoustic oscillations (BAO). Fig. 2.1 compares the theoretically

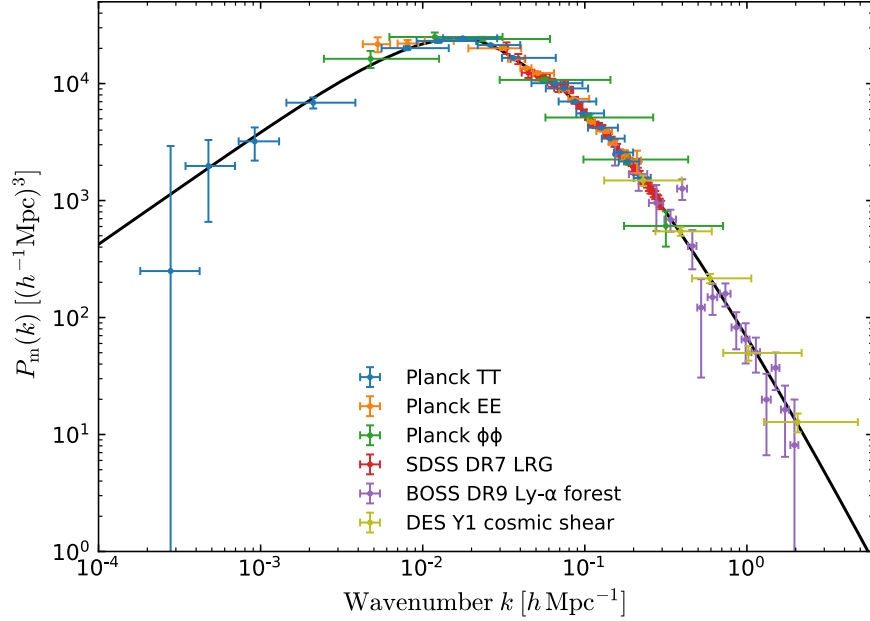


Figure 2.1. The matter power spectrum in the Λ CDM model (black line) compared to observational data from galaxy surveys, CMB, and Lyman- α measurements. Adapted from [66].

predicted matter power spectrum within the Λ CDM model (black line) to measurements from different cosmological probes. As seen in the figure, CMB data provides precise constraints on large-scale modes, while cosmic shear and Lyman- α forest observations are crucial for probing small-scale fluctuations.

2.3.2 Beyond linear evolution

So far we have considered only first order perturbations in matter density and in the metric. This is a good approximation to use when describing the early Universe. Yet, moving to understand late Universe and its structures requires going beyond the linear theory. Luckily, gravity is weak even at late Universe, therefore, it is fine to continue working in an FLRW metric and introduce the non-linearity for matter density. For this purpose, there are several developed tools depending on the problem at hand. As examples: perturbation theory, effective field theory of large-scale structure (EFTofLSS), halo model, and spherical collapse. Simulations also play a big role in the understanding of non-linear regimes.

In the following, we will focus mostly on the spherical collapse model and the halo mass function. The reason is that this framework is relevant for studying galaxy cluster abundance.

Spherical collapse

The *spherical collapse model* describes the formation of dark matter halos from small initial density perturbations. These perturbations, encoded in the metric perturbations of Eq. (2.14), grow proportionally to the linear growth factor, $\delta(a) \propto D_+(a)$. In the early Universe, when perturbations are small, their evolution can be described by linear theory. However, once the overdensity δ exceeds a critical threshold, $\delta_c \approx 1.686$ for a matter-dominated Universe, the region enters a *non-linear regime* and collapses under its own gravity, leading to the formation of a gravitationally bound object—a dark matter halo.

During collapse, the system undergoes violent relaxation and eventually settles into a *virialized state*. At this stage, the halo satisfies the *virial theorem*, which relates the kinetic energy K and gravitational potential energy U as

$$2K + U = 0. \quad (2.42)$$

The *virial radius* of the halo, a characteristic scale defining its size, is given by

$$R_{\text{vir}} = \left(\frac{3M}{4\pi\Delta_c\bar{\rho}} \right)^{1/3}, \quad (2.43)$$

where M is the total halo mass, $\bar{\rho}$ is the mean background density of the Universe, and Δ_c is the critical density contrast at collapse. The value of Δ_c depends on the halo definition used in numerical simulations, with typical values ranging from 176 to 500 (e.g., $\Delta_c = 200$ is commonly used to define halos in the spherical overdensity method).

Halo mass function

The Press-Schechter (PS) approach [67] provides a first-principle derivation of the halo mass function using linear perturbation theory. It predicts the number density of halos of mass M at redshift z , based on the assumption that a region collapses into a halo if its smoothed density contrast exceeds a critical threshold.

Starting from a smoothed density contrast δ_r , a spherical collapse occurs if δ_r exceeds the critical threshold δ_c . The fraction of collapsed regions where $\delta_r > \delta_c$ is given by

$$\begin{aligned} F(> \delta_c) &= 2 \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\delta_c}^{\infty} d\delta_r e^{-\delta_r^2/2\sigma^2} \\ &= 2 \frac{1}{\sqrt{2\pi}} \int_{\nu}^{\infty} d\nu e^{-\nu^2/2}, \quad \text{with } \nu = \frac{\delta_c}{\sigma}. \end{aligned} \quad (2.44)$$

Here, $\sigma(M, z)$ is the variance of the density fluctuations smoothed on a scale corresponding to mass M at redshift z . The factor of 2 was introduced as an *ad hoc* correction to ensure proper mass conservation in the limit $M \rightarrow 0$. This factor was later justified through the excursion set formalism by Bond et al. [68], which interprets structure formation in terms of the first-passage probability of a stochastic density field crossing a critical threshold.

The probability that a density fluctuation crosses the barrier δ_c for the first time at a given smoothing scale determines the number of halos of that mass. This leads to the Press-Schechter mass function

$$\frac{dn(M, z)}{d \ln M} = \frac{\rho_m(t_0)}{M} f_{\text{PS}} \left(\frac{\delta_{\text{cr}}}{\sigma(M, z)} \right) \left| \frac{d \ln \sigma(M, z)}{d \ln M} \right|, \quad (2.45)$$

where the mass function shape is determined by the function

$$f_{\text{PS}}(\nu) = \sqrt{\frac{2}{\pi}} \nu e^{-\nu^2/2}, \quad (2.46)$$

which describes the fraction of mass contained in halos of mass M at a given redshift z .

The variance of the smoothed density field is computed from the linear matter power spectrum $P(k)$ as

$$\sigma^2(M) = \int_0^\infty P(k) \hat{W}^2(k, R) k^2 dk, \quad (2.47)$$

where $\hat{W}(k, R)$ is a window function used to smooth the density field, typically a "spherical top-hat" filter. When computed at $R = 8 h^{-1} \text{Mpc}$, it gives the famous σ_8 parameter which is widely used in cosmology and will be discussed extensively in the next chapters.

While the Press-Schechter model provides a useful framework for understanding halo formation, it has several limitations

- The underlying density field is not necessarily *Gaussian*, and non-linear effects can introduce deviations from the predicted mass function.
- The assumption of *spherical collapse* oversimplifies real structure formation, which is better described by *ellipsoidal collapse* models.
- The *first-crossing condition* does not fully account for environmental effects and correlations between different scales.

To address these issues, modifications such as the *Sheth-Tormen* formalism introduce empirical corrections to better match numerical simulations (see [70]). These models often require calibration against high-resolution N -body simulations to improve their accuracy. An example is the *Tinker mass function* [69], which generalizes the PS form using a fitting function

$$f(\sigma) = A \left[\left(\frac{\sigma}{b} \right)^{-a} + 1 \right] e^{-c/\sigma^2}, \quad (2.48)$$

where the parameters A , a , b , and c are obtained from numerical simulations. Figure 2.2 illustrates how the modified halo mass functions compare with data points from simulations.

Modern approaches to halo mass functions incorporate non-Gaussian initial conditions, baryonic effects, and improved modeling of large-scale structure formation. Combining excursion set theory, ellipsoidal collapse models, and high-resolution simulations allows

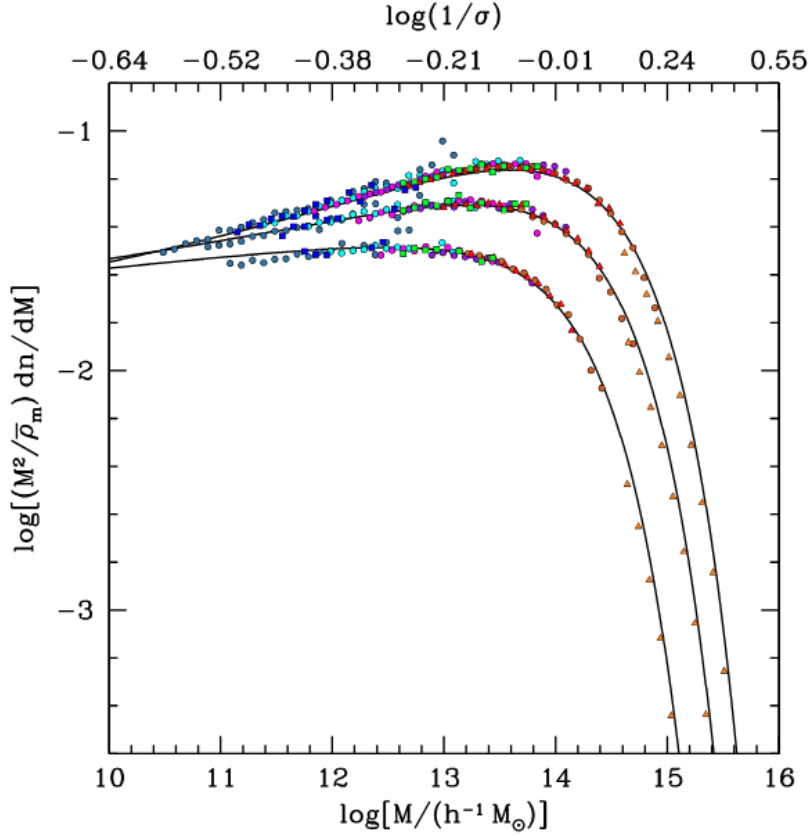


Figure 2.2. Calibrated halo mass functions (lines), as introduced in Eq. (2.48) compared to simulations (data points for different choices (Δ of 200, 800, and 3200)). $M_{\odot}$ refers to solar mass. From [69].

for a more accurate description of the cosmic web. However, these simulations are computationally expensive and sometimes cannot accommodate beyond Λ CDM model.

2.4 Cosmological probes

To study the structure of the Universe and constrain cosmological model parameters, there exist several cosmological probes at different times and scales. In this section, we will review the main probes that were employed in our analyses and their results are described in Sec.

2.4.1 Cosmic microwave background (CMB)

The cosmic microwave background (CMB) is the relic radiation that reaches us today from the epoch of recombination, when the Universe transitioned from being opaque to

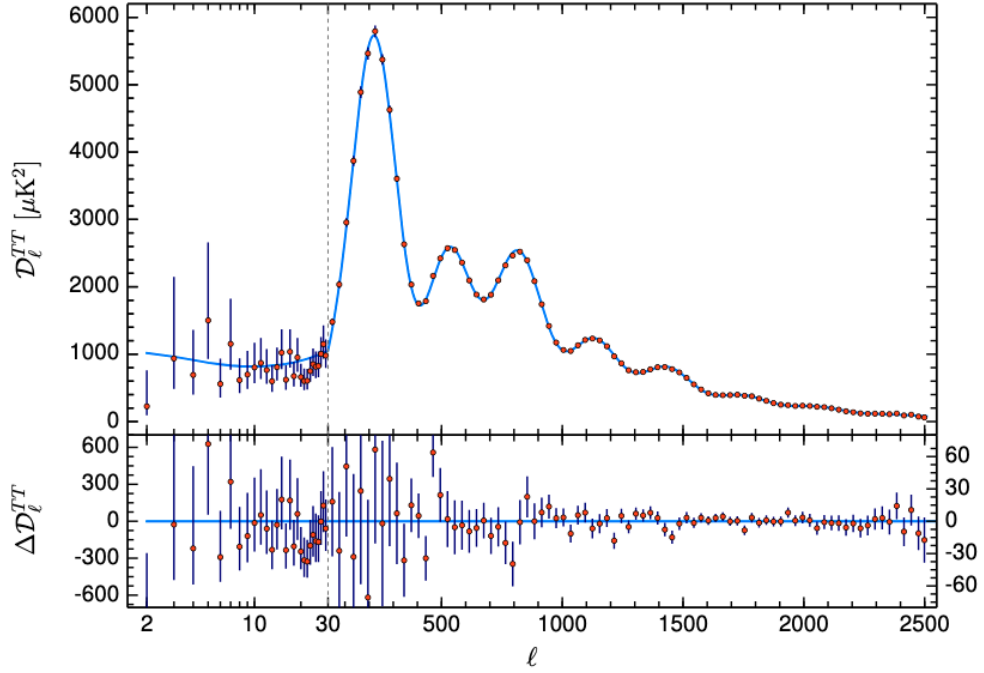


Figure 2.3. Temperature power spectrum from CMB measurements. From [26].

transparent as electrons and protons combined to form neutral hydrogen. This decoupling of photons from matter allows the CMB to serve as a snapshot of the Universe at its early stages, making it one of the most powerful observational tools in cosmology.

The CMB was first detected accidentally in 1964 by Arno Penzias and Robert Wilson [71], an achievement that later earned them the Nobel Prize. Initial observations revealed that the Universe has a nearly uniform temperature of approximately 3 K. However, subsequent, more precise measurements uncovered small anisotropies at the level of $\sim 10^{-5}$. These anisotropies were first detected by COBE in 1992, followed by significantly higher precision observations from WMAP [72] and Planck [26, 73], refining our understanding of cosmological parameters.

The temperature fluctuation in a given direction $\hat{\mathbf{p}}$ on the sky is commonly expressed as

$$\Theta(\hat{\mathbf{p}}) = \frac{T(\hat{\mathbf{p}}) - T_0}{T_0}, \quad (2.49)$$

where $T_0 = (2.7255 \pm 0.0006)$ K is the mean temperature of the CMB. These anisotropies are studied by expanding the temperature and polarization maps in terms of spherical harmonics

$$\Theta(\tau, \mathbf{x}, \hat{\mathbf{p}}) = \sum_{\ell, m} a_{\ell m}(\tau, \mathbf{x}) Y_{\ell m}(\hat{\mathbf{p}}), \quad (2.50)$$

where $Y_{\ell m}(\hat{\mathbf{p}})$ are spherical harmonics, and $a_{\ell m}(\tau, \mathbf{x})$ are the expansion coefficients that quantify the anisotropies. Under the assumption of Gaussian primordial fluctuations,

the statistical properties of these anisotropies can be described by the angular power spectrum C_ℓ

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \delta_{\ell\ell'}^K \delta_{mm'}^K C_\ell, \quad (2.51)$$

where δ^K is the Kronecker delta function, ensuring statistical isotropy, and ℓ corresponds to the angular scale. A commonly used quantity is the dimensionless power spectrum

$$\mathcal{D}_\ell = T_0^2 \ell(\ell+1) C_\ell, \quad (2.52)$$

which is typically plotted to visualize the acoustic peaks in the CMB power spectrum (see Fig. 2.3). This spectrum encodes information about fundamental physical processes in the Universe, such as

- **Baryon Acoustic Oscillations (BAO):** Oscillations in the photon-baryon plasma due to interactions before recombination, imprinting characteristic scales in both the CMB and large-scale structure.
- **Sachs-Wolfe Effect:** The gravitational redshift of CMB photons as they escape potential wells, leading to large-scale temperature fluctuations.
- **Sunyaev-Zeldovich Effect (SZE):** The scattering of CMB photons off hot electrons in galaxy clusters, distorting the CMB spectrum.
- **Doppler shift and Silk Damping:** The velocity of the baryon-photon fluid and diffusion damping that erase fluctuations at small scales.

CMB anisotropies provide a precise measurement of the matter power spectrum, especially at large scales. The connection between the angular power spectrum C_ℓ and the 3D matter power spectrum $P(k)$ is given by

$$C_\ell = \frac{2}{\pi} \int_0^\infty dk k^2 P(k) |\tau_\ell(k)|^2, \quad (2.53)$$

where $\tau_\ell(k)$ is the transfer function encoding the evolution of perturbations. Through these anisotropies, the CMB carries an immense wealth of information about the fundamental properties of the Universe, including its composition, geometry, and evolution. Measurements of the CMB power spectrum allow us to determine key cosmological parameters, such as the Hubble constant, dark matter and dark energy densities, and the primordial fluctuation spectrum, making it a cornerstone of modern cosmology. Other CMB surveys that refined these measurements are SPT-3G [28], ACT [27], the ongoing Simons Observatory survey [35], and the future CMB-S4 [29].

2.4.2 Galaxy cluster abundance

In contrast to the cosmic microwave background (CMB), which probes the early Universe, galaxy clusters serve as a powerful cosmological probe of the Universe at late times. The tiny primordial fluctuations in the matter distribution grow under gravitational instability, eventually forming the largest gravitationally bound structures in the

Universe—galaxy clusters. These clusters consist of hundreds to thousands of galaxies embedded in a massive dark matter halo, along with a hot intracluster medium (ICM) and non-thermal components such as cosmic rays and magnetic fields. The typical total mass of a galaxy cluster lies in the range of $10^{14} - 10^{15} M_{\odot}$, making them critical tracers of large-scale structure formation. Main components of galaxy clusters are:

- **Galaxies ($\sim 1 - 5\%$ of total mass):** Clusters contain a mix of galaxy types. Elliptical galaxies dominate in the dense central regions, characterized by red, passive, and old stellar populations. Spiral galaxies are more prevalent in the outskirts, where environmental effects are weaker. A special category is the *Brightest Cluster Galaxy*, a massive elliptical galaxy typically found near the cluster center, thought to grow via mergers and dynamical processes. For example, see Fig. 2.4, which presents a high-resolution infrared image of a galaxy cluster captured by the James Webb Space Telescope, showcasing intricate details of cluster galaxies.
- **Dark Matter ($\sim 85\%$ of total mass):** The dominant mass component of a galaxy cluster is its dark matter halo, which governs the cluster’s gravitational potential. This is why clusters are often studied in the framework of the *halo mass function* (see Sec. 2.3.2). Clusters serve as an important tool for probing the nature of dark matter.
- **Intracluster Medium (ICM, $\sim 10 - 15\%$ of total mass):** The ICM consists of hot, ionized plasma at temperatures ranging from 10^7 to 10^8 K. The ICM serves in two detection methods of clusters; X-ray observations and thermal Sunyaev-Zel’dovich Effect (tSZE).

Clusters can be detected through multiple observational techniques. 1) *X-ray observations*: the hot ICM emits X-rays through thermal bremsstrahlung radiation, which is then observed using X-ray telescopes such as eROSITA [74] and Chandra [75]. 2) *Optical and near-infrared surveys*: clusters are identified as overdensities of galaxies, often using red-sequence selection methods, for e.g., surveys like the Sloan Digital Sky Survey (SDSS) and Dark Energy Survey (DES) [76]. 3) *Thermal Sunyaev-Zel’dovich Effect (tSZE)*: refers to the scattering of CMB photons off electrons in the ICM. This method is particularly useful for detecting clusters at high redshift, and will be the focus of this thesis.

The tSZE occurs when CMB photons scatter off high-energy electrons in the hot ICM, an interaction known as **inverse Compton scattering**. This boosts the energy of CMB photons, distorting the CMB spectrum and creating a characteristic frequency-dependent signature. Several major surveys have constructed galaxy cluster samples based on the SZE, including: South Pole Telescope (SPT) [77–79], Atacama Cosmology Telescope (ACT) [80], and Planck Satellite [81].

A crucial step in cluster abundance analysis is the accurate calibration of *observable-to-mass* relations. Since the total mass of a galaxy cluster is not directly observable, we rely on indirect probes to infer it. One of the most powerful methods for cluster mass estimation is *weak gravitational lensing*, which provides a direct measurement of



Figure 2.4. Infrared image of the galaxy cluster SMACS 0723, which is 4.6 billion light years away from Earth. Credit: NASA JWST.

the total matter distribution, including dark matter. Weak lensing exploits the subtle distortions in the shapes of background galaxies due to the gravitational potential of the foreground cluster. The observed shear signal is statistically analyzed to reconstruct the projected mass density, which is then used to infer the total mass of the cluster, see, e.g., [82, 83]. This will be discussed in more detail in chapters 7 and 8.

2.4.3 Baryon acoustic oscillations (BAO)

Before recombination, baryons and photons formed a tightly coupled plasma, where two opposing forces governed the dynamics: radiation pressure from photons pushed matter outward, while gravity acted to pull it inward. This interplay gave rise to oscillations in the plasma, known as *baryon acoustic oscillations* (BAO). The maximum distance these waves could travel before recombination is known as the *sound horizon*, which serves as a *standard ruler* in cosmology. This characteristic scale, often referred to as the *drag distance* r_{drag} , is given by

$$r_{\text{drag}} = \int_0^{t_{\text{drag}}} dt' \frac{c_s(t')}{a(t')} = \int_{z_{\text{drag}}}^{\infty} dz' \frac{c_s(z')}{H(z')}, \quad (2.54)$$

where c_s is the sound speed in the baryon-photon plasma, $H(z)$ is the Hubble expansion rate, and z_{drag} is the redshift at which baryons were released from the photon field.

After recombination, as photons decoupled from baryons, these acoustic waves ceased propagating and left a static imprint in the distribution of matter, forming a characteristic scale in the large-scale structure, as illustrated in Fig. 2.5. This imprint is

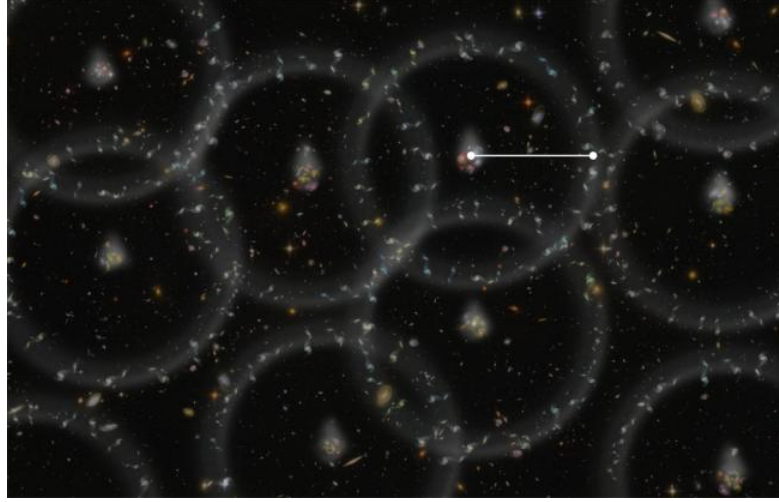


Figure 2.5. Sketch of BAO imprints in the Universe, offering a standard ruler in cosmology. Credit: Zosia Rostomian, Lawrence Berkeley National Laboratory

observable as an enhanced probability of finding galaxy pairs separated by ~ 150 , Mpc, which corresponds to the sound horizon at recombination. BAO can be detected through statistical analyses of galaxy clustering in redshift space, making galaxy surveys an ideal probe for this effect. Additionally, BAO manifest in the cosmic microwave background (CMB) power spectrum, providing complementary constraints on cosmological parameters. Recent measurements of BAO have been obtained from a variety of galaxy surveys, including eBOSS [84], DES Y6 [85], and DESI [56].

2.4.4 Galaxy clustering

Galaxy clustering describes the large-scale distribution of galaxies in the Universe, which is shaped by the gravitational growth of structure. Due to gravity, galaxies do not form a uniform distribution but instead cluster together, forming cosmic structures such as filaments, walls, and voids. To study this distribution, galaxy redshift surveys map the three-dimensional positions of galaxies by measuring their redshifts, which serve as proxies for their distances (see Fig. 2.6). These surveys provide crucial insights into the large-scale structure of the Universe and its evolution over cosmic time.

Theoretically, galaxy clustering is analyzed using statistical tools, with the two-point correlation function and its Fourier counterpart, the power spectrum, being the primary measures. In the linear regime, the galaxy density field traces the underlying matter distribution with a bias parameter, b , which encapsulates the relationship between galaxy and matter overdensities. This is expressed as

$$P_g(k) = b^2 P_m(k), \quad (2.55)$$

where $P_g(k)$ is the galaxy power spectrum, and $P_m(k)$ is the matter power spectrum.

On smaller, nonlinear scales, the clustering signal becomes more complex and requires advanced modeling techniques. One approach is the *halo model*, which describes the matter distribution by considering the properties of dark matter halos that host galaxies, see [86] for a review. Another approach incorporates perturbative methods such as the Effective Field Theory of Large-Scale Structure (EFTofLSS), which systematically accounts for nonlinear corrections and small-scale effects (for a review, see, e.g., [87]). A comprehensive analysis must also include observational effects such as Redshift-Space Distortions (RSD). RSD arises from the peculiar velocities of galaxies, causing distortions in the clustering pattern when observed in redshift space, see [88]. This effect provides a key probe of the growth of cosmic structure and can be used independently as a measurement of $f\sigma_8$, where f is the linear growth rate. Additionally, galaxy clustering measurements capture the BAO signal.

Several galaxy redshift surveys have provided high-precision clustering measurements. These include the SDSS, which consists of BOSS [21, 22, 89] and eBOSS [90], as well as earlier surveys like 2dFGRS [91]. More recent and ongoing surveys include DES [92] and DESI [93]. Looking ahead, the Euclid space mission [33] and the Rubin Observatory’s Legacy Survey of Space and Time (LSST) [34] will significantly improve our understanding of galaxy clustering with unprecedented precision and coverage.

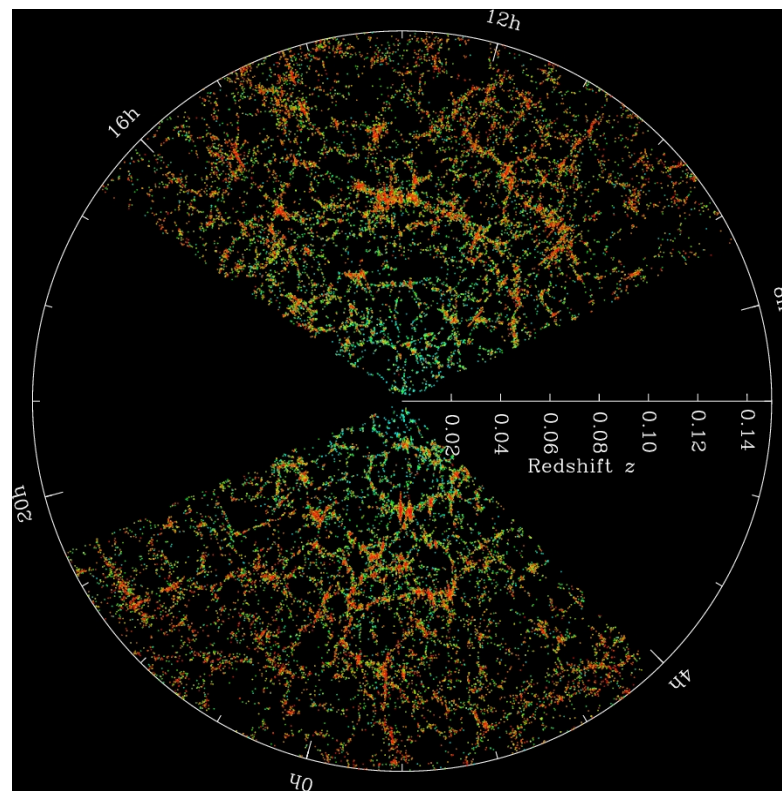


Figure 2.6. Galaxy clustering map in redshift space from SDSS. credit: M. Blanton and the Sloan Digital Sky Survey.

3 Standard Model of Particle Physics

The connection between *particle physics* and *cosmology* is one of the most profound links in modern physics. While particle physics explores the fundamental constituents of matter and their interactions at microscopic scales, cosmology seeks to understand the universe’s large-scale structure and evolution. These two fields converge when studying the early universe, where high-energy conditions resemble those found in particle accelerators. A prime example of this intersection is the study of *dark matter*. Despite overwhelming astrophysical evidence, dark matter remains undetected in laboratory experiments, and its nature cannot be explained within the framework of the Standard Model (SM) of particle physics.

Understanding dark matter necessitates the *extension of the Standard Model*, as none of its known particles exhibit the required properties to account for cosmological observations. This motivates a wide range of *Beyond Standard Model* (BSM) theories, proposing candidates such as Weakly Interacting Massive Particles (WIMPs), axions, sterile neutrinos, and even macroscopic objects like primordial black holes. To explore these possibilities, both cosmological and particle physics tools must be employed.

The SM provides the mathematical and conceptual framework to describe all known fundamental forces—except gravity—through the interactions of quarks, leptons, and gauge bosons. It is an extremely successful theory, accurately predicting experimental results across a vast range of energy scales.

In this chapter, we will review what constitutes the standard model and introduce the main quantities that will be useful in understanding the topic of this thesis: interacting dark sector models.

3.1 The Standard Model in a nutshell

The standard model of particle physics describes the particles and their interactions. Covers three fundamental forces in nature; electromagnetism, weak, and strong interactions. It contains fermions (spin $\frac{1}{2}$), which are divided into two groups; leptons (electron, muon, and tau with their corresponding neutrinos), and quarks (up, down, top, bottom, charm, and strange). Then we have gauge bosons (spin 1), which are interaction mediators; photon (γ), gluon (g), W and Z bosons. And the Higgs boson H (spin 0). See Fig. 3.1.

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{Yukawa}} \quad (3.1)$$

Adding the Lagrangian for each part, we obtain

$$\begin{aligned}
\mathcal{L}_{\text{SM}} = & -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{4}W_{\mu\nu}^i W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu} B^{\mu\nu} \\
& + (D_\mu \Phi)^\dagger (D_\mu \Phi) - V(\Phi) \\
& + \sum_\psi \bar{\psi} i \gamma^\mu D_\mu \psi \\
& - y_e \bar{L} \Phi e_R - y_d \bar{Q} \Phi d_R - y_u \bar{Q} \Phi^\dagger u_R + \text{h.c.}
\end{aligned} \tag{3.2}$$

The first term for gauge fields. Here, $G_{\mu\nu}^a$ is the field strength tensor for the strong interaction (gluons), $W_{\mu\nu}^i$ is the field strength tensor for the weak interaction (W and Z bosons), and $B_{\mu\nu}$ is the field strength tensor for the hypercharge (photon). The second term is for the Higgs field and its potential ($V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$). Where Φ is the Higgs field doublet, D_μ is the covariant derivative, μ^2 is a parameter related to the Higgs mass, and λ is the self-coupling. The third term describes the kinetic terms for fermions. Where ψ represents the various fermion fields (quarks and leptons), γ^μ are the Dirac gamma matrices, and D_μ is the covariant derivative. The last line describes the interactions between the Higgs field and the fermions, giving them mass after electroweak symmetry breaking. Here, y_e, y_d, y_u are the Yukawa coupling constants, L and Q are the left-handed lepton and quark doublets, e_R, d_R, u_R are the right-handed singlets, and $\tilde{\Phi}$ is the charge-conjugated Higgs field.

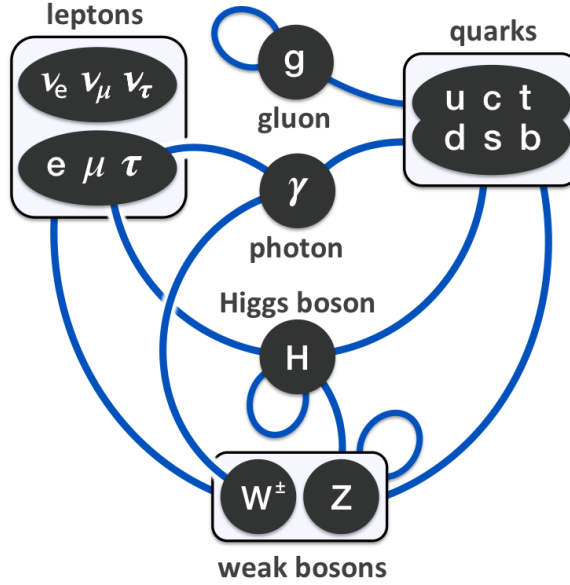


Figure 3.1. A sketch of particle components in the standard model. It contains fermions with two classes; leptons and quarks, gauge bosons for mediating interactions, and the Higgs field. By Eric Drexler.

A fundamental concept in quantum field theory and experimental particle physics is the *scattering amplitude*, which describes the probability amplitude for an interaction between initial and final states. The squared amplitude, $|\mathcal{M}|^2$, directly determines the *cross section* σ , which quantifies the likelihood of a particular interaction occurring.

In a two-body scattering process, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}, \quad (3.3)$$

where s is the Mandelstam variable representing the square of the center-of-mass energy, and $\mathbf{p}_i, \mathbf{p}_f$ denote the initial and final momenta, respectively.

In cosmology, cross sections play a crucial role in determining the relic abundance of dark matter through the Boltzmann equation. And a similar quantity is relevant for interacting dark matter models, which we call the *interaction rate*. This will be discussed in chapter 4.

3.2 Gauge symmetry

Gauge symmetry is a fundamental concept in the Standard Model of particle physics. It refers to a type of symmetry where the laws of physics remain invariant under certain local transformations of fields. These transformations are associated with the internal degrees of freedom of particles. The Standard Model is based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, where

- $SU(3)_C$ describes the strong interaction (Quantum Chromodynamics, QCD) with gluons as gauge fields.
- $SU(2)_L$ describes the weak interaction with W and Z bosons as gauge fields. It acts on left-handed fermions, hence the subscript L .
- $U(1)_Y$ describes the hypercharge boson, which mixes with the neutral component of the $SU(2)_L$ gauge bosons to yield the physical states being the Z boson and the photon.

Gauge symmetries are classified as Abelian or non-Abelian, depending on the commutation properties of their generators.

Abelian gauge symmetry ($U(1)$)

A simple example of gauge symmetry is the Abelian $U(1)$ symmetry, which underlies electromagnetism. It is characterized by a single gauge field A_μ and a local phase transformation of a charged field $\psi(x)$

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x), \quad (3.4)$$

where $\alpha(x)$ is an arbitrary function of spacetime coordinates.

To ensure local invariance, we introduce the gauge field A_μ that transforms as

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha(x). \quad (3.5)$$

The corresponding field strength tensor, also known as the electromagnetic field tensor, is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.6)$$

This formalism naturally leads to Maxwell's equations when coupled to charged particles.

Non-Abelian gauge symmetry ($SU(N)$)

Non-Abelian gauge theories extend this concept to larger symmetry groups such as $SU(N)$, which form the foundation of the weak and strong interactions. These symmetries are described by Yang-Mills theory, where the gauge fields A_μ^a are associated with the generators T^a of the gauge group. The transformation of a matter field $\psi(x)$ in the fundamental representation is given by

$$\psi(x) \rightarrow U(x)\psi(x), \quad (3.7)$$

where $U(x)$ is a local transformation matrix of the form

$$U(x) = e^{i\alpha^a(x)T^a}. \quad (3.8)$$

Unlike in Abelian theories, the generators T^a of a non-Abelian group satisfy the commutation relation

$$[T^a, T^b] = if^{abc}T^c, \quad (3.9)$$

where f^{abc} are the structure constants of the gauge group, encoding its non-trivial algebraic properties.

The field strength tensor in non-Abelian gauge theories, such as Quantum Chromodynamics (QCD), is given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \quad (3.10)$$

where g is the gauge coupling constant. The presence of the additional term $gf^{abc}A_\mu^b A_\nu^c$ leads to self-interactions of the gauge bosons, a key feature distinguishing non-Abelian gauge theories from Abelian ones. A viable model for the dark sector considered in this work (see Sec. 4.2.1) is ruled by a dark non-Abelian gauge symmetry, and the self-interaction is a crucial ingredient for obtaining a temperature-dependence of the interaction rate that can address the S_8 tension.

Part II

Beyond the Standard Models

4 Interacting Dark Sector

The *visible sector*, at the cosmological level, consists of multiple components interacting in the early universe, a complexity mirrored more explicitly at the particle physics level, where numerous particles interact via distinct forces. Inspired by this richness, it is natural to hypothesize that the *dark sector* may also consist of multiple components and interactions, rather than being limited to a single dark matter particle interacting solely through gravity. Exploring this possibility opens new avenues for understanding dark matter and its fundamental properties, and also for addressing unresolved tensions within the standard cosmological model.

A natural extension is to consider an interaction in the dark sector analogous to that of electron–photon Compton scattering in the visible sector. For that, we include a relativistic component to the dark sector, namely, dark radiation (DR). Furthermore, we introduce an interaction between DM (or a fraction of it) with DR, and a possible self-interaction for each component (depending on the model), see 4.1. This model was introduced as a general setup for dark matter in several works, we mention [11, 12]. To study the effect of the interacting dark sector on structure formation, an *Effective Theory of Structure Formation* (ETHOS) was introduced in [13].

In this Chapter, we will start by introducing the ETHOS formalism with its main parameters in Sec. 4.1. Then, we will review three main particle physics models that can describe such an interacting dark sector in Sec. 4.2. Finally, we will show the impact of the non-Abelian model (which we will mostly focus on in this thesis) on the matter power spectrum and the halo mass function in Sec. 4.3.

4.1 ETHOS parameterization

The effective theory of structure formation (ETHOS) offers an easy and comprehensive framework to study models of an interacting dark sector and its impact on structure formation. It offers a number of effective parameters on the cosmological level that can be easily mapped to the underlying particle physics model considered.

