

Skew Parallelogram Nets: A Common Equation behind many Integrable Systems in Discrete Differential Geometry

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Abstract

This thesis is concerned with *skew parallelogram nets*, a system of lattice equations that may be formulated for any unit associative algebra. The system comes with characteristic features of an integrable system such as multidimensional consistency and an associated family. We obtain many objects of discrete differential geometry as reductions, in particular, K-nets, Voss nets, cross-ratio systems and integrable dynamics like the bicycle transformation and polygon recutting.

The Lax representation is linear in the spectral parameter, and paths in the lattice give rise to polynomial dependencies. We discuss factorization of such polynomials and argue that skew parallelogram nets encompass all systems with a generic polynomial representation.

We provide the factorization explicitly for discrete constant curvature surfaces and reveal that they are slices in certain 4D cross-ratio systems. Encompassing the discrete DPW method, this interpretation constructs such surfaces from given discrete holomorphic maps. We discuss a second factorization of CMC nets for which the associated family gives rise to CMC nets in space forms.

Moreover, we introduce a hierarchy of discrete curves by requiring an integer number of Bäcklund transformations to induce an isometry on the curve. We propose versions of this hierarchy in Euclidean space and in 2-dimensional space forms. We identify discrete elastic rods in space and (constrained) elastic curves in the plane as elements of the hierarchy.

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1. Introduction

In this thesis we aim to unify many 2-dimensional integrable systems studied in discrete differential geometry into reductions of a common lattice equation system.

Let us start by introducing the fundamental equations. Consider a map

$$p : \{\text{directed edges of } \mathbb{Z}^n\} \rightarrow \mathcal{A} \quad (1.1)$$

into a unit associative algebra $\mathcal{A}$ for which the values q, r, s, t at the four edges of every elementary quad (as in Figure 1.1) fulfill the two simple equations

$$q + r = s + t, \quad rq = ts. \quad (1.2)$$

We are interested in geometric objects that can be described by such a map p . In fact, in discrete differential geometry maps of this kind appear in the context of discrete surfaces and in the context of transformations of discrete curves and polygons. The specific geometric object depends on the algebra $\mathcal{A}$ and further constraints on the map p . For example, for the algebra of quaternions this equation system describes parallelograms folded along a diagonal, in other words, skew parallelograms. Thus, we call the map p an (n -dimensional) *skew parallelogram net*.

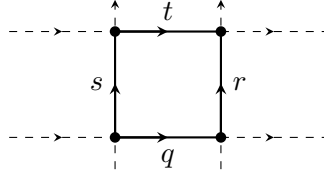


Figure 1.1.: Four values $q, r, s, t \in \mathcal{A}$ associated to the directed edges of an elementary quad of $\mathbb{Z}^2$.

Let us introduce some important concepts. First, observe that (1.2) is equivalent to

$$(\lambda \mathbf{1} + r)(\lambda \mathbf{1} + q) = (\lambda \mathbf{1} + t)(\lambda \mathbf{1} + s) \quad (1.3)$$

holding for every scalar $\lambda \in \mathcal{A}$. Such a description of the equation system via this simple parameter-dependent condition is called a *Lax representation* and λ is called the *spectral parameter*. We will see that one can use the Lax representation to construct

a 1-parameter family of associated solutions of (1.2) to a given solution and we call this the *associated family*. For example, for a discrete surface described by a skew parallelogram net this implies that the surface comes with a deformation which preserves some elementary properties.

Also, we will see that generically the values q, s determine the values r, t uniquely and that one can, therefore, construct a (patch of a) two-dimensional skew parallelogram net from initial values. If we extend the construction to higher dimension there is the ambiguity that (1.2) on different quads yields multiple candidates for some values of p . Now, for skew parallelogram nets all of these candidates coincide and this remarkable property is called *multidimensional consistency*. It implies that (patches of) n -dimensional skew parallelogram nets are constructible from initial data as well. Layers in such a net are $(n - 1)$ -dimensional skew parallelogram nets. Neighboring layers are related by (1.2) on corresponding quads and we say that they are related by a *Bäcklund transformation*.

The aforementioned properties of skew parallelogram nets are characteristic features of a class of equation systems called *integrable systems*. Many such integrable systems come with Lax representations that are polynomial in the spectral parameter. Clearly, our Lax representations (1.3) assigns a linear factor to each edge and, thus, we consider this Lax representation to be linear.

The aim of this thesis is a systematic study of skew parallelogram nets and their geometric applications. We will unify many 2-dimensional integrable systems studied in discrete differential geometry into reductions of this system. In particular, by factorizing polynomial Lax representations into linear factors we will describe some systems of higher order in terms of skew parallelogram nets. We will argue that such a factorization is generically possible and that, therefore, skew parallelogram nets are fundamental to integrable systems of this kind.

Related work

First, let us point out some publications in which we find elementary properties of (1.3) as they are directly concerned with the equation without additional constraints. Skew parallelogram nets for the algebra of quaternions have been studied in [98] as discretization of the (Pohlmeyer-)Lund-Regge equation. In this paper we find extensive study of permutability and multidimensional consistency of (1.3). An associated family obtained by a Sym-formula is introduced. The same nets have also been studied in [55] and [87] in the context of Bäcklund transformations of space curves and the discrete Hashimoto flow. This discrete flow induces a dynamic by evolving periodic curves. Also, the linear Lax representation and its connection to a discrete non-linear Schrödinger equation (NLSE) are observed. Building on the quaternionic case, the observation that the Lax representation and multidimensional consistency of the equation generalize to unit associative algebras has been made in [24, Exercise 6.19 and 6.20].

Next, let us turn to some types of discrete surfaces for which (1.3) appears. Discrete

K-nets (also called *pseudospherical surfaces*) are discretizations of surfaces with constant negative Gaussian curvature. They are built from skew parallelograms and constitute a special case of quaternionic skew parallelogram nets. Their fundamental equation has historically been presented in a slightly different (gauge-equivalent) manner. Work on discrete K-nets from the geometric perspective dates back to [90, 107]. The integrable perspective including their Lax representation and associated family is due to [16, 17]. Bäcklund transformations have originally been studied as natural transformation of K-nets and the sine-Gordon equation (see, for example, [89]). Nets parallel-reciprocal to a K-net are called (discrete) *Voss nets* and they discretize surfaces in conjugate geodesic parametrization [92, 107]. They have a well-known isometric deformation [70, 99, 67] but its explicit description in terms of an associated family has been unknown.

Now, (1.3) is also fundamental for the flat connection of (discrete) *isothermic nets* which are nets with factorizing real cross-ratio [15, 49, 39, 30]. Isothermic nets in arbitrary dimension [95] and in various geometries such as sphere geometries [23] are all described by Moutard nets in quadrics. Moutard nets in quadrics also describe other types of surfaces, for instance, discrete s-CMC nets [57, 71, 12]. Nets with (possibly complex) factorizing cross-ratio in the complex plane are called *cross-ratio systems* and, again, their Lax representation has been shown in [81] to be of the form (1.3). Periodic solutions for cross-ratio systems [3, 4, 52] and isothermic nets [36] induce a dynamic on an initial solution.

For some discretizations of surfaces with constant curvature the relation to (1.3) is more involved since their Lax representation is quadratic in the spectral parameter. For instance, *circular K-nets* [60] are related to 4D cubes of K-nets. Also, for *constant mean curvature (CMC) nets* [17] the specific relations have not been known. However, remarkably, a combinatoric flip allows for a description of the associated family of CMC nets in terms of equally folded parallelogram cubes [62] - which can be understood as factorization on the geometric level since the edges of the CMC nets correspond to diagonals of the skew parallelograms. Smooth CMC nets can be constructed from holomorphic data using the DPW method [40] which generalizes the well-known Weierstraß representation of smooth minimal surfaces. Discrete analogues of Weierstraß representations [85] and the DPW method [53, 84] exist as well. Smooth CMC nets in Euclidean space are known to correspond to minimal surfaces on the sphere $\mathbb{S}^3$ via the Lawson correspondence [74]. A discrete version is introduced in [19].

Skew parallelogram nets can also describe transformations of curves. We already mentioned two dynamics obtained by imposing periodicity on quaternionic parallelogram nets or cross-ratio systems. They have a common special case which is a dynamic on curves in the plane related to the trace of front and rear wheel of a bicycle and is, thus, known as the discrete *bicycle transformation* [105]. It is built from *anti-parallelograms* (or *Darboux butterflies*) which are described by (1.3). Anti-parallelograms also give rise to polygon recutting [2] where the dynamic can be described in terms of refactorization of quaternionic polynomials [68] and the relation to (1.3) is apparent. In fact, quaternionic

polynomials come with a well-known factorization theory [45, 83]. Factorization of such polynomials is also extensively studied in the field of linkage kinematics [76] where it is used to realize rational motions with closed linkages. This has been studied for Euclidean motions [48], hyperbolic motions [94, 93] and, recently, conformal motions [77].

We will also be interested in special curves, specifically (constrained) elastic curves and elastic rods. The study of elastica originally stems from Bernoulli and Euler [42] who described the shape of elastic rods by curves minimizing a bending energy under constrained length. Elastic curves have many interesting properties [37, 42, 73, 106]. In particular, they belong to a hierarchy of commuting energy functionals [72]. This point of view naturally leads to the class of area-constrained elastic curves, which are critical values of the bending energy under constrained length and oriented area [5, 26, 86].

The symplectic gradients of the energy functionals constitute a hierarchy of commuting flows of space curves [35] which serves as geometric approach to the non-linear Schrödinger hierarchy. This hierarchy is also studied in centro-affine space [34]. Every other flow is contained in a plane (if applied to a planar curve) and this sub-hierarchy is known as the modified Korteweg-de Vries (mKdV) hierarchy [44, 78]. Elastic curves are the invariant curves of the Hashimoto flow, while constrained elastic curves are the invariants of the mKdV flow.

A discrete notion of elastic curves in Euclidean space has been introduced in [21]. This integrable discretization relies on the variational approach. Further studies, based on discrete Frenet formulas, can be found in [108]. The latter also demonstrates the relevance of elastic curves in applications by using them for fairing planar curves (see [28] for a smooth version of this algorithm). As indicated in [27, 47], elastic curves are also useful in the design of curved surfaces, and, as demonstrated in [7], for physical simulation of elastic material.

Discrete elastic curves are known to be invariant under a semi-discrete Hashimoto flow [21]. Similarly, discrete area-constrained elastic curves in Euclidean space, characterized as curves that are invariant under a semi-discrete mKdV-flow, have been introduced in [59]. One can also discretize the flow direction. As mentioned above, in [55, 87] a discrete Hashimoto flow for space curves is built from pairs of Bäcklund transformations. The invariant curves of this discrete flow are, again, discrete elastica. Bäcklund and Darboux transformations of discrete curves are also known to provide a (semi-)discrete potential mKdV equation closely related to the discrete sine-Gordon equation [38, 66].

Note that the study of constrained elastic curves has relevance in surface theory since such curves foliate isothermic surfaces with a family of spherical curvature lines in both the smooth [11, 37] and discrete [65] case.

Finally, our discussion of polynomial two-dimensional integrable systems is limited to the scope of discrete differential geometry. It shall be noted there is plenty of literature revolving around polynomial integrable equations that have motivation elsewhere, for instance, in physics. Among others, examples are the motion of a rigid body [80] or

the Ablowitz-Ladik hierarchy [1, 102]. Both are directly connected to the theory of elastic rods and Bäcklund transformations through the discrete NLSE and the discrete Heisenberg magnet [54, 55]. For a comprehensive treatment of this topic, see [103].

Contribution to research

We present the first comprehensive theory of skew parallelogram nets in unit associative algebras. While some properties are straight-forward generalizations from the quaternionic version we provide an in-depth analysis revealing new insights. For instance, we explain how the associated family indeed partitions the set of skew parallelogram nets into 1-parameter families and how the associated family is affected by different parametrizations of the Lax representation.

We then present the systems discussed above (and others) as reductions of skew parallelogram nets. Indeed, the main objective of this thesis is the development of this unified perspective. While some of the systems are well-known, this perspective is new and reveals some previously unknown connections.

First, let us highlight some observations about well-known systems that do not seem to be written in the existing literature:

- By realizing a planar limit of the classical K-net associated family, we observe that every K-net can be unfolded from a net of anti-parallelograms in a plane (see Figure 4.6). This net of anti-parallelograms is a special cross-ratio system and can be seen as geometric interpretation of the Lax representation of K-nets. Also, initial data for periodic K-nets is given by quasi-periodic nets of anti-parallelograms and such nets arise as orbits of polygon recutting or the bicycle transformation. For example, initial data for the pseudosphere can be obtained by polygon recutting applied to a zig-zag.
- We present a formula (5.78) for the construction of the isometric deformation of Voss nets by encoding its kinematic behavior with dual quaternions. Here, the deformation parameter is the spectral parameter. This algebraic description is directly related to the algebraic description of K-nets and initial data for a Voss net is obtained by equipping initial data for a K-net with an additional dual part.
- In Theorem 5.2.1 we present a deformation of nets of anti-parallelograms into non-Euclidean space forms via a Sym-formula. This deformation preserves the local geometry at each vertex. As such it can be related to a fundamental theorem for space form curves since it preserves their curvature. The theorem also immediately yields space form versions of polygon recutting and the bicycle correspondence and it is connected to the Lawson correspondence for discrete CMC surfaces.
- Although algebraic representations of Moutard nets in quadrics in terms of Clifford algebras are known (see [95], [24, Section 6.9]), our construction of an associated

family for such a net and its relation to the Calapso transformation seem to be new.

Also, we will be concerned in detail with discrete surfaces of constant curvature. We will describe them by skew parallelogram nets with special combinatorics:

- We show that surfaces of constant Gaussian curvature (positive or negative) can be described in terms of special 4D cross-ratio systems. These arise as unique extensions of initial 2D cross-ratio systems. By choosing the initial 2D cross-ratio system as a discrete holomorphic map we find a method for the construction of discrete CMC surfaces which simplifies the discrete DPW method. We can also apply this method to construct lattices of breather transformations of negative K-surfaces or lattices of Darboux transformations of CMC surfaces.
- We extend the geometric observation that the associated family of CMC surfaces consists of equally folded parallelogram cubes after a combinatoric flip [62]. We show how this associated family coincides with the associated family of skew parallelogram nets and how this algebraic description yields discrete CMC surfaces in space forms as well.

Finally, we contribute to the theory of flows and elastica by discussing a hierarchy of discrete curves:

- We introduce some elementary discrete curve theory in space forms using a matrix model. It extends known geometric concepts from Euclidean geometry. We propose a notion of discrete (constrained) elastic curves in space forms.
- We propose the hierarchy of n -invariant curves which are invariant under an integer number of Bäcklund transformations. Specifically, this means that the n -th transform of the curve is related to it by an isometry. Differently speaking, these nets arise as parameter lines of skew parallelogram nets with periodicity. We propose versions of this hierarchy for curves in Euclidean space and for plane curves. The second approach directly extends to non-Euclidean geometry. We show how to construct n -invariant curves.
- We show how n -invariant curves describe discrete elastica. Specifically, we find (constrained) elastic curves in space forms as elements of the hierarchy in the plane while we find helices (even in general parametrization) and elastic rods as elements of the hierarchy in space. Our findings extend a result in [55] which already states that curves invariant under a double Bäcklund transformation are elastic rods. We also show the converse, meaning that we show how to construct the invariant sequences of Bäcklund transformations to a given curve.

Overview of chapters

In Chapter 2 we establish notation and some important concepts and definitions of Discrete Differential Geometry.

Chapter 3 introduces *skew parallelogram nets* in the general setting of unit associative algebras. Sections 3.1, 3.2 and 3.3 contain the main definition and elementary properties such as the Cauchy problem, multidimensional consistency, the Lax representation and the associated family. In Section 3.4 we give some details about the structure of the associated family and in Section 3.5 we discuss skew parallelogram nets with a periodicity condition.

Chapter 4 is devoted to the study of our equation system for the algebra of quaternions. Here, our nets consist of skew parallelograms in Euclidean space. By applying the concepts of Chapter 3 to this case, Section 4.2 motivates many notions introduced for our general theory. In Section 4.3 we investigate the interplay of geometry and algebra for skew parallelograms. In the remaining sections we discuss how several known geometric concepts arise as special cases of our theory. This includes K-nets, the curve dynamics of polygon recutting and the bicycle transformation and its generalization to space via quasi-periodic Bäcklund transformations. As we observe in Section 4.7, quaternionic skew parallelogram nets can also describe Moutard nets on the sphere.

Chapter 5 discusses examples of our system in algebras different from the quaternions. We start with the matrix algebra $\mathbb{C}^{2 \times 2}$ which naturally extends the quaternions. In Section 5.2 we use this algebra to describe Moutard nets in hyperbolic space and we find that one can use the associated family to deform nets of anti-parallelograms into Moutard nets in space forms (Theorem 5.2.1) while preserving the local geometry at the vertices. Next, in Section 5.3 and 5.4 we review a natural description of cross-ratio systems in this algebra and an analogous description of isothermic nets in the algebra $\mathbb{H}^{2 \times 2}$. As presented in Section 5.5, one can describe any Moutard net in a quadric using skew parallelogram nets in Clifford algebras and this description comes with an associated family that agrees with the Calapso transformation in the case of isothermic nets. In Section 5.6 we discuss the algebra of dual quaternions for which parallelogram nets include a natural description of Voss nets which also encompasses a formula for the kinematic behavior of their isometric deformation. Finally, in Section 5.7 we remark on two more systems with linear Lax pairs.

The Lax representation of skew parallelogram nets assigns polynomials to paths on the lattice. In Chapter 6 we discuss factorization of such polynomials. We introduce some general theory and provide more details for polynomials over the quaternions and over the matrix algebra $\mathbb{C}^{2 \times 2}$. We motivate the factorization of polynomial systems into skew parallelogram nets. In the remaining chapters we discuss systems of higher order factorized into skew parallelogram nets and, thus, these chapters can be seen as a demonstration of the concept of factorization.

Chapter 7 discusses description of some types of discrete surfaces in terms of skew

parallelogram nets. We consider surfaces with constant Gaussian or constant mean curvature (CMC). Their Lax representations can be interpreted as quadratic in the spectral parameter and we present factorized versions. In particular, the factorization corresponding to surfaces of constant Gaussian curvature yields 4D cross-ratio systems with special properties (which we call C^+ and C^- lattice). We introduce these cross-ratio systems in Section 7.2 and relate them to corresponding surfaces in Section 7.3 and 7.4. For positive Gaussian curvature and CMC surfaces our construction corresponds to the DPW method. In Section 7.5, we give another description of CMC surfaces in terms of quaternionic parallelogram nets using a combinatoric flip. We can apply our framework to find an associated family which yields surfaces in non-Euclidean space forms and this construction corresponds to a Lawson correspondence.

In Chapter 8 we introduce the hierarchy of n -invariant curves in Euclidean space which are curves for which there exist a sequence of Bäcklund transformations that preserve the shape of the curve. Alternatively, one can characterize n -invariant curves as parameter lines of a quasi-periodic skew parallelogram net. We motivate the connection to flow theory by interpreting the sequence of transformations as discrete flow of the curve. In Section 8.3 we show that 1-invariant curves are discrete helices, an observation that extends to curves in general parametrization. In Section 8.4 we show that 2-invariant curves are exactly elastic rods.

In Chapter 9 we discuss a planar version of the hierarchy of n -invariant curves which naturally extends to space forms. We start with discrete curve theory in space forms and propose a notion of discrete constrained elastic curves. We show that curves are constrained elastic if and only if they are 3-invariant (Theorem 9.4.11) by proving this result in Euclidean space in Section 9.5 and extending the result to space forms using the associated family in Section 9.6.

Submitted preprints and contribution in collaboration

The investigation of skew parallelogram nets as general equation system encompassing many other systems has been part of a project joint with Tim Hoffmann and Andrew O. Sageman-Furnas. Some parts of this thesis are summarized in the submitted preprint [61] and parts of the preprint have significant overlap with corresponding parts of this thesis. The present author's work on this project builds on known properties of skew parallelogram nets together with initial ideas of his co-authors. His contribution lies in the systematic incorporation of old and new systems along with their respective details into one framework. With input and advice from the co-authors the author formulated the theory of skew parallelogram nets and wrote the manuscript. During this process the author found new perspectives such as the explicit relations to polygon recutting and kinematics, the description of K-nets that unfold from anti-parallelograms in the plane and the connection between discrete surfaces of constant curvature, 4D cross-ratio systems and the DPW method.

The introduction of discrete constrained elastic curves in space forms and its relation to the hierarchy of n -invariant curves in the plane has been joint work with Tim Hoffmann and Gudrun Szewieczek and a submitted preprint is available [64]. Chapter 9 of this thesis is a shortened version of the preprint. The contributions of the present author are the introduction of the matrix model (contained in Section 9.1, 9.2) for discrete curves in space forms and the investigation of the hierarchy of n -invariant curves (Section 9.4, 9.5 and 9.6). In the preprint we also prove some results using a light-cone model. The corresponding sections have been written by Gudrun Szewieczek and are, thus, only summarized (as in the case of Section 9.3) or not contained at all in this thesis.

2. Preliminaries: Discrete nets

In this thesis we describe discrete curves and surfaces by maps $f : \mathbb{Z}^n \rightarrow V$ where V is some vector space. We call such maps (*discrete*) *nets*. For example, a map $f : \mathbb{Z} \rightarrow \mathbb{R}^2$ describes a polygonal chain in the plane and we interpret it as a discrete curve. Similarly, a map $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ describes a discrete surface. In this chapter, we introduce our notation for such nets and discuss combinatorics. We also introduce some well-known types of nets and some important concepts in Discrete Differential Geometry.

2.1. Notation

Throughout, we consider (subsets of) the n -dimensional integer lattice $\mathbb{Z}^n$ as a graph. We identify $\mathbb{Z}^n$ with its vertex set $\mathcal{V}(\mathbb{Z}^n)$. Its set of undirected edges is given by

$$\mathcal{E}(\mathbb{Z}^n) = \{ \{x, x + e_i\} \mid x \in \mathbb{Z}^n, i \in [n] \}, \quad (2.1)$$

where $e_i \in \mathbb{Z}^n$ denotes the i -th unit vector. We will also consider the set of directed edges

$$\vec{\mathcal{E}}(\mathbb{Z}^n) = \{ (x, x + e_i) \mid x \in \mathbb{Z}^n, i \in [n] \}. \quad (2.2)$$

Consider a vertex based map $f : \mathbb{Z}^n \rightarrow V$. Often, we will be interested in its local properties, for example, properties that only depend on values of f at a vertex and its neighbors. For this we use *shift notation* as in Figure 2.1: For the value $f(x)$ at an arbitrary vertex x we simply write f . The value $f(x + e_i)$ at the adjacent vertex $x + e_i$ is denoted by f_i . Similarly, we write $f_{\bar{i}}$ short for $f(x - e_i)$. We can also encode multiple shifts. For example, $f_{ij} = f_{ji}$ is short for $f(x + e_i + e_j)$ and $f_{\bar{i}\bar{j}kk}$ is short for $f(x - 2e_i + e_j + 2e_k)$.

Now, consider an edge based map $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow V$. We write $p = (p^1, \dots, p^n)$ where $p^j(x) := p(x, x + e_j)$ denotes the value of p at the edge heading from $x \in \mathbb{Z}^n$ in direction j . As also exemplified in Figure 2.1, we use shift notation for edge based maps as well: For an arbitrary $x \in \mathbb{Z}^n$ we write p^j short for $p^j(x)$ and denote $p^j(x + e_i)$ and $p^j(x - e_i)$ by p_i^j and $p_{\bar{i}}^j$ respectively. Often, we will discuss nets of dimension $n \leq 3$. To simplify notation in such cases we denote $u := p^1, v := p^2, w := p^3$ (see, e.g., Figure 3.2).

An (*elementary*) *quad* or *face* of our graph is given by four vertices $x, x + e_i, x + e_i + e_j, x + e_j$. Many equations will be formulated in terms of values at vertices f, f_i, f_{ij}, f_j or

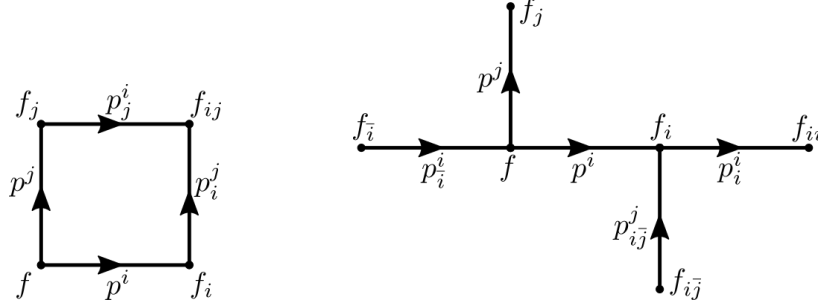


Figure 2.1.: Some examples of shift notation. On the left is an elementary quad.

of values at edges p^i, p^j, p_i^j, p_j^i at a quad. Here, we assumed $i \neq j$. Throughout, we will always assume that different indices i, j are distinct. More formally, the set of oriented faces is

$$\vec{\mathcal{F}}(\mathbb{Z}^n) = \{ (x, x + e_i, x + e_i + e_j, x + e_j) \mid x \in \mathbb{Z}^n, i, j \in [n], i \neq j \}. \quad (2.3)$$

For a face based map $a : \vec{\mathcal{F}}(\mathbb{Z}^n) \rightarrow V$, we write a^{ij} for each value $a(x, x + e_i, x + e_i + e_j, x + e_j)$. Usually, the two values at a face a^{ij} and a^{ji} will be related, for example, by $a^{ji} = -a^{ij}$.

Sometimes we will discuss global properties and constructions. Then, the origin of the lattice plays a special role and we denote it by $\mathbf{0} := (0, \dots, 0) \in \mathbb{Z}^n$.

Often, we investigate nets mapping to a unit associative $\mathbb{K}$ -algebra $V = \mathcal{A}$. For example, we will be interested in nets mapping to the quaternions or to matrix algebras. We denote the unit element of the algebra $\mathcal{A}$ by $\mathbf{1}$ and identify scalars $\lambda \in \mathbb{K}$ with the corresponding multiple of the unit element $\lambda \mathbf{1} \in \mathcal{A}$. In particular, this defines addition of scalars with elements of the algebra.

2.2. Special nets

Some types of nets appear several times throughout this thesis.

Definition 2.2.1 A map $f : \mathbb{Z}^n \rightarrow \mathbb{R}^m$ is called a *circular net* if the four points f, f_i, f_{ij}, f_j of every quad lie on a common circle.

This implies that the four points are also co-planar. Circular nets $\mathbb{Z}^2 \rightarrow \mathbb{R}^3$ describe discrete surfaces in curvature line parametrization. For an overview of circular nets see [24, Section 3.1].

Definition 2.2.2 A map $f : \mathbb{Z}^n \rightarrow \mathbb{R}^m$ is called a *Moutard net* if there exists a face based map $a : \vec{\mathcal{F}}(\mathbb{Z}^n) \rightarrow \mathbb{R}$ such that for all $i, j \in [n]$ with $i \neq j$ we have

$$f_{ij} - f = a^{ij}(f_j - f_i). \quad (2.4)$$

Clearly, we have $a^{ji} = -a^{ij}$. Equation (2.4) is called the *Moutard equation*. It means that the diagonals of the quad are parallel. Note that the quads are self-crossing. Moutard nets are also called *T-nets*. For an overview of Moutard nets see [24, Section 2.3].

Remark 2.2.3 *In Discrete Differential Geometry, an important question is whether one can take a smooth limit of a discrete equation. Equation (2.4) does, in fact, not have a smooth limit. However, in dimension $n = 2$ we can define a net f by flipping every second point of f in one lattice direction, for example, by $\hat{f}(k, l) = (-1)^l f(k, l)$. Then, this flipped net $\hat{f}$ fulfills*

$$f_{12} + f = a^{12}(f_1 + f_2) \quad (2.5)$$

which is also called a discrete Moutard equation and has a smooth limit. We will encounter such a flip for normal maps to discrete surfaces.

Can a quad be both circular and fulfill a Moutard equation? This happens in two cases: There is the degenerate case where one diagonal vanishes (i.e., $f_i = f_j$ or $f_{ij} = f$). Often, this case will be excluded from our considerations. The other case is the case of the anti-parallelogram as in Figure 2.2 on the left. For such a quad the perpendicular bisectors of the diagonals agree and form an axis of symmetry. Therefore, any three points of an anti-parallelogram determine the fourth point uniquely: Two of the points determine one of the diagonals which determines the axis of symmetry and the fourth point can then be obtained by reflecting the third point along this axis.

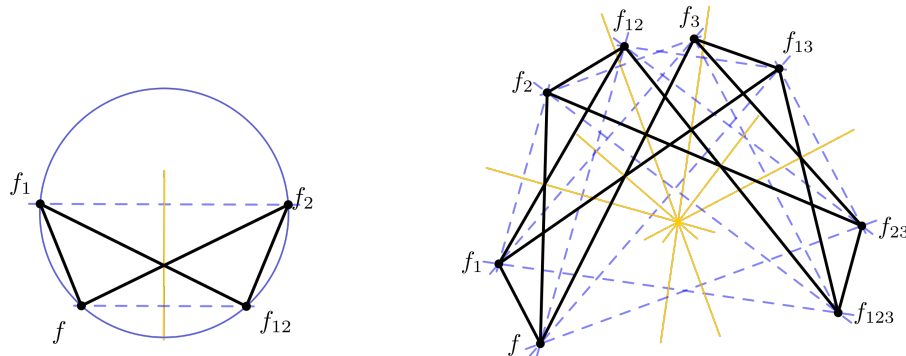


Figure 2.2.: Left: An anti-parallelogram with its axis of symmetry. Right: A combinatorial cube of which every quad is an anti-parallelogram.

An important property is that opposite edges of an anti-parallelogram have the same length. This property is known as the *Chebyshev property*:

Definition 2.2.4 A map $f : \mathbb{Z}^n \rightarrow \mathbb{R}^3$ is called a Chebyshev net if opposite edges of each elementary quad have the same length, i.e., if

$$\|f_{ij} - f_i\| = \|f_j - f\|. \quad (2.6)$$

This property of the edge-lengths is also called *labelling property*:

Definition 2.2.5 A map $\alpha : \mathcal{E}(\mathbb{Z}^n) \rightarrow V$ is said to have the labelling property if its values on opposite edges of each quad agree, i.e., if for $i \neq j$

$$\alpha_j^i = \alpha_i^j. \quad (2.7)$$

We also call such a map an edge labelling.

A quad of a Chebyshev net is not necessarily planar. In fact, it is planar if and only if it is a parallelogram or an anti-parallelogram. All Chebyshev quads (short for: quads of a Chebyshev net) are obtained by folding a parallelogram into space along its diagonal. This is a geometric interpretation in three dimensions since rotation around an axis is a three-dimensional operation. Thus, we consider Chebyshev nets only in three-dimensional ambient space, even though the definition can be extended to higher dimensions as well. We also call Chebyshev quads *skew parallelograms*. A main geometric object studied throughout this thesis will be the *quaternionic skew parallelogram net* (introduced in Chapter 4) which is an algebraic description of Chebyshev nets which fulfill an additional property.

We will now introduce some concepts studied in Discrete Differential Geometry using nets of anti-parallelograms as an example. How can we construct such a net from given initial data? This is called the *initial value problem* or *Cauchy problem* for this net. For example, suppose we want to construct a two-dimensional net $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^m$ consisting of anti-parallelograms from given values $f(k, 0)$ and $f(0, l)$ on the axes. Then, we can iteratively construct the net using the fact that three points of a quad determine the fourth point. Thus, the values on the axes are independent variables that uniquely determine the net. Such initial data is also called *Cauchy data*.

If we want to extend this to three-dimensional nets $f : \mathbb{Z}^3 \rightarrow \mathbb{R}^m$ of anti-parallelograms we find an obstruction in the construction of elementary cubes as in Figure 2.2 on the right: Four initial points f, f_1, f_2 and f_3 determine f_{12}, f_{13} and f_{23} uniquely by completing the corresponding three anti-parallelograms. Completing the remaining three anti-parallelograms gives three more points. Remarkably, one can show that these points all coincide and, thus, give a well-defined eighth point f_{123} of the cube. A proof can be found in [24, Theorem 4.3] and we will also find it as a consequence of Proposition 3.2.1. From this, we conclude that we can construct three-dimensional nets of anti-parallelograms by prescribing values on the axes. This property is called *3D-consistency* and is an indicator of the concept of integrability of the corresponding equations. Since cubes of

dimension $n > 3$ consist of three-dimensional cubes one can check that 3D-consistency implies consistency in any dimension, also called *multidimensional consistency*. Thus, one can construct nets of anti parallelogram nets in any dimension $f : \mathbb{Z}^n \rightarrow \mathbb{R}^m$ from initial data on the axes.

For circular nets and Moutard nets the Cauchy problem and consistency are slightly more complicated since the fourth point of a quad is not uniquely determined by the other three. Still, such nets can be constructed in higher dimension using the fact that, in both cases, seven points of a cube determine a unique eighth point. Chebyshev nets do not have this property: There is no unique eighth point that completes a cube of a Chebyshev net for arbitrary initial values. For this, observe that a suitable eighth point f_{123} is given by intersecting the three spheres around f_{12}, f_{13} and f_{23} with radii $\|f_3 - f\|, \|f_2 - f\|$ and $\|f_1 - f\|$ respectively. However, the intersection of three spheres generically contains two points or no point at all. For the special case of quaternionic skew parallelogram nets (as introduced in Chapter 4) the fourth point of each quad will be unique and applying a result of Section 3.2 will show 3D-consistency of this system.

Finally, consider a net consisting of anti-parallelograms $f : \mathbb{Z}^{n+1} \rightarrow \mathbb{R}^m$. For each fixed $k \in \mathbb{Z}$ the map $\mathbb{Z}^n \rightarrow \mathbb{R}^m$ given by $x \mapsto f(x, k)$ describes a layer of f . Each such layer is a net of anti-parallelograms as well. Neighboring layers are related by anti-parallelograms and are, thus, not independent of each other. One considers two such nets to be related by a *Bäcklund transformation* or *Darboux transformation*. We will stick to the name Bäcklund transformation since they are strongly related to the well-known Bäcklund transformation of K-nets which we consider in Section 4.4. Similarly, other types of nets like circular nets and Moutard nets also have a natural transformation defined by the condition on two nets to be neighboring layers in a corresponding net of higher dimension. For a detailed discussion on these transformations see [24] and references therein.

3. Skew parallelogram nets

In this chapter, we introduce skew parallelogram nets and their properties in unit associative algebras. For the simplest geometric application of this theory the author recommends the reader to also study Section 4.2.

3.1. The skew parallelogram equations and evolution

Let $\mathcal{A}$ be a unit associative algebra over the field $\mathbb{K}$, with $\mathbb{K}$ being the real or complex numbers. Sometimes, we will discuss the case of a matrix algebra since all algebras considered in this thesis can be represented as (subalgebras of) matrix algebras.

Definition 3.1.1 *An n -dimensional skew parallelogram net is a map $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathcal{A}$ such that its values at the four edges of each elementary quad fulfill*

$$p^i + p_i^j = p^j + p_j^i \quad (3.1)$$

and

$$p_i^j p^i = p_j^i p^j. \quad (3.2)$$

We can interpret these equations as both addition and multiplication (in the right order) being path-independent. For brevity we refer to p as a *parallelogram net*. Often we will consider 2-dimensional parallelogram nets for which the equations read $u+v_1 = v+u_2$ and $v_1 u = u_2 v$.

A *primitive map* of p is a vertex based map $f : \mathbb{Z}^n \rightarrow \mathcal{A}$ generated by p and an initial point $f(\mathbf{0})$. In this thesis we consider

- the *additive primitive map* generated by $f_i := f + p^i$,
- the *multiplicative primitive map* generated by $f_i := p^i f$ and
- primitive maps generated from a sandwich product by $f_i := p^i f (p^i)^{-1}$.

Here, we call the operation $x \mapsto axa^{-1}$ *sandwich product* to avoid confusion with conjugation of complex numbers and quaternions.

Equation (3.1) implies that the additive primitive map is well-defined. Provided p^i is invertible, equation (3.2) implies that the other two primitive maps are well-defined.

3. Skew parallelogram nets

Throughout this thesis we relate primitive maps f to various geometric objects, in particular, to curves and surfaces. Nets in Euclidean geometry are usually obtained from the additive primitive map while the other primitive maps can generate nets in quadrics, for example, normals to a surface or nets in a non-Euclidean geometry. The specific objects we get depend on the choice of $\mathcal{A}$ and further constraints on the parallelogram net p . In many examples, p can be seen as a discrete derivative of f which motivates the name *primitive map*.

One can easily construct a new parallelogram net $\tilde{p}$ from a given solution p :

- By a sandwich product: $\tilde{p}^j = cp^jc^{-1}$, $c \in \mathcal{A}$ invertible.
- By scaling: $\tilde{p}^j = sp^j$, $s \in \mathbb{K} \setminus \{0\}$.
- By adding a scalar: $\tilde{p}^j = p^j + r$, $r \in \mathbb{K}$ (recall that we identify $r \in \mathbb{K}$ with $r\mathbf{1} \in \mathcal{A}$).

If two parallelogram nets can be obtained from each other by these operations, we call them *primarily equivalent*. In many cases, primarily equivalent parallelogram nets will describe geometric objects which are immediately related, for example, by a rigid motion. Primary equivalence partitions the set of parallelogram nets and, sometimes, it will be useful to consider representatives of an equivalence class that have special properties.

Proposition and Definition 3.1.2 *We call a parallelogram net p evolvable if its diagonals $p^j - p^i = p_i^j - p_j^i$ for $i \neq j$ are invertible. An evolvable parallelogram net p satisfies the evolution equations*

$$p_j^i = (p^j - p^i)p^i(p^j - p^i)^{-1} \quad (3.3)$$

and

$$p^i = (p_i^j - p_j^i)^{-1}p_j^i(p_i^j - p_j^i). \quad (3.4)$$

Proof. Eliminating p_i^j by plugging (3.1) into (3.2) yields

$$(p^j - p^i)p^i = p_j^i(p^j - p^i) \quad (3.5)$$

and, equivalently,

$$(p_i^j - p_j^i)p^i = p_j^i(p_i^j - p_j^i) \quad (3.6)$$

□

In the simplest case, where $\mathcal{A}$ is commutative, this evolution is the identity. The non-commutative case is more interesting and, thus, we will only consider non-commutative algebras. Still, the evolution has many invariants. For example, if $\mathcal{A}$ is a matrix algebra, the matrices p^j and p_i^j are similar, implying that their characteristic polynomials

coincide. Hence, the coefficients of the characteristic polynomial are invariants of the evolution. In the case where $\mathcal{A}$ is the algebra of quaternions these invariants are (formulas of) the real part and the length. In all cases, every such invariant α is an edge based map $\mathcal{E}(\mathbb{Z}^n) \rightarrow \mathbb{K}$ which has the labelling property meaning that α^i only depends on the i -th coordinate or, equivalently, that $\alpha_j^i = \alpha^i$ for $i \neq j$.

Remark 3.1.3 Consider a matrix algebra $\mathcal{A}$ and assume that the values of p are diagonalizable. As observed in [98], the evolution (3.3) can be linearized: If we write $p^i = Y^i D^i (Y^i)^{-1}$ for a diagonal matrix D^i then D^i has the labelling property and the evolution for Y^i is

$$Y_j^i = Y^i D^i - p^j Y^i. \quad (3.7)$$

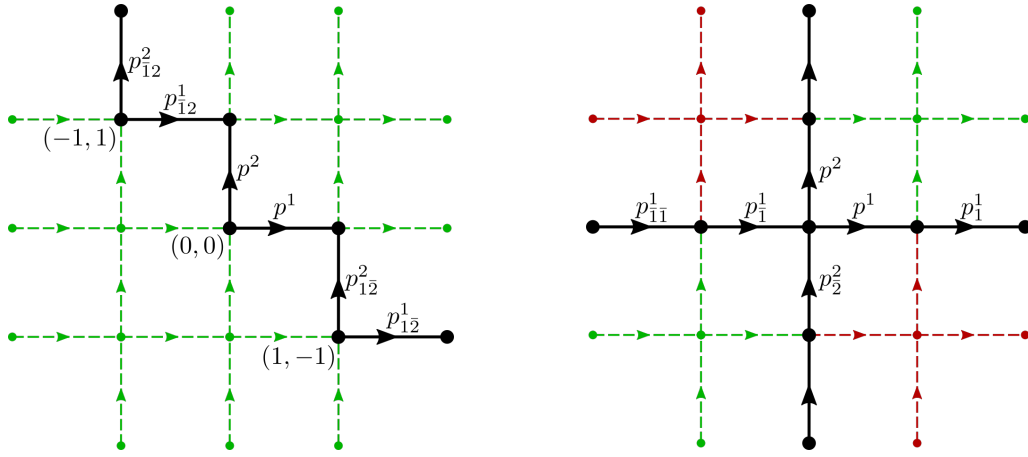


Figure 3.1.: Initial data for two-dimensional parallelogram nets: From a zig-zag (left) we can construct all values of p while from the axes (right) we can only construct the values of p on two quadrants.

Let us turn to the Cauchy problem of parallelogram nets. Consider an evolvable two-dimensional parallelogram net and its values $p^i(k, -k)$ on a zig-zag path as in Figure 3.1 on the left. Now, all values of p to the top right of the path can be obtained from the values on the path using the forwards evolution (3.3). Similarly, all values on the bottom left are determined by values on the path using the backwards evolution (3.4). Therefore, the values on the zig-zag determine the parallelogram net. One can also consider the values on the axes as in Figure 3.1 on the right. Then, only the top right and the bottom left quadrant are determined. This extends to n -dimensional parallelogram nets where, also, only two quadrants are determined. To reconstruct all quadrants from the values on the axes we need a sideways evolution formula to determine p^i, p_i^j from p^j, p_j^i . Surprisingly, this needs special properties of the algebra $\mathcal{A}$ and turns out to be strongly related to the factorization of polynomials as discussed in Chapter 6.

Remark 3.1.4 *The reason why the sideways evolution does not exist in the general setting is that the values of p live on directed edges. It is desirable to find a way to redirect the edges in such a manner that versions of (3.1), (3.2) still hold. In some algebras there is a canonical candidate: Assume there exists a linear involution $f \mapsto f^*$ with the properties $f + f^* \in \mathbb{K}$, $ff^* \in \mathbb{K}$ and $(fg)^* = g^*f^*$ (for example, for quaternions this is the quaternionic conjugate). For evolvable parallelogram nets one can compute that $r^i := p^i + (p^i)^*$ and $s^i := p^i(p^i)^*$ are invariants of the evolution (3.3) and, hence, if $s^i \neq 0$ we have*

$$(p^i)^* + p^j = r^i - p^i + p^j = r^i + p_i^j - p_j^i = p_i^j + (p_j^i)^* \quad (3.8)$$

$$p^j(p^i)^* = \frac{1}{s^i}(p_j^i)^*p_j^ip^j(p^i)^* = \frac{1}{s^i}(p_j^i)^*p_i^jp^i(p^i)^* = (p_j^i)^*p_i^j \quad (3.9)$$

Thus, it is reasonable to define $p(x + e_i, x) := p(x, x + e_i)^*$ since these values fulfill redirected versions of (3.1) and (3.2). The natural extension of the additive primitive map defined by $f(y) = f(x) + p(x, y)$ for adjacent $x, y \in \mathbb{Z}^n$ is only well-defined up to addition of a scalar since going back and forth along an edge or going around closed loops produces such a scalar. Analogously, the natural extension of the multiplicative primitive map is only defined up to scalar multiplication.

