Output Feedback Model Reference Adaptive Control of Piecewise Affine Systems With Parameter Convergence Analysis

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***Abstract*—In this article, the output feedback-based direct model reference adaptive control of piecewise affine systems and its parameter convergence are investigated. Under the slow switching assumption, it is shown that all the closed-loop signals are bounded and the output tracking error is small in the mean square sense. Built upon this result, the estimation error of controller parameters is proved to converge to a residual set if the input signal is sufficiently rich. The relationship between the size of this residual set and the switching frequency is established. Moreover, the convergence of the estimated controller parameters to their nominal values can be achieved for a certain subsystem given that this subsystem is activated for infinitely long time. Simulation results validate the effectiveness of the proposed approach.**

***Index Terms*—Adaptive control, output feedback, piecewise affine (PWA) system.**

I. Introduction

The systems in the real world are mostly hybrid and highly nonlinear. The mixture of the continuous states and discrete modes as well as the system nonlinearity complicates the analysis and control design. Piecewise affine (PWA) system is a powerful tool to model the hybrid systems and approximate nonlinearity. The state-input space of a PWA system is partitioned into convex polytopes, in which the subsystem dynamics are linear. The hyperplanes, which determine how the state-input space is partitioned, characterize the switching mechanism of the hybrid systems.

Since proposed, PWA systems have attracted significant interest. Numerous applications in various areas have been explored like in [1] and [2]. Meanwhile, exploration in theory has also made great progress such as analysis of controllability and observability [3], model predictive control [4], etc. Recent works introduce the adaptation mechanism in PWA systems to counter uncertainties and disturbances. One aspect is the adaptive identification. Due to the hybrid nature of PWA systems, both the switching hyperplane estimation and the subsystem parameter identification have been explored in [5] and [6], respectively. Another aspect lies in adaptive control.

Various adaptive control approaches of switched systems are proposed in the literature. Sang and Tao develop model reference adaptive control (MRAC) of piecewise linear (PWL) systems for state tracking [7] and output tracking [8], respectively. Compared with PWL systems, PWA systems are preferable because they require less prior knowledge about the systems. MRAC approaches of PWA systems for state tracking are studied by Bernardo *et al.* in [9] and Kersting and Buss in [10]. The state or output tracking errors reported in the given references are either asymptotic convergent or small in the mean square sense.

Apart from the tracking performance of the MRAC, studying the parameter convergence is also a topic of major interest [11]. Its importance is depicted in [12] and [13] by showing that a large parameter estimation error may result in bad transient behavior. In light of this, much effort has been devoted to the study of parameter convergence in MRAC of PWL/PWA systems, In [7] and [14], the controller parameter convergence of MRAC for PWL systems is proved under the persistently exciting (PE) condition. For the MRAC of PWA systems shown in [10], the controller and system parameters are proved to converge to the nominal values for direct and indirect cases with PE inputs, respectively.

Note that the referenced previous works require full state feedback. For the output feedback case, the MRAC of PWL systems for output tracking is studied in [15]. Nevertheless, the convergence of the controller parameters remains unexplored. To fill this gap, we investigate the output feedback MRAC for PWA counterparts with special focuses on the analysis of controller parameter convergence. It is a challenge to analyze the effect of the special controller structure for PWA systems on the excitation of the estimated parameters. Besides, the influence of the tracking error, as well as the switching behavior on the parameter convergence, needs to be evaluated. Our main contribution lies in the analysis of parameter convergence in direct MRAC for PWA systems with output feedback. To achieve this, we first extend the controller proposed in [15] to the context of PWA systems and prove that the tracking error is small in the mean square sense under slow switching. Based on this result, we prove that the controller parameter estimation error converges to a bounded set given a PE reference signal. We establish the relationship between the size of the set and the switching frequency. Finally, we show that the convergence of the controller parameters to the nominal values can be achieved in a special case where the trajectory is kept staying in one subsystem for infinitely long.

The rest of this article is structured as follows. The preliminaries and the problem formulation is presented in Section II. In Section III, our proposed control law is depicted. The tracking error as well as parameter convergence are investigated. The approach is validated by a numerical example presented in Section IV, and finally, Section V concludes this article.

II. Preliminaries and Problem Statement

This section gives the definition of the PWA systems. It is also revisited how to derive a PWA system based on the linearization of a nonlinear system at different operating points. The MRAC problem of PWA with output feedback is also formulated.

Consider the piecewise affine system with $s \in \mathbb{N}$ subsystems

$$\dot{x} = A_i x + B_i u + f_i, \quad i = 1, \ldots, s$$
$$y = C^T x, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state of the PWA system and control input, and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times p}$, and $f_i \in \mathbb{R}^n$ denote the unknown system parameters of the ith subsystem. In PWA systems, the state-input space $[x^T u^T]^T \in \mathbb{R}^{n+p}$ is partitioned into s convex regions.

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We use a set of inequalities to define each convex region Ω_i

$$\Omega_i = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+p} | H_i \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \preceq 0 \right\} \tag{2}$$

where each element of $\preceq$ represents an operators $<$ or $\leq$. The hyperplane is determined by each row of H_i. The set of hyperplanes H_i determines the polyhedral region Ω_i.

The indicator function can be utilized to indicate which subsystem is activated.

$$\chi_i(t) = \begin{cases} 1, & \text{if } (x(t), u(t)) \in \Omega_i \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

The PWA system can be obtained by linearizing a nonlinear system at multiple operating points. To compare the properties of PWL and PWA systems, we revisit the derivation as follows. Consider the nonlinear system

$$\dot{x}(t) = g(x(t), u(t)) \tag{4}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state and control input of the nonlinear system, and $g : \mathbb{R}^{n+p} \to \mathbb{R}^n$ denote a smooth nonlinear function. Given a set of operating points (x_i^*, u_i^*), the linearization of g around the operating points is

$$\dot{x} \approx g(x_i^*, u_i^*) + A_i(x - x_i^*) + B_i(u - u_i^*) \tag{5}$$

where $A_i = \frac{\partial g}{\partial x}|_{(x_i^*, u_i^*)}$ and $B_i = \frac{\partial g}{\partial u}|_{(x_i^*, u_i^*)}$. The following PWL system can be obtained by assuming zero equilibrium operating points $g(x_i^*, u_i^*) = 0$ and prior knowledge of the operating points (x_i^*, u_i^*)

$$\dot{x} \approx A_i \Delta x_i + B_i \Delta u_i \tag{6}$$

with $\Delta x_i = x - x_i$ and $\Delta u_i = u - u_i$ denoting the local state and input vector around the operating points. The PWA model can be derived by

$$\dot{x} = A_i x + B_i u + f_i \tag{7}$$

with $f_i = g(x_i^*, u_i^*) - A_i x_i^* - B_i u_i^*$. Both PWL and PWA systems can approximate the nonlinear systems given the same operating points and partitioning of the state-input space. Comparing with PWL systems, PWA systems utilize global state and input by introducing the affine term f_i. They allow nonzero $g(x_i^*, u_i^*)$ and do not require that the operating points are known.

The input–output mapping of the PWA system when the system i is activated is given by

$$y(t) = G_{pi}(s)[u](t) + G_{fi}(s)[1](t) \tag{8}$$

where

$$G_{pi}(s) = k_{pi} \frac{Z_{pi}(s)}{R_{pi}(s)} = C^T (sI - A_i)^{-1} B_i$$

$$G_{fi}(s) = k_{fi} \frac{Z_{fi}(s)}{R_{pi}(s)} = C^T (sI - A_i)^{-1} f_i. \tag{9}$$

The notation $y(t) = G(s)[u](t)$ represents the output in time domain at time t of a system, which is characterized by the transfer function $G(s)$ and input $u(t)$ [16]. Given a reference system

$$y_m(t) = W_m(s)[r](t), \quad W_m(s) = k_m \frac{Z_m(s)}{R_m(s)} \tag{10}$$

where y_m is the reference output trajectory, $W_m(s)$ denotes the transfer function of the reference system, and $r(t)$ represents reference input signal.