4.1.1 Formalism

To obtain the Boltzmann equations for the interacting dark matter (IDM) and DR, one can follow the same procedure as done in Sec. 2.2.2 for Λ CDM components. In Appendix. B, we review all the details of the computation and add more steps for clarity, especially concerning the computation of the drag force term between IDM and DR. For non-relativistic IDM, only the first two moments are relevant: the density contrast δ_{IDM}

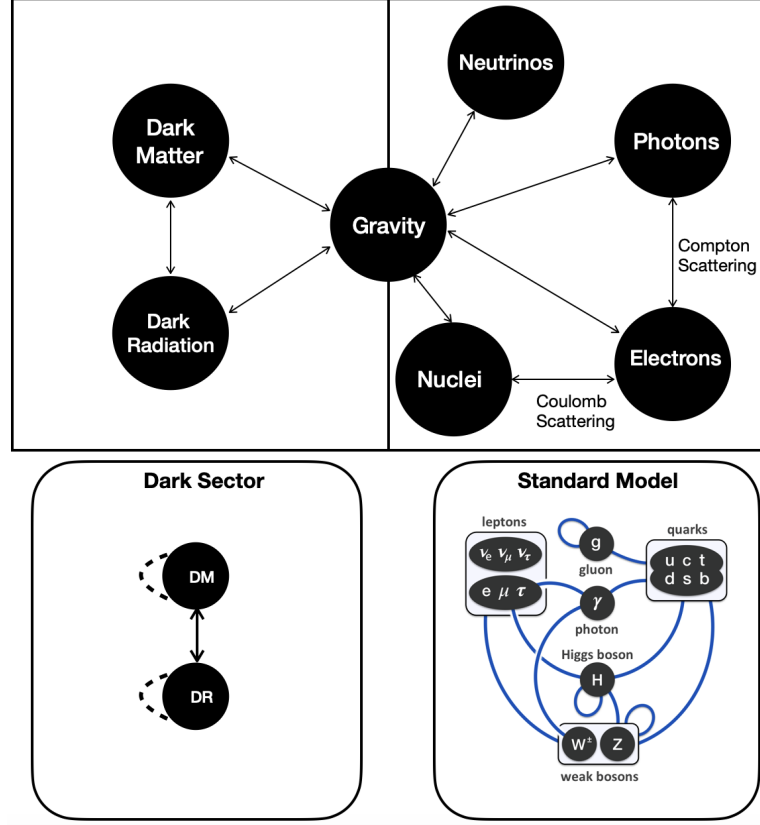


Figure 4.1. A schematic representation of the various components relevant for structure formation and their interactions before recombination, along with their counterparts in particle physics. In this framework, we extend the dark sector by incorporating a relativistic component, dark radiation (DR), and introducing interactions between DR and dark matter (DM) or a fraction of it. Additionally, we consider possible self-interactions for both, DM and DR, depending on the chosen model.

and the velocity divergence θ_{IDM} . Because the scattering process is elastic, the particle number is conserved, meaning the continuity equation for the density is identical to that of CDM. However, the interaction between IDM and DR results in momentum exchange, introducing a drag force term in the Euler equation

$$\dot{\delta}_{\text{IDM}} + \theta_{\text{IDM}} - 3\dot{\phi} = 0, \quad (4.1)$$

$$\dot{\theta}_{\text{IDM}} - c_{\text{IDM}}^2 k^2 \delta_{\text{IDM}} + \mathcal{H} \theta_{\text{IDM}} - k^2 \psi = \Gamma_{\text{IDM-DR}} (\theta_{\text{IDM}} - \theta_{\text{DR}}), \quad (4.2)$$

here ϕ and ψ are gravitational potentials, c_{IDM} is the adiabatic IDM sound speed, a dot denotes a derivative with respect to conformal time. On the right-hand side of Eq. (4.2), θ_{DR} is the DR velocity divergence and $\Gamma_{\text{IDM-DR}}$ is the IDM interaction rate given by

$$\Gamma_{\text{IDM-DR}} = a \frac{1}{16\pi m_\chi^3} \frac{\eta_{\text{DR}}}{3} \int \frac{p_1^2 dp_1}{2\pi^2} p_1^2 \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} (A_0(p_1) - A_1(p_1)) \quad (4.3)$$

p_1 is the amplitude of DR initial momentum and $f_{\text{DR}}^{(0)}(p_1)$ is its distribution function at zero order (Fermi-Dirac distribution for fermions or Bose-Einstein distribution for bosons). m_χ is the IDM mass and $A_l(p)$ are the coefficients of projecting the spin-summed squared amplitude onto the l^{th} Legendre polynomials $P_l(x)$, defined as follows

$$A_l(p) = \frac{1}{2} \int_{-1}^1 d\tilde{\mu} P_l(\tilde{\mu}) \left(\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum |\mathcal{M}|^2 \right) \Big|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi+2p_1m_\chi}}, \quad (4.4)$$

here $\tilde{\mu} = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3$ is the angle between the initial and final momentum of the DR particle. η_χ and η_{DR} are the degrees of freedom for DM and DR particles, respectively. $\sum |\mathcal{M}|^2$ is the squared amplitude of the scattering, where the sum is over spin and color degrees of freedom of final states, and t and u are Mandelstam variables.

The equations for the DR follow that of photons, except that we add here a term for DR self-interaction, for generality. The equations for the density contrast δ_{DR} , velocity divergence θ_{DR} , and higher moments are given by

$$\dot{\delta}_{\text{DR}} + \frac{4}{3}\theta_{\text{DR}} - 4\dot{\phi} = 0, \quad (4.5)$$

$$\dot{\theta}_{\text{DR}} - \frac{1}{4}k^2\delta_{\text{DR}} + k^2\sigma_{\text{DR}}^2 - k^2\psi = \Gamma_{\text{DR-IDM}} (\theta_{\text{DR}} - \theta_{\text{IDM}}), \quad (4.6)$$

$$\dot{\Pi}_{\text{DR},l} + \frac{k}{2l+1} [(l+1)\Pi_{\text{DR},l+1} - l\Pi_{\text{DR},l-1}] = (\alpha_l \Gamma_{\text{DR-IDM}} + \beta_l \Gamma_{\text{DR-DR}}) \Pi_{\text{DR},l}, \quad (4.7)$$

where $\sigma_{\text{DR}} = \Pi_{\text{DR},2}/2$ is the shear stress, $\Pi_{\text{DR},l}$ is the l^{th} moment of the DR multipole hierarchy, and $\Gamma_{\text{DR-IDM}}$ is related to the drag opacity for DR-IDM coupling.

The two interaction rates $\Gamma_{\text{IDM-DR}}$ and $\Gamma_{\text{DR-IDM}}$ can be related using energy-momentum conservation and expanded as power laws in temperature or redshift as follows

$$\begin{aligned} \Gamma_{\text{IDM-DR}} &= \frac{4}{3} \left(\frac{\rho_{\text{DR}}}{\rho_{\text{IDM}}} \right) \Gamma_{\text{DR-IDM}} \\ &= -\frac{4}{3} (\Omega_{\text{DR}} h^2) \sum_n a_n x_{\text{IDM}}(z) \frac{(1+z)^{n+1}}{(1+z_{\text{D}})^n}, \end{aligned} \quad (4.8)$$

and the interaction rate for DR self-interaction can also be written as a power law in redshift

$$\Gamma_{\text{DR-DR}} = (\Omega_{\text{DR}} h^2) x_{\text{DR-DR}}(z) \sum_n b_n \left(\frac{1+z}{1+z_{\text{D}}} \right)^n, \quad (4.9)$$

ρ_{IDM} and ρ_{DR} are, respectively, the energy densities of IDM and DR, and Ω_{DR} is the DR density in units of the critical density. The coefficients a_n and b_n are constants with units of inverse length and encapsulate the details of the microscopic model describing the interactions. Here $x_{\text{IDM}}(z)$ and $x_{\text{DR-DR}}(z)$ are dimensionless functions introduced to capture any deviation from a power law dependence on redshift, and z_{D} is introduced for normalization and chosen to be $z_{\text{D}} = 10^7$.

4.1.2 Parameters

The IDM–DR model is characterized by three main parameters

- ξ_{DR} : the temperature ratio between dark radiation and CMB photons today

$$\xi_{\text{DR}} \equiv \left. \frac{T_{\text{DR}}}{T_{\text{CMB}}} \right|_{z=0}, \quad (4.10)$$

which can equivalently be mapped onto the density parameter of DR as

$$\omega_{\text{DR}} = \Omega_{\text{DR}} h^2 = \frac{\eta_{\text{DR}}}{2} \zeta \xi_{\text{DR}}^4 \Omega_{\gamma} h^2, \quad (4.11)$$

with $\zeta = 1$ for bosonic DR and $\zeta = 7/8$ for fermionic DR. η_{DR} is the number of degrees of freedom for DR. dark radiation also contributes to the effective number of relativistic species N_{eff} , therefore, it can also be parameterized as ΔN_{eff} with

$$\Delta N_{\text{eff}} = \frac{\rho_{\text{DR}}}{\rho_{1\nu}} = \frac{\eta_{\text{DR}}}{2} \zeta \xi_{\text{DR}}^4 \frac{8}{7} \left(\frac{11}{4} \right)^{4/3} \quad (4.12)$$

being the ratio between the DR energy density and the energy density of one neutrino family. When considering an $SU(N)$ dark sector, DR behaves as a fluid. This means, we set $\Delta N_{\text{eff}} = \Delta N_{\text{fluid}}$.

- f_{IDM} : the fraction of interacting dark matter

$$f_{\text{IDM}} \equiv \frac{\Omega_{\text{IDM}}}{\Omega_{\text{IDM}} + \Omega_{\text{CDM}}}. \quad (4.13)$$

Depending on the model considered, all DM can interact or only a fraction. When $f_{\text{IDM}} < 1$ the rest of DM behaves as usual cold dark matter. This parameter enters in the energy density in the equation of the interaction rate as defined in Eq. (4.8)

- a_n : the intensity of the interaction between IDM and DR, as defined in Eq. (4.8).

4.2 Particle physics models for an interacting dark sector

From a particle physics perspective, there can be different realizations for an interacting dark sector. In this section, we will review three examples and their main equations in ETHOS parameterization [13]. In the results part III, we will refer to them as $n = 0, 2, 4$, depending on their redshift dependence in Eq. (4.8). In our first work [Paper I](#), we did a data analysis on all of them, yet, later in [Paper II](#) and [Paper III](#) we focused only on the $SU(N)$ case. In all of the following examples $x_{\text{IDM}} = x_{\text{DR-DR}} = 1$, i.e, all of them follow a power law temperature dependence.

4.2.1 Non-Abelian dark sector ($n = 0$)

A dark sector described by an unbroken $SU(N)$ gauge theory scenario was introduced in [11, 12]. IDM is a Dirac fermion in the fundamental representation of the group and the gauge bosons act as DR. The gauge coupling g_d is assumed to be small and below the confinement scale. Dark radiation in this model is always self-interacting, which makes it behave as a fluid. Therefore, higher moments $l \geq 2$ are strongly suppressed. Eventually we only consider the two Boltzmann equations (4.5) and (4.6), and set $\sigma_{\text{DR}} = 0$. The interaction Lagrangian is given by

$$\mathcal{L}_{\text{int}} = -(D^\mu \chi)^\dagger D_\mu \chi - m_\chi^2 \chi^\dagger \chi, \quad \text{where } D_\mu = \partial_\mu - ig_d T^a \tilde{A}_\mu^a, \quad (4.14)$$

where χ refers to the IDM particle and m_χ is its mass. T^a are the $SU(N)$ generators, as introduced in Sec. 3.2, and $\tilde{A}_\mu^a$ are the non-Abelian gauge fields. The degrees of freedom for IDM and DR are $\eta_\chi = 2N$ and $\eta_{\text{DR}} = 2(N^2 - 1)$, respectively. The squared amplitude for the t-channel scattering $\tilde{\gamma}_d(\mathbf{p}_1) + \chi(\mathbf{p}_2) \leftrightarrow \tilde{\gamma}_d(\mathbf{p}_3) + \chi(\mathbf{p}_4)$ is

$$\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2g_d^4 (s - m_\chi^2)(s + t - m_\chi^2)}{t^2} \quad (4.15)$$

when evaluated in the non-relativistic limit, it gives

$$\left(\frac{1}{\eta_\chi \eta_{\nu_s}} \sum_{\text{spins}} |\mathcal{M}|^2 \right) \bigg|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi^2+2p_1 m_\chi}} \simeq \frac{2g_d^4 m_\chi^2}{p_1^2 (\tilde{\mu} - 1)^2} \quad \text{for } p_1 \ll m_\chi. \quad (4.16)$$

The coefficients for Legendre expansion as defined in Eq. (4.4) are

$$A_0(p_1) = \frac{g_d^4 m_\chi^2 \cot^2\left(\frac{\theta_{\min}}{2}\right)}{2p_1^2}, \quad A_1(p_1) = \frac{g_d^4 m_\chi^2 \left(\cot^2\left(\frac{\theta_{\min}}{2}\right) - 2 \ln \left[\csc^2\left(\frac{\theta_{\min}}{2}\right) \right] \right)}{2p_1^2}, \quad (4.17)$$

leading to

$$\Gamma_{\text{DR-IDM}} = -a \frac{5\alpha_d^2}{2\pi} n_\chi^{(0)} \frac{1}{T_{\text{DR}}^2} \ln \alpha_d^{-1}, \quad \Gamma_{\text{IDM-DR}} = -a(N^2 - 1) \frac{\pi \alpha_d^2}{9} \frac{T_{\text{DR}}^2}{m_\chi} \ln \alpha_d^{-1}, \quad (4.18)$$

$\alpha_d \equiv g_d^2/(4\pi)$ is the fine-structure parameter, and take $\csc^2(\theta_{\min}/2) = \alpha_d^{-1}$. The interaction rate coefficients are

$$a_0 = \frac{5\alpha_d^2}{2\pi} \frac{\tilde{\rho}_{\text{cr}}}{m_\chi \xi_{\text{DR}}^2 T_{\text{CMB},0}^2} \ln \alpha_d^{-1}, \quad a_{n \geq 1} = 0, \quad (4.19)$$

where $\tilde{\rho}_{\text{cr}} \equiv \rho_{\text{cr}}/h^2$.

Note that the choice of the cut-off to regulate the divergence in the amplitude is an ad-hoc, and it only gives part of the leading-order (LO) contribution. In the next

chapter (5), we will show how to treat the divergence correctly by considering the Debye mass for DR, and include corrections to leading- and next-to-leading order contributions. This work was done in [Paper I](#).

4.2.2 Hidden-charged scalar DM ($n = 2$)

In another scenario, DM is a complex scalar charged under a new unbroken dark $U(1)$ interaction mediated by the gauge field $\tilde{A}_\mu$ [94]. The Lagrangian is given by

$$\mathcal{L}_{\text{int}} = -(D^\mu \chi)^\dagger D_\mu \chi - m_\chi^2 \chi^\dagger \chi, \quad \text{where } D_\mu = \partial_\mu - ig_\chi \tilde{A}_\mu, \quad (4.20)$$

where g_χ is the coupling constant. In this case, the degrees of freedom are $\eta_\chi = \eta_{\text{DR}} = 2$. The squared amplitude for $\tilde{\gamma}(\mathbf{p}_1) + \chi(\mathbf{p}_2) \leftrightarrow \tilde{\gamma}(\mathbf{p}_3) + \chi(\mathbf{p}_4)$ is

$$\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4g_\chi^4 [(m_\chi^2 - s)^4 + 2(m_\chi^2 - s)^2 st + (m_\chi^4 + s^2)t^2]}{(m_\chi^2 - s)^2 (s + t - m^2)}. \quad (4.21)$$

In the non-relativistic limit, it becomes

$$\left(\frac{1}{\eta_\chi \eta_{\nu_s}} \sum_{\text{spins}} |\mathcal{M}|^2 \right) \Big|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi^2+2p_1 m_\chi}} \simeq 2g_\chi^4 (1 + \tilde{\mu}^2) \quad \text{for } p_1 \ll m_\chi \quad (4.22)$$

The coefficients for Legendre expansion following Eq. (4.4) are

$$A_0(p_1) = \frac{8g_\chi^4}{3}, \quad A_1(p_1) = 0, \quad A_2(p_1) = \frac{4g_\chi^4}{15}, \quad A_{l \geq 3}(p_1) = 0, \quad (4.23)$$

leading to the following opacities

$$\Gamma_{\text{DR-IDM}} = -a \frac{g_\chi^4}{6\pi m_\phi^2} n_\chi^{(0)}, \quad \Gamma_{\text{IDM-DR}} = \frac{4\rho_{\text{DR}}}{3\rho_\chi} \Gamma_{\text{DR-IDM}}, \quad \alpha_2 = \frac{9}{10}. \quad (4.24)$$

Finally, following Eq. (4.8), this leads to

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = (1 + z_D)^2 \frac{g_\chi^4}{6\pi m_\phi^2} \frac{\tilde{\rho}_{\text{cr}}}{m_\chi}, \quad a_{n \geq 3} = 0, \quad (4.25)$$

4.2.3 DM-DR scattering via a massive mediator ($n = 4$)

The last example, DM interacts with massive sterile neutrinos (ν_s) (considered as DR), via a broken $U(1)$ theory, mediated by a massive vector boson ϕ_μ [95–97]. The Lagrangian is

$$\mathcal{L}_{\text{int}} = -g_\chi \phi_\mu \bar{\chi} \gamma^\mu \chi - \frac{1}{2} g_\nu \phi_\mu \bar{\nu}_s \gamma^\mu \nu_s - \frac{1}{2} m_\phi^2 \phi_\mu \phi^\mu - \frac{1}{2} m_\chi \bar{\chi} \chi. \quad (4.26)$$

We have the scattering $\nu_s(\mathbf{p}_1) + \chi(\mathbf{p}_2) \leftrightarrow \nu_s(\mathbf{p}_3) + \chi(\mathbf{p}_4)$, with $\eta_\chi = \eta_{\nu_s} = 2$. The spin-summed squared matrix is

$$\frac{1}{\eta_\chi \eta_{\nu_s}} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{2g_\chi^2 g_\nu^2}{(m_\phi^2 - t)^2} (t^2 + 2st + 2(m_\chi^2 - s)^2). \quad (4.27)$$

Evaluating the matrix elements

$$\left(\frac{1}{\eta_\chi \eta_{\nu_s}} \sum_{\text{spins}} |\mathcal{M}|^2 \right) \bigg|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi^2+2p_1 m_\chi}} \simeq \frac{8g_\chi^2 g_\nu^2}{m_\phi^4} m_\chi^2 p_1^2 (1 + \tilde{\mu}) \quad \text{for } p_1 \ll m_\phi < m_\chi \quad (4.28)$$

this is in the non-relativistic limit. The Legendre coefficients for this scenario are

$$A_0(p_1) = \frac{8g_\chi^2 g_\nu^2}{m_\phi^4} m_\chi^2 p_1^2, \quad A_1 = (p_1) \frac{A_0(p_1)}{3}, \quad A_{l \geq 2}(p_1) = 0, \quad (4.29)$$

leading to the interaction opacities as follows

$$\Gamma_{\text{DR-IDM}} = -a \frac{3}{2} \frac{\pi g_\chi^2 g_\nu^2}{m_\phi^4} \left(\frac{310}{441} \right) n_\chi^{(0)} T_{\text{DR}}^2, \quad \Gamma_{\text{IDM-DR}} = \frac{4\rho_{\text{DR}}}{3\rho_\chi} \Gamma_{\text{DR-IDM}}, \quad \alpha_{l \geq 2} = \frac{3}{2}, \quad (4.30)$$

and their respective a_n s as defined in Eq. (4.8)

$$a_{n \leq 3} = 0, \quad a_4 = (1 + z_D)^4 \frac{3}{2} \frac{\pi g_\chi^2 g_\nu^2}{m_\phi^4} \frac{\tilde{\rho}_{\text{cr}}}{m_\chi} \left(\frac{310}{441} \right) \xi_{\text{DR}}^2 T_{\text{CDM},0}^2, \quad a_{n \geq 5} = 0, \quad (4.31)$$

4.3 Effect on matter power spectrum and halo mass function

To understand the effect of each model on structure formation, we begin by looking at the ratio between the interaction rate and the Hubble parameter $\Gamma_{\text{IDM-DR}}/\mathcal{H}$ as a function of the redshift z , see Fig. 4.2. The two models $n = 2, 4$ have a similar impact. Their interaction rate has a sharp decrease in time during the radiation-dominated era, which means they affect only modes that enter the horizon before the ratio $\Gamma_{\text{IDM-DR}}/\mathcal{H}$ falls below unity. This leads to a sharp suppression of the power spectrum on small scales that cannot be probed with CMB or LSS data. This explains why these two scenarios cannot explain the S_8 problem as discussed in Sec. 6 (based on results in Paper I and [47]). In contrast, for the $n = 0$ model, the interaction rate scales similarly to the expansion rate of the universe during the radiation-dominated era. Consequently, the ratio $\Gamma_{\text{IDM-DR}}/\mathcal{H}$ remains constant during this period, affecting perturbation modes of various scales as they enter the horizon. During matter domination, this ratio begins to decrease. This behavior leads to a more gradual suppression of the power spectrum and allows for the possibility of supporting lower S_8 values. Additionally, a distinct feature of the non-Abelian dark sector gauge interaction is the presence of dark radiation (DR) self-

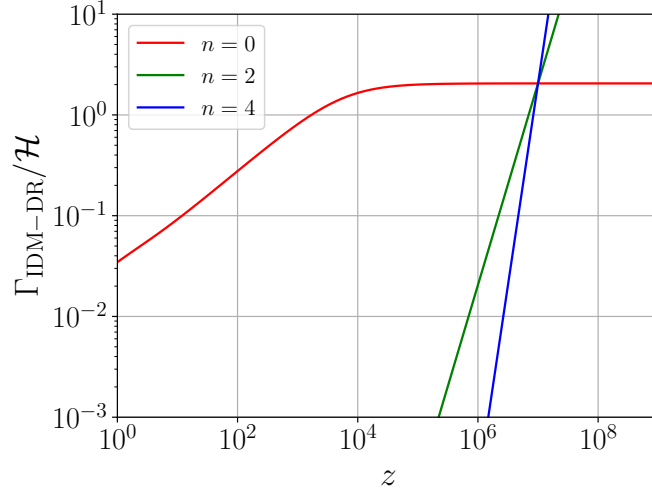


Figure 4.2. The fraction between the interaction rate and hubble matter for different ETHOS cases.

interactions, which cause it to behave as a perfect fluid, suppressing higher multipoles for $l \geq 2$.

After exploring all three cases in [Paper I](#), we decided to focus only on the case $n = 0$ in [Paper II](#) and [Paper III](#). This is because it is phenomenologically richer, can address the S_8 tension, and can be tested with CMB and LSS data. This will also be the case in the following discussion about the impact on the matter power spectrum and halo mass function. Since we are interested only in small deviations from Λ CDM we consider two main limits previously discussed in the literature [[1](#), [45](#)]

- **Weak interaction limit:** In this limit, all of the dark matter interacts with dark radiation ($f_{\text{IDM}} = 1$). To remain consistent with constraints from the cosmic microwave background (CMB) and structure formation, the interaction rate must be less than or comparable to the expansion rate of the universe during radiation domination (RD), i.e., $\Gamma_{\text{IDM-DR}} \leq \mathcal{H}$.
- **Tight coupling limit:** This limit occurs when the interaction rate is significantly larger than the expansion rate during RD, $\Gamma_{\text{IDM-DR}} \gg \mathcal{H}$ (i.e., for a large a_{dark}). This scenario is generally viable only when a small fraction of DM is interacting, meaning $f_{\text{IDM}} \ll 1$.

In both works ([Paper II](#) and [Paper III](#)), the focus is primarily on the tight coupling limit, with a brief exploration of the consequences of relaxing this assumption. This limit serves as the dark sector analog to the interaction between photons and electrons in the visible sector. A mapping between the constraints on the interaction parameter a_{dark} and the fine-structure constant α_{D} , which characterizes the gauge interaction strength in the dark sector, suggests that the weak coupling limit corresponds to $\alpha_{\text{D}} \sim 10^{-10} - 10^{-8}$ [[1](#)]. These values are smaller than any coupling observed in the Standard Model, whereas

values in the tight coupling limit can be comparable to the strength of electroweak interactions. Furthermore, α_D values within the tight coupling regime may also be relevant for DM self-interactions, potentially addressing small-scale structure problems [7, 8].

Through its interaction with a relativistic component (DR), interacting dark matter acquires momentum, enabling it to escape from overdensities and resulting in the suppression of structure formation. The influence of the model parameters is illustrated in Fig. 4.3, where we present the dependence of the matter power spectrum $P(k)$ and the halo mass function (HMF) within the IDM–DR framework, varying one parameter while keeping the other two fixed.

The impact of each ETHOS parameter on the matter power spectrum is illustrated in the top row of Fig. 4.3. The amount of dark radiation, characterized by the temperature ratio ξ_{DR} of DR to the cosmic microwave background (CMB), determines the scale at which the suppression of the matter power spectrum occurs. This arises from two distinct effects. First, DR has a background effect by contributing to ΔN_{fluid} , thereby altering the timing of matter-radiation equality. However, this effect is minimal for the range of ξ_{DR} values considered here. Second, the primary impact stems from the fact that, in the tight coupling limit, IDM and DR behave as a dark plasma and can be treated as a single fluid. During the radiation-dominated era, δ_{IDM} follows δ_{DR} , and both evolve according to a common effective sound velocity, which is predominantly determined by ξ_{DR} (for further details, see [45]). This behavior is analogous to the tightly coupled baryon-photon plasma in the Λ CDM model.

The amount of suppression is determined by the other two parameters: the interaction strength a_{dark} and the fraction of IDM f_{IDM} . Regarding the interaction strength, we observe that for $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}] \gtrsim 6$, the suppression of the matter power spectrum becomes independent of a_{dark} , signaling that the tight coupling limit has been reached. In this tightly coupled regime, f_{IDM} and ξ_{DR} remain the key free parameters of interest. The fraction f_{IDM} also plays a significant role in determining the extent of suppression. It can be truncated at approximately $f_{\text{IDM}} \sim 20\%$, which suppresses the matter power spectrum by over 80% and is likely ruled out by existing small-scale datasets.

For the halo mass function (HMF), we adopt the Tinker N -body simulation-based model [69]. Similar to previous studies on massive neutrinos [98, 99] and ultralight axion-like particles (ULAs) [100], we treat IDM as a biased tracer of cold dark matter, as halo formation is primarily driven by CDM when the IDM fraction is small. The impact of IDM–DR on the HMF is incorporated solely through its effect on the matter power spectrum. In this analysis, we adopt this simplified approach, deferring a more detailed treatment of the IDM–DR influence on halo formation to future work.

The bottom row of Fig. 4.3 illustrates the impact of each ETHOS parameter on the halo mass function (HMF). The gray shaded region represents the range of halo masses included in our samples. Higher values of ξ_{DR} affect the HMF for halos with $M \gtrsim 10^{13} [h^{-1} M_{\odot}]$, which encompasses the mass range accessible to the SPT-3G and CMB-S4 tSZ-selected galaxy cluster samples. In contrast, the effects of a_{dark} and f_{IDM} are significant only for lower-mass halos, $M \lesssim 5 \cdot 10^{13} [h^{-1} M_{\odot}]$, which are too small to be included in the tSZ-selected samples analyzed here. To differentiate between varying

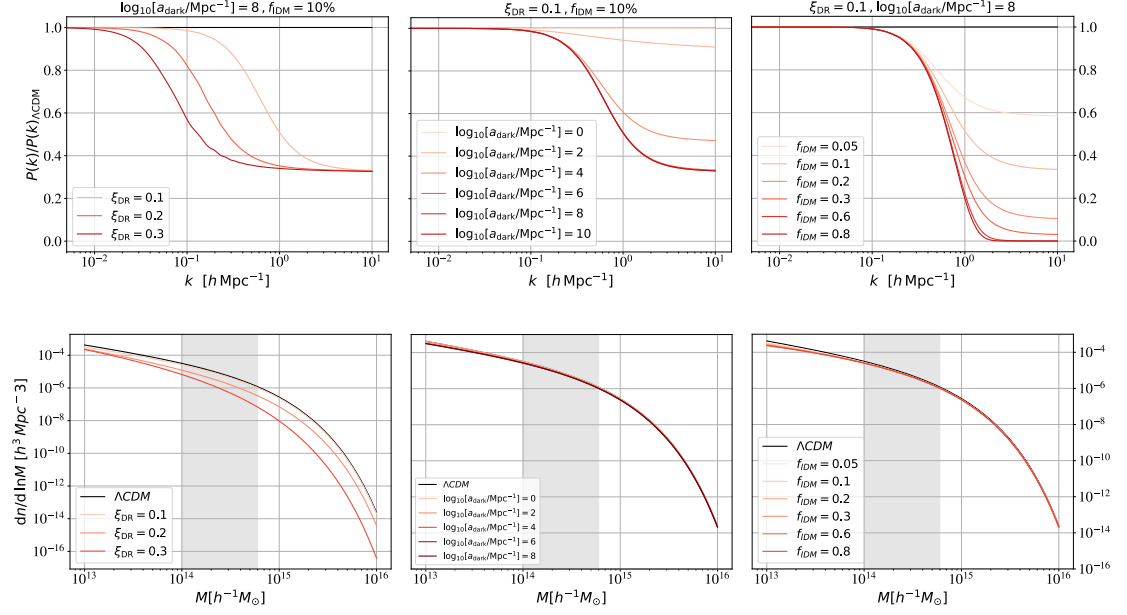


Figure 4.3. The impact of each IDM–DR parameter on the matter power spectrum (top) and the halo mass function (HMF) (bottom). The three columns illustrate the effects of varying the dark radiation temperature ξ_{DR} (left), the IDM–DR interaction strength a_{dark} (middle), and the IDM fraction f_{IDM} (right). For the matter power spectrum, we present the ratio relative to the ΛCDM spectrum. In the HMF plot, the gray shaded region highlights the range of halo masses corresponding to the galaxy cluster samples analyzed in this study. Reprinted from [Paper III](#).

values of a_{dark} and f_{IDM} , it would be necessary to incorporate a sample of galaxy groups at lower masses or utilize complementary non-cluster constraints at smaller scales.

5 Interaction Rate of an $SU(N)$ Dark Sector

As discussed in the previous chapter, the non-Abelian $SU(N)$ gauge theory (corresponding to ETHOS $n = 0$) exhibits a richer phenomenology compared to the other two scenarios considered. In [Paper I](#), we conducted a more in-depth study of this model, refining the computation of the interaction rate by addressing the logarithmic divergence more rigorously. The previous result from the literature [13] is summarized in Sec. 4.2.1. In this chapter, we first provide a detailed description of the underlying particle physics model along with the corrected interaction rate. The second section is dedicated to the technical aspects of the computation, given that the derivation involves nontrivial steps.

5.1 Dark sector with non-Abelian gauge interaction and Debye mass

The interaction between dark matter and dark radiation is similar to the Compton scattering of baryons and photons but with a key distinction; it includes an additional t -channel contribution to the scattering amplitude (see Fig. 5.1) involving the non-Abelian three-gluon vertex. IDM–DR interactions occur in the non-relativistic limit for the IDM particle, $T_{\text{DR}} \ll m_\chi$ (since typical values for m_χ are in the GeV–TeV range while recombination occurs at the eV scale). In this limit, the non-Abelian interaction introduces a significant difference compared to the Abelian (Thomson) scattering process. Specifically, for small scattering angles, the momentum transfer in the t -channel approaches zero, resulting in a strong enhancement of the non-Abelian contributions. As a result, the cross section and consequently the interaction rate would diverge logarithmically when integrating over all angles. This divergence is regulated by Debye screening in the non-Abelian plasma, described by the Debye mass [101]

$$m_D^2 = \frac{1}{3}g_d^2 T_{\text{DR}}^2 (C_A + \frac{1}{2}N_f). \quad (5.1)$$

Here, $C_A = N$ is related to the gauge group and N_f is the number of light fermionic degrees of freedom. While N_f is zero within the minimal IDM–DR model and the relevant temperature range considered here, it is included for the sake of generality. Now, the t -channel propagator needs to be replaced by the hard thermal loop (HTL) resummed propagator [101], which incorporates the Dyson resummation of self-energy diagrams, as illustrated in Fig. 5.2. The HTL approximation captures the leading dependence in the large temperature limit, i.e., for a momentum transfer that is much smaller than T_{DR} ,

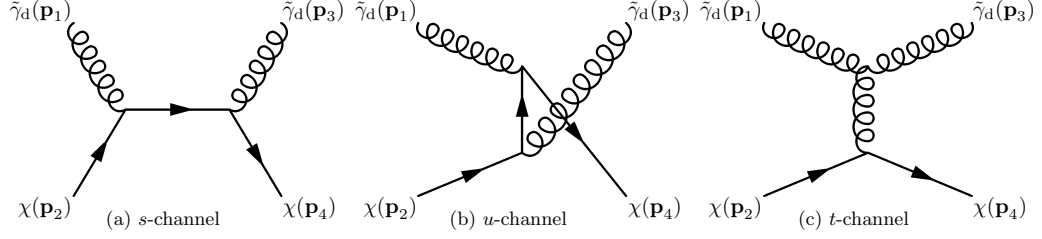


Figure 5.1. Feynman diagrams for IDM-DR scattering in a non-Abelian $SU(N)$ gauge theory. From [Paper I](#).

and considers only the dominant contribution from (ultra-)relativistic particles in the self-energy loop.

The HTL approximation requires replacing the gluon propagator [102] as follows

$$t^{-1} \rightarrow (t - \Pi_{L,T})^{-1}, \quad (5.2)$$

where L and T denote the longitudinal and transverse projections of the gluon self-energy $\Pi_{\mu\nu}$ as described, for instance, in [103]. These components correspond to the electric and magnetic interactions, respectively. Up to corrections that are relatively suppressed by higher powers of T_{DR}/m_χ , only the longitudinal part contributes, and the momentum transfer is dominated by its spatial part $q \simeq (0, \mathbf{q})$, corresponding to the Coulomb limit. Furthermore, the Debye screening is relevant for $|\mathbf{q}| \ll T_{\text{DR}}$. Altogether, this means that to capture the leading dependence on T_{DR}/m_χ , we can approximate $\Pi_L \rightarrow m_D^2$, while neglecting the transverse contribution. An explicit computation taking the Debye-

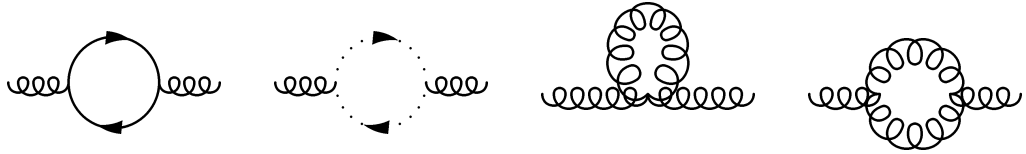


Figure 5.2. Diagrams illustrating contributions to the self-energy of the dark $SU(N)$ gauge bosons (DR), relevant for Debye screening giving rise to the Debye mass m_D . Dotted lines represent ghost propagators. From [Paper I](#).

screening into account yields the interaction rate

$$\Gamma_{\text{IDM-DR}} = -a \frac{\pi}{18} \frac{\alpha_d^2}{m_\chi} \eta_{\text{DR}} \left\{ T_{\text{DR}}^2 \left[\ln \alpha_d^{-1} + c_0 + c_1 g_d + \mathcal{O}(g_d^2) \right] + \mathcal{O} \left(\frac{T_{\text{DR}}^4}{m_\chi^2} \right) \right\}, \quad (5.3)$$

with $\alpha_d = g_d^2/(4\pi)$. The logarithmic term arises from Debye-screening and agrees with previous results [13]. Here we provide also the constant term as well as the first-order

correction with coefficients

$$c_0 = 1 + \ln \left(\frac{6}{2N + N_f} \right) + \ln(4\pi) - 24 \ln(A), \quad \text{and} \quad c_1 = \frac{3\sqrt{2N + N_f}}{4\pi} \sqrt{\frac{3}{2}}, \quad (5.4)$$

with $A \simeq 1.28243$ being the Glaisher-Kinkelin constant. The non-analytic correction linear in g_d arises from averaging over the thermal Bose-Einstein distribution of dark gauge bosons, which exhibits a strong enhancement at small momenta. A similar effect is well known in the computation of various transport rates at finite temperature [101]. While the logarithmic and linear terms receive contributions exclusively from the t -channel process, the constant term c_0 also includes contributions from the interference between the s - and u -channels as well as the t -channel. In contrast, Thomson-like contributions (being the square of the s - and u -channel as well as their interference) are suppressed by a relative factor T_{DR}^2/m_χ^2 . We refer to the next section for details of the computation.

5.2 Details of computing corrections to the interaction rate

The interaction rate in the ETHOS formalism is given by

$$\Gamma_{\text{IDM-DR}} \equiv a \frac{1}{16\pi m_\chi^3} \frac{\eta_{\text{DR}}}{3} \int \frac{p_1^2 dp_1}{2\pi^2} p_1^2 \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} (A_0(p_1) - A_1(p_1)) = \frac{4\rho_{\text{DR}}}{3n_\chi^{(0)} m_\chi} \Gamma_{\text{DR-IDM}}, \quad (5.5)$$

where $f_{\text{DR}}^{(0)}(p_1) = (e^{p_1/T_{\text{DR}}} - 1)^{-1}$ is the unperturbed DR distribution function, $n_\chi^{(0)}$ is the spatially homogeneous DM number density, and $\rho_{\text{DR}} = \eta_{\text{DR}} \zeta \pi^2 T_{\text{DR}}^4 / 30$ is the homogeneous part of the DR energy density (with $\zeta = 1$ for bosonic DR and $\zeta = 7/8$ for fermionic DR). To compute $\Gamma_{\text{IDM-DR}}$, one must first determine the coefficients $A_0(p_1)$ and $A_1(p_1)$ using the general formula for projecting the spin-summed squared matrix element onto the l^{th} Legendre polynomial $P_l(x)$.

$$A_l(p) = \frac{1}{2} \int_{-1}^1 d\tilde{\mu} P_l(\tilde{\mu}) \left(\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum |\mathcal{M}|^2 \right) \Big|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi^2+2p_1 m_\chi}}, \quad (5.6)$$

with $P_0(\tilde{\mu}) = 1$ and $P_1(\tilde{\mu}) = \tilde{\mu}$.

