ALEA, Lat. Am. J. Probab. Math. Stat. **22**, 473–491 (2025)
DOI: 10.30757/ALEA.v22-17

Branching random walk and log-slowly varying tails

Ayan Bhattacharya, Piotr Dyszewski, Nina Gantert and Zbigniew Palmowski

Department of Mathematics, Indian Institute of Technology, Bombay, Powai, Mumbai-400076, Maharashtra, India
E-mail address: ayanbh@math.iitb.ac.in
URL: https://sites.google.com/site/bhattacharya90ayan/home

Mathematical Institute, University of Wroclaw, pl. Grunwaldzki 2, 50-384 Wroclaw, Poland
E-mail address: pdysz@math.uni.wroc.pl
URL: https://sites.google.com/site/piotrdyszewski/

Department of Mathematics, School of Computation, Information and Technology, Technical University of Munich, Boltzmannstraße 3 85748 Garching bei München, Germany
E-mail address: gantert@ma.tum.de
URL: https://www.math.cit.tum.de/math/personen/professuren/gantert/

Department of Applied Mathematics, Faculty of Pure and Applied Mathematics, Wroclaw University of Science and Technology, ul. Hoene-Wrońskiego 13, 50-376 Wroclaw, Poland
E-mail address: zbigniew.palmowski@pwr.edu.pl
URL: http://prac.im.pwr.edu.pl/∼zpalma/

Abstract. We investigate a branching random walk with independent and identically distributed heavy-tailed displacements. The offspring distribution is supercritical and satisfies the Kesten-Stigum condition. Our focus is on the case where the displacement law does not belong to the max-domain of attraction of an extreme value distribution. We demonstrate that when the tails of the displacements are such that the absolute value of their logarithm is a slowly varying function, the extremes of the process can still be effectively analyzed. Specifically, after applying a non-linear transformation, the extremes of the branching random walk converge to a clustered Cox process.

1. Introduction

1.1. *Model and main results.* Consider a system of particles evolving on the real line $\mathbb{R}$ as follows. Initially, the system consists of a single particle located at the origin 0 of $\mathbb{R}$. At each discrete time step $n \in \mathbb{N} = 1, 2, \ldots$, the particles present in the system reproduce independently according to a given reproduction law. Each newly created particle is then shifted independently from its birth position according to a fixed *displacement distribution*, represented by a generic random variable

Received by the editors April 28th, 2024; accepted February 19th, 2025.

2010 *Mathematics Subject Classification.* 60F10, 60J80, 60G50.

Key words and phrases. Branching random walk, slowly varying functions, limit theorems, point processes, extreme values.

The research of PD is partially supported by the National Science Centre, Poland (Sonata, grant number 2020/39/D/ST1/00258). The research of ZP is partially supported by the National Science Centre, Poland (Opus, grant number 2021/41/B/HS4/00599).

X. Let $\mathbf{V}_n \subseteq \mathbb{R}$ denote the point process of the particle positions at time epoch n. The sequence $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$ forms a *branching random walk (BRW)*. We denote the *rightmost particle position* by

$$M_n = \max \mathbf{V}_n.$$

Assuming that the underlying genealogical structure follows a supercritical Galton-Watson process, we study the case of independent and identically distributed (iid) displacements, where the tail of the displacement distribution is given by

$$\mathbb{P}[X > t] = a(t)e^{-L(t)}, \tag{1.1}$$

for $t \in \mathbb{R}$, where $a(\cdot)$ is a bounded, measurable function such that $a(t) \to a > 0$ as $t \to \infty$. The asymptotic behavior of both $\mathbf{V}_n$ and M_n depends on the asymptotic properties of L.

We recall that a function ℓ is *slowly varying* if

$$\lim_{t\to\infty} \ell(\mathrm{c}t)/\ell(t) = 1 \qquad \text{for any } \mathrm{c} > 0.$$

Additionally, we say that a function ℓ is *superlogarithmic* if

$$\lim_{t\to\infty} \log t/\ell(t) = 0,$$

and it is *sublogarithmic* if

$$\lim_{t\to\infty} \ell(t)\log t = 0.$$

Finally, the function ℓ is *regularly varying* with index $r \in (0, 1)$ if

$$\lim_{t\to\infty} \ell(\mathrm{c}t)/\ell(t) = \mathrm{c}^r, \tag{1.2}$$

for any $\mathrm{c} > 0$.

In this paper, we consider the case where L appearing in (1.1) is *slowly varying*. One of our results shows that if the function L is superlogarithmic (a notable example being the lognormal distribution, for which $L(t) = (\log t)^2/2 + \log\log t$), then there exist subexponentially growing sequences $\{b_n\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$ such that

$$(M_n - b_n)/a_n$$

converges in law to a random shift of the Gumbel distribution, i.e., a distribution with a cumulative distribution function (cdf) given by

$$x \mapsto \mathbb{E}\left[\exp\{-We^{-x}\}\right], \tag{1.3}$$

where W is a random variable representing the limit of a martingale associated with the underlying Galton-Watson process. Here, and in what follows, a sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ is said to grow subexponentially if $x_n e^{-\varepsilon n} \to 0$ for any $\varepsilon > 0$.

In the sublogarithmic case, the classical extreme value theory no longer applies. Note that this condition implies that $\mathbb{E}[|X|^\varepsilon] = \infty$ for any positive $\varepsilon > 0$. However, M_n can still be analyzed under a non-linear scale. As our main results show,

$$L(M_n) - n\log m$$

converges in law to a distribution with a cumulative distribution function (cdf) given by (1.3).

As is often the case with heavy-tailed distributions, we are able to analyze not only the rightmost position M_n but also the joint behavior of the rightmost extremes of the BRW $\mathbf{V}_n$. Following the classical approach (see Resnick (2008)), we formulate our main results in terms of point process convergence on the right-point compactification of the real line.

1.2. *Historical background.* Since the pioneering studies in the 1970s of Biggins (1976), Kingman (1975), and Hammersley (1974), branching random walks (BRW) have become an active area of research in probability theory. The BRW is a discrete-time analogue of branching Brownian motion and is thus closely related to the Fisher-Kolmogorov-Petrovsky-Piskunov (FKPP) equation (see Kolmogorov (1937); Fisher (1937); McKean (1975); Bovier (2017) for more details).

Beyond its natural interpretation as a model for population fitness, branching random walks are also connected to fragmentation processes (see Kyprianou et al. (2017); Dadoun (2017)), Mandelbrot cascade measures (see Liu (2000); Barral et al. (2014)), and the so-called smoothing transform (see Alsmeyer et al. (2012)). The FKPP equation, in particular, arises in the study of the extremal positions of the particles. For this reason, the position of the rightmost particle M_n has been one of the central objects of probabilistic analysis in BRW. As one might expect, the asymptotic behavior of M_n is closely tied to the asymptotic properties of the displacement distribution L given in (1.1).

In the classical *light-tailed case*, where the displacement distribution has finite exponential moments, M_n moves at a linear speed and, after an additional logarithmic correction, converges weakly to a random shift of the Gumbel distribution (see Aïdékon (2013)). More precisely, assume that

$$\liminf_{t\to\infty} L(t)/t > 0. \tag{1.4}$$

Then, under some mild technical assumptions, there exist positive constants c_1 and c_2 such that

$$M_n - \mathrm{c}_1 n - \mathrm{c}_2 \log n$$

converges in distribution to a law with the cumulative distribution function given by

$$x \mapsto \mathbb{E}\left[\exp\{-D_\infty e^{-x}\}\right],$$

where D_∞ is the limit of the so-called derivative martingale associated with the BRW. For a self-contained treatment, we refer to Shi (2015). Note that in this case, the local behavior of L affects the limiting distribution.

In the *heavy-tailed case*, where the displacements have no finite exponential moments but possess some finite power moments, one relies on extreme value theory to show that M_n grows faster than linearly but at most exponentially, with a weak limit that is either a random shift of the Gumbel law (see Dyszewski and Gantert (2022)) or the Fréchet law (see Durrett (1983)), depending on the tails of the displacements. More precisely, if L in (1.1) is regularly varying with index $r \in (0,1)$, then for some slowly varying functions ℓ_1 and ℓ_2,

$$(M_n - n^{1/r}\ell_1(n))/(n^{1/r-1}\ell_2(n))$$

converges in law to a distribution with a cumulative distribution function (1.3).

The case where L grows logarithmically essentially reduces to steps with regularly varying tails. If we assume, for simplicity, that $L(t) = \alpha \log t$ for some $\alpha > 0$, then

$$m^{-n/\alpha} M_n$$

converges in law to a random shift of a Fréchet distribution (Durrett, 1983), which is a distribution with a cumulative distribution function (cdf) given by

$$x \mapsto \mathbb{E}\left[\exp\{-W x^{-\alpha}\}\right],$$

where, as before, W is the martingale limit associated with the underlying Galton-Watson process. Here, and in what follows $m > 1$ denotes the mean of the reproduction law.