The evolution equations allow us to construct (quadrants of) a two-dimensional parallelogram net from generic values on the initial zig-zag or on the axes. For this, we verify that the values on a quad obtained from evolution (3.3) indeed fulfill (3.1) and (3.2):

$$\begin{aligned} p^j + p_j^i &= (p^j(p^j - p^i) + (p^j - p^i)p^i)(p^j - p^i)^{-1} = ((p^j)^2 - (p^i)^2)(p^j - p^i)^{-1} \\ &= ((p^i)^2 - (p^j)^2)(p^i - p^j)^{-1} = \dots = p^i + p_i^j \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} p_j^ip^j &= (p^j - p^i)p^i(p^j - p^i)^{-1}p^j = (p^j - p^i)(p^i + p^i(p^j - p^i)^{-1}p^i) \\ &= (p^j - p^i)p^j(p^j - p^i)^{-1}p^i = p_i^jp^i. \end{aligned} \quad (3.11)$$

For (3.4) similar calculations can be made. Many examples in this thesis have been computed in this manner by prescribing initial data on the axes. We saw that the axes and the zig-zag serve as Cauchy data for (quadrants of) two-dimensional parallelogram nets. We will see that this concept extends to more dimensions using the 3D-consistency of parallelogram nets which we discuss in the next section.

As a bibliographic remark, note that the parallelogram equations and evolution have been studied before. We will see throughout this thesis that they appear in different contexts. For example, for the algebra of quaternions they appear as discrete flows on curves in [87, 55] and as discrete Lund-Regge surfaces in [98]. Our approach of presenting a unifying theory in unit associative algebras is new, however, it shall be noted that some of its elementary properties have already been observed in [24, Exercise 6.19 and 6.20].

3.2. Multidimensional consistency and Bäcklund transformations

Multidimensional consistency of parallelogram nets is already well-understood. In particular, in [98] we find a detailed discussion on consistency from the perspective of permutability for parallelogram nets in the quaternions. Another permutability result can be found in [87]. We review a result for unit associative algebras given in [24, Exercise 6.19]:

Proposition 3.2.1 *The evolution (3.3) is 3D-consistent in the following sense: Let (u, v, w) be initial values on a cube as in Figure 3.2. Then, applying (3.3) to the faces of this cube yields two possible formulas for each value u_{23}, v_{13}, w_{12} respectively and the results of both formulas coincide.*

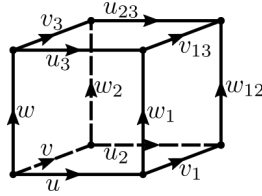


Figure 3.2.: Notation around a cube.

Proof. Consider some initial values u, v, w on three edges in a three-dimensional cube. Assuming invertible diagonals we can use the evolution (3.3) to find two values $\hat{w}_{12}$ and $\tilde{w}_{12}$ which are candidates for w_{12} :

$$\begin{aligned} v_1 &:= (v - u)v(v - u)^{-1}, & w_1 &:= (w - u)w(w - u)^{-1}, \\ \hat{w}_{12} &:= (w_1 - v_1)w_1(w_1 - v_1)^{-1} = ((w_1 - v_1)(w - u))w((w_1 - v_1)(w - u))^{-1}, \\ u_2 &:= (v - u)u(v - u)^{-1}, & w_2 &:= (w - v)w(w - v)^{-1}, \\ \tilde{w}_{12} &:= (w_2 - u_2)w_2(w_2 - u_2)^{-1} = ((w_2 - u_2)(w - v))w((w_2 - u_2)(w - v))^{-1}. \end{aligned} \quad (3.12)$$

To prove $\hat{w}_{12} = \tilde{w}_{12}$ it suffices to show $(w_1 - v_1)(w - u) = (w_2 - u_2)(w - v)$. We have

$$\begin{aligned} (w_1 - v_1)(w - u) &= (w - u)w - (v - u)v(v - u)^{-1}(w - u) \\ &= (v - u)(x_u w - v x_u), \\ (w_2 - u_2)(w - v) &= (w - v)w - (v - u)u(v - u)^{-1}(w - v) \\ &= (v - u)(x_v w - u x_v), \end{aligned} \quad (3.13)$$

3. Skew parallelogram nets

where $x_u := (v - u)^{-1}(w - u)$ and $x_v := (v - u)^{-1}(w - v)$. Note that $x_u - x_v = 1$ and hence

$$\begin{aligned} x_u w - v x_u &= (1 + x_v)w - v(1 + x_v) = x_v w - v x_v + (w - v) \\ &= x_v w - v x_v + (v - u)x_v = x_v w - u x_v. \end{aligned} \quad (3.14)$$

□

This statement on 3D-consistency implies that (quadrants of) three-dimensional nets can be constructed from generic Cauchy data. Note that this also extends to dimension $n > 3$ and, thus, we consider parallelogram nets to be multidimensionally consistent.

The evolution (3.3) requires invertibility of diagonals of p and we can make a statement about consistency without this requirement:

Proposition 3.2.2 *Consider invertible values $u, v, w \in \mathcal{A}$ on the edges of a cube as in Figure 3.2. If the parallelogram equations (3.1), (3.2) are fulfilled on five faces of the cube then they are also fulfilled on the sixth face.*

This result is straight-forward but does not seem to appear in existing literature.

Proof. We can obtain the equations on the top face from the equations on the other faces:

$$\begin{aligned} v_3 + u_{23} &= (-w + v + w_2) + (-w_2 + u_2 + w_{12}) \\ &= (-w + u + w_1) + (-w_1 + v_1 + w_{12}) = u_3 + v_{13} \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} u_{23}v_3 &= u_{23}v_3ww^{-1} = u_{23}w_2vw^{-1} = w_{12}u_2vw^{-1} \\ &= w_{12}v_1uw^{-1} = v_{13}w_1uw^{-1} = v_{13}u_3. \end{aligned} \quad (3.16)$$

Other faces work analogously. □

Finally, we define a natural transformation for parallelogram nets:

Definition 3.2.3 *Two n -dimensional parallelogram nets p and $\tilde{p}$ are said to be related by a Bäcklund transformation if there exists a map $v : \mathbb{Z}^n \rightarrow \mathcal{A}$, such that at every edge in $\vec{\mathcal{E}}(\mathbb{Z}^n)$ we have*

$$p^i + v_i = v + \tilde{p}^i \quad \text{and} \quad v_i p^i = \tilde{p}^i v. \quad (3.17)$$

Consider an $(n + 1)$ -dimensional parallelogram net p and two neighbouring layers of $\mathbb{Z}^{n+1}$, for example, the layers given by points $(x, 0)$ for $x \in \mathbb{Z}^n$ and by points $(x, 1)$ for $x \in \mathbb{Z}^n$ respectively. Then, the two n -dimensional parallelogram nets at these layers are

related by a Bäcklund transformation where $v(x) = p((x, 0), (x, 1))$. In this way every Bäcklund transformation is just a two-layered $(n + 1)$ -dimensional parallelogram net. We call $\tilde{p}$ a *Bäcklund transform* of p . Generically, 3D-consistency implies that one can construct a Bäcklund transform of a given parallelogram net p from p and one initial value $v(0)$.

3.3. Lax representation and associated family

Linear Lax representations for parallelogram nets in the quaternions have appeared in [98, 87, 55] and an associated family is introduced in [98]. For unit associative algebras we find a Lax representation in [24, Exercise 6.20]. In this section, we present a generalized unified approach to the Lax representation that includes these known representations.

Many systems of discrete equations can be formulated in terms of an edge based map P depending on additional parameters: Such a map is called a *Lax representation* if the systems of equations is equivalent to the *compatibility condition*

$$P_i^j P^i = P_j^i P^j. \quad (3.18)$$

Lax representations (or *Lax pairs*) are an important tool in the theory of integrable systems [75]. Throughout this thesis, we will relate many Lax pairs studied in the field of Discrete Differential Geometry to the following Lax representation of parallelogram nets:

Definition 3.3.1 (Lax representation) *To an edge based map $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathcal{A}$ we assign the map $P : \vec{\mathcal{E}}(\mathbb{Z}^n) \times \mathbb{K}^2 \rightarrow \mathcal{A}$ defined by*

$$P(d, \lambda, \mu) := \lambda + \mu p(d). \quad (3.19)$$

Again, recall that addition of scalars and elements of the algebra is defined by identifying $\lambda \in \mathbb{K}$ with $\lambda \mathbf{1} \in \mathcal{A}$.

Proposition 3.3.2 *An edge based net p is a parallelogram net if and only if the Lax representation (3.19) fulfills the compatibility condition (3.18) for all $\lambda, \mu \in \mathbb{K}$.*

Proof. Simply expand

$$\begin{aligned} P_i^j P^i &= (\lambda + \mu p_i^j)(\lambda + \mu p^i) = \lambda^2 + \lambda \mu (p^i + p_i^j) + \mu^2 p_i^j p^i, \\ P_j^i P^j &= (\lambda + \mu p_j^i)(\lambda + \mu p^j) = \lambda^2 + \lambda \mu (p^j + p_j^i) + \mu^2 p_j^i p^j. \end{aligned} \quad (3.20)$$

Hence, $P_i^j P^i = P_j^i P^j$ for all $\lambda, \mu \in \mathbb{K}$ is equivalent to the parallelogram equations (3.1) and (3.2). $\square$

3. Skew parallelogram nets

A key property of parallelogram nets is that this Lax representation is $\mathbb{K}$ -linear in λ and μ . It means that the Lax representation plays the role of a linear factor of a polynomial as we will discuss in Chapter 6. Also, note that if p is a parallelogram net then $P(\cdot, \lambda, \mu)$ is primarily equivalent to p for any fixed λ, μ (with $\mu \neq 0$).

Treating λ and μ as independent variables allows for a global rescaling of P that does not affect the compatibility condition (3.18). Thus, it is reasonable to consider Lax representations that only depend on a single variable such as $\lambda + p^i, 1 + \mu p^i$ or $\lambda + \frac{1}{\lambda} p^i$. However, committing to a specific choice of the representation has some implications on the associated family (defined below). Instead, as a unifying approach, we will consider (3.19) and treat λ and μ as differentiable functions of a parameter $t \in \mathbb{R}$. Then, P is a map $\tilde{\mathcal{E}}(\mathbb{Z}^n) \times \mathbb{R} \rightarrow \mathcal{A}$ with

$$P^i(t) = \lambda(t) + \mu(t)p^i. \quad (3.21)$$

For brevity, the dependence of the variables λ, μ, P^i and Φ (defined below) on the parameter t will usually not be stated explicitly. The implication of the specific choice of functions λ, μ will be discussed in Section 3.4. We will denote by $(P^i)' := \lambda' + \mu'p^i$ the formal derivative with respect to t .

Definition 3.3.3 *Let p be a parallelogram net and let $\lambda, \mu : \mathbb{R} \rightarrow \mathbb{K}$ be two differentiable functions with $\lambda'\mu - \lambda\mu' \neq 0$ for all $t \in \mathbb{R}$ and assume that $P^i(t)$ is invertible at every edge. Then, for any initial invertible $\Phi(\mathbf{0}, t) \in \mathcal{A}$ we call the vertex based map $\Phi : \mathbb{Z}^n \times \mathbb{R} \rightarrow \mathcal{A}$ given by*

$$\Phi_i = P^i \Phi \quad (3.22)$$

a (moving) frame of p . We call the family of edge based nets p_t given by

$$p_t^i = \Phi_i^{-1} (P^i)' \Phi \quad (3.23)$$

the associated family of p .

The name *associated family* for p_t will be motivated below where we will show that p_t is indeed a family of parallelogram nets containing p . The name *frame* for Φ is motivated by relations of Φ to frames of curves and surfaces which we will uncover throughout this thesis.

The regularity condition $\lambda'\mu - \lambda\mu' \neq 0$ will ensure that members of the associated family do not degenerate. Note that it implies that the derivative of the ratio $\frac{\lambda}{\mu}$ does not vanish.

Also, note that for any fixed t the edge based nets P defined by $P^i = \lambda + \mu p^i$ and P' defined by $(P^i)' = \lambda' + \mu'p^i$ are parallelogram nets primarily equivalent to p . Throughout this thesis we will assume $\Phi(\mathbf{0}, t) = \mathbf{1}$ unless stated otherwise. Choosing a different initial $\Phi(\mathbf{0}, t)$ only effects the associated family by a global sandwich product and, thus, the resulting nets are primarily equivalent to the corresponding nets in the original family.

Remark 3.3.4 If $P^i(t)$ is non-invertible for some $t \in \mathbb{R}$ the frame and the associated family are not defined at this value t . In our examples this will occur only at a zero-set of points: In a matrix algebra $\mathcal{A}$ invertibility of P^i is equivalent to non-vanishing determinant

$$\det P^i = \det(\lambda + \mu p^i) \quad (3.24)$$

This determinant vanishes if and only if $-\frac{\lambda}{\mu}$ is an eigenvalue of p^i . Therefore, the associated family is only not defined for the countable set

$$\{t \in \mathbb{R} \mid -\frac{\lambda(t)}{\mu(t)} \text{ is eigenvalue of } p^i(x) \text{ for some } i \in [n], x \in \mathbb{Z}^n\} \subseteq \mathbb{R}. \quad (3.25)$$

We will mostly ignore such cases.

Remarkably, as shown in [98, Theorem 7] for the quaternionic case, the associated family preserves the parallelogram equations:

Proposition 3.3.5 Every net p_t in the associated family (for some λ, μ) of a parallelogram net p is a parallelogram net. An additive primitive map f_t for p_t is given by

$$f_t = \Phi^{-1}\Phi'. \quad (3.26)$$

For any invertible $g^0 \in \mathcal{A}$ a multiplicative primitive map g_t of p_t is given by

$$g_t = \Phi^{-1}\tilde{\Phi}g^0 \quad (3.27)$$

where $\tilde{\Phi}$ denotes the frame of p defined by replacing P^i in (3.22) with $\tilde{P}^i := (P^i)' = \lambda' + \mu' p^i$.

Proof. Consider the vertex based net $f_t = \Phi^{-1}\Phi'$. It is an additive primitive map of p_t since

$$(f_t)_i - f_t = \Phi^{-1}(P^i)^{-1}(P^i\Phi' + (P^i)'\Phi) - \Phi^{-1}\Phi' = \Phi_i^{-1}(P^i)'\Phi = p_t^i. \quad (3.28)$$

Similarly, the vertex based net $g_t = \Phi^{-1}\tilde{\Phi}g^0$ is a multiplicative primitive map of p_t since

$$(g_t)_i g_t^{-1} = \Phi_i^{-1}\tilde{\Phi}_i\tilde{\Phi}^{-1}\Phi = \Phi_i^{-1}\tilde{P}^i\Phi = p_t^i. \quad (3.29)$$

The existence of these primitive maps already imply (3.1) and (3.2) for p_t respectively and, therefore, p_t is a parallelogram net. $\square$

Taking the logarithmic derivative (3.26) to obtain a family of surfaces from its Lax representation is known as the *Sym formula* [104, 9] (also, *Sym-Tafel* or *Sym-Bobenko formula*).

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Note that we can rewrite (3.23) as

$$p_t^i = \Phi^{-1}(P^i)^{-1}(P^i)' \Phi = \Phi_i^{-1}(P^i)'(P^i)^{-1} \Phi_i \quad (3.30)$$

Hence, the values of p_t at edges adjacent to a common vertex x only depend on local variables and one common sandwich product with $\Phi(x)$. In our geometric examples this will imply local rigidity at vertices throughout the associated family (see Proposition 4.2.3).

Remark 3.3.6 *Similar to Remark 3.1.4, we comment on the missing symmetry of our definitions which is induced by edges being directed. On the directed edge $d = (x, x + e_i) \in \vec{\mathcal{E}}(\mathbb{Z}^n)$ lives the linear Lax representation $P(d) = \lambda + \mu p(d)$. If all components of $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ are non-negative we find a path of directed edges $d_1, \dots, d_k \in \vec{\mathcal{E}}(\mathbb{Z}^n)$ connecting $\mathbf{0}$ to x and we have*

$$\Phi(x) = P(d_k) \cdots P(d_1) \quad (3.31)$$

which is a polynomial in λ and μ . To a reversed directed edge $(x + e_i, x) \notin \vec{\mathcal{E}}(\mathbb{Z}^n)$ we can naturally assign the inverse $P(x + e_i, x) := (P(x, x + e_i))^{-1}$ which is not linear in λ and μ . If x has negative components $\Phi(x)$ is no polynomial since such an inverse appears.

However, if we have an involution $f \mapsto f^$ as in Remark 3.1.4 we have*

$$(P(x, x + e_i))^{-1} = \frac{\lambda + \mu(p(x, x + e_i))^*}{P^i \cdot (P^i)^*} \quad (3.32)$$

since $P^i \cdot (P^i)^ = \lambda^2 + \lambda\mu(p^i + (p^i)^*) + \mu^2 p^i (p^i)^*$ is scalar valued. Hence, a scalar multiple of $\Phi(x)$ is still a polynomial.*

For a symmetric definition of the transport of the frame along an edge one could set $P(x + e_i, x) := \lambda + \mu(p(x, x + e_i))^$ and define the frame by $\Phi(y) = P(x, y)\Phi(x)$ for adjacent vertices. Then, the frame Φ is only defined up to scalar multiples since going back and forth along an edge produces a scalar function in λ, μ .*

We conclude this section with a discussion that, generically, any linear Lax representation is equivalent to (3.19) under a gauge:

Lemma 3.3.7 *Let $\tilde{P} : \vec{\mathcal{E}}(\mathbb{Z}^n) \times \mathbb{K}^2 \rightarrow \mathcal{A}$ be an edge based map with $\tilde{P}^i(\lambda, \mu) = \lambda A^i + \mu B^i$ for two maps $A, B : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathcal{A}$ such that the compatibility condition*

$$\tilde{P}_j^i \tilde{P}^j = \tilde{P}_i^j \tilde{P}^i \quad (3.33)$$

holds on each quad. Let $\Psi : \mathbb{Z}^n \times \mathbb{K}^2 \rightarrow \mathcal{A}$ be a corresponding frame defined by the transport $\Psi_i = \tilde{P}^i \Psi$. If every value of A is invertible then

$$\Phi := G\Psi \quad \text{with} \quad G := \Psi(\lambda = 1, \mu = 0)^{-1} \quad (3.34)$$

is a frame of a parallelogram net, meaning that $P^i := \Phi_i \Phi^{-1}$ is of the form (3.19).

Proof. We have

$$\Phi_i = G_i \Psi_i = G(A^i)^{-1}(\lambda A^i + \mu B^i)G^{-1}\Phi = (\lambda + \mu p^i)\Phi \quad (3.35)$$

with $p^i := G(A^i)^{-1}B^iG^{-1}$. Note that p is a parallelogram net. $\square$

Such a transformation $\Psi \mapsto \Phi = G\Psi$ is called a *gauge*. For an associated family we, again, assume λ, μ to be functions $\mathbb{R} \rightarrow \mathbb{K}$. The Sym formula is not affected by the gauge since G does not depend on t and, hence,

$$\Phi^{-1}\Phi' = \Psi^{-1}G^{-1}G\Psi' = \Psi^{-1}\Psi'. \quad (3.36)$$

Therefore, the associated family is independent of the gauge. This principle will be useful many times throughout this thesis.

3.4. Structure of the associated family

In this section we investigate how our generalized approach of considering the Lax representation (3.19) for different functions λ and μ affects the associated family. Also, we will see that associated families partition the set of parallelogram nets. Since we have the functional freedom of choosing λ and μ it might appear as if we can get a lot of different nets from our construction. However, we can show that, up to primary equivalence, each net p_{t_0} is already determined by the ratio $\frac{\lambda(t_0)}{\mu(t_0)}$ (see Figure 3.3 on the left):

Proposition 3.4.1 *For arbitrary functions λ, μ consider p_{t_0} in the associated family of a parallelogram net p . If $\mu(t_0) = 0$ the net p_{t_0} is primarily equivalent to p . If $\mu(t_0) \neq 0$ the net p_{t_0} is primarily equivalent to $\tilde{p}_r$ where $r = \frac{\lambda(t_0)}{\mu(t_0)}$ and $\tilde{p}_t$ is the associated family of p defined by $\tilde{\lambda}(t) = t, \tilde{\mu}(t) = 1$.*

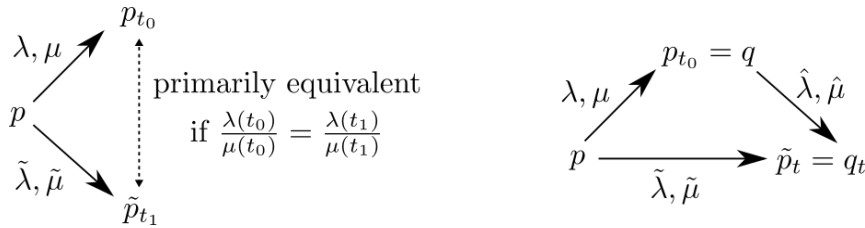


Figure 3.3.: Left: Members of the family depend only on the ratio $\frac{\lambda}{\mu}$. Right: Transitivity of the associated family given by (3.39).

Proof. First, consider t_0 with $\mu(t_0) = 0$. The nets p and p_{t_0} are primarily equivalent since $P^i(t_0) = \lambda(t_0)$ implies $\Phi(x, t_0) = \lambda(t_0)^{x_1 + \dots + x_n} \in \mathbb{K}$ for $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ and,

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hence, at $t = t_0$ we have

$$p_{t_0}^i = \Phi_i^{-1}(P^i)' \Phi = \frac{\lambda'}{\lambda} + \frac{\mu'}{\lambda} p^i. \quad (3.37)$$

Now, consider t_0 with $\mu(t_0) \neq 0$. Note that we have $\tilde{P}^i(t) = t + p^i$ and $(\tilde{P}^i)'(t) = 1$. Then, we have $P^i(t_0) = \mu(t_0)(\frac{\lambda(t_0)}{\mu(t_0)} + p^i) = \mu(t_0)(r + p^i) = \mu(t_0)\tilde{P}^i(r)$ and, hence, $\Phi(x, t_0) = \mu(t_0)^{x_0 + \dots + x_n} \tilde{\Phi}(x, r)$. Now, p_{t_0} and $\tilde{p}_r$ are primarily equivalent, since

$$\begin{aligned} p_{t_0}^i &= \Phi_i^{-1}(P^i)' \Phi = \frac{1}{\mu} \tilde{\Phi}_i^{-1}(\lambda' + \mu' p^i) \tilde{\Phi} \\ &= \frac{1}{\mu} \tilde{\Phi}_i^{-1}(\mu' \tilde{P}^i + \lambda' - r\mu') \tilde{\Phi} = \frac{\mu'}{\mu} + \frac{\lambda' - r\mu'}{\mu} \tilde{p}_r^i. \end{aligned} \quad (3.38)$$

Here, λ, μ, P, Φ are evaluated at t_0 and $\tilde{P}, \tilde{\Phi}$ are evaluated at r . $\square$

Thus, for $\mathbb{K} = \mathbb{R}$ the associated family is a one-parameter family and different choices of λ and μ correspond to different parametrizations of (subsets of) this family. For $\mathbb{K} = \mathbb{C}$ the associated family depends on the path of the ratio $\frac{\lambda}{\mu}$ in the complex plane and we will usually choose this path to be the real axis, the imaginary axis or the unit circle. The next proposition shows that if we apply the associated family twice we still end up in the associated family of the original net parametrized by different functions. It also shows that we can go back to the original net. Thus, we can consider the associated families as equivalence classes within the set of parallelogram nets: Proposition 3.4.1 shows reflexivity while Proposition 3.4.2 shows transitivity and symmetry.

Proposition 3.4.2 *Let $q = p_{t_0}$ be a member of the associated family of a parallelogram net p given by the functions λ, μ and let q_t be the associated family of q given by the functions $\hat{\lambda}, \hat{\mu}$ (see Figure 3.3 on the right). Then,*

1. *the associated family q_t of q agrees with the associated family $\tilde{p}_t$ of p given by the functions*

$$\tilde{\lambda}(t) = \hat{\lambda}(t)\lambda(t_0) + \hat{\mu}(t)\lambda'(t_0), \quad \tilde{\mu}(t) = \hat{\lambda}(t)\mu(t_0) + \hat{\mu}(t)\mu'(t_0), \quad (3.39)$$

2. *if $\frac{\hat{\lambda}(t_1)}{\hat{\mu}(t_1)} = -\frac{\mu'(t_0)}{\mu(t_0)}$ for some $t_1 \in \mathbb{R}$ then q_{t_1} is primarily equivalent to p .*

Proof. The second statement directly follows from the first statement and Proposition 3.4.1 since $\tilde{\mu}(t_1) = 0$. For the first statement consider the moving frames

- Φ of p given by the functions λ and μ , in other words, fulfilling

$$\Phi_i = (\lambda + \mu p^i) \Phi = P^i \Phi \quad (3.40)$$

- Ψ of q given by the functions $\hat{\lambda}$ and $\hat{\mu}$, in other words, fulfilling

$$\Psi_i = (\hat{\lambda} + \hat{\mu}q^i)\Psi \quad (3.41)$$

- $\tilde{\Phi}$ of p given by the functions $\tilde{\lambda}$ and $\tilde{\mu}$, in other words, fulfilling

$$\tilde{\Phi}_i = (\tilde{\lambda} + \tilde{\mu}p^i)\tilde{\Phi} \quad (3.42)$$

In the following consider λ, μ, P, Φ to be evaluated at t_0 and $\hat{\lambda}, \hat{\mu}, \Psi, \tilde{\lambda}, \tilde{\mu}, \tilde{\Phi}$ to be evaluated at t . We calculate

$$\begin{aligned} \Psi_i \Psi^{-1} &= \hat{\lambda} + \hat{\mu}p_{t_0}^i = \hat{\lambda} + \hat{\mu}\Phi_i^{-1}(P^i)' \Phi = \Phi_i^{-1}(\hat{\lambda}P^i + \hat{\mu}(P^i)')\Phi \\ &= \Phi_i^{-1}(\tilde{\lambda} + \tilde{\mu}p^i)\Phi = \Phi_i^{-1}\tilde{\Phi}_i\tilde{\Phi}^{-1}\Phi = (\Phi_i^{-1}\tilde{\Phi}_i)(\Phi^{-1}\tilde{\Phi})^{-1} \end{aligned} \quad (3.43)$$

Therefore, the transport of Ψ and the vertex based map $\Phi^{-1}\tilde{\Phi}$ along each edge agree. Also, we have $\Phi(\mathbf{0}) = \Psi(\mathbf{0}) = \tilde{\Phi}(\mathbf{0}) = 1$ and, hence, both maps also agree at $x = \mathbf{0}$ and we can conclude $\Psi = \Phi^{-1}\tilde{\Phi}$.

Also, we have

$$\hat{\lambda}' + \hat{\mu}'p_{t_0}^i = \hat{\lambda}' + \hat{\mu}'\Phi_i^{-1}(P^i)' \Phi = \Phi_i^{-1}(\hat{\lambda}'P^i + \hat{\mu}'(P^i)')\Phi = \Phi_i^{-1}(\tilde{\lambda}' + \tilde{\mu}'p^i)\Phi. \quad (3.44)$$

These two observations yield

$$q_t^i = \Psi_i^{-1}(\hat{\lambda}' + \hat{\mu}'p_{t_0}^i)\Psi = \Psi_i^{-1}\Phi_i^{-1}(\tilde{\lambda}' + \tilde{\mu}'p^i)\Phi\Psi = \tilde{\Phi}_i^{-1}(\tilde{\lambda}' + \tilde{\mu}'p^i)\tilde{\Phi} = \tilde{p}_t^i. \quad (3.45)$$

□

Some choices of functions will appear throughout this thesis and we can apply the proposition:

Example 3.4.3 With $\lambda(t) = \hat{\lambda}(t) = 1$ and $\mu(t) = \hat{\mu}(t) = t$ we have $\tilde{\lambda}(t) = 1$ and $\tilde{\mu}(t) = t + t_0$.

Example 3.4.4 With $\lambda(t) = \hat{\lambda}(t) = \cos(t)$ and $\mu(t) = \hat{\mu}(t) = \sin(t)$ we have $\tilde{\lambda}(t) = \cos(t + t_0)$ and $\tilde{\mu}(t) = \sin(t + t_0)$.

Example 3.4.5 If we choose $\lambda(t) = e^{\frac{1}{c}t}$, $\mu(t) = e^{-\frac{1}{c}t}$ and $\hat{\lambda}(t) = e^t$ and $\hat{\mu}(t) = ce^{-t}$ for some constant $c \neq 0$ we have $\tilde{\lambda}(t) = e^{\frac{1}{c}t_0}(e^t + e^{-t})$ and $\tilde{\mu}(t) = e^{-\frac{1}{c}t_0}(e^t - e^{-t})$. With $t_0 = t_1 = 0$ equation (3.37) implies $\tilde{p}_{t_1=0} = p$. We will use these functions to parametrize special families of parallelogram nets (see Propositions 4.3.4, 5.1.3 and Theorem 5.2.1).

3.5. Periodicity

We turn to parallelogram nets which fulfill a periodicity condition. Periodic solutions appear in the study of closed curves and surfaces. Also, they appear in the context of integrable dynamics since periodicity often implies the existence of invariants. An example of such a dynamic is the discrete bicycle transformation [105] of curves in the plane and the discrete Hashimoto flow which is its analogue in space [87, 55]. This dynamic is obtained by constructing periodic Bäcklund transformation of periodic curves. The results presented in this section generalize parts of this technique to unit associative algebras. We will review this dynamic again in Section 4.6. Another example of such an integrable dynamic is polygon recutting [2, 68] as presented in Section 4.5.

Definition 3.5.1 *Let p be a parallelogram net. We call it quasi-periodic with period $T \in \mathbb{Z}^n \setminus \{0\}$ if*

$$p^i(x + T) = E^{-1}p^i(x)E \quad (3.46)$$

for some invertible constant $E \in \mathcal{A}$. We call it periodic if $E = 1$.

The associated family preserves quasi-periodicity:

Proposition 3.5.2 *If p is quasi-periodic with period T and constant E then every member of the associated family p_t is quasi-periodic with period T and constant*

$$E_t := \Phi^{-1}(x)E\Phi(x + T). \quad (3.47)$$

Here, 'constant E_t ' means that E_t does not depend on x .

Proof. Note that we have $P^i(x + T) = E^{-1}P^i(x)E$ and $(P^i)'(x + T) = E^{-1}(P^i)'(x)E$. First, we show that E_t (which is well-defined at every vertex) does not depend on the vertex x . This follows from the fact that it is invariant under a shift since

$$\begin{aligned} (E_t)_i &= \Phi^{-1}(x + e_i)E\Phi(x + e_i + T) = \Phi^{-1}(x)(P^i(x))^{-1}EP^i(x + T)\Phi(x + T) \\ &= \Phi^{-1}(x)E\Phi(x + T) = E_t. \end{aligned} \quad (3.48)$$

Then, using $\Phi(x + T) = E^{-1}\Phi(x)E_t$, periodicity of p_t follows from

$$\begin{aligned} p_t^i(x + T) &= \Phi^{-1}(x + e_i + T)(P^i)'(x + T)\Phi(x + T) \\ &= E_t^{-1}\Phi^{-1}(x + e_i)EE^{-1}(P^i)'(x)EE^{-1}\Phi(x)E_t = E_t^{-1}p_t^i(x)E_t. \end{aligned} \quad (3.49)$$

□

From now, we assume that all components of $T = (T_1, \dots, T_n) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ are non-negative. Then, there exists a path of directed edges $d_1, \dots, d_m \in \vec{\mathcal{E}}(\mathbb{Z}^n)$ between x and $x + T$ and we have

$$E_t = \Phi^{-1}(x)E\Phi(x + T) = \Phi^{-1}(x)P(d_m) \cdots P(d_1)\Phi(x). \quad (3.50)$$

Remember that this is well-defined since the product $P(d_m) \cdots P(d_1)$ agrees for all possible paths from x to $x + T$. Now, we set $\lambda(t) = t$, $\mu(t) = 1$ and consider the Lax representation $P^i = t + p^i$ as linear polynomial in t . Then, to each point we assign the polynomial

$$\mathcal{P}(x) := \Phi(x)E_t\Phi(x)^{-1} = E\Phi(x + T)\Phi(x)^{-1} = E(t + p(d_m)) \cdots (t + p(d_1)). \quad (3.51)$$

This polynomial stores information about the periodic net and we will see in Section 4.6 that it plays the role of a monodromy map. We will investigate some of its properties.

First, observe that this polynomial has many invariants: If $\mathcal{A}$ is a matrix algebra the determinant $\det \mathcal{P}(t) = \det E_t$ and the trace $\text{tr } \mathcal{P}(t) = \text{tr } E_t$ are scalar polynomials which do not depend on the vertex. Therefore, all of their coefficients are invariants.

Next, we find that neighboring polynomials are related by a sandwich product since

$$\mathcal{P}(x + e_i)(t + p^i(x)) = E(t + p^i(x + T))E^{-1}\mathcal{P}(x) = (t + p^i(x))\mathcal{P}(x) \quad (3.52)$$

implies

$$\mathcal{P}_i = (t + p^i)\mathcal{P}(t + p^i)^{-1}. \quad (3.53)$$

We can investigate the coefficients C^k of $\mathcal{P}$ given by

$$\mathcal{P}(t) = t^n C^0 + t^{n-1} C^1 + t^{n-2} C^2 + \cdots + t^0 C^n. \quad (3.54)$$

Note that $C^0 = E$. Expanding (3.52) in powers of t gives us equations for these coefficients:

$$\begin{aligned} t^{n+1} : \quad & C^0 = C^0 \\ t^n : \quad & C_i^1 + E p^i = C^i + p^i E \\ & \dots \\ t^{n-k} : \quad & C_i^{k+1} + C_i^k p^i = C^{k+1} + p^i C^k \\ & \dots \\ t^0 : \quad & C_i^n p^i = p^i C^n \end{aligned} \quad (3.55)$$

The first $n + 1$ equations already determine the polynomial by

$$\begin{aligned} C_i^0 &= C^0 \\ C_i^k &= C^k + p^i C^{k-1} - C_i^{k-1} p^i \end{aligned} \quad (3.56)$$

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It is remarkable that this evolution of the polynomial does not need any requirements on the parallelogram net (such as evolvability). From the last equation (which is $C_i^n p^i = p^i C^n$) we can obtain a very local condition on $\mathcal{P}$ and p^i : We can eliminate all coefficients C_i^k by iteratively plugging in (3.56)

$$\begin{aligned} p^i C^n &= C_i^n p^i = C^n p^i + p^i C^{n-1} p^i - C_1^{n-1} (p^i)^2 = \dots \\ &= C^n p^i + p^i C^{n-1} p^i - C^{n-1} (p^i)^2 - p^i C^{n-2} (p^i)^2 + C^{n-2} (p^i)^3 + \dots \end{aligned} \quad (3.57)$$

This can be rearranged to

$$\begin{aligned} &p^i (C^n - C^{n-1} p^i + C^{n-2} (p^i)^2 - \dots + (-1)^n C^0 (p^i)^n) \\ &= (C^n - C^{n-1} p^i + C^{n-2} (p^i)^2 - \dots + (-1)^n C^0 (p^i)^n) p^i. \end{aligned} \quad (3.58)$$

In Section 4.6 we will use this condition to reconstruct p^i from $\mathcal{P}$ using special properties of the quaternions. In fact, we will see that this condition determines quasi-periodic Bäcklund transformations of quasi-periodic parallelogram nets. We will present another application of periodicity in Chapter 8 and 9.

4. Quaternionic skew parallelogram nets

4.1. Quaternions

Here, we give a brief overview over the division algebra of quaternions. For a more extensive treatment see, for example, [41]. The quaternions $\mathbb{H}$ are the real four-dimensional unit algebra generated by the four basis elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1. \quad (4.1)$$

The multiplication is associative but not commutative. Every quaternion can be written as $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and, similar to the complex numbers, has a real and an imaginary part given by

$$\operatorname{Re} q = a, \quad \operatorname{Im} q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \quad (4.2)$$

The real part is a real number but, unlike for complex numbers, the imaginary part, also called vector part, is a quaternion itself. Thus, we will use the short notation $\vec{q} := \operatorname{Im} q$. A quaternion with zero real part is called an imaginary quaternion. The conjugate of a quaternion q is given by

$$\bar{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}. \quad (4.3)$$

This also implies the relations

$$2 \operatorname{Re} q = q + \bar{q}, \quad 2\vec{q} = q - \bar{q} \quad (4.4)$$

and, after a slightly longer computation,

$$q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2 = \|q\|^2 \quad (4.5)$$

where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^4$. Remarkably, the norm is multiplicative, i.e., fulfills $\|pq\| = \|p\|\|q\|$. We will often consider the two and three dimensional unit spheres given by

$$\mathbb{S}^2 = \{q \in \operatorname{Im} \mathbb{H} \mid \|q\| = 1\}, \quad \mathbb{S}^3 = \{q \in \mathbb{H} \mid \|q\| = 1\}. \quad (4.6)$$

Equation (4.5) implies that the inverse of a non-zero quaternion is given by

$$q^{-1} = \frac{\bar{q}}{\|q\|^2} \quad (4.7)$$

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and, thus, the quaternions are a division algebra. We can also find the Euclidean scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^4$ as

$$\langle q_1, q_2 \rangle = \frac{1}{2}(q_1 \bar{q}_2 + q_2 \bar{q}_1) = \operatorname{Re} q_1 \bar{q}_2. \quad (4.8)$$

Often, we will identify imaginary quaternions with vectors $\operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$ by $b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \leftrightarrow (b, c, d)$. Many notions of three-dimensional Euclidean geometry can be written in terms of quaternions. For instance, the product of two imaginary quaternions $q_1, q_2 \in \operatorname{Im} \mathbb{H}$ can be expressed in terms of the three-dimensional scalar product and cross-product as

$$q_1 q_2 = -\langle q_1, q_2 \rangle + q_1 \times q_2. \quad (4.9)$$

Also, a rotation can be expressed using the sandwich product

$$x \mapsto qxq^{-1} \quad (4.10)$$

of an imaginary quaternion $x \in \operatorname{Im} \mathbb{H}$ with some non-zero quaternion $q \in \mathbb{H}$: The vector x gets rotated around the axis $\operatorname{Im} q$ by an angle determined by $\operatorname{Re} q$ and $\|\vec{q}\|$. Specifically, if q is of the form $q = \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \hat{q} \in \mathbb{S}^3$ for some unit imaginary quaternion $\hat{q} \in \mathbb{S}^2$ then α is the angle of rotation. Therefore, for a general quaternion the angle of rotation α is given by

$$\tan \frac{\alpha}{2} = \frac{\|\vec{q}\|}{\operatorname{Re} q}. \quad (4.11)$$

Note that for any two quaternions $x, q \in \mathbb{H}$ a sandwich product as in (4.10) preserves real part and norm of x .

Finally, we will often write quaternions as complex matrices. This is possible since the quaternions are isomorphic to a real sub-algebra of the complex matrix algebra $\mathbb{C}^{2 \times 2}$ via the isomorphism

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto \begin{pmatrix} a - d\mathbf{i} & -c - b\mathbf{i} \\ c - b\mathbf{i} & a + d\mathbf{i} \end{pmatrix} \in \mathbb{C}^{2 \times 2} \quad (4.12)$$

We will identify each quaternion with the corresponding matrix. Then, the real part is related to the trace of the matrix while the norm is related to the determinant:

$$\operatorname{tr} q = 2 \operatorname{Re} q, \quad \det q = \|q\|^2. \quad (4.13)$$

The basis elements are identified via $\mathbf{1} = \sigma_0, \mathbf{i} = -i\sigma_1, \mathbf{j} = -i\sigma_2, \mathbf{k} = -i\sigma_3$ where the matrices σ_i are the well-known Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.14)$$

4.2. Geometric skew parallelogram nets

We call a (skew) parallelogram net $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{H}$ *quaternionic* if it maps to the algebra of quaternions. In this section, we investigate the notions of Chapter 3 for quaternionic parallelogram nets and we study the geometry of their additive primitive maps. More specifically, if $f : \mathbb{Z}^n \rightarrow \mathbb{H}$ is such a primitive map we are interested in the geometry of its projection $\vec{f} : \mathbb{Z}^n \rightarrow \text{Im } \mathbb{H} \cong \mathbb{R}^3$ which is a net in three-dimensional Euclidean space. Nets of this kind have been studied in the context of discrete Lund-Regge surfaces [98] and Bäcklund transformations of discrete curves [87, 55] and, thus, most results presented here are not new. For brevity, we will refer to $\vec{f}$ as the *primitive map* instead of the *projected additive primitive map*.

A diagonal $p^j - p^i$ of an edge-based net p is only non-invertible if $p^i = p^j$. Hence, the only non-evolvable solution of (3.1), (3.2) in the quaternions is the trivial solution $p^i = p^j$ and $p_j^i = p_i^j$. Thus, we will mostly discuss evolvable nets. For a first geometric observation, we can consider the invariants of the evolution (3.3). This evolution is given by a sandwich product and, hence, preserves real part and norm. Therefore, for an evolvable quaternionic parallelogram net p these have the labelling property given by

$$\text{Re } p^i = \text{Re } p_j^i, \quad \|p^i\| = \|p_j^i\|. \quad (4.15)$$

For non-evolvable nets we will still assume real part and norm to be invariants. Non-evolvable nets with this property can be considered as limit cases of evolvable nets.

Proposition 4.2.1 *The primitive map $\vec{f}$ of a quaternionic parallelogram net p is a Chebyshev net.*

Proof. The length of each edge is given by $\|\vec{f}_i - \vec{f}\|^2 = \|\vec{p}^i\|^2 = \|p^i\|^2 - (\text{Re } p^i)^2$ which is an invariant of the evolution (3.3) and, hence, has the labelling property. $\square$

Thus, quaternionic (skew) parallelogram nets indeed describe nets consisting of skew parallelograms/Chebyshev quads. The converse is not true: Not all Chebyshev nets are primitive maps of a parallelogram net, in other words, such Chebyshev nets need to fulfill an additional condition. We will describe this condition in Section 4.3.

Note that the parallelogram net p only defines $\vec{f}$ up to translation since we can choose the initial point $f(\mathbf{0})$ of the additive primitive map. If a second parallelogram net $\tilde{p}$ is primarily equivalent to p their corresponding primitive maps are related by a similarity transformation: A sandwich product of p with a quaternion acts as a rotation on $\vec{f}$, scaling p also scales $\vec{f}$, and adding a scalar does not affect $\vec{f}$.

We can complete the solution of the Cauchy problem for quaternionic parallelogram nets. In fact, we can make the observation of Remark 3.1.4 more explicit:

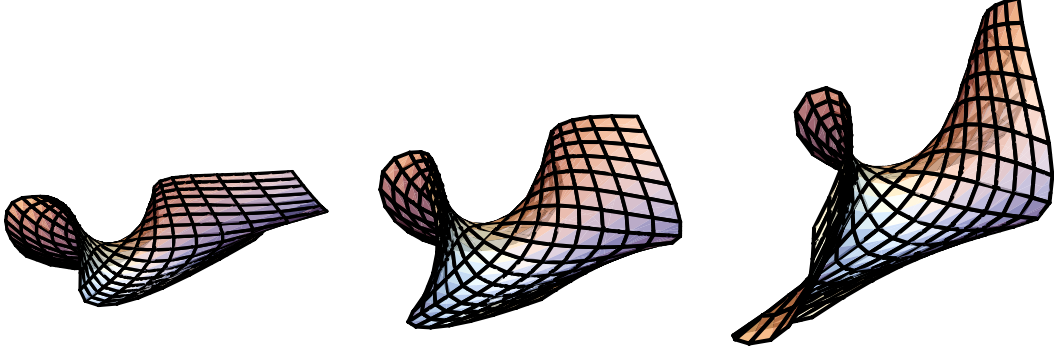


Figure 4.1.: Three quaternionic parallelogram nets which belong to a common associated family.