The initial step in the calculation is to determine the squared amplitude, averaged over initial states and summed over final states. After evaluating the color factors and

the traces of gamma matrices using FORM [104], the result of each contribution is

$$|\overline{\mathcal{M}}_s|^2 = \frac{(N^2-1)}{N^2} \frac{g_d^4}{2} 2 \left(2m_\chi^4 + m_\chi^2(s - m_\chi^2) - \frac{1}{2}(s - m_\chi^2)(u - m_\chi^2) \right) \frac{1}{(s - m_\chi^2)^2}, \quad (5.7)$$

$$|\overline{\mathcal{M}}_u|^2 = \frac{(N^2-1)}{N^2} \frac{g_d^4}{2} 2 \left(2m_\chi^4 + m_\chi^2(u - m_\chi^2) - \frac{1}{2}(s - m_\chi^2)(u - m_\chi^2) \right) \frac{1}{(u - m_\chi^2)^2}, \quad (5.8)$$

$$|\overline{\mathcal{M}}_t|^2 = 2g_d^4(s - m_\chi^2)(m_\chi^2 - u) \frac{1}{t^2} \quad (5.9)$$

$$2\text{Re } \overline{\mathcal{M}}_s \mathcal{M}_u^* = \frac{-1}{N^2} \frac{g_d^4}{2} 2 \left(4m_\chi^4 + m_\chi^2(s - m_\chi^2) + m_\chi^2(u - m_\chi^2) \right) \frac{1}{(s - m_\chi^2)(u - m_\chi^2)}, \quad (5.10)$$

$$2\text{Re } \overline{\mathcal{M}}_t \mathcal{M}_s^* = -g_d^4 \left(-(s - m_\chi^2)^2 - (s - m_\chi^2)t + m_\chi^2(u - m_\chi^2) \right) \frac{1}{(s - m_\chi^2)t}, \quad (5.11)$$

$$2\text{Re } \overline{\mathcal{M}}_t \mathcal{M}_u^* = -g_d^4 \left(-(u - m_\chi^2)^2 - (u - m_\chi^2)t + m_\chi^2(s - m_\chi^2) \right) \frac{1}{(u - m_\chi^2)t}, \quad (5.12)$$

with s , u , and t being the Mandelstam variables. Combining all contributions and evaluating at $t = 2p_1^2(\tilde{\mu} - 1)$ and $s = m_\chi^2 + 2p_1 m_\chi$ (where $\tilde{\mu} = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3$, $\hat{\mathbf{p}}_i$ are unit vectors, and $p_1 = |\mathbf{p}_1|$), we obtain the total squared amplitude

$$\begin{aligned} \left(\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum |\mathcal{M}|^2 \right) \Big|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi^2+2p_1 m_\chi}} &\simeq \frac{2g_d^4 4p_1^2 m_\chi^2}{(2p_1^2(\tilde{\mu} - 1) - m_D^2)^2} - \frac{1}{2N^2} g_d^4 (1 + \tilde{\mu}^2) \\ &+ g_d^4 \left(2 + \frac{m_\chi^2}{p_1^2} + \frac{4m_\chi^2}{2p_1^2(\tilde{\mu} - 1) - m_D^2} \right), \quad \text{for } p_1 \ll m_\chi. \end{aligned} \quad (5.13)$$

Note that we used the corrected gluon propagator, replacing $t \rightarrow t - m_D^2$ in the denominator (see the discussion in Sec. 5.1 for the definition of the Debye mass m_D). The first term in (5.13) arises solely from the t -channel, the second term corresponds to Thomson scattering in the Abelian case, and the last term represents the remaining contributions from the s/u -channels and their interference with the t -channel. The approximation symbol indicates the neglect of terms proportional to $m_\chi p_1^3(\tilde{\mu} - 1)$ in the numerator of the t -channel contribution. These terms are suppressed in the non-relativistic limit $p_1 \lesssim T_{\text{DR}} \ll m_\chi$.

Using the result obtained for the full squared amplitude in (5.13), we compute now the coefficients A_0 and A_1 as defined in Eq. (5.6). We divide the computation of the interaction rate into two parts. We begin with the t -channel term

$$A_0^t(p_1) = g_d^4 m_\chi^2 \frac{8p_1^2}{m_D^2(m_D^2 + 4p_1^2)}, \quad (5.14a)$$

$$A_1^t(p_1) = g_d^4 m_\chi^2 \left[\frac{8p_1^2}{m_D^2(m_D^2 + 4p_1^2)} + \frac{4}{m_D^2 + 4p_1^2} + \frac{1}{p_1^2} \left(\ln\left(\frac{m_D^2}{T_{\text{DR}}^2}\right) - \ln\left(\frac{m_D^2 + 4p_1^2}{T_{\text{DR}}^2}\right) \right) \right], \quad (5.14b)$$

leading to the following contribution to the interaction rate

$$\Gamma_{\text{IDM-DR}}^t = C m_\chi^2 \int \frac{p_1^2 dp_1}{2\pi^2} p_1^2 \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} \times \left[\frac{1}{p_1^2} \left(-\ln\left(\frac{m_D^2}{T_{\text{DR}}^2}\right) - \ln\left(\frac{T_{\text{DR}}^2}{4p_1^2}\right) + \ln\left(\frac{m_D^2}{4p_1^2} + 1\right) \right) - \frac{4}{m_D^2 + 4p_1^2} \right]. \quad (5.15)$$

For convenience, we defined $C = a \frac{1}{16\pi m_\chi^3} \frac{\eta_{\text{DR}}}{3} g_d^4$. This result can be further divided into four parts. The contribution from the first two terms in the integrand of (5.15) can be computed analytically; therefore, we combine them and perform the integration by parts to obtain

$$\begin{aligned} i + ii &= C m_\chi^2 \int \frac{p_1^2 dp_1}{2\pi^2} \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} \left(-\ln\left(\frac{m_D^2}{T_{\text{DR}}^2}\right) - \ln\left(\frac{T_{\text{DR}}^2}{4p_1^2}\right) \right) \\ &= -C m_\chi^2 \frac{1}{6} T_{\text{DR}}^2 \left[\ln\left(\frac{3}{(N + \frac{1}{2}N_f)}\right) + \ln(g_d^{-2}) + 3 + 2 \ln(4\pi) - 24 \ln(A) \right], \end{aligned} \quad (5.16)$$

where A is known as the Glaisher-Kinkelin constant given by $\ln A = 1/12 - \zeta'_R(-1)$ where ζ_R is the Riemann zeta function, such that $A \simeq 1.28243$.

The third term in the integrand of (5.15) cannot be computed analytically. To address this, we begin by simplifying it as follows

$$\begin{aligned} iii &= C m_\chi^2 \int \frac{p_1^2 dp_1}{2\pi^2} \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} \ln\left(\frac{m_D^2}{4p_1^2} + 1\right) \\ &= -C m_\chi^2 \frac{T_{\text{DR}}^2}{6} \int_0^\infty \frac{dx}{2\pi^2} 6 f_{\text{DR}}^{(0)}(x) \left[2x \ln\left(\frac{(2N + N_f)g_d^2}{24} \frac{1}{x^2} + 1\right) - \frac{2(2N + N_f)g_d^2 x}{(2N + N_f)g_d^2 + 24x^2} \right] \\ &\equiv -C m_\chi^2 \frac{T_{\text{DR}}^2}{6} F(g_d). \end{aligned} \quad (5.17)$$

In the second step, we used the change of variable $x = p_1/T_{\text{DR}}$ and integrated by part, and set $f_{\text{DR}}^{(0)}(x) = (e^x - 1)^{-1}$ in terms of x . In the last step, we defined the full term in the integral over x as the function $F(g_d)$.

We want to expand $F(g_d)$ in powers of the coupling g_d . However, a straightforward Taylor expansion of the integrand in terms of g_d^2 would lead to ill-defined integrals. This indicates that $F(g_d)$ does not have an analytic expansion in g_d^2 . By examining the integrand, we find that for large $x \gg 1$, it is exponentially suppressed due to the DR distribution function $f_{\text{DR}}^{(0)} \rightarrow e^{-x}$. However, because of the infrared singularity where $f_{\text{DR}}^{(0)} \rightarrow 1/x$ for small x (related to Bose enhancement), the integral is sensitive to the region $x \lesssim g_d \ll 1$. To address this, we employ a method inspired by the method of regions [105] to obtain the asymptotic behavior for small g_d . The approach involves

splitting the integral into two parts

$$F(g_d) = F_1^\lambda(g_d) + F_2^\lambda(g_d). \quad (5.18)$$

First, we perform the integration from 0 to a certain cutoff λ , chosen such that $g_d \ll \lambda \ll 1$. This choice allows us to expand $f_{\text{DR}}^{(0)}$ for small x . Second, we integrate from λ to infinity. In this part, we can expand the integrand in powers of g_d^2 , as $g_d \ll x$ in the region $\lambda \leq x < \infty$, while the small- x behavior is crucial for $0 \leq x < \lambda$. Ultimately, F should be independent of the arbitrary parameter λ .

For the first part, F_1^λ , we analyze the asymptotic expansion by expanding the Bose-Einstein distribution up to the fifth order and performing the integration from 0 to the cutoff λ .

$$F_1^\lambda(g_d) = \int_0^\lambda \frac{dx}{2\pi^2} 6 \left(\frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} + \frac{x^5}{30240} + \mathcal{O}(x^6) \right) \times \left[2x \ln \left(\frac{(2N + N_f)g_d^2}{24} \frac{1}{x^2} + 1 \right) - \frac{2(2N + N_f)g_d^2 x}{(2N + N_f)g_d^2 + 24x^2} \right]. \quad (5.19)$$

It was not possible to compute this integral analytically for an arbitrary constant λ . As a result, we evaluated it numerically for various values of the cutoff λ to investigate its dependence on this parameter.

	g_d	g_d^2	g_d^3	g_d^4	g_d^5	g_d^6
$1/x$	$\frac{\sqrt{2N+N_f}}{2\pi} \sqrt{\frac{3}{2}}$			$c_1^{(1)}(\lambda)$		$d_1^{(1)}(\lambda)$
$-1/2$		$-\frac{2N+N_f}{16\pi^2}$		$c_2^{(1)}(\lambda)$		$d_2^{(1)}(\lambda)$
$x/12$			$\frac{(2N+N_f)^{3/2}}{576\sqrt{6}\pi}$	$c_3^{(1)}(\lambda)$		$d_3^{(1)}(\lambda)$
$-x^3/729$				$c_4^{(1)}(\lambda)$	$\frac{(2N+N_f)^{5/2}}{466560\sqrt{6}\pi}$	$d_4^{(1)}(\lambda)$

Table 5.1. Contributions to the function $F_1^\lambda(g_d)$ after integration in the small- x region $0 \leq x < \lambda$, defined in (5.19). *Rows:* correspond to powers of x in the expansion of $f_{\text{DR}}^{(0)}(x) = (e^x - 1)^{-1}$ around zero, while *columns:* represent powers of g_d . From [Paper I](#).

By considering all terms appearing in (5.19), we present the results in Tab. 5.1. Interestingly, the result can be expressed as a Taylor series in g_d rather than g_d^2 , contrary to what might have been naively expected. This behavior is characteristic of thermal corrections in the presence of infrared singularities caused by Bose enhancement [101]. From Tab. 5.1, it is evident that up to the sixth order in g_d , all terms with odd powers are independent of the cutoff, while for even powers, only g_d^2 is independent of λ . Additionally, each term in the expansion of the Bose-Einstein distribution (corresponding to the rows in Tab. 5.1) contributes at the fourth and sixth order in g_d in a λ -dependent manner. We also verified that this is the case for g_d^8 and higher orders. The lower subscript in the coefficients $c_i^{(1)}(\lambda)$ and $d_i^{(1)}(\lambda)$ denotes the term in the expansion of the

Bose-Einstein distribution, while the upper index refers to the first part of the integral, i.e., the small- x region $0 \leq x < \lambda$. By construction, the cutoff dependence of these g_d^{2k+4} terms ($k = 0, 1, \dots$) must cancel with the cutoff dependence of the corresponding terms in the second part of the integral. For now, we can summarize the result obtained for the first part as

$$\begin{aligned} F_1^\lambda(g_d) = & \frac{\sqrt{2N+N_f}}{2\pi} \sqrt{\frac{3}{2}} g_d - \frac{2N+N_f}{16\pi^2} g_d^2 + \frac{(2N+N_f)^{3/2}}{576\sqrt{6}\pi} g_d^3 + \sum_i c_i^{(1)}(\lambda) g_d^4 \\ & + \frac{(2N+N_f)^{5/2}}{466560\sqrt{6}\pi} g_d^5 + \sum_i d_i^{(1)}(\lambda) g_d^6 + \mathcal{O}(g_d^7). \end{aligned} \quad (5.20)$$

For the second part, $F_2^\lambda(g_d)$, the expansion is performed in terms of small values of g_d^2 (since $g_d \ll x$ in this region), giving

$$\begin{aligned} F_2^\lambda(g_d) = & \int_\lambda^\infty \frac{dx}{2\pi^2} \frac{6}{e^x - 1} 2x \left(\frac{(2N+N_f)^2}{24^2} \frac{g_d^4}{x^4} - \frac{(2N+N_f)^3}{24^3} \frac{g_d^6}{x^6} \right) + \mathcal{O}(g_d^8) \\ = & c^{(2)}(\lambda) g_d^4 + d^{(2)}(\lambda) g_d^6 + \mathcal{O}(g_d^8). \end{aligned} \quad (5.21)$$

Similarly to the first part, it was not possible to perform the integral for an arbitrary value of λ , so we computed it for various fixed values to determine the structure of the result. This second part is analytic, therefore, Taylor expansion in g_d^2 and integration commute. As the expansion of the integrand begins at fourth order in g_d , the expansion of $F_2^\lambda(g_d)$ starts at the same order and exhibits cutoff dependence. These cutoff dependent terms must cancel with that of the corresponding terms in $F_1^\lambda(g_d)$. The structure of the result suggests that this cancellation occurs, though explicit verification of the cancellation at order g_d^4 is not possible because $F_1^\lambda(g_d)$ includes g_d^4 contributions at all orders in the expansion in x . Nonetheless, the results support the conclusion that contributions up to order g_d^3 in $F_1^\lambda(g_d)$ are cutoff-independent, as confirmed by our explicit calculations. In summary, the method of splitting the integral into two regions allows us to analytically extract the expansion of the function $F(g_d)$ up to order g_d^3 .

$$F(g_d) = \frac{\sqrt{2N+N_f}}{2\pi} \sqrt{\frac{3}{2}} g_d - \frac{2N+N_f}{16\pi^2} g_d^2 + \frac{(2N+N_f)^{3/2}}{576\sqrt{6}\pi} g_d^3 + \mathcal{O}(g_d^4).$$

Based on the previous discussion, we conclude that the t -channel squared amplitude receives corrections at linear order in g_d as well as higher-order terms that can be neglected. We can express this part as

$$iii = -C m_\chi^2 \frac{T_{\text{DR}}^2}{6} \left(\frac{\sqrt{2N+N_f}}{2\pi} \sqrt{\frac{3}{2}} g_d + \mathcal{O}(g_d^2) \right). \quad (5.22)$$

The fourth term in the integrand of (5.15) is also somewhat complex to handle, so we

apply a similar procedure to it as described above.

$$\begin{aligned}
 iv &= C m_\chi^2 \int \frac{p_1^2 dp_1}{2\pi^2} p_1^2 \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} \left(-\frac{4}{m_{\text{D}}^2 + 4p_1^2} \right) \\
 &= C m_\chi^2 \frac{T_{\text{DR}}^2}{6} \int_0^\infty \frac{dx}{2\pi^2} f_{\text{DR}}^{(0)}(x) \left[\frac{12x}{\frac{(2N+N_f)}{24} \frac{g_d^2}{x^2} + 1} + \frac{(2N+N_f)g_d^2}{2x \left(\frac{(2N+N_f)}{24} \frac{g_d^2}{x^2} + 1 \right)^2} \right] \\
 &\equiv C m_\chi^2 \frac{T_{\text{DR}}^2}{6} G(g_d).
 \end{aligned} \tag{5.23}$$

In the second step, we again perform a change of variable ($x = p_1/T_{\text{DR}}$), insert the Debye mass from (5.1), and carry out integration by parts. In the final step, we define the function $G(g_d)$, which cannot be computed analytically. To proceed, we apply the same approach as before, which results in

$$G(g_d) = 1 - \frac{\sqrt{2N+N_f}}{4\pi} \sqrt{\frac{3}{2}} g_d + \frac{9}{16\pi^2} g_d^2 - \frac{3}{128\pi} \sqrt{\frac{3}{2}} g_d^3 + \mathcal{O}(g_d^4). \tag{5.24}$$

Inserting this result into (5.23), we get

$$iv = C m_\chi^2 \frac{T_{\text{DR}}^2}{6} \left(1 - \frac{\sqrt{2N+N_f}}{4\pi} \sqrt{\frac{3}{2}} g_d + \mathcal{O}(g_d^2) \right). \tag{5.25}$$

We can now combine our results from (5.16, 5.22, 5.25) to obtain the full contribution to the IDM–DR interaction rate the t -channel.

$$\begin{aligned}
 \Gamma_{\text{IDM-DR}}^t &= -C m_\chi^2 \frac{T_{\text{DR}}^2}{6} \left(\ln(\alpha_d^{-1}) + \frac{3\sqrt{2N+N_f}}{4\pi} \sqrt{\frac{3}{2}} g_d + 2 + \ln \left(\frac{6}{(2N+N_f)} \right) \right. \\
 &\quad \left. + \ln(4\pi) - 24 \ln(A) + \mathcal{O}(g_d^2) \right).
 \end{aligned} \tag{5.26}$$

The computation of the remaining terms in the squared amplitude given in (5.13), which come from the square of the s/u -channel and interference terms (denoted as s , u , $\dots$), is straightforward.

$$A_0^{s,u,\dots}(p_1) = -\frac{2}{3N^2} g_d^4 + g_d^4 \left[2 + \frac{m_\chi^2}{p_1^2} + \frac{m_\chi^2}{p_1^2} \left(\ln(m_{\text{D}}^2) - \ln(m_{\text{D}}^2 + 4p_1^2) \right) \right], \tag{5.27a}$$

$$A_1^{s,u,\dots}(p_1) = g_d^4 \left[\frac{2m_\chi^2}{p_1^2} + \frac{m_\chi^2}{p_1^2} \left(\ln(m_{\text{D}}^2) - \ln(m_{\text{D}}^2 + 4p_1^2) \right) \right], \quad \text{for } p_1 \gg m_{\text{D}}. \tag{5.27b}$$

By combining these results, the Debye mass cancels out, and the contribution to the IDM–DR interaction rate from the square of the s/u -channel and interference terms is

given by

$$\Gamma_{\text{IDM-DR}}^{s,u,\dots} = C \frac{4\pi^2}{15} T_{\text{DR}}^4 \left(\frac{1}{3N^2} - 1 \right) + C \frac{m_\chi^2}{6} T_{\text{DR}}^2. \quad (5.28)$$

The first term is highly suppressed because we are considering the limit where $m_\chi \gg T_{\text{DR}}$. The second term provides an additional correction to the interaction rate, at the same order as the $\mathcal{O}(g_d^0)$ terms in the expansion used in (5.26). It results from the interference between the t -channel diagram and the s - and u -channel contributions. Combining this with the previous result from the t -channel contribution in (5.26) yields the complete interaction rate for the non-Abelian DM-DR model.

$$\begin{aligned} \Gamma_{\text{IDM-DR}} = & -C \frac{m_\chi^2}{6} \left\{ T_{\text{DR}}^2 \left(\ln(\alpha_d^{-1}) + 1 + \ln \left(\frac{6}{(2N + N_f)} \right) + \ln(4\pi) - 24 \ln(A) \right. \right. \\ & \left. \left. + \frac{3\sqrt{2N + N_f}}{4\pi} \sqrt{\frac{3}{2}} g_d + \mathcal{O}(g_d^2) \right) + \mathcal{O} \left(\frac{T_{\text{DR}}^4}{m_\chi^2} \right) \right\}. \end{aligned} \quad (5.29)$$

Part III

Cosmological Constraints: datasets, analysis methods, and results

6 Galaxy Clustering Constraints

In [Paper I](#), we performed a full-shape analysis of galaxy clustering data from BOSS DR12 [21, 22, 89] in the IDM-DR framework. In this work, we explored all possible models ($n = 0, 2, 4$) for interactions in the dark sector as introduced in [Sec. 4.2](#). To fully incorporate the shape information of the BOSS galaxy clustering data, we employ the CLASS-PT framework [106], following methodologies from [107, 108]. This implementation relies on a perturbative treatment of non-linear corrections in the power spectrum of biased tracers in redshift space [109, 110], leveraging the large-scale bias expansion (see [111] for a comprehensive review) alongside an effective field theory approach to redshift space clustering [112–115]. The novel component of our analysis is the inclusion of the monopole, quadrupole, and hexadecapole galaxy clustering power spectra in redshift space, as provided by DR12 of the BOSS galaxy survey [21, 22]. These data are combined with CMB Planck 2018 legacy measurements [26, 53] and additional BAO-scale observations. Furthermore, we assess the ability of different scenarios to resolve the S_8 tension, drawing comparisons to a prior derived from KiDS-1000 [116].

6.1 The large-scale bias expansion

We use the large-scale bias expansion to establish a connection between the observed galaxy density contrast, δ_g , and the underlying matter distribution. In this framework, the galaxy overdensity is expressed as a linear combination of independent operators, $\mathcal{O}$, as follows [117–119]

$$\delta^g(\mathbf{x}, z) = \sum_{\mathcal{O}} b_{\mathcal{O}}(z) \mathcal{O}(\mathbf{x}, z) + \epsilon(\mathbf{x}, z) + \sum_{\mathcal{O}} \epsilon_{\mathcal{O}}(\mathbf{x}, z) \mathcal{O}(\mathbf{x}, z). \quad (6.1)$$

Each operator is associated with a corresponding bias parameter, $b_{\mathcal{O}}$. We treat these bias parameters as free nuisance parameters in our Bayesian analysis, nevertheless, we should say that they have clear physical interpretations (for a comprehensive review, see [120]), and substantial efforts have been made to refine theoretical constraints [121–123]. The terms ϵ account for the stochasticity inherent in structure formation. The set of operators considered up to third order in perturbation theory is given by

$$\mathcal{O} \in \left\{ \delta, \delta^2, \delta^3, \mathcal{G}_2[\phi_g], \delta\mathcal{G}_2[\phi_g], \mathcal{G}_3[\phi_g], \Gamma_3[\phi_g, \phi_v], \nabla^2\delta \right\}, \quad (6.2)$$

where ϕ_v is the velocity potential and ϕ_g is the gravitational potential. The terms $\mathcal{G}_n$ are Galileon operators of order n , while Γ_3 represents the difference between Galileon terms for density and velocity potentials [118, 119]. The coefficient of the last operator,

$\nabla^2\delta$, represents non-local bias and encapsulates small-scale uncertainties in an effective field theory framework for dark matter clustering [112, 124].

Considering correlations between all pairs of operators, the galaxy-galaxy power spectrum takes the form

$$\begin{aligned}
P^{gg}(z, k) = & (b_1)^2 [P_{\text{lin}}(z, k) + P_{1\text{L}}(z, k)] + b_1 b_2 \mathcal{I}_{\delta^2}(z, k) + 2b_1 b_{\mathcal{G}_2} \mathcal{I}_{\mathcal{G}_2}(z, k) \\
& + \left(2b_1 b_{\mathcal{G}_2} + \frac{4}{5} b_1 b_{\Gamma_3}\right) \mathcal{F}_{\mathcal{G}_2}(z, k) + \frac{1}{4} (b_2)^2 \mathcal{I}_{\delta^2\delta^2}(z, k) \\
& + (b_{\mathcal{G}_2})^2 \mathcal{I}_{\mathcal{G}_2\mathcal{G}_2}(z, k) + b_2 b_{\mathcal{G}_2} \mathcal{I}_{\delta^2\mathcal{G}_2}(z, k) + P_{\nabla^2\delta}(z, k) + P_{\varepsilon\varepsilon}(z, k),
\end{aligned} \tag{6.3}$$

here P_{lin} is the linear matter power spectrum and $P_{1\text{L}}$ denotes the one-loop contribution. For clarity, the redshift dependence of all bias parameters is omitted. Explicit expressions for $\mathcal{I}_{\delta^2}, \mathcal{I}_{\mathcal{G}_2}, \mathcal{F}_{\mathcal{G}_2}, \mathcal{I}_{\delta^2\delta^2}, \mathcal{I}_{\mathcal{G}_2\mathcal{G}_2}$, and $\mathcal{I}_{\delta^2\mathcal{G}_2}$ can be found in [106]. For an extension of the large-scale bias expansion to multiple tracers, see [125]. The final two terms in Eq. (6.3) correspond to a correction term proportional to $\langle \nabla^2\delta\delta \rangle \sim k^2 P_{\text{lin}}$ and a stochastic contribution scaling as k^0 and k^2 .

In redshift space, the full galaxy-galaxy power spectrum is expressed using a multipole expansion with respect to the angle between the Fourier wavevector and the line-of-sight direction, $\hat{\mathbf{z}}$. The explicit forms of the monopole, quadrupole, and hexadecapole used in this analysis can be found in [106]. The bias parameters and structure adopted here follow [126]. The power spectrum terms are computed using CLASS-PT [106], which leverages the FFT-Log algorithm from [127, 128] to enhance computational efficiency. CLASS-PT also accounts for Alcock-Paczynski (AP) distortions [129], which arise from assuming a fiducial cosmology for converting angular and redshift differences into distances, and incorporates IR resummation techniques [130, 131] to improve accuracy.

6.2 Datasets and statistics

In this section, we start by introducing the different datasets considered in this work. Then, we explain the statistics used to compare the interacting dark sector models with ΛCDM and to quantify the S_8 tension in those models.

6.2.1 Datasets description

For this work, we consider the following datasets:

- **Planck 2018:** We combine CMB anisotropies from Planck 2018 high- ℓ TT+TE+EE, low- ℓ TT+EE and lensing data [26, 53].
- **BAO(+RSD):** BAO measurements from the 6dFGS at $z = 0.106$ [132], SDSS at $z = 0.15$ [133] and from SDSS-III DR12 [21]. We also include eBOSS DR14 Lyman- α auto-correlation [134] and cross-correlation with quasars [135] (see also [136]). When *not* including FS in the analysis, we also add the measurement of $f\sigma_8$ from redshift space distortion (RSD) of BOSS DR12 [21].

- **KiDS**: we implement the KiDS-1000 likelihood from [116] as a split-normal likelihood with $S_8 = 0.759^{+0.024}_{-0.021}$, similar to [14].
- **Full-shape (FS)**: galaxy clustering data from BOSS DR12 from different cuts of the sky (NGC and SGC) [21, 22, 89], divided in two redshift slices between $0.2 < z < 0.5$ (z_1) and $0.5 < z < 0.75$ (z_3). The likelihoods¹ are constructed using the covariance matrix estimated from a set of 2048 mock catalogs from ‘MultiDark-Patchy’ [137, 138] and the window(free) function of [139, 140]. We use the information from the power spectrum monopole, quadrupole and hexadecapole (up to $k_{\text{max}} = 0.2h \text{ Mpc}^{-1}$), real-space power spectrum proxy from [141, 142] (up to $k_{\text{max}} = 0.4h \text{ Mpc}^{-1}$), the BAO post-reconstructed spectrum from [143], similarly to what was done in [126] and based on [144]. For the priors in the bias and counter-term parameters from the large-scale bias expansion, introduced in Sec. 6.1, we used the same setup as described in [126], with each bias and stochastic parameters being fit separately in each dataset. We explicitly checked that none of the bias terms reaches the prior bounds. We note that here we use FS information only in combination with Planck, therefore, the impact of prior volume effects on the posterior of cosmological parameters is not relevant.

In conclusion, we compare three main setups

- Planck + BAO + RSD (When FS is included, we do not include RSD from BOSS.),
- Planck + BAO + FS,
- Planck + BAO + FS + KiDS.

We stress that the inclusion of KiDS in the analysis is not to provide a full joint analysis of KiDS with other datasets, because that would require a model-specific KiDS likelihood. Instead, the KiDS prior serves as an assessment of how well the IDM-DR models can account for the low- S_8 trend seen by KiDS. Additionally, combining datasets that disagree is, of course, troublesome, for e.g., the case of Planck and KiDS for Λ CDM. As we will show this later, this is not the case for KiDS and Planck + BAO + FS for an interacting dark sector.

For the Bayesian analysis, we use MontePython [145, 146] to run the Markov chain Monte Carlo. MontePython calls CLASS [147] to solve the Boltzmann system of equations. The non-linear power spectra are calculated using CLASS-PT. We note that CLASS-PT is implemented on top of CLASS v.2.6, while the ETHOS parameterization is included only in CLASS v.2.9 [148]. Therefore, we modify CLASS-PT to include the ETHOS model.

In our analyses we explore the following parameter space

$$\{\omega_b, \omega_{\text{cdm}}, \log(10^{10} A_s), n_s, \tau_{\text{reio}}, H_0, \xi_{\text{DR}}, a_{\text{dark}}\}. \quad (6.4)$$

¹https://github.com/oliverphilcox/full_shape_likelihoods.

We fix the other ETHOS parameter; the fraction of IDM f_{IDM} , to a certain value for each scenario. We consider as our benchmark *fiducial* scenario the *dark fluid* limit $\Gamma_{\text{DR-DR}} \rightarrow \infty$ for DR self-interaction, with $f_{\text{IDM}} = 1$. This can be achieved only for the scenario $n = 0$ ($SU(N)$ theory). Then we discuss the same setup but with a mixed cold and interacting DM, once we fix f_{IDM} to 0.1, and then we consider it as a free parameter. We also consider the *free-streaming* scenario $\Gamma_{\text{DR-DR}} \rightarrow 0$ of DR without self-interaction, for all scenarios $n = 0, 2, 4$.

We apply the Gelman-Rubin criterion [149] to assess the convergence of the chains, requiring $(R - 1) < 0.02$. For the parameter a_{dark} , the chains are sampled in logarithmic coordinates with the prior $\log_{10}(a_{\text{dark}}) < 20$, following the approach in [47]. It is important to note that the IDM model is also affected by volume effects when $\xi_{\text{DR}} \rightarrow 0$. A detailed examination of these volume effects, for instance using the profile likelihood method (see e.g. [150, 151]), remains an interesting avenue for further study. In our specific case, as discussed later, these volume effects are not significant since a key result of our analysis indicates that ξ_{DR} deviates from zero when incorporating the BOSS likelihood (particularly when considering the low- S_8 region and/or including KiDS). Moreover, for model comparison, we rely on statistical indicators rooted in frequentist statistics (see Sec. 6.2.2), which are inherently independent of volume effects. It is also necessary to put an upper limit on a_{dark} ; otherwise, in the limit $\xi_{\text{DR}} \rightarrow 0$, arbitrarily large values of a_{dark} are permissible. For ξ_{DR} , we impose a conservative prior of $\xi_{\text{DR}} < 1$, ensuring consistency within the CMB constraints on N_{eff} .

6.2.2 Statistics for model comparison and tension quantification

To assess how well a given model fits the data in comparison to ΛCDM , we evaluate the χ^2 difference. The χ^2 difference for a model $\mathcal{M}$ relative to ΛCDM is defined as

$$\Delta\chi^2_{\mathcal{M},\text{data}} = \chi^2_{\text{min},\mathcal{M},\text{data}} - \chi^2_{\text{min},\Lambda\text{CDM},\text{data}}, \quad (6.5)$$

where χ^2_{min} represents the best-fit value. It is important to note that $\Delta\chi^2$ depends strongly on a single point in the posterior, namely the best-fit value. Identifying the global minimum in an MCMC analysis can be challenging, depending on the likelihood topology. In some cases, multiple independent runs were required, each consisting of six chains and $\mathcal{O}(10^6)$ points. Using $\Delta\chi^2$, we define the Akaike Information Criterion (AIC) as

$$\Delta\text{AIC}_{\mathcal{M},\text{data}} = \Delta\chi^2_{\mathcal{M},\text{data}} + 2(N_{\mathcal{M}} - N_{\Lambda\text{CDM}}), \quad (6.6)$$

where N denotes the number of degrees of freedom in the model. The AIC provides a means of penalizing models with excessive free parameters, following Occam's razor principle.

To quantify the level of tension within a dataset for a given model (where the number of free parameters remains unchanged), we use the 'difference in maximum a posteriori' (DMAP) as defined in [152]

$$Q_{\text{DMAP}}^{\mathcal{M},\text{data}} = \chi^2_{\text{min},\mathcal{M}}(\text{w/ data}) - \chi^2_{\text{min},\mathcal{M}}(\text{w/o data}). \quad (6.7)$$

The degree of tension between a dataset and other datasets can be quantified by taking the square root of $Q_{\text{DMAP}}^{\mathcal{M}, \text{data}}$. Specifically, the tension of a model $\mathcal{M}$ with KiDS is given by

$$\text{Tension} = \sqrt{Q_{\text{DMAP}}^{\mathcal{M}, \text{KiDS}}} = \sqrt{\chi_{\min, \mathcal{M}}^2(\text{w/ KiDS}) - \chi_{\min, \mathcal{M}}^2(\text{w/o KiDS})}. \quad (6.8)$$

The statistical approach used here to evaluate both the tension with KiDS data and the preference of a model over ΛCDM is analogous to the methodology employed in [153] for analyzing how different models address the H_0 tension relative to SH0ES.

6.3 Results

In this section, we present results for various models within the ETHOS framework. We examine three data combinations as outlined in Sec. 6.2.1: Planck + BAO + RSD, Planck + BAO + FS, and Planck + BAO + FS + KiDS. It is important to note that the posteriors including KiDS should be interpreted cautiously: the KiDS result is implemented as a Gaussian around the S_8 value measured for ΛCDM , rather than as a full model-dependent likelihood. We include this for comparative purposes and to identify preferred parameter space directions when considering the low S_8 value reported by KiDS.

We begin in Sec. 6.3.1 by discussing the fiducial case, where all of dark matter (DM) interacts with dark radiation, with DR behaving as a dark fluid. Section 6.3.2 is dedicated to scenarios with varying interacting dark matter (IDM) fractions, and Sec. 6.3.3 explores the case of free-streaming DR for all scenarios ($n = 0, 2, 4$). The main results are summarized in Tab. 6.1.

6.3.1 Fiducial (dark fluid, $f_{\text{IDM}} = 1$, $n = 0$) scenario

We begin by presenting the results for the fiducial scenario, where DR is modeled as a dark fluid. In this scenario, all DM interacts with DR ($f_{\text{IDM}} = 1$), and the temperature dependence of the interaction rate matches that of the Hubble rate ($n = 0$).

We present the posterior results for S_8 for ΛCDM (dashed) and the fiducial model (solid) in the right panel of Fig. 6.1. The discrepancy between the red dashed line and the gray band, which represents the nominal 1σ KiDS constraints, illustrates the tension between Planck + BAO and weak-lensing data within ΛCDM . Combining Planck and BAO with FS does not shift the posterior to lower S_8 values (blue dashed line), indicating that Planck data dominate the posterior.

For the fiducial IDM model, the Planck + BAO + RSD dataset (red solid) favors lower values of S_8 compared to ΛCDM . In addition to the expected broadening of the posterior, due to the inclusion of additional free parameters, there is also a noticeable skew towards smaller S_8 values. Incorporating FS further enhances this skewed tail and induces a slight shift toward lower S_8 , in agreement with [126]. Notably, the Planck + BAO + FS dataset are compatible with KiDS in the context of IDM, making it reasonable to combine these datasets. The tension in S_8 is reduced from 2.9σ in ΛCDM

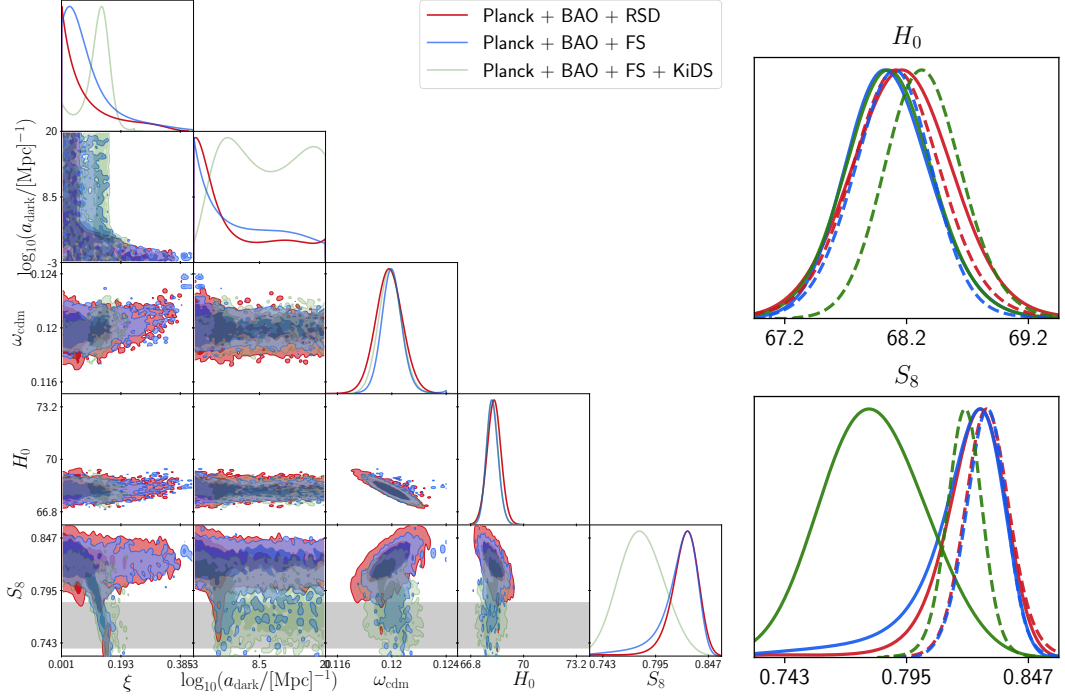


Figure 6.1. Left: Posterior distributions for parameters in the fiducial scenario with $f_{\text{IDM}} = 1$ for three different dataset combinations: Planck + BAO + RSD (red), Planck + BAO + FS (blue) and Planck + BAO + FS + KiDS (green). Gray bands refer to the KiDS 1σ bounds on S_8 . Right: posteriors of S_8 and H_0 for the fiducial model (solid) and for ΛCDM (dashed) for the same dataset combinations.

to 1.5σ in the fiducial IDM scenario (see Tab. 6.1). As we will discuss in Sec. 6.3.2, models with a lower fraction of IDM provide an even greater reduction in this tension.

An important consideration is the potential degeneracy between the bias parameters used in the FS analysis and shifts in S_8 . However, we explicitly verified that the observed shifts in S_8 are not trivially absorbed by b_1 or other bias parameters.

In the left side of Fig. 6.1, we present the combined posteriors for the fiducial IDM cosmology. The S_8 vs. ξ_{DR} panel reveals a distinct feature: a preferred direction in parameter space that allows for a significant reduction in S_8 . When including KiDS data (light green), this region becomes the most favored, yielding at the 1σ level a DR temperature of

$$\xi_{\text{DR}} = 0.117^{+0.026}_{-0.016} \quad (\text{Planck} + \text{BAO} + \text{FS} + \text{KiDS}, f = 1). \quad (6.9)$$

Additionally, for the Planck + BAO + FS dataset, the IDM model exhibits $\Delta\chi^2 = -5.5$ compared to ΛCDM (see Tab. 6.1), indicating a *mild* preference for IDM with $f_{\text{IDM}} = 1$ according to the AIC criterion. When KiDS data is included, this preference strengthens significantly, with $\Delta\chi^2 = -11.7$, demonstrating a *strong* preference for IDM over ΛCDM .