The problem we would like to solve is formulated as follows: given a PWA system (1) with known subsystem partition Ω_i, design a feedback control law based on the output feedback, such that the plant output $y(t)$ tracks the reference trajectory $y_m(t)$.

Assumption: The assumptions are summarized as follows, which apply to the entire work.

1) $Z_{pi}(s)$ is a monic strict Hurwitz polynomial of degree m.
2) The sign of $k_{pi}, i \in \{1, 2, \ldots, s\}$ is assumed to be known.
3) $Z_m(s)$ and $R_m(s)$ are monic strict Hurwitz polynomials.
4) $Z_{pi}(s)$ and $R_{pi}(s), i \in \{1, 2, \ldots, s\}$ are coprime.
5) $Z_{fi}(s)/Z_{pi}(s), i \in \{1, 2, \ldots, s\}$ is proper.
6) The relative degree of the plant is equal to that of the reference model.
7) The number of switches $N(T)$ within time interval $[t, t+T)$ satisfies $N(T) \leq C + \mu T, \forall t, T \geq 0$ for some positive constants C, μ.
8) Each polyhedral region Ω_i only depends on y and u and is assumed to be known.

Assuming $Z_{pi}(s)$ and $Z_m(s)$ to be strict Hurwitz requires the reference model and each subsystem of the PWA to be in minimum phase. The reference model is also stable since $R_m(s)$ is assumed to be strict Hurwitz. The term strict Hurwitz polynomial implies that the real parts of the roots are strictly negative.

The assumption on the number of the switches $N(T)$ over a time interval with the length of T constraints the average frequency of the switches among subsystems, which is characterized by μ. A small μ reveals slow switching. Limiting the frequency of switches is essential to ensure the closed-loop stability and is widely adopted in the area of adaptive control for switched systems [7], [8], [15].

Definition (small in the mean square sense [17]). Let $x : [0, \infty) \mapsto \mathbb{R}^n$ with $x \in \mathcal{L}_{2e}$, and consider the set

$$\mathcal{S}(\mu) = \left\{ x | \int_t^{t+T} x^T(t) x(t) \mathrm{d}\tau \leq c_0 \mu T + c_1 \forall t, T \geq 0 \right\}$$

for a given positive constant μ, where $c_0, c_1 \geq 0$ are some finite constants, and c_0 is independent of μ. x is said to be μ-small in the mean square sense, if $x \in \mathcal{S}(\mu)$.

III. Controller Design

In this section, the plant parameters are first assumed to be known in order to derive the nominal controller. The nominal controller parameters are determined by solving algebraic matching equations. Then, the adaptive controller is discussed in case where the plant parameters are unknown.

A. Nominal Control Design

Consider the feedback control law for the ith subsystem

$$u(t) = \theta_{1i}^{*\,T} \frac{\alpha(s)}{\Lambda(s)}[u](t) + \theta_{2i}^{*\,T} \frac{\alpha(s)}{\Lambda(s)}[y](t) + \theta_{3i}^* y(t) + c_{0i}^* r(t) + d_{0i}^* \tag{11}$$

where $c_{0i}^*,\ d_{0i}^*,\ \theta_{3i}^* \in \mathbb{R}^1,\ \theta_{1i}^*,\ \theta_{2i}^* \in \mathbb{R}^{n-1}$ represent the nominal controller parameters to be designed, $\alpha(s) = [s^{n-2}, s^{n-3}, \ldots, s, 1]^T$, and $\Lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$, which can be designed by user, e.g., $\Lambda(s) = s^{n-1} + \lambda_{n-2} s^{n-2} + \cdots + \lambda_1 s + \lambda_0$. Inserting the control law into (8) yields the closed-loop behavior of the ith subsystem

$$y(t) = G_{ci}(s)[r](t) + F_{ci}(s)[1](t) \tag{12}$$

with

$$G_{ci}(s) = \frac{k_{pi} Z_{pi} c_{0i}^* \Lambda}{R_{pi}(\Lambda - \theta_{1i}^{*\,T} \alpha) - k_{pi} Z_{pi}(\theta_{2i}^{*\,T} \alpha + \theta_{3i}^* \Lambda)} \tag{13}$$

representing the transfer function that relates the input and output signals and

$$F_{ci}(s) = \frac{k_{pi} Z_{pi} \Lambda d_{0i}^* + k_{fi} Z_{fi}(\Lambda - \theta_{1i}^{*\,T} \alpha)}{R_{pi}(\Lambda - \theta_{1i}^{*\,T} \alpha) - k_{pi} Z_{pi}(\theta_{2i}^{*\,T} \alpha + \theta_{3i}^* \Lambda)} \tag{14}$$

denoting the behavior caused by the affine term. The control goal is that the output of the closed-loop system tracks the output of the reference model. So, we let the transfer function of the closed-loop system equal to the one of the reference model and use the final value theorem to enforce the affine term to decay to zero

$$G_{ci}(s) = k_m \frac{Z_m(s)}{R_m(s)}$$
$$\lim_{s\to 0} F_{ci}(s) = 0 \tag{15}$$

which leads to the matching equation

$$R_{pi}(\Lambda - \theta_{1i}^{*\,T}\alpha) - k_{pi}Z_{pi}(\theta_{2i}^{*\,T}\alpha + \theta_{3i}^{*}\Lambda) = Z_{pi}\Lambda_0 R_m$$
$$k_{pi}Z_{pi}\Lambda d_{0i}^{*} + k_{fi}Z_{fi}(\Lambda - \theta_{1i}^{*\,T}\alpha)|_{s\to 0} = 0. \tag{16}$$

Here, c_{0i} is chosen as $c_{0i} = \frac{k_m}{k_{pi}}$ and $\Lambda(s) = \Lambda_0(s)Z_m(s)$. The nominal control parameters are obtained by solving the algebraic matching equations.

Remark 1: Since the relative degree of the plant is equal to that of the reference model and $Z_{pi}(s)$ and $R_{pi}(s)$ are coprime, the left and right sides of the first equation in (16) have the same degree $2n-1$ without cancellation. This ensures the uniqueness of its solution.

Remark 2: Note that the final value theorem is utilized to eliminate the biasing effect of the affine term. The conditions of the final value theorem are that the nonzero roots of the denominator of F_{ci} must have negative real parts and it must not have more than one zero pole, which requires that Z_{pi}, Λ_0, and R_m are strictly Hurwitz polynomials.

Remark 3: The second equation of (15) can be expanded as

$$\lim_{s\to 0} \frac{k_{pi}Z_m}{R_m}(d_{0i}^{*} + \frac{k_{fi}Z_{fi}(\Lambda - \theta_{1i}^{*\,T}\alpha)}{k_{pi}Z_{pi}\Lambda_0 Z_m}) = 0. \tag{17}$$

Since Z_{fi}/Z_{pi} is proper, the second summand in the brackets is also proper. To ensure the existence of the solution of d_{0i}^{*}, it requires that $Z_m|_{s\to 0} \neq 0, R_m|_{s\to 0} \neq 0$ and $k_{pi}Z_{pi}\Lambda_0 Z_m|_{s\to 0} \neq 0$. These are achieved by applying the assumptions Z_{pi}, R_m, Z_m being strict Hurwitz polynomials and designing Λ_0 to be strict Hurwitz. Simplifying (17) further gives the second equation of (16).