1.3. *Organisation of the paper.* The article is organized as follows. In Section 2, we introduce the main terms and notations. Section 3 presents the precise statements of our main results, Theorem 3.2 and Theorem 3.6. Section 4 provides estimates for random walks, which are used in the proofs presented in Section 5.

2. Preliminaries

To state our main results precisely, we first introduce some basic notations, assumptions, and facts. Throughout the article, we write $f(t) = o(g(t))$ for functions $f, g\colon \mathbb{R} \to \mathbb{R}$ if $f(t)/g(t) \to 0$ as $t \to \infty$. We also use c to denote a generic constant whose specific value is unimportant. Note that the actual value of c may change from line to line.

One of our standing assumptions is that the genealogical structure is independent of the displacements, and thus we introduce both components separately.

2.1. *Galton-Watson process.* We begin with the underlying Galton-Watson process, which we introduce by explicitly describing the corresponding random tree. Let $\mathbb{U} = \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \mathbb{Z}_+^k$, where $\mathbb{Z}_+ = \{1, 2, \dots\}$ is the set of all possible labels. For $x, y \in \mathbb{U}$, we write $x \le y$ if $y_{|k} = x$ for some $k \ge 1$, and we define $|x| = n$ if $x \in \mathbb{Z}_+^n$. Additionally, for $x \in \mathbb{U}$, we write $x_0 = \emptyset$ and x_k for the projection of x onto the kth coordinate for $k \le |x|$.

Consider a family $\{N(x) : x \in \mathbb{U}\}$ of iid random variables taking values in $\mathbb{Z}_+ \cup \{0\}$. We interpret $N(x)$ as the number of children of the particle labeled by $x \in \mathbb{U}$. The aforementioned random tree is defined as

$$\mathbb{T} = \{x \in \mathbb{U} \,:\, x_k \le N(x_{k-1}) \text{ for all } 1 \le k \le |x|\}\,.$$

The tree $\mathbb{T}$ can be given a structure by placing an edge between any two vertices $x, y \in \mathbb{T}$ such that $y = (x, k)$ with $k \le N(x)$.

Note that $\mathbb{T}$ is the genealogical tree corresponding to the Galton-Watson process

$$Z_n = \#\mathbb{T}_n, \qquad \mathbb{T}_n = \{x \in \mathbb{T} \,:\, |x| = n\},$$

and that $Z_1 = N(\emptyset)$. In what follows, we impose conditions on Z_1 that allow us to control the growth of the process $\{Z_n\}_{n\in\mathbb{N}}$.

Assumption 2.1. We assume that the Galton-Watson process $\{Z_n\}_{n\in\mathbb{N}}$ is supercritical, i.e., $\mathbb{E}[Z_1] = m \in (1, \infty)$. Furthermore, we assume that $\mathbb{E}[Z_1 \log^+(Z_1)] < \infty$.

The first condition ensures that the process $\{Z_n\}_{n\in\mathbb{N}}$ survives with positive probability. More precisely, we have

$$\mathbb{P}[Z_n \ge 1, \forall n \ge 1] > 0.$$

Since our considerations become void when the population dies out, we can restrict our analysis to the event of survival and work under the conditional probability

$$\mathbb{P}^*[\,\cdot\,] = \mathbb{P}[\,\cdot \mid Z_n \ge 1, \forall n \ge 1].$$

While our main results can be recast under $\mathbb{P}^*$, it is more convenient to work with $\mathbb{P}$ in the proofs.

The second condition in Assumption 2.1 ensures the strict exponential growth of Z_n. To describe this precisely, note that the sequence $\{m^{-n} Z_n\}_{n\in\mathbb{N}}$ forms a non-negative $\mathbb{P}$-martingale and therefore converges to

$$\lim_{n\to\infty} m^{-n} Z_n = W \quad \mathbb{P} - \text{a.s.} \tag{2.1}$$

It turns out that, under Assumption 2.1, $\mathbb{P}^*[W > 0] = 1$, and the above convergence occurs in $L^1(\mathbb{P})$ (see Athreya and Ney (1972, Theorem I.10.1)). The distribution of the limiting random variable W exhibits self-similarity, which translates to an analogous property for the limiting measure of $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$.

For $x \in \mathbb{T}_1$, let $Z_n(x) = \#\{y \in \mathbb{T} \,:\, y \ge x,\ |y| = n + |x|\}$. Conditioned on $x \in \mathbb{T}_1$, the sequence $\{Z_n(x)\}_{n\in\mathbb{N}}$ is distributed as $\{Z_n\}_{n\in\mathbb{N}}$, and thus

$$\lim_{n\to\infty} m^{-n} Z_n(x) = W_x \quad \mathbb{P} - \text{a.s.}$$

Using $Z_n = \sum_{|x|=1} Z_{n-1}(x)$, we infer that

$$W = \frac{1}{m} \sum_{|x|=1} W_x. \tag{2.2}$$

The key feature of this formula is that the W_x's are iid copies of W, independent of the length of the sum, $N = Z_1 = \#\mathbb{T}_1$. The random variable W serves as a shift parameter in the directing measure for the limit of $\mathbf{V}_n$.

2.2. *Displacements and related point process* $\mathbf{V}_n$. We now turn our attention to the displacements in our model. Consider a family $\{X_y : y \in \mathbb{U} \setminus \{\emptyset\}\}$ of iid random variables, where each X_y represents the shift of particle y from its place of birth. Thus, the position $V(x)$ of the particle $x \in \mathbb{T}$ can be expressed as

$$V(x) = \sum_{y \in (\emptyset, x]} X_y,$$

where $(\emptyset, x] = \{y \in \mathbb{T} : y \leq x,\ y \neq \emptyset\}$. The position of the rightmost particle

$$M_n = \max_{|x|=n} V(x) \tag{2.3}$$

is the maximum of Z_n dependent random walks with step distribution X.

To analyze different order statistics of $\{V(x)\}_{|x|=n}$, we study the point process

$$\mathbf{V}_n = \sum_{|x|=n} \delta_{V(x)}.$$

In what follows, $\mathbf{V}_n$ is treated as a random element of the space of non-negative Radon measures (see Chapter 3 in Resnick (2008) and Section 6.1.3 in Resnick (2007)). Let $\overline{\mathbb{R}} = (-\infty, \infty]$ be a homomorphic image of $(0, 1]$, and denote by $\mathbb{M}(\overline{\mathbb{R}})$ the space of measures on $\overline{\mathbb{R}}$ that charge finite mass to each compact subset of $\overline{\mathbb{R}}$.

Let $C_c^+(\overline{\mathbb{R}})$ denote the space of non-negative bounded continuous functions with compact support. We say that a sequence of point measures $\{\nu_n\}_{n \in \mathbb{N}}$ converges vaguely to a measure ν on $\overline{\mathbb{R}}$ if

$$\lim_{n \to \infty} \nu_n(f) = \nu(f)$$

for all $f \in C_c^+(\overline{\mathbb{R}})$, where $\nu(f) = \int f \, \mathrm{d}\nu$. By Resnick (2008, Proposition 3.17), $\mathbb{M}(\overline{\mathbb{R}})$ equipped with the vague topology is a Polish space.

In this paper we study the weak convergence of $\mathbf{V}_n$ as a random element of $\mathbb{M}(\overline{\mathbb{R}})$. We recall that measures $\mathbf{\Theta}_n$ converges weakly to $\mathbf{\Theta}$ in $\mathbb{M}(\overline{\mathbb{R}})$ if

$$\mathbb{E}(\mathbf{\Theta}_n(f)) = \mathbb{E}\Big(\int f(x) \mathbf{\Theta}_n(\mathrm{d}x) \Big) \to \mathbb{E}(\mathbf{\Theta}(f)) = \mathbb{E}\Big(\int f(x) \mathbf{\Theta}(\mathrm{d}x) \Big)$$

for all bounded continuous real valued functions f on $\mathbb{M}(\overline{\mathbb{R}})$. We shall use $\Rightarrow$ to denote weak convergence of random elements. Moreover, a sequence of point processes $\mathbf{\Theta}_n$ converges weakly to $\mathbf{\Theta}$ in $\mathbb{M}(\overline{\mathbb{R}})$ if and only if

$$\lim_{n \to \infty} \mathbb{E}[\exp\{-\mathbf{\Theta}_n(f)\}] = \mathbb{E}[\exp\{-\mathbf{\Theta}(f)\}]$$

for all $f \in C_c^+(\overline{\mathbb{R}})$ (see Resnick (2008, Proposition 3.19)). We shall use Laplace functionals to obtain the weak limit of $\mathbf{V}_n$.