From Fig. 6.1, we observe that the posteriors of Planck + BAO + FS (blue) favor slightly larger values of a_{dark} and ξ_{DR} compared to those for Planck + BAO + RSD (red), while also exhibiting a narrower distribution for ω_{cdm} and H_0 . This suggests that the FS analysis provides additional valuable information within the IDM framework. Beyond constraining certain cosmological parameters, the FS data also induces a significant shift in the ETHOS parameter values, pushing them further from the ΛCDM limit—specifically, favoring a higher IDM temperature ξ_{DR} and a stronger interaction.

The contributions of each likelihood to the total χ^2 are presented in Tab. 6.2. Notably, when incorporating Planck and FS data, the analysis of IDM-DR model in the fiducial scenario yields an improvement in χ^2 relative to ΛCDM , consistent with the expected reduction due to the additional free parameters. This expected decrease is accounted for in the ΔAIC statistics (see Tab. 6.1), which does not strongly favor either model when weak lensing data is excluded but exhibits a slight preference for IDM-DR.

It is important to highlight that when weak lensing data (KiDS) is included in the likelihood for ΛCDM , the χ^2 value increases by 8.5, with the dominant contribution coming from KiDS itself, alongside a partial contribution from FS. This reflects the S_8 tension within ΛCDM . In contrast, for IDM-DR with $f_{\text{IDM}} = 1$, the inclusion of KiDS results in a much smaller change in χ^2 , increasing by only 2.2, primarily due to contributions from Planck high- ℓ . Additionally, the FS χ^2 improves by approximately 2 when KiDS is included. These results suggest that IDM-DR provides a significantly better fit to KiDS and FS data compared to ΛCDM , with only a minor deterioration in the fit to Planck high- ℓ data. This indicates that a future analysis incorporating data from the Atacama Cosmology Telescope (ACT) and the South Pole Telescope (SPT) would be a promising avenue for further investigation.

An important complementary probe for scenarios that suppress the power spectrum is the Lyman- α forest. In this study, we focus on the BOSS FS analysis and leave a quantitative investigation incorporating Lyman- α data for future work. Nevertheless, it is worth noting that for the $n = 0$ scenario considered here, the suppression of the power spectrum is relatively flat. This occurs because the DM-DR interaction rate follows a time dependence similar to that of the Hubble rate, allowing the interaction to remain active as a broad range of scales enter the horizon [45] (see Sec 4.3). This feature also explains why Lyman- α data are less critical for this scenario. Indeed, as shown in [154], Planck data impose stronger constraints on IDM-DR compared to HIREES and MIKE Lyman- α observations [155]. A similar conclusion was drawn in [156] based on BOSS Lyman- α observations [157, 158], where a conservative approach was adopted regarding astrophysical uncertainties. Additionally, if only a fraction of DM interacts with DR, the suppression of the power spectrum is more gradual [43, 45]. This possibility will be explored in the next section.

It is also important to discuss the H_0 parameter. The posterior distributions for H_0 in both ΛCDM and IDM-DR are displayed in the right panel of Fig. 6.1. We find that incorporating FS data leads to improved constraints on H_0 for IDM compared to Planck + BAO + RSD. However, the peak position remains unchanged and does not shift toward higher H_0 values, as would be required to alleviate the tension with SH0ES [159].

This indicates that IDM-DR model does not resolve the H_0 tension, in agreement with previous findings [153]. Although the tension could be mitigated by imposing a lower bound prior on N_{eff} [47], this comes at the cost of degrading the overall fit.

From our analysis, we obtain:

$$H_0 = 68.11^{+0.36}_{-0.37} \quad (\text{Planck} + \text{BAO} + \text{FS, for IDM}). \quad (6.10)$$

This result remains consistent across different values of f_{IDM} or n , which will be examined in the following sections.

	Planck + BAO			+ FS			+ FS + KiDS			$\sqrt{Q_{\text{DMAP}}^{\text{KiDS}}}$
	χ^2	$\Delta\chi^2$	ΔAIC	χ^2	$\Delta\chi^2$	ΔAIC	χ^2	$\Delta\chi^2$	ΔAIC	
ΛCDM	2791.6	-	-	3563.1	-	-	3571.6	-	-	2.9σ
Fluid:										
Fiducial	2790.3	-1.3	2.7	3557.7	-5.5	-1.5	3559.9	-11.7	-7.7	1.5σ
$f_{\text{IDM}} = 0.1$	2789.5	-2.1	1.9	3560.4	-2.7	1.3	3562.0	-9.6	-5.6	1.3σ
$f_{\text{IDM}} = 0.01$	2791.1	-0.5	3.5	3561.5	-1.6	2.4	3566.9	-4.7	-0.7	2.3σ
f_{IDM} free	2790.3	-1.3	4.7	3558.0	-5.2	0.8	3561.4	-10.2	-4.2	1.9σ
Free-streaming:										
$n = 0$	2790.2	-1.4	2.6	3559.0	-4.1	-0.1	3559.7	-11.9	-7.9	0.8σ
$n = 2$	2790.0	-1.6	2.4	3560.7	-2.4	1.6	3567.7	-3.9	0.1	2.6σ
$n = 4$	2790.7	-0.9	3.1	3559.9	-3.2	0.8	3568.0	-3.6	0.4	2.8σ

Table 6.1. Values of χ^2 , $\Delta\chi^2$, ΔAIC and $\sqrt{Q_{\text{DMAP}}^{\text{KiDS}}}$ for the best-fit point of different models considered. When considering the fluid scenario in the case $n = 0$. The fiducial and free-streaming scenarios correspond to $f_{\text{IDM}} = 1$.

6.3.2 Mixed cold and interacting DM

A scenario in which only a fraction of DM interacts with DR is well-motivated, particularly in the context of a WIMP dark sector. An example is the Partially Acoustic Dark Matter (PacDM) model proposed in [43], which consists of two matter fields, χ_1 and χ_2 , where only χ_2 interacts with DR [160]. In this framework, χ_1 remains collisionless and constitutes the dominant component of DM. The suppression of χ_2 clustering leads to a reduction in the S_8 parameter. We analyze the scenario in which only 10% of DM interacts with DR. Notably, in this case, the ratio of IDM to CDM densities is similar in magnitude to the ratio between baryons and DM. As shown in Tab. 6.1, the S_8 tension is significantly reduced to 1.3σ . Furthermore, when KiDS data is included, we get $\Delta\chi^2 = -9.6$ (with $\Delta\text{AIC} = -5.6$), indicating a preference for this model over ΛCDM . The combined posteriors for $f_{\text{IDM}} = 0.1$ are presented in Fig. 6.2.

Compared to the scenario where 100% of DM interacts with DR, the 10% case offers reduction in S_8 through DM-DR interaction and avoids excessive suppression that would conflict with Planck and BAO data. It is also worth noting that in the 10% scenario, a strong DM-DR interaction (i.e., large a_{dark}) is allowed when compared to the results shown in Fig. 6.1 with KiDS data. This behavior is reminiscent of the *dark*

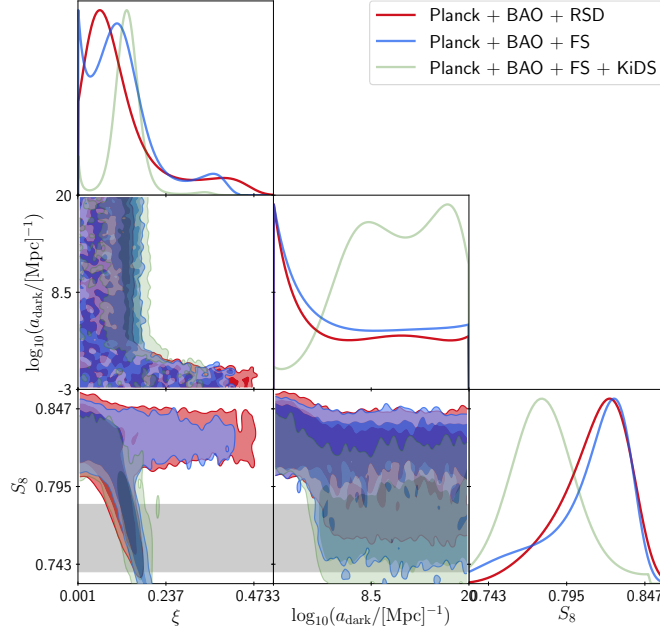


Figure 6.2. Same as Fig 6.1 but with $f_{\text{IDM}} = 0.1$.

plasma and *weakly interacting* scenarios discussed in Sec. 4.1 and [45]: when IDM makes up 100% of DM, a relatively weak DM-DR interaction strength is slightly preferred, where the interaction rate is comparable to the Hubble rate as relevant modes enter the horizon. However, when only 10% of DM interacts, the interaction can be much stronger, resulting in tight coupling of IDM with DR, akin to the dark plasma scenario. Our parameterization covers both extremes, and the difference in the posteriors for a_{dark} between Fig. 6.1 and Fig. 6.2 can be clearly associated with the weakly interacting and dark plasma limits, respectively.

Furthermore, we emphasize that when considering only the FS likelihood, there is *no* preference for a 10% IDM fraction ($\Delta\text{AIC} = 1.3$). The contributions of each likelihood to the total χ^2 for the 10% IDM model are detailed in Tab. 6.2. However, when including KiDS data, the FS likelihood exhibits a strong preference for 10% IDM, despite a slight increase in χ^2 from Planck data. The preferred value of the DR temperature in the presence of KiDS is

$$\xi_{\text{DR}} = 0.140^{+0.026}_{-0.022} \quad (\text{Planck} + \text{BAO} + \text{FS} + \text{KiDS}, f = 0.1). \quad (6.11)$$

In the left side of Fig. 6.3, we present the MCMC results for the scenario in which IDM constitutes only 1% of DM. In this case, the suppression of the power spectrum is mild and insufficient to shift S_8 toward the KiDS reference value. As a result, the S_8 tension is only marginally alleviated to 2.3σ , with $\Delta\chi^2 = -4.7$ when KiDS is included, and no significant preference for this model is observed.

The right side of Fig. 6.3 depicts the analysis where we marginalize over f_{IDM} . In this

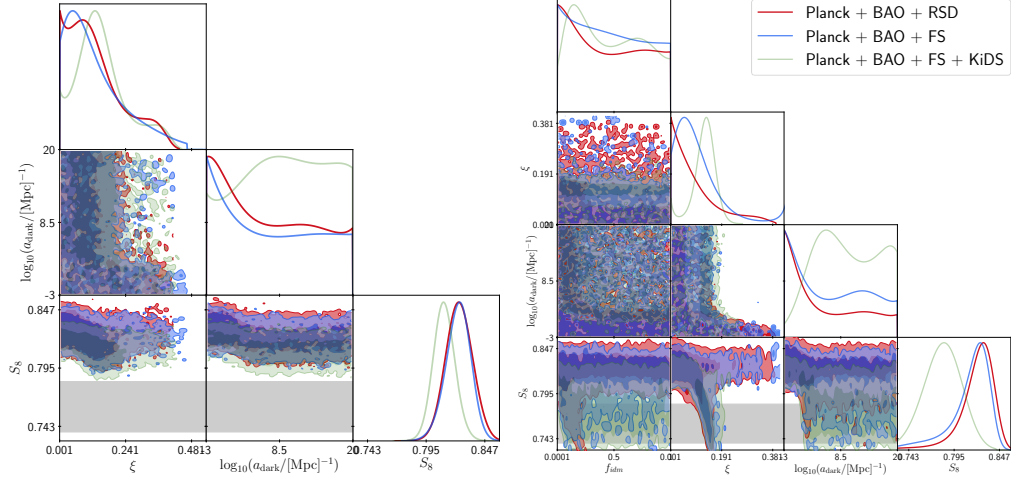


Figure 6.3. MCMC results for IDM fraction $f_{\text{IDM}} = 0.01$ (left) and for f_{IDM} as a free parameter (right).

case, we find that the S_8 tension is reduced to 1.9σ . When incorporating FS data, we obtain $\Delta\chi^2 = -5.2$ and $\Delta\text{AIC} = 0.8$, indicating *no* clear preference for any specific value of f_{IDM} . We note that fluctuations in the Monte Carlo minimization procedure for χ^2 lead to a slightly larger χ^2 value for the free f_{IDM} case compared to the fixed $f_{\text{IDM}} = 1$ case, despite the latter being nested within the former. Given that total χ^2 fluctuations on the order of ~ 0.5 are expected, we mitigated this uncertainty by running multiple independent chains, as described earlier. These fluctuations do not affect the model comparison or the overall conclusion that IDM significantly alleviates the S_8 tension. When including KiDS data, we observe a substantial improvement with $\Delta\chi^2 = -10.2$ and $\Delta\text{AIC} = -4.2$, providing strong evidence in favor of IDM over ΛCDM . The limiting case $f_{\text{IDM}} = 0$, which corresponds to ΛCDM with additional DR, is *not* preferred over other parameter values when KiDS data is included. In fact, there is a slight preference for nonzero values of f_{IDM} , with a peak around $f_{\text{IDM}} \sim 10\%$, as previously discussed, which aligns well with KiDS constraints.

6.3.3 Free-streaming DR

We now turn our attention to the scenario involving free-streaming for DR. Here, we assume negligible self-interaction rate of DR as given in Eq. (4.9), and solve the entire hierarchy of moments for the Boltzmann system up to $l = 17$ using CLASS.

The free-streaming case is particularly pertinent in scenarios such as sterile neutrinos for DR ($n = 4$) or when DM is charged under an Abelian $U(1)$ symmetry ($n = 2$). We also explore the case with $n = 0$ where the temperature dependence of the interaction is negligible.

The MCMC results for the free-streaming cases with $n = 0, 2, 4$ are shown in Fig. 6.4. It is evident that only the $n = 0$ scenario reaches lower values of S_8 . The cases with $n = 2$

and $n = 4$ exhibit a strong temperature dependence, resulting in a sharp transition from a regime where $\Gamma/\mathcal{H} \gg 1$ to $\Gamma/\mathcal{H} \ll 1$. Consequently, structure formation is significantly suppressed on scales entering the horizon before this transition. In contrast, the DM-DR interaction for $n = 0$ remains nearly constant during radiation domination, leading to a gradual suppression of the power spectrum.

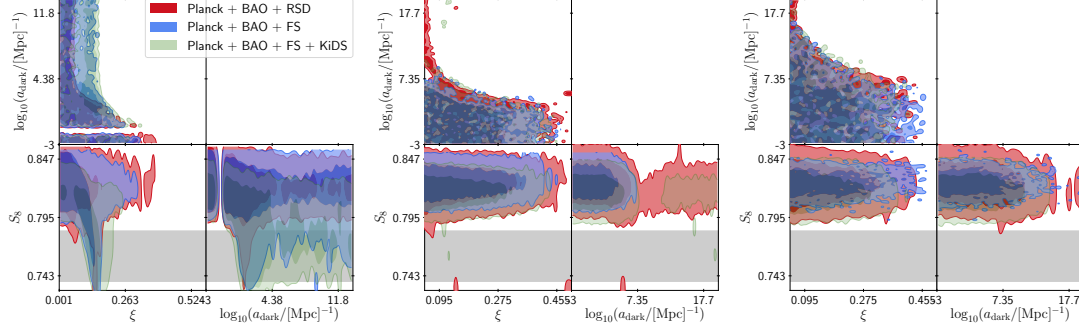


Figure 6.4. Posteriors for the free-streaming DR case for all examples $n = 0, 2, 4$ (left, middle, right).

	Λ CDM		Fiducial		$f_{\text{IDM}} = 0.1$	
	+FS	+FS+KiDS	+FS	+FS+KiDS	+FS	+FS+KiDS
Planck High ℓ TT-TE-EE	2354.3	2352.7	2348.0	2351.6	2348.7	2355.8
Planck Low ℓ EE	395.8	396.5	396.1	396.2	396.8	395.9
Planck Low ℓ TT	22.8	23.1	23.0	23.4	23.3	23.5
Planck lensing	9.0	8.9	9.2	8.7	9.4	9.5
BAO	9.8	9.8	10.0	9.9	9.8	10.9
FS	771.3	772.7	771.4	769.5	772.5	766.4
KiDS	-	7.9	-	0.4	-	0.0
Total	3563.1	3571.6	3557.7	3559.9	3560.4	3562.0

Table 6.2. The contribution of various likelihoods, as outlined in Sec. 6.2.1, to the total χ^2 of the best-fit models for Λ CDM, the fiducial model ($f_{\text{IDM}} = 1$), and the $f_{\text{IDM}} = 0.1$ model. The χ^2 values are rounded to the first decimal place, which results in the KiDS χ^2 appearing as zero and, in some cases, causes the total sum to differ slightly from the sum of the individual contributions.

We observe that only the scenario where $n = 0$ leads to a lower value of S_8 . In this scenario, the tension with KiDS data is minimized to just 0.8σ . However, the Q_{DMAP} statistic, reliant on finding minima, is not ideally suited for MCMC, hence the precise numerical value should be interpreted cautiously. MCMC convergence requires extended sampling time according to the Gelman-Rubin criterion for this scenario, necessitating multiple chains. Nevertheless, the findings from Tab. 6.1 consistently indicate a discernible trend: IDM mitigates the tension between KiDS and other probes to approximately the 1σ level, particularly when $n = 0$ and $f_{\text{IDM}} \gtrsim 0.1$.

Interestingly, for $n = 2, 4$, incorporating FS information enhances the upper constraint

on the interaction strength a_{dark} , even for relatively small values of ξ_{DR} . This enhancement comes from the scale-dependence of power suppression in cases with $n = 2, 4$, and the sensitivity of FS data to smaller scales compared to Planck. Additionally, unlike $n = 0$, Lyman- α data can provide additional information for $n = 2, 4$ [47]. However, a comprehensive analysis combining Lyman- α with Planck and FS data exceeds the scope of this study.

6.4 Mapping of cosmological constraints on the model parameter space

The DM-DR interaction rate, Eq (5.3), can be expressed in terms of the ETHOS parameterization (Eq. (4.8)). By neglecting contributions that are suppressed in the non-relativistic limit $T_{\text{DR}} \ll m_\chi$, we obtain $a_n = 0$ for $n > 0$ and define $a_{\text{dark}} \equiv a_0$.²

$$\begin{aligned} a_{\text{dark}} &= \frac{\pi}{12} \frac{\alpha_d^2}{m_\chi} \frac{1}{\xi_N^2} \frac{T_{\gamma,0}^2}{\Omega_\gamma h^2} \left[\ln \alpha_d^{-1} + c_0 + c_1 g_d + \mathcal{O}(g_d^2) \right] \\ &= 0.91 \cdot 10^9 \text{Mpc}^{-1} \left(\frac{\alpha_d}{10^{-4}} \right)^2 \left(\frac{100 \text{ GeV}}{m_\chi} \right) \left(\frac{0.1}{\xi_N} \right)^2 \left[\ln \alpha_d^{-1} - (1.34 + \ln N) + 0.413 \sqrt{N} g_d \right], \end{aligned} \quad (6.12)$$

where $\xi_N = T_{\text{DR}}/T_\gamma$ (see below), $T_{\gamma,0}$ represents the CMB temperature today, and $\Omega_\gamma h^2$ is its density parameter. We used natural units in the second line to convert the rate into the units used in previous sections.

The DR energy density is related to its temperature via

$$\rho_{\text{DR}} = \frac{\pi^2}{30} \eta_{\text{DR}} \xi_N^4 T_\gamma^4. \quad (6.13)$$

Within the parameterization adopted for the MCMC analysis, the DR temperature explicitly appears only through ρ_{DR} . However, for the input parameter ξ used in previous sections, we assumed a fiducial value of $\eta_{\text{DR}}^{\text{fid}} = 2$, whereas for the dark $SU(N)$ model, we have $\eta_{\text{DR}} = 2(N^2 - 1)$. The two cases can be related by requiring that they correspond to the same DR energy density, which leads to the mapping $\xi_N = \xi(N^2 - 1)^{-1/4}$. Here, ξ_N corresponds to the actual temperature of the $SU(N)$ dark plasma, whereas ξ is the MCMC parameter used in previous sections. Alternatively, the DR energy density can be parameterized in terms of N_{eff} , or N_{fluid} for $SU(N)$ as

$$\Delta N_{\text{fluid}} = \frac{\rho_{\text{DR}}}{2 \frac{\pi^2}{30} \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} T_\gamma^4} \simeq 1.11 \cdot 10^{-3} \eta_{\text{DR}} \left(\frac{\xi_N}{0.15} \right)^4. \quad (6.14)$$

²This result differs by a factor of 2 from Eq. (37) in [13]. However, we verified agreement in the logarithmic contribution to $\Gamma_{\text{IDM-DR}}$ with the correct Eq. (36) in [13], from which the prefactor in a_0 quoted here is obtained. The discrepancy arises from a missing factor of 2 in the left-hand Eq. (36) in [13].

If the dark sector was in thermal equilibrium with the visible sector (i.e., SM particles) above some decoupling temperature T_{dec} , its temperature can be determined by entropy conservation

$$\xi_N^{\text{thermal}} = \left(\frac{g_{*s}^{\text{vis}}(T) g_{*s}^d(T_{\text{dec}})}{g_{*s}^{\text{vis}}(T_{\text{dec}}) g_{*s}^d(T_{\text{DR}})} \right)^{1/3} \simeq 0.33 \left(\frac{\eta_{\text{DR}} + \eta_\chi}{\eta_{\text{DR}}} \right)^{1/3} \left(\frac{106.75}{g_{*s}^{\text{vis}}(T_{\text{dec}})} \right)^{1/3}, \quad (6.15)$$

where $s^{\text{vis}} = \frac{\pi^2}{45} g_{*s}^{\text{vis}} T^3$ and $s^d = \frac{\pi^2}{45} g_{*s}^d T_d^3$ denote the entropy densities of the visible and dark sectors, respectively. The second line assumes $T_{\text{dec}} > m_\chi$ and that $T \ll m_e, T_{\text{DR}} \ll m_\chi$ during the epoch when perturbation modes relevant for CMB and LSS enter the horizon. For $T_{\text{dec}} < m_\chi$, the only modification is the absence of the second factor in the final expression.

This can be compared to the DR abundance required to alleviate the S_8 tension, which is approximately $\xi_N|_{S_8} \sim 0.1(10\xi^{\text{MCMC}})(N^2 - 1)^{-1/4}$ (see Sec. 6.3). If the dark sector was initially in thermal equilibrium with the visible sector, resolving the S_8 tension would require additional heavy particles in the thermal bath to ensure $g_{*s}^{\text{vis}}(T_{\text{dec}}) \gg 106.75$. A more plausible alternative is that the dark sector was never in thermal contact, allowing for arbitrarily small values of ξ_N .

The $SU(N)$ gauge interaction exhibits a stronger coupling at low energies due to the operation of the coupling constant, governed by the β function

$$\beta(g_d) = -\frac{11}{3} C_A \frac{g_d^3}{16\pi^2} + \mathcal{O}(g_d^5), \quad (6.16)$$

where $C_A = N$, and we assume that there are no additional light degrees of freedom apart from gauge bosons for the relevant temperature regime $T_{\text{DR}} < m_\chi$. The condition that the confinement scale Λ_c does not exceed a maximum value Λ_c^{max} translates into an upper bound on α_d at a reference scale μ_{ref}

$$\alpha_d < \frac{3}{11} \frac{4\pi}{C_A} \frac{1}{\ln \frac{\mu_{\text{ref}}}{\Lambda_c^{\text{max}}}} = 0.083 \frac{3}{N} \frac{1}{1 + \ln[(1 + z^{\text{horizon entry}})/10^6]}, \quad (6.17)$$

where we estimate the confinement scale using the Landau pole from the one-loop beta function. The reference scale is taken to be the typical IDM–DR scattering energy, which corresponds to the DR temperature at horizon entry, $\mu_{\text{ref}} = T_{\text{DR}}^{\text{horizon entry}} = \xi_N T_{\gamma,0} (1 + z^{\text{horizon entry}})$, while we set $\Lambda_c^{\text{max}} = T_{\text{DR},0} = \xi_N T_{\gamma,0}$.

Additionally, constraints from galaxy ellipticity observations, specifically from NGC720 [161], impose a bound on the long-range interaction strength

$$\sqrt{\frac{1}{2} C_F \alpha_d} \lesssim 0.01 \frac{1}{\sqrt{f}} \left(\frac{m_\chi}{300 \text{ GeV}} \right)^{3/2}. \quad (6.18)$$

We use a factor of $C_F/2 = (N^2 - 1)/(4N)$ compared to the Abelian case, which arises from the color factor in the $\chi\chi \rightarrow \chi\chi$ scattering process. This factor accounts for the averaging over the dark $SU(N)$ degrees of freedom of one of the incoming χ particles,

while summing over all others, ensuring the correct scaling of the scattering rate for a single χ particle. Furthermore, if only a fraction f_{IDM} of dark matter is interacting, we adjust the bound accordingly by rescaling it to reflect the reduced scattering rate.

Self-interaction cross-sections of the order of cm^2/g are relevant in addressing small-scale structure issues, such as the core-cusp problem [7]. The self-scattering cross-section for $SU(N)$ theory can be estimated as [161]

$$\frac{\sigma}{m_\chi} = \frac{C_F}{2} \frac{8\pi\alpha_d^2}{m_\chi^3 v^4} \sim 1 \frac{\text{cm}^2}{\text{g}} \frac{C_F}{2} \left(\frac{\alpha_d}{2.5 \cdot 10^{-5}} \right)^2 \left(\frac{100 \text{ GeV}}{m_\chi} \right)^3 \left(\frac{30 \text{ km/s}}{v} \right)^4, \quad (6.19)$$

where v denotes the relative velocity.

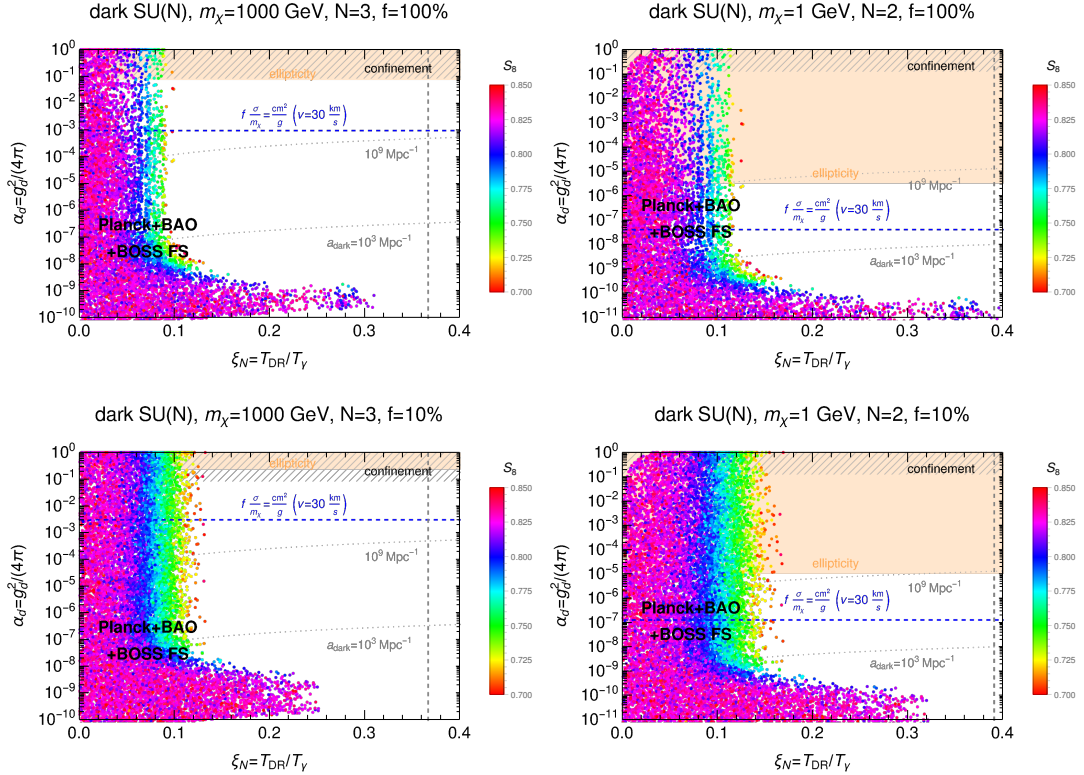


Figure 6.5. Results for the dark fine-structure constant α_d as a function of the dark sector temperature ratio $\xi_N = T_{\text{DR}}/T_\gamma$. Colored points represent IDM–DR models that are allowed at the 95% C.L. by Planck+BAO+BOSS FS data, with colors indicating the corresponding values of S_8 . The hatched region is excluded based on the requirement that the gauge force remains unconfined until today, while the orange shaded region is ruled out by constraints from the ellipticity of NGC 720. The blue dashed line represents a self-interaction cross-section of $f \times \sigma/m_\chi = 1 \text{ cm}^2/\text{g}$. The vertical gray dashed line denotes the DR temperature expected if the dark sector was in thermal equilibrium with the Standard Model in the early Universe. Dotted contour lines correspond to $\log_{10}(a_{\text{dark}}[\text{Mpc}^{-1}]) = 3, 9$. For illustration, we display different values of $m_\chi = 1000, 1 \text{ GeV}$, $N = 3, 2$, and $f_{\text{IDM}} = 1, 0.1$.

We use the result in Eq. (6.12) for the IDM–DR interaction rate to translate the

Planck+BAO+FS constraints derived in Sec. 6.3.1 and Sec. 6.3.2 into the parameter space of the dark fine-structure constant, $\alpha_d = g_d^2/(4\pi)$, versus the DR temperature, ξ_N . The corresponding results are presented in Fig. 6.5, assuming the following values for illustration; $m_\chi = 1$ TeV with $N = 3$ (left panels) and $m_\chi = 1$ GeV with $N = 2$ (right panels). Additionally, we display results for scenarios where the interacting DM fraction is $f_{\text{IDM}} = 100\%$ (upper panels) and $f_{\text{IDM}} = 10\%$ (lower panels). Each data point represents a set of model parameters consistent with Planck + BAO + FS constraints at the 95% confidence level. The points are color-coded based on their corresponding S_8 values.

For DR temperatures above $\xi_N \gtrsim 0.1 - 0.15$, stringent constraints are imposed on the gauge coupling, limiting it to $\alpha_d \lesssim 10^{-8} - 10^{-10}$. Stronger interactions in this regime would significantly suppress structures on scales that are tightly constrained by CMB and LSS data. However, for lower DR temperatures, these constraints on α_d become significantly weaker, as the DR energy density is too small to notably impact the power spectrum.

The region of parameter space that aligns with the KiDS results [116], which report $S_8 = 0.759^{+0.024}_{-0.021}$, corresponds to DR temperatures around $\xi_N \sim 0.1$. Slightly lower values of ξ_N are preferred for $N = 3$ compared to $N = 2$, due to the increased number of gauge bosons in the former case. Furthermore, for $f_{\text{IDM}} = 100\%$, there is a general tendency for smaller values of α_d to be favored for points within the KiDS S_8 range, though larger values remain viable. In the case of $f_{\text{IDM}} = 10\%$, the interaction strength is only constrained from below, requiring $\alpha_d \gtrsim 10^{-8}(10^{-9})$ for $m_\chi = 1000(1)$ GeV to maintain compatibility with KiDS results, consistent with the limiting scenario of a tightly coupled dark plasma.

This mapping allows us to compare complementary constraints arising from confinement and the effects of long-range interactions on elliptical galaxies, as discussed earlier. The corresponding exclusion regions are depicted as hatched and orange-shaded areas in Fig. 6.5, respectively. It is evident that ellipticity constraints impose stronger limits for low-mass dark matter, while confinement constraints become more relevant for heavier masses in the TeV range. Nevertheless, these constraints remain compatible with regions in parameter space that are allowed by cosmological data from Planck + BAO + FS and exhibit S_8 values favored by KiDS. Furthermore, the viable parameter space aligns with a self-interaction cross-section of approximately $1 \text{ cm}^2/\text{g}$, which is relevant for addressing small-scale structure anomalies.

As mentioned earlier, the DR temperature favored by KiDS is notably lower than what would be expected if the dark and visible sectors had been in thermal equilibrium in the early Universe. This suggests that the dark sector was never in thermal contact with the Standard Model. A viable production mechanism for dark sector particles under these conditions is freeze-in [162]. In the simplest scenario, freeze-in can occur solely via gravitational interactions [163], naturally leading to a significantly lower dark sector temperature compared to the Standard Model, depending on the reheating temperature. The subsequent evolution of the dark sector hinges on whether the dark gauge coupling is strong enough to establish thermal equilibrium [164]. If so, the annihilation process

$\chi\chi \rightarrow A_d A_d$ can further deplete the density of χ particles, with freeze-out within the dark sector determining the DM abundance [161]. A similar evolutionary pathway occurs if the dark and visible sectors interact via higher-dimensional operators suppressed by a large energy scale. While such interactions would not alter the structure formation phenomenology discussed here, they could set the initial DR temperature and DM abundance via UV freeze-in [165], followed by dark sector thermalization and freeze-out [166]. In all these scenarios, the DR temperature and DM abundance are linked to the reheating temperature, which in turn is related to the scale of inflation and thus to the tensor-to-scalar ratio r . Notably, gravitational production generally requires a high reheating or inflation scale, implying a lower bound on r , which could be probed by upcoming CMB B-mode polarization experiments [167]. A more detailed investigation of DM and DR production mechanisms is left for future work.

Main Results

- A scenario of $n = 0$ with mixed CDM and IDM is better suited for addressing the S_8 tension.
- The S_8 tension can be reduced from 2.9σ (for Λ CDM) to $\sim 1\sigma$ for IDM–DR models with $n = 0$, $f_{\text{IDM}} \sim 10\%$, and DR temperature $\xi_{\text{DR}} \sim 0.1$.
- $n = 2, 4$ models cannot address the tension.
- We computed interaction rates including higher-order corrections (Debye screening) and used it to perform a mapping between the constraints on the cosmological parameter (the interaction rate a_{dark}) to the particle physics parameter (fine-structure α_d).
- The S_8 solution requires a dark fine-structure constant $\alpha_d \gtrsim 10^{-8}(10^{-9})$ for $m_\chi = 1000(1)$ GeV.
- These values are consistent with upper bounds from elliptical galaxy asphericity and allow for self-interaction cross-sections relevant to small-scale structure problems.

7 Galaxy Cluster Abundance Constraints

In this study, [Paper II](#), we conduct an analysis of the IDM–DR model using the abundance of galaxy clusters detected through the thermal Sunyaev-Zeldovich effect (tSZE) in the South Pole Telescope (SPT) survey data [168]. The clusters are confirmed using the MCMF algorithm [169, 170], applied to optical/NIR follow-up data. This analysis of cluster abundance is supplemented by weak lensing data from the Dark Energy Survey (DES) Year 3 [171–173] and targeted weak-lensing measurements from the Hubble Space Telescope (HST) [174]. The cluster sample consists of 1,005 clusters, derived from the combined SPT-SZ, SPTpol ECS, and SPTpol 500d surveys [77–79, 175], spanning a redshift range of 0.25 – 1.78. The weak-lensing data provide insight into cluster masses and enable empirical constraints on observable-mass relations. We employ weak-lensing data from DES Year 3 for 688 clusters with redshifts $z < 0.95$ and HST data for 39 clusters in the redshift range 0.6 – 1.7. This combination of datasets was recently analyzed in [25], providing competitive constraints on Λ CDM, the dark energy equation of state, and the sum of neutrino masses. In the present work, we follow the same data analysis framework.

7.1 Cluster catalog and weak-lensing datasets

In this section, we give a summary of the main datasets employed in this analysis. we cover first the SPT cluster catalog and then the DES Y3 and HST weak-lensing data used for the mass calibration. Further details can be found in [176].

7.1.1 SPT cluster catalog

The SPT cluster catalog is built from three multi-frequency mm-wave surveys: SPT-SZ, SPTpol ECS, and SPTpol 500d [77–79, 175], covering a total of 5,270 deg² of the southern sky. In regions where the SPT-SZ and SPTpol 500d surveys overlap, we use only the deeper SPTpol 500d data. Cluster candidates are selected based on the observed tSZE *detection significance*, $\hat{\zeta}$, which is the maximum signal-to-noise ratio (SNR) obtained by scanning over a range of angular sizes. Optical and NIR data are used to confirm the clusters, providing ,first, the observed *richness*, $\hat{\lambda}$, representing the weighted count of passive galaxies, and second, the cluster redshift, z . The MCMF algorithm [169, 177] was employed to increase the sample size by $\sim 50\%$, allowing confirmation of lower $\hat{\zeta}$ candidates while maintaining high purity. This method has also been used to create the largest all-sky X-ray cluster sample [178], the deepest Planck tSZE cluster sample [179], and the largest ACT tSZE cluster sample [180].

The catalog includes only clusters at redshifts $z > 0.25$, as lower redshift clusters are excluded due to the filtering of the SPT maps, which removes atmospheric noise but also diminishes signals from low-redshift clusters.

The overlap region between SPT clusters and DES Y3 weak-lensing data spans $3,567 \text{ deg}^2$ (about 70% of the SPT area). The MCMF confirmation process [169] provides cluster redshifts, richness $\hat{\lambda}$, and optical centers. A tSZE-selected SPT cluster candidate is confirmed if its richness exceeds a threshold, $\lambda_{\min}$, which varies with redshift to maintain consistent purity ($\sim 98\%$) across all redshifts [177, 181]. At higher redshifts ($z > 1.1$), where DES Y3 data is limited, we use data from the Wide-Field Infrared Survey Explorer (WISE) [182] for confirmation, applying the same MCMF method to determine $\lambda_{\min}(z)$.

To ensure roughly consistent sample purity across different SPT surveys, we apply varying detection significance and richness thresholds. Clusters are selected based on the following criteria:

$$\begin{aligned}\hat{\xi} &> 4.25 / 4.5 / 5 \text{ (500d / SZ / ECS)}, \\ \hat{\lambda} &> \hat{\lambda}_{\min}(z), \\ z &> 0.25.\end{aligned}\tag{7.1}$$

For the $1,327 \text{ deg}^2$ not covered by DES Y3, we select clusters with $\xi > 5$, yielding a candidate list with purity $\gtrsim 95\%$. Confirmation and redshift determination are performed using targeted optical observations, such as PISCO [183]. More details are available in the original catalog publication [77].