Proposition 4.2.2 *Let p be an evolvable parallelogram net where the second diagonal $p_j^i - \bar{p}^j$ does not vanish. Then, it fulfills the sideways evolution*

$$\begin{aligned} p^i &= (p_j^i - \bar{p}^j) p_j^i (p_j^i - \bar{p}^j)^{-1}, \\ p_i^j &= (p_j^i - \bar{p}^j) p^j (p_j^i - \bar{p}^j)^{-1}. \end{aligned} \quad (4.16)$$

Proof. Since real part and norm have the labelling property equations (3.1), (3.2) are equivalent to (compare (3.8), (3.9))

$$\begin{aligned} p^i + \bar{p}^j &= p_j^i + \bar{p}_i^j, \\ p^i \bar{p}^j &= \bar{p}_i^j p_j^i. \end{aligned} \quad (4.17)$$

Analogously to (3.3), (3.4), these are solvable into (4.16). $\square$

Thus, we know how to reconstruct quaternionic two-dimensional parallelogram nets from initial values on the axes.

Next, we investigate how the geometry evolves in the associated family. Recall that the associated family is a real one-parameter family and different choices of the functions λ and μ correspond to different parametrizations of (subsets of) the family.

Now, an (oriented) line in Euclidean space is parametrized by $t \mapsto a + td$ where $a \in \mathbb{R}^3$ is a point and $d \in \mathbb{S}^2$ is the direction of the line. Consider the line connecting adjacent points f and f_i on our net (which we will call *geodesic edge* in Chapter 9). The direction of this line is given by $\frac{\bar{p}^i}{\|\bar{p}^i\|} = \frac{\bar{f}_i - \bar{f}}{\|f_i - f\|}$ and, hence, such a direction is associated to every edge.

Proposition 4.2.3 *At each vertex the associated family acts as a common rotation on the set of directions of edges incident to this vertex.*

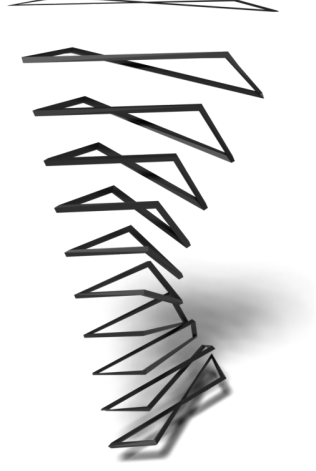


Figure 4.2.: A skew parallelogram and its associated family.

In a sense, the geometry at each vertex is rigid throughout the associated family. In particular, all angles between edges stay the same. In Figure 4.2 we can see a single skew parallelogram and its associated family. The edge-lengths change but the angles between edges stay the same.

Proof. The associated family (3.23) at edges incident to a common vertex $x \in \mathbb{Z}^n$ can be written (equivalently to (3.30)) as

$$p_t^i = \Phi^{-1}(P^i)^{-1}(P^i)'\Phi, \quad (p_t^i)_{\bar{i}} = \Phi^{-1}(P^i)_{\bar{i}}^{-1}(P^i)_{\bar{i}}'\Phi \quad (4.18)$$

which is a combination of a sandwich product with $\Phi = \Phi(x)$ and a logarithmic derivative. Expanding the logarithmic derivative of $P^i = \lambda + \mu p^i$ yields

$$\begin{aligned} (P^i)^{-1}(P^i)' &= (P^i)'(P^i)^{-1} \\ &= \frac{(\lambda + \mu \operatorname{Re} p^i)(\lambda' + \mu' \operatorname{Re} p^i) + \mu\mu'\|\vec{p}^i\|^2}{(\lambda + \mu \operatorname{Re} p^i)^2 + \mu^2\|\vec{p}^i\|^2} + \frac{\lambda\mu' - \mu\lambda'}{(\lambda + \mu \operatorname{Re} p^i)^2 + \mu^2\|\vec{p}^i\|^2}\vec{p}^i. \end{aligned} \quad (4.19)$$

Therefore, the logarithmic derivative does not affect the direction (except for the global sign of $\lambda\mu' - \mu\lambda'$ which we disregard for brevity) and the direction of an edge in the family is given by the original direction of this edge rotated by Φ . $\square$

4. Quaternionic skew parallelogram nets

For example, if the directions incident to a point are coplanar the net is said to have *planar vertex stars*. Parallelogram nets with this property turn out to be K-nets (see Section 4.4) and Proposition 4.2.3 implies that K-nets stay K-nets in the associated family (for example, see Figure 4.6). While this has been known for K-nets, Proposition 4.2.3 for our more general class of nets seems to be new.

For later use, note that (4.19) implies

$$\operatorname{Re} p_t^i = \frac{(\lambda + \mu \operatorname{Re} p^i)(\lambda' + \mu' \operatorname{Re} p^i) + \mu\mu' \|\vec{p}^i\|^2}{(\lambda + \mu \operatorname{Re} p^i)^2 + \mu^2 \|\vec{p}^i\|^2} \quad (4.20)$$

and

$$\|\vec{p}_t^i\| = \frac{|\lambda\mu' - \mu\lambda'|}{(\lambda + \mu \operatorname{Re} p^i)^2 + \mu^2 \|\vec{p}^i\|^2} \|\vec{p}^i\|. \quad (4.21)$$

We conclude this section with the geometric interpretation of (quasi-)periodicity.

Proposition 4.2.4 *If a quaternionic parallelogram net p is quasi-periodic with period T then its primitive map $\vec{f}$ is quasi-periodic in the sense that $\vec{f}$ and $\tilde{f}$ defined by $\vec{f}(x) = \tilde{f}(x + T)$ are related by a Euclidean motion.*

Proof. If p is quasi-periodic then $p^i(x + T) = E^{-1}p^i(x)E$ for some non-zero $E \in \mathbb{H}$ and, hence, the edges of $\vec{f}$ and $\tilde{f}$ are related by a common rotation with E . Therefore, $\vec{f}$ and $\tilde{f}$ are related by a Euclidean motion. $\square$

Example 4.2.5 *A screw motion is a combination of a rotation around an axis and a translation in direction of the axis. A screw motion around the $\mathbf{k}$ -axis can be written as $f \mapsto qfq^{-1} + h\mathbf{k}$ with $q = \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\mathbf{k}$. Two such screw motions given by q_1, h_1 and q_2, h_2 commute since they share an axis. Thus, we can define a two-dimensional net $f : \mathbb{Z}^2 \rightarrow \operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$ from an initial point $f(\mathbf{0})$ by the condition that each point $f(m, n)$ is obtained by applying a first screw motion m times and a second screw motion n times to $f(\mathbf{0})$:*

$$f(m, n) = q_2^n q_1^m f(\mathbf{0}) q_1^{-m} q_2^{-n} + (h_1 m + h_2 n) \mathbf{k}. \quad (4.22)$$

This gives a discrete cylinder as in Figure 4.3. We define an edge-based net $p = (u, v)$ by

$$u = \frac{h_1}{\tan \frac{\alpha_1}{2}} + f_1 - f, \quad v = \frac{h_2}{\tan \frac{\alpha_2}{2}} + f_2 - f. \quad (4.23)$$

One can compute that this fulfills the parallelogram equations (3.1), (3.2) and that

$$u_1 = q_1 u q_1^{-1}, \quad v_1 = q_1 v q_1^{-1}, \quad u_2 = q_2 u q_2^{-1}, \quad v_2 = q_2 v q_2^{-1}. \quad (4.24)$$

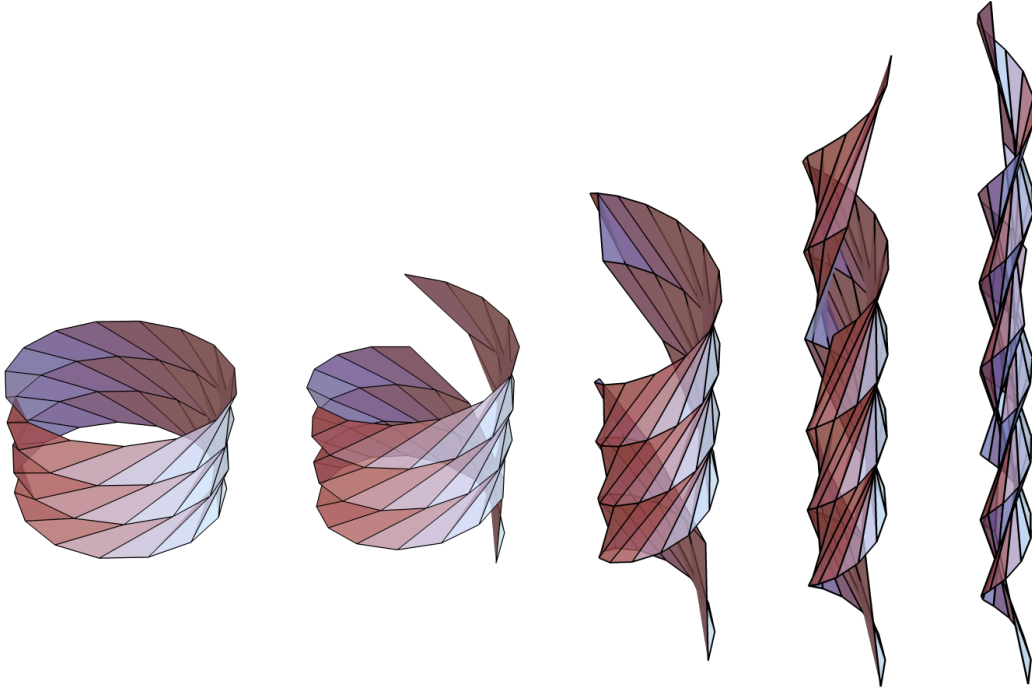


Figure 4.3.: A cylinder and members of its associated family. The rightmost net consists of anti-parallelograms.

Hence, the parallelogram net p is doubly quasi-periodic with periods $T_1 = (1, 0)$ and $T_2 = (0, 1)$ and so are the members of its associated family. Note that each parameter line $m \mapsto f(m, n)$ becomes a closed regular polygon on a circle if $h_1 = 0$ and $\alpha_1 = \frac{2\pi i}{k}$ for $k \in \mathbb{N}$. The circle breaks and becomes a helix in the associated family. Also, note that there exists a member in the associated family consisting entirely of anti-parallelograms.

In the next section we dive deeper into the geometry of skew parallelograms.

4.3. Skew parallelograms

The geometry of a skew parallelogram is well understood. For previous work, we refer, again, to [98, 87, 55] and, for skew parallelograms in the context of K-nets, to [24, Chapter 4.2.1]. For example, as stated in [24, Lemma 4.7], a skew parallelogram is symmetric under a 180° rotation around the line that connects the midpoints of its diagonals. Here, we will review an algebraic description of skew parallelograms in $\mathbb{R}^3 \cong \text{Im } \mathbb{H}$.

One can take a parallelogram and fold it along a diagonal. The resulting quad is not necessarily planar but one important property of the parallelogram gets preserved: Opposite edges have the same length, in other words, it is a Chebyshev quad. In fact, all generic Chebyshev quads can be obtained this way. We can describe such a quad f, f_1, f_{12}, f_2 using quaternions: Its edge vectors $f_{12} - f_2$ and $f_{12} - f_1$ are obtained by rotating $f_1 - f$ and $f_2 - f$ around the diagonal $f_2 - f_1$ and, hence, we have

$$\begin{aligned} f_{12} - f_2 &= (r + f_2 - f_1)(f_1 - f)(r + f_2 - f_1)^{-1}, \\ f_{12} - f_1 &= (r + f_2 - f_1)(f_2 - f)(r + f_2 - f_1)^{-1} \end{aligned} \quad (4.25)$$

for some $r \in \mathbb{R}$. If $r = 0$ we rotate by an angle of π and we get an anti-parallelogram.

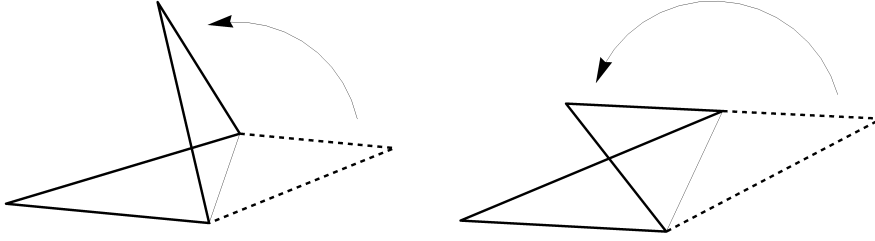


Figure 4.4.: Folding a parallelogram.

By defining

$$u = a + f_1 - f, \quad u_2 = a + f_{12} - f_2, \quad v = b + f_2 - f, \quad v_1 = b + f_1 - f \quad (4.26)$$

for some $a, b \in \mathbb{R}$ with $r = b - a$ these equations become

$$\begin{aligned} u_2 &= (v - u)u(v - u)^{-1}, \\ v_1 &= (v - u)v(v - u)^{-1} \end{aligned} \quad (4.27)$$

which is just the evolution (3.3) for a single quad. This is, by the same argumentation as in Section 3.1, equivalent to the parallelogram equations

$$u + v_1 = v + u_2, \quad v_1 u = u_2 v \quad (4.28)$$

for this quad. Thus, generically, quads with such an algebraic description and Chebyshev quads are the same. There are a few special cases which we exclude from our considerations:

- Actual parallelograms do not have such an algebraic description as they arise in the limit $r \rightarrow \infty$.

- All points of the quad can be collinear. In this case r is not unique and some geometric arguments fall apart. Still for a prescribed number r many of the algebraic formulas still hold. Note that in this case all edge values (4.26) commute.
- In the degenerate case $f_1 = f_2$ our argumentation falls apart. This case corresponds to non-evolvability and, as in Section 4.2, Chebyshev quads of this type can be considered a limit case.

To understand how r can be calculated from the points we can rewrite (4.25) to

$$\begin{aligned} r(f_{12} - f_2 - f_1 + f) &= (f_2 - f_1)(f_1 - f) - (f_{12} - f_2)(f_2 - f_1) \\ &= \text{Im}(-(f_2 - f_1)f - f_{12}(f_2 - f_1)) \end{aligned} \quad (4.29)$$

Here, the second equality follows from the fact that the LHS is imaginary and, hence, taking the imaginary part does not affect the equation. We can use (4.9) to get

$$r(f_{12} - f_2 - f_1 + f) = -(f_{12} - f) \times (f_2 - f_1), \quad (4.30)$$

which is the equation governing Chebyshev nets given by [98, Equation (3.5)]. It allows us to characterize Chebyshev nets coming from parallelogram nets:

Proposition 4.3.1 *Let $f : \mathbb{Z}^n \rightarrow \mathbb{R}^3$ be a Chebyshev net and $r : \vec{\mathcal{F}}(\mathbb{Z}^n) \rightarrow \mathbb{R}$ be the face based map fulfilling*

$$r^{ij}(f_{ij} - f_j - f_i + f) = -(f_{ij} - f) \times (f_j - f_i). \quad (4.31)$$

The net f is the projected additive primitive map of a quaternionic parallelogram net if and only if there exists an edge labelling $r : \mathcal{E}(\mathbb{Z}^n) \rightarrow \mathbb{R}$ such that

$$r^{ij} = r^j - r^i \quad (4.32)$$

and in this case such a parallelogram net p is given by $p^i = r^i + f_i - f$.

Recall that r^{ij} denotes the value of r at the quad (f, f_i, f_{ij}, f_j) . Clearly, we have $r^{ij} = -r^{ji}$.

Proof. This follows directly from above argumentation: If we assume (4.32) then the edge-based map p given by $p^i = r^i + f_i - f$ is a parallelogram net. Conversely, if f is the primitive map of a parallelogram net p then on each quad (4.25) is just the evolution (3.3) and we have $r^{ij} = \text{Re } p^j - \text{Re } p^i$. $\square$

For example, for a two-dimensional parallelogram net this condition is equivalent to

$$r_{12}^{12} + r^{12} = r_1^{12} + r_2^{12}. \quad (4.33)$$

4. Quaternionic skew parallelogram nets

This system is equivalent to the system defined by [98, Equation (3.16)] which Schief also calls a *generalized discrete Lund-Regge system*.

We now turn to a special choice of the real parts. Again, consider a single Chebyshev quad. We have the one-parameter freedom of choosing a, b with $r = b - a$. For $r \neq 0$ there is one particularly interesting choice: There is a unique a such that $\|u\| = \|v\|$ since

$$a^2 + \|\vec{u}\|^2 = \|u\|^2 = \|v\|^2 = (r + a)^2 + \|\vec{v}\|^2 = a^2 + 2ra + r^2 + \|\vec{v}\|^2 \quad (4.34)$$

is equivalent to

$$a = \frac{\|\vec{u}\|^2 - \|\vec{v}\|^2 - r^2}{2r}. \quad (4.35)$$

Such u, v with $\sigma := \frac{1}{\|u\|} = \frac{1}{\|v\|}$ allow for a remarkable construction of vertex normals to the quad: Choose an initial $g \in \text{Im}(\mathbb{H})$ orthogonal to the edges $\vec{u}$ and $\vec{v}$ and with $\|g\|^2 = 1$ (i.e., $g \in \mathbb{S}^2$). These properties determine g up to sign. Now, define

$$g_1 := \sigma u g, \quad g_2 := \sigma v g, \quad g_{12} := \sigma u_2 g_2 = \sigma v_1 g_1. \quad (4.36)$$

By construction, we have $\|g_1\| = \|g_2\| = \|g_{12}\| = 1$. Now, g_1 is imaginary and orthogonal to u and v_1 since

$$\text{Re } g_1 = -\sigma \langle u, g \rangle = 0 \quad (4.37)$$

and

$$\langle u, g_1 \rangle = \text{Re } \bar{u} g_1 = \frac{1}{\sigma} \text{Re } g = 0 \quad (4.38)$$

and

$$\begin{aligned} \langle v_1, g_1 \rangle &= -\text{Re}(v_1 g_1) = -\sigma \text{Re}((v - u)v(v - u)^{-1}ug) \\ &= -\frac{\sigma \text{Re}((v - u)(v\bar{v}u - v\bar{u}u)g)}{\|v - u\|^2} = \frac{\text{Re}((v - u)^2 g)}{\sigma \|v - u\|^2} = -\frac{\langle (v - u)^2, g \rangle}{\sigma \|v - u\|^2} = 0. \end{aligned} \quad (4.39)$$

The last equation follows from the fact that for any quaternion q we have $q^2 \in \text{span}_{\mathbb{R}}\{1, q\}$ and, therefore, since $g \in \text{Im } \mathbb{H}$ is orthogonal to $v - u$ it is also orthogonal to $(v - u)^2$. Analogously, one can show that g_2 and g_{12} are imaginary and orthogonal to incident edges. Now, since $q^{-1} = -q$ for $q \in \mathbb{S}^2$ we can rewrite (4.36) into

$$u = -\frac{1}{\sigma} g_1 g, \quad v = -\frac{1}{\sigma} g_2 g, \quad u_2 = -\frac{1}{\sigma} g_{12} g_2, \quad v_1 = -\frac{1}{\sigma} g_{12} g_1. \quad (4.40)$$

Taking the imaginary part gives

$$\begin{aligned} f_1 - f &= -\frac{1}{\sigma} g_1 \times g, & f_2 - f &= -\frac{1}{\sigma} g_2 \times g, \\ f_{12} - f_2 &= -\frac{1}{\sigma} g_{12} \times g_2, & f_{12} - f_1 &= -\frac{1}{\sigma} g_{12} \times g_1. \end{aligned} \quad (4.41)$$

This is called *Lelievre representation*. One can find such a representation for all nets where every point shares a plane with all adjacent points (also called *A-nets*, see [24, Theorem 2.42.] and references therein). Note that, so far, we have only considered this representation for a single quad.

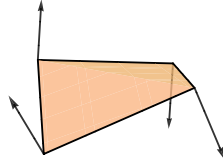


Figure 4.5.: The normals g of a parallelogram.

One can consider the dihedral angle at an edge of the tetrahedron given by the four points. We measure it as the angle $\delta \in [0, \pi]$ between adjacent normals:

$$\delta^1 := \angle(g, g_1), \quad \delta^2 := \angle(g, g_2), \quad \delta_2^1 := \angle(g_2, g_{12}), \quad \delta_1^2 := \angle(g_1, g_{12}) \quad (4.42)$$

The real parts of (4.40) give

$$a = \frac{1}{\sigma} \cos \delta^1 = \frac{1}{\sigma} \cos \delta_2^1, \quad b = \frac{1}{\sigma} \cos \delta^2 = \frac{1}{\sigma} \cos \delta_1^2 \quad (4.43)$$

and, therefore, these angles agree for opposite edges, i.e., $\delta^1 = \delta_2^1$ and $\delta^2 = \delta_1^2$. The norm of (4.41) gives

$$\sigma = \frac{\sin \delta^1}{\|\vec{u}\|} = \frac{\sin \delta^2}{\|\vec{v}\|} \quad (4.44)$$

The value σ is called the *folding parameter* of the skew parallelogram (see [55, 62]). This quantity also plays a role in mechanism science where it is known as the *Bennett ratio* (see, e.g., [79, equation (11.69)]).

Let us investigate how these observations extend to some special cases:

- If all points of the quad are collinear there is no unique initial normal g . Instead there is a one-parameter family of normals to this quad. All other arguments extend to this case.

- The important case of an anti-parallelogram can be considered to be the limit case $r \rightarrow 0$. We have $a, b \rightarrow \infty$ and $\|u\| = \|v\| \rightarrow \infty$ and, thus $\sigma \rightarrow 0$. All normals g, g_1, g_2, g_{12} agree up to sign and, thus, $\delta^1, \delta^2 \in \{0, \pi\}$. The Lelievre representation (4.41) does not work anymore, however, (4.44) holds trivially.

We have assigned a special real part to each edge of the quad. Now, consider a parallelogram net where the real part $\text{Re } p^i$ at each edge is this special real part. This is equivalent to $\sigma = \frac{1}{\|p^i\|}$ being constant everywhere. Thus, we can rescale the net to $\sigma = 1$.

Definition 4.3.2 *We call a quaternionic parallelogram net p equally folded if $\|p^i\| = 1$ for all edges. We call it zero-folded if $\text{Re } p^i = 0$ for all edges.*

Equally folded parallelogram nets have already been introduced in [98, Equations (3.16), (3.17)] where they are called *discrete Lund-Regge surfaces*.

Equally folded nets have constant folding parameter $\sigma = 1$. Zero-folded nets have constant folding parameter $\sigma = 0$ and every quad is an anti-parallelogram. In fact, zero-folded nets describe nets of anti-parallelograms in $\mathbb{R}^3$ which we discussed in Section 2.2. Nets which are primarily equivalent to an equally folded or zero-folded net also have constant folding parameter. However, the converse is not true: The folding parameter only determines the real parts up to sign (via $\frac{1}{\sigma^2} = a^2 + \|\vec{u}\|^2$) and, hence, the special real parts induced on an edge by incident faces do not need to agree.

Remark 4.3.3 *Often, it is reasonable to prescribe the sign of the real parts in each lattice direction: Consider a quad of an equally folded parallelogram net with small edge lengths $\|\vec{u}\|, \|\vec{v}\|$. This situation naturally appears if one takes a smooth limit. Then, the squared real parts a^2, b^2 approximately agree with $\frac{1}{\sigma^2} = 1$. Now, if the real parts a, b have the same sign then $r = b - a$ is very small and the quad is close to an anti-parallelogram. If the real parts have opposite sign then $r = b - a$ is very big compared to the diagonal $v - u$ and the quad is close to a parallelogram. This second case is desirable on a discrete version of a surface. On the other hand, the first case can also be reasonable, for example, for Bäcklund transformations that discretize flows of curves as in Section 4.6.*

Finally, note that (4.43) implies that for positive real part neighboring normals g, g_i along this edge almost agree while for negative real part they only almost agree after swapping the sign of one normal $g_i \mapsto -g_i$. Thus, a sign swap along edges with negative real part is natural to obtain a normal map admitting a limit (compare Remark 2.2.3).

Equally folded nets come in families:

Proposition 4.3.4 *Using the functions $\lambda(t) = e^t, \mu(t) = ce^{-t}$ for some $c \in \mathbb{R} \setminus \{0\}$ each net q_t in the associated family of a zero-folded net q is equally folded. Every equally folded net p with $\vec{p}^i \neq 0$ can be represented this way.*

This result seems to be new. In particular, [98, Theorem 7] states that members of the family of a generalized Lund-Regge surface (in our language: a quaternionic parallelogram net) are again generalized Lund-Regge surfaces. Now, our result shows that, using the right parametrization, (almost all) members of the family of a Lund-Regge surface (in our language: an equally folded parallelogram net) are again Lund-Regge surfaces.

Proof. If q is a zero-folded parallelogram net we calculate that

$$\|q_t^i\|^2 = \|\Phi^{-1}(P^i)^{-1}(P^i)'\Phi\|^2 = \frac{\|e^t - ce^{-t}q^i\|^2}{\|e^t + ce^{-t}q^i\|^2} = \frac{e^{2t} + c^2e^{-2t}\|q^i\|^2}{e^{2t} + c^2e^{-2t}\|q^i\|^2} = 1. \quad (4.45)$$

If p is equally folded we can find a zero-folded net q in its associated family: Consider $q := p_{t_0=0}$ in the associated family obtained from $\lambda(t) = e^{\frac{1}{c}t}, \mu(t) = e^{-\frac{1}{c}t}$. The net q is zero-folded which can be observed by plugging $\|p^i\|^2 = 1$ into (4.20). Next, consider the net $q_{t_1=0}$ in the family of q with $\tilde{\lambda}(t) = e^t, \tilde{\mu}(t) = ce^{-t}$ (compare Example 3.4.5). It is primarily equivalent to p which is a direct application of the second statement of Proposition 3.4.2. More explicitly, the functions in (3.39) are $\tilde{\lambda}(t) = e^t + e^{-t}, \tilde{\mu} = e^t - e^{-t}$ and, hence, equation (3.37) yields $q_{t_2=0} = p$. Thus, we found p in the associated family of the zero-folded net q . The condition $\tilde{p}^i \neq 0$ excludes some very degenerate cases where the associated families are not well-defined. $\square$

We will use this to parametrize associated families of equally folded nets. Note that, depending on the sign of c , we only obtain positive or negative ratios $\frac{\lambda(t_0)}{\mu(t_0)}$ and, thus, we only parametrize 'half' of the associated family. In both cases we are missing the special ratio $\frac{\lambda(t_0)}{\mu(t_0)} = 0$ which corresponds to a second zero-folded net: Plugging $\lambda(t_0) = 0$ and $\operatorname{Re} q^i = 0$ into (4.20) shows that $\operatorname{Re} q_{t_0}^i = \frac{\mu'(t_0)}{\mu(t_0)}$ is constant and, therefore, q_{t_0} is primarily equivalent to a zero-folded net.

4.4. K-nets

K-nets, also known as pseudospherical nets are a well-known discretization of surfaces with constant negative Gaussian curvature. They have originally been studied in [107, 90] and have been developed further in [16, 17]. We will study their close relation to parallelogram nets. We will rediscover many known properties of K-nets from the perspective of parallelogram nets. Also, we present a Lax representation for K-nets which is natural from the perspective of parallelogram nets. It is different from the classical Lax pair but related to it by gauge equivalence.

Definition 4.4.1 *A K-net (or pseudospherical net) is a Chebyshev net $f : \mathbb{Z}^n \rightarrow \mathbb{R}^3$ with planar vertex stars, i.e., where every point f and all its adjacent points $f_i, f_{\bar{i}}$ lie in a common plane.*

4. Quaternionic skew parallelogram nets

As before, we assume that the four points of each quad are not collinear and that no quad is an actual parallelogram. Parallelogram nets provide an algebraic description for K-nets:

Proposition 4.4.2 *The additive primitive map of a quaternionic parallelogram net p is a K-net if and only if p has planar vertex stars in $\text{Im } \mathbb{H} \cong \mathbb{R}^3$ meaning that at every vertex $x \in \mathbb{Z}^n$ all incident values $\vec{p}^i(x)$ and $\vec{p}_i^*(x)$ for $i \in [n]$ lie in a common two-dimensional subspace of $\mathbb{R}^3$. Every K-net is the primitive map of a parallelogram net p and (up to primary equivalence) p is equally folded or zero-folded.*

Proof. The first statement immediately follows from the fact that the points of a vertex star lie in a common plane if and only if their differences lie in a two-dimensional subspace. For the second statement, let f be a K-net. The quads are skew parallelograms and induce a special real part on edges as in Section 4.3. We will show that different quads induce the same real part on a common edge. For this, consider two neighboring quads of f . We consider normals and a folding parameter g, σ and $\hat{g}, \hat{\sigma}$ to each quad respectively as explained in Section 4.3. Then, (4.41) for their common edge reads

$$f_i - f = -\frac{1}{\sigma} g_i \times g = -\frac{1}{\hat{\sigma}} \hat{g}_i \times \hat{g} \quad (4.46)$$

Since the K-net has planar vertex stars the vertex normal at each point is given by the direction orthogonal to this star and, thus, we have $\hat{g} = \pm g$ and $\hat{g}_i = \pm g_i$. Since $\sigma \geq 0$ equation (4.46) implies $\hat{\sigma} = \sigma$ and $\hat{g}_i = g_i, \hat{g} = g$ or $\hat{g}_i = -g_i, \hat{g} = -g$. In both cases the real part assigned to this edge (4.43) agrees for both quads. Denoting this real part by r^i the net p given by $p^i = r^i + f_i - f$ is a parallelogram net. Also, since σ agrees for neighbouring quads it is constant everywhere and, thus, p is equally folded (up to a global rescaling). For the special case of anti-parallelograms, note that the vertex stars at both points define two planes. Both planes coincide if and only if one of the quads is planar which is the case of an anti-parallelogram. Then, the other quad is also an anti-parallelogram and we find the net p defined by $p^i = f_i - f$ to be a zero-folded parallelogram net. $\square$

Thus, quaternionic parallelogram nets with the special property explained in the proposition describe K-nets and, for brevity, we will refer to them as K-nets as well.

For a global view on the normal map, note that the planar vertex stars of a K-net define each normal g up to sign and using (4.36) we can choose the signs such that $g_i = \sigma p^i g$ holds along each edge. Thus, the normal map is a multiplicative primitive map of the (scaled) K-net and we will study this normal map in Section 4.7.

As discussed in Section 4.2, Proposition 4.2.3 directly implies

Proposition 4.4.3 *Each member of the associated family of a K-net is, again, a K-net.*

From Proposition 4.3.4 we know that every K-net can be found as member of the associated family of a zero-folded K-net p . Now, remarkably, a zero-folded K-net has planar quads and planar vertex stars and is, therefore, completely contained in a plane. Without loss of generality we can assume this plane to be the i, j -plane of the quaternions. If we use the matrix representation (4.12) for quaternions each value p^i has only off diagonal entries and, with $\lambda = e^t$ and $\mu = e^{-t}$, the Lax representation (3.19) takes the form

$$P^i = \begin{pmatrix} \lambda & \frac{1}{\lambda} a^i \\ -\frac{1}{\lambda} \bar{a}^i & \lambda \end{pmatrix}, \quad \lambda = e^t. \quad (4.47)$$

This is a Lax representation for K-nets. While the original net p is zero-folded and consists of anti-parallelograms, the equally folded nets in the family unfold into non-planar K-nets as exemplified in Figure 4.6. To compare this representation with the classical K-net representation we denote its two-dimensional version (for K-nets $\mathbb{Z}^2 \rightarrow \mathbb{R}^3$) by

$$U = \begin{pmatrix} \lambda & \frac{1}{\lambda} a \\ -\frac{1}{\lambda} \bar{a} & \lambda \end{pmatrix}, \quad V = \begin{pmatrix} \lambda & \frac{1}{\lambda} b \\ -\frac{1}{\lambda} \bar{b} & \lambda \end{pmatrix}, \quad \lambda = e^t \quad (4.48)$$

where U, a are defined on edges $(x, x + e_1)$ and V, b are defined on edges $(x, x + e_2)$.

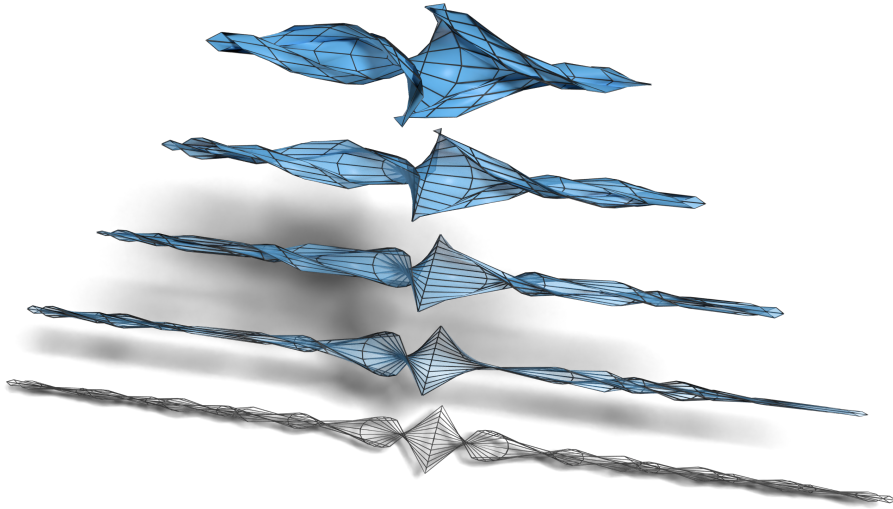


Figure 4.6.: A zero-folded K-net unfolds into its associated family. This example shows an Amsler surface.

We will show that our Lax representation (4.48) is gauge equivalent to the classical K-net Lax representation. This implies that our associated family coincides with the

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known associated family. The classical Lax representation for K-nets given by [17, equation (45),(46)] is of the form

$$L = \begin{pmatrix} x & i\lambda \\ i\lambda & \bar{x} \end{pmatrix}, \quad M = \begin{pmatrix} 1 & \frac{y}{\lambda} \\ \frac{-\bar{y}}{\lambda} & 1 \end{pmatrix}, \quad \lambda = e^t. \quad (4.49)$$

If Ψ is a solution $\Psi_1 = L\Psi, \Psi_2 = M\Psi$ then a K-net is given by the Sym formula $f = 2 \operatorname{Im}(\Psi^{-1}\Psi')$.

Proposition 4.4.4 *The classical K-net Lax representation (4.49) is gauge equivalent to the Lax representation (4.48).*

Proof. In the Lax representation L, M given by (4.49) we can replace $\lambda = e^t$ with $\lambda^2 = e^{2t}$ and rescale L and M with λ and $\frac{1}{\lambda}$, respectively. We obtain

$$\hat{L} := \frac{1}{\lambda} L(\lambda^2) = \begin{pmatrix} \frac{x}{\lambda} & i\lambda \\ i\lambda & \frac{\bar{x}}{\lambda} \end{pmatrix}, \quad \hat{M} := \lambda M(\lambda^2) = \begin{pmatrix} \lambda & \frac{y}{\lambda} \\ \frac{-\bar{y}}{\lambda} & \lambda \end{pmatrix}. \quad (4.50)$$

A gauge with $G(k, l) = \mathbf{i}^k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}^k$ gives

$$U = G_1 \hat{L} G^{-1} = \begin{pmatrix} \lambda & \frac{a}{\lambda} \\ \frac{-\bar{a}}{\lambda} & \lambda \end{pmatrix}, \quad V = G_2 \hat{M} G^{-1} = \begin{pmatrix} \lambda & \frac{b}{\lambda} \\ \frac{-\bar{b}}{\lambda} & \lambda \end{pmatrix}, \quad (4.51)$$

where for even k we have $a = -i\bar{x}, b = y$ and for odd k we have $a = -ix, b = -\bar{y}$. This is exactly (4.48).

None of these transformations affects the compatibility condition, i.e., $L_2 M = M_1 L$ is equivalent to $U_2 V = V_1 U$. Also the Sym formulas coincide up to scaling and real part, since $2 \operatorname{Im}(\Psi^{-1}\Psi') = \operatorname{Im}(\Phi^{-1}\Phi')$, where Ψ belongs to the pair L, M and Φ belongs to U, V . $\square$

We already discussed that the normal map arises as multiplicative primitive map. In fact, with $\Phi(\mathbf{0}) = 1$ formula (3.27) with $g_t(\mathbf{0}) = g^0 = \mathbf{k}$ produces the normal map g_t for K-nets f_t in the associated family since $\mathbf{k}$ is, by construction, normal to the vertex star of f_t at the origin $\mathbf{0}$. Now, for the Lax representation (4.47) we have

$$(P^i)' \mathbf{k} = (e^t - e^{-t} p^i) \mathbf{k} = \mathbf{k} (e^t + e^{-t} p^i) = \mathbf{k} P^i \quad (4.52)$$

and, hence, the normal map is also given by

$$g_t = \Phi^{-1} \mathbf{k} \Phi \quad (4.53)$$

which justifies calling Φ a *frame*. In the classical Lax representation (4.49) the normal map is given by $\hat{g}_t = \Psi^{-1} \mathbf{k} \Psi$. The two normal maps are related by $\hat{g}_t(k, l) = (-1)^k g_t(k, l)$. This swap of signs corresponds to the swap of signs in Remark 4.3.3 and $\hat{g}_t$ is the geometric normal map which admits a smooth limit.

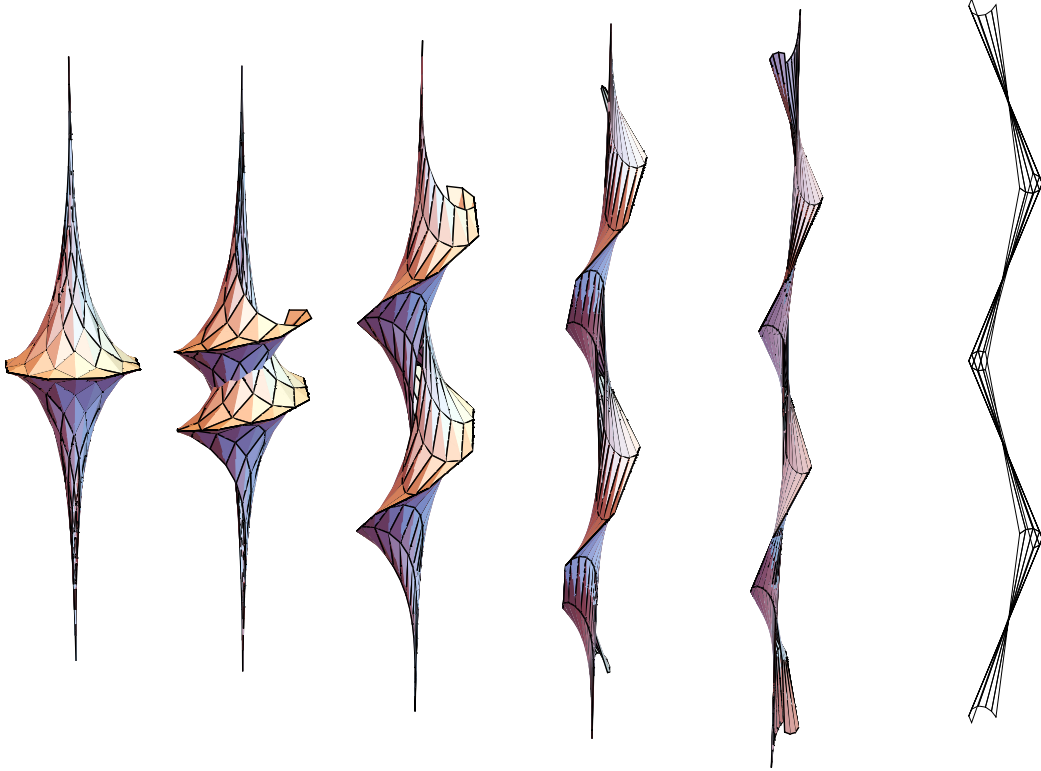


Figure 4.7.: The pseudosphere and members of its associated family. The right-most member is zero-folded and contained in a plane.

Example 4.4.5 Using our representation we can construct (all) K-nets by prescribing a net of anti-parallelograms in the plane. Such a two-dimensional net $p = (u, v)$ is determined by its values $u(m, 0), v(0, n)$ on the axes. For example, choosing constant $u(m, 0) = a$ and $v(0, n) = b$ for two quaternions $a, b \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ (with $\|a\| \neq \|b\|$) yields the associated family of the Amsler surface shown in Figure 4.6.

Example 4.4.6 We call a K-net rhombic if all edge-lengths $\|\vec{f}_i - \vec{f}\|$ coincide. In this case, the quads are rhombi folded along a diagonal. Let $p = (u, v)$ be a two-dimensional net of anti-parallelograms in the plane such that $\|u\| = l$ and $\|v\| = \frac{1}{l}$ for some constant $l \in \mathbb{R}$. Then, as consequence of (4.21), the member $p_{t=0}$ of the associated family (with $\lambda = e^t, \mu = e^{-t}$) is a rhombic K-net with edge-length $\|\vec{f}_i - \vec{f}\| = \frac{2l}{1+t^2}$.

Some well-known K-nets arise as Bäcklund transformations of other, very simple, K-nets. For example, Figure 4.7 shows the associated family of the *pseudosphere* which is a Bäcklund transformations of the straight line. Here, the straight line is the trivial

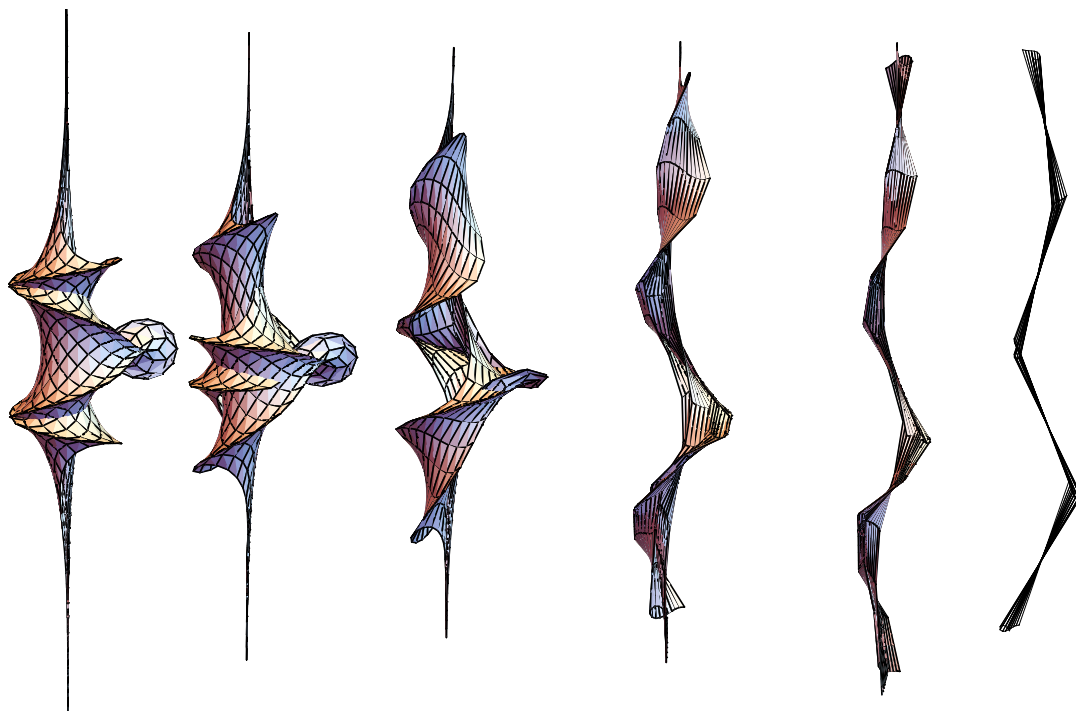


Figure 4.8.: The Kuen surface and members of its associated family. The right-most member is zero-folded and contained in a plane.

K-net given by $u(m, n) = r + a\mathbf{i}$ and $v(m, n) = s + b\mathbf{i}$ for some constants $r, s, a, b \in \mathbb{R}$. A second Bäcklund transformation applied to the pseudosphere yields the *Kuen surface* as in Figure 4.8.