2.3. *Scaling and a randomly shifted decorated Poisson process.* Aiming to provide centring and scaling for $\mathbf{V}_n$ we define the shift and scaling for point measures as follows. For $\nu \in \mathbb{M}(\overline{\mathbb{R}})$ given via $\nu = \sum_i \delta_{x_i}$ let the the shift by $x \in \mathbb{R}$ and scaling by $y > 0$ be defined as

$$\mathfrak{T}_x \nu = \sum_i \delta_{x_i + x}, \qquad \mathfrak{S}_y \nu = \sum_i \delta_{y x_i}.$$

We will find sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

$$\mathfrak{S}_{a_n^{-1}} \mathfrak{T}_{-b_n} \mathbf{V}_n = \sum_{|x|=n} \delta_{(V(x)-b_n)/a_n} \tag{2.4}$$

converges weakly in $\mathbb{M}(\overline{\mathbb{R}})$ to a non-trivial limit $\mathbf{V}$. By the choice of the space, this provides a description of the positions of the particles in the $n-$th generation in the vicinity of the rightmost one.

To describe this limit we need the following definition of a randomly shifted decorated Poisson process (SDPPP) from Subag and Zeitouni (2015).

Definition 2.2. A point process $\mathbf{\Theta}$ is called a randomly shifted decorated Poisson process if there exists a Radon measure μ, a point process $\mathbf{\Lambda}$ and a random variable S such that

$$\mathbf{\Theta} = \sum_k \mathfrak{T}_{\xi_k + S} \mathbf{\Lambda}_k,$$

where $\sum_k \delta_{\xi_k}$ is a Poisson point process with intensity μ, $\{\mathbf{\Lambda}_k\}_{k\in\mathbb{N}}$ are iid copies of $\mathbf{\Lambda}$ such that $\sum_k \delta_{\xi_k}$, $\{\mathbf{\Lambda}_k\}_{k\in\mathbb{N}}$ and S are independent. In this case one writes $\mathbf{\Theta} \sim \mathrm{SDPPP}(\mu, \mathbf{\Lambda}, S)$.

To provide a detailed description of the limit let $\{A_k\}_{k\in\mathbb{N}}$ be a collection of iid random variables distributed as

$$\mathbb{P}[A_k = j] = \frac{1}{v} \sum_{l=0}^{\infty} m^{-l} \mathbb{P}[Z_l = j], \qquad j \in \mathbb{N},$$

where v is the normalising constant

$$v = \sum_{l=0}^{\infty} m^{-l} \mathbb{P}[Z_l > 0]. \tag{2.5}$$

Let $\{\ell_k\}_k$ be a Poisson point process on $\mathbb{R}$ with intensity $e^{-x}\mathrm{d}x$. Our limiting process $\mathbf{V}$ of (2.4) can be written as

$$\mathbf{V} = \sum_k A_k \delta_{\ell_k - \log(vW)}. \tag{2.6}$$

Note that $\mathbf{V}$ is $\mathrm{SDPPP}(e^{-x}\mathrm{d}x, A_1\delta_0, \log(vW))$. Moreover $\mathbf{V}$ is a cluster Cox process with Laplace functional given via

$$f \mapsto \mathbb{E}\left[\exp\left\{-\sum_{l=0}^{\infty} m^{-l} W \int \left(1 - e^{-f(x)Z_l}\right) e^{-x}\mathrm{d}x\right\}\right]$$

for measurable $f\colon \mathbb{R} \to [0, +\infty)$.

3. Main results

3.1. *Statements.* To provide the normalization for $\mathbf{V}_n$, we need to introduce the conditions imposed on the step distribution. We work in two cases namely sub- and sup-logarithmic regimes. We begin with the latter, since in this case the classical extreme value theory applies. As we will see later, in the latter case, extreme value theory does not apply.

Assumption 3.1. We assume that the random variable X has a distribution of the form

$$\mathbb{P}[X > t] = a(t)e^{-L(t)}, \qquad t \in \mathbb{R},$$

where $a(t) \to a > 0$, $L(\cdot)$ is C^2 function such that L' is slowly varying and

$$\lim_{x\to\infty} \frac{L''(x)}{L'(x)^2} = 0, \qquad \lim_{x\to\infty} \frac{L(x)}{xL'(x)\sqrt{\log\log(L(x))}} = \infty \tag{3.1}$$

and the function $x \mapsto x^{-1/3}L(x)$ is eventually decreasing, that is, there exists x_0 such that $x_1^{-1/3}L(x_1) \le x_2^{-1/3}L(x_2)$ for all $x_1 \le x_2$ such that $x_1 \ge x_0$. Assume moreover the following left tail condition

$$\mathbb{P}[X < -t] \le t^{-\varepsilon}.$$

for some $\varepsilon > 0$ and sufficiently large $t > 0$.

We denote

$$b_n = \inf\{t \ge 0 \,:\, \mathbb{P}[X > t] \le m^{-n}\}, \qquad a_n = 1/L'(b_n). \tag{3.2}$$

This is the first main result of this paper.

Theorem 3.2 (The suplogarithmic case)**.** *Under Assumptions 2.1 and 3.1, one has for the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ given in (3.2) the convergence*

$$\mathfrak{S}_{a_n^{-1}}\mathfrak{T}_{-b_n}\mathbf{V}_n \Rightarrow \mathbf{V}$$

in $\mathbb{M}(\overline{\mathbb{R}})$ as $n \to \infty$, where $\mathbf{V}$ is defined in (2.6).

Remark 3.3. We note that the appearance of $\sqrt{\log\log(L(x))}$ in the second condition in (3.1) might be an artefact of our proof. The first condition in (3.1) implies in particular that L grows faster than logarithmic. Assumption 3.1 ensures that $e^{-L(x)}$ is a von-Mises function which is used in the proof.

Note that under Assumption 3.1 both b_n and a_n grow faster than any polynomial. For example, in the case of lognormal displacements where

$$L(t) = ((\log t)^2 + 2\log\log t)/2,$$

one has

$$b_n = \exp\left\{\sqrt{2n\log m - 2\log(2n\log m)}\right\}(1 + o(1)) \qquad \text{and} \qquad a_n = b_n/\sqrt{2n\log m}(1 + o(1)).$$

Taking b_n and a_n as in (3.2) gives

$$m^n\mathbb{P}[X > b_n + a_n x] \to e^{-x}$$

as $n \to \infty$ for any $x \in \mathbb{R}$ and furthermore the maximum of m^n independent copies of $(X - b_n)/a_n$ converges in law to the Gumbel law. Note that at this point one uses the limiting relations in Assumption 3.1. We use this limit theorem in the proof of Theorem 3.2.

Results in terms of point process convergence allow to treat joint convergence of the extremes of $\{V(x)\}_{|x|=n}$ by standard arguments (see Bhattacharya et al. (2017, Section 4.6) for details). In particular the following fact holds true.

Corollary 3.4 (Weak convergence of the normalized rightmost position)**.** *Under Assumptions 2.1 and 3.1, for the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ given in (3.2) and M_n given via (2.3), we have*

$$\mathbb{P}\left[M_n \le b_n + a_n x\right] = \mathbb{P}\left[\mathfrak{S}_{a_n^{-1}}\mathfrak{T}_{-b_n}\mathbf{V}_n(x, +\infty] = 0\right] \to \mathbb{E}\left[e^{-vWe^{-x}}\right]$$

for any real x, as $n \to \infty$, where v is defined in (2.5).

To analyse the sublogarithmic case we introduce the following assumption.

Assumption 3.5. We assume that the random variable X is non-negative and has distribution of the form

$$\mathbb{P}[X > t] = e^{-L(t)}, \qquad t \in \mathbb{R},$$

where $L(\cdot)$ is slowly varying such that $L(x) = o(\log x)$ as $x \to \infty$.

To study $\mathbf{V}_n$ under the above conditions we need to use a non-linear scale. For $\nu \in \mathbb{M}(\overline{\mathbb{R}})$ given via $\nu = \sum_i \delta_{x_i}$ let

$$\mathfrak{L}\nu = \sum_i \delta_{L(x_i)}.$$

Our second main result is the following theorem.

Theorem 3.6 (The sublogarithmic case)**.** *Under Assumptions 2.1 and 3.5,*

$$\mathfrak{T}_{-n\log m}\mathfrak{L}\mathbf{V}_n \Rightarrow \mathbf{V}$$

in $\mathbb{M}(\overline{\mathbb{R}})$ *as* $n \to \infty$ *where* $\mathbf{V}$ *is the point process given in* (2.6)*.*

As it is the case for Theorem 3.2, we may infer limit theorems for the extremes getting the following corollary.

Corollary 3.7 (Weak convergence of the normalized rightmost position)**.** *Under Assumptions 2.1 and 3.5, for* M_n *given via* (2.3)*, as* $n \to \infty$*, we have*

$$\mathbb{P}\left[L(M_n) \leq n\log m + x\right] = \mathbb{P}\left[\mathfrak{T}_{-n\log m}\mathfrak{L}\mathbf{V}_n(x, +\infty] = 0\right] \to \mathbb{E}\left[e^{-vWe^{-x}}\right]$$

for any real x*.*

Remark 3.8. The above result is similar to the limit theorem for random walks with increments having log-slowly varying tails where one has to apply a non-linear scale (see Darling (1952, Theorem 4.1)).