The final sample consists of 1,005 confirmed clusters with tSZE significance, richness, and redshift measurements. Clusters within the DES Y3 region also have richness estimates and optically determined center positions based on the MCMF algorithm applied to DES Y3 or WISE data. The sample significance and richness as a function of redshift are shown in Fig. 7.1

7.1.2 DES Y3 weak-lensing data

The DES Y3 dataset comprises three years of survey observations, covering approximately $5,000 \text{ deg}^2$ of the southern sky. The data were collected using the 570-megapixel Dark Energy Camera (DECam) mounted on the Blanco 4-meter telescope at the Cerro Tololo Inter-American Observatory (CTIO) in Chile. The shape catalog was generated using the METACALIBRATION method [184, 185]. Detailed discussions about systematics and calibration are provided in the DES Y3 catalog papers: [186] for the photometric dataset, [187] for point-spread function modeling, and [188, 189] for image and survey simulations. After applying the selection criteria for source galaxies, the DES Y3 shear catalog includes approximately 100 million galaxies across a solid angle of $4,143 \text{ deg}^2$, with an effective source galaxy density of $5\text{-}6 \text{ arcmin}^{-2}$.

A total of $3,567 \text{ deg}^2$ of the DES Y3 dataset overlaps with the SPT surveys. We arrange lensing source galaxies in four tomographic bins consistent with the $3 \times 2 \text{ pt}$ analysis [18],

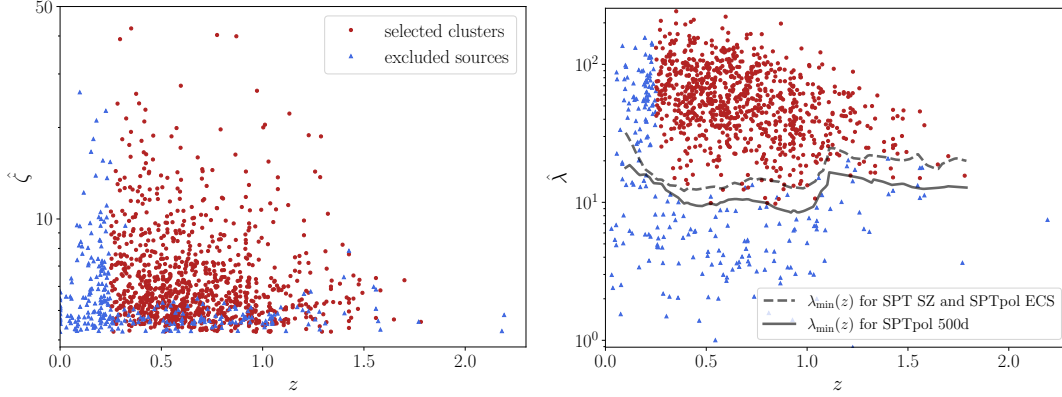


Figure 7.1. The distribution of galaxy clusters and contaminating sources identified in the SPT surveys [77–79] is shown in terms of the observables: tSZE significance ($\hat{\zeta}$), optical or NIR richness ($\hat{\lambda}$), and redshift (z). Clusters confirmed to meet the selection criteria in Eq.(7.1) are represented by red points and constitute the sample analyzed in this study. The blue points, located at lower $\hat{\zeta}$ (left) and $\hat{\lambda}$ (right), correspond to contaminants from the original tSZE-selected candidate list as well as confirmed clusters at $z < 0.25$. Most confirmed clusters are positioned well above the $\lambda_{\min}(z)$ MCMF selection threshold and the majority of the confirmed sample is at redshifts $z < 1$. Additionally, the feature in the $\lambda_{\min}(z)$ selection at $z = 1.1$ corresponds to the inclusion of WISE data at higher redshifts. Reprinted from [Paper II](#).

with redshift distributions calibrated using self-organizing maps [190]. We extract a weak-lensing shear profile for each SPT cluster within the radial range $500 h^{-1} \text{ kpc} < R < \frac{3.2}{1+z} h^{-1} \text{ Mpc}$, centered on the optical cluster center determined by MCMF. We restrict Weak-lensing data from DES Y3 to clusters with redshifts $z < 0.95$, resulting in 688 cluster shear profiles derived from a sample of 555,912 source galaxies. Figure 7.2 displays the constructed averaged matter profiles for confirmed MCMF SPT clusters. Key sources of uncertainty, including cluster member contamination, shear profile miscentering, shear and photo- z calibration, halo mass modeling, and the impact of large-scale structure, are discussed in the cluster cosmology methods paper for this dataset combination [176].

7.1.3 High-redshift HST weak-lensing data

At higher redshifts ($z \gtrsim 0.9$), we use HST weak-lensing data, because DES Y3 loses its constraining power. HST provides precise targeted measurements of shear profiles for clusters owing to its high-resolution imaging. We have weak-lensing measurements for 39 clusters in the redshift range $0.6 - 1.7$ from [174, 191–194]. These references offer a detailed description of the dataset, and how it is measured. Fig. 7.2 shows the constructed averaged matter profiles.

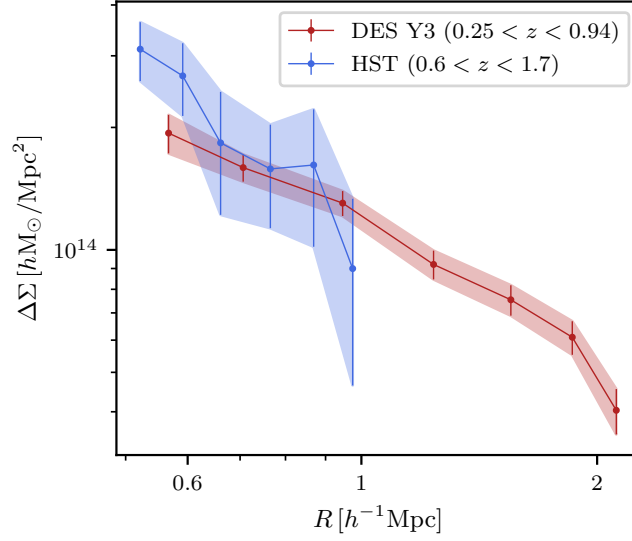


Figure 7.2. The average matter profiles, $\Delta\Sigma(R)$, for MCMF-confirmed SPT-selected clusters are derived from DES Y3 data for 688 clusters (red) and from HST data for 39 clusters (blue). The error bars represent the 1σ uncertainty region. From [Paper II](#).

7.2 Analysis Method

In this section, we outline the analytical approach employed for the abundance and mass calibration likelihoods. We first introduce the relations between observables and mass, followed by a discussion on the method used to extract mass information from weak-lensing measurements. Subsequently, we present the calibration of the weak-lensing mass to halo mass mean relation ($M_{\text{WL}}-M_{\text{halo}}$), which serves as the basis for modeling systematic uncertainties in weak-lensing mass calibration. Finally, we detail the components of the pipeline, including the relevant likelihoods and parameter priors. This approach follows closely the framework presented in [176].

7.2.1 Observable–mass relations

In the matched filter detection technique adopted by SPT [195], each cluster is assigned a sky position and scale (core radius) that maximizes the detection significance. As a result, this significance exhibits a certain level of bias. To account for this bias, the observed significance $\hat{\zeta}$ is linked to an unbiased or intrinsic tSZE significance ζ as follows [196]

$$P(\hat{\zeta}|\zeta) = \mathcal{N}\left(\sqrt{\zeta^2 + 3}, 1\right), \quad (7.2)$$

where $\mathcal{N}(a, \sigma)$ denotes a Gaussian distribution with mean a and standard deviation σ . This normal distribution arises from Gaussian noise in the survey maps. The bias correction factor of 3 accounts for noise bias introduced during the matched-filter search

for peaks, considering three parameters: two for the position on the sky and one for the effective core radius of the tSZE signature.

the scaling relation between the unbiased SPT detection significance ζ and mass is modeled as a power law in both mass and the redshift-dependent dimensionless Hubble parameter ($E(z) \equiv H(z)/H_0$)

$$\begin{aligned} \langle \ln \zeta \rangle = & \ln \zeta_0 + \zeta_M \ln \left(\frac{M_{200c}}{3 \cdot 10^{14} h^{-1} \text{M}_\odot} \right) \\ & + \zeta_z \ln \left(\frac{E(z)}{E(0.6)} \right). \end{aligned} \quad (7.3)$$

The parameters ζ_0 , ζ_M , and ζ_z represent the amplitude, mass trend, and redshift trend of the scaling relation, respectively. The quantity M_{200c} refers to the halo mass within a region where the overdensity is 200 times the critical density at the cluster redshift. The pivot redshift and mass are set at $z = 0.6$ and $M_{200c} = 3 \cdot 10^{14} h^{-1} \text{M}_\odot$, respectively. Observed and simulated clusters have structural variations due to factors such as the time elapsed since their last major merger; this introduces scatter in the observable at any given mass and redshift. We model the intrinsic scatter of ζ around the mean relation as a log-normal distribution with mass- and redshift-independent RMS (root mean square) variation, $\sigma_{\ln \zeta}$.

Since the depths of the various SPT surveys differ, characterized by the parameter γ_{field} , we rescale ζ_0 for each field as

$$\zeta_{0,\text{field}} = \gamma_{\text{field}} \zeta_0. \quad (7.4)$$

This variation in depth also impacts the redshift evolution parameter ζ_z , which we rescale for each SPT survey. A more detailed discussion of the survey information and parameter rescaling can be found in [176].

The noise in the observed richness $\hat{\lambda}$ of a given cluster is represented by a Poisson sampling of an intrinsic richness λ . For $\lambda > 10$, it is appropriate to use a log-normal approximation for the Poisson distribution as follows

$$P(\ln \hat{\lambda} | \ln \lambda) = \mathcal{N}(\ln \lambda, 1/\lambda). \quad (7.5)$$

For modeling the intrinsic richness, we use the same approach as for detection significance

$$\begin{aligned} \langle \ln \lambda \rangle = & \ln \lambda_0 + \lambda_M \ln \left(\frac{M_{200c}}{3 \cdot 10^{14} h^{-1} \text{M}_\odot} \right) \\ & + \lambda_z \ln \left(\frac{1+z}{1.6} \right). \end{aligned} \quad (7.6)$$

The richness–mass scaling relation parameters λ_0 , λ_M , and λ_z correspond to the amplitude, mass trend, and redshift trend, respectively. The log-normal scatter of the intrinsic richness at a given mass and redshift around the mean richness is modeled by its RMS

variation, $\sigma_{\ln \lambda}$.

We have two distinct measurements of richness: DES Y3 data for clusters with redshift $z < 1.1$ and WISE data for clusters with redshift $z > 1.1$. We establish separate observable–mass relations for each and transition from DES-based to WISE-based richness at $z = 1.1$.

7.2.2 Mass calibration

We empirically calibrate the observable–mass relations in Eqs. (7.3) and (7.6) along with their intrinsic scatter using weak-lensing data. We provide a brief overview of the key steps here, and refer to [176] for more details.

For each cluster, the weak-lensing (WL) observable is the reduced tangential shear, $g_t(R)$, which consists of a single shear profile that merges all tomographic bins of WL source galaxies. This observable is then used to determine the WL halo mass (M_{WL}) by fitting the reduced shear profile to a Navarro-Frenk-White (NFW) profile [197] for the mass distribution within the radial range $500 h^{-1} \text{ kpc} < R < \frac{3.2}{1+z} h^{-1} \text{ Mpc}$.

Halos do not align perfectly with the NFW model individually, therefore, the inferred WL mass, M_{WL} , is not an exact determination of the halo mass. To account for discrepancies between M_{WL} and the halo mass, we establish a M_{WL} – M_{halo} mean relation [198–200]

$$\left\langle \ln \left(\frac{M_{WL}}{2 \cdot 10^{14} h^{-1} M_{\odot}} \right) \right\rangle = \ln M_{WL_0} + M_{WL_M} \ln \left(\frac{M_{200c}}{2 \cdot 10^{14} h^{-1} M_{\odot}} \right), \quad (7.7)$$

where M_{200c} represents the halo mass. Here, $\ln M_{WL_0}$ is the logarithmic mass bias at $M_{200c} = 2 \cdot 10^{14} h^{-1} M_{\odot}$, and M_{WL_M} is the mass trend in this bias. The log-normal scatter around the mean M_{WL} is described by its variance

$$\ln \sigma_{\ln WL}^2 = \ln \sigma_{\ln WL_0}^2 + \sigma_{\ln WL_M}^2 \ln \left(\frac{M_{200c}}{2 \cdot 10^{14} h^{-1} M_{\odot}} \right). \quad (7.8)$$

To calibrate the free parameters of the M_{WL} – M_{halo} relation, we generate synthetic cluster shear maps by applying the source redshift distribution, cluster miscentering, and cluster member contamination to halo mass maps from numerical simulations (detailed below) [200].

A parameter p with its standard deviation Δp can also be expressed as

$$p = \mathcal{N}(\bar{p}, \Delta p) = \bar{p} + \Delta p \mathcal{N}(0, 1), \quad (7.9)$$

where $\bar{p}$ is the mean value and Δp is the uncertainty of the parameter p derived from simulations. To accurately model the redshift-dependent uncertainty of the logarithmic mass bias $\ln M_{WL_0}$, we express the scatter in this parameter, $\Delta \ln M_{WL_0}(z)$, as a linear combination of two redshift-dependent functions:

$$\Delta \ln M_{WL_0}(z) = \Delta_1 \ln M_{WL_0}(z) + \Delta_2 \ln M_{WL_0}(z). \quad (7.10)$$

We use mass maps from full-physics hydrodynamical simulations paired with gravity-only simulations having identical initial conditions to establish a relationship between M_{WL} and the gravity-only mass, M_{halo} . This method accounts for baryonic effects while leveraging more robust HMF predictions from gravity-only simulations, such as [69], including emulators like [201–203]. Specifically, we apply this method to the Magneticum [204–207] and the Illustris TNG simulations [208–213]. The resulting constraints on the $M_{\text{WL}}-M_{\text{halo}}$ relation show some variability, which we attribute to uncertainties in modeling baryonic effects. To account for this, we inflate the uncertainty on all parameters of the $M_{\text{WL}}-M_{\text{halo}}$ relation by the additional uncertainty listed in Table 2 of [200], which amounts to 0.02 or 2% for the amplitude $\ln M_{\text{WL}_0}$. This level of systematic uncertainty is smaller than the current statistical uncertainty in the lensing dataset, which ensures that our analysis is not overly dependent on the accuracy of the Magneticum or Illustris TNG simulations.

HST-39 data is a targeted weak-lensing dataset for clusters. Due to the high resolution of HST and the selection of lensing sources via color cuts, cluster member contamination and miscentering effects are much smaller than in the DES Y3 analysis, and therefore not modeled here. Miscentering for this sample has been accounted for as part of the mass bias estimation, as described in [174, 214]. Finally, the HST $M_{\text{WL}}-M_{\text{halo}}$ mean relation includes only the first term in Eq. (7.7) ($\ln M_{\text{WL}_0}$) and its scatter ($\ln \sigma_{\ln \text{WL}_0}^2$) [191].

In summary, the observable–mass relations (ζ –mass– z and λ –mass– z) along with the $M_{\text{WL}}-M_{\text{halo}}$ relation are power-law relations as described in Eqs. (7.3), (7.6), and (7.7), respectively. All have intrinsic scatter modeled as log-normal. We establish a covariance matrix that connects all scatter parameters and account for potential correlations among them. We introduce the parameters $\rho_{\zeta, \text{WL}}$, $\rho_{\zeta, \lambda}$, and $\rho_{\text{WL}, \lambda}$ to represent the correlations between significance and weak-lensing, significance and richness, and weak-lensing and richness, respectively.

7.2.3 Likelihood

galaxy clusters are essentially rare, discrete events that occur randomly and independently in space. The Poisson distribution is well-suited for modeling count data when the events occur with a known average rate but randomly with respect to their spatial location. Therefore, we model the cluster population as a set of independent Poisson realizations of the halo mass function. In the following, we give a review of the likelihood used in the analysis, see [176] for more details.

The probability of observing k halos, given an expected rate μ , follows a Poisson distribution

$$P(k|\mu) = \frac{\mu^k e^{-\mu}}{k!} \Rightarrow \ln P(k|\mu) = k \ln \mu - \mu + \text{const.} \quad (7.11)$$

To construct the likelihood, we divide the observable space into finely spaced bins (e.g., in redshift, SPT detection significance, etc.), ensuring that each bin contains at most

one event. This leads to the likelihood function,

$$\ln \mathcal{L} = \sum_i \ln \mu_i - \sum_j \mu_j, \quad (7.12)$$

where the first sum runs over bins that contain observed events, while the second sum extends over all bins.

Now, taking the limit where bins become infinitesimally small (dx), the expected differential number of events is given by $d\mu = \frac{d\mu}{dx} dx$. Consequently, the likelihood function can be rewritten as

$$\ln \mathcal{L} = \sum_i \ln \left(\frac{d\mu}{dx} dx \right) \Big|_{x_i} - \int \frac{d\mu}{dx} dx = \sum_i \ln \frac{d\mu}{dx} \Big|_{x_i} - \int \frac{d\mu}{dx} dx + \text{const.} \quad (7.13)$$

In this differential form, the index i now runs over individual events, whereas previously it indexed bins. Therefore, Eq. (7.13) represents the unbinned Poisson likelihood, which does not depend on any explicit binning scheme.

The log-likelihood expression for a multi-observable cluster sample is given by

$$\begin{aligned} \ln \mathcal{L}(\mathbf{p}) &= \sum_i \ln \frac{d^4 N(\mathbf{p})}{d\hat{\zeta} d\hat{\lambda} d\mathbf{g}_t dz} \Big|_{\hat{\zeta}_i, \hat{\lambda}_i, \mathbf{g}_{t,i}, z_i} \\ &\quad - \int \cdots \int d\hat{\zeta} d\hat{\lambda} d\mathbf{g}_t dz \frac{d^4 N(\mathbf{p})}{d\hat{\zeta} d\hat{\lambda} d\mathbf{g}_t dz} \Theta_s(\hat{\zeta}, \hat{\lambda}, z) + \text{const.} \end{aligned} \quad (7.14)$$

Here, Θ_s represents the survey selection function, defined by the lower-limit thresholds imposed on the observables $\hat{\zeta}$, $\hat{\lambda}$, and z , as detailed in Sec. 7.1.1. The lensing data consists of tangential shear profiles, denoted by $\mathbf{g}_t$. The differential cluster abundance in observable space is given by

$$\begin{aligned} \frac{d^4 N(\mathbf{p})}{d\hat{\zeta} d\hat{\lambda} d\mathbf{g}_t dz} &= \int d\Omega_s \iiint dM d\zeta d\lambda dM_{\text{WL}} P(\mathbf{g}_t | M_{\text{WL}}, \mathbf{p}) P(\hat{\zeta} | \zeta) P(\hat{\lambda} | \lambda) \\ &\quad P(\zeta, \lambda, M_{\text{WL}} | M, z, \mathbf{p}) \frac{d^2 N(\mathbf{p}, M, z)}{dM dV} \frac{d^2 V(\mathbf{p}, z)}{dz d\Omega_s}, \end{aligned} \quad (7.15)$$

where $\frac{d^2 N(\mathbf{p}, M, z)}{dM dV}$ represents the halo mass function, $\frac{d^2 V(\mathbf{p}, z)}{dz d\Omega_s}$ denotes the differential volume, and Ω_s is the survey footprint. The relations between observed and intrinsic parameters, $P(\hat{\zeta} | \zeta)$ and $P(\hat{\lambda} | \lambda)$, are defined in Eqs. (7.2) and (7.5). The distribution $P(\zeta, \lambda, M_{\text{WL}} | M, z, \mathbf{p})$ follows from Eqs. (7.3), (7.6), (7.7), and (7.8).

Finally, the lensing likelihood $P(\mathbf{g}_t | M_{\text{WL}}, \mathbf{p})$ is modeled as a product of independent Gaussian probabilities in each radial bin i of the tangential reduced shear profile for a given cluster:

$$P(\mathbf{g}_t | M_{\text{WL}}, \mathbf{p}) = \prod_i \left(\sqrt{2\pi} \Delta g_{t,i} \right)^{-1} e^{-\frac{1}{2} \left(\frac{g_{t,i} - g_{t,i}(M_{\text{WL}}, \mathbf{p})}{\Delta g_{t,i}} \right)^2}, \quad (7.16)$$

where $\Delta g_{t,i}$ represents the shape noise.

7.3 Pipeline

The likelihood is implemented as a Python module within the **CosmoSIS** framework¹ [215]. To explore the parameter space, we use the nested sampler **MultiNest** [216–218]. The matter power spectrum is computed with the Boltzmann solver **CLASS** [63, 64]. For comparison, we include an analysis that incorporates CMB anisotropies from Planck 2018 [26] and BAO data from BOSS DR12 [21]. Previous studies have analyzed IDM–DR using Planck and BAO data in [1, 47]. We consider the following dataset combinations

- SPT-cluster×WL: a sample of 1,005 tSZ-selected clusters from SPT surveys [77–79, 175], confirmed with the MCMF algorithm and complemented with weak-lensing mass information from DES Y3 [171–173] and HST [174], as described in Sec. 7.1.
- CMB+BAO: CMB data from Planck 2018 (TT, TE, EE + lowE) [26] and BAO measurements from BOSS DR12 [21].

For the SPT-cluster×WL analysis, we run the full log-likelihood with the cluster abundance and the weak-lensing mass calibration simultaneously, as given in Eq. (7.14). All priors for the fit parameters are listed in Table 7.1. As discussed in Sec. 7.2, we apply informative, Gaussian priors to the weak-lensing mass calibration parameters introduced in Eqs. (7.7)–(7.8), with separate treatment for DES Y3 data and HST data, as described in Sec. 7.2.2. The observable–mass– z parameters introduced in Eqs. (7.2)–(7.6) are fully constrained by the mass calibration likelihood, and we apply flat, uninformative priors to them. As explained in Sec. 7.2, we consider two richness–mass scaling relations for the datasets (DES Y3 and WISE) at different redshift regimes, and thus apply separate priors to each. We define flat priors for the correlation coefficients in the covariance matrix, ensuring that the resulting correlation matrix remains non-singular across all parameter combinations. Finally, we apply priors to the cosmological parameters. When analyzing galaxy cluster abundance alone, we define Gaussian priors on $\Omega_b h^2$ and n_s based on Planck results, and a broad prior on h incorporating all current constraints [$\mathcal{N}(0.7, 0.05)$]. This is because clusters are not sensitive to the baryon density or the spectral index, and they do not provide strong constraints on the Hubble parameter by themselves. When CMB and BAO data are included, we apply only flat priors to the aforementioned parameters and marginalize over the optical depth of reionization, τ .

The upper bound on the flat prior for ξ_{DR} is derived from Planck constraints on ΔN_{eff} for relativistic extra species, translated to ξ_{DR} using Eq. (4.12) (see also [1, 47]). This choice does not significantly impact our results, as the constraints we derive on DR interacting with DM are well below this upper limit. Additionally, as explained in Sec. 4.3, we focus on the tight coupling limit, where the interaction intensity is set to a high enough value that our results are independent of its precise value. Concretely,

¹<https://cosmosis.readthedocs.io/>

Parameter	Description	Informative Prior
DES Y3 cluster lensing		
$\Delta_1 \ln M_{\text{WL}_0}(z)$	bias uncertainty as $f(z)$	$\mathcal{N}(0, 1)$
$\Delta_2 \ln M_{\text{WL}_0}(z)$	bias uncertainty as $f(z)$	$\mathcal{N}(0, 1)$
M_{WL_M}	mass trend of bias	$\mathcal{N}(1.029, 0.006)$
$\ln \sigma_{\ln \text{WL}_0}^2$	normalization of scatter	$\mathcal{N}(0, 1)$
$\sigma_{\ln \text{WL}_M}^2$	mass trend of scatter	$\mathcal{N}(-0.226, 0.040)$
HST cluster lensing		
$\ln M_{\text{WL}_0}$	amplitude of bias	$\mathcal{N}(0, 1)$
$\ln \sigma_{\ln \text{WL}_0}^2$	amplitude of scatter	$\mathcal{N}(0, 1)$
tSZE ζ -mass- z parameters		
$\ln \zeta_0$	amplitude	$\mathcal{U}(0.3, 2.0)$
ζ_M	mass trend	$\mathcal{U}(1.2, 2.0)$
ζ_z	redshift trend	$\mathcal{U}(-1.0, 1.5)$
$\sigma_{\ln \zeta}$	intrinsic scatter	$\mathcal{U}(0.05, 0.5)$
γ_{ECS}	depth of SPTpol ECS	$\mathcal{U}(0.9, 1.2)$
DES λ -mass- z parameters (used for $z < 1.1$)		
$\ln \lambda_0$	amplitude	$\mathcal{U}(3.0, 4.0)$
λ_M	mass trend	$\mathcal{U}(0.7, 1.5)$
λ_z	redshift trend	$\mathcal{U}(-1.0, 0.8)$
$\sigma_{\ln \lambda}$	intrinsic scatter	$\mathcal{U}(0.05, 0.5)$
WISE λ -mass- z parameters (used for $z > 1.1$)		
$\ln \lambda_0$	amplitude	$\mathcal{U}(3.0, 5.0)$
λ_M	mass trend	$\mathcal{U}(0.7, 1.5)$
λ_z	redshift trend	$\mathcal{U}(-4.0, 0.8)$
$\sigma_{\ln \lambda}$	intrinsic scatter	$\mathcal{U}(0.05, 0.5)$
Correlation coefficients		
$\rho_{\zeta, \text{WL}}$	tSZE ζ –WL	$\mathcal{U}(-0.5, 0.5)$
$\rho_{\zeta, \lambda}$	tSZE ζ – λ	$\mathcal{U}(-0.5, 0.5)$
$\rho_{\text{WL}, \lambda}$	WL– λ	$\mathcal{U}(-0.5, 0.5)$
Cosmology		
ξ_{DR}	DR temperature ratio	$\mathcal{U}(0.001, 0.5)$
Ω_{m}	matter density parameter	$\mathcal{U}(0.1, 0.5)$
$\Omega_{\text{b}} h^2$	baryon density parameter	$\mathcal{N}(0.02236, 0.00015)$
h	Hubble parameter	$\mathcal{N}(0.7, 0.05)$
$\ln 10^{10} A_s$	amplitude of $P(k)$	$\mathcal{U}(2.0, 5.0)$
n_s	scalar spectral index	$\mathcal{N}(0.9649, 0.0044)$

Table 7.1. Priors on parameters used in the analysis of the SPT cluster abundance, with DES Y3 and HST-39 lensing data. The parameters of the lensing models are primarily constrained by priors derived from simulations, as detailed in Sec. 7.2, and contribute to the empirical calibration of other observable–mass relations. Here, $\mathcal{U}$ denotes a uniform (flat) distribution, while $\mathcal{N}(a, \sigma)$ represents a Gaussian distribution with mean a and standard deviation σ . Reprinted from [Paper II](#).

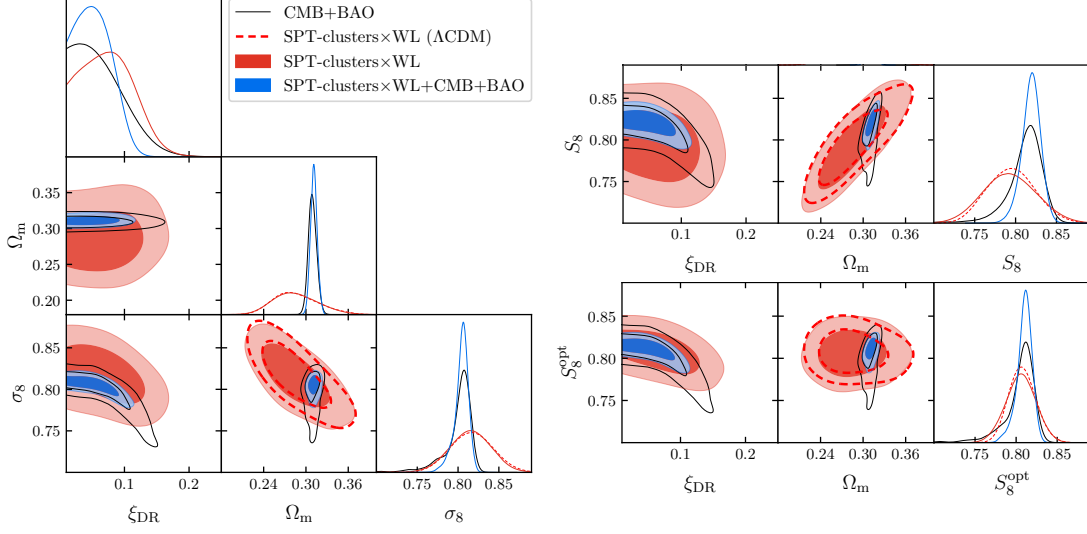


Figure 7.3. Posterior distributions from the analysis of the IDM–DR model using galaxy cluster abundance data, specifically SPT tSZE-selected clusters from the SPT-SZ, SPTpol ECS, and SPTpol 500d surveys, combined with weak lensing mass calibration data from DES Y3 and HST (shown in red). Additionally, we include results incorporating CMB (Planck 2018) and BAO (BOSS DR12) data (shown in blue) and compare them to constraints from CMB+BAO alone (black lines). For reference, we also show the analysis of cluster data within the Λ CDM model from [25] (dashed red lines). On the left, we display constraints on ξ_{DR} , Ω_m , and σ_8 at the 68% credible interval (CI) along with 95% upper limits. On the right, we present constraints on the commonly used parameter combination $S_8 = \sigma_8(\Omega_m/0.3)^{0.5}$ and $S_8^{\text{opt}} = \sigma_8(\Omega_m/0.3)^{0.2}$, which is the most relevant combination for galaxy cluster count analyses. From Paper II.

we use $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}] = 8$. To simplify the analysis and avoid parameter projection effects, we fix the IDM fraction to $f_{\text{IDM}} = 10\%$. The matter power spectrum and HMF are relatively insensitive to this precise value within the relevant mass regime of the cluster sample studied here; for further discussion, you can go back to Sec. 4.3. A complementary analysis, where both a_{dark} and f_{IDM} are treated as free parameters, is discussed in Sec 7.4.3.

7.4 Results

We present constraints on cosmological parameters within the IDM–DR model using galaxy cluster abundance data from the SPT sample, with masses informed by weak-lensing measurements from DES Y3 and HST. Furthermore, we examine the interaction between the IDM–DR model and massive neutrinos, and include the results of an analysis with a_{dark} and f_{IDM} as free parameters. The complete plot, including all observable–mass parameters used in the analysis, is presented in Appendix C.2.

	IDM-DR model			Λ CDM model
	CMB+BAO	SPT-clusters $\times$ WL	SPT-clusters $\times$ WL+CMB+BAO	SPT-clusters $\times$ WL
ξ_{DR}	< 0.130	< 0.166	< 0.098	...
ΔN_{eff}	< 0.010	< 0.027	< 0.003	...
Ω_{m}	0.312 ± 0.005	0.285 ± 0.032	0.311 ± 0.005	0.286 ± 0.032
σ_8	0.801 ± 0.019	0.815 ± 0.025	0.804 ± 0.009	0.817 ± 0.026
S_8	0.817 ± 0.021	0.793 ± 0.032	0.819 ± 0.011	0.795 ± 0.029
S_8^{opt}	0.809 ± 0.020	0.805 ± 0.018	0.810 ± 0.009	0.807 ± 0.016

Table 7.2. Parameter constraints for the IDM-DR and Λ CDM models (marginal mean at 68% CI, or upper limit at 95% CI) from galaxy cluster analysis using SPT cluster sample with DES Y3 and HST weak-lensing data. $S_8^{\text{opt}} = \sigma_8(\Omega_{\text{m}}/0.3)^{0.2}$ corresponds to the combination that cluster counts are most sensitive to. Results for Λ CDM are from [25]. Reprinted from Paper II.

7.4.1 Cosmological constraints in the IDM-DR model

We present our main results in Fig. 7.3 and summarize the constraints on model parameters in Table 7.2. The left side of Fig. 7.3 shows the marginalized posteriors for σ_8 , Ω_{m} , and ξ_{DR} . In the IDM-DR model, SPT-clusters $\times$ WL indicate a slight decrease in σ_8 with increasing ξ_{DR} . A similar trend was observed in previous analyses of CMB and galaxy clustering data [1, 47], which can be attributed to the suppression of the fluctuation amplitude due to the IDM-DR interaction, shifting to larger scales as ξ_{DR} increases. From the SPT-clusters $\times$ WL data, we derive an upper limit on the DR temperature ratio of $\xi_{\text{DR}} < 0.166$ at 95% credibility. Including clusters with CMB and BAO measurements significantly improves the constraint on the temperature ratio, yielding $\xi_{\text{DR}} < 0.098$ (95% credibility), compared to $\xi_{\text{DR}} < 0.130$ from CMB+BAO alone.

This upper bound on ξ_{DR} corresponds to an upper limit on ΔN_{eff} , as defined in Eq. (4.12). For an $SU(3)$ theory, this leads to $\Delta N_{\text{eff}} < 0.003$, which is about three times tighter than the constraint from CMB+BAO alone. It is important to say that this constraint applies to DR tightly coupled to IDM in the fluid limit. For comparison, this bound is approximately 100 times stronger than the constraint on free-streaming, non-interacting extra radiation from Planck CMB data combined with BAO ($\Delta N_{\text{eff}} < 0.28$ [30]). This stronger constraint is due to the significant impact of the DR density on the halo mass function (HMF) via its coupling to IDM, which is not the case for free-streaming DR.

As presented in Table 7.2 and can be seen in Fig. 7.3, the constraints on Ω_{m} and σ_8 from SPT-clusters $\times$ WL in the IDM-DR model are generally consistent with those obtained in the Λ CDM model [25]. The right side of Fig. 7.3 displays constraints on S_8 and S_8^{opt} . In light of the S_8 tension, our results are consistent, within the uncertainty range, with both early-time measurements (e.g., from Planck [26]) and late-time measurements (e.g., cosmic shear measurements from the joint analysis of KiDS and DES data [19]). In particular, when combining the cluster data with the CMB and BAO measurements, the constraining power on S_8 improves by a factor of two compared to CMB+BAO alone. The optimal choice of parameter combination is $S_8^{\text{opt}} = \sigma_8(\Omega_{\text{m}}/0.3)^{0.2}$. As shown in

the lower right of Fig. 7.3, the degeneracy between S_8^{opt} and Ω_m is fully broken. From Table 7.2, one can read that the uncertainty in S_8^{opt} is reduced by about 40% compared to S_8 when considering SPT-clusters×WL in the IDM–DR model. Additionally, the errors on both S_8^{opt} and S_8 are only degraded by approximately 10% within IDM–DR compared to Λ CDM. We also highlight that the posteriors for σ_8 , S_8^{opt} , and S_8 exhibit a shift toward lower values, as seen in Fig. 7.3, leaving room for addressing the S_8 tension within the IDM–DR framework.

There is also a slight preference for a non-zero value of ξ_{DR} . Although not statistically significant (within 2σ), we verified that this indication remains robust under several variations in the analysis, including the consideration of neutrino masses (see Sec. 7.4.2). This hint also persists when treating a_{dark} and f_{IDM} as free model parameters, as discussed in Sec. 7.4.3. Future cluster abundance measurements and complementary datasets will be crucial in further investigating this hint.

7.4.2 Effect of neutrinos

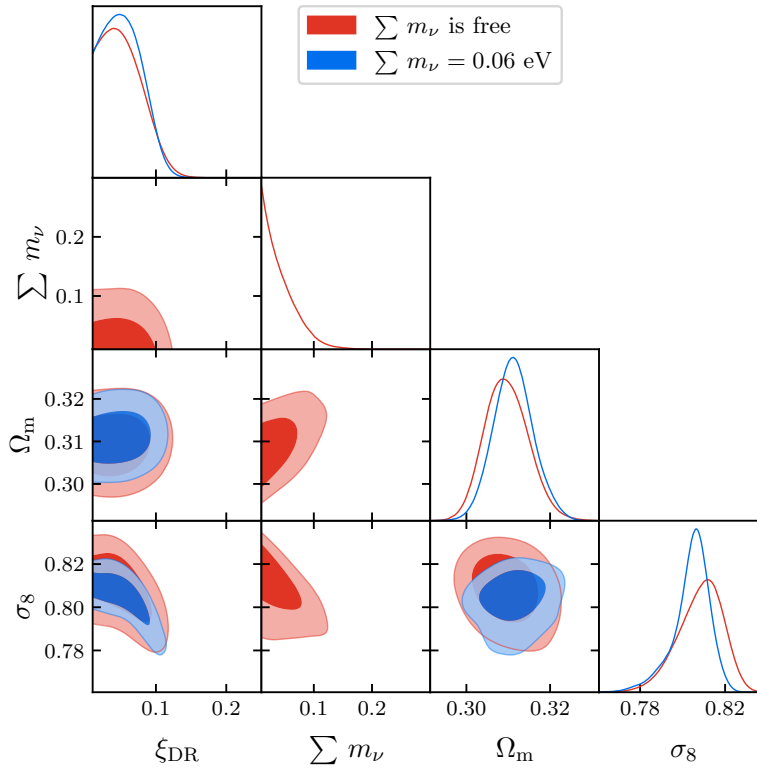


Figure 7.4. Comparison between the combined posteriors obtained when fixing the sum of neutrino masses to $\sum m_\nu = 0.06$ eV (blue), and when marginalizing over it (red). We show results for the combination SPT-clusters×WL+CMB+BAO. From Paper II.

Massive neutrinos also suppress structure formation, therefore, we investigate the interplay of massive neutrinos with the suppression from IDM–DR interactions. To determine if we can distinguish between the two effects, we perform an analysis including the sum of the neutrino masses $\sum m_\nu$ as a free parameter. We focus on the combination of clusters with CMB anisotropies, since cluster data alone have been found to provide only weak sensitivity to neutrino masses (see, for e.g. [23]).

A comparison between fixing the sum of neutrino masses to $\sum m_\nu = 0.06 \text{ eV}$ and allowing it to vary is shown in Fig. 7.4.

We compare a case where we vary the sum of neutrino masses, and a case where we fix its value to $\sum m_\nu = 0.06 \text{ eV}$. The later choice is the minimum value for neutrino masses in the normal hierarchy as indicated by neutrino flavor oscillation experiments, for a review see [219]. We present the results in Fig. 7.4. We find that the sum of neutrino masses does not significantly affect the constraints on ξ_{DR} , showing no degeneracy between $\sum m_\nu$ and ξ_{DR} . Additionally, we find that neutrino mass bounds are not significantly degraded within the IDM–DR framework compared to ΛCDM . Specifically, we obtain an upper bound of $\sum m_\nu < 0.096 \text{ eV}$ when using the data combination SPT-clusters $\times$ WL+CMB+BAO, which is more stringent than the limit of $\sum m_\nu < 0.13 \text{ eV}$ from CMB+BAO [26].