B. Error Model

We rewrite the nominal control law for the PWA system as

$$u(t) = \sum_{i=1}^{s} \chi_i \theta_i^{*T}\omega(t) \tag{18}$$

where $\theta_i^{*} = [\theta_{1i}^{*\,T}, \theta_{2i}^{*\,T}, \theta_{3i}^{*}, c_{0i}^{*}, d_{0i}^{*}]^T$ is the control parameter vector and $\omega = [\omega_1^{\,T}, \omega_2^{\,T}, y, r, 1]^T$ with

$$\omega_1 = \frac{\alpha(s)}{\Lambda(s)}[u](t), \quad \omega_2 = \frac{\alpha(s)}{\Lambda(s)}[y](t). \tag{19}$$

Applying the nominal controller, the closed-loop system can be written in a state-space form as

$$\dot{x}_c = A_{ci}x_c + B_{ci}r + f_{ci}$$
$$y = C_c^T x_c \tag{20}$$

where $x_c = [x^T, \omega_1^T, \omega_2^T]^T$ and $C_c^T = [C^T, 0^T]$.

Lemma 1: For the closed-loop system (20), the equation $C_c^T A_{ci}^{-1} f_{ci} = 0$ holds $\forall i \in \{1, 2, \ldots, s\}$.

Proof: The effect of the ith closed-loop affine term can be expressed by $F_{ci}(s)\frac{1}{s} = C_c^T(sI - A_{ci})^{-1} f_{ci}\frac{1}{s}$. Recalling (15) yields $\lim_{s\to 0} C_c^T(sI - A_{ci})^{-1} f_{ci} = 0$ and it follows $C_c^T A_{ci}^{-1} f_{ci} = 0$. □

We study the tracking error behavior when the nominal controller is applied. The following theorem states the smallness property of the tracking error.

Theorem 1: Let the PWA system (1) with known subsystem partitioning Ω_i be controlled by the output feedback nominal controller (11). There exists $\mu_0 \in \mathbb{R}^+$ such that $\forall \mu \in (0, \mu_0)$, the output tracking error $e = y - y_m \in \mathcal{S}(\mu)$. Furthermore, $\lim_{t\to\infty} \sup_{\tau > t} |e(\tau)| \leq c\bar{r} + d$ for $|r(t)| \leq \bar{r}$ and some constants $c, d \in \mathbb{R}^+$.

Theorem 1 reveals that the tracking error exists even if the nominal control parameters are utilized and the matching equations for every subsystem hold. Once the system switches, the output deviates from the reference one, the deviation decays to zero provided that the trajectory stays in the subsystem for sufficiently long time (characterized by μ) until the next switch occurs.

C. Adaptive Control Design

Now consider the case where the plant parameters are unknown. In this case, the nominal control parameters cannot be determined by solving matching equations. The estimation of the controller parameters is utilized to implement the adaptive controller

$$u(t) = \sum_{i=1}^{l} \chi_i \theta_i^{\,T}\omega(t) \tag{21}$$

where $\theta_i = [\theta_{1i}^{\,T}, \theta_{2i}^{\,T}, \theta_{3i}, c_{0i}, d_{0i}]^T$ denotes the estimated parameter vector for the ith subsystem. The output of the system can then be expressed by the output of the reference system perturbed by the error of control parameters $\tilde{\theta}_i = \theta_i - \theta_i^{*}$ and the transient terms η, Δ (see Appendix A for the detailed derivation) caused by switching.

$$y(t) = W_m[r](t) + \sum_{i=1}^{s} \chi_i \rho_i^{*} W_m\left[\sum_{i=1}^{s} \chi_i \tilde{\theta}_i^T \omega\right](t) + \eta(t) + \Delta(t)$$
$$= W_m\left[r + \sum_{i=1}^{s} \chi_i \rho_i^{*} \tilde{\theta}_i^T \omega\right](t) + \eta(t) + \Delta(t) \tag{22}$$

with $\rho_i^{*} = \frac{1}{c_{0i}^{*}}$. Define the estimation error for the ith subsystem as

$$\epsilon_i(t) = e(t) + \rho_i(t)\xi_i(t) \tag{23}$$

where

$$\xi_i(t) = \theta_i^T(t)\zeta(t) - W_m(s)[\theta_i^T\omega](t)$$
$$\zeta(t) = W_m(s)[\omega](t), \tag{24}$$

with $e(t) = y - y_m$ the tracking error. The following update law is proposed:

$$\dot{\theta}_i(t) = -\chi_i \mathrm{Pr}\left[\frac{\mathrm{sign}[k_{pi}]\Gamma_i\epsilon_i(t)\zeta(t)}{m^2(t)}\right]$$
$$\dot{\rho}_i(t) = -\chi_i \mathrm{Pr}\left[\frac{\gamma_i\epsilon_i(t)\xi_i(t)}{m^2(t)}\right] \tag{25}$$

where $\mathrm{Pr}[\cdot]$ is the projection operator to constraint the parameters within a bounded convex set, which is known as prior information. $\Gamma_i = \Gamma_i^T > 0$ and $\gamma_i > 0$ are adaptation gains, $m(t)$ is a dynamic normalizing signal defined by $m^2 = 1 + m_s$ with

$$\dot{m}_s(t) = -\delta_0 m_s + u^2 + y^2, \quad m_s(0) = 0 \tag{26}$$

where δ_0 is a nonnegative constant. The following theorem describes the property of the tracking error in adaptive case.

Theorem 2: Let the PWA system (1) with known subsystem partitioning Ω_i be controlled by the output feedback controller (21) with the adaptation law (25). There exists $\mu_0 \in \mathbb{R}^+$ such that $\forall \mu \in (0, \mu_0)$, the output tracking error $e \in \mathcal{S}(\mu)$.

Remark 4: Compared with the counterpart for PWL systems [15], the controller (21) introduces a constant term in ω to cancel out the biasing effect caused by the affine term. Since the affine term can also be viewed as input disturbance [18], nonequilibrium offset [19], actuator failure [20], and system damage [21], the controller for each subsystem has the common structure as the output feedback-based controllers proposed in [18, section 4] and [20, chapter 4]. Note that these two cases exhibit either no switching or switching only once, and thus, the disturbance or actuator failure compensation error decays to zero as $t \to \infty$. This further gives asymptotic output tracking. Different from this result, the tracking error e in the PWA context is small in the mean square sense due to the switch-dependent property of η as well as Δ, as discussed in Theorem 1.

D. Control Parameter Convergence

Now we study the convergence of the control parameters. We extend the analysis method for linear systems in [17, p. 757] to the PWA systems. In particular, the proposed controller (21) contains a constant term, which is reflected in ω or equivalently ζ. The effect of this controller structure on the PE property of ω, ζ needs to be specifically analyzed. Furthermore, the tracking error e is small in the mean square sense, whose influence on the parameter convergence needs to be discussed. In addition, how the switching frequency affects the parameter convergence remains to be explored. The following theorem shows our result.