Here, there is an ambiguity with Bäcklund transformations: In Definition 3.2.3 we define a Bäcklund transformation via the parallelogram equations. However, unless explicitly stated, we will only consider two n -dimensional K-nets to be related by a Bäcklund transformation if they are adjacent layers of an $(n+1)$ -dimensional K-net. This means that the condition of planar vertex stars constitutes an additional requirement on the transformation.

Constructing Bäcklund transformations of K-nets is particularly simple on the level of zero-folded nets: If the original zero-folded net and initial edges of Bäcklund transformations are contained in a common plane the resulting zero-folded net yields Bäcklund transformations of K-nets in the associated family. In particular, the zero-folded straight line is given by $u(m, n) = a\mathbf{i}$ and $v(m, n) = b\mathbf{i}$ for constants $a, b \in \mathbb{R}$ and a Bäcklund transformation is determined by one initial edge in the $\mathbf{i}, \mathbf{j}$ -plane.

Finally, one can consider K-nets that fulfill a periodicity condition. For example, the pseudosphere is periodic as it is a surface of revolution. Periodic K-nets have been studied

extensively in [16]. Note that Proposition 3.5.2 implies that the zero-folded planar K-net corresponding to a periodic K-net must be quasi-periodic as well. Such quasi-periodic nets of anti-parallelograms in the plane also appear in the following section, however, the geometric motivation will be completely different.

4.5. Polygon recutting

Polygon recutting has been introduced in [2]. Its algebraic description has been investigated in [68] and we will show how it fits into the theory of parallelogram nets. The results within the framework of polygon recutting are not new, however, we point out a connection to periodic K-nets which, to the knowledge of the author, has not been observed.

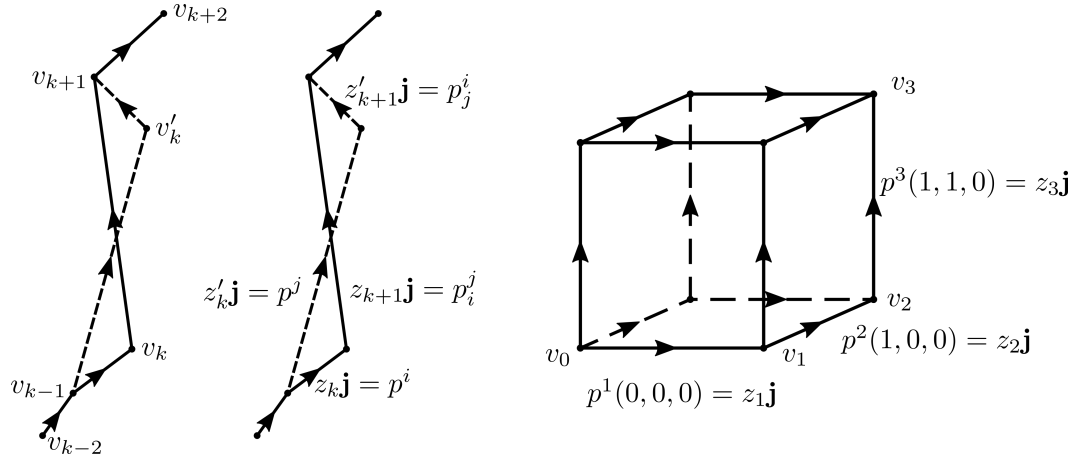


Figure 4.9.: Left: Polygon recutting at vertex k . Right: The orbit of the finite polygon $v_0, \dots, v_3$ corresponds to a three-dimensional parallelogram cube.

Polygon recutting is an action on a polygon in the plane: A polygon is given by a sequence $(v_i \in \mathbb{C})_{i \in \mathbb{Z}}$. Now, recutting at vertex k is defined by replacing $(v_i)_{i \in \mathbb{Z}}$ with a new polygon $(v'_i)_{i \in \mathbb{Z}}$ such that 1. $v'_i = v_i$ for all $i \neq k$ and 2. the new value v'_k is obtained by reflection of the point v_k on the perpendicular bisector of the diagonal (v_{k-1}, v_{k+1}) . As shown in Figure 4.9 on the left, the four points v_{k-1}, v_k, v_{k+1} and v'_k form an anti-parallelogram. Thus, our equations for the description of anti-parallelograms in a plane also describe polygon recutting.

We will review the algebraic description given in [68] and see how it matches our description. Polygon recutting at a vertex k can be expressed in terms of its edges

$z_k = v_k - v_{k-1}$ by (see [68, equation (19)])

$$\begin{aligned} z'_k + z'_{k+1} &= z_k + z_{k+1}, \\ z'_k \bar{z}'_{k+1} &= z_k \bar{z}_{k+1}. \end{aligned} \quad (4.54)$$

If we identify $\mathbb{C}$ with $\text{span}_{\mathbb{R}}\{1, \mathbf{k}\} \subseteq \mathbb{H}$ this is equivalent to

$$(1 + \mu z'_{k+1} \mathbf{j})(1 + \mu z'_k \mathbf{j}) = (1 + \mu z_{k+1} \mathbf{j})(1 + \mu z_k \mathbf{j}). \quad (4.55)$$

After a conjugation, which reverses the order, and with $t = -\mu$ this equation agrees with [68, equation (9)]. If one defines $p^i = z_k \mathbf{j}, \dots$ as in Figure 4.9 on the left (4.55) becomes our compatibility condition (3.18) (with $\lambda \equiv 1$) for quaternionic parallelogram nets in the $\mathbf{i}, \mathbf{j}$ -plane.

With more involved combinatorics one can express polygon recutting in terms of parallelogram nets even globally. For a finite sequence $v_0, \dots, v_n$ consider an n -dimensional cube in a parallelogram net such that

$$p^k(e_1 + \dots + e_{k-1}) = z_k \mathbf{j}. \quad (4.56)$$

Recutting of the polygon then corresponds to taking different paths through the n -dimensional cube in the parallelogram net (see, e.g., Figure 4.9 on the right). In other words, the n -dimensional cube corresponds to the orbit of the polygon under recutting. Computing this orbit from the initial polygon can be achieved using the sideways evolution (4.16) or the concept of factorization of polynomials which we discuss in Chapter 6.

This system becomes more interesting if one requires quasi-periodicity of the polygon: Consider an infinite sequence $(v_i \in \mathbb{C})_{i \in \mathbb{Z}}$ where for some $n \in \mathbb{N}$ the vertex v_{i+n} is related to v_i by a Euclidean motion ψ , i.e., where $v_{i+n} = \psi(v_i)$ for all i . In the quaternionic setting we write this quasi-periodicity as $v_{i+n} \mathbf{j} = E^{-1} v_i \mathbf{j} E + F$ for some constant quaternions E, F . Then, consider an n -dimensional parallelogram net with (4.56) and the quasi-periodicity $p^k(x + T) = E^{-1} p^k(x) E$ with the period $T = e_1 + \dots + e_n$. Now, polygons obtained by recutting (v_k) correspond to periodic paths in this parallelogram net.

Note that recutting acts as the identity if the edges adjacent to the vertex have the same length. Indeed, if $|z_k| = |z_{k+1}|$ we have $v'_k = v_k$. This has a strong implication on the dimension of the smallest parallelogram net that can cover the orbit of recutting. In fact, this dimension is the number of edge-lengths of the polygon $M = \#\{|z_k| \mid k \in \mathbb{Z}\}$. For example, recutting of a polygon with $M = 2$ generates a two-dimensional parallelogram net. Remarkably, the initial data (as described in the previous section) of a periodic K-net with constant edge-lengths is of this type. Conversely, recutting of a polygon with $M = 2$ corresponds to the construction of initial data for a quasi-periodic K-net. Figure 4.10 shows a parallelogram net obtained by recutting of a zig-zag. Up to tweaking initial parameters, this net agrees with the zero-folded net in Figure 4.7 on the right and its associated family contains the pseudosphere.

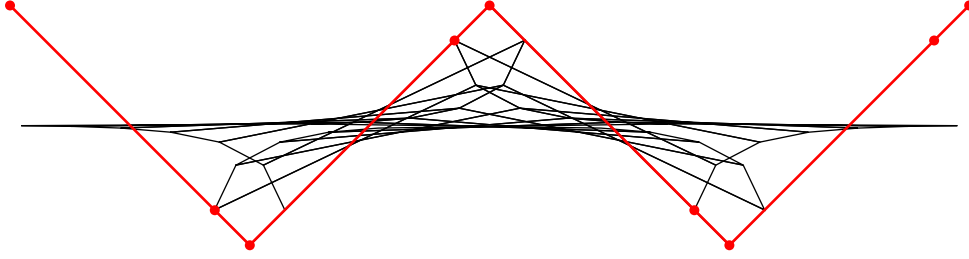


Figure 4.10.: A zig-zag path (red) with alternating edge-lengths and a part of its orbit - which forms a two-dimensional parallelogram net in a plane.

4.6. Bäcklund transformations of curves

Periodic Bäcklund transformations of discrete curves with periodicity have been studied in the context of discrete flows [87, 55] and, for planar curves, in the context of a discrete bicycle transformation [105]. We review the well-known algorithm which constructs such transformations. In particular, we present a slightly generalized version since we study quasi-periodic transformations of quasi-periodic curves in space. This generalization is motivated by its application in Chapter 8 and 9 where we use it to construct a special hierarchy of curves.

We will be concerned with a two-dimensional quaternionic parallelogram net $p = (u, v) : \mathcal{E}(\mathbb{Z}^2) \rightarrow \mathbb{H}$ and its additive primitive map $f : \mathbb{Z}^2 \rightarrow \mathbb{H}$. We interpret each parameter line $k \mapsto f(k, l_0)$ as discrete curve in $\mathbb{H}$. We will say that neighboring curves are related by a Bäcklund transformation since the corresponding parameter lines of the parallelogram net $k \mapsto u(k, l_0)$ (which, trivially, are 1-dimensional parallelogram nets) are related by a Bäcklund transformation as in Definition 3.2.3.

Let us begin with some initial observations about such Bäcklund transformations. To avoid some degenerate cases we will assume that at no quad we have both $\operatorname{Re} u = \operatorname{Re} v$ and $\|u\| = \|v\|$. In particular, this implies that the parallelogram net is evolvable and we can consider the evolution (3.3) which reads

$$v \mapsto v_1 = (u - v)v(u - v)^{-1}. \quad (4.57)$$

We will write the quaternion v as $v = r + \eta q$ with $r = \operatorname{Re} v$, $\eta = \|\vec{v}\|$ and $q \in \mathbb{S}^2$. Both r and η are invariant under the evolution and we call these quantities the *Bäcklund parameters*. We can express the evolution of the imaginary unit quaternion q by the fractional linear map

$$\begin{aligned} T_u^{r, \eta} : \mathbb{S}^2 &\rightarrow \mathbb{S}^2, \\ q &\mapsto q_1 = (u - v)q(u - v)^{-1} = (u - r - \eta q)q(u - r - \eta q)^{-1} \\ &= ((u - r)q + \eta)(-\eta q + u - r)^{-1}. \end{aligned} \quad (4.58)$$

Thus, the unit quaternion q evolves by

$$q_1 = T_u^{r,\eta}(q). \quad (4.59)$$

The map $T_u^{r,\eta}$ is linear when written in homogeneous coordinates in $\mathbb{HP}^1$, i.e., by identifying $\mathbb{HP}^1 \ni (p, q) \cong pq^{-1} \in \mathbb{H}$. Then, the map is given by

$$q \cong \begin{pmatrix} q \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} u-r & \eta \\ -\eta & u-r \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} \cong T_u^{r,\eta}(q). \quad (4.60)$$

We will make no distinction between the map $T_u^{r,\eta}$ and its matrix representation in $\mathbb{HP}^1$. The matrix $T_u^{r,\eta}$ belongs to the set of matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbb{H}^{2 \times 2}. \quad (4.61)$$

As described with more detail in [87], this set becomes an algebra isomorphic to the biquaternions $\mathbb{C}^{2 \times 2}$ when equipped with the scalar multiplication

$$\lambda M := (\operatorname{Re}_{\mathbb{C}}(\lambda)I + \operatorname{Im}_{\mathbb{C}}(\lambda)J)M, \quad \lambda \in \mathbb{C}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.62)$$

The matrix J corresponds to the imaginary unit $i \in \mathbb{C}$ and, in particular, commutes with all elements of the set. Complex trace and determinant are given by

$$\frac{1}{2} \operatorname{tr} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \operatorname{Re} A + \operatorname{Re} Bi, \quad \det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = |A|^2 - |B|^2 + 2\langle A, B \rangle_{4i}. \quad (4.63)$$

where $\langle A, B \rangle_4 = \operatorname{Re}(A \cdot \bar{B})$ denotes the standard scalar product of the quaternions. The trace free component of $M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ is $\begin{pmatrix} \vec{A} & \vec{B} \\ -\vec{B} & \vec{A} \end{pmatrix}$ and we will denote it by $\vec{M}$.

Note, that $\det M = 0$ is equivalent to either $B^{-1}A \in \mathbb{S}^2$ or $A = B = 0$. If $\det M \neq 0$, the matrix M defines a map $\mathbb{S}^2 \rightarrow \mathbb{S}^2$:

$$q \cong \begin{pmatrix} q \\ 1 \end{pmatrix} \mapsto M \begin{pmatrix} q \\ 1 \end{pmatrix} \cong (A - Bq)q(A - Bq)^{-1} \in \mathbb{S}^2. \quad (4.64)$$

It is well defined, since $A = Bq$ for a $q \in \mathbb{S}^2$ implies $\det M = 0$. If $\vec{M} = 0$ this map acts as the identity. If $B = 0$ it acts as a conjugation: $q \mapsto AqA^{-1}$. In particular, our evolution $T_u^{r,\eta}$ defines such a map since $\det T_u^{r,\eta} = \|u - r\|^2 - \eta^2 + 2(\operatorname{Re} u - r)i \neq 0$.

Matrices with $B = 0$ are isomorphic to the quaternions. Our map $T_u^{r,\eta}$ is just a representation of $(-r + i\eta) + u$. Hence, it fulfills the parallelogram equations (3.1) and (3.2). Also, note that the Lax matrix for Bäcklund transformations presented in [87] is a representation of $\lambda + iu$ and, thus, coincides with our Lax representation (3.19).

Now, we are interested in quasi-periodic parallelogram nets (as in Section 3.5) with period $T = (n, 0)$ and we express the periodicity by

$$u(k+n, l) = E^{-1}u(k, l)E, \quad v(k+n, l) = E^{-1}v(k, l)E. \quad (4.65)$$

for some $E \in \mathbb{H}$. This corresponds to a quasi-periodicity of each curve $f(\cdot, l_0)$ given by

$$f(k+n, l_0) = E^{-1}f(k, l_0)E + E_T \quad (4.66)$$

for some $E, E_T \in \mathbb{H}$. In most applications we will project to Euclidean 3-space $\text{Im } \mathbb{H} \cong \mathbb{R}^3$ for which quasi-periodicity is exactly periodicity of $p = (u, v)$ up to rotation or, equivalently, periodicity of f up to Euclidean motion.

Now, let us fix some initial l_0 and consider the curve $f(k) := f(k, l_0)$ and the corresponding (1-dimensional) parallelogram net $u(k) := u(k, l_0)$. The rotation given by E composed with the evolution of $q(k)$ along each edge of the curve can be written as a linear map in $\mathbb{HP}^1$ by

$$M^{r, \eta} := \begin{pmatrix} A & B \\ -B & A \end{pmatrix} := \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} T_{u(n-1)}^{r, \eta} \cdots T_{u(0)}^{r, \eta} \quad (4.67)$$

and we call it the *monodromy* of the curve. It transports $q(k) := q(k, l_0)$ along the curve. The central observation is that the quasi-periodicity implies that $q(0)$ is a fixed point of the monodromy. Conversely, any fixed point of the monodromy gives rise to a quasi-periodic Bäcklund transformation of the curve.

To make a statement about existence and uniqueness of such transformations we exclude some degenerate cases. For this, we call a pair of numbers $r \in \mathbb{R}$ and $\eta \in \mathbb{R}_{\geq 0}$ an *admissible pair of Bäcklund parameters* if $\det M^{r, \eta} \neq 0$ and $\det \vec{M}^{r, \eta} \neq 0$. Here, the conditions $\det M \neq 0$ ensures that M gives a well-defined map $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ which (as argued in [55]) is a Möbius transformation of $\mathbb{S}^2$. The condition $\det \vec{M} \neq 0$ excludes cases where this Möbius transformations has only one or infinitely many fixed points.

Theorem 4.6.1 *Let $f : \mathbb{Z} \rightarrow \mathbb{H}$ be quasi-periodic (as in (4.66)). Then, for any admissible pair of Bäcklund parameters r, η there exist exactly two Bäcklund transformations of f that preserve its quasi-periodicity.*

The proof immediately follows from our considerations above together with the following lemma applied to the monodromy $M^{r, \eta}$.

Lemma 4.6.2 *Consider a fractional linear map $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ defined by the matrix*

$$M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbb{H}^{2 \times 2} \quad (4.68)$$

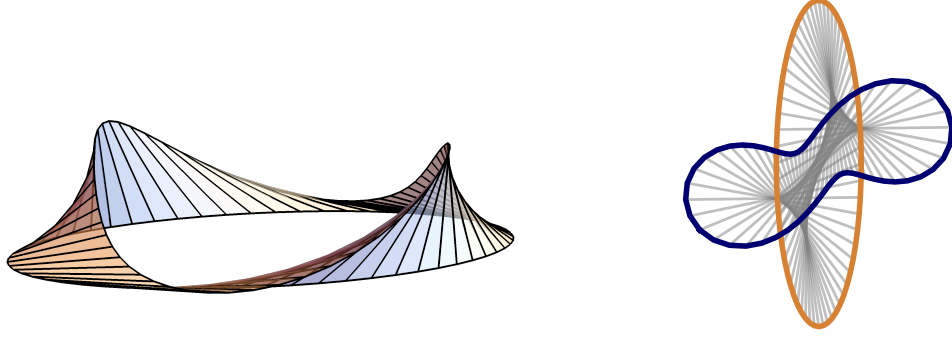


Figure 4.11.: Periodic Bäcklund transformations of an ellipse. On the right, the transform is contained in a plane (see Remark 4.6.7).

with $\det M \neq 0$ and $\det \vec{M} \neq 0$. It has exactly two fixed points and they are given by

$$q = (\beta + \vec{B})^{-1}(\alpha + \vec{A}), \quad (4.69)$$

where $\alpha, \beta \in \mathbb{R}$ fulfill $\alpha + i\beta = \pm \sqrt{\det \vec{M}}$.

Proof. We need to find every $q \in \mathbb{S}^2$ such that

$$(A - Bq)q = q(A - Bq). \quad (4.70)$$

Two quaternions commute if and only if their imaginary parts are parallel. This means that there exists $\alpha, \beta \in \mathbb{R}$ such that

$$A - Bq = (\operatorname{Re} A - \alpha) + (\beta - \operatorname{Re} B)q, \quad (4.71)$$

which can be solved into (4.69). The equations $\operatorname{Re} q = 0$ and $\|q\| = 1$ can be written as

$$\begin{aligned} 0 &= \|\beta + \vec{B}\|^2 \operatorname{Re} q = \operatorname{Re}((\beta - \vec{B})(\alpha + \vec{A})) = \beta\alpha + \langle \vec{B}, \vec{A} \rangle, \\ 0 &= \|\beta + \vec{B}\|^2 - \|\alpha + \vec{A}\|^2 = \beta^2 - \alpha^2 + \|\vec{B}\|^2 - \|\vec{A}\|^2 \end{aligned} \quad (4.72)$$

These are equivalent to the complex equation

$$(\alpha + i\beta)^2 = \|\vec{B}\|^2 - \|\vec{A}\|^2 - 2i\langle \vec{B}, \vec{A} \rangle = \det \vec{M}. \quad (4.73)$$

Hence q is a fixed point if and only if it fulfills (4.69) with α, β of this form. $\square$

Let us clarify some relations to our investigation of quasi-periodic parallelogram nets in Section 3.5. By using our identification with the biquaternions we saw that the matrix $T_u^{r,\eta}$ corresponds to the biquaternion $(-r + i\eta) + u$. Thus, by treating the monodromy as

polynomial in the complex parameter $t = -r + i\eta$ it coincides with the polynomial (3.51) and we can write $M^{r,\eta} = \mathcal{P}(0, l_0)$. Equation (3.53) now reads

$$M_2^{r,\eta} = T_v^{r,\eta} M^{r,\eta} (T_v^{r,\eta})^{-1} \quad (4.74)$$

where $M_2^{r,\eta}$ is the monodromy of the Bäcklund transform $f(\cdot, l_0 + 1)$. Thus, as in Section 3.5, we can observe that the polynomials $\det \mathcal{P} = \det M^{r,\eta}$ and $\text{tr } \mathcal{P} = \text{tr } M^{r,\eta}$ are invariant under a Bäcklund transformation. Expanding the polynomial $\mathcal{P} = M^{r,\eta} = \sum_{i=0}^n t^i C^{n-i}$ in terms of coefficients we find that the coefficients evolve by (3.56).

Remark 4.6.3 *As observed in [55], in the case of a closed curve this evolution of the coefficients yields a geometric invariant: We have $C^0 = E = \mathbf{1}$ and $C^1 = \sum_{k=0}^n u(k) = 0$. Thus, (3.56) implies that the coefficient C^2 is constant. This coefficient is called area vector since for any $N \in \mathbb{R}^3$ the quantity $\langle \vec{C}^2, N \rangle_3$ can be computed to be the enclosed area of the closed curve $\vec{f}$ projected into the plane $N^\perp$.*

For the case $r = 0$ (which can always be achieved using primary equivalence) it is straight-forward to relate the entries A and B of the monodromy to the coefficients C^i of the polynomial: They are given by taking real and imaginary part of the biquaternion $\mathcal{P}(t = i\eta) = M^{r,\eta} = A + iB$ and we have

$$\begin{aligned} A &= C^n - \eta^2 C^{n-2} + \eta^4 C^{n-4} - \dots, \\ B &= \eta C^{n-1} - \eta^3 C^{n-3} + \eta^5 C^{n-5} - \dots \end{aligned} \quad (4.75)$$

It becomes apparent that the central condition relating the monodromy to the fixed point (4.70) is exactly (3.58) which we computed for general quasi-periodic parallelogram nets.

Now, from an initial curve f one can iterate the construction of periodic Bäcklund transformations to obtain a quasi-periodic parallelogram net. In each step one can choose between the two possible transformations. An interesting question is whether one of the choices is more natural. In fact, if we choose the same Bäcklund parameters in every step it turns out that one of the Bäcklund transformations goes back to the previous curve. Thus, going 'forward' yields a unique evolution of the curve:

Theorem 4.6.4 (Algorithm) *Let $f(\cdot, 0) : \mathbb{Z} \rightarrow \mathbb{H}$ be quasi-periodic (as in (4.66)), set $u(k, 0) := f(k+1, 0) - f(k, 0)$ and let r, η be an admissible pair of Bäcklund parameters of the monodromy of f . Then, we obtain an evolution of f by iterating the following steps, starting from $l = 0$:*

1. *If $l = 0$, choose $q(0, 0)$ as one of the two fixed points of the monodromy of $f(\cdot, 0)$. If $l > 0$, choose $q(0, l)$ as the unique fixed point of the monodromy of $f(\cdot, l)$ which is distinct from $-q(0, l-1)$.*
2. *Use the evolution (3.3) to construct the subsequent curve $u(\cdot, l+1)$ (and corresponding $f(\cdot, l+1)$) from the initial $v(0, l) := r + \eta q(0, l)$.*

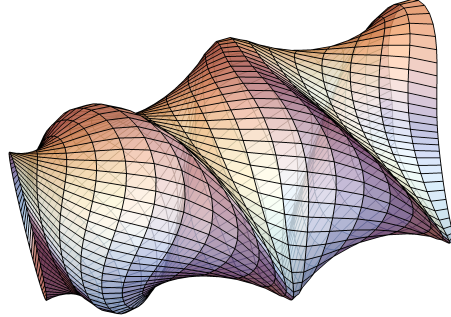


Figure 4.12.: The evolution in Theorem 4.6.4 applied to a discrete ellipse.

Proof. The assumptions of Theorem 4.6.1 on admissibility of the Bäcklund parameters r, η are always fulfilled since $\det M^{r, \eta}$ and $\det \vec{M}^{r, \eta}$ are invariant. Thus, in every step the monodromy has two fixed points. Now, one of the fixed points must be $-q(0, l-1)$ since going back to the previous curve yields a Bäcklund transformation with parameters r, η fulfilling all requirements. We conclude that there is one unique fixed point distinct from $-q(0, l-1)$ in every step. $\square$

Together with the analogous construction for $l < 0$ (starting from the other fixed point) this evolution constructs a quasi-periodic two-dimensional parallelogram net $p = (u, v)$ with additive primitive map f . As exemplified in Figure 4.12, one can apply this to a closed curve to obtain a discrete surface with the topology of a cylinder.

For implementation of the algorithm and for some applications presented in Chapter 8 and 9 the following lemma will be helpful since it allows us to compute the fixed point in each step explicitly from (4.69):

Lemma 4.6.5 *Throughout the evolution described in Theorem 4.6.4 the quantities α, β determining the fixed point as in (4.69) are constant.*

Proof. Since $\det \vec{M}^{r, \eta}$ is constant we only have to show that the sign of $\sqrt{\det \vec{M}^{r, \eta}}$ that corresponds to going 'forward' is constant as well. For this, consider a Bäcklund transformation with initial fixed point q of the monodromy $M^{r, \eta}$. We will show how in the next step $-q$ is obtained from (4.69) as a fixed point of the monodromy $M_2^{r, \eta}$.

The off-diagonal entries of (4.74) yield $A_2 + B_2q = qB + A$. By writing q as in (4.69)

we obtain

$$\begin{aligned} A_2 + B_2 q &= q(\operatorname{Re} B + \vec{B}) + \vec{A} + \alpha - \alpha + \operatorname{Re} A = q(\operatorname{Re} B + \vec{B}) + (\beta + \vec{B})q - \alpha + \operatorname{Re} A \\ &= q\vec{B} + \vec{B}q + (\beta + \operatorname{Re} b)q - \alpha + \operatorname{Re} A = (\beta + \operatorname{Re} B)q + (\alpha + \operatorname{Re} A), \end{aligned} \quad (4.76)$$

where in the last step we use $q\vec{B} + \vec{B}q = 2\operatorname{Re}(\vec{B}q) = 2\alpha$. Comparing both sides of this equation yields

$$-q = (-\beta + \vec{B}_2)^{-1}(-\alpha + \vec{A}_2) \quad (4.77)$$

and, hence, the pair $-\alpha, -\beta$ gives us the fixed point $-q$ which corresponds to the Bäcklund transformation going back to the previous curve. The other fixed point of $M_2^{r,\eta}$ must then be obtained from (4.69) by $\alpha_2 = \alpha, \beta_2 = \beta$. $\square$

For the case where the Bäcklund parameters are not constant we review [87, Theorem 2] which directly follows from our considerations in Section 3.2.

Proposition 4.6.6 (Permutability) *If f_2 is a Bäcklund transform of a curve f with parameters r, η and f_3 is a Bäcklund transform of f with parameters $\hat{r}, \hat{\eta}$ then there exists a unique Bäcklund transform f_{23} of f_2 with parameters $\hat{r}, \hat{\eta}$ which is also a Bäcklund transform of f_3 with parameters r, η . This curve f_{23} can be computed from f, f_2 and f_3 using the evolution (3.3).*

We conclude this section by remarking that our construction for periodic Bäcklund transformations has a reduction to the plane (see Figure 4.11 on the right). This is known as the discrete bicycle transformation [105] and we will be concerned more with this planar reduction in Chapter 9.

Remark 4.6.7 *Assume that we have a quasi-periodic curve contained in the $\mathbf{i}, \mathbf{j}$ -plane which is given by $u(\cdot, 0) \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ and $E \in \operatorname{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ or $E \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$. We find that each coefficient C^k of the monodromy is in contained in $\operatorname{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ or $\operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ depending on the parity of k . Then, by (4.75) we get $A \in \operatorname{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ and $B \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ (or the other way around) and we find $\det \vec{M} = \|A\|^2 - \|B\|^2 \in \mathbb{R}$ to be real. Depending on the sign of $\det \vec{M}$ either α or β vanish and in one of those cases we have $q \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$. Hence, for Bäcklund parameters $r = 0$ and η for which the sign of $\det \vec{M}$ yields this aforementioned case the corresponding Bäcklund transform is contained in the plane as well.*

4.7. Moutard nets and spherical geometry

We have mostly been concerned with the additive primitive map and Euclidean geometry. However, we already saw that the multiplicative primitive map corresponding to a K-net is its normal map, which is spherical. In fact, it is known that the normal maps of

K-nets are Moutard nets on the sphere $\mathbb{S}^2$ (see [24, Theorem 4.8]). More generally, with $g(\mathbf{0}) \in \mathbb{S}^3$ the multiplicative primitive map of an equally folded net is a map $g : \mathbb{Z}^n \rightarrow \mathbb{S}^3$ and has also been related to Moutard nets [98]. In this section, we review this close relation between equally folded parallelogram nets and Moutard nets on the sphere. As a consequence of Proposition 4.3.4 we find that one can interpret the associated family as a deformation of Euclidean anti-parallelograms into spherical anti-parallelograms and this interpretation seems to be new.

First, we introduce some basics about Moutard nets on a sphere. These can also be found in [24, Section 4.1]. Recall Definition 2.2.2 of Moutard nets via the Moutard equation

$$f_{ij} - f = a^{ij}(f_j - f_i). \quad (4.78)$$

If all points of a Moutard net are contained in a sphere, we have $\|f\| = 1$ and, hence,

$$\begin{aligned} 1 &= \langle f_{ij}, f_{ij} \rangle = \langle f + a^{ij}(f_j - f_i), f + a^{ij}(f_j - f_i) \rangle \\ &= 1 + 2a^{ij}\langle f, f_j - f_i \rangle + (a^{ij})^2 \langle f_j - f_i, f_j - f_i \rangle. \end{aligned} \quad (4.79)$$

We will assume that diagonals in every quad do not vanish which implies $a^{ij} \neq 0$. Then, a^{ij} is uniquely given by the equation

$$a_{ij} = \frac{-2\langle f, f_j - f_i \rangle}{\langle f_j - f_i, f_j - f_i \rangle} = \frac{\langle f, f_i - f_j \rangle}{1 - \langle f_i, f_j \rangle} \quad (4.80)$$

implying that the fourth point f_{ij} of every quad is determined by the other three points by

$$f_{ij} = f + \frac{\langle f, f_i - f_j \rangle}{1 - \langle f_i, f_j \rangle} (f_j - f_i). \quad (4.81)$$

On a Moutard net on the sphere the scalar products $\langle f, f_i \rangle$ have the labelling property:

$$\langle f_{ij}, f_j \rangle = \langle f, f_j \rangle + \frac{\langle f, f_i - f_j \rangle}{1 - \langle f_i, f_j \rangle} (1 - \langle f_i, f_j \rangle) = \langle f, f_i \rangle. \quad (4.82)$$

This scalar product encodes the spherical distance between two points and, hence, the distance between neighboring points has the labelling property as well. Also, diagonals of each quad are parallel and, thus, we can interpret each quad as a spherical anti-parallelogram (see Figure 4.14 on the left for an anti-parallelogram on $\mathbb{S}^2$). Now, we will give an algebraic description of such Moutard nets using quaternions.

Lemma 4.7.1 *A net $f : \mathbb{Z}^n \rightarrow \mathbb{S}^3 \subseteq \mathbb{H}$ is a Moutard net if and only if*

$$(f_{ij} - f)(\bar{f}_j - \bar{f}_i) = -2\langle f, f_j - f_i \rangle. \quad (4.83)$$

Proof. If f is a Moutard net we use the Moutard equation (4.78) and (4.80) to compute

$$\begin{aligned} (f_{ij} - f)(\bar{f}_j - \bar{f}_i) &= a^{ij}(f_j - f_i)\overline{(f_j - f_i)} \\ &= a^{ij}\langle f_j - f_i, f_j - f_i \rangle = -2\langle f, f_j - f_i \rangle. \end{aligned} \quad (4.84)$$

Conversely, (4.83) directly implies that the diagonals $f_{ij} - f$ and $f_j - f_i$ are parallel and, hence, f is a Moutard net. $\square$

Now, we describe the Moutard net as a parallelogram net.

Proposition 4.7.2 *A net $f : \mathbb{Z}^n \rightarrow \mathbb{S}^3 \subseteq \mathbb{H}$ is a Moutard net if and only if it is the multiplicative primitive map of an equally folded quaternionic parallelogram net.*

Proof. Consider the edge based net p defined by $p^i = f_i f^{-1} = f_i \bar{f}$. For this net the multiplicative equation (3.2) always holds since

$$p_i^j p^i = f_{ij} \bar{f}_i f_i \bar{f} = f_{ij} \bar{f} = f_{ij} \bar{f}_j f_j \bar{f} = p_j^i p^j. \quad (4.85)$$

For the additive equation (3.1) consider $f_i \bar{f} + f \bar{f}_i = 2 \operatorname{Re}(f_i \bar{f}) = 2\langle f_i, f \rangle$ which yields

$$\begin{aligned} p_j^i + p^j &= f_{ij} \bar{f}_j + f_j \bar{f} = (f_{ij} - f) \bar{f}_j + 2\langle f, f_j \rangle, \\ p_i^j + p^i &= f_{ij} \bar{f}_i + f_i \bar{f} = (f_{ij} - f) \bar{f}_i + 2\langle f, f_i \rangle. \end{aligned} \quad (4.86)$$

Hence, (3.1) is equivalent to (4.83). We conclude that f is a Moutard net if and only if p is a parallelogram net which yields the claim. $\square$

Thus, equally folded parallelogram nets connect objects in Euclidean and spherical geometry: They have an additive primitive map with Euclidean interpretation and a multiplicative primitive map with spherical interpretation.

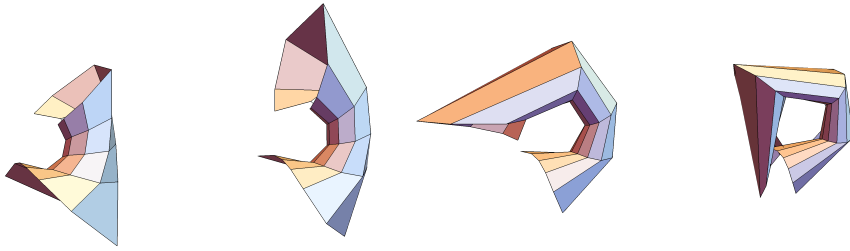


Figure 4.13.: Members in $\mathbb{S}^3$ of the associated family of the cylinder under a stereographic projection.

Example 4.7.3 *The parallelogram net describing the cylinder discussed in Example 4.2.5 is primarily equivalent to an equally folded net q . Thus, the multiplicative primitive map g of q is a Moutard net on $\mathbb{S}^3$. In Figure 4.13 we see this map g and members of its associated family under a stereographic projection. Here, we also applied a flip of signs $g(k, l) \mapsto (-1)^k g(k, l)$ of every second point. As explained in Remark 2.2.3, such a flip is necessary to describe surfaces with a smooth limit.*

Now, we will apply our theory of parallelogram nets to spherical geometry. First, note that two equally folded nets p and $\tilde{p}$ are primarily equivalent if $\tilde{p}^i = ap^i a^{-1}$ for some $a \in \mathbb{S}^3$. We will show that isometries interact with primary equivalence of parallelogram nets in a similar manner to the Euclidean case. An isometry of $\mathbb{S}^3$ is given by a map

$$\mathbb{S}^3 \rightarrow \mathbb{S}^3, \quad f \mapsto afb \quad (4.87)$$

for some $a, b \in \mathbb{S}^3$. If two Moutard nets f and $\tilde{f}$ in $\mathbb{S}^3$ are related by an isometry via $\tilde{f} = afb$ we have

$$\tilde{p}^i = \tilde{f}_i \tilde{f}^{-1} = af_i b b^{-1} f^{-1} a^{-1} = ap^i a^{-1} \quad (4.88)$$

and, hence, the corresponding parallelogram nets are primarily equivalent. Conversely, if equally folded parallelogram nets p and $\tilde{p}$ are primarily equivalent via $\tilde{p}^i = ap^i a^{-1}$ then respective multiplicative primitive maps $f, \tilde{f}$ are related by the isometry $\tilde{f} = afb$ with $b = f^{-1} a^{-1} \tilde{f}$. For this, note that b is constant since

$$b_i = f_i^{-1} a^{-1} \tilde{f}_i = f^{-1} (p^i)^{-1} a^{-1} \tilde{p}^i \tilde{f} = f^{-1} (p^i)^{-1} a^{-1} ap^i a^{-1} \tilde{f} = b. \quad (4.89)$$

Proposition 4.7.4 *A Moutard net $f : \mathbb{Z}^n \rightarrow \mathbb{S}^3$ is quasi-periodic with period $T \in \mathbb{Z}^n$ in the sense that $f(x + T) = af(x)b$ for some $a, b \in \mathbb{S}^3$ if and only if the corresponding parallelogram net p is quasi-periodic with $p^i(x + T) = ap^i(x)a^{-1}$.*

Proof. This follows directly from our considerations above by considering the Moutard net $\tilde{f}$ defined by $\tilde{f}(x) = f(x + T)$ and the parallelogram net $\tilde{p}$ defined by $\tilde{p}^i(x) = p^i(x + T)$. $\square$

Now, remember that the associated family given in Proposition 4.3.4 assigns a family of equally folded nets to a zero-folded net. Via the corresponding primitive maps it relates nets of Euclidean anti-parallelograms to nets of spherical anti-parallelograms. We can show that the Euclidean case arises in the limit $t \rightarrow \infty$ (meaning $\frac{\mu(t)}{\lambda(t)} = ce^{-2t} \rightarrow 0$) obtained in the associated family p_t if we rescale the sphere $\mathbb{S}^3 \mapsto -\frac{e^{2t}}{2c}\mathbb{S}^3$. Indeed, (3.27) (with $\Phi(\mathbf{0}) = \tilde{\Phi}(\mathbf{0}) = f_t(\mathbf{0}) = \mathbf{1}$) under this rescaling becomes $f_t = -\frac{e^{2t}}{2c}\Phi^{-1}\tilde{\Phi}$ and with $P^i = e^t + ce^{-t}p^i$ and $\tilde{P}^i = e^t - ce^{-t}p^i$ we get

$$\begin{aligned} (f_t)_i - f_t &= -\frac{e^{2t}}{2c}(\Phi_i^{-1}\tilde{\Phi}_i - \Phi^{-1}\tilde{\Phi}) = -\frac{e^{2t}}{2c}\Phi_i^{-1}(\tilde{P}^i - P^i)\tilde{\Phi} \\ &= e^t \Phi_i^{-1} p^i \tilde{\Phi} \xrightarrow[t \rightarrow \infty]{} p^i \end{aligned} \quad (4.90)$$

since $e^{-t}P^i \rightarrow \mathbf{1}$ and $e^{-t}\tilde{P}^i \rightarrow \mathbf{1}$. This allows for an interpretation of the associated family as deformation of Euclidean anti-parallelograms into spherical anti-parallelograms. We will expand on this connection and its relevance in Theorem 5.2.1 and an application can be found in Chapter 9.

Let us turn to the special case of the normal map (4.53) to a K-net. It arises in the case where all points are contained in the two-dimensional sphere $\mathbb{S}^2$.

Proposition 4.7.5 *A net $g : \mathbb{Z}^n \rightarrow \mathbb{S}^3 \subseteq \text{Im } \mathbb{H}$ is a Moutard net in $\mathbb{S}^2$ (i.e., $\text{Re } g = 0$) if and only if it is the multiplicative primitive map of an equally folded parallelogram net corresponding to a K-net.*

Proof. If g is a Moutard net in $\mathbb{S}^2$ then the map p given by $p^i = g_i \bar{g} = -g_i g$ is a parallelogram net by Proposition 4.7.2. The edges of its projected additive primitive map f are given by

$$f_i - f = \text{Im } p^i = -\text{Im } g_i g = -g_i \times g. \quad (4.91)$$

Therefore, at every vertex all incident edges of f are orthogonal to g and, hence, f has planar vertex stars and is a K-net. Conversely, the multiplicative primitive map of an equally folded parallelogram net belonging a K-net is its normal map in $\mathbb{S}^2$ and it is a Moutard net by Proposition 4.7.2. $\square$

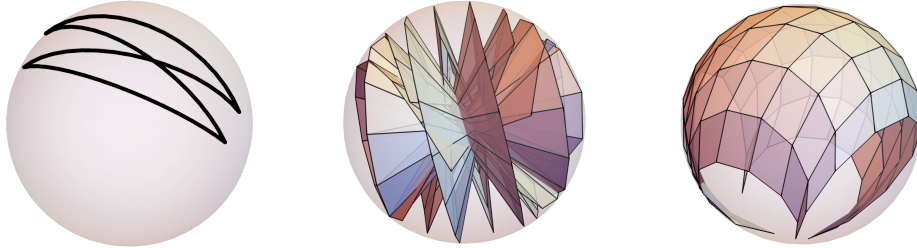


Figure 4.14.: Left: A spherical anti-parallelogram on $\mathbb{S}^2$. Middle: The normals of an Amsler surface. Right: Flipped normals of an Amsler surface.

Note that (4.91) is the well-known Lelievre representation of K-nets. In Figure 4.14 in the middle and on the right we see the normal map of an Amsler surface on $\mathbb{S}^2$. As before, a flip of the form $g(k, l) \mapsto (-1)^k g(k, l)$ produces consistently oriented normals that admit a limit.

We conclude this section by reviewing an observation from [98] which shows how an equally folded parallelogram net p (which does not belong to a K-net) also gives rise to a net $\tilde{f} : \mathbb{Z}^n \rightarrow \mathbb{R}^3$ with planar vertex stars: For this, consider a the multiplicative primitive map g of p which is a Moutard net in $\mathbb{S}^3$. Note that the conjugate net $\hat{g} := \bar{g}$ is

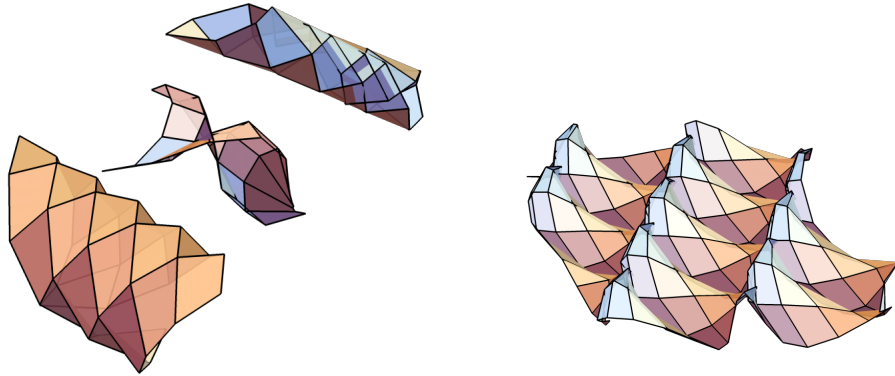


Figure 4.15.: Left: Two cylinders and their mid-surface. Right: A larger piece of the same mid-surface.

also a Moutard net in $\mathbb{S}^3$ giving rise to a second parallelogram net $\hat{p}$ with $\hat{p}^i = \hat{g}_i \bar{\hat{g}} = \bar{g}_i g$. The projected additive primitive maps $f, \hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^3$ of p and $\hat{p}$ respectively are given by

$$\begin{aligned} f_i - f &= \text{Im } p^i = \text{Im}(g_i \bar{g}) = +(\text{Re } g) \bar{g}_i - (\text{Re } g_i) \bar{g} - \bar{g}_i \times \bar{g}, \\ \hat{f}_i - \hat{f} &= \text{Im } \hat{p}^i = \text{Im}(\bar{g}_i g) = -(\text{Re } g) \bar{g}_i + (\text{Re } g_i) \bar{g} - \bar{g}_i \times \bar{g}. \end{aligned} \quad (4.92)$$

Hence, their mid-surface

$$\tilde{f} := \frac{f + \hat{f}}{2} \quad (4.93)$$

is given by

$$\tilde{f}_i - \tilde{f} = -\bar{g}_i \times \bar{g}. \quad (4.94)$$

At every vertex the vector $\bar{g}$ is perpendicular to every edge of $\tilde{f}$ and, therefore, $\tilde{f}$ has planar vertex stars with a normal map given by $\bar{g}$. An example can be seen in Figure 4.15.