This result demonstrates that cluster abundance measurements combined with CMB anisotropies can distinguish between the effects of massive neutrinos and dark sector interactions within the IDM–DR model. This can be attributed to the following reasons: first, the suppression for each case occurs at different scales. For neutrinos, the effect is related to their free-streaming length, corresponding to $k \sim 10^{-2} [h \text{ Mpc}^{-1}]$ for viable neutrino masses. In contrast, for IDM–DR, the suppression occurs at different scales depending on the value of ξ_{DR} (as described in Sec. 4.3 and demonstrated in Fig. 4.3).

Additionally, the effects on the CMB differ. Neutrino masses alter the background evolution by contributing to the matter density at $z \lesssim 100$ while behaving as radiation at earlier times, leading to a distinct impact on the CMB anisotropy spectrum. In contrast, IDM–DR with viable values of ξ_{DR} modifies only the perturbations. Moreover, the power suppression due to IDM–DR occurs mostly before recombination, while for massive neutrinos, it is imprinted afterwards. This implies a different impact on the growth history and, consequently, on the redshift-dependence of the power suppression.

7.4.3 Varying a_{dark} and f_{IDM}

The primary results in this work are based on the tight coupling limit, defined as $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}] > 8$, with the fraction of interacting dark matter fixed to $f_{\text{IDM}} = 0.1$. This choice is motivated by the observation that the halo mass function (HMF) is primarily sensitive to the dark radiation (DR) temperature within the galaxy cluster mass range probed by SPT, as detailed in Sec. 4.3. For completeness, we also conduct an analysis where both a_{dark} and f_{IDM} are treated as free parameters. We apply flat priors on them; $\mathcal{U}(0.0, 10.0)$ for $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}]$ and $\mathcal{U}(0.001, 1.0)$ for f_{IDM} .

As shown in Fig. 7.5, cluster abundance data, along with CMB+BAO data, exhibit only weak sensitivity to a_{dark} and f_{IDM} . Importantly, the constraint on the DR temper-

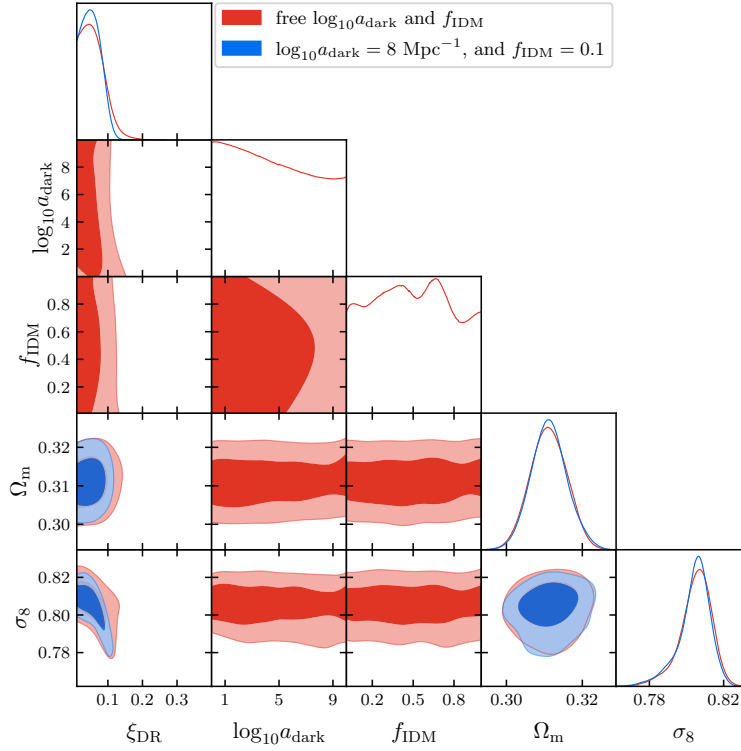


Figure 7.5. Similar to Fig. 7.3 while including a_{dark} and f_{IDM} as free parameters in the analysis (red). We compare it to the case discussed in the main text where both parameters are fixed (blue). We present only the case for data combination SPT-clusters×WL+CMB+BAO.. From [Paper II](#).

ature ξ_{DR} remains consistent with the results obtained when a_{dark} and f_{IDM} are fixed, as presented in Sec. 7.4.1. This demonstrates that the constraints on ξ_{DR} are robust against variations in the ETHOS parameters over a broad range. Interestingly, the slight preference for a non-zero ξ_{DR} observed in the main text persists even when a_{dark} and f_{IDM} are allowed to vary.

Additionally, we performed an analysis where $\log_{10} [a_{\text{dark}}/\text{Mpc}^{-1}] = 8$ was fixed (i.e., within the tight coupling limit) while treating f_{IDM} as a free parameter. The results were found to be very similar to those shown in Fig. 7.5.

Main Results

- We used the state-of-art cluster abundance analysis for an IDM–DR model using SPT cluster sample combined with DES Y3 and HST weak-lensing data.
- Cluster data is mostly sensitive to the ETHOS parameter; DR temperature ratio, and put an upper bound on it of $\xi_{\text{DR}} < 17\%$ at 95% limit.
- Adding CMB and BAO measurements from Planck 2018 and BOSS DR12, respectively, gives; $\xi_{\text{DR}} < 17\%$ at 95% limit, a 30% improvement over CMB+BAO alone.
- We found a preference for a nonzero dark radiation temperature ratio at 2σ .
- Slightly lower S_8 values are possible ($S_8 = 0.793 \pm 0.032$ (68% CI)) in the IDM–DR model compared to Λ CDM, which offers a potential solution to the tension.
- SPT galaxy clusters are better at constraining a different parameter combination $S_8^{\text{opt}} = \sigma_8(\Omega_{\text{m}}/0.3)^{0.2} = 0.805 \pm 0.018$ (68% CI).
- Varying the sum of neutrino masses does not significantly impact the constraints on ξ_{DR} , indicating no degeneracy between $\sum m_\nu$ and ξ_{DR} .
- Current cluster data is not able to constrain the other two ETHOS parameters; f_{IDM} and a_{dark} . Nevertheless, including them in the analysis does not change the conclusion concerning constraints on ξ_{DR} and S_8 .

8 Forecasting Constraints from Future Cluster Samples

In Chapter 7, we introduced the galaxy cluster abundance analysis from the SPT $\times$ DES (and HST) in the IDM-DR framework (based on Paper II). In this chapter, we introduce the work done in Paper III where we forecast the sensitivity of ongoing and future cluster surveys to an IDM-DR model. This forecast assesses the potential of future experiments to distinguish between an IDM-DR and a Λ CDM model. We generate mock tSZ-selected cluster catalogs for two upcoming surveys: SPT-3G [220, 221] and CMB-S4 [29]. For both cluster samples, the mass calibration is based on next-generation weak-lensing (ngWL) data of the quality anticipated from the ongoing Euclid mission or the forthcoming Rubin Observatory. We create mock data

we start by introducing the surveys in Sec. 8.1, then explain how to generate mock data in Sec. 8.2. The analysis method is very similar to the previous work, main quantities and their equations were described in the previous chapter in Sec. 7.2. Nevertheless, there are a few changes that were necessary due to a difference in the size of the samples. we discuss that in Sec. 8.3 with the details of what enters the pipeline. Finally, we show results in Sec. 8.4.

8.1 Survey descriptions

In this section, we outline the key features of the tSZE cluster surveys and the next-generation weak lensing (ngWL) surveys.

8.1.1 Galaxy Cluster Surveys (SPT-3G and CMB-S4)

We analyze two mm-wave surveys that, when combined with optical and near-infrared (NIR) follow-up observations, can generate extensive tSZE-selected cluster samples:

- **SPT-3G:** This mm-wave survey is conducted using the South Pole Telescope (SPT) equipped with a third-generation wide-field camera featuring over 15,000 polarization-sensitive bolometers operating at 95, 150, and 220 GHz [222]. Since 2018, the survey collected data across several thousand square degrees until the end of 2024. SPT-3G aims to achieve noise levels of a few μ K-arcmin in each frequency band, representing a tenfold improvement over the SPT-SZ 2500d survey and a two- to threefold enhancement over the SPT-pol 500d survey [223, 224]. For our analysis, we adopt a simplified model of the SPT-3G $\times$ ngWL survey (based on available information in 2023), covering $4,000 \text{ deg}^2$ divided into two regions:

a deep field spanning $1,500 \text{ deg}^2$ and a shallower field covering $2,500 \text{ deg}^2$. The tSZE-selected clusters used in this forecast span a redshift range of $z \in [0.25, 2]$.

- **CMB-S4:** This upcoming Stage-4 mm-wave survey will use multiple telescopes located at the South Pole and in Chile. Designed to map approximately 70% of the sky, CMB-S4 will employ over 500,000 polarization-sensitive bolometers to achieve noise levels below $\mu\text{K-arcmin}$ at central CMB frequencies, with coverage extending from 20 to 270 GHz [30]. The survey is expected to produce a tSZE-selected cluster sample more than ten times larger than the current largest catalog from ACT [30, 80]. For our forecast, we follow the assumptions outlined in a previous study [225], which estimated the detection of 75,000 tSZE-selected clusters within a wide survey area covering 50% of the sky and reaching redshifts beyond 2.

8.1.2 Next-Generation Weak Lensing (ngWL)

Ongoing and future surveys, such as the Euclid mission and the Legacy Survey of Space and Time (LSST) at the Vera C. Rubin Observatory, will deliver a new generation of weak-lensing (ngWL) data. With their extensive sky coverage and advanced instrumentation, these surveys will produce an order of magnitude more WL data than currently available, with significantly improved quality. For this work, we focus specifically on ngWL data from the Euclid mission. Operated by the European Space Agency, the Euclid Wide Survey will cover $15,000 \text{ deg}^2$ of the extragalactic sky over six years. Its imaging and spectroscopy surveys in visible and near-infrared bands are designed to enhance photometric redshift (photo- z) estimation, leading to more precise weak-lensing measurements [33]. Euclid will measure weak-lensing shear profiles of galaxies up to redshift 2, with an average galaxy number density of 30 arcmin^{-2} .

8.2 Generating mock data

To create a mock catalog for each survey, we first compute the matter power spectrum for a given cosmological model (ΛCDM or IDM-DR) using the Boltzmann solver CLASS [63, 64]. The matter power spectrum is then used to derive the halo mass function. We adopt the simulation-based Tinker HMF [69] in the mass range $M \in [10^{13}, 10^{16}] h^{-1} \text{ M}_{\odot}$.

We generate mock data for two benchmarks based on different cosmological models. The first is based on the IDM-DR model with parameters chosen to be compatible with Planck 2018 data, yielding a lower value of S_8 due to the interaction of IDM with DR, similar to those reported by weak-lensing shear measurements. Here, we compare with the recent joint analysis of DES Y3 and KiDS-1000 [19]. The second benchmark is based on a ΛCDM model, with input values chosen as the mean parameter posteriors from Planck 2018 temperature and polarization anisotropies without CMB lensing [26]. The cosmological parameters for each benchmark are listed in Table 8.1. In both cases, we consider a zero curvature geometry with three neutrino species: two massless and one massive neutrino with $m_{\nu} = 0.06 \text{ eV}$.

Param	Benchmark Models	
	$Mock_{\text{IDM-DR}}$	$Mock_{\Lambda\text{CDM}}$
ξ_{DR}	0.112	0.000
Ω_{m}	0.3068	0.3166
$\Omega_{\text{b}}h^2$	0.02550	0.02236
h	0.6800	0.6736
$\ln(10^{10}A_{\text{s}})$	3.030	3.045
n_{s}	0.9660	0.9649
τ_{reio}	0.0513	0.0544

Table 8.1. Cosmological parameters for the two benchmark models analyzed in this study. Both cases assume zero spatial curvature and a fixed sum of neutrino masses at $m_{\nu} = 0.06$ eV. Reprinted from [Paper III](#).

To generate the cluster and WL datasets, we start by using the HMF over the mass range $M \in [10^{13}, 10^{16}] h^{-1} \text{M}_{\odot}$ and the survey redshift range, along with the per-field survey solid angle Ω_{s} . We estimate the expected number of halos in the surveyed volume as a function of mass and redshift: $\langle N(M, z) \rangle = \frac{dN(\mathbf{p}, M, z)}{dM} \frac{dV(\mathbf{p}, z)}{dz} dM dz$. Here, the first term represents the HMF, which describes the comoving abundance of halos, and the second term is the differential volume, which depends on the survey solid angle. By generating a Poisson realization of this model, we iterate through mass and redshift, producing the corresponding number of halos within each mass-redshift cell.

Each observed halo of mass M_{200c} and redshift z is then assigned intrinsic observables ζ and λ , and their observable counterparts $\hat{\zeta}$ and $\hat{\lambda}$, as described in Sec. 7.2.1. We generate the observables using the eight ζ - and λ -mass- z scaling relation parameters, which we fix to the mean value in the Gaussian priors obtained in the mass calibration constraints, listed in Table 8.2. To produce the final cluster catalogs for each survey, we apply the survey selection cuts as follows

$$\begin{aligned}
\hat{\zeta} &> \hat{\zeta}_{\min}, \\
\hat{\lambda} &> \hat{\lambda}_{\min}(z), \\
z_{\min} &< z < z_{\max}.
\end{aligned} \tag{8.1}$$

For the redshift-dependent richness threshold $\hat{\lambda}_{\min}(z)$, we adopt the values obtained for the recent SPT×DES ΛCDM analysis [176].

For SPT-3G, the entire survey region is covered by ngWL surveys Euclid and Rubin, with a sky coverage of $\Omega_{\text{s}} = 4,000 \text{ deg}^2$ and a redshift range of $0.25 < z \lesssim 2$. We apply two significance thresholds: $\hat{\zeta}_{\min} = 5$ for the shallower field and $\hat{\zeta}_{\min} = 4.5$ for the deep field. The deep field is characterized by a depth parameter $\gamma_{\text{field}} = 3.5$, while the shallower field has $\gamma_{\text{field}} = 1.5$. Eventually, we obtain approximately 4,000 clusters for $Mock_{\text{IDM-DR}}$ and 6,400 clusters for $Mock_{\Lambda\text{CDM}}$.

The overlapping region between CMB-S4 and the ngWL survey Euclid covers approximately $\Omega_s = 10, 100 \text{ deg}^2$, with an even larger overlap for Rubin. We adopt the solid angle of the Euclid overlap and a redshift range of $0.1 < z \lesssim 2$. While the gains from using the Rubin overlap are marginal, we do not explore them further in this analysis. Although CMB-S4 achieves tSZE sensitivities capable of detecting clusters at redshifts beyond $z > 2$, we limit our analysis to clusters with $z \lesssim 2$. This is because cluster cosmology at $z > 2$ has not yet been extensively studied, and we focus instead on the better-understood regime of $z < 2$. For CMB-S4, we set the significance threshold to $\hat{\zeta}_{\min} = 5$ and the field depth parameter to $\gamma_{\text{field}} = 4.0$. With these specifications, the survey includes approximately 24,000 clusters for *Mock*_{IDM-DR} and 32,000 clusters for *Mock* _{Λ CDM}.

In addition to the tSZE observables, a mass estimate, M_{WL} , is assigned to each selected cluster halo based on Eqs. (7.7) and (7.8). Then, we provide each mock cluster with a mock shear profile consistent with the NFW model, which is used to fit the shear profiles during the analysis. The lensing source galaxies are distributed across ten tomographic redshift bins spanning the redshift range $0 - 2.2$. We assign profile measurement uncertainties based on shape noise, assuming an intrinsic shear variance of 0.3^2 and a lensing source galaxy surface density of 30 arcmin^{-2} . We compute the amplitude of the shear profiles for each tomographic bin using the appropriate Σ_{crit} factor, with the weighting factor set to zero for bins at redshifts lower than the cluster redshift. Cluster member contamination of the source galaxy sample is included, consistent with measurements derived from tSZE-selected clusters within the SPT $\times$ DES shear dataset. The priors on the weak lensing to halo mass relation are provided in Table 8.2 and discussed in more detail in Sec. 8.3.1. When generating the mock data, we adopt the central values of each parameter, applying this relation to mock clusters across all redshifts.

8.3 Analysis and pipeline

The Markov chain Monte Carlo (MCMC) analysis is conducted using **CosmoSIS** [215] with the Polychord sampler [226, 227]. We verify that the results are consistent across all posterior distributions by cross-checking with the MultiNest sampler [216–218].

In the previous work Paper II and recent analyses of cluster abundance informed by weak lensing [23, 176], the full log-likelihood, which includes both cluster abundance and weak lensing mass calibration, has been iterated simultaneously. This approach is optimal because the weak-lensing mass calibration depends on the distance–redshift relation, which is sensitive to cosmological parameters such as Ω_m . However, for the forecast analysis conducted here, where both cluster datasets and Planck 2018 anisotropy data are included, the likelihood can be partitioned, resulting in computational efficiency. In a related study [228], it was demonstrated that the dependence of mass calibration on cosmology for the cluster samples analyzed here is negligible, supporting the validity of this separation.

Consequently, we perform the mass calibration independently, as detailed in Sec. 8.3.1, and use the resulting parameter posteriors as priors for a stand-alone cluster abundance

likelihood analysis, as described in Sec. 8.3.2.

$M_{\text{WL}} - M_{200c}$	Priors	
$\ln M_{\text{WL}0}$	$\mathcal{N}(-0.05, 0.01)$	
$M_{\text{WL}M}$	$\mathcal{N}(1.029, 0.009)$	
$\ln \sigma_{\ln \text{WL}0}^2$	$\mathcal{N}(\ln(0.212)^2, (0.110)^2)$	
$\sigma_{\ln \text{WL}M}^2$	$\mathcal{N}(-0.226, 0.300)$	
$\zeta - M_{200c} - z$	Constraints	
	CMB-S4 $\times$ ngWL	SPT-3G $\times$ ngWL
$\ln \zeta_0$	$\mathcal{N}(0.960, 0.025)$	$\mathcal{N}(0.96, 0.031)$
ζ_M	$\mathcal{N}(1.500, 0.038)$	$\mathcal{N}(1.50, 0.041)$
ζ_z	$\mathcal{N}(0.500, 0.099)$	$\mathcal{N}(0.50, 0.17)$
$\sigma_{\ln \zeta}$	$\mathcal{N}(0.200, 0.017)$	$\mathcal{N}(0.200, 0.026)$
$\lambda - M_{200c} - z$	Constraints	
	CMB-S4 $\times$ ngWL	SPT-3G $\times$ ngWL
$\ln \lambda_0$	$\mathcal{N}(4.250, 0.016)$	$\mathcal{N}(4.25, 0.02)$
λ_M	$\mathcal{N}(1.000, 0.029)$	$\mathcal{N}(1.000, 0.031)$
λ_z	$\mathcal{N}(0.000, 0.070)$	$\mathcal{N}(0.00, 0.12)$
$\sigma_{\ln \lambda}$	$\mathcal{N}(0.200, 0.009)$	$\mathcal{N}(0.20, 0.02)$

Table 8.2. Parameters for the scaling relations, including priors and constraints. The priors on the weak lensing-to-halo mass relation are adopted from the SPT $\times$ DES analysis [176], with uncertainties reduced by a factor of two. Constraints on the parameters of the tSZE significance and richness observable–mass scaling relations are derived from the WL mass calibration. Here, $\mathcal{N}(a, \sigma)$ denotes a Gaussian distribution with mean a and standard deviation σ . Reprinted from Paper III.

8.3.1 WL mass calibration analysis

To perform the weak lensing (WL) mass calibration, the mock shear profiles are analyzed in the same manner as the observed DES-based shear profiles used in the recent SPT $\times$ DES analysis [176]. These profiles are corrected for contamination by cluster member galaxies and fitted with a projected NFW density profile. The fit excludes both the cluster infall region dominated by the two-halo term and the central $500h^{-1}\text{kpc}$, where the cluster profile is particularly sensitive to feedback effects, hydrodynamical modeling, and the choice of cluster center. This fitting process yields a WL mass estimate, M_{WL} , for each cluster. As previously discussed in Sec. 8.2, the M_{WL} measurement is a biased and noisy estimator of the underlying halo mass, M_{200c} . We account for this source of systematic and stochastic uncertainty by marginalizing over the uncertainties in the parameters of the weak lensing-to-halo mass relation described in Eqs. (7.7) and (7.8).

This method accounts for the mismatch between complex cluster morphologies and the

simple NFW shear profile used to estimate M_{WL} , as well as the impact of uncertainties in cluster mis-centering and contamination of the shear source galaxy sample by cluster member galaxies. Additionally, systematic uncertainties in the photometric redshifts of the source sample and in the shear measurements themselves are included. Finally, there are further systematics arising from the uncertainties in the subgrid physics used in the hydrodynamic simulations that constrain the $M_{\text{WL}} - M_{\text{halo}}$ relation.

For the WL calibration employed here, we adopt the systematic requirements for the source galaxy photometric redshifts and shear measurements from the ngWL surveys. These requirements are stringent compared to existing shear datasets from, e.g., DES, making these systematics subdominant in our mass calibration analysis. For the remaining systematic uncertainties associated with model mismatch, mis-centering, member galaxy contamination, and hydrodynamical subgrid physics, we find that uncertainties in the latter to be the most significant with approximately 2% [229]. Since it is unclear how much improvement to expect in these systematics before the actual samples are analyzed, we adopt a conservative approach with systematic uncertainties of about 1%, and an optimistic approach where the systematics are reduced by a factor of ten to approximately 0.2%. As discussed in the companion work [228], the ngWL datasets are primarily shape-noise dominated under both conservative and optimistic scenarios, making the exact level of systematic uncertainties relatively unimportant for our forecasts. Consequently, we adopt the conservative approach with a 1% uncertainty on $\ln M_{\text{WL}_0}$. The mass trend in the logarithmic bias and scatter parameters of the weak lensing-to-halo mass relation are taken from the recent SPT×DES analysis, with uncertainties reduced by a factor of two [176]. All parameter priors are listed in Table 8.2.

Given that conducting the WL mass calibration on a cluster-by-cluster basis would be time-consuming for the samples analyzed here, we use a subset of 1000 randomly selected clusters for mass calibration in each sample. The shape noise is adjusted to retain the expected information content from an analysis of each full sample (further details can be found in [228]). This analysis yields tight posterior distributions for the parameters of the observable–mass relations, as shown in Table 8.2 for SPT-3G×ngWL and CMB-S4×ngWL. These parameter constraints are then used as Gaussian priors in the cluster abundance analysis described in the following section. Note that to account for potential halo mass shape changes due to IDM and their impact on the WL mass calibration, separate structure formation simulations could be employed. For the forecasts presented here, we assume that the WL systematic uncertainties are similar for both IDM and CDM.

8.3.2 Cosmology analysis

We employ the likelihood for cluster abundance as described previously in Sec. 7.2.3, along with the likelihood for the CMB Planck 2018 TT, TE, EE data available in CosmoSIS. While similar analyses using Planck data for this model have already been conducted [1, 47], we include it here for comparison. Consequently, we perform a cosmological analysis for the following datasets

Parameter	Prior
ξ_{DR}	$\mathcal{U}(0.001, 0.5)$
Ω_{m}	$\mathcal{U}(0.2, 0.4)$
$\Omega_{\text{b}}h^2$	$\mathcal{U}(0.018, 0.027)(\times \mathcal{N}(c, 0.00015))$
h	$\mathcal{U}(0.6, 0.8)(\times \mathcal{N}(d, 0.006))$
$\ln(10^{10}A_{\text{s}})$	$\mathcal{U}(1.0, 4.0)$
n_{s}	$\mathcal{U}(0.94, 1.00)$
τ_{reio}	$\mathcal{U}(0.04, 0.08)$

Table 8.3. List of priors on parameter used in the analysis. The parameter τ_{reio} is considered only when adding CMB data, and Gaussian priors on $\Omega_{\text{b}}h^2$ and h are added when using cluster data alone, with mean c standing for the value of $\Omega_{\text{b}}h^2$ and d for the mean value of h as specified for either $\text{Mock}_{\text{IDM-DR}}$ or $\text{Mock}_{\Lambda\text{CDM}}$, respectively. The $\mathcal{U}$ stands for a uniform distribution. Reprinted from [Paper III](#).

- Planck 2018 (TT, TE, EE)
- SPT-3G (tSZE clusters) $\times$ ngWL (WL masses)
- CMB-S4 (tSZE clusters) $\times$ ngWL (WL masses)

Hereafter, we refer to these datasets as Planck, SPT-3G $\times$ ngWL, and CMB-S4 $\times$ ngWL, respectively. Additionally, we consider combining Planck with either SPT-3G $\times$ ngWL or CMB-S4 $\times$ ngWL to evaluate the enhanced constraining power provided by future cluster data.

We use the constraints on the observable–mass– z scaling relation parameters from the WL mass calibration as priors in the cosmology analysis, they are summarized in Table 8.2. We explore the usual parameters of the ΛCDM model and include the parameter ξ_{DR} for the IDM–DR model. We use flat priors on these parameters, as shown in Table 8.3. Since cluster counts are not sensitive to $\Omega_{\text{b}}h^2$ and provide only weak constraints on h , we apply Gaussian priors to these parameters when using cluster data alone, as listed in Table 8.3.

The prior on ξ_{DR} is derived from the constraint on extra (non-interacting) radiation, ΔN_{eff} , from Planck, which can be interpreted as an upper limit in the interacting case. As previously mentioned, we choose the tight coupling limit, fixing the interaction intensity to a sufficiently large value so that our results are independent of its precise value. Specifically, we use $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}] = 8$. For simplicity and to avoid parameter projection effects, we also fix the IDM fraction to $f_{\text{IDM}} = 10\%$. As discussed in Sec. 4.3, the matter power spectrum and halo mass function (HMF) are relatively insensitive to the exact value of f_{IDM} within the relevant regime for the cluster samples analyzed here. However, an analysis with a_{dark} and f_{IDM} as free parameters is presented in Sec. 8.4.4.

8.4 Results and discussion

We present the results of forecasting the sensitivity to an interacting dark sector using galaxy cluster samples identified via the tSZE by either SPT-3G or CMB-S4, in combination with weak-lensing mass calibration from a Euclid- or Rubin-like ngWL survey.

Our Bayesian analysis is performed within the IDM-DR and CDM frameworks for the two benchmark models specified in Table 8.1, with results detailed in Sec. 8.4.1 and Sec. 8.4.2, respectively. Additionally, in Sec. 7.4.2, we explore the interplay between these models and massive neutrinos. While our Bayesian inference runs include all previously mentioned parameters, we focus on presenting results for ξ_{DR} , Ω_m , and S_8 , with the complete results for all parameters available in Appendix C.3.

Cosmological probes constrain the parameter combination $\sigma_8(\Omega_m/0.3)^\alpha$, where α varies depending on the dataset and survey characteristics. For the surveys considered in this work, we find that CMB-S4 provides the strongest constraints for $\alpha = 0.18$, while SPT-3G is more sensitive for $\alpha = 0.2$. Nevertheless, to facilitate comparisons with previous shear analyzes and to examine the associated tension, we present our results in terms of $S_8 = \sigma_8(\Omega_m/0.3)^{0.5}$.

8.4.1 IDM-DR benchmark

We discuss here the IDM-DR benchmark model, chosen for its compatibility with Planck 2018 data and its ability to produce a low value of S_8 due to interactions within the dark sector. This scenario is pertinent if the indications of low S_8 values from weak lensing shear and current cluster measurements are confirmed. The results of the analysis using *Mock*_{IDM-DR} data are presented in Fig. 8.1 for the dataset combinations outlined in Sec 7.2. We summarize in Table 8.4 The parameter values at 68% credibility or upper limits at 95% credibility level (CL), along with their relative errors.

We note that all input values are recovered within 1σ for the results presented in this study, demonstrating self-consistency in theoretical description, mock data generation, and likelihood analysis. In particular, the temperature of dark radiation, parameterized by ξ_{DR} , is recovered and tightly constrained from galaxy cluster catalogs with a relative error of 14% for CMB-S4×ngWL and 42% for SPT-3G×ngWL. This indicates that cluster abundance can differentiate the IDM-DR benchmark model from Λ CDM, corresponding to the limit $\xi_{\text{DR}} \rightarrow 0$. Hence, both SPT-3G and CMB-S4 cluster counts will be able to test if dark sector interactions account for the S_8 tension when combined with ngWL mass measurements.

The capability of CMB-S4×ngWL in constraining ξ_{DR} is significantly higher than that of SPT-3G×ngWL. This difference can be attributed to the larger catalog size, where CMB-S4 can identify around 24,000 clusters compared to 4,000 clusters for SPT-3G. Furthermore, the extensive overlapping region of the CMB-S4 and ngWL surveys leads to improved mass calibration, reducing uncertainty in the observable–mass relations (see Table 8.2). Both cluster samples offer comparable constraining power on S_8 with a precision of about 1% for CMB-S4×ngWL and 2% for SPT-3G×ngWL (see Table 8.4). As illustrated in Fig. 8.1, combining cluster catalogs with Planck temperature and polariza-

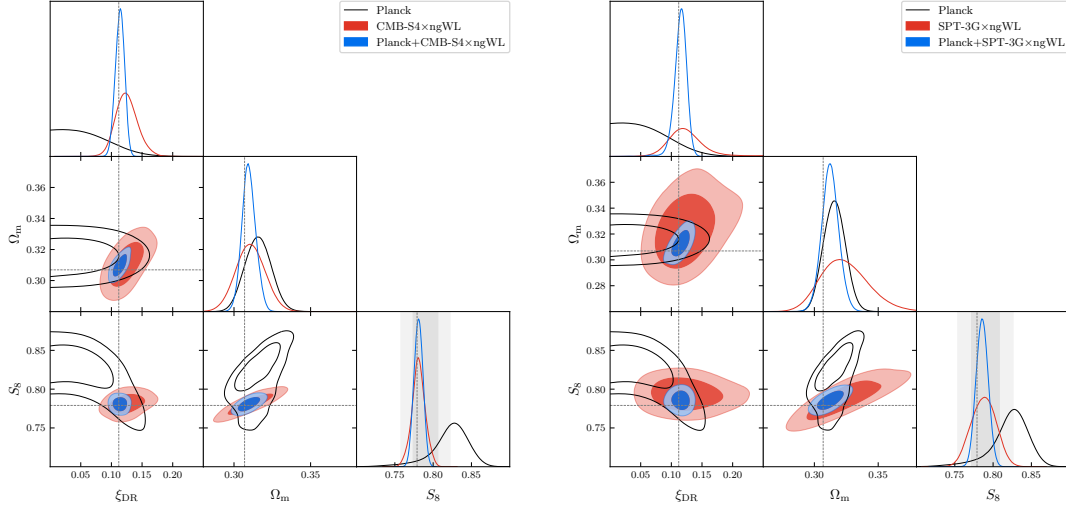


Figure 8.1. Posterior distributions for the IDM-DR model, assuming the $Mock_{IDM-DR}$ benchmark model. *Left:* we present the forecast for CMB-S4 $\times$ ngWL and *right* for SPT-3G $\times$ ngWL. We show the Red contours from cluster abundance only in red, and from a combination with Planck 2018 in blue. For comparison, the black lines represent current constraints from Planck 2018 only, the gray shaded area refers to the results from the joint analysis of DES-Y3 and KiDS-1000 [19], while the dotted lines denote the input values of the benchmark model used to generate the mock data. From Paper III.

tion anisotropy likelihoods significantly enhances the parameter constraints, particularly for the smaller SPT-3G cluster sample.

Importantly, current Planck 2018 data only provide an upper limit of $\xi_{DR} \leq 0.13$ (see Table 8.4), and are unable to distinguish the IDM-DR benchmark model from Λ CDM (see black contour lines in Fig. 8.1). The same limitation applies to combinations of Planck with BAO or BOSS galaxy clustering data [1, 47]. Therefore, future cluster abundance measurements will substantially increase the sensitivity to interacting dark sectors. We conclude that CMB-S4 cluster catalogs, together with ngWL mass calibration, will be capable of decisively testing IDM-DR as an explanation for the S_8 tension.

8.4.2 Λ CDM benchmark

Now, we consider the Λ CDM benchmark, with parameters compatible with Planck 2018 and a correspondingly high value of S_8 . This is relevant if the current hints of low S_8 from weak lensing shear and other measurements are not confirmed in future analyses. In this case, we can examine how future cluster abundance data will enable improved constraints on the IDM-DR model.

We present the results of Bayesian inference for $Mock_{\Lambda CDM}$ in Fig. 8.2, and the parameter values at 68% credibility or upper limits at 95% CL in Table 8.4.

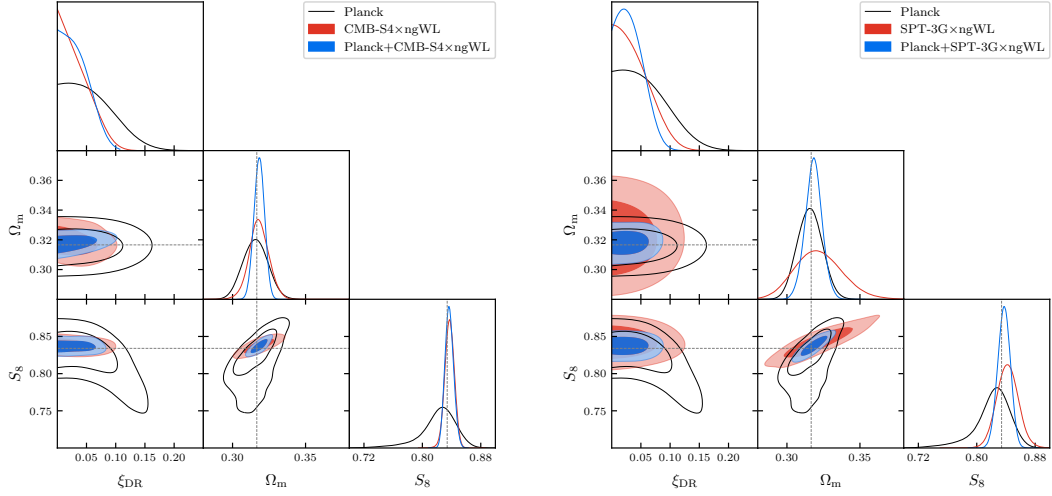


Figure 8.2. Same as Fig. 8.1, but for the $Mock_{\Lambda\text{CDM}}$ benchmark model, which corresponds to a ΛCDM model with Planck 2018 values. From Paper III.

In this scenario, we find that cluster abundances put tighter constraints on ξ_{DR} of 0.08 and 0.09 for CMB-S4×ngWL and SPT-3G×ngWL, respectively, at 95% CL. The degeneracy between S_8 and ξ_{DR} present in Planck 2018 CMB data is lifted, and the possibility of a lower value of S_8 due to dark sector interactions is excluded.

This implies that if ΛCDM provides an accurate description of our Universe, future cluster counts will improve the constraints on ξ_{DR} over Planck 2018 combined with BOSS, BAO, and galaxy clustering by about a factor of 1.5-2. This corresponds to an improvement in the upper limit on the abundance of dark radiation $\Delta N_{\text{fluid}} \propto \xi_{\text{DR}}^4$ by a factor of 5-10. Using Eq. (4.12), the upper bound on ξ_{DR} translates to an upper bound on ΔN_{fluid} of 0.0014 and 0.0023 for CMB-S4×ngWL and SPT-3G×ngWL, respectively, for $SU(3)$ theory.

Moreover, future cluster abundance data will enable a reduction in the uncertainty on S_8 by about a factor of two compared to CMB data from Planck, with a relative error of 0.7% – 1.7% for clusters and 2.4% for Planck. For the optimal parameter combination that is most precisely constrained by cluster counts, the relative error is even smaller, ranging from 0.5% to 0.8%.

This implies that future cluster catalogs will be capable of detecting even tiny contributions from interacting DR, making them a sensitive probe of dark sector interactions, regardless of whether the S_8 tension is confirmed or excluded with future measurements.

8.4.3 Effect of neutrinos

Massive neutrinos suppress structure formation below their free-streaming scale. This suppression differs from that within the IDM–DR model, occurring at a distinct length

	Planck		<i>Mock</i> _{ΛCDM}				<i>Mock</i> _{IDM-DR}			
			CMB-S4×ngWL		SPT-3G×ngWL		CMB-S4×ngWL		SPT-3G×ngWL	
	constraint	error	constraint	error	constraint	error	constraint	error	constraint	error
ξ_{DR}	< 0.13	-	< 0.08	-	< 0.087	-	$0.119^{+0.015}_{-0.018}$	13.8%	0.12 ± 0.05	42%
Ω_{m}	0.315 ± 0.008	2.5 %	$0.316^{+0.008}_{-0.004}$	1.8%	$0.321^{+0.012}_{-0.018}$	4.7%	$0.309^{+0.011}_{-0.008}$	3.1%	$0.312^{+0.024}_{-0.008}$	10%
S_8	0.827 ± 0.021	2.4%	$0.838^{+0.005}_{-0.007}$	0.7%	$0.837^{+0.019}_{-0.009}$	1.7%	$0.780^{+0.009}_{-0.007}$	1.03%	$0.791^{+0.011}_{-0.021}$	2%

Table 8.4. Summary of parameter values at 68% confidence level (CL) or 95% CL upper limits for the Planck 2018 TT-TE-EE dataset, as well as the forecasted CMB-S4 (tSZ clusters)×ngWL and SPT-3G (tSZ clusters)×ngWL datasets, assuming either the *Mock*_{ΛCDM} or *Mock*_{IDM-DR} benchmark models. From [Paper III](#).