Theorem 3: Let the PWA system (1) with known subsystem partitioning Ω_i be controlled by the output feedback controller (21) with the adaptation law (25). If the reference signal r is sufficiently rich of order $2n$ with distinct frequencies and activates all the subsystems repeatedly, i.e., $\forall i \in \{1,\ldots,s\}$ and $\forall t_s \in \mathbb{R}^+$, there exists $t_d > t_s$ and $\delta t \in \mathbb{R}^+$ such that $\chi_i(t) = 1$ for $t \in [t_d, t_d + \delta t)$ and if the projection in (25) is not activated, then $|e|$ and $|\tilde{\theta}_i|$ converge to a residual set

$$\mathcal{S}_{\theta_i} = \left\{ e \in \mathbb{R},\ \tilde{\theta}_i \in \mathbb{R}^{2n+1} \middle| |e| + |\tilde{\theta}_i| \le c_0(\nu_0 + \sqrt{\mu}) \right\}$$

for some positive constants $c_0, \nu_0 \in \mathbb{R}^+$ and $\mu \in (0, \mu_0)$.

In our article, we first decompose ζ into ζ_m and ζ_e. ζ_m can be further decomposed into one component depending on input frequencies and one constant term representing zero frequency. These constitute the excitation source. ζ_e contains all the error terms and is proved to be $\mathcal{S}(\mu)$. We show that its effect on the excitation can be eliminated by carefully balancing the switching frequency μ and excitation level α_0 of ζ_m. Finally, we establish the relationship between the switching frequency μ and the size of the bounded set $\mathcal{S}_{\theta_i}$ by expressing $|\theta_i|$ in terms of an inequality of μ.

Theorem 3 indicates that the bound of the residual set relates to the switching frequency. Fast switching results in a large residual set. The convergence to the nominal value is, however, possible and discussed as follows.

Corollary 1: Let the PWA system (1) with known subsystem partitioning Ω_i be controlled by the output feedback controller (21) with the adaptation law (25) without projection. The reference signal r is sufficiently rich of order $2n$. If for a certain $i \in \{1,\ldots,s\}$ and a certain time instant $t_0 \ge 0$, we have $\chi_i(t) = 1\ \forall t \in [t_0, \infty)$, then $e(t) \to 0, \tilde{\theta}_i(t) \to 0$ as $t \to \infty$.

As Corollary 1 shows, if the output trajectory is kept staying in a certain subsystem, the periodic deviations caused by switching are avoided and this further results in both the convergence of the tracking error and control parameter estimation error.

Corollary 1 depicts a special case of the PWA system, which can be interpreted as the system with a sudden change in the dynamics such as aircraft suffering from damage [21]. The controller parameters for the new system can converge to the nominal values by superposing proper PE probing signals to the reference signal. The convergence can improve the transient behavior, and thus, the results of Corollary 1 are of great importance for the real applications.

IV. Numerical Validation

A numerical example taken from [22] is utilized to validate the proposed control algorithm. The plant parameters of the PWA system are given by

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -2.5 & -1 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 1 \\ -1.5 & -1 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix},\ f_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix},\ f_3 = \begin{bmatrix} 0 \\ -0.3 \end{bmatrix}$$

with the common input matrix $B = [0, 1.5]^T$. The sign of each subsystem is 1 and known as prior. The switching hyperplanes depend on the system output and are given by

$$\Omega_1 = \{y \in \mathbb{R} | -2 \le y \le 2\}$$
$$\Omega_2 = \{y \in \mathbb{R} | y > 2\}$$
$$\Omega_3 = \{y \in \mathbb{R} | y < -2\}.$$

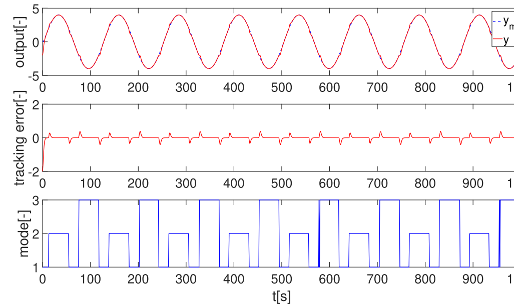

Fig. 1. Output tracking performance with input signal $r = 4\sin(0.05t)$, the nominal controller is applied.

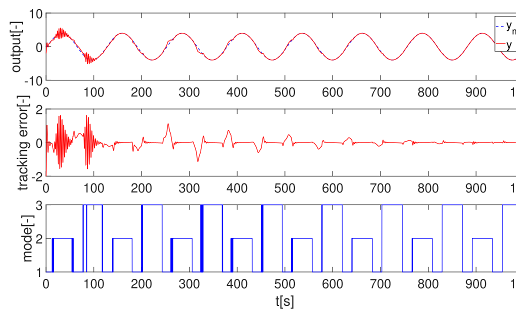

Fig. 2. Output tracking performance with input signal $r = 4\sin(0.05t)$, the proposed adaptive controller is applied.

The reference model is chosen as

$$W_m = \frac{1}{(s+1)^2}. \tag{27}$$

The relative degree of the reference system is 2, which is equal to the one of all the subsystems of the PWA system. Selecting $\Lambda(s) = \frac{1}{s+1}$ and $\alpha = 1$, the nominal control parameters are obtained by matching equations (15)

$$\theta_1^* = [-1, 1.33, -0.67, 0.67, -0.53]^T$$
$$\theta_2^* = [-1, 1.67, 1, 0.67, -0.27]^T$$
$$\theta_3^* = [-1, 1, 0.33, 0.67, 0.4]^T$$

Given an input signal $r = 4\sin(0.05t)$, the output tracking performance of the closed-loop system by applying the nominal controller is displayed in Fig. 1. It shows that the output tracking error exists even when the nominal control parameters are employed. When the system switches, the output of the closed-loop system deviates from the output of the reference system, as depicted in (34). This deviation vanishes given a sufficiently slow switching. The overall output tracking error over the whole time interval is thus small in the mean square sense.

Given the adaptation gains $\Gamma_i = \gamma_i = 10$, the output tracking performance of the adaptive system is displayed in Fig. 2. It can be seen that the desired performance is achieved by applying the adaptive controller. The deviation from the reference output occurs due to the switches among subsystems. The smallness of the tracking error in the mean square sense validates the theory derivation. Compared with the adaptive controller, the nominal controller exhibits a better transient performance. This motivates us to study the convergence property of the controller parameters. To validate the control parameter convergence, the input signal is required to be sufficiently rich of order 4. Define the input signal $r = \sin(0.9t) + \sin(0.1t) + \bar{r}$, with a periodic offset signal

$$\bar{r}(t) = \begin{cases} 4, & 1000 + kT \le t < 3000 + kT\text{s} \\ -4, & 4000 + kT \le t < 6000 + kT\text{s} \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

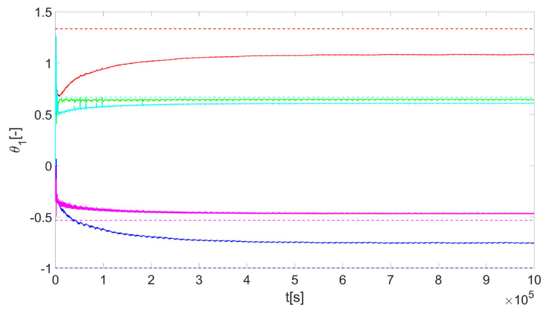

Fig. 3. Control parameters converge to a residual set around the nominal values with slow switching.

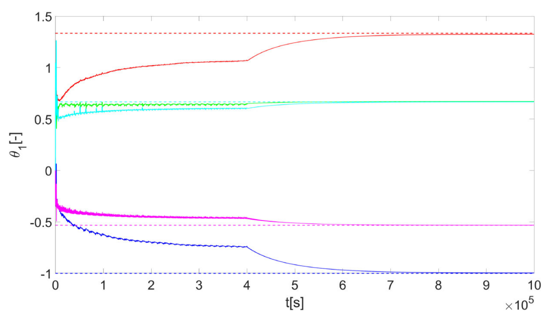

Fig. 4. Control parameters converge to the nominal values for $\chi_1 = 1$ after 40000 s.

where $k \in \mathbb{N}$ and $T = 6000$s. $\bar{r}$ drives the trajectory into all subsystems periodically. In Fig. 3, the dashed lines represent the nominal control parameters and the solid lines the adaptive control parameters. It reveals that the adaptive control parameters converge to a set around the nominal values under slow switching. Since the convergence of $\tilde{\theta}_i$ is similar, so only $\tilde{\theta}_1$ is displayed for clarity.