5. Skew parallelogram nets in other algebras

5.1. The matrix algebra $\mathbb{C}^{2 \times 2}$

Here, we investigate parallelogram nets in the matrix algebra $\mathbb{C}^{2 \times 2}$ which, in this generality, have not been considered before. This algebra is a natural extension of our previous considerations since it arises as complexification of the quaternions: Quaternions with complex coefficients, also called *biquaternions*, are isomorphic to $\mathbb{C}^{2 \times 2}$ and an isomorphism is given by (4.12). We will extend some quaternionic results to this more general algebra and apply this to two geometric examples in Section 5.2 and 5.3. Generally speaking, most results of the quaternionic setup can be stated almost analogously as long as they do not rely on invertibility of non-zero elements. For example, the invariants of the evolution (3.3) are given by the characteristic polynomial of p^i and are, hence, given in terms of its eigenvalues or in terms of the trace and determinant

$$\mathrm{tr} p^i = \mathrm{tr} p_j^i, \quad \det p^i = \det p_j^i. \quad (5.1)$$

Note that trace and determinant generalize the notion of (two times) the real part and (squared) norm of quaternions. We will still denote the trace-free part of a matrix by $\bar{A} := A - \frac{1}{2} \mathrm{tr} A \mathbf{1}$. Similarly, the adjugate matrix

$$\mathrm{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (5.2)$$

generalizes the quaternionic conjugate $q \mapsto \bar{q}$. As extension of (4.4), (4.5) it fulfills

$$A + \mathrm{adj} A = \mathrm{tr} A \mathbf{1}, \quad A \mathrm{adj} A = \mathrm{adj} A A = \det A \mathbf{1}. \quad (5.3)$$

Provided invertibility of diagonals we have the forwards and backwards evolution (3.3), (3.4). Similarly, the sideways evolution for quaternions (4.16) also extends to $\mathbb{C}^{2 \times 2}$ by replacing the quaternionic conjugate with the adjugate matrix. Recall that for quaternions non-invertibility of a diagonal $p^i - p^j$ (or $p_j^i - \mathrm{adj} p^j$) always implied that the quads are of a special degenerate form, i.e. fulfill $p^i = p^j$ (or $p_j^i = \mathrm{adj} p^j$). In $\mathbb{C}^{2 \times 2}$ this is not the case and more complicated non-evolvable cases can appear.

Example 5.1.1 For a trivial example of a non-evolvable parallelogram net consider an arbitrary map $f : \mathbb{Z}^n \rightarrow \mathbb{C}$. It can be described by the non-evolvable parallelogram net p given by

$$p^i = \begin{pmatrix} 0 & f_i - f \\ 0 & 0 \end{pmatrix} \quad (5.4)$$

One can calculate that the associated family acts as identity in the sense that every p_t is primarily equivalent to p .

The following notions from the quaternionic theory work analogously, although the geometric interpretation becomes less clear:

Definition 5.1.2 We call a parallelogram net $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{C}^{2 \times 2}$ equally folded if $\det p^i = 1$ for all edges. We call it zero-folded if $\text{tr } p^i = 0$ for all edges.

Differently phrased, equally folded nets are parallelogram nets in the matrix group

$$SL(2, \mathbb{C}) = \{A \in \mathbb{C}^{2 \times 2} \mid \det A = 1\}. \quad (5.5)$$

For an evolvable zero-folded quad the evolution (3.3) implies (using $\text{adj } A = -A$ if $\text{tr } A = 0$) that

$$p^i + p_i^j = (p^i(p^j - p^i) + (p^j - p^i)p^j)(p^j - p^i)^{-1} = \frac{\det p^i - \det p^j}{\det(p^j - p^i)}(p^i - p^j). \quad (5.6)$$

This implies that diagonals of the quad are multiples of each other and we can interpret quads of the additive primitive map as complexified anti-parallelograms. An almost literal copy of Proposition 4.3.4 also holds in $\mathbb{C}^{2 \times 2}$:

Proposition 5.1.3 Using the functions $\lambda(t) = e^t, \mu(t) = ce^{-t}$ for some $c \in \mathbb{C} \setminus \{0\}$ a net in the associated family of a zero-folded net is equally folded. Every equally folded net p with $\det \bar{p}^i \neq 0$ can be represented this way.

Note that the condition $\det \bar{p}^i \neq 0$ is stronger than $\bar{p}^i \neq 0$.

Proof. The proof is also almost a literal copy of the proof of Proposition 4.3.4 if one replaces each real part with half the trace and each squared norm with the determinant. For this, note that (4.19) and (4.20) also hold in $\mathbb{C}^{2 \times 2}$. Again, the condition $\det \bar{p}^i \neq 0$ excludes some cases where the associated family is not defined. $\square$

Now, consider a multiplicative primitive map $f : \mathbb{Z}^n \rightarrow SL(2, \mathbb{C})$ of an equally folded parallelogram net p and observe that $\text{tr}(f_i \text{adj } f) = \text{tr } p^i$ has the labelling property. This

value can be seen as an inner product of f and f_i : The Euclidean scalar product of quaternions extends to the complex-valued symmetric bilinear form

$$\langle A, B \rangle_{\mathbb{C}} := \frac{1}{2} \operatorname{tr}(A \operatorname{adj} B). \quad (5.7)$$

The geometry of f is described by the following generalization of Proposition 4.7.2:

Proposition 5.1.4 *A net $f : \mathbb{Z}^n \rightarrow SL(2, \mathbb{C})$ with invertible diagonals $f_j - f_i$ is the multiplicative primitive map of an equally folded parallelogram net $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{C}^{2 \times 2}$ if and only if it fulfills a Moutard equation*

$$f_{ij} - f = a^{ij}(f_j - f_i) \quad (5.8)$$

for some map $a^{ij} : \{\text{faces of } \mathbb{Z}^n\} \rightarrow \mathbb{C}$.

Proof. All considerations leading up to Proposition 4.7.2 and its proof itself translate directly by replacing the scalar product with $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ and the quaternionic conjugate with the adjugate matrix. In particular, the algebraic characterization (4.83) of Moutard nets now reads

$$(f_{ij} - f) \operatorname{adj}(f_j - f_i) = -2 \langle f, f_j - f_i \rangle_{\mathbb{C}}. \quad (5.9)$$

Note that, unlike in the quaternionic case, the requirement for $f_j - f_i$ to be invertible is stronger than the requirement for f_j and f_i to be distinct. $\square$

5.2. Moutard nets and hyperbolic geometry

Moutard nets on the sphere $\mathbb{S}^3$ and their spherical geometry as in Section 4.7 can be considered as special case of Proposition 5.1.4. Similarly, we will now consider Moutard nets on the hyperboloid and their hyperbolic geometry. While Moutard nets in quadrics are well-known (see, also, Section 5.5), we introduce a deformation of nets of anti-parallelogram from Euclidean space into spherical and hyperbolic space (Theorem 5.2.1) which seems to be new.

For this, consider the real vector space of Hermitian matrices

$$\operatorname{span}_{\mathbb{R}}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} = \{A \in \mathbb{C}^{2 \times 2} \mid A = \bar{A}^T\} \subseteq \mathbb{C}^{2 \times 2} \quad (5.10)$$

where σ_i are the Pauli matrices (4.14). The determinant

$$\det(a\sigma_0 + b\sigma_1 + c\sigma_2 + d\sigma_3) = \det \begin{pmatrix} a+d & b-ci \\ b+ci & a-d \end{pmatrix} = a^2 - b^2 - c^2 - d^2 \quad (5.11)$$

defines a real quadratic form and the corresponding real symmetric bilinear form given by (5.7) has signature $(1, 3, 0)$. A two-sheeted hyperboloid is then given by

$$\mathbb{H}^3 := \{A \in \mathbb{C}^{2 \times 2} \mid A = \bar{A}^T, \det A = 1\} \quad (5.12)$$

The forward sheet is given by the points with $\operatorname{tr} A > 0$ and the backward sheet by $\operatorname{tr} A < 0$. To each pair of points $A, B \in \mathbb{H}^3$ we assign the distance

$$d(A, B) = \operatorname{arcosh} \langle A, B \rangle_{\mathbb{C}}. \quad (5.13)$$

This is the hyperboloid model of three-dimensional hyperbolic geometry. The map

$$\mathbb{H}^3 \rightarrow \operatorname{Im} \mathbb{H} \cong \mathbb{R}^3, \quad A \mapsto \frac{i}{2 + \operatorname{tr} A} (\operatorname{tr} A - 2A) \quad (5.14)$$

is a stereographic projection mapping the forward sheet of the hyperboloid to the Poincaré ball model of hyperbolic space.

We are interested in Moutard nets on the hyperboloid. By Proposition 5.1.4 they arise as multiplicative primitive maps of equally folded parallelogram nets. The labelling property of $\frac{1}{2} \operatorname{tr} p^i = \frac{1}{2} \operatorname{tr}(f_i \operatorname{adj} f) = \langle f_i, f \rangle_{\mathbb{C}}$ implies that opposite edges have the same hyperbolic length. We will refer to them as hyperbolic anti-parallelograms and an example of such a quad is visible in Figure 5.1. In Chapter 9 we will dive deeper into their hyperbolic geometry.

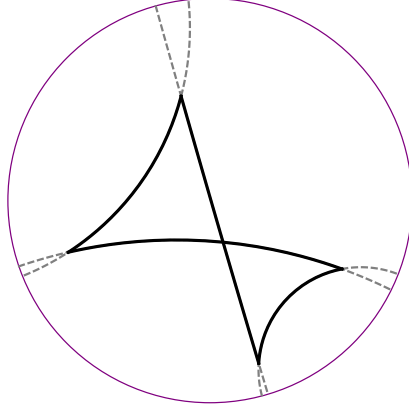


Figure 5.1.: A quad of a Moutard net on $\mathbb{H}^2$ in the Poincaré disc model.

Recall that nets in the associated family of a parallelogram net p depend on the ratio $\frac{\lambda}{\mu}$. If we consider the quaternions as special complex matrices we can also consider members of the associated family of p for complex parameter $\frac{\lambda}{\mu} \notin \mathbb{R}$. These are parallelogram nets but not quaternionic. Remarkably, in this manner, the associated family can map

to hyperbolic space. As motivation, observe that the matrix $a + ibq$ for an imaginary quaternion q and $a, b \in \mathbb{R}$ is Hermitian. Hence, the Lax representation $P^i = e^t + ie^{-t}p^i$ of a zero-folded quaternionic parallelogram net is Hermitian.

Now, let p_t be the associated family of a parallelogram net p given by functions $\lambda(t) = e^t$, $\mu(t) = ce^{-t}$. Remember (3.27), stating that the multiplicative primitive map f_t with $f_t(\mathbf{0}) = \mathbf{1}$ of p_t is given by

$$f_t = \Phi^{-1}\tilde{\Phi} \quad (5.15)$$

where $\tilde{\Phi}$ denotes the frame of p defined by replacing $P^i = e^t + ce^{-t}p^i$ with $\tilde{P}^i := (P^i)' = e^t - ce^{-t}p^i$. For imaginary $c \in i\mathbb{R}$ the map f_t turns out to live in hyperbolic space $\mathbb{H}^3$:

Theorem 5.2.1 *Consider a zero-folded quaternionic parallelogram net p and its associated family p_t given by functions $\lambda(t) = e^t$, $\mu(t) = ce^{-t}$ for some $c \neq 0$ and $\Phi(\mathbf{0}) = \mathbf{1}$. For every $t \in \mathbb{R}$ the multiplicative primitive map f_t of p_t given by (5.15) is*

1. a Moutard net on the sphere $\mathbb{S}^3$ for $c \in \mathbb{R}$ and
2. a Moutard net on the hyperboloid $\mathbb{H}^3$ for $c \in i\mathbb{R}$.

All Moutard nets f on $\mathbb{S}^3$ and $\mathbb{H}^3$ with $f(\mathbf{0}) = \mathbf{1}$ and non-vanishing edges and diagonals can be described in this manner.

Proof. There is nothing to show for $\mathbb{S}^3$ since every operation stays within the quaternions and the result already follows from Proposition 4.3.4 and our considerations in Section 4.7. For $\mathbb{H}^3$ we need to show that $f_t(x)$ is Hermitian for $x \in \mathbb{Z}^n$. For this, consider a path of neighbouring vertices $x_0 = \mathbf{0}, x_1, \dots, x_k = x \in \mathbb{Z}^n$ and the corresponding path of edges $d_1 = (x_0, x_1), \dots, d_k = (x_{k-1}, x)$ connecting the origin to x . We assume w.l.o.g. that all edges are directed forwards, i.e., that $d_i \in \tilde{\mathcal{E}}(\mathbb{Z}^n)$ (for edges directed backwards the argument still holds if we invert corresponding quantities). Then, (5.15) reads

$$\begin{aligned} f_t(x) &= P(d_1)^{-1} \dots P(d_k)^{-1} \tilde{P}(d_k) \dots \tilde{P}(d_1) \\ &= \frac{1}{\det \Phi(x)} \tilde{P}(d_1) \dots \tilde{P}(d_k) \tilde{P}(d_k) \dots \tilde{P}(d_1) \end{aligned} \quad (5.16)$$

where we used $\text{adj } P(d_i) = e^t - ce^{-t}p(d_i) = \tilde{P}(d_i)$. Now, since $\tilde{P}(d_i)$ is Hermitian for $c \in i\mathbb{R}$ we can conclude $\bar{f}_t^T(x) = f_t(x)$ and, hence, $f_t(x) \in \mathbb{H}^3$. Proposition 5.1.3 and 5.1.4 imply that f_t is a Moutard net.

Also, these propositions imply that any such Moutard net can be obtained in such a manner (with some $c \in i\mathbb{R}$) as multiplicative primitive map f_t in the associated family p_t of a zero-folded parallelogram net p . We have to show that p is quaternionic. We will make an iterative argument. Consider a vertex x and a path $d_1, \dots, d_k$ from $\mathbf{0}$ to x as

above and assume that $p(d_1), \dots, p(d_{k-1}) \in \mathbb{H}$ are quaternions. Then, (5.16) holds and $f_t(x)$ and $P(d_1), \dots, P(d_{k-1})$ are Hermitian and, hence,

$$\frac{\tilde{P}(d_k)\tilde{P}(d_k)}{\det P(d_k)} = \frac{(e^{2t} - c^2 e^{-2t} \det p(d_k))\mathbf{1} - 2cp(d_k)}{e^{2t} + c^2 e^{-2t} \det p(d_k)} \quad (5.17)$$

is also Hermitian. Now, the trace being real implies that $\det p(d_k)$ is real as well and, therefore, the trace-free matrix $cp(d_k)$ is Hermitian. Then, $p(d_k)$ is a quaternion. Iterating this argument proves that all values of p are quaternions. Note that, again, for edges directed backwards the argument works analogous if we invert corresponding quantities. $\square$

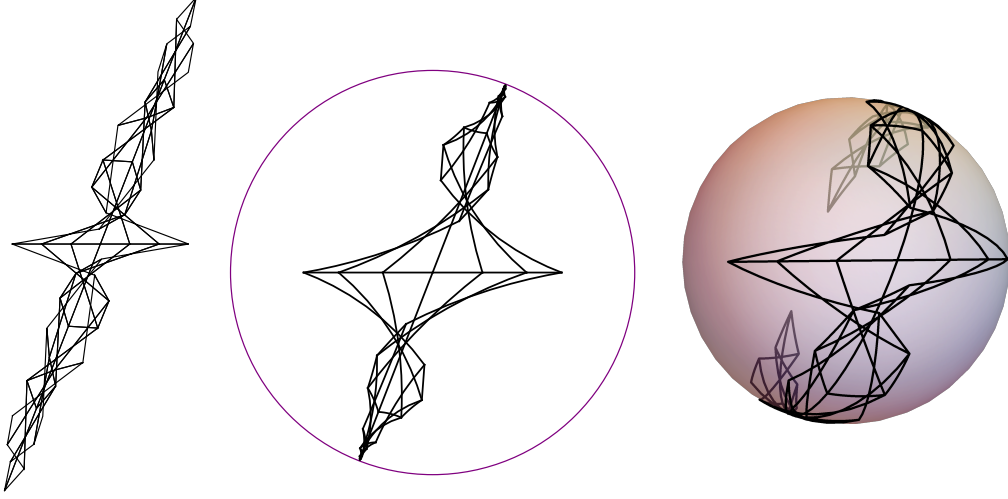


Figure 5.2.: The anti-parallellograms belonging to an Amsler surfaces (compare Figure 4.6) deformed into hyperbolic and spherical geometry.

Thus, we can deform nets of Euclidean anti-parallellograms into spherical and hyperbolic space. Remember Proposition 4.2.3 which states that the local geometry at each vertex stays rigid in the Euclidean associated family (given by the additive primitive map). We will show a similar statement for the associated family in Theorem 5.2.1.

For this, we need to discuss some space form geometry. Three points lie on a common geodesic in $\mathbb{S}^3$ or $\mathbb{H}^3$ if they are linearly dependent. We define the *geodesic edges* of a Moutard net f in $\mathbb{S}^3$ or $\mathbb{H}^3$ to be the geodesics connecting adjacent points f and f_i . A

parametrization of the geodesic edge connecting f to f_i is given by

$$\alpha \mapsto (\cos \alpha + \sin \alpha \frac{\vec{p}^i}{\sqrt{\det p^i}})f \quad (5.18)$$

in the spherical case and by

$$\alpha \mapsto (\cosh \alpha + \sinh \alpha \frac{\vec{p}^i}{\sqrt{-\det p^i}})f \quad (5.19)$$

in the hyperbolic case. Here, the quantity $\vec{p}^i f$ is an element of the tangent space of the space form at f and can be interpreted as direction of the geodesic edge. Our inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ induces the corresponding Riemannian metric on $\mathbb{S}^3$ and $\mathbb{H}^3$ and the angle between adjacent geodesic edges connecting f to f_i and f_j respectively is given by

$$\cos \alpha^{ij} = \frac{\langle \vec{p}^i f, \vec{p}^j f \rangle_{\mathbb{C}}}{\sqrt{\langle \vec{p}^i f, \vec{p}^i f \rangle_{\mathbb{C}} \langle \vec{p}^j f, \vec{p}^j f \rangle_{\mathbb{C}}}}. \quad (5.20)$$

Clearly, an analogous formula holds for 'backwards' geodesic edges connecting f to $f_{\bar{i}}$.

Proposition 5.2.2 *The associated family in Theorem 5.2.1 preserves the angle between adjacent geodesic edges.*

Proof. Since $\det f = 1$ we have $\langle \vec{p}^i f, \vec{p}^j f \rangle_{\mathbb{C}} = \frac{1}{2} \text{tr}(\vec{p}^i f \text{adj } f \vec{p}^j) = \langle \vec{p}^i, \vec{p}^j \rangle_{\mathbb{C}}$. Now, as discussed in the proof of Proposition 4.2.3, throughout the associated family $\vec{p}^i$ and $\vec{p}^j$ get scaled and are subject to a common sandwich product with the frame Φ . The sandwich product does not affect our inner product and the scaling gets eliminated in (5.20) as well. Thus, the angle remains constant. $\square$

This thesis contains two applications of this associated family: In Section 7.5 we will apply this to nets of constant mean curvature (CMC) and in Chapter 9 we will apply this to discrete curves and relate it to a corresponding fundamental theorem.

We also remark that this can be applied to other constructions built from anti-parallelograms like initial data for K-nets (see Figure 5.2) or the curve dynamics of the bicycle transformation and polygon recutting. In particular, we have seen that the associated family preserves quasi-periodicity (see Proposition 3.5.2 and 4.7.4 for $\mathbb{S}^3$, an analogous statement can be made for $\mathbb{H}^3$) and, thus, the associated family immediately yields space form versions of both curve dynamics.

Finally, note that in the spherical case all operations happen within the algebraically closed set of quaternions. This makes the hyperbolic case more complicated since Hermitian matrices are not closed under multiplication. In Chapter 9 we will model two-dimensional hyperbolic space $\mathbb{H}^2$ by Hermitian matrices with no σ_3 component. These belong to the algebra of split-quaternions which is closed under multiplication making the corresponding parallelogram net split-quaternionic.

5.3. Cross-ratio systems

We will give an overview over cross-ratio system and the corresponding Hirota systems which are well known [22, 82, 14]. The cross-ratio of four points $a, b, c, d \in \mathbb{C}$ is defined as

$$cr(a, b, c, d) = \frac{a - b}{b - c} \frac{c - d}{d - a}. \quad (5.21)$$

We consider vertex based maps $z : \mathbb{Z}^n \rightarrow \mathbb{C}$ in the complex plane and their edge vectors $d^i := z_i - z$. For simplicity, we will assume the four points of each quad to be pairwise distinct.

Definition 5.3.1 *A map $z : \mathbb{Z}^n \rightarrow \mathbb{C}$ is called an (n -dimensional) cross-ratio system if there exists an edge labelling $\alpha : \mathcal{E}(\mathbb{Z}^n) \rightarrow \mathbb{C}$, i.e., $\alpha_j^i = \alpha^i$ for $i \neq j$ such that*

$$cr(z, z_i, z_{ij}, z_j) = \frac{d^i d_j^i}{d^j d_i^j} = \frac{(\alpha^i)^2}{(\alpha^j)^2}. \quad (5.22)$$

A two-dimensional cross-ratio system with $cr(z, z_1, z_{12}, z_2) = \frac{(\alpha^1)^2}{(\alpha^2)^2} = -1$ is called a discrete holomorphic map.

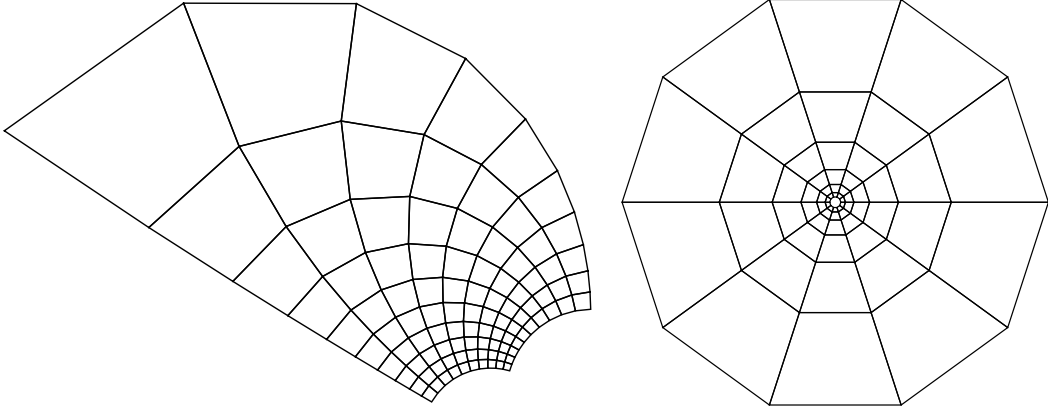
The edge labelling (5.22) is sometimes called factorizing. A cross-ratio system is a map with factorizing cross-ratios.

Example 5.3.2 *A grid of parallelograms is given by $z(l, m) := hl + km$ for constant $h, k \in \mathbb{C}$. It is a cross-ratio system with constant cross-ratio $\frac{h^2}{k^2}$. With $h = 1, k = i$ we obtain the discrete holomorphic square grid. Möbius transformations $x \mapsto \frac{ax+b}{cx+d}$ preserve the cross-ratio and, thus, cross-ratio systems. A Möbius transformation of the square grid can be seen in Figure 5.3 on the left.*

Example 5.3.3 *Similarly, for constant $h, k \in \mathbb{C}$ the discrete exponential map $z(l, m) = e^{hl+km}$ is a cross-ratio system with constant cross-ratio $\frac{1-\cosh h}{1-\cosh k}$. A periodic example with $h = \operatorname{arcosh}(2 - \frac{\phi}{2})$ (where ϕ is the golden ratio) and $k = \frac{2\pi}{10}i$ can be seen in Figure 5.3 on the right.*

In a cross-ratio system, we have

$$\frac{(\alpha^j)^2}{d^j} + \frac{(\alpha^i)^2}{d_j^i} = \frac{(\alpha^j)^2}{d^j d_i^j} (d_i^j + d^i) = \frac{(\alpha^j)^2}{d^j d_i^j} (d_i^j + d^j) = \frac{(\alpha^i)^2}{d^i} + \frac{(\alpha^j)^2}{d_i^j}. \quad (5.23)$$


 Figure 5.3.: Two-dimensional cross-ratio systems with cross-ratio -1 .

Hence, a map $z^* : \mathbb{Z}^n \rightarrow \mathbb{C}$ is defined up to translation by

$$(d^*)^i = z_i^* - z^* = \frac{(\alpha^i)^2}{d^i}. \quad (5.24)$$

This map z^* is called the *dual* of z and it is a cross-ratio system with the same labelling α . We will often express cross-ratio systems in terms of a special vertex based map:

Lemma and Definition 5.3.4 *For each cross-ratio system z there exists a unique map $s : \mathbb{Z}^n \rightarrow \mathbb{C}$ with $s(\mathbf{0}) = 1$ and*

$$d^i = \alpha^i s_i s. \quad (5.25)$$

and we call this map a Hirota system.

Proof. The Hirota system is well-defined if the transport $s_i = \frac{d^i}{\alpha^i s}$ is path-independent. For each quad this condition reads $\frac{d_i^j}{\alpha^j} \frac{\alpha^i}{d^i} s = \frac{d_j^i}{\alpha^i} \frac{\alpha^j}{d^j} s$ which is exactly (5.22). Hence, s is well-defined for cross-ratio systems. $\square$

Note that the dual z^* is given by $(d^i)^* = \frac{\alpha^i}{s s_i}$. Also the closing condition $d^i + d_i^j = d^j + d_j^i$ written in terms of s reads

$$\frac{s_{ij}}{s} = \frac{\alpha^j s_j - \alpha^i s_i}{\alpha^j s_i - \alpha^i s_j}. \quad (5.26)$$

Some reductions of cross-ratio systems have an elegant formulation in terms of the Hirota system:

Example 5.3.5 Consider a two-dimensional cross-ratio system z with cross-ratio -1 and labelling $\alpha^1 = 1, \alpha^2 = i$. Also, consider a partition of $\mathbb{Z}^2$ into black and white vertices such that each edge connects a black and a white vertex. Now, assume that

$$s(l, m) \in \mathbb{R} \quad \text{at black vertices}, \quad s(l, m) \in \mathbb{S}^1 \quad \text{at white vertices.} \quad (5.27)$$

Then, at every black vertex x there is a circle with center $z(x)$ and radius $|s(x)|$ contain-

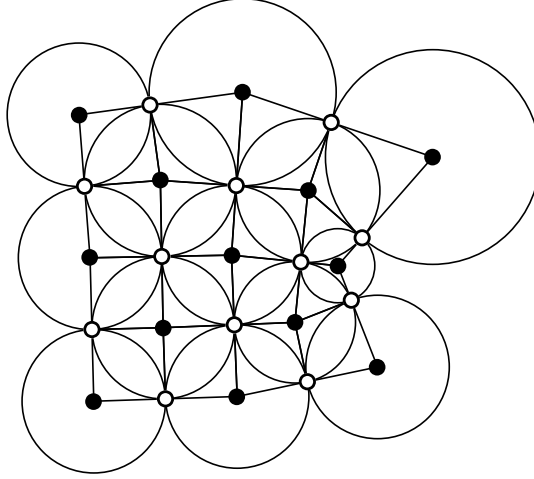


Figure 5.4.: An orthogonal circle pattern.

ing every neighbouring white points $z(y)$: The length of incident edges is $|z(y) - z(x)| = |d(x, y)| = |\alpha(x, y)s(x)s(y)| = |s(x)|$. Similarly, one can see that at every white point opposite pairs of circles touch and that the two pairs of touching circles intersect orthogonally. This is called an orthogonal circle pattern (see [100, 13]) and an example can be seen in Figure 5.4.

Now, we will write cross-ratio systems as a reduction of parallelogram nets:

Proposition 5.3.6 An edge based map

$$p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{C}^{2 \times 2}, \quad p^i = \begin{pmatrix} 0 & d^i \\ -\frac{(\alpha^i)^2}{d^i} & 0 \end{pmatrix} \quad (5.28)$$

is an evolvable parallelogram net if and only if d is the edge map of a cross-ratio system with labelling α .

Here, we only consider maps α with $\alpha^i \neq 0$ and $\alpha^i \neq \alpha^j$ for $i \neq j$ which for a cross-ratio system already follows from the fact that no two points in a quad coincide.

Proof. The multiplicative parallelogram equation (3.2) reads

$$(\alpha^i)^2 \frac{d_i^j}{d^i} = (\alpha^j)^2 \frac{d_j^i}{d^j} \quad \text{and} \quad (\alpha_i^j)^2 \frac{d^i}{d_j^j} = (\alpha_j^i)^2 \frac{d^j}{d_i^i}. \quad (5.29)$$

If α has the labelling property both agree with (5.22). Hence, for a cross-ratio system the parallelogram equations are fulfilled and we can calculate evolvability using $z_i \neq z_j$. Conversely, in a parallelogram net the labelling property of α follows from the fact that $\alpha^i := \sqrt{\det p^i}$ is an invariant of the parallelogram evolution. The additive parallelogram equation (3.1) implies that d can be integrated into a vertex based map and the multiplicative equation implies (5.22). $\square$

We will often write the parallelogram net (5.28) in terms of the Hirota system s :

$$p^i = \begin{pmatrix} 0 & \alpha^i s s_i \\ -\frac{(\alpha^i)}{s s_i} & 0 \end{pmatrix} \quad (5.30)$$

Let us apply some of our theory to cross-ratio systems. First, note that the map

$$f = \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} \quad (5.31)$$

is an additive primitive map of the parallelogram net p . Also, our Lax representation (3.19) (with $\lambda = 1$) is gauge equivalent to the original linear Lax representation for cross-ratio systems established in [81]. For the associated family, note that p is zero-folded and, thus, Propositions 5.1.3 and 5.1.4 apply: Its associated family consists of equally folded nets and their multiplicative primitive maps are Moutard nets in $SL(2, \mathbb{C})$. Now, we make a remarkable observation: The Lax representation $P^i = \lambda + \frac{1}{\lambda} p^i$ of (5.28) is nothing but a complexified (i.e., biquaternionic) version of the K-net Lax representation (4.47): A parallelogram net $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{C}^{2 \times 2}$ corresponds to a cross-ratio system if and only if it lives in the $\mathbf{i}, \mathbf{j}$ -plane of the biquaternions. In this sense, one can understand its quads as complexified anti-parallelograms. Then, the additive primitive map of a member of the associated family p_t can be interpreted as a complexified K-net. Many observations for K-nets extend naturally to cross-ratio systems in this manner. For example, a natural normal (4.53) still arises from (3.27) as multiplicative primitive map of p_t . Each such normal g_t is orthogonal to the vertex star in the sense that $\langle g_t, p^i \rangle_{\mathbb{C}} = 0$ holds for all incident edges. This interpretation of an associated family of cross-ratio systems seems to be new.

As converse to our observation, we can also interpret K-nets as associated family of special cross-ratio systems: If the net (5.28) is quaternionic it corresponds to a K-net. As visible in (5.30), this reduction is very elegant in terms of the Hirota map: p is quaternionic if and only if

$$\alpha^i \in \mathbb{R} \quad \text{and} \quad s \in \mathbb{S}^1. \quad (5.32)$$

This relation will be crucial for our study of the connection of cross-ratio systems to further discrete surfaces in Chapter 7.

5.4. Isothermic nets

Isothermic nets are a well-known generalization of cross-ratio systems (with real cross-ratio) into higher dimensional space [15, 51, 95]. We will review their Lax representation which belongs to parallelogram nets in $\mathbb{H}^{2 \times 2}$. In particular, this Lax representation is known to play a role in the construction of a Calapso transformation for isothermic nets [49, 50, 33, 31].

By defining the notion of a cross-ratio for quaternions we can consider nets of factorizing cross-ratio in $\mathbb{R}^3 \cong \text{Im } \mathbb{H}$ or $\mathbb{R}^4 \cong \mathbb{H}$. Many concepts of Section 5.3 extend to this case. The cross-ratio of four quaternions $a, b, c, d \in \mathbb{H}$ is defined as

$$cr(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1} \in \mathbb{H}. \quad (5.33)$$

It is real if and only if the four point lie on a circle (see, e.g., [46]). In this case, $cr(a, b, c, d)$ agrees with the cross-ratio obtained by identifying the plane containing the points with the complex plane $\mathbb{C}$ and can also be computed with the reordered formula

$$(d - a)^{-1}(c - d)(b - c)^{-1}(a - b) = cr(a, b, c, d) \in \mathbb{R}. \quad (5.34)$$

We will consider vertex based maps $f : \mathbb{Z}^n \rightarrow \mathbb{H}$ and, again, we denote their edge vectors by $d^i := f_i - f$. We will assume the four points of each quad to be pairwise distinct.

Definition 5.4.1 *A net $f : \mathbb{Z}^n \rightarrow \mathbb{H}$ is called an isothermic net if there exists an edge labelling $\alpha : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{R} \cup i\mathbb{R}$, i.e., $\alpha_j^i = \alpha^i$ for $i \neq j$ such that*

$$cr(f, f_i, f_{ij}, f_j) = d^i(d_i^j)^{-1}d_j^i(d^j)^{-1} = \frac{(\alpha^i)^2}{(\alpha^j)^2} \in \mathbb{R}. \quad (5.35)$$

By requiring α^i to be real or imaginary we require the cross-ratio to be real. If all points lie in a common plane we have a cross-ratio system as in Section 5.3 with real cross-ratio.

The isothermic (non-commutative) version of (5.23) is given by

$$\begin{aligned} (\alpha^j)^2(d^j)^{-1} + (\alpha^i)^2(d_j^i)^{-1} &= (\alpha^j)^2(d^j)^{-1}(d_i^j + d^i)(d_i^j)^{-1} \\ &= (\alpha^j)^2(d^j)^{-1}(d_j^i + d^j)(d_i^j)^{-1} = (\alpha^i)^2(d^i)^{-1} + (\alpha^j)^2(d_i^j)^{-1}. \end{aligned} \quad (5.36)$$

In the last step, we use the reordered version $(d^j)^{-1}d_i^j(d_i^j)^{-1}d^i = \frac{(\alpha^i)^2}{(\alpha^j)^2}$ of (5.35). Hence, a map $f^* : \mathbb{Z}^n \rightarrow \mathbb{H}$ is defined up to translation by

$$(d^*)^i = f_i^* - f^* = \frac{(\alpha^i)^2}{d^i}. \quad (5.37)$$

This map f^* is called the *dual* of f and it is an isothermic net with the same labelling α . Analogous to cross-ratio systems, we can write isothermic nets as a reduction of parallelogram nets:

Proposition 5.4.2 *An edge based map*

$$p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{H}^{2 \times 2}, \quad p^i = \begin{pmatrix} 0 & d^i \\ -\frac{(\alpha^i)^2}{d^i} & 0 \end{pmatrix} \quad (5.38)$$

for $\alpha : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{R} \cup i\mathbb{R}$ and $d : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{H}$ is an evolvable parallelogram net if and only if d is the edge map of an isothermic net with labelling α .

Again, we only consider maps α with $\alpha^i \neq 0$ and $\alpha^i \neq \alpha^j$ for $i \neq j$. Also, note that the inverse of a matrix $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \mathbb{H}^{2 \times 2}$ exists if $b, c \neq 0$ and is given by $A = \begin{pmatrix} 0 & c^{-1} \\ b^{-1} & 0 \end{pmatrix}$. The *Study determinant* $\det A = \|b\|^2 \|c\|^2$ is invariant under a sandwich product $A \mapsto BAB^{-1}$. A more general inverse is explained in Remark 5.4.3.

Proof. This is analogous to the proof of Proposition 5.3.6. Here, (3.2) reads

$$(\alpha^i)^2 d_i^j (d^i)^{-1} = (\alpha^j)^2 d_j^i (d^j)^{-1} \quad \text{and} \quad (\alpha_i^j)^2 (d_i^j)^{-1} d^i = (\alpha_j^i)^2 (d_j^i)^{-1} d^j \quad (5.39)$$

If α has the labelling property then the first equation agrees with (5.35) and the second equation is the reordered version. Hence, for an isothermic net (5.38) is a parallelogram net. For the converse, we again have that α^i is an edge labelling since it is an invariant of the evolution. Equation (3.1) implies that d can be integrated into a vertex based map and it is isothermic due to (3.2). $\square$

Remark 5.4.3 *As explained in [50], the inverse of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{H}^{2 \times 2}$ generically exists: Specifically, one can assign the so-called Study determinant*

$$\det A = |a|^2 |d|^2 + |b|^2 |c|^2 - \bar{a}b\bar{d}c - \bar{c}d\bar{b}a. \quad (5.40)$$

If $\det A \neq 0$ the inverse of A is given by

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} |d|^2 \bar{a} - \bar{c}d\bar{b} & |b|^2 \bar{c} - \bar{a}b\bar{d} \\ |c|^2 \bar{b} - \bar{d}c\bar{a} & |a|^2 \bar{d} - \bar{b}a\bar{c} \end{pmatrix}. \quad (5.41)$$

A natural Lax representation for isothermic nets is given by (3.19) with $\lambda = 1$ and $\mu \mapsto -\mu$ which becomes

$$P^i = \mathbf{1} + \mu \begin{pmatrix} 0 & -d^i \\ \frac{(\alpha^i)^2}{d^i} & 0 \end{pmatrix}. \quad (5.42)$$

One can easily check that this is gauge equivalent to the Lax representation for isothermic nets given in [15]. On the other hand, after a gauge with $G = \begin{pmatrix} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix}$ we get

$$\hat{P}^i := GP^iG^{-1} = \mathbf{1} + \begin{pmatrix} 0 & -d^i \\ \hat{\mu} \frac{(\alpha^i)^2}{d^i} & 0 \end{pmatrix} \quad (5.43)$$

with $\hat{\mu} := \mu^2$. This is known to be a *flat connection* of discrete isothermic nets. It is still a linear Lax representation for isothermic nets, however, it is not of the form (3.19). We can define a moving frame Ψ by the transport $\Psi_i = \hat{P}^i \Psi$ and by setting $\Psi(\mathbf{0}) = \begin{pmatrix} 1 & -f(\mathbf{0}) \\ 0 & 1 \end{pmatrix}$. For another gauge we set $\hat{G} = \Psi^{-1}|_{\hat{\mu}=0} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$. Note that this is an application of our general approach (3.34) for gauging of a general linear Lax representation into the form (3.19). We get

$$\hat{G}_i \hat{P}^i \hat{G}^{-1} = \mathbf{1} + \hat{\mu} \begin{pmatrix} -f_i(d^i)^* & f_i(d^i)^* f \\ -(d^i)^* & (d^i)^* f \end{pmatrix} = \mathbf{1} + \hat{\mu} \begin{pmatrix} f_i \\ 1 \end{pmatrix} (d^i)^* \begin{pmatrix} -1 & f \end{pmatrix} =: \mathbf{1} + \hat{\mu} \hat{p}^i \quad (5.44)$$

and $\hat{p}$ is a parallelogram net. This parallelogram net $\hat{p}$ is called a *retraction form* since its image and kernel are given by

$$\text{im } \hat{p}^i = \begin{pmatrix} f_i \\ 1 \end{pmatrix} \mathbb{H}, \quad \ker \hat{p}^i = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}. \quad (5.45)$$

These are homogeneous coordinates of the points f_i and f in $\mathbb{H}\mathbb{P}^1$ respectively: The projective line over the quaternions $\mathbb{H}\mathbb{P}^1$ is defined by equipping $\mathbb{H}^2 \setminus \{(0,0)\}$ with the equivalence relation

$$\begin{pmatrix} c \\ d \end{pmatrix} \sim \begin{pmatrix} a \\ b \end{pmatrix} \quad :\Leftrightarrow \quad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} q = \begin{pmatrix} aq \\ bq \end{pmatrix} \text{ for some } q \in \mathbb{H} \setminus \{0\}. \quad (5.46)$$

The elements of $\mathbb{H}\mathbb{P}^1$ are the equivalence classes and they yield homogeneous coordinates of $\mathbb{H} \cup \infty$ via

$$\begin{pmatrix} a \\ b \end{pmatrix} \cong ab^{-1}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cong \infty. \quad (5.47)$$

Now, note that the second column of $\Psi_{\hat{\mu}=0}^{-1}$ contains the homogeneous coordinates $\begin{pmatrix} f \\ 1 \end{pmatrix}$ of the isothermic net f . In fact, remarkably, this second column of Ψ^{-1} contains homogeneous coordinates of an isothermic net for every $\hat{\mu} \in \mathbb{R}$ as well. The family of isothermic nets $f_{\hat{\mu}}$ obtained this way is called the *Calapso transformation* of f . We will find another version of the Calapso transformation in Section 5.5.

Further investigation of the connection between isothermic nets and parallelogram nets can be a subject of future work. For example, it should be possible to relate work on periodic Darboux transformations [36] to quasi-periodic parallelogram nets as we discussed in Section 3.5.

5.5. Moutard nets in quadrics and Clifford algebras

We have already explored Moutard nets in $\mathbb{S}^3$ in Section 4.7 and in $\mathbb{H}^3$ in Section 5.2. We described them by parallelogram nets in the quaternions and biquaternions. Here, we present a more general setting: Moutard nets can be described by parallelogram nets in Clifford algebras. We will see that they arise naturally as additive or multiplicative primitive maps. While related linear Lax representations for Moutard nets in terms of Clifford algebras exist (see [95] or [24, Section 6.9]), this description seems to be new. This section has some overlap with Section 4.7 since a lot of the theory works completely analogous to the quaternionic setting.

We consider nets in the space $\mathbb{R}^{p,q}$ which is $\mathbb{R}^{p+q}$ equipped with the inner product

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}. \quad (5.48)$$

A net $f : \mathbb{Z}^n \rightarrow \mathbb{R}^{p,q}$ is in a quadric if $\langle f, f \rangle = \kappa$ for some global constant $\kappa \in \mathbb{R}$.

Recall Definition 2.2.2, stating that $f : \mathbb{Z}^n \rightarrow \mathbb{R}^{p,q}$ is called a Moutard net if its quadrilaterals fulfill a Moutard equation, i.e., if there exists $a : \vec{\mathcal{F}}(\mathbb{Z}^n) \rightarrow \mathbb{R}$ such that for all $i, j \in [n]$

$$f_{ij} - f = a^{ij}(f_j - f_i). \quad (5.49)$$

For simplicity, we will only consider nets where no diagonal $f_{ij} - f$ or $f_j - f_i$ vanishes.