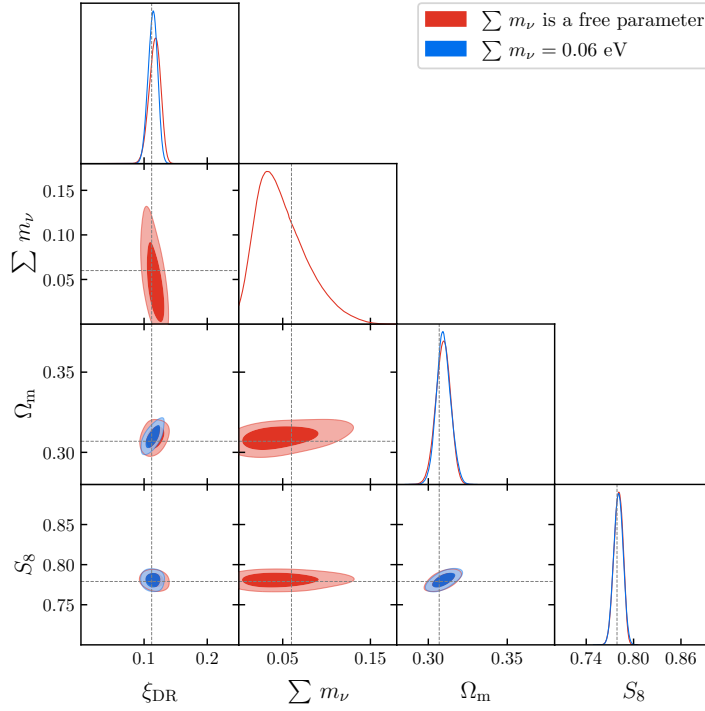


Figure 8.3. Comparison of constraints from CMB-S4 × ngWL+Planck for the *Mock*_{IDM-DR} benchmark model, with the sum of neutrino masses fixed at 0.06 eV (blue) and treated as a free parameter (red). The dotted lines indicate the *true* values of the benchmark model. From [Paper III](#).

scale. Moreover, neutrino masses influence the background evolution after recombination, unlike IDM–DR which has an influence at the perturbation level, more discussion is given in 7.4.2. Although the effects of massive neutrinos are distinct from those of the IDM–DR model, we perform an analysis including the sum of neutrino masses as a free parameter to assess whether it impacts the constraint on ξ_{DR} . As in the previous work, we only consider the combination of clusters with CMB measurements because clusters alone cannot provide constraints on neutrino masses.

The results of this analysis are shown in Fig. 8.3. We find that the effect of neutrinos is distinguishable from that of IDM–DR, and the input value of ξ_{DR} is well recovered with a slightly larger posterior. Fig. 8.3 also shows that the degeneracy between $\sum m_\nu$ and ξ_{DR} is very weak. The projected upper bound on $\sum m_\nu$ of 0.103 eV at 95% CL arises primarily from CMB data. This value is very similar to the one obtained in the previous work, where we used real data from SPT ($\sum m_\nu < 0.096$ eV); note that there we included BAO measurements as well.

8.4.4 Varying a_{dark} and f_{IDM}

As in the previous work, we present results when allowing the fraction of interacting dark matter, f_{IDM} , and the intensity of IDM–DR interaction, a_{dark} , to vary alongside other cosmological parameters. As previously discussed (see Fig. 4.3), the HMF is predominantly sensitive to ξ_{DR} and cannot constrain the other two ETHOS parameters.

Since the amount of dark radiation (DR) determines the suppression scale of the matter power spectrum and the IDM fraction governs the magnitude of this suppression (see Sec. 4.3), these two parameters are independent and exhibit no degeneracy. Fixing f_{IDM} and a_{dark} in the analysis is advantageous as it accelerates the analysis runs.

Nonetheless, we adopt a conservative approach to assess the impact of fixing f_{IDM} and a_{dark} . We introduce f_{IDM} and $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}]$ as free parameters in the analysis, applying flat priors of $\mathcal{U}(0.001, 1.0)$ and $\mathcal{U}(0.0, 10.0)$, respectively. As presented in Fig. 8.4, we recover the input value of ξ_{DR} , and observe no degeneracy with f_{IDM} or a_{dark} . The parameter f_{IDM} is unconstrained and spans the entire prior range, whereas clusters provide a lower bound on $\log_{10}[a_{\text{dark}}/\text{Mpc}^{-1}]$ of approximately 1.

In conclusion, while the weak-lensing mass calibration informed by cluster abundance is highly sensitive to the dark sector temperature parameter ξ_{DR} , it lacks sensitivity to small scales ($k \gtrsim 0.5 h \text{ Mpc}^{-1}$) to constrain the other two ETHOS parameters, f_{IDM} and a_{dark} . Additional datasets sensitive to smaller scales are required for this purpose.

8.4.5 Assessing the scale sensitivity and comparison to other probes

To explore potential deviations from the ΛCDM paradigm, it is useful to quantify the dominant range of length scales to which a given measurement is sensitive, as well as the level of precision. In this section, we address this question for galaxy cluster abundance (from current and future surveys), CMB+BAO, as well as weak-lensing shear data and galaxy clustering with their correlation. Fig. 8.5 presents an approximate evaluation of the scale sensitivity for SPT-clusters×WL, CMB+BAO, and their combination. We

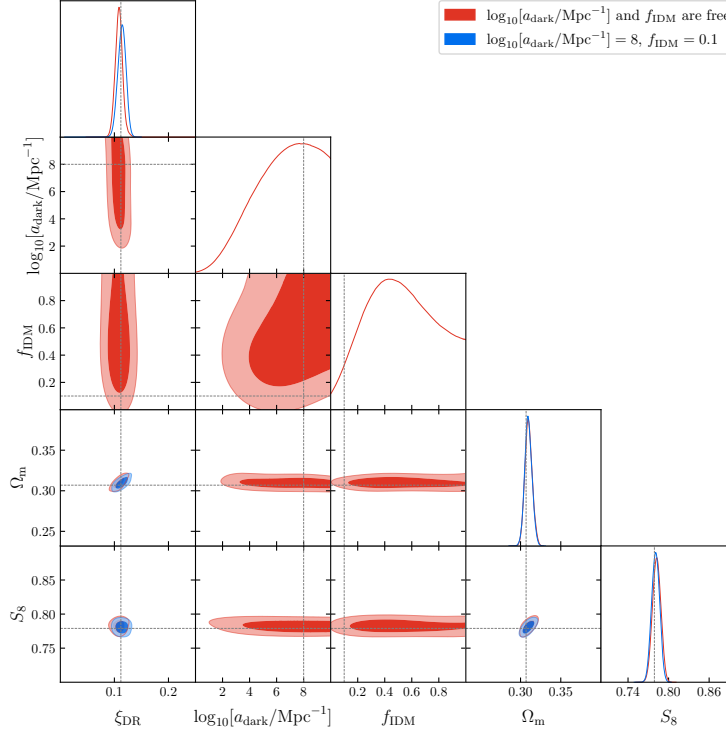


Figure 8.4. Posterior distributions of parameters from the analysis of Planck+CMB-S4+ngWL in the case *Mock*_{IDM-DR} benchmark model, marginalizing over $\ln_{10}[a_{\text{dark}}/\text{Mpc}^{-1}]$ and f_{IDM} (in red), and compared to the case of fixing them (in blue). From [Paper III](#).

determine the linear matter power spectrum for each set of model parameters contained in the converged samples of our IDM-DR framework analysis. We then show the variation of the power spectrum around its mean value within 1σ for each wavenumber, with $\Delta P(k) = (P(k)_{\text{mean}} \pm \text{stdev})/P(k)_{\text{mean}}$.

As anticipated, cluster abundance measurements are most sensitive to larger scales, specifically $k \in [0.02, 0.08] h/\text{Mpc}$. This implies that they are less impacted by non-linear cosmic evolution and small-scale effects like baryonic feedback. CMB+BAO data exhibit strong sensitivity to very large scales with small uncertainty. Moreover, the sensitivity of CMB+BAO improves when combined with galaxy cluster data (as shown in yellow in Fig. 8.5), especially around $k \sim 0.1 h/\text{Mpc}$, which is particularly relevant for probing the IDM-DR model. These datasets lose sensitivity at smaller scales, beyond $k \gtrsim 0.1 h/\text{Mpc}$. This explains why cluster and CMB+BAO measurements are able to constrain ξ_{DR} , which governs the scale at which the suppression of the matter power spectrum occurs. As shown in Fig. 1 of [3], larger values of ξ_{DR} shift this suppression to larger scales. This scale dependence also accounts for the relatively weak sensitivity of both cluster and CMB+BAO data to IDM-DR parameters such as a_{dark} and f_{IDM} . Constraining these parameters requires datasets more sensitive to smaller scales, such

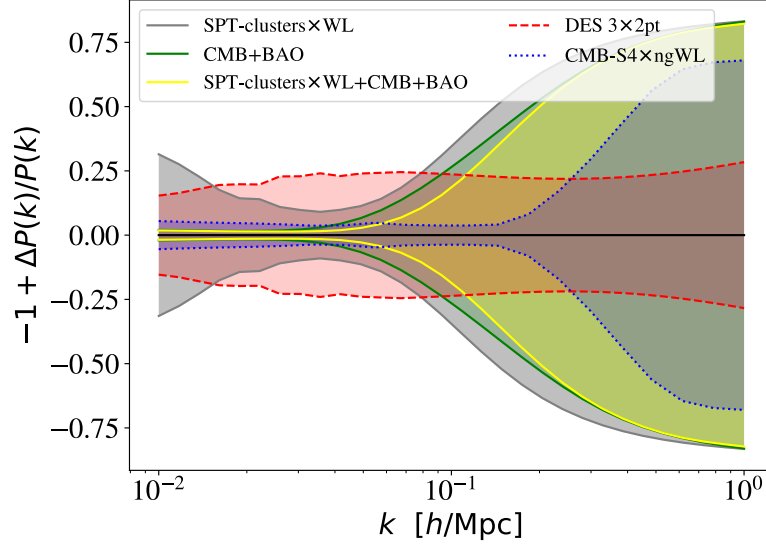


Figure 8.5. A comparison for the scale sensitivity of various data by showing the variance (1σ region) of the matter power spectrum as a function of wave number k , inferred from converged samples of model parameters. We define $\Delta P(k) = P(k) \pm \text{stdev}$, where $P(k)$ is the mean value of the matter power spectrum. We have here; SPT-clusters $\times$ WL, CMB+BAO and their combination, we also show corresponding results from weak lensing shear measurements (DES $3 \times 2\text{pt}$ results computed in the ΛCDM model from [18]) as well as the forecast for future cluster abundance measurements combined with next-generation weak-lensing data (CMB-S4 $\times$ ngWL). From Paper II.

as weak-lensing shear measurements, galaxy clustering, and their cross-correlations (see DES Y3 $3 \times 2\text{pt}$ results for ΛCDM in Fig. 8.5). A joint analysis of cluster abundance and cosmic shear therefore offers a promising approach for probing the entire IDM–DR parameter space, with clusters providing sensitivity to the DR temperature ξ_{DR} and $3 \times 2\text{pt}$ functions offering sensitivity to the fraction of interacting DM f_{IDM} and/or the interaction strength a_{dark} .

Additionally, it is worth noting that the sensitivity of cluster data will significantly improve with ongoing and future surveys, owing to their larger samples and improved systematics. This is illustrated in blue in Fig. 8.5 for clusters that could be detected by a CMB-S4 survey combined with next-generation weak-lensing data from Euclid or Rubin.

Main Results

- We performed a galaxy cluster abundance analysis by generating mockdata to ongoing and future cluster surveys (SPT-3G and CMB-S4), with mass information from next-generation weak-lensing (ngWL) data.
- Future cluster abundance data will differentiate between Λ CDM and IDM-DR models.
- These datasets will significantly improve sensitivity to dark sector interactions compared to current cluster data (SPT-clusters $\times$ WL, see previous chapter) and Planck 2018 CMB data.
- For an IDM-DR benchmark model; DR temperature, ξ_{DR} , can be recovered with 14% precision (CMB-S4 $\times$ ngWL) and 42% precision (SPT-3G $\times$ ngWL).
- When we include sum of neutrino masses, $\sum m_\nu$, in the analysis, the value of ξ_{DR} is recovered. This shows that the effect of neutrinos is distinct from IDM-DR model.
- For a Λ CDM benchmark model; future cluster data will improve the upper limit to $\xi_{\text{DR}} \sim 0.08$ (CMB-S4 $\times$ ngWL) and $\xi_{\text{DR}} \sim 0.09$ (SPT-3G $\times$ ngWL).
- These cluster samples will constrain structure formation at Mpc scales with a precision of $0.7 - 2\%$ in S_8 .
- Ongoing and future cluster data also will *not* be able to fully constrain f_{IDM} and a_{dark} .
- Combining with weak lensing shear and galaxy clustering data, with their correlation (3×2 pt functions), can add constraining power at smaller scales. This offers a possibility to constraint the other two ETHOS parameters and is crucial for testing if IDM-DR model resolves the S_8 tension.

9 Conclusions and Outlook

This thesis is aimed to contribute to our understanding of dark matter properties with addressing the S_8 cosmological tension. We go beyond the Λ CDM paradigm by exploring an interacting dark sector, where DM interacts with dark radiation. This interaction is similar to Compton scattering in ordinary matter between photons and electrons. Such a model has an impact on structure formation and can be tested using cosmological probes.

In [Paper I](#), we explored constraints from full shape galaxy clustering data (BOSS DR12) on an interacting dark sector. We considered all three particle physics models introduced in [Chapter 4](#), $n = 0, 2, 4$. We found that only the $n = 0$ case, referring to a non-Abelian $SU(N)$ theory, is relevant for addressing the S_8 tension and is richer phenomenologically. Specifically, the fiducial case where an IDM fraction of $f_{\text{IDM}} \sim 10\%$, and a dark radiation temperature parameter of $\xi_{\text{DR}} \sim 0.1$, can reduce the S_8 tension from 2.9σ (as seen in Λ CDM) to approximately 1σ . Another key result is regulating the divergence in the interaction rate by considering the Debye mass of DR and considering all channels' contributions (s , u , and t). The final result is a corrected interaction rate of IDM–DR with leading and next-to-leading order contributions. Eventually, we used this result to establish a mapping between cosmological constraints on the interaction rate a_{dark} and the particle physics parameter, α_d , which is known as the dark fine-structure constant. Resolving the S_8 tension requires $\alpha_d \gtrsim 10^{-8}$ for a dark matter mass of $m_\chi = 1000$ GeV and $\alpha_d \gtrsim 10^{-9}$ for $m_\chi = 1$ GeV. These values remain consistent with upper bounds from elliptical galaxy asphericity and permit self-interaction cross-sections relevant to small-scale problems.

The results from the first work lead to focus only on the $SU(N)$ theory in the next two works. In [Paper II](#), we performed the state-of-the-art cluster abundance analysis for an IDM–DR model using the SPT cluster sample, combined with DES Y3 and HST weak-lensing data. Cluster data are primarily sensitive to the ETHOS parameter ξ_{DR} , the dark radiation temperature ratio, and place an upper bound of $\xi_{\text{DR}} < 17\%$ at the 95% confidence level. Including CMB and BAO measurements from Planck 2018 and BOSS DR12, respectively, strengthens this constraint, yielding $\xi_{\text{DR}} < 17\%$ at 95% confidence, a 30% improvement over CMB+BAO alone. We found a preference for a nonzero dark radiation temperature ratio at the 2σ level. Additionally, the IDM–DR model allows for slightly lower S_8 values ($S_8 = 0.793 \pm 0.032$ at 68% confidence), offering a potential resolution to the S_8 tension compared to Λ CDM. The SPT galaxy clusters provide tighter constraints on an alternative parameter combination, yielding $S_8^{\text{opt}} = \sigma_8(\Omega_m/0.3)^{0.2} = 0.805 \pm 0.018$ at 68% confidence. However, current cluster data do not constrain the other two ETHOS parameters, f_{IDM} and a_{dark} . Nevertheless, we included these parameters in the analysis even though they do not alter the conclusions

regarding the constraints on ξ_{DR} and S_8 .

In [Paper III](#), we performed a galaxy cluster abundance analysis by generating mock data for ongoing and future cluster surveys (SPT-3G and CMB-S4), incorporating mass information from next-generation weak-lensing (ngWL) data. Future cluster abundance measurements will be able to distinguish between Λ CDM and IDM-DR models. These datasets will significantly enhance sensitivity to dark sector interactions compared to current cluster data (SPT-clusters $\times$ WL) and Planck 2018 CMB data. For an IDM-DR benchmark model, the dark radiation temperature ratio ξ_{DR} can be recovered with a precision of 14% using CMB-S4 $\times$ ngWL and 42% using SPT-3G $\times$ ngWL. For a Λ CDM benchmark model, future cluster data will improve the upper limit to $\xi_{\text{DR}} \sim 0.08$ (CMB-S4 $\times$ ngWL) and $\xi_{\text{DR}} \sim 0.09$ (SPT-3G $\times$ ngWL). These cluster samples will provide constraints on structure formation at Mpc scales with a precision of 0.7–2% in S_8 . However, ongoing and future cluster data alone will still be unable to fully constrain the ETHOS parameters f_{IDM} and a_{dark} . In both works, [Paper II](#) and [Paper III](#), we explored the interplay between IDM-DR model and massive neutrinos. The results showed no degeneracy between ξ_{DR} and the sum of masses for neutrinos $\sum m_\nu$, with a slight inflation of ξ_{DR} posterior without significant change to the constraint on it or to the overall conclusion concerning the S_8 tension.

Future research will benefit from combining these data with weak-lensing shear and galaxy clustering measurements, along with their cross-correlations (3×2 pt functions), to improve constraints at smaller scales. This approach could provide a pathway for refining constraints on the remaining ETHOS parameters and is essential for assessing whether the IDM-DR model can fully resolve the S_8 tension. Additionally, leveraging these datasets will enable the exploration of alternative models to further investigate the properties of dark matter and address other cosmological tensions. One such extension of the model studied here is the stepped or Wess-Zumino dark radiation (WSDR) model [[230](#), [231](#)], which offers a unified theoretical framework for simultaneously addressing both the S_8 and the H_0 tensions.

Another promising direction involves a more detailed investigation of the non-Abelian $SU(N)$ dark sector and its production mechanisms. Previous studies consistently indicate a dark sector temperature ratio of $\xi_{\text{DR}} \sim 10\%$ (in the tight coupling limit), implying that it was never in thermal equilibrium with the visible sector. Consequently, a standard freeze-out scenario is not viable. Freeze-out here refers to when DM particles stop annihilating efficiently to SM particles as their interaction rate drops below the cosmic expansion rate, setting their relic abundance. Instead, alternative mechanisms must be considered. One possibility is a minimal setup in which freeze-in occurs solely through gravitational interactions [[163](#)], naturally resulting in a significantly cooler dark sector, contingent on the reheating temperature. A similar scenario arises if interactions between the dark and visible sectors are mediated by higher-dimensional operators suppressed by a large energy scale, which preserves the structure formation phenomenology discussed here. In this case, the initial DR temperature and IDM abundance could be set via UV freeze-in [[165](#)], followed by dark sector thermalization and freeze-out [[166](#)].

Appendices

A Conventions and Astrophysical Constants

Conventions

Throughout this thesis, we take the metric signature $(-, + + +)$.

We use the following convention for the Fourier Transform

$$f(\mathbf{x}) = \int \frac{d^3 \mathbf{x}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}} \tilde{f}(\mathbf{k}), \quad \tilde{f}(\mathbf{k}) = \int d^3 \mathbf{k} e^{-i\mathbf{k}\mathbf{x}} f(\mathbf{x}) \quad (\text{A.1})$$

The number density of a fluid is defined as

$$n(x, t) \equiv g_s \int_{-\infty}^{\infty} \frac{d^3 p}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t), \quad (\text{A.2})$$

and its velocity is given by

$$v(x, t) \equiv g_s \frac{1}{n(x, t)} \int_{-\infty}^{\infty} \frac{d^3 p}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) \frac{\mathbf{p}}{E}, \quad (\text{A.3})$$

and finally, the fluid's density is defined as

$$\rho(\mathbf{x}, t) = g_s \int \frac{d^3 p}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t) E(p) \quad (\text{A.4})$$

Astrophysical constants

Parsec: $1\text{pc} = 3.0856 \times 10^{18} \text{ cm}$

Solar mass: $M_{\odot} = 1.989 \times 10^3 \text{ kg} = 1.116 \times 10^{57} \text{ GeV}/c^2$

Redshift at equality: $z_{\text{eq}} = -1 + \frac{\Omega_{\text{m}} h^2}{\Omega_{\text{r}} h^2} \approx 3400$

Scale factor at equality: $a_{\text{eq}} = 4.15 \times 10^{-5} (\Omega_{\text{m}} h^2)^{-1} \approx 2.94 \times 10^{-4}$

Wavenumber at equality: $k_{\text{eq}} = 0.073 \Omega_{\text{m}} h^2 \text{Mpc}^{-1} \approx 0.01 h/\text{Mpc}$

B Boltzmann Equations for an Interacting Dark Sector

This appendix is dedicated to the details of computing Boltzmann equations for IDM and DR at linear order perturbations. The procedure is similar to previous Boltzmann equations introduced for Λ CDM in Sec. 2.2.2. Yet, there are different conventions and definitions. We checked that the final results lead to the consistent equations with the standard approach (for e.g. in Modern Cosmology book [49]). This section follows closely the detailed appendix in [13] with more details and extra steps in the calculation for clarity.

B.1 Generalities and Setup

We start by considering an interacting dark matter component (IDM, denoted χ) described by a distribution function $f_\chi(\mathbf{x}, \mathbf{P}, \tau)$, and a DR component described by a distribution function $f_{\text{DR}}(\mathbf{x}, \mathbf{P}, \tau)$. Here $\mathbf{P}$ denotes the canonical conjugate variable to $\mathbf{x}$, and τ is the conformal time as defined in Eq. (2.7). We consider having a $2 \leftrightarrow 2$ interaction between IDM and DR, $\tilde{\gamma}\chi \leftrightarrow \tilde{\gamma}\chi$, and a self-interaction for DR, $\tilde{\gamma}\tilde{\gamma} \leftrightarrow \tilde{\gamma}\tilde{\gamma}$. The two coupled Boltzmann equations describing the evolution of IDM and DR are

$$\frac{df_\chi}{d\lambda} = C_{\chi\tilde{\gamma}\leftrightarrow\chi\tilde{\gamma}}[f_\chi, f_{\text{DR}}], \quad \frac{df_{\text{DR}}}{d\lambda} = C_{\chi\tilde{\gamma}\leftrightarrow\chi\tilde{\gamma}}[f_\chi, f_{\text{DR}}] + C_{\tilde{\gamma}\tilde{\gamma}\leftrightarrow\tilde{\gamma}\tilde{\gamma}}[f_{\text{DR}}], \quad (\text{B.1})$$

where λ is an affine parameter that describes the trajectory of the observer, $C_{\dots}$ refers to collision terms. The space-time metric is defined in a conformal Newtonian gauge

$$ds^2 = a^2(\tau)[-(1 + 2\psi)d\tau^2 + (1 - 2\phi)d\mathbf{x}^2]. \quad (\text{B.2})$$

Note that the convention here is different than the one introduced previously in Eq. (2.14). We have here a factor of 2 in front of the gravitational potentials ψ and ϕ , and a different sign convention.

we want to write the equations in Eq. (B.1) in terms of the conformal time. We start by introducing the four-momentum $P^\mu \equiv \frac{dx^\mu}{d\lambda}$, where $x^\mu = (\tau, \mathbf{x})$ is the trajectory of the observer. Using Eq. (B.2), we find

$$\frac{d}{d\lambda} = \frac{d\tau}{d\lambda} \frac{d}{d\tau} = P^0 \frac{d}{d\tau} = \frac{E(1 - \psi)}{a} \frac{d}{d\tau}, \quad (\text{B.3})$$

by using the dispersion relation $g_{\mu\nu}P^\mu P^\nu = -m^2$ and defining the energy as $E =$

$\sqrt{p^2 + m^2}$, where $p = |\mathbf{p}|$, and $p^2 = g_{ij}P^iP^j$.

For IDM, we get a similar equation as CDM, except that it has a collision term

$$\frac{\partial f_\chi}{\partial \tau} + \frac{p}{E} \hat{p}^i \frac{\partial f_\chi}{\partial x^i} + p \frac{\partial f_\chi}{\partial p} \left[-\mathcal{H} + \frac{\partial \phi}{\partial \tau} - \frac{E}{p} \hat{p}^i \frac{\partial \psi}{\partial x^i} \right] = \frac{a}{E} (1 + \psi) C_{\chi\bar{\chi} \leftrightarrow \chi\bar{\chi}}[\mathbf{p}]. \quad (\text{B.4})$$

For DR, we introduce a comoving momentum $q \equiv ap$ and comoving energy $\epsilon \equiv aE$. With this, the Boltzmann equation for DR becomes

$$\frac{\partial f_{\text{DR}}}{\partial \tau} + \frac{q}{\epsilon} \hat{q}^i \frac{\partial f_{\text{DR}}}{\partial x^i} + q \frac{\partial f_{\text{DR}}}{\partial q} \left[\frac{\partial \phi}{\partial \tau} - \frac{\epsilon}{q} \hat{q}^i \frac{\partial \psi}{\partial x^i} \right] = \frac{a^2}{\epsilon} (1 + \psi) \left(C_{\chi\bar{\chi} \leftrightarrow \chi\bar{\chi}}\left[\frac{\mathbf{q}}{a}\right] + C_{\bar{\chi}\bar{\chi} \leftrightarrow \bar{\chi}\bar{\chi}}\left[\frac{\mathbf{q}}{a}\right] \right). \quad (\text{B.5})$$

We will now further simplify these equations by introducing distributions functions for DR and IDM that are nearly isotropic and homogeneous.

• Dark Radiation

We parameterize the deviation from perfect equilibrium for a DR distribution function as follows

$$f_{\text{DR}}(\mathbf{x}, \mathbf{q}, \tau) = f_{\text{DR}}^{(0)}[1 + \Theta_{\text{DR}}(\mathbf{x}, \mathbf{q}, \tau)], \quad (\text{B.6})$$

where $f_{\text{DR}}^{(0)}$ is the homogeneous and isotropic part of DR distribution function, which is either a Fermi-Dirac (for fermions) or Bose-Einstein (for bosons). $\Theta_{\text{DR}}(\mathbf{x}, \mathbf{q}, \tau)$ represents a thermal perturbation. Note that Θ_{DR} here is defined differently from the case of photons as given in Eq. (2.17).

The first-order Boltzmann equation in Fourier space is

$$f_{\text{DR}} \left[\frac{\partial \Theta_{\text{DR}}}{\partial \tau} + i \frac{q}{\epsilon} k \mu \Theta_{\text{DR}} \right] + q \frac{\partial f_{\text{DR}}^{(0)}}{\partial q} \left[\frac{\partial \phi}{\partial \tau} - i \frac{\epsilon}{q} k \mu \psi \right] = \frac{a^2}{\epsilon} C^{(1)} \left[\frac{\mathbf{q}}{a} \right], \quad (\text{B.7})$$

with $\mu \equiv \hat{q} \cdot \hat{k}$, and the unitary vector $\hat{k} = \mathbf{k}/k$. $C^{(1)}$ contains the first order interaction term for both DR-IDM and DR-DR interactions.

Considering only scalar fluctuations allows us to expand the μ -dependence of Θ in Legendre Polynomials as follows

$$\Theta_{\text{DR}}(k, \hat{q}, q, \tau) = \sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(k, q, \tau) P_l(\mu). \quad (\text{B.8})$$

Substituting this expression in Eq. (B.7) and integrating both sides with $\frac{1}{2(-i)^l} \int_{-1}^1 d\mu P_l(\mu)$ gives

$$\begin{aligned} f_{\text{DR}}^{(0)} \left[\frac{\partial F_l}{\partial \tau} + k \frac{q}{\epsilon} \left(\frac{l+1}{2l+1} F_{l+1} - \frac{l}{2l+1} F_{l-1} \right) \right] + q \frac{\partial f_{\text{DR}}^{(0)}}{\partial q} \left[\frac{\partial \phi}{\partial \tau} \delta_{l0} + \frac{k}{3} \frac{\epsilon}{q} \psi \delta_{l1} \right] \\ = \frac{a^2}{\epsilon} \frac{1}{2(-i)^l} \int_{-1}^1 d\mu P_l(\mu) C^{(1)} \left[\frac{\mathbf{q}}{a} \right]. \end{aligned} \quad (\text{B.9})$$

We make use of the recurrence relation for Legendre Polynomials $P_l(\mu)$

$$(l+1)\mathcal{P}_{l+1}(\mu) = (2l+1)\mu\mathcal{P}_l(\mu) - l\mathcal{P}_{l-1}(\mu), \quad (\text{B.10})$$

and the orthogonality relation

$$\int_{-1}^1 d\mu \mathcal{P}_l(\mu) \mathcal{P}_{l'}(\mu) = \delta_{ll'} \frac{2}{2l+1}. \quad (\text{B.11})$$

Now the result can be further simplified by taking $\epsilon = q$ for massless particles, and defining the temperature fluctuation as

$$\nu_l \equiv -4 \frac{F_l}{\frac{\partial \ln f_{\text{DR}}^{(0)}}{\partial \ln q}}. \quad (\text{B.12})$$

This leads to

$$\begin{aligned} \frac{\partial \nu_l}{\partial \tau} + k \left(\frac{l+1}{2l+1} \nu_{l+1} - \frac{l}{2l+1} \nu_{l-1} \right) - 4 \left[\frac{\partial \phi}{\partial \tau} \delta_{l0} + \frac{k}{3} \psi \delta_{l1} \right] \\ = -\frac{a^2}{q} \frac{2}{(-i)^l} \frac{1}{\frac{\partial f_{\text{DR}}^{(0)}}{\partial \ln q}} \int_{-1}^1 d\mu P_l(\mu) C^{(1)} \left[\frac{\mathbf{q}}{a} \right]. \end{aligned} \quad (\text{B.13})$$

• Non-relativistic Interacting Dark Matter

Now we want to get the equations for IDM density and velocity fluctuations, in order to do that we take the moments of Eq. (B.4), keeping only the leading order terms in the non-relativistic limit $p/E \sim p/m_\chi \ll 1$. The first moment is

$$\begin{aligned} \int \frac{d^3 p}{(2\pi)^3} \left[\frac{\partial f_\chi}{\partial \tau} + \frac{p}{E} \hat{p}^i \frac{\partial f_\chi}{\partial x^i} + p \frac{\partial f_\chi}{\partial p} \left[-\mathcal{H} + \frac{\partial \phi}{\partial \tau} - \frac{E}{p} \hat{p}^i \frac{\partial \psi}{\partial x^i} \right] \right] \\ = \int \frac{d^3 p}{(2\pi)^3} \left[\frac{a}{E} (1 + \psi) C_{\chi \tilde{\gamma} \leftrightarrow \chi \tilde{\gamma}}[\mathbf{p}] \right]. \end{aligned} \quad (\text{B.14})$$

We use the definitions of IDM bulk velocity and total number density

$$\mathbf{v}_\chi \equiv \frac{\eta_\chi}{n_\chi} \int \frac{d^3 p}{(2\pi)^3} f_\chi(\mathbf{p}) \frac{p \hat{\mathbf{p}}}{E}, \quad n_\chi \equiv \eta_\chi \int \frac{d^3 p}{(2\pi)^3} f_\chi(\mathbf{p}), \quad (\text{B.15})$$

to obtain

$$\frac{\partial n_\chi}{\partial \tau} + 3\mathcal{H}n_\chi + \frac{\partial(n_\chi v_\chi^i)}{\partial x^i} - 3 \frac{\partial \phi}{\partial \tau} n_\chi = 0. \quad (\text{B.16})$$

We expand the number density of IDM as

$$n_\chi(\mathbf{x}, \tau) \equiv n_\chi^{(0)}(\tau) [1 + \delta_\chi(\mathbf{x}, \tau)]. \quad (\text{B.17})$$

Keeping only the first order pieces and performing a Fourier transform yields the equation

$$\dot{\delta}_\chi + \theta_\chi - 3\dot{\phi} = 0, \quad (\text{B.18})$$

with $\theta_\chi \equiv i\mathbf{k} \cdot \mathbf{v}_\chi$ is the divergence of the IDM velocity. The collision term is zero here because the elastic scattering alone does not change the number density of DM. This equation is the same as for CDM.

To get an equation for IDM bulk velocity, we compute the first moment. Multiplying Eq. (B.4) by $\int \frac{d^3p}{(2\pi)^3} \frac{p\hat{\mathbf{p}}}{E}$

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{\mathbf{p}}}{E} \left[\frac{\partial f_\chi}{\partial \tau} + \frac{p}{E} \hat{p}^i \frac{\partial f_\chi}{\partial x^i} + p \frac{\partial f_\chi}{\partial p} \left[-\mathcal{H} + \frac{\partial \phi}{\partial \tau} - \frac{E}{p} \hat{p}^i \frac{\partial \psi}{\partial x^i} \right] \right] \\ = \int \frac{d^3p}{(2\pi)^3} \frac{p\hat{\mathbf{p}}}{E} \left[\frac{a}{E} (1 + \psi) C_{\chi\tilde{\gamma} \leftrightarrow \chi\tilde{\gamma}}[\mathbf{p}] \right]. \end{aligned} \quad (\text{B.19})$$

The LHS of the equation at the first order gives

$$\frac{\partial(n_\chi^{(0)} \mathbf{v}_\chi)}{\partial \tau} + \nabla \left[\frac{\delta p_\chi}{m_\chi} \right] + 4\mathcal{H}n_\chi^{(0)} \mathbf{v}_\chi + n_\chi^{(0)} \nabla \psi, \quad (\text{B.20})$$

where δp_χ stands for the dark matter pressure perturbation. This term is usually negligible in the case of CDM, because non-relativistic particles have a negligible pressure compared to their energy density. For the case of IDM it is important to keep it, because due to the interaction with DR (which is relativistic) IDM acquires momentum. Assuming that it is adiabatic, means it can be written as $\delta p_\chi = c_\chi^2 n_\chi^{(0)} m_\chi \delta_\chi$, where c_χ is the IDM sound speed. Now, multiplying the previous equation with Nabla and using that $\partial(a^3 n_\chi^{(0)})/\partial \tau = 0$, it becomes

$$\frac{\partial(\nabla \cdot \mathbf{v}_\chi)}{\partial \tau} + c_\chi^2 \nabla^2 \delta_\chi + \mathcal{H} \nabla \cdot \mathbf{v}_\chi + \nabla^2 \psi. \quad (\text{B.21})$$

In Fourier space the equation for the IDM bulk velocity with both sides becomes

$$\dot{\theta}_\chi - c_\chi^2 k^2 \delta_\chi + \mathcal{H} \theta_\chi - k^2 \psi = \frac{a(1 + \psi) \eta_\chi}{n_\chi^{(0)}} \int \frac{d^3p}{(2\pi)^3} \frac{p(i\mathbf{k} \cdot \hat{\mathbf{p}})}{E^2} C_{\chi\tilde{\gamma} \leftrightarrow \chi\tilde{\gamma}}[\mathbf{p}]. \quad (\text{B.22})$$

The IDM speed can be computed via

$$c_\chi^2 = \frac{\dot{P}_\chi}{\dot{\rho}_\chi} = \frac{\dot{n}_\chi^{(0)} T_\chi + n_\chi^{(0)} \dot{T}_\chi}{m_\chi \dot{n}_\chi^{(0)}} = \frac{3\mathcal{H}T_\chi - \dot{T}_\chi}{3\mathcal{H}m_\chi} = \frac{T_\chi}{m_\chi} \left(1 - \frac{\dot{T}_\chi}{3\mathcal{H}T_\chi} \right). \quad (\text{B.23})$$

B.2 Structure of the IDM-DR collision term

For the elastic scattering process $\tilde{\gamma}(\mathbf{p}_1) + \chi(\mathbf{p}_2) \leftrightarrow \tilde{\gamma}(\mathbf{p}_3) + \chi(\mathbf{p}_4)$ the general form of the collision term is

$$C_{\chi\tilde{\gamma}\leftrightarrow\chi\tilde{\gamma}}[\mathbf{p}_1] = \frac{1}{2}\eta_\chi \int d\Pi_2 d\Pi_3 d\Pi_4 \left(\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum_{\text{states}} |\mathcal{M}|^2 \right) (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4) F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4), \quad (\text{B.24})$$

$|\mathcal{M}|^2$ is the squared amplitude of the scattering, η_{DR} is the number of degrees of freedom for DR. P_i represents the i^{th} four-momentum, whereas $\mathbf{p}_i$ represents the three-momentum ($p_i = |\mathbf{p}_i|$). The Lorentz-invariant phase-space measure is shortly written as

$$d\Pi_i = \frac{d^3 p_i}{(2\pi)^3 2E_i}, \quad (\text{B.25})$$

and

$$F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) \approx f_\chi(\mathbf{p}_4) f_{\text{DR}}(\mathbf{p}_3) (1 \pm f_{\text{DR}}(\mathbf{p}_1)) - f_\chi(\mathbf{p}_2) f_{\text{DR}}(\mathbf{p}_1) (1 \pm f_{\text{DR}}(\mathbf{p}_3)), \quad (\text{B.26})$$

where the sign (+) is for bosons and the sign (−) is for fermions. Note that they decided to keep here the Bose enhancement/Pauli blocking of DR for completeness, but we will see in what comes that it actually vanishes at first order perturbation. In some standard books (for e.g., [49]) these terms are neglected from the beginning of the computation.