To show the parameter convergence stated in Corollary 1, the trajectory of the closed-loop system must be kept within a certain subsystem of the PWA system from a certain time instant t_0 on. Here, we remove $\bar{r}$ from r at $t_0 = 40000$s, which leads to $\chi_1 = 1, \chi_2 = \chi_3 = 0\, \forall t \in [t_0, \infty)$, the adaptive control parameters θ_1 converge to the nominal control values θ_1^*, as shown in Fig. 4. The simulation validates the theory derivation.

V. Conclusion

In this article, we have developed the output feedback-based direct MRAC for PWA systems for output tracking and explored the controller parameter convergence. With the proposed approach, all the signals in the closed-loop are bounded and the output tracking error is small in the mean square sense with sufficiently slow switching. If the input signal is sufficiently rich, the control parameters converge to a residual set around the nominal values for slow switches. The approach is based on input/output information and is limited to be applied to the single-input–single-output PWA systems. To overcome this limitation, future work can be an extension to the general case, where partial state feedback is available. It is also an interesting topic to study the problem, where the region partitions depend on the system state, and thus, are unknown. How to avoid frequent switching and sliding mode will also be studied in future work.

Appendix A
Proof of Theorem 1

Proof: Let $(C_c, A_{ci_k}, B_{ci_k}, f_{ci_k})$ denote the active system over time interval $[t_k, t_{k+1})$, $k \in \mathbb{Z}^+$, $i_k \in \{1, 2, \ldots, s\}$. The trajectory of y over time interval $[t_k, t_{k+1})$ is given by

$$y(t) = \int_{t_k}^{t} C_c^T \Phi_c(t,\tau) B_{ci_k} r(\tau) \mathrm{d}\tau + C_c^T \Phi_c(t, t_k) x_c(t_k) + \int_{t_k}^{t} C_c^T \Phi_c(t,\tau) f_{ci_k} \mathrm{d}\tau \tag{29}$$

where $\Phi_c(t,\tau)$ denotes the associated closed-loop state transition matrix. The matching equation (15) ensures

$$\int_{t_k}^{t} C_c^T \Phi_c(t,\tau) B_{ci_k} r(\tau)\mathrm{d}\tau = \int_{t_k}^{t} C_m^T \Phi_m(t,\tau) B_m r(\tau) \mathrm{d}\tau \tag{30}$$

which yields the tracking error at time $t \in [t_k, t_{k+1})$

$$\begin{aligned} e(t) &= y(t) - y_m(t) \\ &= C_c^T \Phi_c(t,t_k) x_c(t_k) - C_m^T \Phi_m(t,t_k) x_m(t_k) \\ &\quad + \int_{t_k}^{t} C_c^T \Phi_c(t,\tau) f_{ci_k} \mathrm{d}\tau. \end{aligned} \tag{31}$$

The eigenvalues of A_{ci} depend on Λ, Z_{pi}, R_m, so A_{ci} is stable and

$$\eta_{i_k}(t) \triangleq C_c^T \Phi_c(t,t_k) x_c(t_k) - C_m^T \Phi_m(t,t_k) x_m(t_k) \tag{32}$$

is exponentially decaying. Furthermore, because of the matching equation, we have

$$\Delta_{i_k}(t) \triangleq \int_{t_k}^{t} C_c^T \Phi_c(t,\tau) f_{ci_k} \mathrm{d}\tau \tag{33}$$

which is the deviation caused by the affine term, decays to zero exponentially. The general expression of the tracking error e over an arbitrary time interval $[t, t+T)$ is

$$e(t) = \eta(t) + \Delta(t) \tag{34}$$

with $\eta(t) = \eta_{i_k}(t)$ and $\Delta(t) = \Delta_{i_k}(t)$ when $t \in [t_k, t_{k+1})$. It is proved in [15] that there exists $\mu_1 > 0$ such that $\forall \mu \in [0, \mu_1)$, $\eta \in \mathcal{S}(\mu)$. This indicates that if the switching is sufficiently slow, the error term η is small in the mean square sense. Following the same concept, there exists $\mu_2 > 0$, such that $\forall \mu \in [0, \mu_2)$, $\Delta \in \mathcal{S}(\mu)$, which together with (34) leads to $e \in \mathcal{S}(\mu)\ \forall \mu \in [0, \mu_0)$ with $\mu_0 = \min\{\mu_1, \mu_2\}$.

From (32)–(34), we have $|e| \leq |\eta| + |\Delta|$ with $|\eta| \leq \|C_c\|\|\Phi_c\| \max_k |x_c(t_k)| + \|C_m\|\|\Phi_m\| \max_k |x_m(t_k)|$ and $|\Delta| \leq \max_k \int_{t_k}^{t_{k+1}} \|C_c\|\|\Phi_c\|\|f_{ci}\| \mathrm{d}\tau$. Based on the slow switching assumption and [23, Theorem 2], we have $\|\Phi_c(t)\| \leq \lambda_c \mathrm{e}^{-\alpha_c t}$ for some $\lambda_c, \alpha_c > 0$. For the reference system, we have $\|\Phi_m(t)\| \leq \lambda_m \mathrm{e}^{-\alpha_m t}$ for some $\lambda_m, \alpha_m > 0$. These lead to $|x_m(t)| \leq c_m \bar{r} + \epsilon_m$ and $|x_c(t)| \leq c_c \bar{r} + d_c + \epsilon_c$ for some $c_m, c_c, d_c > 0$ and exponentially decaying terms ϵ_m, ϵ_c, which in turn gives $\lim_{t\to\infty} \sup_{\tau > t} |e(\tau)| \leq c\bar{r} + d$ for some $c, d > 0$. □

Appendix B
Proof of Theorem 2

Proof: From (23) and (24), it can be derived that

$$\epsilon_i = \rho_i^* \tilde{\theta}_i^T \zeta + \tilde{\rho}_i \xi_i + \eta_i + \Delta_i. \tag{35}$$

Consider the Lyapunov function

$$V(\tilde{\theta}, \tilde{\rho}) = \sum_{i=1}^{s} \chi_i (|\rho_i^*| \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i + \gamma_i^{-1} \tilde{\rho}_i^2). \tag{36}$$

Taking the piecewise derivative of V along the trajectories of (25) yields

$$\dot{V} = -\sum_{i=1}^{s} \chi_i \frac{2\epsilon_i}{m^2}(\rho_i^* \tilde{\theta}_i^T \zeta + \tilde{\rho}_i \xi_i). \tag{37}$$

Inserting (35) into (37) yields

$$\dot{V} = -2\sum_{i=1}^{s} \frac{\chi_i \epsilon_i^2}{m^2} + 2\sum_{i=1}^{s} \frac{\chi_i \epsilon_i}{m^2}(\eta_i + \Delta_i). \tag{38}$$

Integrating over an arbitrary interval $[t, t+T)$, in which multiple switching may occur, yields

$$\int_t^{t+T} (\frac{\epsilon}{m})^2 \mathrm{d}t \le (V(t) - V(t+T)) + \frac{1}{m^2}\int_t^{t+T} (\eta + \Delta)^2 \tag{39}$$

with $\epsilon = \sum_{i=1}^{s} \chi_i \epsilon_i$. Because $\eta + \Delta \in \mathcal{S}(\mu)$, it follows $\frac{\epsilon}{m} \in \mathcal{S}(\mu)$. The rest of the proof can be divided into several steps as follows.