Now, consider a Moutard net in a quadric. Plugging (5.49) into $\kappa = \langle f_{ij}, f_{ij} \rangle$ yields

$$a^{ij}(\kappa - \langle f_i, f_j \rangle) = \langle f, f_i - f_j \rangle. \quad (5.50)$$

Hence, generically (i.e., if $\langle f_i, f_j \rangle \neq \kappa$) the fourth point f_{ij} of every quad is determined by the other three points by

$$f_{ij} = f + \frac{\langle f, f_i - f_j \rangle}{\kappa - \langle f_i, f_j \rangle} (f_j - f_i). \quad (5.51)$$

Note that $\langle f, f_i \rangle$ has the labelling property since

$$\langle f_j, f_{ij} \rangle = \langle f_j, f \rangle + \frac{\langle f, f_i - f_j \rangle}{\kappa - \langle f_i, f_j \rangle} \langle f_j, f_j - f_i \rangle = \langle f, f_i \rangle. \quad (5.52)$$

For an algebraic description we turn to the Clifford algebra $Cl_{p,q}(\mathbb{R})$ which we introduce briefly. For a more elaborate overview see, for example, [6]. Consider multiplication of vectors in the standard basis $e_1, \dots, e_{p+q}$ of $\mathbb{R}^{p,q}$ subject to the relations

$$e_i e_j = -e_j e_i \quad \text{for } i \neq j, \quad e_i^2 = -\langle e_i, e_i \rangle. \quad (5.53)$$

This generates a unit associative algebra which is the Clifford algebra $Cl_{p,q}(\mathbb{R})$. Its dimension is 2^{p+q} and a basis is given by elements of the form $e_{i_1} \cdots e_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. This includes the empty product which is the unit element $\mathbf{1}$. Clifford algebras are graded algebras. For example, scalars are multiples of $\mathbf{1}$ and have grade zero. Similarly, vectors in $\mathbb{R}^{p,q}$ are contained in the algebra as linear combinations of $e_1, \dots, e_{p+q}$ and have grade one. Two such vectors $a, b \in \mathbb{R}^{p,q}$ fulfill

$$-a^2 = \langle a, a \rangle, \quad ab + ba = -2\langle a, b \rangle. \quad (5.54)$$

We can use the Clifford product to describe Moutard nets.

Lemma 5.5.1 *A net f in a quadric is a Moutard net if and only if*

$$(f_{ij} - f)(f_j - f_i) = 2\langle f, f_j - f_i \rangle. \quad (5.55)$$

Proof. If f is a Moutard net we get this equation from (5.49), (5.50) and $-(f_j - f_i)^2 = 2(\kappa - \langle f_i, f_j \rangle)$. Conversely, if the product of the diagonals of f is a scalar the diagonals are parallel and f is a Moutard net. $\square$

Now, we describe the Moutard net as a parallelogram net.

Proposition 5.5.2 *A net f in a quadric of $\mathbb{R}^{p,q}$ is a Moutard net if and only if $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{R}^{p,q} \subseteq Cl_{p,q}(\mathbb{R})$ defined by $p^i = f_i - f$ is a parallelogram net in $Cl_{p,q}(\mathbb{R})$.*

Proof. While the additive equation (3.1) holds by construction, the multiplicative equation (3.2) reads

$$\begin{aligned} 0 &= p_j^i p^j - p_i^j p^i = (f_{ij} - f_j)(f_j - f) - (f_{ij} - f_i)(f_i - f) \\ &= f_{ij}(f_j - f_i) + (f_j - f_i)f = (f_{ij} - f)(f_j - f_i) - 2\langle f, f_j - f_i \rangle \end{aligned} \quad (5.56)$$

and is equivalent to (5.55). Here, we used $f_i^2 = f_j^2 = -\kappa$. $\square$

The Lipschitz group is the group of products $a = a_1 \cdots a_k \in Cl_{p,q}(\mathbb{R})$ of invertible vectors $a_i \in \mathbb{R}^{p,q}$. These are exactly the elements of the algebra which preserve the grade

of a vector b under the sandwich product $b \mapsto aba^{-1}$. Any element of the Lipschitz group is either graded even or odd, depending on the number of vectors in the product.

Since $p^i = f_i - f$ is of grade one the Lax representation $\lambda + \mu p^i$ is not in the Lipschitz group. The associated family (3.23) does not preserve the grade and p_t^i is not in the vector space which makes geometric interpretation difficult. However, there is a more natural construction which resolves this issue: We can also assign a parallelogram net to a Moutard net where p^i is a bi-vector, i.e., an element of grade two.

Proposition 5.5.3 *A net f in a quadric of $\mathbb{R}^{p,q}$ is a Moutard net if and only if $p : \vec{\mathcal{E}}(\mathbb{Z}^n) \rightarrow \mathbb{R}^{p,q} \subseteq Cl_{p,q}(\mathbb{R})$ defined by $p^i = f_i f$ is a parallelogram net in $Cl_{p,q}(\mathbb{R})$.*

If $\kappa \neq 0$ every f is invertible and we have

$$f_i = p^i f^{-1} = -\frac{1}{\kappa} p^i f \quad (5.57)$$

and f is a multiplicative primitive map of $-\frac{1}{\kappa} p$.

Proof. The multiplicative equation (3.2) always holds, since

$$p_j^i p^j = f_{ij} f_i f_j f = -\kappa f_{ij} f = f_{ij} f_j f_j f = p_j^i p^j \quad (5.58)$$

For the additive equation (3.1) consider

$$\begin{aligned} p_j^i + p^j &= f_{ij} f_j + f_j f = (f_{ij} - f) f_j - 2\langle f, f_j \rangle \\ p_i^j + p^i &= f_{ij} f_i + f_i f = (f_{ij} - f) f_i - 2\langle f, f_i \rangle. \end{aligned} \quad (5.59)$$

Hence, (3.1) is equivalent to (5.55). $\square$

Now, the Lax representation $\lambda + \mu p^i$ is a bi-vector in the Lipschitz group. We can consider the edge values p_t^i in the associated family (3.23). The frame Φ is in the Lipschitz group and, hence, a sandwich product with Φ preserves the grade. Then, $p_t^i = \Phi^{-1}(P^i)^{-1}(P^i)' \Phi$ is indeed a bi-vector again. As corresponding vertex based net we define the vector valued map $f_t := \Phi^{-1} f \Phi$.

Proposition 5.5.4 *Let f be a Moutard net in a quadric and let p be the parallelogram net defined by $p^i = f_i f$ with a frame Φ . Then, the family of nets defined by $f_t := \Phi^{-1} f \Phi$ is a family of Moutard nets in the same quadric as f .*

Proof. Since Φ is in the Lipschitz group f_t is of grade one, i.e., in the vector space $\mathbb{R}^{p,q}$. Also, we have $f_t^2 = \Phi^{-1} f^2 \Phi = -\kappa$ and, hence, f_t is in the same quadric as f . We will show that the edge based net q_t defined by $q_t^i = (f_t)_i f_t$ is a parallelogram net. Then, by Proposition 5.5.3, f_t is a Moutard net. Specifically, it will turn out that q_t is of the form

$q_t^i = r + sp_t^i$ where p_t is in the associated family of p and $s, r \in \mathbb{R}$ are globally constant. Then, q_t is primarily equivalent to p_t and, hence, also a parallelogram net.

We calculate

$$q_t^i = (f_t)_i f_t = \Phi_i^{-1} f_i \Phi_i \Phi^{-1} f \Phi = \Phi_i^{-1} f_i (\lambda + \mu f_i f) f \Phi = \Phi_i^{-1} (\mu \kappa^2 + \lambda f_i f) \Phi \quad (5.60)$$

and

$$r + sp_t^i = r \Phi_i^{-1} P^i \Phi + s \Phi_i^{-1} (P^i)' \Phi = \Phi_i^{-1} ((r\lambda + s\lambda') + (r\mu + s\mu') f_i f) \Phi. \quad (5.61)$$

These coincide if we choose $r = \frac{\lambda\lambda' - \mu\mu'\kappa^2}{\lambda'\mu - \lambda\mu'}$ and $s = \frac{\mu^2\kappa^2 - \lambda^2}{\lambda'\mu - \lambda\mu'}$. Note that r and s are globally constant, but depend on the parameter t . $\square$

Clearly, each member f_t of the associated family only depends on the fraction $\frac{\lambda(t)}{\mu(t)}$ and, hence, we can set $\lambda(t) = 1$ and $\mu(t) = t$. Note that r and s in the proof simplify to $r = t\kappa^2$ and $s = 1 - t^2\kappa^2$.

We will demonstrate that this associated family is the Calapso transformation as discussed in Section 5.4 by turning to the representation of isothermic nets in the lightcone of $\mathbb{R}^{4,1}$: Let $g : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ be an isothermic net with edge labelling α . It is known (see, e.g., [24, Section 4.3.5]) that there is a map $s : \mathbb{Z}^2 \rightarrow \mathbb{R}$ such that

$$f : \mathbb{Z}^2 \rightarrow \mathbb{R}^{4,1}, \quad f = \frac{1}{s} \begin{pmatrix} g \\ \frac{1}{2}(-1 + \|g\|^2) \\ \frac{1}{2}(+1 + \|g\|^2) \end{pmatrix} \quad (5.62)$$

is a Moutard net in the lightcone which is the quadric given by $\langle f, f \rangle = 0$. The net f is called the *Moutard lift* of g . It fulfills $f^2 = -\langle f, f \rangle = 0$ and

$$f_i f + f f_i = -2\langle f, f_i \rangle = \frac{\|g_i - g\|^2}{ss_i} = (\alpha^i)^2. \quad (5.63)$$

Conversely, every Moutard net in the light-cone corresponds to an isothermic net. Thus, Proposition 5.5.4 assigns a family of nets to each isothermic net and its members are isothermic again.

Proposition 5.5.5 *Let f be the Moutard lift of an isothermic net g with labelling α . Then, the family g_t of isothermic nets obtained from $f_t = \Phi^{-1} f \Phi$ as in Proposition 5.5.4 with $\Phi_i = (1 + t f_i f) \Phi$ has the following properties:*

1. *The isothermic nets g_t have cross-ratio*

$$\frac{(\alpha^i)^2}{(\alpha^j)^2} \frac{1 + t(\alpha^j)^2}{1 + t(\alpha^i)^2}. \quad (5.64)$$

2. Each vertex star of g_t is obtained by applying the common Möbius transformation $\Phi(t)$ to the corresponding vertex star of g .

Here, by *vertex star* we mean a point g and its neighbours $g_i, g_{\bar{i}}$. These properties characterize the Calapso transformation (see [50]). It coincides with the transformation described in Section 5.4. Note that the following proof extends to nets in the lightcone of other vector spaces as well and, thus, analogous statements can be made.

Proof. The inverse of each transport $1 + tf_i f$ is given by

$$(1 + tf_i f)^{-1} = \frac{1 + tf f_i}{1 + t(\alpha^i)^2}. \quad (5.65)$$

This implies

$$(f_t)_i f_t = \Phi^{-1}(1 + tf_i f)^{-1} f_i (1 + tf_i f) f \Phi = \frac{1}{1 + t(\alpha^i)^2} \Phi^{-1} f_i f \Phi \quad (5.66)$$

and taking the scalar part yields

$$(\alpha_t^i)^2 = -2\langle (f_t)_i, f_t \rangle = \frac{-2\langle f_i, f \rangle}{1 + t(\alpha^i)^2} = \frac{(\alpha^i)^2}{1 + t(\alpha^i)^2}. \quad (5.67)$$

This implies (5.64).

Also, for a point and its neighbours we have

$$\begin{aligned} f_t &= \Phi^{-1} f \Phi, \\ (f_t)_i &= \Phi_i^{-1} f_i \Phi_i = \Phi^{-1}(1 + tf_i f)^{-1} f_i (1 + tf_i f) \Phi = \frac{1}{1 + t(\alpha^i)^2} \Phi^{-1} f_i \Phi, \\ (f_t)_{\bar{i}} &= \Phi_{\bar{i}}^{-1} f_{\bar{i}} \Phi_{\bar{i}} = \Phi^{-1}(1 + tf f_{\bar{i}}) f_{\bar{i}} (1 + tf f_{\bar{i}})^{-1} \Phi = \frac{1}{1 + t(\alpha_{\bar{i}}^i)^2} \Phi^{-1} f_{\bar{i}} \Phi. \end{aligned} \quad (5.68)$$

Note that scalar multiples in this model are identified and that a sandwich product with a member of the Lipschitz group acts as a Möbius transformation. This implies the second statement. $\square$

5.6. Voss nets and kinematics

In this section, we investigate parallelogram nets in the dual quaternions $\mathbb{DH}$. While quaternions describe rotations in three dimensional space, dual quaternions describe Euclidean displacements. Thus, parallelogram nets in the dual quaternions can be considered as nets of displacements. We will consider a special case where the associated family yields an isometric deformation of a surface. Our observations are based on the

description of linkages in [48]. As canonical example we will discuss Voss nets. While the kinematic properties of the deformation of Voss nets are well-known [92, 91] we obtain an algebraic description of this deformations that does not seem to exist in the literature.

We will briefly introduce dual quaternions. For more context on dual quaternions and line geometry see [8, 88]. A dual quaternion is of the form $q = a + \epsilon b$ where $a, b \in \mathbb{H}$ are quaternions and ϵ is a scalar fulfilling $\epsilon^2 = 0$. Addition is defined componentwise and the associative multiplication of two dual quaternions $q_1 = a_1 + \epsilon b_1$ and $q_2 = a_2 + \epsilon b_2$ is given by

$$q_1 q_2 = a_1 a_2 + \epsilon(a_1 b_2 + b_1 a_2) \quad (5.69)$$

One can extend many properties of quaternions to dual quaternions. Consider a dual quaternion $q = a + \epsilon b$ with $a, b \in \mathbb{H}$. Then, a is called the *primal part* and b is called the *dual part* of q . The *conjugate* is given by $\bar{q} = \bar{a} + \epsilon \bar{b}$. The *scalar part* of q is given by $\text{Re } q = \text{Re } a + \epsilon \text{Re } b = \frac{1}{2}(q + \bar{q})$ and the *vector part* is given by $\vec{q} = \vec{a} + \epsilon \vec{b} = \frac{1}{2}(q - \bar{q})$. A quantity analogous to the squared norm of quaternions is given by $q\bar{q} = \bar{q}q = a\bar{a} + \epsilon(a\bar{b} + b\bar{a}) = \|a\|^2 + \epsilon 2\langle a, b \rangle$. If this squared norm does not vanish q has an inverse given by

$$q^{-1} = a^{-1}(1 - \epsilon b a^{-1}) \quad (5.70)$$

Also, we call $q_\epsilon := a - \epsilon b$ the ϵ -conjugate.

We are mostly interested in quaternions where the squared norm is real, called *Study quaternions*: In fact, the set $\mathbb{DH}^\times := \{q \in \mathbb{DH} \mid q\bar{q} \in \mathbb{R} \setminus \{0\}\}$ is a group under multiplication.

We will describe a line through two points $f_1, f_2 \in \mathbb{R}^3 \cong \text{Im } \mathbb{H}$ by the dual quaternion

$$l := f_2 - f_1 + \epsilon(f_1 \times f_2) \quad (5.71)$$

These are called *Plücker coordinates* for the straight line. They fulfill $l \in \mathbb{DH}^\times$ and $\text{Re } l = 0$. In fact, every dual quaternion with these properties describes a straight line and the Plücker coordinates of a straight line are unique up to multiplication with a real number. A point f lies on the line given by a dual quaternion l if and only if

$$(1 + \epsilon f)l_\epsilon = l(1 + \epsilon f) \quad (5.72)$$

To us, the most important property of dual quaternions is that they can be used to describe rigid motions in Euclidean space. In fact, if we embed Euclidean space $\mathbb{R}^3$ into the dual quaternions by $f \mapsto 1 + \epsilon f$ then a dual quaternion $q \in \mathbb{DH}^\times$ acts as a rigid motion on $\mathbb{R}^3$ via the map

$$1 + \epsilon f \mapsto q_\epsilon(1 + \epsilon f)q^{-1} \quad (5.73)$$

and all rigid motions can be described in this manner. Remarkably, composition of two rigid motions corresponds to multiplication of corresponding dual quaternions. The rigid motion induced by $q = a + \epsilon b \in \mathbb{DH}^\times$ is a translation if $a \in \mathbb{R}$. It is a rotation if $a \notin \mathbb{R}$ and $\operatorname{Re} b = 0$ and the axis of the rotation is the line given by $\vec{q} = \vec{a} + \epsilon \vec{b}$. Analogously to (4.11), the angle of rotation depends on $\operatorname{Re} a$. We denote the set of dual quaternions inducing rotations by

$$\mathcal{R} := \{q = a + \epsilon b \in \mathbb{DH} \mid q\bar{q} \in \mathbb{R}, \vec{a} \neq 0, \operatorname{Re} b = 0\} \quad (5.74)$$

Now, consider a two-dimensional parallelogram net $p : \vec{\mathcal{E}}(\mathbb{Z}^2) \rightarrow \mathcal{R}$. The sandwich product $x \mapsto qxq^{-1}$ of dual quaternions preserves $\operatorname{Re} x$ and $x\bar{x}$. Therefore, the invariants of the evolution (3.3) are the scalar part $\operatorname{Re} p^i$ and the squared norm $p^i \bar{p}^i$ and they have the labelling property. Note that, the evolution (3.3) is closed in the set $\mathcal{R}$ since the condition to be in $\mathcal{R}$ can be expressed in terms of the invariants. Hence, such parallelogram nets can be constructed from initial data. Also, note that the primal part of p is a quaternionic parallelogram net.

Let us turn to a kinematic interpretation. Consider the Lax representation (3.19) given by $P^i(t) = \lambda(t) + \mu(t)p^i$ for two functions $\lambda, \mu : \mathbb{R} \rightarrow \mathbb{R}$. Then,

$$1 + \epsilon f \mapsto (P^i)_\epsilon^{-1} (1 + \epsilon f) P^i \quad (5.75)$$

describes a rotation around the axis $\vec{p}^i$ and the angle of rotation depends on t . The compatibility condition $P_j^i P_j^j = P_i^j P_i^i$ implies that composition of these rotations along paths on the lattice is path-independent. The corresponding moving frame Φ with $\Phi_i = P^i \Phi$ and $\Phi(\mathbf{0}) = \mathbf{1}$ assigns a motion to every point which is given by

$$1 + \epsilon f \mapsto \Phi_\epsilon^{-1} (1 + \epsilon f) \Phi \quad (5.76)$$

One can build a physical realization of this by realizing each rotation with a rotational joint. In fact, the combination of such rotational joints along a path from $\mathbf{0}$ to $x \in \mathbb{Z}^2$ realizes the motion $\Phi(x)$ as a linkage. A realization of two distinct paths from $\mathbf{0}$ to $x \in \mathbb{Z}^2$ yields a closed linkage. With the parametrization $P^i = t - p^i$ (i.e., $\lambda(t) = t, \mu(t) = -1$) the frame $\Phi(x)$ becomes polynomial in t and, thus, $\Phi(x)$ is called a *motion polynomial*. Realization of such polynomial motions as closed linkages by splitting them into compositions of rotation is investigated in [48].

As discrete differential geometric application we are interested in the case where the lines serving as axes of rotation make a discrete surface. For this, we will consider a two-dimensional parallelogram net $p = (u, v)$ such that at each quad the four lines given by u, v, u_2, v_1 intersect in one point F^{12} . The resulting discrete surface $F : \vec{\mathcal{F}}(\mathbb{Z}^2) \rightarrow \mathbb{R}^3$ is combinatorially dual to our lattice $\mathbb{Z}^2$: A point $F^{12} = F^{21}$ of the surface belongs to a quad of the parallelogram net. As a motivation for this construction note that this condition on p is consistent with the evolution (3.3):

Lemma 5.6.1 *The evolution (3.3) preserves incidence in the following sense: If two lines given by dual quaternions u and v intersect in a point F then the line given by $\hat{u} = (v - u)u(v - u)^{-1}$ also intersects F .*

Proof. This can be shown by repeated application of (5.72) and its ϵ -conjugated version $(1 - \epsilon f)l = l_\epsilon(1 - \epsilon f)$:

$$\begin{aligned} (1 + \epsilon F)\hat{u}_\epsilon &= (1 + \epsilon F)(v_\epsilon - u_\epsilon)u_\epsilon(v_\epsilon - u_\epsilon)^{-1} = (v - u)(1 + \epsilon F)u_\epsilon(v_\epsilon - u_\epsilon)^{-1} \\ &= (v - u)u(1 + \epsilon F)(v_\epsilon - u_\epsilon)^{-1} = (v - u)u((v_\epsilon - u_\epsilon)(1 - \epsilon F))^{-1} \\ &= (v - u)u((1 - \epsilon F)(v - u))^{-1} = (v - u)u(v - u)^{-1}(1 + \epsilon F) \\ &= \hat{u}(1 + \epsilon F). \end{aligned} \tag{5.77}$$

□

Our frame $\Phi(x)$ defines a motion at every vertex $x \in \mathbb{Z}^2$ which we can apply to the corresponding quad of the surface F . In this manner, four motions $\Phi, \Phi_1, \Phi_2, \Phi_{12}$ are assigned to each point F^{12} . These motions are related by rotation around the respective axes $\vec{u}, \vec{v}, \vec{u}_2, \vec{v}_1$ which intersect in the point F^{12} . Hence, these motions agree on the point F^{12} . Thus, applying Φ yields a deformation F_t of the surface which is given by the non-symmetric formula

$$1 + \epsilon F_t^{12} = \Phi_\epsilon^{-1}(1 + \epsilon F^{12})\Phi \tag{5.78}$$

with

$$\begin{aligned} \Phi_\epsilon^{-1}(1 + \epsilon F^{12})\Phi &= (\Phi_1)_\epsilon^{-1}(1 + \epsilon F^{12})\Phi_1 \\ &= (\Phi_2)_\epsilon^{-1}(1 + \epsilon F^{12})\Phi_2 = (\Phi_{12})_\epsilon^{-1}(1 + \epsilon F^{12})\Phi_{12}. \end{aligned} \tag{5.79}$$

Every quad remains rigid under this deformation and, in this sense, the deformation is *isometric*.

Generally, the construction of such parallelogram nets from initial data still has to be investigated. Instead, we will turn to an interesting subclass: For this, we denote the set of dual quaternions describing 180° -rotations around an axis in the $\mathbf{i}, \mathbf{j}$ -plane by

$$\mathcal{O} := \{a\mathbf{i} + b\mathbf{j} + c\epsilon\mathbf{k} \in \mathbb{DH} \mid a, b, c, \in \mathbb{R}, a^2 + b^2 \neq 0\} \subseteq \mathcal{R}. \tag{5.80}$$

Elements of this set correspond to lines in the $\mathbf{i}, \mathbf{j}$ -plane and up to multiplication with a scalar each such line determines an element of $\mathcal{O}$.

Now, consider a parallelogram net $p = (u, v) : \vec{\mathcal{E}}(\mathbb{Z}^2) \rightarrow \mathcal{O}$. For regularity, we will assume that lines given by adjacent values of p are not parallel. Then, adjacent lines intersect and Lemma 5.6.1 implies that the four lines of each quad intersect in a common point F^{12} . Therefore, using the evolution (3.3) one can construct examples of such parallelogram nets from any generic initial data on the axes or on a zig-zag.

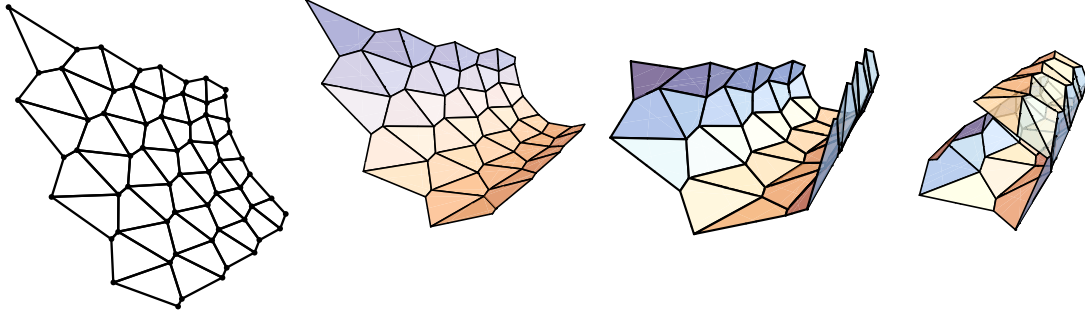


Figure 5.5.: An isometric deformation given by (5.78) of an *anti-Voss-net* which is a net where opposite angles at every vertex are supplementary.

The discrete net F defined by the intersection points is contained in the plane, however, under the deformation (5.78) this net unfolds into three-dimensional space (see Figure 5.5). By construction, the quads of F are planar and, hence, the quads of F_t are planar as well. Note that the primal part of p contains the information about the direction of each line. As we discussed in Section 4.4, the quads of the additive primitive map of this primal part are anti-parallelograms. Opposite inner angles of anti-parallelograms are equal and, hence, opposite angles at every point F^{12} are equal or supplementary (depending on the position of neighboring intersection points $F_i^{12}, F_{\bar{i}}^{12}$). Let us summarize these observations:

Proposition 5.6.2 *Let $p : \vec{\mathcal{E}}(\mathbb{Z}^2) \rightarrow \mathcal{O}$ be a parallelogram net. Then, the corresponding net of intersection points F and the nets F_t given by the isometric deformation (5.78)*

1. *have planar faces*
2. *and opposite angles at every vertex are equal or supplementary.*

In the special case where all pairs of opposite angles are equal (and not supplementary) they belong to a well-known class of nets:

Definition 5.6.3 *A discrete net $F : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ is called a Voss net if it has planar faces and if opposite angles at every vertex are equal.*

Such discrete Voss nets are known as discretization of surfaces in conjugate geodesic parametrization [92]. Their isometric deformation (see [70, 99, 67, 69]) and their connection to K-nets (see [107]) are well-understood. In particular, the primal part of our parallelogram net describes a K-net as in Section 4.4. The Voss net is parallel-reciprocal

to this K-net and it is only uniquely determined from the K-net if one prescribes additional information about the position of each edge. In our setup, this information is stored in the dual part of the parallelogram net. Matching our observations, the definition of discrete Voss nets has a natural generalization that includes vertices where opposite angles are supplementary [69] (called *anti-V-hedral vertices*). A net consisting of such vertices is seen in Figure 5.5.

The author believes that all (generalized) Voss nets and their deformations can be described by a parallelogram net as in Proposition 5.6.2. We will sketch a possible proof without giving details: Voss nets are *flat-foldable* meaning that their deformation contains a planar member F . The edges of F are lines in a plane and by representing the lines by dual quaternions we obtain a net $p : \vec{\mathcal{E}}(\mathbb{Z}^2) \rightarrow \mathcal{O}$. The lines define each value of p up to a scalar multiple. Since F is reciprocal parallel to a K-net (in this case: a net of anti-parallelograms) we can scale each value of p in a way such that its primal part gives the edges of the K-net. This already implies that the primal part is a parallelogram net. The parallelogram equations (3.1),(3.2) are invariant under a translation in the plane. Translating the intersection point F of four lines corresponding to a quad into the origin yields that (3.1),(3.2) hold for the full dual quaternions. Hence, p is a parallelogram net and it determines the Voss net and its deformation as in Proposition 5.6.2.

Finally, since parallelogram nets are multidimensionally consistent we can extend our considerations to a lattice of higher dimension, for example, we can consider a parallelogram net $p : \vec{\mathcal{E}}(\mathbb{Z}^3) \rightarrow \mathcal{O}$. The three-dimensional lattice consists of elementary cubes and since the corresponding net of intersection points F is defined on the faces of the cubes the net F consists of elementary octahedra. The faces of each elementary octahedron of F stay rigid in our isometric deformation. An octahedron which allows for such a deformation is called a *Bricard octahedron* as discovered in [29] (for more details, see [101, 43]).

5.7. Other linear Lax pairs

In this section, we showcase two more examples of linear Lax pairs which can be gauged into our Lax representation (3.19) for parallelogram nets. Exploring their connection to our theory in a more meaningful manner remains the subject of future work.

Discrete O surfaces have been established in [96] and constitute a class of surfaces which includes many well-known integrable surface classes, in particular, isothermic nets, minimal nets and nets of constant (Gaussian or mean) curvature. The Lax pair for discrete O surfaces is given in [96, Theorem 6.2] by the λ -dependent pair $(U, V) :$

$\vec{\mathcal{E}}(\mathbb{Z}^2) \rightarrow \mathbb{R}^{(3+n) \times (3+n)}$ with

$$\begin{aligned} U &= \mathbf{1} + \frac{2}{1 - \alpha\lambda} \begin{pmatrix} \alpha\lambda \mathbf{X} \mathbf{X}^T & \lambda \mathbf{X} \underline{H} \\ \underline{H}^T \mathbf{X}^T & \lambda \underline{H}^T \underline{H} \end{pmatrix} \in \mathbb{R}^{(3+n) \times (3+n)} \\ V &= \mathbf{1} + \frac{2}{1 - \beta\lambda} \begin{pmatrix} \beta\lambda \mathbf{Y} \mathbf{Y}^T & \lambda \mathbf{Y} \underline{K} \\ \underline{K}^T \mathbf{Y}^T & \lambda \underline{K}^T \underline{K} \end{pmatrix} \in \mathbb{R}^{(3+n) \times (3+n)}, \end{aligned} \quad (5.81)$$

where $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^3$ are column vectors, $\underline{H}, \underline{K} \in \mathbb{R}^n$ are row vectors and $\alpha, \beta \in \mathbb{R}$ have the labelling property. Each O surface corresponds to a solution of the compatibility condition $V_1 U = U_2 V$ for (5.81). Now, observe that $(1 - \alpha\lambda)U$ and $(1 - \beta\lambda)V$ are linear in λ and, thus, of the form

$$(1 - \alpha\lambda)U = a + \lambda b, \quad (1 - \beta\lambda)V = c + \lambda d \quad (5.82)$$

for $a, b, c, d \in \mathbb{R}^{(3+n) \times (3+n)}$ based on corresponding edges. Then, by application of Lemma 3.3.7 we see that the Lax pair (5.81) of O surfaces is gauge equivalent to our Lax representation (3.19) of parallelogram nets.

The theories of parallelogram nets and O surfaces both unify a large class of integrable systems and, thus, a better understanding of their connection is a natural goal for future research.

A similar gauge equivalence can be observed for *indefinite affine spheres* which have been introduced in [20]. Discrete indefinite affine spheres are maps $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ with the following properties:

1. Vertex stars are planar, i.e., every point f and its neighbours $f_{\bar{1}}, f_{\bar{2}}, f_1, f_2$ are in a common plane and
2. On all faces the discrete affine normal line defined by connecting the midpoints of diagonals $\frac{1}{2}(f_{12} + f)$ and $\frac{1}{2}(f_1 + f_2)$ intersect the origin $\mathbf{0} \in \mathbb{R}^3$.

At each point consider the matrix

$$R = \begin{pmatrix} (f_1 - f)^T \\ (f_2 - f)^T \\ f^T \end{pmatrix} \in \mathbb{R}^{3 \times 3}. \quad (5.83)$$

This can be considered as a frame of the surface. The transport of this frame given in [20, Theorem 3.3] comes with a natural dependence on a parameter λ . Writing this transport in terms of a pair $(U, V) : \vec{\mathcal{E}}(\mathbb{Z}^2) \rightarrow \mathbb{R}^{3 \times 3}$ fulfilling $R_1 = UR, R_2 = VR$ yields

$$\begin{aligned} U &= \begin{pmatrix} \frac{H_1-1}{(H-1)H_1} + \lambda A_1 & \lambda \frac{A_1}{H-1} H & \lambda 2A_1 \\ H-1 & H & 2(H-1) \\ 1 & 0 & 1 \end{pmatrix} =: a + \lambda b, \\ V &= \begin{pmatrix} H & H-1 & 2(H-1) \\ \frac{1}{\lambda} \frac{B_2}{H-1} H & \frac{H_2-1}{(H-1)H_2} + \frac{1}{\lambda} B_2 & \frac{1}{\lambda} 2B_2 \\ 0 & 1 & 1 \end{pmatrix} =: \frac{1}{\lambda}(c + \lambda d) \end{aligned} \quad (5.84)$$

where $H = H^{12}$ is a face based map and A and B are vertex based maps.

Note that b and c are matrices with rank equal to 1 and, as such, not invertible. Thus, Lemma 3.3.7 cannot be applied immediately. However, after a reparametrization $\lambda \mapsto \hat{\lambda} = \lambda - 1$ the lemma can be applied to the pair

$$U = a + b + \hat{\lambda}b, \quad \lambda V = c + d + \hat{\lambda}d \quad (5.85)$$

and, hence, the Lax pair for indefinite affine spheres (5.84) is gauge equivalent to the Lax pair (3.19) for parallelogram nets.

6. Universal factorization

A general observation made several times throughout this thesis is that on every path of consecutive edges $d_1, \dots, d_k \in \tilde{\mathcal{E}}(\mathbb{Z}^n)$ lives the product

$$(\lambda + p(d_k)) \cdots (\lambda + p(d_1)). \quad (6.1)$$

which is a polynomial in λ . The coefficients are elements of the non-commutative algebra $\mathcal{A}$ while the variable λ is a scalar and commutes with elements of $\mathcal{A}$. In this chapter, we discuss factorization of such polynomials and relate it to (skew) parallelogram nets.

6.1. Polynomials over rings

Here, we present some known general theory following [76]. We consider polynomials $F = \sum_{k=0}^n t^k a_k$ in an indeterminate t where the coefficients a_k lie in some ring $\mathcal{R}$. We assume that t commutes with all elements of the ring. We define the sum and product of two polynomials $F = \sum_{k=0}^n t^k a_k, G = \sum_{k=0}^m t^k b_k$ by

$$F + G = \sum_{k=0}^{\max\{m,n\}} t^k (a_k + b_k) \quad (6.2)$$

and

$$FG = \sum_{k=0}^{m+n} t^k c_k, \quad c_k = \sum_{i+j=k} a_i b_j \quad (6.3)$$

respectively. With these two operations the polynomials form a ring. In our applications $\mathcal{R} = \mathcal{A}$ is a unit associative algebra and t will be a scalar (in $\mathbb{R}$ or in $\mathbb{C}$). Scalars commute with all elements and, hence, are contained in the center $Z(\mathcal{R})$ of the ring. Thus, the assumption on the variable t to commute with the coefficients is reasonable.

We want to be able to evaluate the polynomial for elements of the ring. Following the notation of [68], we define left evaluation ev_F^l and right evaluation ev_F^r of F at $x \in \mathcal{R}$ by

$$\text{ev}_F^l(x) := \sum_{k=0}^n x^k a_k, \quad \text{ev}_F^r(x) := \sum_{k=0}^n a_k x^k \quad (6.4)$$

For elements of the center $x \in Z(R)$ we have $\text{ev}_F^l(x) = \text{ev}_F^r(x) =: \text{ev}_F(x)$ and the evaluation is a ring homomorphism:

$$\text{ev}_{FG}(x) = \text{ev}_F(x)\text{ev}_G(x), \quad \text{ev}_{F+G}(x) = \text{ev}_F(x) + \text{ev}_G(x). \quad (6.5)$$

For general elements $x \in \mathcal{R}$ of the ring right and left evaluation are not homomorphisms.

Now, consider $F = \sum_{k=0}^n t^k a_k$ with $a_n \neq 0$. Then, a_n is called the *leading coefficient* and if $a_n = 1$ the polynomial is called *monic*. The number n is the *degree* of F which we denote by $\deg F = n$. We turn to polynomial division:

Proposition 6.1.1 *Let F, G be polynomials with $\deg G < \deg F$ such that the leading coefficient of G is invertible. Then, there exist unique polynomials Q_r, R_r and Q_l, R_l with $\deg R_r, \deg R_l < \deg G$ such that*

$$F = Q_l G + R_l = G Q_r + R_r \quad (6.6)$$

Proof. One proves this using the well-known Euclidean division for polynomials: We will demonstrate this for left division, right division works analogously. For this, let g be the leading coefficient of G and set $Q_0 := 0$ and $R_0 := F$. We define polynomials Q_i, R_i by iterating the following steps as long as $\deg R_i \geq \deg G$:

1. define $s_i := t^{\deg R_i - \deg G} r_i g^{-1}$ where r_i is the leading coefficient of R_i ,
2. $Q_{i+1} := Q_i + s_i$,
3. $R_{i+1} := R_i - s_i G$.

In every iteration we have $F = Q_i G + R_i$ and $\deg R_i$ is strictly decreasing. Hence, this algorithm stops after $k < \infty$ iterations and we set $Q_l := Q_k$, $R_l := R_k$. These fulfill the requirements of the theorem. For uniqueness, assume that $\tilde{Q}_l$ and $\tilde{R}_l$ also fulfill the requirements. Then, we have

$$(\tilde{Q}_l - Q_l)G = R_l - \tilde{R}_l. \quad (6.7)$$

Now, if $\tilde{Q}_l - Q_l \neq 0$, then the degree of the LHS is larger or equal to $\deg G$ which contradicts the assumption that the degree of the RHS is smaller than $\deg G$. Hence, we have $\tilde{Q}_l = Q_l$ and $\tilde{R}_l = R_l$. $\square$

Division with remainder is an important tool for factorization. We call G a *right factor* of F if there exists a polynomial Q such that $F = QG$. This is equivalent to a vanishing remainder R_l . We call $h \in \mathcal{A}$ a *right root* of F if $\text{ev}_F^r(h) = 0$. Of course, left factors and roots can be defined analogously. Linear right factors are closely related to right roots:

Proposition 6.1.2 *The polynomial $t - h$ is a right factor of F if and only if h is a right root of F .*

Proof. If $t - h$ is a right factor of F we can write $F = Q(t - h)$ for some $Q = \sum_{i=0}^n t^i q_i$. We have

$$F = \sum_{i=0}^n t^i q_i (t - h) = \sum_{i=0}^n t^{i+1} q_i - \sum_{i=0}^n t^i q_i h \quad (6.8)$$

and, therefore,

$$\text{ev}_F^r(h) = \sum_{i=0}^n q_i h^{i+1} - \sum_{i=0}^n q_i h h^i = 0. \quad (6.9)$$

Now, for the converse, assume that h is a right root of F . Using polynomial division we can find Q, R such that $F = Q(t - h) + R$ and $\deg R = 0$, meaning that $R = t^0 r_0$ is a constant polynomial. Right evaluation yields

$$0 = \text{ev}_F^r(h) = \text{ev}_{Q(t-h)}^r(h) + \text{ev}_R^r(h) = 0 + r_0 \quad (6.10)$$

since $Q(t - h)$ has $t - h$ as a right factor and, thus, h as a right root by above argumentation. Hence, $F = Q(t - h)$ and $t - h$ is a right factor of F . This proof works analogously for left factors and roots. $\square$

In the following sections, we will discuss existence and uniqueness of a factorization of a polynomial $F = \sum_{k=0}^n t^k a_k$ into linear factors

$$F = a_n(t - h_n) \cdots (t - h_1), \quad h_i \in \mathcal{R}. \quad (6.11)$$

Specifically, we will review well-known results for quaternions and introduce a new result for the complex matrix ring $\mathbb{C}^{2 \times 2}$. Similar factorization results exist for dual quaternions [48], split quaternions [94, 93] and, recently, conformal geometric algebra [77].

6.2. Quaternionic polynomials

If the coefficients a_k of the polynomial $F = \sum_{k=0}^n t^k a_k$ are quaternions, i.e., if $a_k \in \mathbb{H}$, we call F a *quaternionic polynomial*. In this section, we review the well-known theory of factorization of such polynomials which has originally been studied in [45, 83] (see also [68]).

First, we consider the special case of real polynomials $\mathbb{R}[t]$: Since real numbers are contained in the quaternions real polynomials are contained in the quaternionic polynomials. The set of roots of a real polynomial consists of real roots and pairs of complex roots related by complex conjugation. The real roots correspond to real linear factors

while each pair of complex roots $\mu, \bar{\mu} \in \mathbb{C}$ corresponds to a (monic) real quadratic factor $X \in \mathbb{R}[t]$ of the form

$$X = (t - \mu)(t - \bar{\mu}) = t^2 - 2 \operatorname{Re} \mu + |\mu|^2. \quad (6.12)$$

The complex linear factors $t - \mu$ and $t - \bar{\mu}$ are not quaternionic polynomials. However, a quaternionic factorization of X is given by

$$X = (t - h)(t - \bar{h}) = t^2 - 2 \operatorname{Re} h + \|h\|^2 \quad (6.13)$$

for any quaternion h contained in the set $\{h \in \mathbb{H} \mid \operatorname{Re} h = \operatorname{Re} \mu, \|h\| = |\mu|\}$. This set contains the quaternionic roots of X and forms a two-dimensional sphere. Thus, if a quaternionic polynomial F has a quadratic factor of the form (6.13) then $h, \bar{h}$ are called *spherical roots* of F . We conclude that factorization of real polynomials over the quaternions can be highly non-unique. However, spherical roots are a special case which we can avoid if we assume a quaternionic polynomial to not have real polynomial factors.

To a general quaternionic polynomial F , we assign the *norm polynomial* $F\bar{F} \in \mathbb{R}[t]$. If F has a factorization of the form (6.11) then the norm polynomial is

$$F\bar{F} = \|a_n\|^2 X_n \cdots X_1 \quad (6.14)$$

where $X_i = (t - h_i)(t - \bar{h}_i) \in \mathbb{R}[t]$ is the norm polynomial of the linear factor $t - h_i$.

On the other hand, since the norm polynomial of a general polynomial F is real it can be factorized into n quadratic factors of the form (6.12). For this, note that every real root of $F\bar{F}$ must be a root of F and $\bar{F}$ and, hence, it is a double root. Every quadratic factor of the norm polynomial corresponds to a right factor of F :

Theorem 6.2.1 *Let F be a quaternionic polynomial with no real polynomial factors. Then, for every monic quadratic real factor X of the norm polynomial $F\bar{F}$ there exists a unique right factor $t - h$ of F with $X = (t - h)(t - \bar{h})$.*

Proof. Using polynomial division we know that there are polynomials Q, R such that $F = QX + R$ and $\deg R \leq 1$. We compute (using $X \in \mathbb{R}[t]$)

$$F\bar{F} = (QX + R)(\bar{X}\bar{Q} + \bar{R}) = (Q\bar{Q}X + Q\bar{R} + R\bar{Q})X + R\bar{R}. \quad (6.15)$$

Since, by assumption, X is a factor of $F\bar{F}$ it is also a factor of $R\bar{R}$ (unless $R = 0$ but this contradicts the assumption that F has no real factor). Therefore, we have $\deg R = 1$ and we can write $R = c(t - h)$ for some $c, h \in \mathbb{H}$. Overall, we have $R\bar{R} = c\bar{c}(t - h)(t - \bar{h}) = \|c\|^2 X$ and

$$F = QX + R = Q(t - \bar{h})(t - h) + c(t - h). \quad (6.16)$$

Therefore, $t - h$ is a right factor of F . For uniqueness, note that every common right factor $t - \hat{h}$ of F and X must be a right factor of R and, hence, must agree with $t - h$. $\square$

By iterative application of the theorem we can factorize the polynomial F into the form (6.11). There are $n!$ possible orders of the n quadratic factors X_i of $F\bar{F}$ and every order yields one factorization of F . If the quadratic factors X_i are different then the factorizations are different as well.

As we will motivate below, we are also interested in factorizations of the form

$$F = a_0(1 - th_n) \cdots (1 - th_1), \quad h_i \in \mathbb{H}. \quad (6.17)$$

Such factorizations are covered by Theorem 6.2.1 as well. In fact, either we can apply the result directly by using the observation that $1 - th$ is a right factor if and only if $t - h^{-1}$ is a right factor (for $h \neq 0$) or we can apply a reparametrization: A factorization of F into the form (6.17) corresponds to a factorization of $G(t) := t^n F(\frac{1}{t})$ into the form (6.11).