Remark 3.9. Observe that the limiting point process $\mathbf{V}$ given in (2.6) as a randomly shifted decorated Poisson process is superposable (see Brunet and Derrida (2011, Section 3.2)), which means that a union of independent copies of $\mathbf{V}$ when viewed from the point of the rightmost particle has the same law as $\mathbf{V}$ viewed from the position of the rightmost particle. As in Dyszewski and Gantert (2022, Section 3) this can be also seen directly by appealing to (2.2) which implies that

$$\sum_{k=1}^{Z_1} \mathbf{V}^{(k)} \stackrel{d}{=} \mathfrak{T}_{\log m}\mathbf{V},$$

where $\mathbf{V}^{(k)}$ are iid copies of $\mathbf{V}$ independent of Z_1. If we thus define

$$\mathfrak{Y}\sum_i \delta_{x_i} = \begin{cases} \sum_i \delta_{x_i - \max_j x_j} & \text{if } \max_j x_j < \infty \\ o & \text{otherwise,} \end{cases}$$

where o denotes the null measure, we see that

$$\mathfrak{Y}\sum_{k=1}^{Z_1} \mathbf{V}^{(k)} \stackrel{d}{=} \mathfrak{Y}\mathbf{V}.$$

Furthermore, if the branching is deterministic then $\mathbf{V}$ is exponentially stable (see Maillard (2013) and Dyszewski and Gantert (2022, Remark 3.5) for details).

3.2. *Idea of the proofs.* In the proof of Theorem 3.2 given in Section 5, we use the principle of one big jump to approximate $\mathbf{V}_n$ via

$$\mathbf{T}_n = \sum_{|x|=n} \delta_{T(x)}, \quad T(x) = \max_{y \in (\emptyset, x]} X_y.$$

In other words, we use the fact that asymptotically two displacements coming from the same parent can not be large simultaneously. Next, we use a stopping line argument which is based on grouping vertices into groups determined by these stopping lines. We consider

$$\mathcal{T}_n = \left\{ |v| \le n \ : \ \exists |x| = n, \ x \ge v, \ \max_{y<v} X_y \ll b_n, \ X_v \approx b_n \right\},$$

with a rigorous definition given in (5.1). One then approximates $\mathbf{T}_n$ via

$$\sum_{v \in \mathcal{T}_n} E_{n-|v|}(v) \delta_{X_v}, \qquad E_n(v) = \# \left\{ |x| = n + |v| \ : \ x \ge v \right\}.$$

The main advantage of the above representation is that $\{E_{n-|v|}\}_{v \in \mathcal{T}_n}$ are independent given $\mathcal{T}_n$. Now one shows two facts. Firstly,

$$\sum_{v \in \mathcal{T}_n} \delta_{(X_v - b_n)/a_n} \Rightarrow \sum_k \delta_{\ell_k - \log(vW)},$$

where $\sum_k \delta_{\ell_k}$ is a Poisson point process with intensity $e^{-x}\mathrm{d}x$. Secondly,

$$\{E_{n-|v|}(v)\}_{v \in \mathcal{T}_n} \Rightarrow \{A_k\}_{k \in \mathbb{N}}.$$

This in turn allows us to infer that $\mathfrak{S}_{a_n^{-1}} \mathfrak{T}_{-b_n} \mathbf{T}_n \Rightarrow \mathbf{V}$ which further implies $\mathfrak{S}_{a_n^{-1}} \mathfrak{T}_{-b_n} \mathbf{V}_n \Rightarrow \mathbf{V}$. To the best of our knowledge this is a first instance of an application of stopping-line argument in the study of a BRW with heavy tailed displacements. We believe that this approach is robust and allows for an efficient analysis of a BRW in the presence of heavy tails in other cases.

The proof of Theorem 3.6 goes along the same lines as the arguments used for Theorem 3.2. It turns out that the stopping line argument can be used in this case as well with the stopping line of the form

$$\mathcal{S}_n = \Big\{ |v| \le n \ : \ \exists |x| = n, \ x \ge v, \ \max_{y<v} L(X_y) \ll n \log m, \ L(X_v) \approx n \log m \Big\}.$$

4. Random walk estimates

In this section, we present large deviation estimates for random walks with steps distributed as in Assumption 3.1 or Assumption 3.5. We begin with some auxiliary estimates on the truncated exponential moments for random variables with logarithmically slowly varying tails. These estimates are later used to establish deviation estimates for the corresponding random walk. In the next two lemmas, we assume that

$$\mathbb{P}[X > t] = e^{-L(t)},$$

for a slowly varying function $L(\cdot)$.

Lemma 4.1. *Suppose that for some $\xi \in (0,1)$, the function $t \mapsto t^{-\xi} L(t)$ is eventually decreasing. Then for any $\gamma \in (0,1)$ and $y > 0$ one has*

$$\mathbb{E}\left[e^{\gamma L(y) X / y} \mathbb{1}_{\{X \le y\}} \right] \le 1 + (1 + o(1)) L(y)^{(1 - 1/\xi)/2}.$$

Proof: Recall from the assumption before the lemma. Write $s = \gamma L(y)/y$. Firstly note that we have a simple bound

$$\mathbb{E}\left[e^{sX} \mathbb{1}_{\{X < 0\}} \right] \le \mathbb{P}\left[X < 0 \right].$$

Secondly, by integrating by parts we get

$$\mathbb{E}\left[e^{sX}\mathbb{1}_{\{X\in[0,y]\}}\right] = \int_0^y se^{st}\mathbb{P}[X>t]\mathrm{d}t + \mathbb{P}[X\geq 0] - e^{sy}\mathbb{P}[X>y]$$
$$\leq \int_0^y se^{st}\mathbb{P}[X>t]\mathrm{d}t + \mathbb{P}[X\geq 0].$$

For the integral we have

$$\int_0^y se^{st}\mathbb{P}[X>t]\mathrm{d}t = \int_0^1 \gamma L(y)e^{\gamma tL(y)-L(ty)}\mathrm{d}t.$$

Put $\varepsilon=(1/\xi-1)/2$ and take $m_y=L(y)^{-1-\epsilon}$ and $M_y=L(y/L(y)^{1+\varepsilon})/L(y)$. We can write the integral on the right-hand side of the last display as a sum of integrals over the intervals $(0,m_y)$, $[m_y,M_y)$ and $[M_y,1)$. The first one is easily bounded above by $(1+o(1))L(y)^{-\varepsilon}$. The second one can be estimated above by

$$L(y)\exp\{(\gamma-1)L(y/L(y)^{1+\varepsilon})\}.$$

Note that under the proviso concerning the monotonicity, for sufficiently large y,

$$L(y/L(y)^{1+\varepsilon})\geq L(y)^{(1-\xi)/2}.$$

Finally, for the third interval we use the above inequality in combination with the estimate $L(ty)\geq t^{\xi}L(y)\geq tL(y)$ for $t\in(0,1)$. In this way we derive the following inequality

$$\int_{M_y}^1 L(y)e^{\gamma tL(y)-L(ty)}\mathrm{d}t \leq \int_{M_y}^1 L(y)e^{(\gamma-1)tL(y)}\mathrm{d}t \leq (1-\gamma)^{-1}e^{(\gamma-1)L(y)^{(1-\xi)/2}}.$$

Combining all the above estimates yields our claim. □

Lemma 4.2. *Let Assumption 3.1 be in force. Then for any $\gamma\in(0,1)$ there exists x sufficiently large such that for any $y>x/2$ and $z\leq x/2$ one has*

$$\mathbb{E}\left[e^{\gamma L(y)X/y}\mathbb{1}_{\{X\in(y,x-z]\}}\right] \leq \mathrm{c}e^{-L(y)}e^{(\gamma L(y)/y-L'(y)/3)(x-z)+yL'(y)/3} + e^{(\gamma-1)L(y)}.$$

Proof: Put $s=\gamma L(y)/y$. By yet another appeal to the integration by parts formula,

$$\mathbb{E}\left[e^{sX}\mathbb{1}_{X\in(y,x-z]}\right] = \int_y^{x-z} se^{st}\mathbb{P}[X>t]\mathrm{d}t + e^{sy}\mathbb{P}[X>y]$$
$$-e^{s(x-z)}\mathbb{P}[X>x-z] \leq \int_y^{x-z} se^{st}\mathbb{P}[X>t]\mathrm{d}t + e^{sy}\mathbb{P}[X>y]. \quad (4.1)$$

The second term present in the last display is equal to $\exp\{(\gamma-1)L(y)\}$. To estimate the integral, we recall that $s=\gamma L(y)/y$ and we write

$$\int_y^{x-z} se^{st}\mathbb{P}[X>t]\mathrm{d}t = \int_y^{x-z} se^{st}e^{-L(t)}\mathrm{d}t = \int_y^{x-z}\gamma\frac{L(y)}{y}e^{\gamma L(y)t/y-L(t)}\mathrm{d}t.$$