Step 1: Express the input and output signals in terms of $\tilde{\theta}_{i_k}^T \omega$. Based on (22), y over the time interval $[t_k, t_{k+1})$ is expressed as

$$y(t) = W_m[r + \rho_{i_k}^* \tilde{\theta}_{i_k}^T \omega](t) + \eta_{i_k}(t) + \Delta_{i_k}(t). \tag{40}$$

Ignoring the effect of the exponentially decaying terms η_{i_k}, Δ_{i_k}, the control signal u can be expressed by

$$u(t) = G_{pi_k}^{-1} W_m[r + \rho_{i_k}^* \tilde{\theta}_{i_k}^T \omega](t) - G_{pi_k}^{-1}[G_{fi_k}[1]](t) \tag{41}$$

$G_{pi_k}^{-1} G_{fi_k}$ and $G_{pi_k}^{-1} W_m$ are stable and proper. Define a fictitious normalizing signal $m_f^2 = e^{-\delta(t-t_k)} m(t_k) + \|u\|^2 + \|y\|^2$, it follows from (40), (41), and [17, Lemma 3.3.2] that

$$m_f^2 \le c + ce^{-\delta(t-t_k)} m^2(t_k) + c\|\tilde{\theta}_{i_k}^T \omega\|^2 \tag{42}$$

where $\|\cdot\|$ denotes, for the purpose of clarity, the $\mathcal{L}_{2\delta}$-norm over $[t_k, t)$, $\delta \in (0, \delta_0]$, $c \ge 0$ denotes any finite constant.

Step 2: Use the swapping lemma to establish the boundedness of $\|\tilde{\theta}_{i_k}^T \omega\|$. The following inequality is obtained by applying the swapping lemma [17]:

$$\begin{aligned}\|\tilde{\theta}_{i_k}^T \omega\| &\le ce^{-\delta(t-t_k)} m^2(t_k) + \frac{c}{\alpha_0}(m_f + \|\dot{\theta}_{i_k} m_f\|) \\ &\quad + c\alpha_0^{n^*}(\|\epsilon_{i_k}\| + \|\dot{\theta}_{i_k} m_f\| + \|\eta_{i_k}\| + \|\Delta_{i_k}\|)\end{aligned} \tag{43}$$

for some $\alpha_0 > 0$. Since $\delta \in (0, \delta_0]$, we have $m \le m_f$, thus

$$\begin{aligned}\|\tilde{\theta}_{i_k}^T \omega\| &\le ce^{-\delta(t-t_k)} m^2(t_k) + \frac{c}{\alpha_0}(m_f + \|\dot{\theta}_{i_k} m_f\|) \\ &\quad + c\alpha_0^{n^*}(\|\frac{\epsilon_{i_k}}{m} m_f\| + \|\dot{\theta}_{i_k} m_f\| + \|\eta_{i_k} m_f\| + \|\Delta_{i_k} m_f\|).\end{aligned} \tag{44}$$

Step 3: Prove the boundedness of closed-loop signals. From (42) and (44), it follows:

$$m_f^2 \le c + ce^{-\delta(t-t_k)} m^2(t_k) + c\alpha_0^{2n^*} \|\tilde{g}_{i_k} m_f\|^2 \tag{45}$$

for large α_0 with $\tilde{g}_{i_k}^2 = (\frac{\epsilon_{i_k}}{m})^2 + \dot{\theta}_{i_k}^2 + \eta_{i_k}^2 + \Delta_{i_k}^2$. Now consider an arbitrary time interval $[t, t+T)$, within which switches occur at time instants $t \le t_{k_1}, t_{k_2}, \ldots, t_{k_N} \le t+T$. The normalizing signal m_f over this interval is then expressed by

$$m_f^2 \le c + ce^{-\delta T} m_q^2 + c\int_t^{t+T} e^{-\delta(t-\tau)} \tilde{g}^2(\tau) m_f^2(\tau) \mathrm{d}\tau \tag{46}$$

where $m_q = \max\{m(t_{k_1}), \ldots, m(t_{k_N})\}$ and $\tilde{g}^2 = (\frac{\epsilon}{m})^2 + \sum_{i=1}^{s} \chi_i \dot{\theta}_i^2 + \eta^2 + \Delta^2 \tilde{g} \in \mathcal{S}(\mu)$. Applying the Bellman–Gronwall Lemma yields

$$\begin{aligned}m_f^2 &\le ce^{-\delta T}(1 + m_q^2) e^{c\int_t^{t+T} \tilde{g}^2(\tau)\mathrm{d}\tau} \\ &\quad + c\delta \int_t^{t+T} e^{-\delta(t-s)} e^{c\int_s^t \tilde{g}^2(\tau)\mathrm{d}\tau} \mathrm{d}s.\end{aligned} \tag{47}$$

To obtain the boundness of m_f, $c\mu < \delta$ should be hold for some positive constant c. This condition can be achieved by letting μ sufficiently small, which implies slow switching. Since $m_f \in \mathcal{L}_\infty$, following from [17, Lemma 6.8.1], it can be concluded that $u, y, \omega, m \in \mathcal{L}_\infty$.

Step 4: Study the property of the tracking error. It follows from (24) and the boundedness of ω that $\xi_i, \zeta, \in \mathcal{L}_\infty$. $\frac{\epsilon}{m} \in \mathcal{S}(\mu)$ together with $m \in \mathcal{L}_\infty$ yields $\epsilon \in \mathcal{S}(\mu)$. From (23), we write the general expression for e

$$e = \sum_{i=1}^{s} \chi_i(\epsilon_i - \rho_i \xi_i) = \epsilon - \sum_{i=1}^{s} \chi_i \rho_i \xi_i. \tag{48}$$

With the boundedness of ρ_i, ξ_i and $\epsilon \in \mathcal{S}(\mu)$, we can conclude that $e \in \mathcal{S}(\mu)$. □

Appendix C
Proof of Theorem 3

Proof: First, remove the subscript i for simplicity and we show that ζ is PE.

$$\zeta(t) = W_m(s)\begin{bmatrix}[\omega_1](t) \\ [\omega_2](t) \\ [y](t) \\ [r](t) \\ [1](t)\end{bmatrix} = W_m(s)\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)}[u](t) \\ \frac{\alpha(s)}{\Lambda(s)}[y](t) \\ [y](t) \\ [r](t) \\ [1](t)\end{bmatrix}. \tag{49}$$

Inserting (8) into (49) and substituting u yields

$$\zeta(t) = \zeta_m + \zeta_e \tag{50}$$

where

$$\zeta_m = \underbrace{W_m(s)(\underbrace{\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)} G_p^{-1} W_m(s) \\ \frac{\alpha(s)}{\Lambda(s)} W_m(s) \\ W_m(s) \\ 1 \\ 0\end{bmatrix}}_{H(s)}[r](t) + \underbrace{\begin{bmatrix}0 \\ 0 \\ 0 \\ 0 \\ 1\end{bmatrix}}_{H_f(s)}[1](t))}_{z}$$

$$\zeta_e = W_m(s)(\underbrace{\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)} G_p^{-1} \\ \frac{\alpha(s)}{\Lambda(s)} \\ 1 \\ 0 \\ 0\end{bmatrix}}_{H_e(s)}[e](t) + \underbrace{\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)} G_p^{-1} G_f \\ 0 \\ 0 \\ 0 \\ 0\end{bmatrix}}_{H_{fe}(s)}[1](t)). \tag{51}$$