The zeroth-order distribution function of DR can be approximated to be Fermi-Dirac (+) or Bose-Einstein (−)

$$f_{\text{DR}}^{(0)}(\mathbf{p}) = \frac{1}{e^{(p/T_{\text{DR}})} \pm 1}. \quad (\text{B.27})$$

We can now rewrite Eq. (B.26) by taking $f_{\text{DR}}^{(0)}(\mathbf{p}_3)$ and $f_{\text{DR}}^{(0)}(\mathbf{p}_1)$ as prefactors

$$F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = f_{\text{DR}}^{(0)}(\mathbf{p}_1) f_{\text{DR}}^{(0)}(\mathbf{p}_3) \left[f_\chi(\mathbf{p}_4) \left(e^{(p_1/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_3) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_1) \right) - f_\chi(\mathbf{p}_2) \left(e^{(p_3/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_1) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_3) \right) \right] \quad (\text{B.28})$$

The delta function can be split into a space part and a time part as

$$\delta^4(...) = \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(E_1 + E_2 - E_3 - E_4) \quad (\text{B.29})$$

Next, after performing the integration over $\mathbf{p}_4$, we can further simplify the time component of the delta function by using $E_1 \approx p_1$, $E_3 \approx p_3$, $E_2 \approx m_\chi + \frac{p_2^2}{2m_\chi}$, $E_4 \approx m_\chi + \frac{(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)^2}{2m_\chi}$, and expanding the delta function for little exchanged momentum

(see [49])

$$\delta\left(p_1 + \frac{p_2^2}{2m_\chi} - p_3 - \frac{(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)^2}{2m_\chi}\right) \simeq \delta(p_1 - p_3) + \frac{(\mathbf{p}_1 - \mathbf{p}_3) \cdot \mathbf{p}_2}{m_\chi} \frac{\partial \delta(p_1 - p_3)}{\partial p_3}. \quad (\text{B.30})$$

With this split in delta function, we now have two parts in the collision term (Eq. (B.24)). The first term is

$$\begin{aligned} I = & \frac{\pi}{2m_\chi} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \eta_\chi \int d\Pi_2 d\Pi_3 \left(\frac{1}{\eta_{\text{DR}} \eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) \delta(p_1 - p_3) f_{\text{DR}}^{(0)}(\mathbf{p}_3) \\ & \times \left[f_\chi(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3) \left(e^{(p_1/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_3) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_1) \right) \right. \\ & \left. - f_\chi(\mathbf{p}_2) \left(e^{(p_3/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_1) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_3) \right) \right]. \end{aligned} \quad (\text{B.31})$$

One can approximate $f_\chi(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3) \simeq f_\chi(\mathbf{p}_2)$, since $p_2 \gg p_1, p_3$. Now, we perform the integration over p_3

$$\begin{aligned} I = & \frac{\pi}{2m_\chi} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \eta_\chi \int d\Pi_2 f_\chi(\mathbf{p}_2) \int d\Omega_3 \int dp_3 \frac{p_3^2}{(2\pi)^3 2p_3} \left(\frac{1}{\eta_{\text{DR}} \eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) \delta(p_1 - p_3) f_{\text{DR}}^{(0)}(\mathbf{p}_3) \\ & \times \left[e^{(p_1/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_3) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_1) - e^{(p_3/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_1) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_3) \right] \end{aligned} \quad (\text{B.32})$$

using that $\mathbf{p}_3 = p_3 \hat{\mathbf{p}}_3$, we get

$$I = \frac{p_1}{8m_\chi (2\pi)^2} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \eta_\chi \int d\Pi_2 f_\chi(\mathbf{p}_2) \int d\Omega_3 \left(\frac{1}{\eta_{\text{DR}} \eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) [\Theta_{\text{DR}}(p_1 \hat{\mathbf{p}}_3) - \Theta_{\text{DR}}(\mathbf{p}_1)]. \quad (\text{B.33})$$

This is similar to the results in standard book where Bose enhancement/Pauli blocking does not appear at first-order perturbation.

The matrix element can be evaluated at $t = 2p_1^2(\tilde{\mu} - 1)$ with $\tilde{\mu} = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3$ and $s = m_\chi + 2p_1 m_\chi (1 - (p_2/m_\chi) \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)$. Since the dependence on the incoming scattering $\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2$ is multiplied by $p_2/m_\chi \ll 1$, it will lead to second order terms which can be neglected here. For the same reason one can neglect the dependence of the matrix element on p_2 , which allows us to perform the integral over p_2 and get

$$I = \frac{p_1}{16m_\chi^2 (2\pi)^2} n_\chi^{(0)} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \int d\Omega_3 \left(\frac{1}{\eta_{\text{DR}} \eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) [\Theta_{\text{DR}}(p_1 \hat{\mathbf{p}}_3) - \Theta_{\text{DR}}(\mathbf{p}_1)]. \quad (\text{B.34})$$

Now the matrix element dependence on $\tilde{\mu}$ can be expanded in Legendre Polynomials

$$\left(\frac{1}{\eta_\chi \eta_{\text{DR}}} \sum |\mathcal{M}|^2 \right) \Big|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi+2p_1 m_\chi}} = \sum_{n=0}^{\infty} (2n+1) A_n(p_1) P_n(\tilde{\mu}). \quad (\text{B.35})$$

Using the expansion for Θ_{DR} in Legendre Polynomials (Eq. (B.8)) with the definition of number density, we get

$$I = \frac{p_1}{16m_\chi^2 2\pi^2} n_\chi^{(0)} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \int_0^{2\pi} d\phi \int_{-1}^1 \frac{1}{2} d\tilde{\mu} \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) P_n(\tilde{\mu}) \right] \times \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) \left(P_l(\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{k}}) - P_l(\mu) \right) \right], \quad (\text{B.36})$$

with $\mu = \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{k}}$. We perform the azimuthal integration using the identity

$$\int_0^{2\pi} d\phi P_l(\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{k}}) = 2\pi P_l(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3) P_l(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{k}}), \quad (\text{B.37})$$

which leads to

$$I = \frac{p_1}{16m_\chi^2 \pi} n_\chi^{(0)} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \int_0^{2\pi} d\phi \int_{-1}^1 \frac{1}{2} d\tilde{\mu} \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) P_n(\tilde{\mu}) \right] \times \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) \left(P_l(\tilde{\mu}) P_l(\mu) - P_l(\mu) \right) \right]. \quad (\text{B.38})$$

The last step is to use the orthogonality relation of Legendre polynomials to perform the $\tilde{\mu}$ integral as follows (we split part I into two terms; we denote the first term as i and the second as ii)

$$\begin{aligned} i &= \int_{-1}^1 \frac{1}{2} d\tilde{\mu} \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) P_n(\tilde{\mu}) \right] \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\tilde{\mu}) P_l(\mu) \right] \\ &= \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) \right] \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) \right] \int_{-1}^1 \frac{1}{2} d\tilde{\mu} P_n(\tilde{\mu}) P_l(\tilde{\mu}) \\ &= \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) \right] \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) \right] \delta_{nl} \frac{1}{2n+1} \\ &= \sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) A_l(p_1), \end{aligned} \quad (\text{B.39})$$

$$\begin{aligned}
 ii &= \int_{-1}^1 \frac{1}{2} d\tilde{\mu} \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) P_n(\tilde{\mu}) \right] \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) \right] \\
 &= \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) \right] \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) \right] \int_{-1}^1 \frac{1}{2} d\tilde{\mu} P_n(\tilde{\mu}) P_0(\tilde{\mu}) \\
 &= \left[\sum_{n=0}^{\infty} (2n+1) A_n(p_1) \right] \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) \right] \delta_{n0} \frac{1}{2n+1} \\
 &= \sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) A_0(p_1),
 \end{aligned} \tag{B.40}$$

with $P_0(\tilde{\mu}) = 1$.

Eventually, one gets the result for the first part in the collision term in

$$I = \frac{p_1}{16m_\chi^2\pi} n_\chi^{(0)} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \left[\sum_{l=0}^{\infty} (-i)^l (2l+1) F_l(p_1) P_l(\mu) (A_l(p_1) - A_0(p_1)) \right]. \tag{B.41}$$

We still have to compute the second part. Since it has the quantity $\mathbf{p}_2/m_\chi$ which is too small, we need to keep only the zeroth order in the function F i.e

$$\begin{aligned}
 &\left(e^{(p_1/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_3) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_1) \right) - \left(e^{(p_3/T_{\text{DR}})} (\Theta_{\text{DR}}(\mathbf{p}_1) + 1) \pm \Theta_{\text{DR}}(\mathbf{p}_3) \right) \\
 &\simeq e^{(p_1/T_{\text{DR}})} - e^{(p_3/T_{\text{DR}})},
 \end{aligned} \tag{B.42}$$

which leads to the following form

$$\begin{aligned}
 II &= \frac{\pi\eta_\chi}{2m_\chi} f_{\text{DR}}^{(0)}(\mathbf{p}_1) \int d\Pi_2 f_\chi(\mathbf{p}_2) \int d\Pi_3 f_{\text{DR}}^{(0)}(\mathbf{p}_3) \left(\frac{1}{\eta_{\text{DR}}\eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) \\
 &\quad \times \frac{(\mathbf{p}_1 - \mathbf{p}_3) \cdot \mathbf{p}_2}{m_\chi} \frac{\partial \delta(p_1 - p_3)}{\partial p_3} \left(e^{(p_1/T_{\text{DR}})} - e^{(p_3/T_{\text{DR}})} \right).
 \end{aligned} \tag{B.43}$$

Next, one needs to perform the integration over $\mathbf{p}_2$, which is straightforward since the matrix element does not depend on it at leading order. And then we need to integrate by part to perform the p_3 integration, the only non vanishing term is when the derivative acts on the exponential. But it's more convenient to combine the zero order distribution function for $\mathbf{p}_3$ and $\mathbf{p}_1$ with the exponential terms as follows

$$\frac{f_{\text{DR}}^{(0)}(\mathbf{p}_3)}{f_{\text{DR}}^{(0)}(\mathbf{p}_1)} = \left(e^{(p_1/T_{\text{DR}})} - e^{(p_3/T_{\text{DR}})} \right) = f_{\text{DR}}^{(0)}(\mathbf{p}_3) - f_{\text{DR}}^{(0)}(\mathbf{p}_1), \tag{B.44}$$

this allows to rewrite the second term as (using also the definition of the bulk velocity

in Eq. (B.15))

$$II = \frac{\pi n_\chi^{(0)}}{4m_\chi^2} \int d\Omega_3 \int \frac{dp_3 p_3^2}{(2\pi)^3 2p_3} \left(\frac{1}{\eta_{\text{DR}} \eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) (\mathbf{p}_1 - \mathbf{p}_3) \cdot \mathbf{v}_\chi \frac{\partial \delta(p_1 - p_3)}{\partial p_3} \times \left(f_{\text{DR}}^{(0)}(\mathbf{p}_3) - f_{\text{DR}}^{(0)}(\mathbf{p}_1) \right), \quad (\text{B.45})$$

performing the integration by part over p_3 leads to

$$II = -\frac{p_1 n_\chi^{(0)}}{16m_\chi^2 (2\pi)^2} \frac{\partial f_{\text{DR}}^{(0)}(\mathbf{p}_1)}{\partial \ln p_1} \int d\Omega_3 \left(\frac{1}{\eta_{\text{DR}} \eta_\chi} \sum_{\text{states}} |\mathcal{M}|^2 \right) \Big|_{\substack{t=2p_1^2(\tilde{\mu}-1) \\ s=m_\chi+2p_1m_\chi}} (\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_3) \cdot \mathbf{v}_\chi. \quad (\text{B.46})$$

Finally, we can use the following relations: $\hat{\mathbf{p}}_1 \cdot \mathbf{v}_\chi = \mu v_\chi = P_1(\mu) v_\chi$ and $\hat{\mathbf{p}}_3 \cdot \mathbf{v}_\chi = P_1(\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{k}}) v_\chi$. Then, using the identity in Eq. (B.37) and Eq. (B.35), we get

$$II = -\frac{p_1 n_\chi^{(0)}}{16m_\chi^2 \pi} \frac{\partial f_{\text{DR}}^{(0)}(\mathbf{p}_1)}{\partial \ln p_1} P_1(\mu) (A_0(p_1) - A_1(p_1)) v_\chi. \quad (\text{B.47})$$

Combining the two parts in Eqs (B.41) and (B.47) and using the definition of ν_l in Eq.(B.12), we get the full collision term

$$C_{\chi\tilde{\gamma} \leftrightarrow \chi\tilde{\gamma}}[\mathbf{p}_1] = \frac{p_1 n_\chi^{(0)}}{16m_\chi^2 (2\pi)^2} \frac{\partial f_{\text{DR}}^{(0)}(\mathbf{p}_1)}{\partial \ln p_1} \left[\frac{1}{4} \sum_{l=1}^{\infty} (-i)^l (2l+1) \nu_l(p_1) P_l(\mu) (A_0(p_1) - A_l(p_1)) - P_1(\mu) (A_0(p_1) - A_1(p_1)) v_\chi \right]. \quad (\text{B.48})$$

B.3 IDM equations

We previously reached the density equation given by Eq. (B.18), and we also had an equation for the bulk velocity given by Eq. (B.22) in which the RHS needs further computation. Notice that the momentum appearing in this term is the incoming DM momentum, whereas that appearing in the collision term is the momentum of the incoming DR particle. Using the conservation of the total momentum [49], one can find the following relation between the two integrals of the collision term

$$\int \frac{d^3p}{(2\pi)^3} \frac{p_2}{m_\chi} \hat{\mathbf{p}}_2 C_{\chi(\mathbf{p}_2)\tilde{\gamma} \leftrightarrow \chi\tilde{\gamma}}[\mathbf{p}_2] = -\frac{\eta_{\text{DR}}}{\eta_\chi} \int \frac{d^3p}{(2\pi)^3} \hat{\mathbf{p}}_1 C_{\chi\tilde{\gamma}(\mathbf{p}_1) \leftrightarrow \chi\tilde{\gamma}}[\mathbf{p}_1]. \quad (\text{B.49})$$

Using this identity with the result of the first order collision term in Eq. (B.48), we get at first order (the ψ can be neglected)

$$\begin{aligned} \frac{-a}{16\pi m_\chi^3} \eta_{\text{DR}} \int \frac{d^3 p_1}{(2\pi)^3} p_1^2 (i\mathbf{k} \cdot \hat{\mathbf{p}}_1) \frac{\partial f_{\text{DR}}^{(0)}(\mathbf{p}_1)}{\partial p_1} \left[\frac{1}{4} \sum_{l=1}^{\infty} (-i)^l (2l+1) \nu_l(p_1) P_l(\mu) (A_0(p_1) - A_l(p_1)) \right. \\ \left. - P_1(\mu) (A_0(p_1) - A_1(p_1)) v_\chi \right], \end{aligned} \quad (\text{B.50})$$

with $i\mathbf{k} \cdot \hat{\mathbf{p}}_1 = ik\mu = ikP_1(\mu)$. The integral over $\mathbf{p}_1$ can be written in a more detailed form as

$$\begin{aligned} \frac{-a}{16\pi m_\chi^3} \eta_{\text{DR}} \int_0^{2\pi} d\phi \int_{-1}^1 \frac{d\mu}{2} ik P_1(\mu) \int \frac{dp_1 p_1^2}{(2\pi)^2 \pi} \frac{\partial f_{\text{DR}}^{(0)}(\mathbf{p}_1)}{\partial p_1} \\ \times \left[\frac{1}{4} \sum_{l=1}^{\infty} (-i)^l (2l+1) \nu_l(p_1) P_l(\mu) (A_0(p_1) - A_l(p_1)) - P_1(\mu) (A_0(p_1) - A_1(p_1)) v_\chi \right]. \end{aligned} \quad (\text{B.51})$$

Now, one needs to perform the integration over the solid angle where the integration over ϕ gives a 2π , and the integration over μ is done with the use of the orthogonality relation for the Legendre polynomials (Eq (B.11)). In the following, we split Eq. (B.51) into two terms ($T1$ and $T2$). We start with the computation for $T1$

$$\begin{aligned} T1 &= ik \frac{1}{4} \sum_{l=1}^{\infty} (-i)^l (2l+1) \nu_l(p_1) (A_0(p_1) - A_l(p_1)) \frac{1}{2} \int_{-1}^1 d\mu P_1(\mu) P_l(\mu) \\ &= ik \frac{1}{4} \sum_{l=1}^{\infty} (-i)^l (2l+1) \nu_l(p_1) (A_0(p_1) - A_l(p_1)) \delta_{1l} \frac{1}{2l+1} \\ &= \frac{1}{4} ik \nu_1(p_1) (A_0(p_1) - A_1(p_1)) \\ &= \frac{1}{3} \theta_{\text{DR}} (A_0(p_1) - A_1(p_1)), \end{aligned} \quad (\text{B.52})$$

with $\nu_1 = (4/3)iv_{\text{DR}}$ and $\theta_{\text{DR}} = ikv_{\text{DR}}$. The second term $T2$

$$\begin{aligned} T2 &= ik (A_0(p_1) - A_1(p_1)) v_\chi \frac{1}{2} \int_{-1}^1 d\mu P_1(\mu) P_1(\mu) \\ &= ik v_\chi (A_0(p_1) - A_1(p_1)) \delta_{11} \frac{1}{3} \\ &= \frac{1}{3} \theta_\chi (A_0(p_1) - A_1(p_1)), \end{aligned} \quad (\text{B.53})$$

with $\theta_\chi = ikv_\chi$. Putting everything together we get the RHS of Eq. (B.22)

$$-a \frac{1}{16\pi m_\chi^3} \frac{\eta_{\text{DR}}}{3} [\theta_{\text{DR}} - \theta_\chi] \int \frac{p_1^2 dp_1}{2\pi^2} p_1^2 \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} (A_0(p_1) - A_1(p_1)) . \quad (\text{B.54})$$

Then, they define the DM drag opacity as

$$\Gamma_\chi \equiv a \frac{1}{16\pi m_\chi^3} \frac{\eta_{\text{DR}}}{3} \int \frac{p_1^2 dp_1}{2\pi^2} p_1^2 \frac{\partial f_{\text{DR}}^{(0)}(p_1)}{\partial p_1} (A_0(p_1) - A_1(p_1)) = \frac{4\rho_{\text{DR}}}{3n_\chi^{(0)} m_\chi} \Gamma_{\text{DR-IDM}} . \quad (\text{B.55})$$

Finally, the two equations for IDM are

$$\dot{\delta}_\chi + \theta_\chi - 3\dot{\phi} = 0 \quad (\text{B.56})$$

$$\dot{\theta}_\chi - c_\chi^2 k^2 \delta_\chi + \mathcal{H} \theta_\chi - k^2 \psi = \Gamma_\chi [\theta_\chi - \theta_{\text{DR}}] . \quad (\text{B.57})$$

Note that in Sec. 4.1, we referred to Γ_χ as $\Gamma_{\text{IDM-DR}}$.

B.4 DR equations

The main interest of this work is getting the interaction rate between IDM and DR, which is given in Eq. (B.55). Therefore, we will not give the same focus on the DR self-interaction, especially since, physically, the main effect of DR self-interaction is to suppress its free-streaming effect. We review here briefly the main quantities in getting DR Boltzmann equations.

A general form of DR-DR interaction in the thermal approximation at first-order reads

$$C_{\tilde{\gamma}\tilde{\gamma}\leftrightarrow\tilde{\gamma}\tilde{\gamma}}^{(1)}[\mathbf{p}_1] = p_1 \frac{\partial f_{\text{DR}}^{(0)}(\mathbf{p}_1)}{\partial \ln p_1} \Lambda_{\tilde{\gamma}\tilde{\gamma}\leftrightarrow\tilde{\gamma}\tilde{\gamma}}(\mathbf{p}_1) \frac{1}{4} \sum_{l=1}^{\infty} (-i)^l (2l+1) P_l(\mu) \nu_l (1 - G_l(\mathbf{p}_1)) , \quad (\text{B.58})$$

substituting this to the general equation for DR (given in Eq. (B.13)), we get

$$\begin{aligned} \frac{\partial \nu_l}{\partial \tau} + k \left(\frac{l+1}{2l+1} \nu_{l+1} - \frac{l}{2l+1} \nu_{l-1} \right) - 4 \left[\frac{\partial \phi}{\partial \tau} \delta_{l0} + \frac{k}{3} \psi \delta_{l1} \right] \\ = -a \frac{1}{16\pi m_\chi^2} n_\chi^{(0)} \left[\left(A_0\left(\frac{q}{a}\right) - A_l\left(\frac{q}{a}\right) \right) \nu_l - \frac{4}{3} i v_\chi \left(A_0\left(\frac{q}{a}\right) - A_1\left(\frac{q}{1}\right) \right) \delta_{l1} \right] \\ - a \Lambda_{\tilde{\gamma}\tilde{\gamma}\leftrightarrow\tilde{\gamma}\tilde{\gamma}}\left(\frac{q}{a}\right) \left(1 - G_l\left(\frac{q}{a}\right) \right) . \end{aligned} \quad (\text{B.59})$$

Now, we use the definition of ν_l , given in Eq. (B.12), to get the hierarchy of moments for DR. For $l = 0$, we have the energy perturbation δ_{DR}

$$\delta_{\text{DR}}(k, \tau) = \frac{\int dq q^3 f_{\text{DR}}^{(0)}(q, \tau) F_0(k, q, \tau)}{\int dq q^3 f_{\text{DR}}^{(0)}(q, \tau)} , \quad (\text{B.60})$$

$l = 1$ gives the velocity divergence of DR θ_{DR}

$$\theta_{\text{DR}}(k, \tau) = \frac{3}{4} \frac{\int dq q^3 f_{\text{DR}}^{(0)}(q, \tau) F_1(k, q, \tau)}{\int dq q^3 f_{\text{DR}}^{(0)}(q, \tau)}, \quad (\text{B.61})$$

and finally, we have higher moments $l \geq 2$ denoted as $\Pi_l(k, \tau)$

$$\Pi_{\text{DR}}(k, \tau) = \frac{\int dq q^3 f_{\text{DR}}^{(0)}(q, \tau) F_l(k, q, \tau)}{\int dq q^3 f_{\text{DR}}^{(0)}(q, \tau)}. \quad (\text{B.62})$$

The hierarchy of DR Boltzmann equations reads

$$\dot{\delta}_{\text{DR}} + \frac{4}{3}\theta_{\text{DR}} - 4\dot{\phi} = 0, \quad (\text{B.63})$$

$$\dot{\theta}_{\text{DR}} + k^2(\sigma_{\text{DR}} - \frac{1}{4}\delta_{\text{DR}}) - k^2\psi = \Gamma_{\text{DR-IDM}}(\theta_{\text{DR}} - \theta_{\chi}), \quad (\text{B.64})$$

$$\dot{\Pi}_{\text{DR},l} + \frac{k}{2l+1}((l+1)\Pi_{\text{DR},l+1} - l\Pi_{\text{DR},l-1}) = (\alpha_l \Gamma_{\text{DR-IDM}} + \beta_l \Gamma_{\text{DR-DR}})\Pi_{\text{DR},l}. \quad (\text{B.65})$$

Here $\sigma_{\text{DR}}(k, \tau) = \Pi_{\text{DR},2}(k, \tau)/2$ is the DR shear perturbation. α_l are l -dependent coefficients which contain information about the angular dependence of the IDM-DR scattering cross section, given by

$$\alpha_l \equiv \frac{\int dp p^4 \frac{\partial f_{\text{DR}}^{(0)}(p)}{\partial p} [A_0(p) - A_l(p)]}{\int dp p^4 \frac{\partial f_{\text{DR}}^{(0)}(p)}{\partial p} [A_0(p) - A_1(p)]}. \quad (\text{B.66})$$

The DR self-scattering opacity can be computed with

$$\Gamma_{\text{DR-DR}} = \frac{a\eta_{\text{DR}}}{4\rho_{\text{DR}}} \int \frac{p^2 dp}{2\pi^2} p^2 \frac{\partial f_{\text{DR}}^{(0)}(p)}{\partial p} \Lambda_{\tilde{\gamma}\tilde{\gamma} \leftrightarrow \tilde{\gamma}\tilde{\gamma}}(p). \quad (\text{B.67})$$

And finally, the angular coefficients for DR self-scattering is defined as

$$\beta_l \equiv \frac{\int dp p^4 \frac{\partial f_{\text{DR}}^{(0)}(p)}{\partial p} \Lambda_{\tilde{\gamma}\tilde{\gamma} \leftrightarrow \tilde{\gamma}\tilde{\gamma}}(p)(1 - G_l(p))}{\int dp p^4 \frac{\partial f_{\text{DR}}^{(0)}(p)}{\partial p} \Lambda_{\tilde{\gamma}\tilde{\gamma} \leftrightarrow \tilde{\gamma}\tilde{\gamma}}(p)}. \quad (\text{B.68})$$

The coefficients $\{\alpha_l, \beta_l\}$ encapsulate the details of the microscopic model. We can use certain limits to simplify them. Regarding DR, we consider two limits

- strongly self-interacting DR, also known as *dark fluid*, leads to suppressed higher moments $\Pi_{\text{DR},l} \rightarrow 0$ for all $l \geq 2$. This limits correspond to $\Gamma_{\text{DR-DR}} \rightarrow \infty$, which means the parameters α_l, β_l are irrelevant,
- free-streaming DR, with negligible or no self-interaction, i.e., $\Gamma_{\text{DR-DR}} \rightarrow 0$. In this

case, we set $\alpha_l = 1$ while β_l is irrelevant. Note that DR still interacts with IDM.

C Full Plots

C.1 Galaxy clustering BOSS DR12

In this appendix, we present the results of the full MCMC run for the bias parameters in the fiducial model ($n = 0$ and $f_{\text{IDM}} = 1$). We verified that other models exhibit similar behavior. It is useful to examine whether the bias parameters are being adjusted to accommodate S_8 .

Figure C.1 displays the posterior distributions for all cosmological and bias parameters in the fiducial model, where the index i in $b^{(i)}$ corresponds to NGC z_3 , SGC z_3 , NGC z_1 , and SGC z_1 as described in Sec. 6.2.1. Notably, bias parameters other than b_1 , b_2 , and b_{G_2} enter the likelihood quadratically and are analytically marginalized over.

We find no significant shift in the bias parameters when comparing results from Planck + BAO + FS with those from Planck + BAO + FS + KiDS. This suggests that the inclusion of KiDS cannot be trivially absorbed by the bias parameters. While S_8 changes, there is no strong degeneracy between S_8 and any of the bias parameters.

C.2 SPT-clusters $\times$ WL

Figure C.2 presents an extended version of Figure 7.3, illustrating the constraints derived from SPT-clusters $\times$ WL data within the IDM-DR model. This figure includes all cosmological model parameters, along with the complete set of flat-prior parameters relevant to cluster abundance and weak-lensing mass calibration, which are jointly varied in our main analysis. Additionally, the plot demonstrates consistency with Λ CDM, as shown by the contours from [232]. Reprinted from Paper II.

We notice no significant change in the posteriors compared to the Λ CDM case. Importantly, there is no strong degeneracy between these parameters and the IDM-DR model parameter, ξ_{DR} .

C.3 CMB-S4-clusters $\times$ ngWL

Figure C.3 displays the marginalized posterior distributions for all parameters considered in the CMB-S4 $\times$ ngWL analysis for $Mock_{\text{IDM-DR}}$, with priors specified in Tables 8.2 and 8.3. As illustrated in the right subplot of Figure C.3, a degeneracy exists between certain cosmological parameters and the parameters governing the observable-mass scaling relation, particularly those describing the mass and redshift dependence of the tSZE cluster detection significance, ζ .

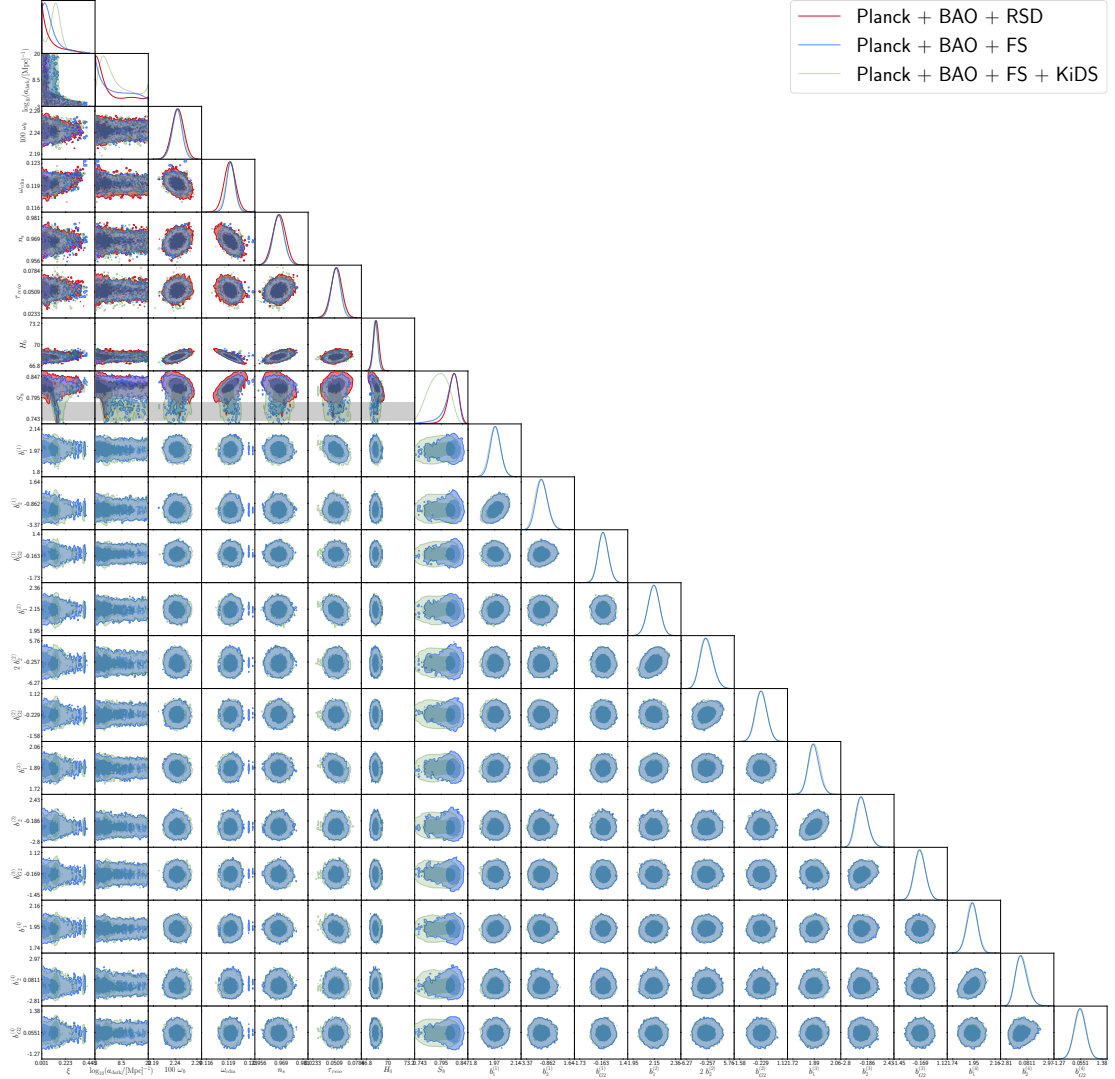


Figure C.1. Posteriors for all cosmological and bias parameters for the fiducial model.

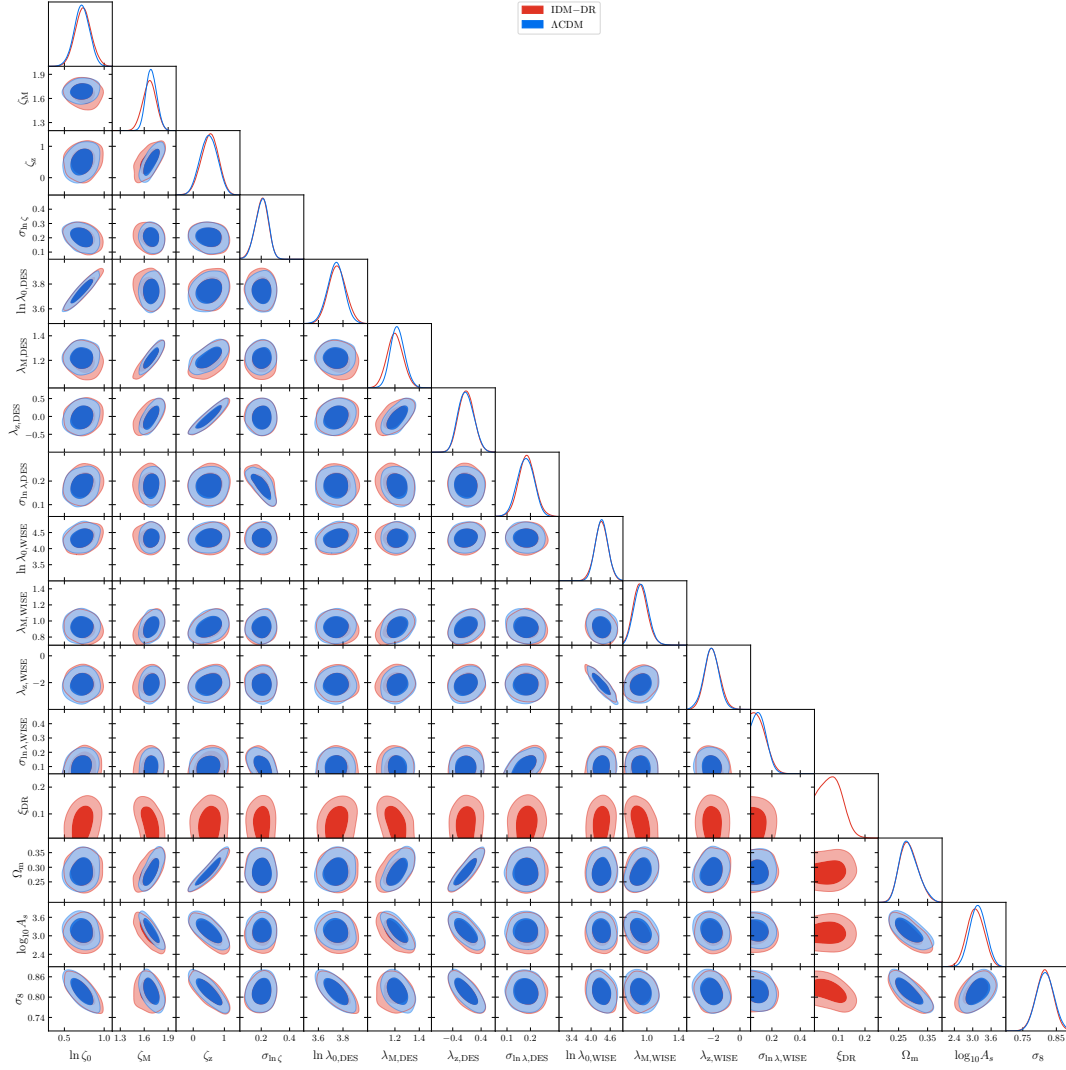


Figure C.2. Posteriors of all tSZ observable–mass relations parameters employed in the analysis of SPT-clusters $\times$ WL. We compare IDM–DR model (red) to Λ CDM model (blue). From [25].

One notable degeneracy arises between ξ_{DR} and ζ_{M} , which can be attributed to the suppression of the halo mass function (HMF) for massive halos at higher dark sector temperature ratios (see Fig. 4.3). This suppression is counterbalanced by an increased value of ζ_{M} (see Eq. 7.3). A potential strategy to mitigate this degeneracy is to improve constraints on ζ_{M} by incorporating a broader mass range, such as including low-mass clusters detected via X-ray emission.

A weaker degeneracy is observed between ξ_{DR} and ζ_{z} , following a similar underlying mechanism. In contrast, a much stronger degeneracy exists between ζ_{z} and Ω_{m} . A promising approach to tightening constraints on ζ_{z} , thereby improving the constraints on Ω_{m} and ξ_{DR} , is to extend the redshift range of the cluster and weak-lensing datasets. Expanding the low-redshift cluster sample could be particularly beneficial. Additionally, CMB lensing data offer significant potential for constraining the masses of high-redshift clusters, where ground-based ngWL samples lose sensitivity.

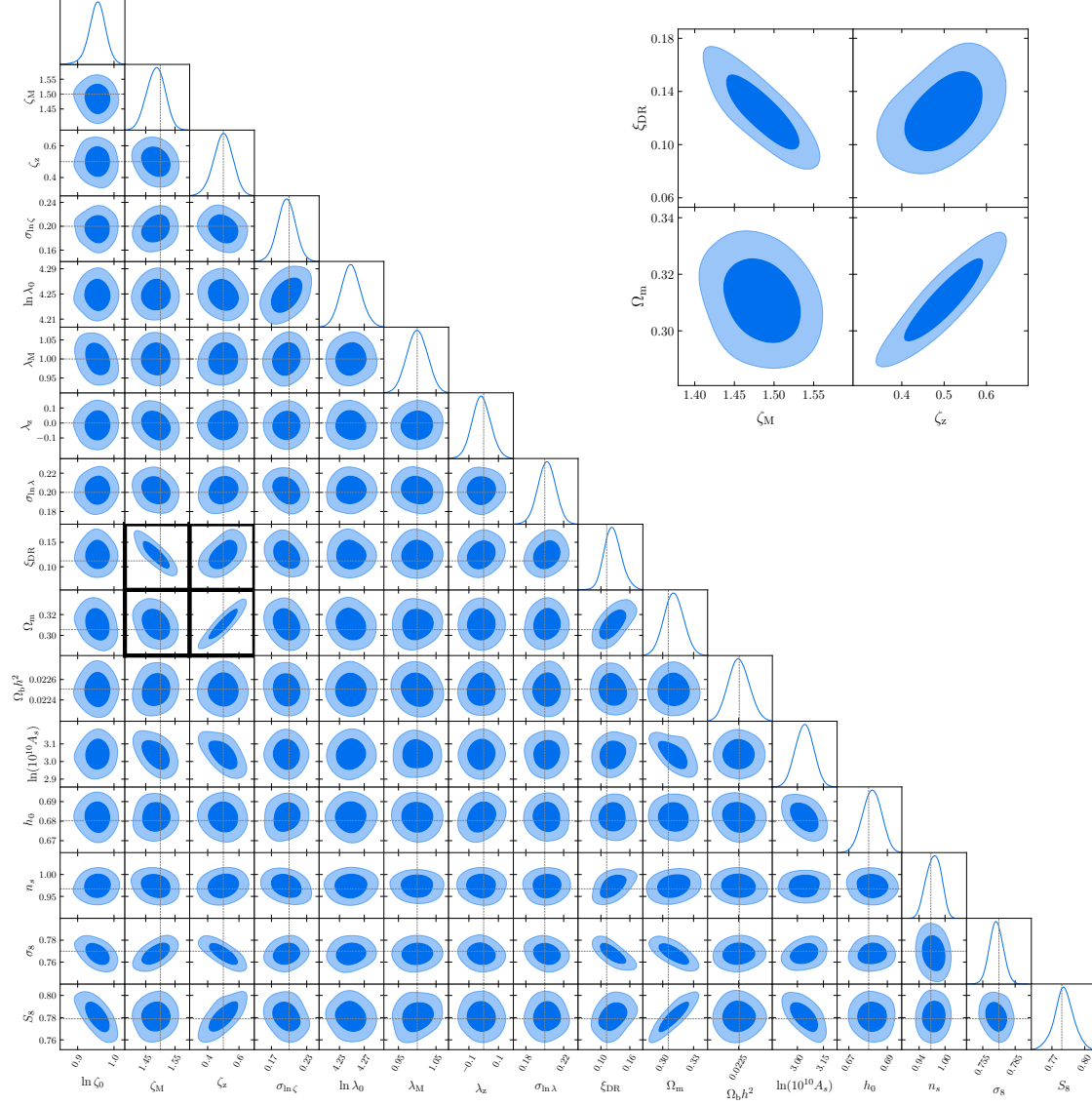


Figure C.3. Results for posterior distributions of all parameters employed in the CMB-S4 $\times$ ngWL analysis for the benchmark model *Mock*_{IDM-DR}. The inset in the figure emphasizes the degeneracies between tSZE observable–mass scaling relation parameters, ζ_M and ζ_z , and the cosmological parameters, Ω_m and ξ_{DR} . From [Paper III](#).

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