Now, we are interested in polynomials where all values h_i as in (6.17) lie in a common plane, by convention, the $\mathbf{i}, \mathbf{j}$ -plane. Then, the polynomial F belongs to a special sub-ring of the ring of quaternionic polynomials:

Definition 6.2.2 *We call a quaternionic polynomial special if all its odd coefficients are in the $\mathbf{i}, \mathbf{j}$ -plane while its even coefficients are in the $\mathbf{1}, \mathbf{k}$ -plane.*

An equivalent characterization for a polynomial F to be special is the condition $F(-t) = -\mathbf{k}F(t)\mathbf{k}$. We are concerned with factorization into the form (6.17) since a linear factor $1 - th$ can be special itself (if h is contained in the $\mathbf{i}, \mathbf{j}$ -plane). A factor of the form $t - h$ cannot be special. Factorization of special polynomials is explained in [68]:

Proposition 6.2.3 *A special quaternionic polynomial F with no real polynomial factors has a factorization of the form (6.17) into special linear factors if and only if all roots $\mu \in \mathbb{C}$ of the norm polynomial $F\bar{F} \in \mathbb{R}[t]$ lie on the imaginary axis.*

Proof. If F has such a factorization, then $F\bar{F}$ consists of quadratic factors of the form $X_i = 1 + t^2\|h_i\|^2$ and, thus, its roots $\mu_{1/2} = \pm i\frac{1}{\|h_i\|} \in i\mathbb{R} \subseteq \mathbb{C}$ are on the imaginary axis.

The converse follows from the fact, that our construction of a right factor remains in the sub-ring of special polynomials and, thus, by iterative application we can find a special factorization. For this, note that polynomial division of special polynomials produces a quotient and remainder which are special. Specifically, the proof of Theorem 6.2.1 states that a right factor $t - h^{-1}$ is obtained from each monic quadratic factor X of $F\bar{F}$ by applying polynomial division to obtain $F = GX + R$ with $R = c(t - h^{-1})$. Now, in our case the roots of X are imaginary and, hence, $X = t^2 + \frac{1}{\|h\|^2}$ is special. Then, $R = -ch^{-1}(1 - th)$ is special and, therefore, $1 - th$ is a special right factor of F . $\square$

6.3. Matrix polynomials

In this section, we present a factorization result for the complex matrix algebra $\mathbb{C}^{2 \times 2}$ which does not seem to appear in the literature. We, then, obtain the well-known Theorem 6.2.1 for quaternions as a consequence. There is one advantage: The right factor is given as an explicit formula (instead of as a result of polynomial division). This can be useful in application.

We call a polynomial $F = \sum_{k=0}^n t^k a_k$ with coefficients $a_k \in \mathbb{C}^{2 \times 2}$ a *matrix polynomial*. Equivalently, a matrix polynomial is a matrix $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with four polynomials over the complex numbers $A, B, C, D \in \mathbb{C}[t]$ as entries. Note that we, again, identify scalars $c \in \mathbb{C}$ with the corresponding multiple of the identity $c\mathbf{1} \in \mathbb{C}^{2 \times 2}$. In this manner, we also identify the indeterminate t for complex polynomials and matrix polynomials.

For a preliminary computation consider a matrix polynomial F with a right root $h \in \mathbb{C}^{2 \times 2}$ such that $F = G(t - h)$. Then,

$$F \operatorname{adj}(t - h) = G(t - h) \operatorname{adj}(t - h) = G \det(t - h). \quad (6.18)$$

The eigenvalues $\mu_1, \mu_2 \in \mathbb{C}$ of h are the roots of $\det(t - h)$ and by plugging in μ_i we obtain

$$(\mu_i - h) \operatorname{adj} F(\mu_i) = 0, \quad i = 1, 2. \quad (6.19)$$

Thus, non-vanishing columns of $\operatorname{adj} F(\mu_i)$ are eigenvectors of h . Each matrix $\operatorname{adj} F(\mu_i)$ yields at most one eigenvector (up to scaling) since $\det F(\mu_i) = \det G(\mu_i) \det(\mu_i - h) = 0$. If taking a column of each matrix determines a basis of eigenvectors Y we can write h in its diagonalized form

$$h = Y \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} Y^{-1} \quad (6.20)$$

We formulate the existence of such a basis for a general matrix polynomial F :

Definition 6.3.1 *Let F be a matrix polynomial. We call two roots $\mu_1, \mu_2 \in \mathbb{C}$ of $\det F$ independent if $\ker F(\mu_1) \cap \ker F(\mu_2) = \{(0, 0)\}$.*

Independence means that the four rows of the two matrices $F(\mu_1), F(\mu_2)$ are not all multiples of each other. Then, for two independent roots we find linear independent columns y_1 and y_2 of $\operatorname{adj} F(\mu_1)$ and $\operatorname{adj} F(\mu_2)$ respectively and the matrix $Y = (y_1 \ y_2)$ is invertible.

Remark 6.3.2 *Note that if $\mu_1 = \mu_2$ the pair is not independent and a corresponding right root of F cannot be reconstructed using (6.20). This includes all non-diagonalizable*

right roots. On the other hand, if such a right root is diagonalizable then it is a multiple of the identity and we have $F(\mu_1) = F(\mu_2) = 0$. Thus, the scalar $\mu_1 = \mu_2$ is not only a root of $\det F$ but also of F itself. In particular, if μ_i is a scalar root of F it is not independent of any other root μ_j of $\det F$.

Theorem 6.3.3 *Let F be a matrix polynomial with two independent roots $\mu_1, \mu_2 \in \mathbb{C}$ of $\det F$. Then, F has a unique right root h such that μ_1, μ_2 are the eigenvalues of h . For a corresponding basis Y it is given by (6.20).*

Proof. Uniqueness follows from the above argumentation since any right root h with independent eigenvalues μ_1, μ_2 is determined by (6.20). Note that since the columns of each $\text{adj } F(\mu_i)$ are linearly dependent the specific choice of the basis Y does not affect h .

For existence we need to show that (6.20) determines a right factor. We have the eigenvalue equations $hy_i = \mu_i y_i$ which immediately imply (6.19). Therefore, the polynomial $F \text{adj}(t - h)$ vanishes at μ_1, μ_2 and, hence, has $\det(t - h)$ as scalar factor. This defines a polynomial G with

$$F \text{adj}(t - h) = G \det(t - h) = G(t - h) \text{adj}(t - h) \quad (6.21)$$

and we conclude $F = G(t - h)$ which implies that h is a right root of F . $\square$

This theorem allows us to find right factors of the polynomial as long as the independence condition is met. We can factorize the polynomial into linear factors if the condition is met in every step. Under the (possibly strong) condition that every pair of roots is independent in every step of the factorization we have $\frac{(2n)!}{2^n}$ possible factorizations belonging to the possibilities of choosing the pairs of roots. In most applications we only consider refactorizations which respect the pairs of roots of $\det F$, in other words, which are obtained by reordering a prescribed set of pairs of roots. Then, as in the quaternionic case, there are $n!$ such factorizations.

For example, a generic quadratic polynomial F has a total of six factorizations, belonging to the different possibilities to choose pairs out of the four roots of its determinant $\det F$. If one such factorization is given by

$$F = a_2(t - h_2)(t - h_1) \quad (6.22)$$

then there is a unique refactorization

$$F = a_2(t - \hat{h}_2)(t - \hat{h}_1) \quad (6.23)$$

obtained by permuting the two pairs of roots of $\det F$. This means that $\hat{h}_1$ has the same eigenvalues as h_2 and that $\hat{h}_2$ has the same eigenvalues as h_1 . In fact, it can be explicitly

computed to be

$$\begin{aligned}\hat{h}_1 &= (h_2 - \text{adj } h_1)h_2(h_2 - \text{adj } h_1)^{-1}, \\ \hat{h}_2 &= (h_2 - \text{adj } h_1)h_1(h_2 - \text{adj } h_1)^{-1}.\end{aligned}\tag{6.24}$$

We will discuss the obvious connection to the evolution (4.16) in the next section.

By identifying quaternions with corresponding complex matrices as in (4.12) we can consider quaternionic polynomials to be a special case of matrix polynomials. We can write a quaternionic polynomial as

$$F = \begin{pmatrix} A - Di & -C - Bi \\ C - Bi & A + Di \end{pmatrix}\tag{6.25}$$

where $A, B, C, D \in \mathbb{R}[t]$ are real polynomials. Then, the determinant polynomial is the norm polynomial $\det F = F\bar{F} = A^2 + B^2 + C^2 + D^2$ and its roots come in pairs related by complex conjugation. Using Theorem 6.3.3, we give a second proof of Theorem 6.2.1. This proof implies that every right root of a quaternionic polynomial with no real polynomial factors can be calculated using (6.20).

Proof. [of Theorem 6.2.1] Each monic quadratic factor of $\det F$ is of the form (6.12) and we set $\mu_1 = \mu$ and $\mu_2 = \bar{\mu}$. We will show that these are independent roots of $\det F$ by finding a basis Y given by a column of $\text{adj } F(\mu)$ and $\text{adj } F(\bar{\mu})$ respectively. For this, note that

$$\text{adj } F = \begin{pmatrix} A - Di & -C - Bi \\ C - Bi & A + Di \end{pmatrix}.\tag{6.26}$$

Consider the matrices

$$\begin{pmatrix} A_0 - D_0i & -\bar{C}_0 - \bar{B}_0i \\ C_0 - B_0i & \bar{A}_0 + \bar{D}_0i \end{pmatrix}, \quad \begin{pmatrix} -C_0 - B_0i & -\bar{A}_0 + \bar{D}_0i \\ A_0 + D_0i & -\bar{C}_0 + \bar{B}_0i \end{pmatrix}\tag{6.27}$$

where $A_0 := A(\mu), B_0 := B(\mu), \dots$. The first matrix consists of the first column of $\text{adj } F(\mu)$ and the second column of $\text{adj } F(\bar{\mu})$ while the second matrix consists of the second column of $\text{adj } F(\mu)$ and the negative of the first column of $\text{adj } F(\bar{\mu})$. Both matrices are quaternions. Also, one of them does not vanish: If both vanish we have $A_0 = B_0 = C_0 = D_0 = 0$ and, therefore, $\mu, \bar{\mu}$ are scalar roots of F . Then, $X = (t - \mu)(t - \bar{\mu})$ is a real factor of F which contradicts our assumptions. We define Y to be this non-vanishing matrix. As a quaternion, it is invertible. Therefore, $\mu_1 = \mu$ and $\mu_2 = \bar{\mu}$ are independent and Theorem 6.3.3 gives a unique right factor $t - h$. Since h is given by (6.20) and Y is a quaternion the right root h is also a quaternion. $\square$

We conclude this section with an example showing how one can also consider non-quaternionic factorizations of quaternionic polynomials.

Example 6.3.4 *As a consequence of the two factorization theorems, we look at the quaternionic polynomial*

$$F := t^2 + t \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}. \quad (6.28)$$

We know that we can find a quaternionic factorization, e.g.,

$$F = (t - \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix})(t - \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}). \quad (6.29)$$

By a different pairing of the roots we can also find a factorization which is not quaternionic, e.g.,

$$\mathcal{P} = (t - \begin{pmatrix} 0 & 1+i \\ -1-i & 0 \end{pmatrix})(t - \begin{pmatrix} 0 & 1-i \\ -1+i & 0 \end{pmatrix}). \quad (6.30)$$

While this second factorization is not quaternionic it has the shape (5.28) of a Lax matrix of a cross-ratio system. We will present an application of these different factorizations in Section 7.3.

6.4. Skew parallelogram nets and universal factorization

Factorization of polynomials as a tool in discrete geometry has been used many times, for example, for polygon recutting [68], for discretization of the motion of a rigid body [80] or for finding closed linkages in kinematics [48]. Here, we relate our theory of parallelogram nets to factorization and sketch some applications. First, note that by setting $\lambda(t) = t$ and $\mu = -1$ the compatibility condition (3.18) of the Lax representation of a parallelogram net reads

$$(t - p_i^j)(t - p^i) = (t - p_j^i)(t - p^j). \quad (6.31)$$

The two sides correspond to two factorizations of the same quadratic polynomial. Similarly, on an n -dimensional cube there are $n!$ paths of consecutive edges $d_1, \dots, d_n \in \vec{\mathcal{E}}(\mathbb{Z}^n)$ connecting $(0, \dots, 0) \in \mathbb{Z}^n$ to $(1, \dots, 1) \in \mathbb{Z}^n$. If a parallelogram net p is defined on this cube then the product of linear factors $P^i = t - p^i$ along each path yields a factorization of the same polynomial

$$\Phi(1, \dots, 1)\Phi(\mathbf{0})^{-1} = (t - p(d_n)) \cdots (t - p(d_1)) \quad (6.32)$$

and this polynomial is the transport of the frame Φ . Over the quaternions these are exactly the $n!$ possible factorizations:

Corollary 6.4.1 *A quaternionic polynomial F of degree n with no real polynomial factors has $n!$ possible factorizations of the form (6.11). The linear factors can be assigned to the directed edges of an n -dimensional cube such that each path from $(0, \dots, 0)$ to $(1, \dots, 1)$ corresponds to one factorization of F and such that the corresponding roots h fulfill the parallelogram equations (3.1) and (3.2). If all roots $\mu \in \mathbb{C}$ of $\det F$ are distinct then all $n!$ factorizations are different and the parallelogram net defined on the cube is evolvable.*

An analogous statements holds for the factorization of generic matrix polynomials over $\mathbb{C}^{2 \times 2}$ if we only consider the $n!$ factorizations corresponding to a prescribed pairing of the roots of $\det F$. Thus, generally speaking, different paths on a parallelogram net correspond to different factorizations of a polynomial and vice versa.

Sometimes (for example in Sections 4.4, 4.5, Chapter 9), we are concerned with parallelogram nets in the quaternionic $\mathbf{i}, \mathbf{j}$ -plane and we have seen that the corresponding quaternionic polynomials are special in the sense of Definition 6.2.2. In [68] factorization of such polynomials is used to show that polygon recutting is integrable.

Let us consider other applications of this connection. First, (re-)factorization of polynomials gives us another perspective on the Cauchy problem. Specifically, the re-factorization (6.24) is completely analogous to the sideways evolution (4.16). Recall that sideways evolution depends on properties of the specific associative algebra. In fact, the sideways evolution corresponds exactly to re-factorization of quadratic polynomials in this algebra. Also, remember that a diagonalization as in (6.20) linearizes the evolution (3.3) as stated in Remark 3.1.3.

Second, note that the coefficients of a polynomial can have geometric interpretation. For example, as we saw in Section 4.6, a result from [87] states that the t^{n-2} -coefficient corresponding to a closed curve is the area-vector of the curve. In Chapter 8 and 9, we will see that one can assign a polynomial to a special class of curves which describes a flow of the curve. Factorization of the polynomial then gives a discrete version of this flow as a sequence of Bäcklund transformations.

Finally, consider a polynomial Lax pair, i.e., a pair L, M of polynomials in a variable t where L is assigned to edges $(x, x + e_1) \in \tilde{\mathcal{E}}(\mathbb{Z}^2)$ and M is assigned to edges $(x, x + e_2) \in \tilde{\mathcal{E}}(\mathbb{Z}^2)$. This Lax pair is supposed to fulfill the compatibility condition $L_2 M = M_1 L$ on every quad. Many integrable systems have a description in terms of such a pair (see [24]) and we will explore some of them in Chapter 7. Now, let us assume that $\deg L = n$, $\deg M = m$ and that the polynomials are quaternionic such that $F := L_2 M = M_1 L$ has no real polynomial factors. Then, each polynomial L (or M) defines an n -dimensional (or m -dimensional) cube of a parallelogram net. On each quad the polynomial F defines an $(n + m)$ -dimensional parallelogram cube. In fact, the whole system is contained in a two-dimensional sub-lattice of an $(n + m)$ -dimensional parallelogram net. Thus, our theory introduced in Chapter 3 can be applied and yields a notion of multidimensional-consistency, an associated family and results for quasi-periodic nets.

Generically, this concept can be extended to polynomial Lax pairs in other algebras in which similar factorization results exist (like $\mathbb{C}^{2 \times 2}$). However, in non-generic cases factorization may be impossible. For example, if a polynomial has a factorization of the form (6.11) with a non-invertible leading coefficient a_n then all of its coefficients a_i are non-invertible. Hence, by contraposition, if a polynomial has a non-invertible leading coefficient a_n while some other coefficient a_i is invertible then it does not have such a factorization. Such a polynomial is given by the transition matrix for minimal surfaces in [58] which is a special case of the CMC transport matrix (which we will discuss in Section 7.4). Thus, we cannot find a factorization which relates minimal surfaces to parallelogram nets as special case of the CMC net factorization presented in Chapter 7. Sometimes an issue of this type can be repaired by a reparametrization of the polynomial (as we have seen in Section 5.7). Possible relations of minimal surfaces to parallelogram nets (using, e.g., reparametrization) remain the subject of future research.

In Chapter 7 we will present explicit factorizations of some examples of surfaces in which L and M are quadratic matrix polynomials. To every edge we will associate a quad of a parallelogram net and to every quad of the surface we will associate a 4-dimensional cube.

We conclude this chapter with the remark that the idea to factorize polynomial systems (appearing in Discrete Differential Geometry) into one unifying linear system seems to be new. The following chapters of this thesis all present polynomial systems factorized into skew parallelogram nets and can, thus, be interpreted as application of this universal factorization.

7. Discrete surfaces of constant curvature

In this chapter, we discuss how several discrete surface classes of constant curvature can be described by parallelogram nets. We have already explored this for K-nets in Section 4.4. Here, the surface classes will be, in a sense, of higher order: Edges of the surface will correspond to diagonals in our parallelogram nets. The corresponding Lax representation is quadratic since it arises as product of two linear factors (3.19). This leads to more involved combinatorics of the parallelogram nets: In Sections 7.2 - 7.4 we find surfaces described by slices of 4D cross-ratio systems and in Section 7.5 we see how two-layered three-dimensional parallelogram nets give rise to pairs of surfaces after a combinatoric flip.

7.1. Definitions

First, we review some well-known types of surfaces of constant curvature. Note that there is not one unique way of defining curvature for a discrete quad net. However, for nets with special properties discrete curvatures have been defined (for example, in [97, 18]). We will use the theory of *edge-constraint nets* introduced in [62] since it encapsulates all surfaces considered in this chapter. For a short summary we follow [60, Definition 9]:

Definition 7.1.1 *The pair (f, n) given by two maps $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ and $n : \mathbb{Z}^2 \rightarrow \mathbb{S}^2$ is called an edge-constraint net if the values along each edge fulfill*

$$f_i - f \perp n_i + n. \quad (7.1)$$

We call n the Gauß map or (vertex) normal map of f . A face normal is a unit vector $N \perp \text{span}_{\mathbb{R}}\{n_{12} - n, n_2 - n_1\}$. For a quad with face normal N we define the Gauß curvature K by

$$K := \frac{\det(n_{12} - n, n_2 - n_1, N)}{\det(f_{12} - f, f_2 - f_1, N)} \quad (7.2)$$

and the mean curvature H by

$$H := \frac{1}{2} \frac{\det(f_{12} - f, n_2 - n_1, N) + \det(n_{12} - n, f_2 - f_1, N)}{\det(f_{12} - f, f_2 - f_1, N)} \quad (7.3)$$

The curvatures are invariant under the choice of N .

It is known (see [62, Theorem 3.37]) that K-nets with their (flipped) normal map have constant negative Gauß curvature $K < 0$ if their edge-lengths $\|f_1 - f\|$ and $\|f_2 - f\|$ are constant. Due to having planar vertex stars, K-nets are considered to be in asymptotic line parametrization.

On the other hand, circular nets (as in Definition 2.2.1) are considered to be discrete surfaces in curvature line parametrization. They are naturally equipped with an edge-parallel Gauß map: Consider a normal map n such that neighbouring normals along an edge n and n_i are related by a reflection in the perpendicular bisector plane of the edge (f, f_i) . For circular nets, such a Gauß map is well-defined for any initial normal $n(\mathbf{0}) \in \mathbb{S}^2$. Then, the pair (f, n) is edge-constraint (due to the symmetry) and the maps f and n are edge-parallel, i.e., they fulfill $n_i - n \parallel f_i - f$.

We can define a discrete analogue of surfaces of constant negative Gauß curvature in curvature line parametrization:

Definition 7.1.2 *A circular net (with a corresponding normal map n) is called a circular K-net (short: cK-net) if every quad has the same negative Gauß curvature $K < 0$.*

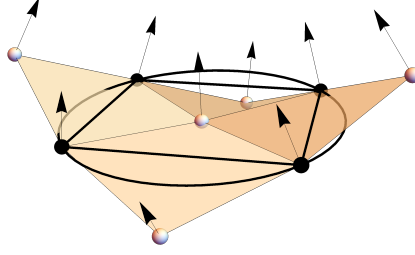


Figure 7.1.: A quad of a cK-net given by diagonals of a rhombic K-net.

Example 7.1.3 *We follow [60, Example 2]: Let $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ be a rhombic K-net, i.e., a K-net for which all edge-lengths coincide. Now, consider a partition of $\mathbb{Z}^2$ into black and white vertices such that each edge connects a black and a white vertex. If we connect the black (or white) vertices by diagonals of the K-net they form an new lattice which yields a cK-net: As exemplified in Figure 7.1, the four black vertices of a quad are neighbours to a common white vertex in the K-net. Now, the property of planar vertex stars implies that this quad is planar. Since the K-net is rhombic all black vertices have the same distance to the white vertices and, thus, the quad is also circular. The (flipped) K-net normals at the black vertices are circular normals: Neighbouring normals are connected by a diagonal of the K-net and due to the symmetry of a skew rhombus they are mirrored along this diagonal. One can calculate that the resulting edge-constraint net has constant curvature $K < 0$.*

We find cK-nets as special parallelogram nets in Section 7.3.

We will also be interested in CMC-nets. Here, we give a geometric definition (see [17, 33]). For this, recall the definition of isothermic nets in Section 5.4.

Definition 7.1.4 *An isothermic net f is called a constant mean curvature net (short: CMC-net) if it has an isothermic dual f^* at constant distance to f .*

Thus, there is a constant $H \in \mathbb{R} \setminus \{0\}$ such that $\|f^* - f\| = \frac{1}{H}$. CMC nets are circular and the natural corresponding normal map is given by $n = H(f^* - f)$. As indicated by the notation, the constant H is the mean curvature (7.3) of the surface. The theory of discrete CMC surfaces also encompasses circular nets with constant positive Gaussian curvature: For every pair of primal and dual CMC surfaces the middle surface $\frac{1}{2}(f + f^*)$ has constant Gaussian curvature $K > 0$. Thus, these two types of constant curvature surfaces are described within one system of equations.

Finally, note that the definition of CMC nets naturally extends to spherical and hyperbolic geometry (see [10, 32, 84]). If we represent points in $\mathbb{S}^3$ by quaternions as in Section 4.7 and points in $\mathbb{H}^3$ by Hermitian matrices as in Section 5.2 we can compute the cross-ratio of four points via

$$cr(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1}. \quad (7.4)$$

Analogous to Section 5.4, an isothermic net $f : \mathbb{Z}^2 \rightarrow \mathbb{S}^3$ (or $\mathbb{H}^3$) is a net with factorizing cross-ratio $cr(f, f_1, f_{12}, f_2) = \frac{(\alpha^1)^2}{(\alpha^2)^2}$. Then, Definition 7.1.4 extends to non-Euclidean space forms: The net f is called a CMC net if it has a dual $f^* : \mathbb{Z}^2 \rightarrow \mathbb{S}^3$ (or $\mathbb{H}^3$) (in the sense of (5.37)) at constant distance of f . Here, the distance between f and f^* is encoded in the quantity $\langle f, f^* \rangle_{\mathbb{C}}$ and, thus, this quantity is constant for CMC-nets. If it vanishes one can consider f to be a minimal net with normal $n := f^*$. A corresponding curvature theory can be developed.

In Sections 7.4 and 7.5 we will see that CMC nets arise from parallelogram nets in two different ways.

7.2. Surfaces from 4D cross-ratio systems

We will now turn to surfaces that appear in the associated family of cross-ratio systems. In Section 5.3, we saw that computing the associated family p_t of the parallelogram net p associated to a cross-ratio system (5.28) yields a parallelogram net of complex matrices. We argued that one can interpret its additive primitive map f_t as complexified version of a K-net with a corresponding normal map (4.53). We discussed that actual K-nets correspond to the case where p is quaternionic. In this case, all points and all normals of the associated family are quaternions as well.

In this section we introduce two reductions of the cross-ratio system that, at some points, give quaternions in the associated family. To the knowledge of the author, these reductions are new. They are motivated by the factorization of polynomials corresponding to surfaces as will be explained in Section 7.3 and 7.4.

The combinatorics of our system become more involved: We consider a 4D cross-ratio system $z : \mathbb{Z}^4 \rightarrow \mathbb{C}$. The vertices of the form $(i, j, i, j) \in \mathbb{Z}^4$ form a new $\mathbb{Z}^2$, which we call the *diagonal net* $D \subset \mathbb{Z}^4$. Naturally, as shown in Figure 7.2, on its edges live the Lax matrices

$$\begin{aligned} L &= \left(\lambda + \frac{1}{\lambda} p_1^3\right) \left(\lambda + \frac{1}{\lambda} p^1\right), \\ M &= \left(\lambda + \frac{1}{\lambda} p_2^4\right) \left(\lambda + \frac{1}{\lambda} p^2\right). \end{aligned} \quad (7.5)$$

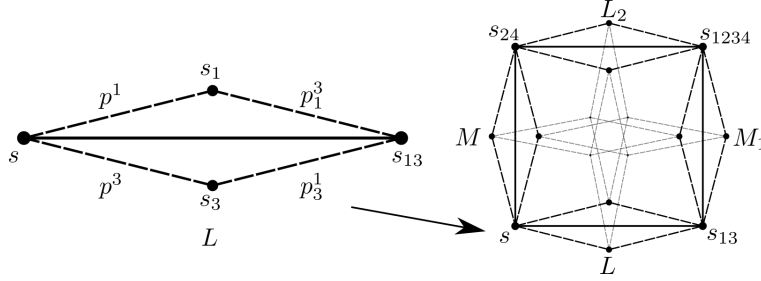


Figure 7.2.: A combinatoric 4D cross-ratio cube with the Lax representation L, M of its diagonal net D .

We introduce two cases where the additive primitive map f_t of the associated family will turn out to be quaternionic for points of the diagonal net, i.e., where $f_t(x) \in \mathbb{H}$ for $x \in D$.

Definition 7.2.1 *Let $z : \mathbb{Z}^4 \rightarrow \mathbb{C}$ be a cross-ratio system with edge labelling α such that the corresponding Hirota system s fulfills*

$$s_1(x) \bar{s}_3(x) = 1, \quad s_2(x) \bar{s}_4(x) = 1 \quad (7.6)$$

for every $x \in D$. We call f

- a C^+ lattice if for all $x \in D$

$$\alpha^1(x) \bar{\alpha}^3(x) = -1, \quad \alpha^2(x) \bar{\alpha}^4(x) = -1. \quad (7.7)$$

- a C^- lattice if for all $x \in D$

$$\alpha^1(x) = \bar{\alpha}^3(x), \quad \alpha^2(x) = \bar{\alpha}^4(x). \quad (7.8)$$

We will see that these C^+ and C^- lattices correspond to discrete surfaces of constant positive and negative Gaussian curvature respectively.

We will use the following important property of the Hirota system s :

Lemma 7.2.2 *On a C^+ lattice we have real $s(x) \in \mathbb{R}$ for all $x \in D$. On a C^- lattice we have unitary $s(x) \in \mathbb{S}^1$ for all $x \in D$.*

Proof. We use (5.26) for a C^+ lattice and its first diagonal direction: If $s \in \mathbb{R}$, we have

$$\bar{s}_{13} = \bar{s} \frac{\bar{\alpha}^1 \bar{s}_1 - \bar{\alpha}^3 \bar{s}_3}{\bar{\alpha}^1 \bar{s}_3 - \bar{\alpha}^3 \bar{s}_1} = s \frac{\frac{-1}{\alpha^3 s_3} - \frac{-1}{\alpha^1 s_1}}{\frac{-1}{\alpha^3 s_1} - \frac{-1}{\alpha^1 s_3}} = s \frac{\alpha^1 s_1 - \alpha^3 s_3}{\alpha^1 s_3 - \alpha^3 s_1} = s_{13}. \quad (7.9)$$

and, hence, $s_{13} \in \mathbb{R}$. The second direction works analogously and, therefore, from $s(\mathbf{0}) = 1$ the claim follows for all $x \in D$.

Similarly, using (5.26) for a C^- lattice and $s \in \mathbb{S}^1$ yields

$$\bar{s}_{13} = \bar{s} \frac{\bar{\alpha}^1 \bar{s}_1 - \bar{\alpha}^3 \bar{s}_3}{\bar{\alpha}^1 \bar{s}_3 - \bar{\alpha}^3 \bar{s}_1} = \frac{1}{s} \frac{\frac{\alpha^3}{s_3} - \frac{\alpha^1}{s_1}}{\frac{\alpha^3}{s_1} - \frac{\alpha^1}{s_3}} = \frac{1}{s} \frac{\alpha^3 s_1 - \alpha^1 s_3}{\alpha^3 s_3 - \alpha^1 s_1} = \frac{1}{s_{13}} \quad (7.10)$$

and, hence, $s_{13} \in \mathbb{S}^1$ and the claim follows with the same argumentation as for the C^+ lattice. $\square$

So far, we do not know that examples of C^+ and C^- lattices actually exist. With the next proposition we solve this problem by showing how to construct such lattices from initial data. For this, we will call the set

$$\{x \in \mathbb{Z}^4 \mid x_k = 0 \text{ for } k \neq i, j\} \subseteq \mathbb{Z}^4 \quad (7.11)$$

for $i \neq j \in [n]$ the (i, j) -coordinate plane. Our result will imply that the C^+ and C^- lattice reductions are consistent in the following sense: If the conditions (7.6) and (7.7) (or (7.8)) hold on the $(1, 3)$ - and the $(2, 4)$ -coordinate plane then they hold on the whole lattice. Note that these coordinate planes correspond to the coordinate axes of the diagonal lattice.

Proposition 7.2.3 *Every two-dimensional cross-ratio system can be extended into a unique C^+ and a unique C^- lattice.*

Proof. Consider a two-dimensional cross-ratio system $z(i, j, 0, 0)$ in the $(1, 2)$ -coordinate plane with labelling α^1, α^2 and Hirota system s . We will construct a C^- lattice from this initial data. First, the labelling property of α together with (7.8) determines α on the whole net: For example, for α^3 we have

$$\alpha^3(i, j, k, l) = \alpha^3(k, 0, k, 0) = \bar{\alpha}^1(k, 0, k, 0) = \bar{\alpha}^1(k, 0, 0, 0). \quad (7.12)$$

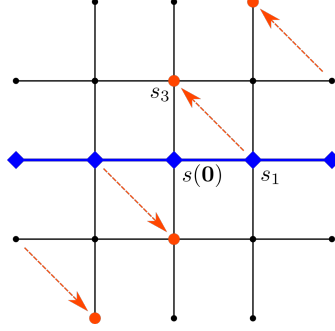


Figure 7.3.: In the $(1,3)$ -coordinate plane one can construct the map s only from $s(i, 0, 0, 0)$ (blue) by using (7.6) as indicated by the arrows (orange) and (5.26) to complete each quad.

Next, equations (7.6) and (5.26) determine s on the $(1,3)$ -coordinate plane, as seen in Figure 7.3, and, analogously, on the $(2,4)$ -coordinate plane. Then, s is determined on all four axes of $\mathbb{Z}^4$ and can be evolved to the whole 4D lattice. Now, the corresponding 4D cross-ratio system z is determined via (5.25) by α , s and $z(\mathbf{0})$.

Finally, we need to prove consistency of (7.6). Consider a point $x \in D$ where

$$s\bar{s} = 1, \quad s_3\bar{s}_1 = 1, \quad s_4\bar{s}_2 = 1. \quad (7.13)$$

Then, one step of (5.26), using the available replacement rules just as in (7.9), yields

$$s_{13}\bar{s}_{13} = s_{34}\bar{s}_{12} = s_{14}\bar{s}_{23} = 1, \quad (7.14)$$

and a second step yields

$$s_{123}\bar{s}_{134} = s_{124}\bar{s}_{234} = 1. \quad (7.15)$$

Hence, (7.6) is consistent.

The construction of a C^+ lattice and its consistency proof work exactly the same using (7.7) in place of (7.8). Here, after one step we have

$$s_{13} = \bar{s}_{13}, \quad s_{34} = \bar{s}_{12}, \quad s_{14} = \bar{s}_{23}. \quad (7.16)$$

□

Now, we prove that we can indeed get real surfaces in the associated family (3.23) for special functions λ . For this, we show that L, M as in (7.5) and their derivatives L', M' with respect to t are quaternions, possibly after a gauge. For $\Phi(\mathbf{0}), f_t(\mathbf{0}) \in \mathbb{H}$ this implies that $\Phi(x)$ and $f_t(x)$ are quaternions for all $x \in D$ since $\Phi_{13} = L\Phi$ and

$$(f_t)_{13} - f_t = \Phi_{13}^{-1}\Phi'_{13} - \Phi^{-1}\Phi' = \Phi_{13}^{-1}(L\Phi' + L'\Phi - L\Phi') = \Phi_{13}^{-1}L'\Phi \in \mathbb{H}. \quad (7.17)$$

Here, we only consider the first lattice direction of the diagonal net since both directions work the same. Consequently, we will only investigate L and L' . For this, we compute

$$L = \begin{pmatrix} \lambda & \frac{\alpha^3 s_{13} s_1}{\lambda} \\ \frac{-\alpha^3}{\lambda s_{13} s_1} & \lambda \end{pmatrix} \begin{pmatrix} \lambda & \frac{\alpha^1 s_1 s}{\lambda} \\ \frac{-\alpha^1}{\lambda s_1 s} & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 - \frac{1}{\lambda^2} \alpha^3 \alpha^1 \frac{s_{13}}{s} & s_1(\alpha^1 s + \alpha^3 s_{13}) \\ -\frac{1}{s_1} (\frac{\alpha^3}{s_{13}} + \frac{\alpha^1}{s}) & \lambda^2 - \frac{1}{\lambda^2} \alpha^3 \alpha^1 \frac{s}{s_{13}} \end{pmatrix}. \quad (7.18)$$

Also, note that (5.26) implies

$$s_1(\alpha^1 s + \alpha^3 s_{13}) = s_3(\alpha^3 s + \alpha^1 s_{13}). \quad (7.19)$$

Proposition 7.2.4 *In a C^- lattice the points $f_t(x)$ for $x \in D$ on the diagonal net of the associated family with $\lambda = e^t \in \mathbb{R}$ are quaternions.*

Proof. Here, L as in (7.18) is a quaternion. This can be seen using (7.6), (7.8), $s, s_{13} \in \mathbb{S}^1$ and, for the off-diagonal entries, (7.19). Also, if L is a quaternion for all t , so is its derivative L' . $\square$

Proposition 7.2.5 *In a C^+ lattice the points $f_t(x)$ for $x \in D$ on the diagonal net of the associated family with $\lambda = e^{it} \in \mathbb{S}^1$ are quaternions.*

Proof. A gauge $\tilde{\Phi} := G\Phi$ of the frame Φ defined by L and M as in (7.18) with $G := \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$ and multiplication with $\frac{1}{\beta}$ defined by $\beta^2 := -\alpha^1 \alpha^3 \in \mathbb{S}^1$ does not affect the compatibility condition or the associated family. We get

$$\tilde{L} := \frac{1}{\beta} G_{13} L G^{-1} = \begin{pmatrix} \frac{\lambda^2}{\beta} + \frac{\beta}{\lambda^2} \frac{s_{13}}{s} & \frac{s_1}{\beta s} (\alpha^1 s + \alpha^3 s_{13}) \\ -\frac{s_{13}}{\beta s_1} (\frac{\alpha^3}{s_{13}} + \frac{\alpha^1}{s}) & \frac{\lambda^2}{\beta} \frac{s_{13}}{s} + \frac{\beta}{\lambda^2} \end{pmatrix}. \quad (7.20)$$

We see that this is a quaternion using $s, s_{13} \in \mathbb{R}, \lambda, \beta \in \mathbb{S}^1$ and, for the off-diagonal entries, (7.19). Again, its derivative $\tilde{L}'$ is also a quaternion. $\square$

Thus, in both cases the projected additive primitive map $\vec{f}_t : D \rightarrow \text{Im } \mathbb{H} \cong \mathbb{R}^3$ can be interpreted as a discrete surface. The natural normal map $n_t := \Phi^{-1} \mathbf{k} \Phi$ as in (4.53) also maps to the quaternions $n_t : D \rightarrow \mathbb{S}^2$: This is clear for the C^- lattice since Φ is a quaternion. For the C^+ lattice this follows from the fact that the gauge $\tilde{\Phi} = G\Phi$ does not affect n_t since G commutes with $\mathbf{k}$ (both are diagonal matrices). In both cases, an even better candidate for a geometric normal is obtained if we first apply another gauge $\Phi(i, j, i, j) \mapsto \mathbf{i}^{i+j} \Phi(i, j, i, j)$: This yields - similar to K-nets - a flipped normal $n_t(i, j, i, i) \mapsto (-1)^{i+j} n_t(i, j, i, i)$. Under this gauge the transport matrix (7.18) becomes independent of λ in its diagonal entries. Then, [62, Theorem 5.1] shows that f_t with this new normal map is edge-constraint.

In the next sections, we explore the surfaces obtained from this construction.

7.3. Circular nets with constant negative Gaussian curvature

Now, we will see how cK-nets can arise as special parallelogram nets. They will turn out to be of two types: The *short edge* case which will correspond to a special 4D K-net (as discussed in [60]) and the *long edge* case which will correspond to the C^- lattice. We will need to prove gauge equivalence of our Lax representation with the known Lax representation for cK-nets. The latter is given in [62, 60] by

$$\begin{aligned} L &= \begin{pmatrix} \frac{1}{t_{(1)}} \frac{l}{r} + t_{(1)} l r_1 & i(\lambda - \frac{r r_1}{\lambda}) \\ i(\lambda - \frac{1}{\lambda r r_1}) & \frac{1}{t_{(1)}} \frac{r}{l} + t_{(1)} \frac{1}{l r_1} \end{pmatrix}, \\ M &= \begin{pmatrix} \frac{1}{t_{(2)}} \frac{m}{r} + t_{(2)} m r_2 & i(\lambda - \frac{r r_2}{\lambda}) \\ i(\lambda - \frac{1}{\lambda r r_2}) & \frac{1}{t_{(2)}} \frac{r}{m} + t_{(2)} \frac{1}{m r_2} \end{pmatrix}, \end{aligned} \quad (7.21)$$

where $r \in \mathbb{S}^1$ is a function on vertices, $t_{(1)}, t_{(2)} \in \mathbb{C}$ are edge labellings which are either real or unitary and where $l, m \in \mathbb{C}$ are functions on edges such that

$$\sqrt{\frac{\cos(\arg r_i - \arg t_{(i)}) + \cos(\arg r - \arg t_{(i)})}{\cos(\arg r_i + \arg t_{(i)}) + \cos(\arg r + \arg t_{(i)})}} \quad (7.22)$$

is the absolute value of l and m for $i = 1, 2$ respectively. If L and M fulfill $M_1 L = L_2 M$ they define a frame Φ by $\Phi_1 = L\Phi$, $\Phi_2 = M\Phi$. This defines a family of nets and corresponding normals by

$$f_t = \Phi^{-1} \Phi', \quad n_t = \Phi^{-1} \mathbf{k} \Phi, \quad \lambda = e^t. \quad (7.23)$$

Then, f_0 is a cK-net and the members of its family f_t still have constant negative Gaussian curvature. All cK-nets have such a description.

This algebraic description corresponds to a special 4D cross-ratio system:

Proposition 7.3.1 *The Lax representations (7.21) for cK-nets and (7.5) for 4D cross-ratio systems are gauge equivalent where, specifically,*

1. *the case of real $t_{(i)}$ corresponds to a cross-ratio system with unitary $s \in \mathbb{S}^1$, real $\alpha^i \in \mathbb{R}$ and where for all $x \in D$*

$$\alpha^3(x) = \frac{1}{\alpha^1(x)}, \quad \alpha^4(x) = \frac{1}{\alpha^2(x)} \quad (7.24)$$

2. *the case of unitary $t_{(i)}$ corresponds to a C^- -lattice with unitary α^i .*

Proof. We only consider one lattice direction, since both directions are analogous. Gauging the Lax matrix L in (7.18) with $G(i, j, i, j) = (-i)^{i+j}$ and $\lambda^2 \rightarrow \lambda$ yields that $G_{13}LG^{-1}$ is of the form (7.21) with

$$t_{(1)} = \frac{1}{\alpha^1} = \alpha^3, \quad r = \begin{cases} s, & \text{for even } i+j, \\ \frac{1}{s}, & \text{for odd } i+j \end{cases}, \quad l = \begin{cases} \frac{1}{is_1}, & \text{for even } i+j, \\ is_1, & \text{for odd } i+j \end{cases}. \quad (7.25)$$

Note that (7.22) is equivalent to (5.26) for the cross-ratio quad corresponding to the cK-net edge. Now, the case of real $t_{(i)}$ corresponds to $\alpha^i \in \mathbb{R}$, $s \in \mathbb{S}^1$ which is the case of a product of K-net Lax matrices. The case of unitary $t_{(i)}$ corresponds to a C^- lattice since (7.8) is fulfilled.

For the converse, we can apply the same gauge backwards. $\square$

The first case corresponds to a 4D K-net where the additional condition (7.24) means that the skew parallelogram corresponding to each edge of the cK-net is a rhombus (compare this to rhombic K-nets as in Example 4.4.6). Recall that (up to global rescaling) the parallelogram net p corresponding to a K-net $\vec{f}$ is equally folded, i.e., $\|\vec{p}^i\| = 1$. This induces an upper bound $\|\vec{f}_i - \vec{f}\| = \|\vec{p}^i\| \leq 1$ on the edge-lengths of the K-net and, hence, an upper bound $\|\vec{f}_{ij} - \vec{f}\| \leq 2$ on each diagonal. Edges of the cK-net correspond to diagonals of the K-net and, therefore, the edge-lengths of the cK-net are bounded and we call this case the *short edge* case.

On the other hand, the edge-lengths in the second case exceed this bound and we call it the *long edge* case.

The proposition implies that the associated family (3.23) of the 4D cross-ratio system agrees with the cK-net family and, hence, we found a description of cK-nets in terms of cross-ratio systems (and, thus, parallelogram nets).

Remark 7.3.2 *Note that we only covered cK-nets where all edges belong to the first case or where all edges belong to the second case. The statement naturally extends to mixed cK-nets as well. Such a cK-net corresponds to a cross-ratio system where the corresponding reductions are mixed. Such a mixed reduction is still consistent with the cross-ratio evolution (as, e.g., given for the Hirota system by (5.26)). With a calculation similar to the proof of Proposition 7.2.3 one can show that the short edge reduction stays preserved under evolution along a long edge and vice versa.*

It shall be noted that numerical experiments suggest that cK-nets with long edges do not have a smooth limit to a surface of constant Gaussian curvature. However, they can be used for the construction of special Bäcklund transformations, so called *breather* solutions (see [96, 60] or, for a treatment in the continuous setting, [89]). Every edge of such a cK-net yields one breather transformation and Proposition 7.2.3 allows us to construct lattices with such edges from given 2D cross-ratio systems. This means that we can construct lattices of surfaces where neighboring surfaces are related by breather

transformations. Specifically, application of this lattice of transformations to a K-net (or cK-net) yields a 6D (or 8D) cross-ratio system. In particular, for its construction we use the mixed consistency explained in Remark 7.3.2. Figure 7.4 on the left shows a lattice of transformations of the straight line generated by the square grid as discrete holomorphic map.