To prove the PE property of ζ, we start by showing that z is PE. The autocovariance of z is given by

$$R_z(0) = \underbrace{\frac{1}{2\pi} H_f(0) H_f(0)^T}_{R_{z1}(0)} + \underbrace{\frac{1}{2\pi}\sum_{l=1}^{2n} F_r(\Omega_l) H(-j\Omega_l) H(j\Omega_l)^T}_{R_{z2}(0)} \tag{52}$$

where $F_r(\Omega_l)$ denotes the spectral peak associated with frequency $\Omega_l, l \in \{1, \ldots, 2n\}$. Note that the constant input 1 in ω leads to an unit spectral peak at zero frequency (see $R_{z1}(0)$), while the frequencies contained in r build $2n$ distinct peaks $F_r(\Omega_l)$ (see $R_{z2}(0)$). We rewrite $H(s)$ as

$$H(s) = \begin{bmatrix} H_-(s) \\ 0 \end{bmatrix} \tag{53}$$

with

$$H_-(s) = \begin{bmatrix} \frac{\alpha(s)}{\Lambda(s)} G_p^{-1} W_m(s) \\ \frac{\alpha(s)}{\Lambda(s)} W_m(s) \\ W_m(s) \\ 1 \end{bmatrix}. \tag{54}$$

It is proved in [17] that $[H_-(j\Omega_1), H_-(j\Omega_2), \ldots H_-(j\Omega_{2n})]$ are linearly independent.

If $R_z(0)$ is positive definite, then equation

$$\mathcal{X}^T R_z(0)\mathcal{X} = 0 \tag{55}$$

only has solution $\mathcal{X} = 0, \mathcal{X} \in \mathbb{R}^{2n+1}$. Because $R_{z1}(0)$ and $R_{z2}(0)$ are positive semi definite, we have

$$\mathcal{X}^T R_{z1}(0)\mathcal{X} \geq 0, \quad \mathcal{X}^T R_{z2}(0)\mathcal{X} \geq 0 \tag{56}$$

which together with (55) implies that

$$\mathcal{X}^T R_{z1}(0)\mathcal{X} = 0, \quad \mathcal{X}^T R_{z2}(0)\mathcal{X} = 0 \tag{57}$$

only if $\mathcal{X} = 0$. Suppose $\mathcal{X} = [C^T, d]^T$ with $C \in \mathbb{R}^{2n}, d \in \mathbb{R}$. From $\mathcal{X}^T R_{z2}(0)\mathcal{X} = 0$ follows

$$L^T C = 0 \tag{58}$$

with $L = [H_-(j\Omega_1), H_-(j\Omega_2), \ldots H_-(j\Omega_{2n})$. Because L has full rank, C must be 0. From $\mathcal{X}^T R_{z1}(0)\mathcal{X} = 0$ follows d must be 0, which implies the positive definiteness of $R_z(0)$, thus z is PE, which together with $\zeta_m = W_m[z](t)$ yields ζ_m being PE. Hence, there exists $T_0 > 0, \alpha_0 > 0$ such that

$$\frac{1}{T_0}\int_t^{t+T_0} \zeta_m(\tau)\zeta_m^T(\tau)\mathrm{d}\tau \geq \alpha_0 I, \forall t \geq 0. \tag{59}$$

Next, we would like to prove that ζ is also PE. Note that

$$\begin{aligned}&\frac{1}{nT_0}\int_t^{t+nT_0} \zeta(\tau)\zeta^T(\tau)\mathrm{d}\tau \\ &\geq \frac{1}{2nT_0}\int_t^{t+nT_0} \zeta_m(\tau)\zeta_m^T(\tau)\mathrm{d}\tau - \frac{1}{nT_0}\int_t^{t+nT_0} \zeta_e(\tau)\zeta_e^T(\tau)\mathrm{d}\tau\end{aligned} \tag{60}$$

where n is an arbitrary positive integer. Because $G_p(s)$ and $W_m(s)$ have the same relative degree, $W_m(s)H_e(s)$ is strictly proper. $G_f(s)$ is also proper, which implies $W_m(s)H_{fe}(s)$ is strictly proper. Because it is established that $e \in \mathcal{S}(\mu)$, we have $W_m(s)H_e(s)[e](t) \in \mathcal{S}(\mu)$. Considering $W_m(s)H_{fe}(s)[1](t) \in \mathcal{L}_\infty$, we have $\zeta_e \in \mathcal{S}(\mu)$, which together with the PE property of ζ_m yields

$$\frac{1}{nT_0}\int_t^{t+nT_0} \zeta(\tau)\zeta^T(\tau)\mathrm{d}\tau \geq \frac{\alpha_0}{2}I - (K_0\mu + \frac{C_0}{nT_0})I \tag{61}$$

for some $C_0, K_0 \geq 0$. If n is chosen such that $C_0 < \frac{\alpha_0}{8}nT_0$, then for $K_0\mu < \frac{\alpha_0}{8}$, we have

$$\frac{1}{nT_0}\int_t^{t+nT_0} \zeta(\tau)\zeta^T(\tau)\mathrm{d}\tau \geq \frac{\alpha_0}{4}I. \tag{62}$$

So ζ is PE.

Insert (35) into (25) yields

$$\dot{\tilde{\theta}}_i(t) = -\chi_i \mathrm{sign}[k_{pi}]\Gamma_i(\frac{\rho_i^*\zeta\zeta^T}{m^2}\tilde{\theta}_i + \frac{\tilde{\rho}_i\xi_i\zeta}{m^2} + \frac{(\eta_i+\Delta_i)\zeta}{m^2}). \tag{63}$$

The homogeneous part of (63) is exponentially stable and consider $\frac{\zeta}{m} \in \mathcal{L}_\infty$

$$\begin{aligned}|\tilde{\theta}_i| &\leq \beta_0 e^{-\beta_2(t-t_k)} + \beta_1\int_{t_k}^t e^{-\beta_2(t-\tau)}(\frac{|\tilde{\rho}_i\xi_i|}{m} + \frac{|\eta_i|+|\Delta_i|}{m})\mathrm{d}\tau \\ &\leq \beta_0 e^{-\beta_2(t-t_k)} + \bar{\beta} + \beta_1\int_{t_k}^t e^{-\beta_2(t-\tau)}(\frac{|\eta_i|+|\Delta_i|}{m})\mathrm{d}\tau\end{aligned} \tag{64}$$

where $\beta_0, \beta_1, \beta_2 \in \mathbb{R}^+$ are some positive constants, $\bar{\beta} = \frac{\beta_1}{\beta_2}\sup_t \frac{|\tilde{\rho}_i\xi_i|}{m}$. Because $\eta_i, \Delta_i \in \mathcal{S}(\mu)$, we apply [17, Corollary 3.3.3] and have

$$\beta_1\int_{t_k}^t e^{-\beta_2(t-\tau)}(\frac{|\eta_i|+|\Delta_i|}{m})\mathrm{d}\tau \leq \beta'(\sqrt{C} + \sqrt{K\mu}) \tag{65}$$

for some constants $C, K \in \mathbb{R}^+$ with $\beta' = 2\sqrt{\frac{\beta_1^2}{\beta_2}\frac{e^{\beta_2}}{1-e^{-\beta_2}}}$. This implies that $\tilde{\theta}_i$ converges to a residual set