By the mean value theorem and regular variation of L', for sufficiently large x,

$$L(t)-L(y)=(t-y)L'(\theta_t)\geq(t-y)L'(y)/3, \qquad t\in(y,x-z),\ \theta_t\in(y,t).$$

Therefore

$$\int_y^{x-z}\gamma\frac{L(y)}{y}e^{\gamma L(y)t/y-L(t)}\mathrm{d}t \leq \mathrm{c}e^{-L(y)}e^{(\gamma L(y)/y-L'(y)/3)(x-z)+yL'(y)/3}.$$

Inserting the last equation into (4.1) completes the proof. □

4.1. *The suplogarithmic case.* Let $\{S_n\}_{n\in\mathbb{N}}$ be a random walk generated by X. That is, $S_0 = 0$ and for $n \geq 1$, $S_n = X_1 + X_2 + \ldots + X_n$, where $\{X_k\}_{k\in\mathbb{N}}$ are iid copies of X with a law satisfying Assumption 3.1. Let

$$N_n = \max_{k\leq n} X_k.$$

In what follows, we suppose that Assumption 3.1 is in force. Write

$$x_n = x_n(K) = b_n + Ka_n, \qquad y_n = (1-\delta)b_n,$$

where $K \in \mathbb{R}$, $\delta \in (0,1)$ are fixed and

$$b_n = \inf\{t \geq 0 \,:\, \mathbb{P}[X > t] \leq m^{-n}\}, \qquad a_n = 1/L'(b_n).$$

We recall that for two function $f(n)$ and $g(n)$ we write $f(n) = o(g(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$.

Lemma 4.3. *Suppose that for some $\xi \in (0,1)$, the function $t \mapsto t^{-\xi}L(t)$ is eventually decreasing. Then, for any $K \in \mathbb{R}$ and $\delta \in (0,1)$,*

$$\mathbb{P}[S_n > x_n,\ N_n \leq y_n] = o\left(m^{-n}\right).$$

Proof: Take $s = \gamma L(y_n)/y_n$ for some $\gamma \in (0,1)$ and consider the following upper bound

$$\mathbb{P}[S_n > x_n,\ N_n \leq y_n] \leq e^{-sx_n}\mathbb{E}\left[e^{sX}\mathbb{1}_{\{X\leq y_n\}}\right]^n$$

which follows form Chebyshev's inequality. An appeal to Lemma 4.1 yields that for some universal constant c,

$$\mathbb{E}\left[e^{sX}\mathbb{1}_{\{X\leq y_n\}}\right] \leq 1 + cn^{(1-1/\xi)/2}. \tag{4.2}$$

We use the inequality $1 - x \leq e^{-x}$, $x \in \mathbb{R}$, to conclude that

$$\mathbb{P}[S_n > x_n,\ N_n \leq y_n] \leq \exp\left\{-\gamma L(y_n)x_n/y_n + Cn^{(3-1/\xi)/2}\right\}. \tag{4.3}$$

Note that

$$\lim_{n\to\infty} \gamma(L(y_n)x_n)/(y_nL(b_n)) = \gamma/(1-\delta)$$

so taking γ sufficiently close to 1 we can get for sufficiently large n,

$$\mathbb{P}[S_n > x_n,\ N_n \leq y_n] \leq \exp\left\{-n\log m/(1-\delta/2) + Cn^{(3-1/\xi)/2}\right\}$$

which yields our claim. □

Let

$$z_n = \frac{Tb_n\log n}{L(b_n)}$$

for some (sufficiently large) constant $T > 0$ that depends on $\delta \in (0,1)$. Note that under Assumption 3.1, $z_n = o(b_n)$ and $a_n = o(z_n)$.

Lemma 4.4. *Let Assumption 3.1 be in force. Then there exists sufficiently small $\delta \in (0,1)$ and sufficiently large $T > 0$ such that for any value of $K \in \mathbb{R}$,*

$$\mathbb{P}[S_{n-1} + X > x_n,\ N_{n-1} \leq y_n,\ X \in (y_n, x_n - z_n]] = o\left(nm^{-n}\right).$$

Proof: Take $s = \gamma L(y_n)/y_n$ for some $\gamma \in (0,1)$. Use the following simple estimate:

$$\mathbb{P}[S_{n-1} + X > x_n,\ N_{n-1} \leq y_n,\ X \in (y_n, x_n - z_n]] \leq$$
$$\exp\{-\gamma L(y_n)x_n/y_n\}\mathbb{E}\left[e^{\gamma L(y_n)X/y_n}\mathbb{1}_{\{X\in(y_n,x_n-z_n]\}}\right]\mathbb{E}\left[e^{sX}\mathbb{1}_{\{X\leq y_n\}}\right]^n.$$

The last factor is already bounded in the proof of Lemma 4.3. To bound the second one we appeal to Lemma 4.2. Using the fact that

$$L(y_n) = L(b_n) - \delta b_nL'(\theta_n).$$

for some $\theta_n \in ((1-\delta)b_n, x_n)$ combined with regular variation of L' we arrive at

$$\mathbb{P}[S_{n-1} + X > x_n,\ N_{n-1} \le y_n,\ X \in (y_n, x_n - z_n]] \le$$
$$\mathrm{c}\exp\left\{-L(b_n) + (\delta/(1-\delta) - (1+\delta)/3)\, b_n L'(b_n) - \gamma L(y_n) z_n/(2y_n)\right\}.$$

If δ is small enough the last bound is smaller than

$$\mathrm{c}\exp\{-L(b_n) - \gamma L(y_n) z_n/(2y_n)\}$$

which is $o(nm^{-n})$ provided that T is sufficiently large. This implies the claim. $\square$

5. Proofs of main results

In this section, we present the proofs of our main results. The arguments used in the corresponding proofs of the main results differ slightly in technical aspects, but they are based on the same idea. Namely, we approximate $\mathbf{V}_n$ using the point process

$$\mathbf{T}_n = \sum_{|x|=n} \delta_{T(x)}, \quad T(x) = \max_{y \in (\emptyset, x]} X_y.$$

Next, we show that $\mathbf{T}_n$ converges to the desired limit.

5.1. *The suplogarithmic case.* We now present the proof of Theorem 3.2. Throughout this subsection, Assumption 2.1 and Assumption 3.1 remain in force. Recall that

$$b_n = \inf\{t \ge 0 \ :\ \mathbb{P}[X > t] \le m^{-n}\}, \qquad z_n = T b_n \log n / L(b_n)$$

and

$$y_n = (1-\delta) b_n, \quad a_n = 1/L'(b_n), \qquad x_n = x_n(K) = b_n + K a_n.$$

We first consider the following subsets of the set of particles in the n-th generation $\mathbb{T}_n$:

$$\mathcal{A}_n^{(1)} = \{x \in \mathbb{T}_n \ :\ T(x) \le y_n\},$$
$$\mathcal{A}_n^{(2)} = \{x \in \mathbb{T}_n \ :\ \exists v_1, v_2 \le x,\ v_1 \ne v_2,\ \min\{X_{v_1}, X_{v_2}\} \ge y_n\},$$
$$\mathcal{A}_n^{(3)} = \{x \in \mathbb{T}_n \setminus (\mathcal{A}_n^{(1)} \cup \mathcal{A}_n^{(2)}) \ :\ T(x) \le b_n - z_n\}.$$

We aim to show that the particles in $\mathcal{A}_n = \mathcal{A}_n^{(1)} \cup \mathcal{A}_n^{(2)} \cup \mathcal{A}_n^{(3)}$ do not contribute to the limit.

Lemma 5.1. *Let*

$$\mathbf{V}_n^{\mathcal{A}_n} = \sum_{x \in \mathcal{A}_n} \delta_{V(x)}.$$

Then, under Assumptions 2.1 and 3.1,

$$\mathfrak{S}_{a_n^{-1}} \mathfrak{T}_{-b_n} \mathbf{V}_n^{\mathcal{A}_n} \Rightarrow o,$$

weakly in $\mathbb{M}(\overline{\mathbb{R}})$, where o denotes the null measure.

Proof: It suffices to show that for any $K \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{V}_n^{\mathcal{A}_n}[[x_n(K), +\infty]] = 0$$

in probability. We appeal to the decomposition $\mathcal{A}_n = \mathcal{A}_n^{(1)} \cup \mathcal{A}_n^{(2)} \cup \mathcal{A}_n^{(3)}$ and treat the sets $\mathcal{A}_n^{(j)}$ one by one. For the first one, we write

$$\mathbb{P}[\exists x \in \mathcal{A}_n^{(1)} \ :\ V(x) > x_n] \le m^n \mathbb{P}[S_n > x_n,\ N_n \le y_n].$$

Invoke Lemma 4.3 to get

$$\mathbb{P}[S_n > x_n,\ N_n \le y_n] = o(m^{-n}).$$

To treat $\mathcal{A}_n^{(2)}$, simply note that

$$\mathbb{P}[\mathcal{A}_n^{(2)} \neq \emptyset] \leq m^n e^{-2L(\delta b_n)} = o(1).$$

Finally, for $\mathcal{A}_n^{(3)}$, we argue similarly to $\mathcal{A}_n^{(2)}$, but instead of using Lemma 4.3, we use Lemma 4.4. □

Lemma 5.2. *Suppose that Assumptions 2.1 and 3.1 hold. Let*

$$M_n^{(A)} = \max\{|V(x) - T(x)| \,:\, x \in \mathbb{T}_n \setminus \mathcal{A}_n\}.$$

Then, for any $\delta \in (0,1)$ and $T > 0$, we have

$$\lim_{n\to\infty} M_n^{(A)}/a_n = 0$$

in probability.

Proof: Take $\varepsilon > 0$ and write

$$\mathbb{P}\left[M_n^{(A)} > \varepsilon a_n\right] \leq n m^n \mathbb{P}[X > b_n - z_n]\mathbb{P}[|S_{n-1}| > \varepsilon a_n].$$

The logarithm of the second term can be expressed as follows:

$$\log \mathbb{P}[X > b_n - z_n] = L(b_n) - z_n L'(\theta_n)$$

for some $\theta_n \in (b_n - z_n, b_n)$. Since $z_n L'(\theta_n) = o(\log n)$, it is sufficient to consider a generous upper bound

$$\mathbb{P}[|S_{n-1}| > \varepsilon a_n] \leq n\mathbb{P}[|X_1| > \varepsilon a_n/n] \leq n\left(\frac{\varepsilon a_n}{n}\right)^{-\varepsilon}.$$

Since a_n grows faster than any polynomial, this yields

$$\mathbb{P}\left[M_n^{(A)} > \varepsilon a_n\right] \leq C/n$$

and secures the claim. □

Using the above lemma, we can approximate $\mathbf{V}_n$ via $\mathbf{T}_n$. We present the analysis of the latter, followed by an approximation lemma. Consider the stopping line

$$\mathcal{T}_n = \left\{|v| \leq n \,:\, \exists x \in \mathbb{T}_n \setminus \mathcal{A}_n,\, x \geq v,\, \max_{y<v} X_y \leq b_n - z_n,\, X_v > b_n - z_n\right\}. \tag{5.1}$$

For $v \in \mathbb{U}$ and $n \in \mathbb{N}$, consider

$$E_n(v) = \#\{|x| = n + |v| \,:\, x \geq v\}.$$

Then, for any $f \in C_c^+((-\infty,\infty])$, for sufficiently large n, on the event $\{\mathcal{A}_n^{(2)} = \emptyset\}$, we have

$$\sum_{|x|=n} f\left(\frac{T(x) - b_n}{a_n}\right) = \sum_{v\in\mathcal{T}_n} f\left(\frac{X_v - b_n}{a_n}\right) E_{n-|v|}(v).$$

Note that $E_{n-|v|}(v)$ denotes the number of descendants of the vertex v in the n-th generation. The sum on the right-hand side can be linked to yet another point process on $(-\infty,\infty] \times \mathbb{N}$, given by

$$\mathbf{N}_n = \sum_{x\in\mathcal{T}_n} \delta_{(X_x - b_n)/a_n} \otimes \delta_{n-|x|}.$$

To state a limiting result for the latter, define a measure on $\mathbb{N}$ by

$$\rho(\cdot) = \sum_{j=0}^{\infty} m^{-j}\delta_j(\cdot),$$

and consider a Poisson point process $\mathbf{N}$ on $\mathbb{R} \times \mathbb{N}$ given by

$$\mathbf{N} = \sum_k \delta_{\ell_k - \log(aW)} \otimes \delta_{\iota_k},$$

where $\{(\ell_k, \iota_k)\}_k$ is a Poisson point process with intensity $e^{-x}\,\mathrm{d}x\,\rho(\mathrm{d}j)$. Then, $\mathbf{N}$ has a Laplace functional of the form

$$\mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}(\mathrm{d}x,\mathrm{d}j)}\right] = \mathbb{E}\left[\exp\left\{-W\int\left(1-e^{-f(x,l)}\right)e^{-x}\,\mathrm{d}x\,\rho(\mathrm{d}l)\right\}\right].$$

Proposition 5.3. *Under Assumption 2.1 and Assumption 3.1, we have*

$$\mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}_n(\mathrm{d}x,\mathrm{d}j)}\right] \to \mathbb{E}\left[\exp\left\{-W\int\left(1-e^{-f(x,l)}\right)e^{-x}\,\mathrm{d}x\,\rho(\mathrm{d}l)\right\}\right]$$

for any continuous, nonnegative function $f\colon(-\infty,\infty]\times\mathbb{N}$ such that $f(x,i)=0$ for sufficiently large $|x|$. In particular, $\mathbf{N}_n$ converges weakly in $\mathbb{M}((-\infty,\infty]\times\mathbb{N})$ to $\mathbf{N}$.

Proof: For a function f as in the statement, we have

$$\mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}_n(\mathrm{d}x,\mathrm{d}j)}\right] = \mathbb{E}\left[\exp\left\{-\sum_{|v|\le n} f\left(\frac{X_v-b_n}{a_n}, n-|v|\right)\right\}\right].$$

Conditioning on the Galton-Watson process, we get

$$\mathbb{E}\left[\prod_{k=1}^{n}\mathbb{E}\left[\exp\left\{-f\left(\frac{X-b_n}{a_n}, n-k\right)\right\}\right]^{Z_k}\right].$$

Using the facts that

$$m^n\mathbb{P}\left[\frac{X-b_n}{a_n}\in\mathrm{d}x\right]\to^{\nu} e^{-x}\,\mathrm{d}x$$

and that for any fixed l,

$$m^{-n}Z_{n-l}\to m^{l}W,$$

we infer that for any fixed l,

$$\mathbb{E}\left[\exp\left\{-f\left(\frac{X-b_n}{a_n}, l\right)\right\}\right]^{Z_{n-l}} \to \exp\left\{-m^{-l}W\int\left(1-e^{-f(x,l)}\right)e^{-x}\,\mathrm{d}x\right\},$$

since $f(\cdot,l)\in C_c^{+}((-\infty,\infty])$. Combining this with the assumption concerning the support of f and using a standard approximation of an infinite product by finite ones, we get

$$\begin{aligned}\mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}_n(\mathrm{d}x,\mathrm{d}j)}\right] &= \mathbb{E}\left[\prod_{k=1}^{n}\mathbb{E}\left[\exp\left\{-f\left(\frac{X-b_n}{a_n}, n-k\right)\right\}\right]^{Z_k}\right]\\ &\to \exp\left\{-\sum_{l=0}^{\infty} m^{-l}W\int\left(1-e^{-f(x,l)}\right)e^{-x}\,\mathrm{d}x\right\}\end{aligned}$$

as n tends to infinity. □

We use the last proposition to prove a limit theorem for

$$\mathbf{N}_n^{*} = \sum_{x\in\mathcal{T}_n}\delta_{(X_x-b_n)/a_n}\otimes\delta_{E_{n-|x|}(x)}.$$

The proof relies on the observation that the random variables $\{E_{n-|x|}(x)\}_{x\in\mathcal{T}_n}$ are independent and independent of $\{X_x\}_{x\in\mathcal{T}_n}$ given $\mathcal{T}_n$. Furthermore,

$$\mathbb{P}[E_{n-|x|}(x)\in\cdot\,|\,x\in\mathcal{T}_n] = \mathbb{P}[Z_{n-k}\in\cdot\,|\,Z_{n-k}>0]\quad\text{for } |x|=k.$$

If we consider iid copies $\{Z^{(v)}\}_{v\in\mathbb{U}}$ of the underlying Galton-Watson process $Z=\{Z_k\}_{k\in\mathbb{N}}$, we can conclude that

$$\mathbf{N}_n^{*} = \sum_{x\in\mathcal{T}_n}\delta_{(X_x-b_n)/a_n}\otimes\delta_{E_{n-|x|}(x)} \overset{d}{=} \sum_{x\in\mathcal{T}_n}\delta_{(X_x-b_n)/a_n}\otimes\delta_{Z^{(x)}_{n-|x|}}.$$

We use this representation in the proof of the next proposition. Define

$$\rho^*(\mathrm{d}k) = \sum_{j=0}^{\infty} m^{-j}\mathbb{P}[Z_j \in \mathrm{d}k],$$

and a Poisson point process $\mathbf{N}^*$ on $\mathbb{R}\times\mathbb{N}$ given by

$$\mathbf{N}^* = \sum_k \delta_{\ell_k - \log(aW)} \otimes \delta_{\iota_k^*},$$

where $\{(\ell_k, \iota_k^*)\}_k$ is a Poisson point process with intensity $e^{-x}\,\mathrm{d}x\,\rho^*(\mathrm{d}j)$. Then, $\mathbf{N}^*$ has a Laplace functional of the form

$$\mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}^*(\mathrm{d}x,\mathrm{d}j)}\right] = \mathbb{E}\left[\exp\left\{-W\int\left(1-e^{-f(x,l)}\right)e^{-x}\,\mathrm{d}x\,\rho^*(\mathrm{d}l)\right\}\right].$$

Proposition 5.4. *Under Assumption 2.1 and Assumption 3.1, we have*

$$\mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}_n^*(\mathrm{d}x,\mathrm{d}j)}\right] \to \mathbb{E}\left[e^{-\int f(x,j)\mathbf{N}^*(\mathrm{d}x,\mathrm{d}j)}\right]$$

for any continuous, nonnegative function $f\colon(-\infty,\infty]\times\mathbb{N}$ such that $f(x,i)=0$ for sufficiently large $|x|$. In particular, $\mathbf{N}_n^*$ converges weakly in $\mathbb{M}((-\infty,\infty]\times\mathbb{N})$ to $\mathbf{N}^*$.