$$|\tilde{\theta}_i| \leq c(\nu + \sqrt{\mu}) + \epsilon_t \tag{66}$$

where $|\cdot|$ denotes any vector norm, $\mu \in (0, \mu_0)$, $\nu = \frac{\bar{\beta}}{\beta'\sqrt{K}} + \sqrt{\frac{C}{K}}$, $c = \beta'\sqrt{K}$, and ϵ_t is an exponentially decaying term. Invoking Lemma 3.3.2 of [17], we have from $e = y - W_m[r]$ and (22) that

$$|e| \leq \max_i |\rho_i^*| \|W_m(s)\|_{2\delta}\|\tilde{\theta}_i^T\omega\|_{2\delta} + \bar{d} \tag{67}$$

with $\bar{d} = \sup_t(|\eta| + |\Delta|)$ and $\|W_m(s)\|_{2\delta}$ denoting the δ-shifted H_2 norm of $W_m(s)$ for some $\delta > 0$. Inserting (66) into the $\mathcal{L}_{2\delta}$-norm $\|\tilde{\theta}_i^T\omega\|_{2\delta}$ in (67) leads to

$$|e| \leq \bar{\omega}(c(\nu + \sqrt{\mu})) + \bar{d} + \epsilon' \tag{68}$$

for $\bar{\omega} = \max_i |\rho_i^*| \|W_m(s)\|_{2\delta}\frac{\sup_t|\omega|}{\sqrt{\delta}}$ and ϵ' being a decaying to zero term. Combining (66) and (68), we have that $|e|$ and $|\tilde{\theta}_i|$ converge to the residual set

$$\mathcal{S}_{\theta_i} = \left\{e \in \mathbb{R}, \tilde{\theta}_i \in \mathbb{R}^{2n+1} \big| |e| + |\tilde{\theta}_i| \leq c_0(\nu_0 + \sqrt{\mu})\right\}$$

for $c_0 = c(1 + \bar{\omega}), \nu_0 = \nu + \frac{\bar{d}}{c_0}$. □

Appendix D
Proof of Corollary 1

Proof: Since the system output remains in the ith subsystem, we focus on the ith subsystem and remove the subscript i for simplicity.

Let $\eta = \eta_c + \eta_m$ with $\eta_c = C_c^T\Phi_c(t, t_0)x_c(t_0)$ and $\eta_m = -C_m^T\Phi_m(t, t_0)x_m(t_0)$. η_c and η_m satisfy

$$\begin{aligned}\dot{\omega}_c &= A_c\omega_c, \quad \omega_c(t_0) = x_c(t_0) \\ \eta_c &= C_c^T\omega_c\end{aligned} \tag{69}$$

and

$$\begin{aligned}\dot{\omega}_m &= A_m\omega_m, \quad \omega_m(t_0) = x_m(t_0) \\ \eta_m &= -C_m^T\omega_m\end{aligned} \tag{70}$$

respectively. Besides, Δ satisfies the equation

$$\dot{\omega}_\delta = A_c\omega_\delta + F_c, \quad \omega_\delta(t_0) = 0$$
$$\Delta = C_c^T\omega_\delta. \tag{71}$$

Define the Lyapunov-like function

$$V = |\rho^*|\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \gamma^{-1}\tilde{\rho}^2 + \omega_c^T P_c\omega_c + \omega_m^T P_m\omega_m + (\omega_\delta - A_c^{-1}F_c)^T P_c(\omega_\delta - A_c^{-1}F_c). \tag{72}$$

Since A_m and A_c are stable, there exist positive definite matrices P_c and P_m such that

$$A_c^T P_c + P_c A_c = -\gamma_c I, \quad A_m^T P_m + P_m A_m = -\gamma_m I \tag{73}$$

for some constants $\gamma_c, \gamma_m > 0$ to be chosen. Take the derivative of V and insert (35), (69), (70), (71), and (73), we have

$$\dot{V} = -\frac{2\epsilon^2}{m^2} + \frac{2\epsilon\eta_c}{m^2} - \gamma_c|\omega_c|^2 + \frac{2\epsilon\eta_m}{m^2} - \gamma_m|\omega_m|^2 + \frac{2\epsilon\Delta}{m^2} - \gamma_c|\overline{\omega}_\delta|^2 \tag{74}$$

where $\overline{\omega}_\delta = \omega_\delta - A_c^{-1}F_c$. Substituting η_c, η_m, and Δ with (69)–(71) and invoking Lemma 1, it follows:

$$\dot{V} \leq -\frac{2\epsilon^2}{m^2} + \frac{2}{m^2}|\epsilon||C_c^T||\omega_c| - \gamma_c|\omega_c|^2 + \frac{2}{m^2}|\epsilon||C_m^T||\omega_m| - \gamma_m|\omega_m|^2 + \frac{2}{m^2}|\epsilon||C_c^T||\overline{\omega}_\delta| - \gamma_c|\overline{\omega}_\delta|^2$$
$$= -\frac{\epsilon^2}{2m^2} + \phi_1 + \phi_2 + \phi_3 \tag{75}$$

where

$$\phi_1 = \frac{\epsilon^2}{2m^2} + \frac{2}{m^2}|\epsilon||C_c^T||\omega_c| - \gamma_c|\omega_c|^2$$
$$= -\frac{\epsilon^2 + (\epsilon - 4|C_c^T||\omega_c|)^2}{4m^2} - |\omega_c|^2(\gamma_c - \frac{4|C_c^T|^2}{m^2}) \tag{76}$$

$$\phi_2 = -\frac{\epsilon^2}{2m^2} + \frac{2}{m^2}|\epsilon||C_m^T||\omega_m| - \gamma_m|\omega_m|^2$$
$$= -\frac{\epsilon^2 + (\epsilon - 4|C_m^T||\omega_m|)^2}{4m^2} - |\omega_m|^2(\gamma_m - \frac{4|C_m^T|^2}{m^2}) \tag{77}$$

and

$$\phi_3 = -\frac{\epsilon^2}{2m^2} + \frac{2}{m^2}|\epsilon||C_c^T||\overline{\omega}_\delta| - \gamma_c|\overline{\omega}_\delta|^2$$
$$= -\frac{\epsilon^2 + (\epsilon - 4|C_c^T||\overline{\omega}_\delta|)^2}{4m^2} - |\overline{\omega}_\delta|^2(\gamma_c - \frac{4|C_c^T|^2}{m^2}). \tag{78}$$

We obtain $\phi_1, \phi_2, \phi_3 \leq 0$ by choosing $\gamma_c \geq 4|C_c^T|^2$ and $\gamma_m \geq 4|C_m^T|^2$, which indicates $\dot{V} \leq 0$.

It follows that $\tilde{\theta}, \tilde{\rho} \in \mathcal{L}_\infty$ and $\frac{\epsilon}{m} \in \mathcal{L}_2$. Following the derivation of Theorem 2 yields $\tilde{g} \in \mathcal{L}_2$ and $\omega, m, \xi, \zeta \in \mathcal{L}_\infty$, which together with (35) and $\eta, \Delta \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ gives $\epsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. From (25), we have $\dot{\theta} \in \mathcal{L}_2$, and thus, $\xi \in \mathcal{L}_2$. It follows from (34) and $\epsilon, \xi \in \mathcal{L}_2, \rho \in \mathcal{L}_\infty$ that $e \in \mathcal{L}_2$, which combined with $\dot{e} \in \mathcal{L}_\infty$ reveals $e(t) \to 0$ as $t \to 0$.

Since ζ is PE, the homogeneous part of (63) is exponentially stable, which together with $\xi, \eta, \delta \in \mathcal{L}_2$ implies $\tilde{\theta}(t) \to 0$ as $t \to \infty$. □

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