Proof: We have

$$\int f(x,j)\mathbf{N}_n^*(\mathrm{d}x,\mathrm{d}j) \overset{d}{=} \sum_{x\in\mathcal{T}_n} f\left(\frac{X_x-b_n}{a_n}, Z_{n-|x|}^{(x)}\right).$$

On the event $\{\mathcal{A}_n^{(2)}=\emptyset\}$, for sufficiently large n,

$$\sum_{x\in\mathcal{T}_n} f\left(\frac{X_x-b_n}{a_n}, Z_{n-|x|}^{(x)}\right) = \sum_{|x|\le n} f\left(\frac{X_x-b_n}{a_n}, Z_{n-|x|}^{(x)}\right).$$

With this last representation in hand, we can consider

$$f^*(x,j) = -\log\mathbb{E}[\exp\{-f(x,Z_j)\}]$$

and write, by conditioning on the X_x's,

$$\mathbb{E}\left[\exp\left\{-\int f(x,j)\mathbf{N}_n^*(\mathrm{d}x,\mathrm{d}j)\right\}\right] = \mathbb{E}\left[\exp\left\{-\int f^*(x,j)\mathbf{N}_n(\mathrm{d}x,\mathrm{d}j)\right\}\right].$$

Since the function f^* satisfies the hypothesis of Proposition 5.3, we conclude the proof. □

We are now ready to prove the main proposition.

Proposition 5.5. *Under Assumption 2.1 and Assumption 3.1, we have*

$$\mathfrak{S}_{a_n^{-1}}\mathfrak{T}_{-b_n}\mathbf{T}_n \Rightarrow \mathbf{V},$$

where $\mathbf{V}$ is a point process given in (2.6).

Proof: Fix $f\in C_c^+(-\infty,\infty]$. As noted previously, for sufficiently large n, on the event $\{\mathcal{A}_n^{(2)}=\emptyset\}$,

$$\sum_{|x|=n} f\left(\frac{T(x)-b_n}{a_n}\right) = \sum_{v\in\mathcal{T}_n} f\left(\frac{X_v-b_n}{a_n}\right)E_{n-|v|}(v).$$

The last sum is equal to

$$\int f(x)j\,\mathbf{N}_n^*(\mathrm{d}x,\mathrm{d}j).$$

By an appeal to Proposition 5.4,

$$\int f(x)j\,\mathbf{N}_n^*(\mathrm{d}x,\mathrm{d}j) \to^d \int f(x)j\,\mathbf{N}^*(\mathrm{d}x,\mathrm{d}j).$$

The result follows, since by construction

$$\int f(x)j\,\mathbf{N}^*(\mathrm{d}x,\mathrm{d}j) \overset{d}{=} \int f(x)\,\mathbf{V}(\mathrm{d}x).$$

□

We are now in a position to give a final touch to the proof of our first main result.

Proof of Theorem 3.2: At this point, it suffices to approximate $\mathbf{V}_n$ through $\mathbf{T}_n$. Fix $\varepsilon > 0$. Any function f in $C_c((-\infty,\infty])$ is uniformly continuous. Hence, we have

$$\omega_f(\delta) = \sup\{|f(x)-f(y)| \,:\, |x-y| < \delta\} \to 0$$

as $\delta \to 0^+$. Note that

$$\left|\int f(x)(\mathbf{V}_n - \mathbf{T}_n)(\mathrm{d}x)\right| \le \int f(x)\mathbf{V}_n^{(\mathcal{A})}(\mathrm{d}x) + \omega_f\left(M_n^{(\mathcal{A})}\right)\mathbf{T}_n\left[\operatorname{supp}(f) + \left(-M_n^{(\mathcal{A})}, M_n^{(\mathcal{A})}\right)\right].$$

By an appeal to Lemma 5.1, the first term vanishes. The second term also vanishes by Proposition 5.5 and Lemma 5.2.

□

5.2. *The sublogarithmic case.* The arguments follow along the lines of the proof of Theorem 3.2. Therefore, we omit some of the simpler steps. Firstly, recall the generalized inverse of L, given by

$$L^{-1}(y) = \inf\{s \in \mathbb{R} \,:\, L(s) \ge y\}.$$

Since L is right-continuous, $L(x) \ge y$ if and only if $x \ge L^{-1}(y)$. Consider

$$\mathcal{B}_n = \{x \in \mathbb{T}_n \,:\, L(T(x)) \le 3n(\log m)/4\}.$$

We first show that the positions of the particles in $\mathcal{B}_n$ do not contribute to the limit of $\mathbf{V}_n$.

Lemma 5.6. *Under Assumption 2.1 and Assumption 3.5, we have*

$$\sum_{x\in\mathcal{B}_n} \delta_{L(V(x))-n\log m} \Rightarrow o$$

in $\mathbb{M}(\overline{\mathbb{R}})$.

Proof: As in the proof of Lemma 5.1, it is sufficient to show that for any $C \in \mathbb{R}$,

$$\mathbb{P}[L(S_n) > n\log m + C,\ L(N_n) \le 3n(\log m)/4] = o(m^{-n}).$$

We once again apply Lemma 5.1. Denote $\ell_n = L^{-1}(3n(\log m)/4)$ and $\varpi_n = L^{-1}(n\log m + C)$. Note that $\ell_n = o(\varpi_n)$. Consider $s = n\log m/(2\ell_n)$ and write

$$\mathbb{P}[L(S_n) > n\log m + C,\ L(N_n) \le 3n(\log m)/4] \le e^{-s\varpi_n}\mathbb{E}\left[e^{sX}\mathbb{1}_{\{X\le\ell_n\}}\right]^n. \tag{5.2}$$

We now present a suitable bound for the integral that appears on the right-hand side. First, note that

$$\mathbb{E}\left[e^{sX}\mathbb{1}_{\{X\le\ell_n/n\}}\right] \le m.$$

For any $\varepsilon > 0$, consider

$$\begin{aligned}\mathbb{E}\left[e^{sX}\mathbb{1}_{\{\ell_n/n\le x\le\varepsilon\ell_n\}}\right] &\le \int_{\ell_n/n}^{\varepsilon\ell_n} se^{sy}\mathbb{P}[X>y]\mathrm{d}y + e\mathbb{P}[X>\ell_n/n] \\ &\le \int_{1/n}^{\varepsilon} n\log m e^{s\ell_n t - L(\ell_n t)}\mathrm{d}t + O(1).\end{aligned}$$

Using Potter bounds,

$$L(\ell_n t) \geq t^{\delta} L(\ell_n) \geq \varepsilon^{\delta-1} t L(\ell_n)$$

for $t \in [1/n, \varepsilon]$. If we plug this into the integral, we get, with sufficiently small ε,

$$\mathbb{E}\left[e^{sX} \mathbb{1}_{\{\ell_n/n \leq x \leq \varepsilon\ell_n\}}\right] = o(1).$$

Finally, for the last part, write

$$\mathbb{E}\left[e^{sX} \mathbb{1}_{\{\varepsilon\ell_n \leq x \leq \ell_n\}}\right] \leq \int_{\varepsilon\ell_n}^{\ell_n} s e^{sy} \mathbb{P}[X > y] \mathrm{d}y + e^{\varepsilon s \ell_n} \mathbb{P}[X > \varepsilon\ell_n] = \int_{\varepsilon}^{1} n \log m e^{s\ell_n t - L(\ell_n t)} \mathrm{d}t + o(1),$$

and the last integral converges to 0 exponentially fast due to the uniform slow variation of L. This yields

$$\mathbb{E}\left[e^{sX} \mathbb{1}_{\{x \leq \ell_n\}}\right] \leq C.$$

Returning to (5.2), we get

$$\mathbb{P}[L(S_n) > n \log m + C,\ L(N_n) \leq n(\log m)/2] \leq \exp\left\{-n \log m \frac{\varpi_n}{2\ell_n} + Cn\right\}.$$

Since $\varpi_n/\ell_n \to \infty$, our claim follows.

□

We now state the result that allows us to approximate $\mathbf{V}_n$ via $\mathbf{T}_n$. Define

$$M_n^{(B)} = \max\{L(V(x)) - L(T(x)) \ :\ |x| = n,\ L(T(x)) > 3n(\log m)/4\}.$$

Lemma 5.7. *Let Assumption 2.1 and Assumption 3.5 be satisfied. Then, as $n \to \infty$,*

$$M_n^{(B)} \to 0$$

in probability.

Proof: We recall that $\{S_n\}_{n\in\mathbb{N}}$ is a random walk generated by X and $N_n = \max_{k\leq n} X_k$. Using Potter bounds, for any $\delta > 0$, $x \in \mathbb{R}$ and $y > 0$,

$$L(x+y) - L(x) \leq CL(x)\left(\left(1 + \frac{y}{x}\right)^{\delta} - 1\right).$$

The right-hand side, by the Bernoulli inequality, is further bounded by $CL(x)\delta\frac{y}{x}$. Thus, for $S_n^o = S_n - N_n$, we have, for sufficiently large n,

$$\{L(S_n) - L(N_n) > \varepsilon,\ N_n > L^{-1}(3n(\log m)/4)\} \subseteq \left\{S_n^o > \frac{\varepsilon s_1}{\delta \log s_1}\right\},$$

where $s_1 = L^{-1}(3n(\log m)/4)$. Now we show that if $L(x) = o(\log x)$, then

$$\lim_{n\to\infty} \frac{s_1/\log s_1}{s_2} = \infty \qquad \text{where } s_2 = L^{-1}(2n(\log m)/3). \tag{5.3}$$

By the Karamata representation,

$$L(t) = c(t) \exp\left\{\int_1^t \frac{\varepsilon(x)}{x} \mathrm{d}x\right\}$$

for $c(t) \to c > 0$ and $\varepsilon(x) \to 0$. Since $L(x) = o(\log x)$, we must have

$$\varepsilon(z) \leq \frac{1}{\log z}$$

for sufficiently large z. Then,

$$\frac{3n(\log m)}{4} \leq L(s_1) = L\left(\frac{s_2}{2}\right)\frac{c(s_1)}{c(s_2/2)}\exp\left\{\int_{s_2/2}^{s_1}\frac{\varepsilon(z)}{z}\mathrm{d}z\right\}$$
$$\leq L\left(\frac{s_2}{2}\right)\frac{c(s_1)}{c(s_2/2)}\frac{\log s_1}{\log(s_2/2)}.$$

This leads to

$$(1+o(1))\frac{9}{8} \leq \frac{\log s_1}{\log s_2},$$

which means that for sufficiently large n, by the monotonicity of the function $x \mapsto \frac{x}{\log x}$,

$$\frac{s_2^{1/8}}{9(\log s_2)/8} \leq \frac{s_1/\log s_1}{s_2}.$$

Since s_2 grows faster than exponentially in n, this secures (5.3). We infer that

$$\mathbb{P}\left[L(S_n) - L(N_n) > \varepsilon,\ N_n > L^{-1}(3n(\log m)/4)\right] \leq$$
$$\mathbb{P}\left[N_n > L^{-1}(3n(\log m)/4), S_n^o > nL^{-1}(2n(\log m)/3)\right] \leq Cn^2 m^{-n(2/3+3/4)}.$$

The lemma now follows by a first moment argument since for any $\varepsilon > 0$,

$$\mathbb{P}[M_n^{(B)} > \varepsilon] \leq m^n\mathbb{P}\left[L(S_n) - L(N_n) > \varepsilon,\ N_n > L^{-1}(3n(\log m)/4)\right] \leq n^2 m^{-5n/12}.$$

□

Proof of Theorem 3.6: We apply the same arguments as in the proof of our first result. This time, we consider the stopping line

$$\mathcal{S}_n = \left\{|v| \leq n\ :\ \exists x \in \mathbb{T}_n \setminus \mathcal{B}_n,\ x \geq v,\ \max_{y<v} L(X_y) \leq 3n(\log m)/4,\ L(X_v) > 3n(\log m)/4\right\}.$$

Now, for any compactly supported and continuous f, with high probability,

$$\sum_{|x|=n} f\left(L(T(x)) - n\log m\right) = \sum_{v\in\mathcal{T}_n} f\left(L(X_v) - n\log m\right) E_{n-|v|}(v).$$

One then uses the same procedure, taking into account that

$$m^n\mathbb{P}[L(X) - n\log m \in \mathrm{d}x] \to^{\nu} e^{-x}\,\mathrm{d}x.$$

□

Acknowledgement

We are thankful to the anonymous referee for his comments and suggestions which improved the readability of the paper.

References

Aïdékon, E. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, **41** (3A), 1362–1426 (2013). MR3098680.

Alsmeyer, G., Biggins, J. D., and Meiners, M. The functional equation of the smoothing transform. *Ann. Probab.*, **40** (5), 2069–2105 (2012). MR3025711.

Athreya, K. B. and Ney, P. E. *Branching processes.* Die Grundlehren der mathematischen Wissenschaften, Band 196. Springer-Verlag, New York-Heidelberg (1972). MR373040.

Barral, J., Kupiainen, A., Nikula, M., Saksman, E., and Webb, C. Critical Mandelbrot cascades. *Comm. Math. Phys.*, **325** (2), 685–711 (2014). MR3148099.

Bhattacharya, A., Hazra, R. S., and Roy, P. Point process convergence for branching random walks with regularly varying steps. *Ann. Inst. Henri Poincaré Probab. Stat.*, **53** (2), 802–818 (2017). MR3634275.

Biggins, J. D. The first- and last-birth problems for a multitype age-dependent branching process. *Advances in Appl. Probability*, **8** (3), 446–459 (1976). MR420890.

Bovier, A. *Gaussian processes on trees. From spin glasses to branching Brownian motion*, volume 163 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2017). ISBN 978-1-107-16049-1. MR3618123.

Brunet, E. and Derrida, B. A branching random walk seen from the tip. *J. Stat. Phys.*, **143** (3), 420–446 (2011). MR2799946.

Dadoun, B. Asymptotics of self-similar growth-fragmentation processes. *Electron. J. Probab.*, **22**, Paper No. 27, 30 (2017). MR3629871.

Darling, D. A. The influence of the maximum term in the addition of independent random variables. *Trans. Amer. Math. Soc.*, **73**, 95–107 (1952). MR48726.

Durrett, R. Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete*, **62** (2), 165–170 (1983). MR688983.

Dyszewski, P. and Gantert, N. The extremal point process for branching random walk with stretched exponential displacements. *ArXiv Mathematics e-prints* (2022). arXiv: 2212.06639.

Fisher, R. A. The wave of advance of advantageous genes. *Annals of Eugenics*, **7** (4), 355–369 (1937). DOI: 10.1111/j.1469-1809.1937.tb02153.x.

Hammersley, J. M. Postulates for subadditive processes. *Ann. Probability*, **2**, 652–680 (1974). MR370721.

Kingman, J. F. C. The first birth problem for an age-dependent branching process. *Ann. Probability*, **3** (5), 790–801 (1975). MR400438.

Kolmogorov, A. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Moscow Univ. Bull. Math.*, **1**, 1–25 (1937).

Kyprianou, A., Lane, F., and Mörters, P. The largest fragment of a homogeneous fragmentation process. *J. Stat. Phys.*, **166** (5), 1226–1246 (2017). MR3610212.

Liu, Q. On generalized multiplicative cascades. *Stochastic Process. Appl.*, **86** (2), 263–286 (2000). MR1741808.

Maillard, P. A note on stable point processes occurring in branching Brownian motion. *Electron. Commun. Probab.*, **18**, no. 5, 9 (2013). MR3019668.

McKean, H. P. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.*, **28** (3), 323–331 (1975). MR400428.

Resnick, S. I. *Heavy-tail phenomena. Probabilistic and statistical modeling.* Springer Series in Operations Research and Financial Engineering. Springer, New York (2007). ISBN 978-0-387-24272-9; 0-387-24272-4. MR2271424.

Resnick, S. I. *Extreme values, regular variation and point processes.* Springer Series in Operations Research and Financial Engineering. Springer, New York (2008). ISBN 978-0-387-75952-4. MR2364939.

Shi, Z. *Branching random walks*, volume 2151 of *Lecture Notes in Mathematics*. Springer, Cham (2015). ISBN 978-3-319-25371-8; 978-3-319-25372-5. Lecture notes from the 42nd Probability Summer School held in Saint Flour, 2012. MR3444654.

Subag, E. and Zeitouni, O. Freezing and decorated Poisson point processes. *Comm. Math. Phys.*, **337** (1), 55–92 (2015). MR3324155.