Remark 7.3.3 *We can find an equation on our lattice which is remarkably similar to the sine-Gordon equation for discrete K-nets given in [24, Theorem 4.19]. For this, we assign to a 4D cross-ratio system a map h based on edges of the diagonal net by $h^{13} = \frac{s_3}{s_1}$ and $h^{24} = \frac{s_{24}}{s}$. Here, h^{ij} denotes the value at an edge $(x, x + e_i + e_j)$ of the diagonal net. Now, (5.26) yields*

$$\frac{s_{13}}{s} = \frac{\alpha^3 h^{13} - \alpha^1}{\alpha^3 - \alpha^1 h^{13}} \quad (7.26)$$

and with $\kappa := \frac{\alpha^1}{\alpha^3}$ the equation $\frac{s_{1234}}{s_{13}} \frac{s_{13}}{s} = \frac{s_{1234}}{s_{24}} \frac{s_{24}}{s}$ can be written in terms of h as

$$h_{13}^{24} \frac{h^{13} - \kappa}{1 - \kappa h^{13}} = h^{24} \frac{h_{24}^{13} - \kappa}{1 - \kappa h_{24}^{13}}. \quad (7.27)$$

In the case of a 4D K-net we have real $\kappa \in \mathbb{R}$ and unitary $h \in \mathbb{S}^1$. If we write $e^{i\varphi^{13}} = h^{13}$ and $e^{i\varphi^{24}} = h^{24}$ the equation can be written as

$$e^{i(\varphi_{13}^{24} + \varphi^{13} - \varphi^{24} - \varphi_{24}^{13})} = \frac{1 - \kappa e^{i\varphi^{13}}}{1 - \kappa e^{-i\varphi^{13}}} \frac{1 - \kappa e^{-i\varphi_{24}^{13}}}{1 - \kappa e^{i\varphi_{24}^{13}}}. \quad (7.28)$$

While this equation is clearly related to the sine-Gordon equation, it is, in a sense, weaker. Indeed, the sine-Gordon equation is one quad based equation for a vertex based map and, as such, determines the values of this map from initial data on the axes. On the other hand, (7.28) is one quad based equation for the edge based map h and, as such, does not determine the values of h from initial data on the axes.

Finally, note that our results can be understood as application of factorization as discussed in Chapter 6. However, for this we did not use a factorization theorem since the factors are explicitly given as Lax matrices of cross-ratio systems. Still, from the perspective of factorization we can make an interesting observation: For this, note that in the C^- lattice the Lax matrix L as in (7.18) is a quaternionic polynomial in the real parameter λ . Besides the factorization into cross-ratio edges presented here, we know from Theorem 6.2.1 that there must also be a quaternionic factorization of L . We saw the phenomenon of these different factorizations in Example 6.3.4. An implication is that between a surface and its breather transformation there exist two real surfaces belonging to the two possible quaternionic factorizations. These surfaces are additive

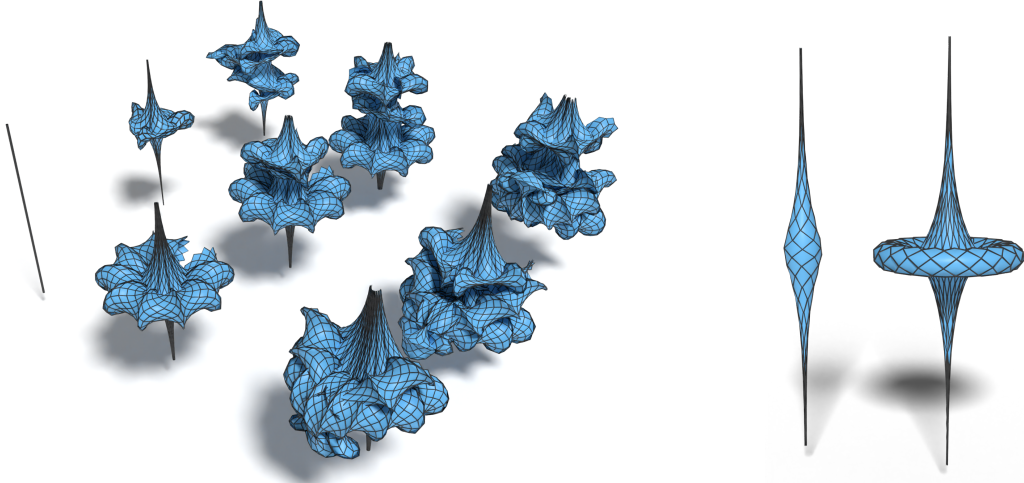


Figure 7.4.: Left: A lattice of breather transformations. Right: Factors of a breather transformation.

primitive maps of quaternionic parallelogram nets (and, as such, Chebyshev nets) and further geometric properties are still being investigated. As far as we know, this is the first time one has been able to factorize a breather transformation into real solutions, albeit with different surface geometry. In Figure 7.4 on the right we see the two real surfaces that combinatorially lie between the straight line and a breather transform.

7.4. CMC nets and a DPW method

Similar to the previous section, we will prove gauge equivalence of our Lax representation for the C^+ lattice to a known Lax representation of CMC nets. Again, this implies that a C^+ lattice represents a CMC net and that the associated families of C^+ lattices and CMC nets coincide. The Lax representation for CMC as in [17] is given by the matrices

$$L = \begin{pmatrix} a & \lambda b + \frac{1}{\lambda b} \\ -\frac{\bar{b}}{\lambda} - \frac{\lambda}{b} & \bar{a} \end{pmatrix}, \quad M = \begin{pmatrix} d & \lambda e + \frac{1}{\lambda e} \\ -\frac{\bar{e}}{\lambda} - \frac{\lambda}{e} & \bar{d} \end{pmatrix}, \quad \lambda = e^{it}. \quad (7.29)$$

If they fulfill the compatibility condition $M_1 L = L_2 M$ they define a frame Φ by $\Phi_1 = L\Phi$, $\Phi_2 = M\Phi$ and a family of nets $f_t = \Phi^{-1}\Phi'$ with normals $n_t = \Phi^{-1}\mathbf{k}\Phi$. Then, the net f_0 has constant positive Gaussian curvature and $f_0 \pm \frac{1}{2}n_0$ is a pair of primal and dual CMC nets in isothermic parametrization. For $t \neq 0$ they are still nets of constant curvature in different parametrizations.

Proposition 7.4.1 *The Lax representations (7.29) for CMC and positive Gaussian curvature nets and (7.5) for the C^+ lattice are gauge equivalent.*

Proof. Again, we only consider one lattice direction as both work the same. A gauge of the C^+ lattice matrix $\tilde{L}$ as in (7.20) with $G(i, j, i, j) = \mathbf{j}^{i+j}$ and $\lambda^2 \rightarrow \lambda$ yields that $\sqrt{\frac{s}{s_{13}}} G_{13} \tilde{L} G^{-1}$ is of the form (7.29) with

$$b = \frac{1}{\beta \sqrt{\hat{s}_{13} \hat{s}}}, \quad a = -\frac{\sqrt{\hat{s} \hat{s}_1}}{\sqrt{\hat{s}_{13} \beta}} \left(\alpha^3 \hat{s}_{13} + \frac{\alpha^1}{\hat{s}} \right), \quad \hat{s} := \begin{cases} s, & i+j \text{ even} \\ \frac{1}{s}, & i+j \text{ odd} \end{cases} \quad (7.30)$$

Conversely, from a solution to (7.29) we can recover α from the fact that $\pm i\alpha^1, \pm i\alpha^3$ are the zeroes of $\det L$. They can be chosen such that (7.7) holds. The values of s can be recovered from (7.30) and yield a C^+ lattice. $\square$

Remark 7.4.2 *As a continuation of Remark 7.3.3, observe that (7.27) also holds for the C^+ lattice with real negative $\kappa \in \mathbb{R}^-$ and real positive $h \in \mathbb{R}^+$ (here, $h^{24} > 0$ can be obtained using (5.26)). With $e^{r^{13}} = h^{13}$ and $e^{r^{24}} = h^{24}$ the equation can be written as*

$$e^{r_{13}^{24} + r^{13} - r_{24}^{24} - r_{24}^{13}} = \frac{1 - \kappa e^{r^{13}}}{1 - \kappa e^{-r^{13}}} \frac{1 - \kappa e^{-r_{24}^{13}}}{1 - \kappa e^{r_{24}^{13}}}. \quad (7.31)$$

This seems to be an ingredient for a discrete sinh-Gordon equation holding on the C^+ lattice. However, as in the sine-Gordon case, (7.31) does not determine h from initial data on the axes. A further investigation of this equation will be the subject of future work.

By Proposition 7.2.3 we can construct CMC nets by uniquely extending a 2D cross-ratio system, in other words, a discrete holomorphic map. An example can be seen in Figure 7.5. Constructing surfaces and harmonic maps from holomorphic maps is usually achieved with the DPW method [40]. Our method appears to be equivalent to the discrete DPW method for CMC nets given in [84, 53]. We do not give a proof of this, however, we give an overview of the discrete DPW method and sketch the connection:

Theorem 7.4.3 (Discrete DPW [84, 53]) *Any isothermic CMC net $f : \mathbb{Z}^2 \rightarrow \mathbb{R}^3$ can be obtained by the following method:*

1. *Choose a holomorphic map $z : \mathbb{Z}^2 \rightarrow \mathbb{C}$ with corresponding labelling $|\alpha^i| \neq 1$ and cross-ratio $\frac{(\alpha^1)^2}{(\alpha^2)^2} < 0$ and set*

$$L^- := 1 + \frac{1}{\lambda} p^1, \quad M^- := 1 + \frac{1}{\lambda} p^2 \quad (7.32)$$

for p given by (5.28).

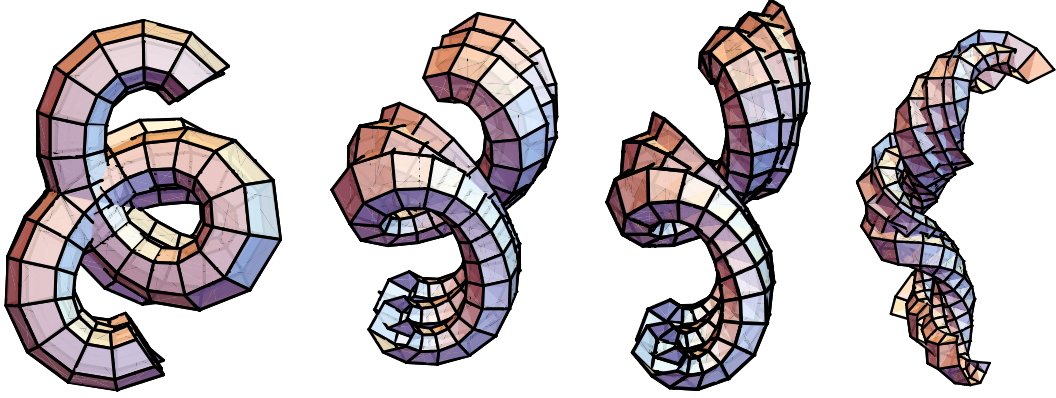


Figure 7.5.: A CMC surface and members of its associated family obtained from the exponential map in Example 5.3.3 as initial data.

2. Solve the initial value problem

$$\Psi_1^- = L^- \Psi^-, \quad \Psi_2^- = M^- \Psi^-, \quad \Psi(0, 0) = \mathbf{1}. \quad (7.33)$$

3. Split

$$\Psi^- = B^+ \Psi. \quad (7.34)$$

4. Use the Sym-Bobenko formula to define

$$f_t = \text{Im}(\Psi^{-1} \Psi + \frac{1}{2} \Psi^{-1} \mathbf{k} \Psi), \quad \lambda = e^{it} \quad (7.35)$$

and set $f = f_0$.

Step 3. needs further explanation: Clearly, Ψ^- is a Laurent polynomial, i.e., $\Psi^-(\lambda) = \sum_{k=-m}^n \lambda^k C_k$. Projectivizing a subset of Laurent polynomials with special properties (for details, see [84, 53]) yields a group. Ψ^- belongs to the subset of this group with $n = 0$ and B^+ belongs to the subset with $m = 0$. Ψ belongs to the subset where each value $\Psi(\lambda)$ for $\lambda \in \mathbb{S}^1$ is a quaternion. Splitting Ψ^- into Laurent polynomial with these properties is known as a discrete version of an Iwasawa decomposition.

Performing this split has two main ingredients which directly correspond to the construction of a C^+ lattice as in the proof of Proposition 7.2.3: First, applying a resplitting of the form $B^- B^+ = \tilde{B}^+ \tilde{B}^-$ corresponds to taking different paths in the 4D lattice which can be achieved by the evolution of cross-ratio systems, e.g., in terms of the Hirota system (5.26) or in terms of the parallelogram evolution (4.16). Second, extension of the form $B^- = B^+ B$ where $B(\lambda)$ is a quaternion corresponds to condition (7.6).

Thus, understanding the discrete DPW method in terms of a reduction of cross-ratio systems on a 4D lattice yields new insights into this method. In particular, the application of our version of this method seems to be much simpler and more constructive than previous versions. Instead of explicit re-splitting of the matrices we simply evolve the holomorphic map directly into a 4D cross-ratio system.

Finally, the reduction to a C^+ lattice is 3D consistent: Proposition 7.2.3 states that the evolution (5.26) preserves the reduction through the lattice and, hence, one can naturally extend any 3D cross-ratio system to a 6D cross-ratio system that has C^+ lattices as four-dimensional layers. The corresponding diagonal net is three-dimensional and in its associated family the layers constitute CMC surfaces related by a natural Darboux transformation (as in [51]). Consequently, a lattice of Darboux transformations can be obtained by extending a 4D cross-ratio system into an 8D lattice and such an example is visible in Figure 7.6.

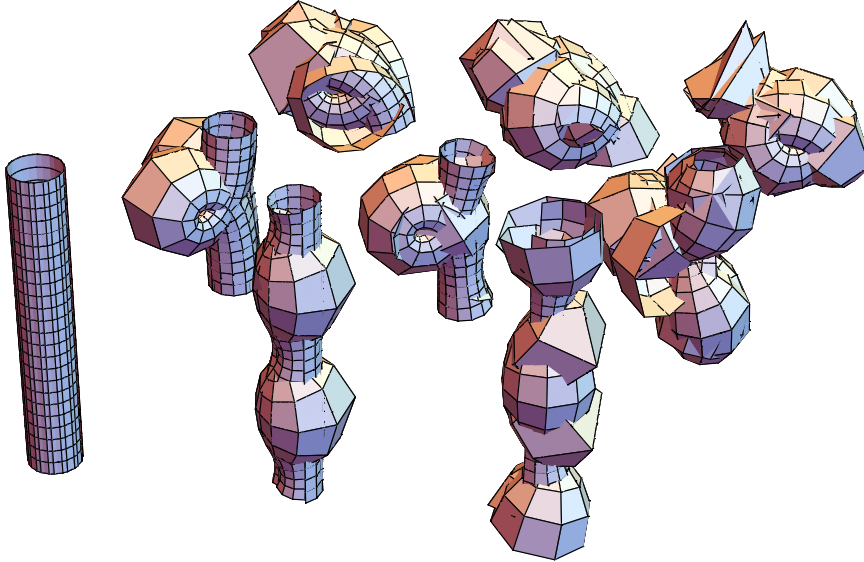


Figure 7.6.: A lattice of transformations of the cylinder.

7.5. CMC nets via flip-over combinatorics

A different connection between CMC nets and parallelogram nets has already been established in [62]: A 3D-cube of a CMC net and its dual form an equally folded quaternionic parallelogram cube after a combinatoric flip. The isothermic case corresponds to a zero-folded cube, i.e., to a cube of anti-parallelograms. This connection was an important

motivation to research skew parallelogram nets. Here, we expand on this connection and, in particular, cover CMC nets in non-Euclidean geometry.

We will consider a primal CMC net f and its dual f^* to form a two-layered three-dimensional net $f : \mathbb{Z}^2 \times \{0, 1\} \rightarrow \mathbb{R}^3$ and in accordance with our shift notation we denote $f_3 := f^* = f(\cdot, \cdot, 1)$. In slight disagreement with our previous use of shift notation we will only use the short notation f for points of the first layer $f(\cdot, \cdot, 0)$. Such a two-layered net consists of elementary cubes where each cube consists of a primal and dual CMC quad. We can obtain a second two-layered net $g : \mathbb{Z}^2 \times \{0, 1\} \rightarrow \mathbb{R}^3$ by a combinatoric flip as in Figure 7.7 on the left: We define

$$g(k, l, 0) := \begin{cases} f(k, l, 0), & \text{if } k + l \text{ is even} \\ f(k, l, 1), & \text{if } k + l \text{ is odd} \end{cases}, \quad g(k, l, 1) := \begin{cases} f(k, l, 1), & \text{if } k + l \text{ is even} \\ f(k, l, 0), & \text{if } k + l \text{ is odd} \end{cases}. \quad (7.36)$$

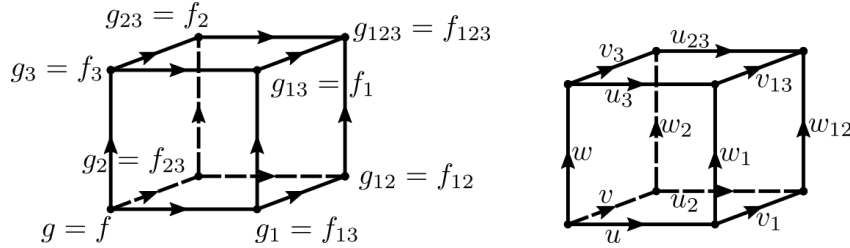


Figure 7.7.: Combinatoric flip at a point $f = f(k, l, 0)$ with even $k + l$.

Both nets consist of the same points but edges of one net correspond to the diagonals of the other. In this section, we always consider nets f and g to be related by this flip. Recall that quads of a Moutard net in $\mathbb{S}^3$ and $\mathbb{H}^3$ can be considered to be non-Euclidean versions of anti-parallelograms since diagonals are parallel and opposite edges have the same length. In this section, we refer to such quads as anti-parallelograms. In Chapter 9 we will call them *Darboux butterflies*.

Proposition 7.5.1 *A two-layered net f forms a pair of primal and dual isothermic CMC surfaces in Euclidean, spherical or hyperbolic space if and only if all quads of the corresponding net g are anti-parallelograms in the corresponding space form.*

Before proving this we compute the cross-ratios of f given that g consists of anti-parallelograms. As before, we identify Euclidean space with imaginary quaternions $\mathbb{R}^3 \cong \text{Im } \mathbb{H}$, spherical space $\mathbb{S}^3$ with unit quaternions and hyperbolic space $\mathbb{H}^3$ with Hermitian matrices with determinant 1. Remember that in every case there is a parallelogram net p corresponding to g : In Euclidean space it is given by $p^i = g_i - g$ and it is zero-folded and in non-Euclidean space it is given by $p^i = g^i g^{-1}$ and it is equally folded.

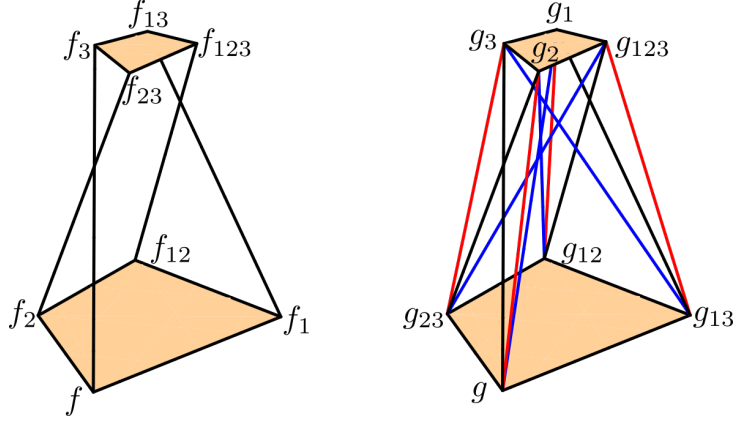


Figure 7.8.: Left: A cube f with of primal and dual CMC quads. Right: The corresponding cube g (at even $k + l$) of anti-parallellograms.

Proposition 7.5.2 *Let g be a cube of anti-parallellograms in Euclidean or non-Euclidean space and let $p = (u, v, w)$ (as in Figure 7.7 on the right) be the corresponding parallellogram net. Then, the cross-ratio of the top and bottom faces of f are given by*

$$cr(f, f_1, f_{12}, f_2) = cr(f_3, f_{13}, f_{123}, f_{23}) = \frac{\det u - \det w}{\det v - \det w} \quad (7.37)$$

in Euclidean space and by

$$cr(f, f_1, f_{12}, f_2) = cr(f_3, f_{13}, f_{123}, f_{23}) = \frac{\text{tr}(u - w)}{\text{tr}(v - w)} \quad (7.38)$$

in spherical space $\mathbb{S}^3$ and hyperbolic space $\mathbb{H}^3$.

Proof. We will calculate the cross-ratio

$$cr(g, g_{13}, g_{12}, g_{23}) = (g - g_{13})(g_{13} - g_{12})^{-1}(g_{12} - g_{23})(g_{23} - g)^{-1} \quad (7.39)$$

which corresponds to the bottom or top quad of f depending on if $k + l$ is even or odd. The other quad works analogously. We will use that both in the Euclidean and in the non-Euclidean case the cross-ratio in terms of $p = (u, v, w)$ is given by

$$cr(g, g_{13}, g_{12}, g_{23}) = (w_1 - \text{adj } u)(v_1 - w_1)^{-1}(u_2 - w_2)(w_2 - \text{adj } v)^{-1}. \quad (7.40)$$

To see this, note that in the Euclidean case we have $\text{adj } p^i = -p^i$ and in the non-Euclidean case we have $(p^i)^{-1} = \text{adj } p^i$ and

$$cr(g, g_{13}, g_{12}, g_{23}) = (u^{-1}w_1^{-1} - \mathbf{1})(\mathbf{1} - v_1w_1^{-1})^{-1}(u_2w_2^{-1} - \mathbf{1})(\mathbf{1} - v^{-1}w_2^{-1})^{-1}. \quad (7.41)$$

Now, the evolution (3.3) for w_1 reads $w_1 = (w - u)w(w - u)^{-1}$ and yields (using $w^2 = (\text{tr } w)w - \det w$)

$$\begin{aligned} (v_1 - w_1)(w_1 - \text{adj } u)^{-1} &= (v_1(w - u) - (w - u)w)((w - u)w - \text{adj } u(w - u))^{-1} \\ &= ((u + v_1)w - w^2 - v_1u)(\det u - \det w + \text{tr}(w - u)w)^{-1}. \end{aligned} \quad (7.42)$$

Analogously, we get

$$(u_2 - w_2)(w_2 - \text{adj } v)^{-1} = ((v + u_2)w - w^2 - u_2v)(\det v - \det w + \text{tr}(w - v)w)^{-1}. \quad (7.43)$$

The first factors of both equations coincide and, hence, we have

$$cr(g, g_{13}, g_{12}, g_{23}) = (\det u - \det w + \text{tr}(w - u)w)(\det v - \det w + \text{tr}(w - v)w)^{-1}. \quad (7.44)$$

Now, using $\text{tr } p^i = 0$ in the Euclidean case and $\det p^i = 1$ in the non-Euclidean case the formula simplifies to (7.37) and (7.38). $\square$

Another ingredient in the proof of Proposition 7.5.1 will be the well-known concept of a *Koenigs net* as in [25]: A net f is called a Koenigs net if there exists a second dual net f^* such that corresponding edges and non-corresponding diagonals of the nets are parallel:

$$f_1^* - f^* \parallel f_1 - f, \quad f_2^* - f^* \parallel f_2 - f, \quad f_{12}^* - f^* \parallel f_1 - f_2, \quad f_1^* - f_2^* \parallel f_{12} - f. \quad (7.45)$$

Isothermic nets are a special case and the isothermic dual is a Koenigs dual. Now, we will use the following important observation: A two-layered net f consists of a primal and dual Koenigs net if and only if the corresponding net g is a Moutard net.

Proof. [of Proposition 7.5.1] If f consist of a primal and dual isothermic CMC net we know that g is a Moutard net. In non-Euclidean geometry this already proves the statement. In Euclidean geometry it remains to be shown that the edge-lengths of g have the labelling property. However, the points of a each cube are known to lie on a common sphere [17, Proposition 8] and, as discussed in Section 4.7, edge-lengths of a Moutard net on a sphere have the labelling property.

Conversely, if all quads of g are anti-parallelograms Proposition 7.5.2 implies that both layers of f are isothermic. Since the layers are Koenigs dual they are also isothermic dual. Now, since corresponding points have constant distance they form a pair of isothermic primal and dual CMC nets. $\square$

Remember that the associated family of a zero-folded net g yields a family of equally folded parallelogram nets. Indeed, as explored in Theorem 5.2.1, the associated family

deforms nets of Euclidean anti-parallelograms into spherical and hyperbolic space. We can apply this to our two-layered nets:

Corollary 7.5.3 *Let f be a Euclidean isothermic CMC net and g the corresponding two-layered net of anti-parallelograms. Application of the associated family as in Theorem 5.2.1 to g yields families of isothermic CMC nets in spherical and hyperbolic space. All spherical and hyperbolic isothermic CMC nets can be obtained by this construction.*

This connection between Euclidean and non-Euclidean CMC nets is closely related to the *Lawson correspondence* which states that to every minimal surface in $\mathbb{S}^3$ there exists an isometric Euclidean CMC net (see [74]). Indeed, our family contains a minimal surface in $\mathbb{S}^3$: The net f_t in the spherical associated family of a Euclidean CMC net f is minimal if $\langle f_t, f_t^* \rangle = \operatorname{Re} w_t = 0$ (with $w_t = f_t^* f_t^{-1}$ or $w_t = f_t (f_t^*)^{-1}$ depending on the vertex). Now, in the associated family as in Theorem 5.2.1 we have $\lambda(t) = e^t$, $\mu(t) = ce^{-t}$. Then, (4.20) implies

$$\langle f_t, f_t^* \rangle = \operatorname{Re} w_t = \frac{e^{2t} - c^2 e^{-2t} \|f^* - f\|^2}{e^{2t} + c^2 e^{-2t} \|f^* - f\|^2} = \frac{e^{4t} - \frac{c^2}{H^2}}{e^{4t} + \frac{c^2}{H^2}} \quad (7.46)$$

and this vanishes for $t = \frac{1}{2} \log \frac{|c|}{|H|}$. A discrete Lawson correspondence can be found in [19]. In [19] and [84] we find two more constructions of non-Euclidean CMC nets from a Euclidean CMC net. In both cases the nets are obtained by integration of the Lax representation (7.29). The similarity of their integration formulas to (5.15) and the gauge equivalence of the Lax representations (which we will show in Proposition 7.5.5) suggest that both constructions coincide with our associated family. We will not give explicit proof of this.

Note that we can apply Proposition 5.2.2 to our construction, which states that angles between geodesic edges are preserved. Under the combinatoric flip (7.36) these invariant angles at a point f_t are of two types:

- the angle between the geodesic edge connecting f_t to f_t^* and the geodesic edge connecting f_t to $(f_t^*)_i$,
- the angle between the geodesic edge connecting f_t to $(f_t^*)_i$ and the geodesic edge connecting f_t to $(f_t^*)_j$.

Obviously, we can also consider geodesic edges 'going backward' by replacing $(f_t^*)_i$ with $(f_t^*)_{\bar{i}}$.

Finally, the additive primitive map of the associated family of a Euclidean isothermic CMC net yields nets in Euclidean space which are generally not isothermic. Proposition 7.5.5 will show that this family coincides with the known associated family of discrete CMC in Euclidean space. In [62] it is shown that the non-isothermic nets in this family still have constant mean curvature.

Remark 7.5.4 Equation (7.40) has another interpretation for the Euclidean associated family (given by the additive primitive map). A comparison with the evolution equations (3.3), (4.16) shows that the normal transport quaternions as introduced in [62, Definition 3.26] of the CMC quad are given by

$$t^1 = u - \text{adj } w_1, \quad t_1^2 = v_1 - w_1, \quad t_2^1 = u_2 - w_2, \quad t^2 = v - \text{adj } w_2. \quad (7.47)$$

Thus, we have

$$\text{adj } t^1 (t_1^2)^{-1} t_2^1 (\text{adj } t^2)^{-1} = \frac{\text{tr}(u - w)}{\text{tr}(v - w)}. \quad (7.48)$$

In [63] this quantity is called the spin cross-ratio of the quad. Unlike the normal cross-ratio it stays real and factorizing throughout this associated family.

We will show that our Lax representation and the classical Lax representation of CMC are gauge equivalent. This implies that the associated families obtained by applying the Sym formula $f_t = \Phi^{-1} \Phi'$ to the respective frames coincide. For this, recall Proposition 3.4.1 which states that the specific choice of functions λ and μ does not affect the associated family (up to primary equivalence). For our zero-folded two-layered three-dimensional parallelogram net $p = (u, v, w)$ we consider the Lax representation

$$U = \cos(t) + \sin(t) u, \quad V = \cos(t) + \sin(t) v, \quad W = \cos(t) + \sin(t) w. \quad (7.49)$$

Proposition 7.5.5 The representation (7.49) and the classic representation for CMC (7.29) are gauge equivalent.

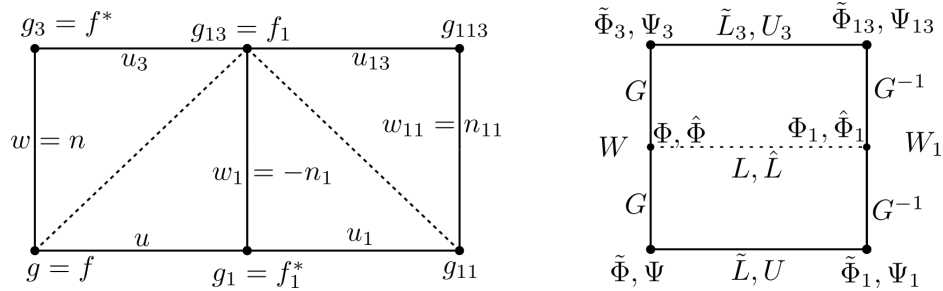


Figure 7.9.: Left: Combinatorics of the two-layered parallelogram net g and the corresponding CMC net f at even $k + l$. Right: Combinatorics at even $k + l$ of the various frames and Lax matrices used in the proof.

Proof. First, consider a frame Φ defined by the Lax matrices (7.29) with $\lambda = e^{it}$. From it, we also get the CMC net f with normal map n and dual f^* . The corresponding

two-layered net g has the CMC net as diagonals (see Figure 7.9). The net g has edges $u, v, w \in \text{Im}(\mathbb{H})$, where $w = (-1)^{k+l}n$. We introduce a two-layered frame $\tilde{\Phi}$ by

$$\tilde{\Phi} = \begin{cases} G^{-1}\Phi, & \text{if } k+l \text{ is even} \\ G\Phi, & \text{if } k+l \text{ is odd} \end{cases}, \quad \tilde{\Phi}_3 = \begin{cases} G\Phi, & \text{if } k+l \text{ is even} \\ G^{-1}\Phi, & \text{if } k+l \text{ is odd} \end{cases}, \quad (7.50)$$

where $G = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}$. Now, it is easy to check the equality $g = \tilde{\Phi}^{-1}\tilde{\Phi}'$ and, hence, $\tilde{\Phi}$ is a frame for g . At even $k+l$, we can calculate

$$\begin{aligned} \tilde{L} &= \tilde{\Phi}_1\tilde{\Phi}^{-1} = GLG = \begin{pmatrix} \lambda a & \lambda b + \frac{1}{\lambda b} \\ -\frac{\bar{b}}{\lambda} - \frac{\lambda}{b} & \frac{\bar{a}}{\lambda} \end{pmatrix} \\ &= \frac{\lambda + \frac{1}{\lambda}}{2} \begin{pmatrix} a & b + \frac{1}{b} \\ -\bar{b} - \frac{1}{b} & \bar{a} \end{pmatrix} + \frac{\lambda - \frac{1}{\lambda}}{2i} \begin{pmatrix} ia & ib - \frac{i}{b} \\ i\bar{b} - \frac{i}{b} & -i\bar{a} \end{pmatrix} \\ &= \cos(t)\tilde{L}(1) + \sin(t)\tilde{L}(i). \end{aligned} \quad (7.51)$$

With similar calculations we can show

$$\tilde{L} = \cos(t)\tilde{L}(1) + \sin(t)\tilde{L}(i), \quad \tilde{M} = \cos(t)\tilde{M}(1) + \sin(t)\tilde{M}(i) \quad (7.52)$$

at all points of both layers. Also, for the third direction, we have

$$\tilde{\Phi}_3\tilde{\Phi}^{-1} = \begin{cases} G^2 & = \cos(t) + \sin(t)\mathbf{k}, & \text{if } k+l \text{ is even} \\ G^{-2} & = \cos(t) - \sin(t)\mathbf{k}, & \text{if } k+l \text{ is odd} \end{cases}. \quad (7.53)$$

We can normalize the coefficient of $\cos(t)$ of $\tilde{L}, \tilde{M}$ with the gauge $\Psi := \tilde{\Phi}(1)^{-1}\tilde{\Phi}$:

$$\begin{aligned} U &:= \Psi_1\Psi^{-1} = \cos(t) + \sin(t)\tilde{\Phi}(1)_1^{-1}\tilde{L}(i)\tilde{\Phi}(1) = \cos(t) + \sin(t)u, \\ V &:= \Psi_2\Psi^{-1} = \cos(t) + \sin(t)\tilde{\Phi}(1)_2^{-1}\tilde{M}(i)\tilde{\Phi}(1) = \cos(t) + \sin(t)v, \\ W &:= \Psi_3\hat{\Psi}^{-1} = \cos(t) + (-1)^{k+l}\sin(t)\Phi(1)^{-1}\mathbf{k}\Phi(1) = \cos(t) + \sin(t)w. \end{aligned} \quad (7.54)$$

The last equality for U and V follows from the fact that the Sym formula at $t = 0$ applied to Ψ gives g which has u, v as edges, but we can also directly compute the Sym formula for a Lax matrix of the form $\cos(t) + \sin(t)x$ to give x and, thus, x needs to agree with u (or v).

Hence, we have shown that g is a two-layered zero-folded parallelogram net and that its Lax representation can be directly obtained from the CMC Lax matrices.

For the converse, consider a two-layered zero-folded parallelogram net g , such that $n := (-1)^{k+l}w \in \mathbb{S}^2$ with a two-layered frame Ψ given by the Lax representation U, V, W .

The gauge $\tilde{\Phi} := (\mathbf{k} + n)\Psi$ yields

$$\begin{aligned}\tilde{\Phi}_3\tilde{\Phi}^{-1} &= (\mathbf{k} + n)(\cos(t) + (-1)^{k+l}\sin(t)n)(\mathbf{k} + n)^{-1} \\ &= \cos(t) + (-1)^{k+l}\sin(t)\mathbf{k} = \begin{cases} G^2, & \text{if } k+l \text{ is even} \\ G^{-2}, & \text{if } k+l \text{ is odd} \end{cases}, \end{aligned} \quad (7.55)$$

with G as above. Also the λ -dependency of $\tilde{L}$ can be expressed as

$$\tilde{L} = \tilde{\Phi}_1\tilde{\Phi}^{-1} = (\mathbf{k} + n_1)\left(\frac{\lambda + \frac{1}{\lambda}}{2} + \frac{\lambda - \frac{1}{\lambda}}{2i}u\right)(\mathbf{k} + n)^{-1} =: \begin{pmatrix} \lambda\hat{a} + \frac{1}{\lambda}\hat{d} & \lambda\hat{b} + \frac{1}{\lambda}\hat{c} \\ -\frac{1}{\lambda}\hat{b} - \lambda\hat{c} & \frac{1}{\lambda}\hat{a} + \lambda\hat{d} \end{pmatrix}. \quad (7.56)$$

At even $k + l$ the Lax matrix assigned to the diagonal $u + w_1$ is the product

$$G^{-2}\tilde{L} = \begin{pmatrix} \hat{a} + \frac{1}{\lambda^2}\hat{d} & \hat{b} + \frac{1}{\lambda^2}\hat{c} \\ -\frac{1}{\lambda}\hat{b} - \lambda^2\hat{c} & \hat{a} + \lambda^2\hat{d} \end{pmatrix}. \quad (7.57)$$

At $t = 0$ we have $G = 1$ and we can use the Sym formula to compute

$$\begin{aligned}\tilde{\Phi}_1(G^{-2}\tilde{L})'\mathbf{k}\tilde{\Phi} &= \tilde{\Phi}_{13}(G^{-2}\tilde{L})'\tilde{\Phi}^{-1}\tilde{\Phi}\mathbf{k}\tilde{\Phi} = (u_2 + w)w \\ &= w_1(u + w_1) = -\tilde{\Phi}_1\mathbf{k}\tilde{\Phi}_1^{-1}\tilde{\Phi}_{13}(G^{-2}\tilde{L})'\tilde{\Phi} = -\tilde{\Phi}_1\mathbf{k}(G^{-2}\tilde{L})'\tilde{\Phi}. \end{aligned} \quad (7.58)$$

Now, at $t = 0$, we have $(G^{-2}\tilde{L})'\mathbf{k} = -\mathbf{k}(G^{-2}\tilde{L})'$ which implies that $(G^{-2}\tilde{L})'$ is purely off-diagonal, i.e., $\hat{d} = 0$.

Now we can introduce the one-layered frame $\hat{\Phi} := G\tilde{\Phi}$ for even $k + l$ and $\hat{\Phi} := G^{-1}\tilde{\Phi}$ for odd $k + l$. This yields Lax matrices of the form

$$\hat{L} = \hat{\Phi}_1\hat{\Phi}^{-1} = \begin{pmatrix} \hat{a} & \lambda\hat{b} + \frac{1}{\lambda}\hat{c} \\ -\frac{1}{\lambda}\hat{b} - \lambda\hat{c} & \hat{a} \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} \hat{d} & \lambda\hat{e} + \frac{1}{\lambda}\hat{f} \\ -\frac{1}{\lambda}\hat{e} - \lambda\hat{f} & \hat{d} \end{pmatrix}. \quad (7.59)$$

The λ^2 coefficient of $\hat{L}_2\hat{M} = \hat{M}_1\hat{L}$ yields $\hat{b}_2\hat{f} = \hat{e}_1\hat{c}$ and $\hat{c}_2\hat{e} = \hat{f}_1\hat{b}$ which implies that a vertex variable x with $x_1 = \frac{\hat{x}}{\hat{b}\hat{c}}, x_2 = \frac{\hat{x}}{\hat{e}\hat{f}}$ is well-defined. A final gauge $\Phi := \begin{pmatrix} \sqrt{x} & 0 \\ 0 & \sqrt{\bar{x}} \end{pmatrix}\hat{\Phi}$ yields

$$L = \Phi_1\Phi^{-1} = \begin{pmatrix} \sqrt{x_1} & 0 \\ 0 & \sqrt{\bar{x}_1} \end{pmatrix}\hat{L}\begin{pmatrix} \frac{1}{\sqrt{x}} & 0 \\ 0 & \frac{1}{\sqrt{\bar{x}}} \end{pmatrix} = \begin{pmatrix} a & \lambda b + \frac{1}{\lambda}\frac{1}{b} \\ -\frac{1}{\lambda}\bar{b} - \lambda\frac{1}{\bar{b}} & \bar{a} \end{pmatrix}, \quad (7.60)$$

where $a := \frac{\sqrt{x_1}}{\sqrt{x}}\hat{a}$, $b := \frac{\sqrt{\bar{b}}}{\sqrt{\bar{e}}}$. The same calculation can be done for $M = \Phi_2\Phi^{-1}$ and L, M have exactly the form (7.29). $\square$

8. Discrete flows and their invariant curves

In this chapter, we are concerned with discrete curves in Euclidean space. We will consider discrete flows of such curves built from Bäcklund transformations and study their invariant curves. This includes the discrete Hashimoto flow and its invariants which are known to be elastic rods [55].

We introduce a new perspective by showing that elastic rods belong to a hierarchy of curves which is directly related to the construction of quasi-periodic parallelogram nets in Section 4.6.

8.1. Discretization of curves and flows

In this section, we review some known theory for curves and flows.

We briefly introduce some notions from smooth theory as they give important context for the discrete theory. A curve is a smooth map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$. For some diffeomorphism σ the curve $\tilde{\gamma} = \gamma \circ \sigma$ is called a *reparametrization* of γ since both curves share the image set $\gamma(\mathbb{R}) = \tilde{\gamma}(\mathbb{R})$. A curve is called *regular* if $\|\gamma'\| \neq 0$. Such a curve can always assumed to be *parametrized by arc-length* which means $\|\gamma'\| = 1$ since there is always a reparametrization with this property. Note, that this will not be possible for discrete curves since reparametrization is not defined. Now, consider a family of curves given by a smooth map $\gamma : \mathbb{R} \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$. We call this a *flow* of the curve $\gamma(\cdot, 0)$. One studies a flow in terms of the vectorfield

$$\dot{\gamma} = \frac{d}{dt}\gamma(s, t). \quad (8.1)$$

We always denote the derivative in flow direction by $\dot{\gamma}$ and the derivative in curve direction by $\gamma' = \frac{d}{ds}\gamma(s, t)$.

Examples of flows are the *tangent flow* of the curve given by

$$\dot{\gamma} = \gamma' \quad (8.2)$$

and the *Hashimoto flow* (also called smoke-ring, Heisenberg or vortex filament flow) given by

$$\dot{\gamma} = \gamma' \times \gamma''. \quad (8.3)$$

Both flows are known to preserve arc-length parametrization and commute.

In a similar manner we define a discrete curve as a map $f : \mathbb{Z} \rightarrow \mathbb{R}^3 \cong \text{Im } \mathbb{H}$. We call it *regular* if any three consecutive points are pairwise distinct. An edge tangent vector is given by the difference vector $u := f_1 - f$ and we call the curve *arc-length parametrized* if $\eta := \|u\|$ is constant. Unlike in the smooth case a non arc-length parametrized curve can not be reparametrized into an arc-length parametrized curve. Studying general curves is more difficult than studying arc-length parametrized curves and, thus, in most cases we restrict ourselves to arc-length parametrized curves.

We introduce a vertex tangent vector for arc-length parametrized curves (see [56]) by the harmonic mean

$$T = 2(u_{\bar{1}}^{-1} + u^{-1})^{-1} = \frac{\eta^2(u_{\bar{1}} + u)}{\eta^2 + \langle u_{\bar{1}}, u \rangle}. \quad (8.4)$$

A flow of a discrete curve can again be defined by a vector field along the curve. The tangent flow for the discrete curve is given by

$$\dot{f} = F^T := T \quad (8.5)$$

This is semi-discrete setting with a discrete curve and a smooth flow direction, i.e., the points of the discrete curve evolve smoothly under the flow.

Similarly, the semi-discrete Hashimoto flow (see [21]) is given by

$$\dot{f} = F^H := 2 \text{Im}(u_{\bar{1}}^{-1} + u^{-1})^{-1} u = \frac{\eta^2 u_{\bar{1}} \times u}{\eta^2 + \langle u_{\bar{1}}, u \rangle}. \quad (8.6)$$

Again, both flows preserve arc-length parametrization and commute.

One can discretize the flow direction as well by using sequences of Bäcklund transformations. For this, we will consider the curves and their Bäcklund transforms to be parameter lines in a two-dimensional quaternionic parallelogram net $p = (u, v) : \mathcal{E}(\mathbb{Z}^2) \rightarrow \mathbb{H}$. Specifically, we consider a (projected) additive primitive map $f : \mathbb{Z}^2 \rightarrow \text{Im } \mathbb{H} \cong \mathbb{R}^3$ and the curves are its parameter lines $f(\cdot, l_0)$. As above, $u = f_1 - f \in \text{Im } \mathbb{H}$ is the edge tangent vector and, thus, we have vanishing real part $\text{Re } u = 0$ in curve direction. However, in transformation direction the real part $\text{Re } v$ does not necessarily vanish. Note that this setting is identical to Section 4.6 (before we introduce periodicity) with the additional condition $\text{Re } u = 0$.

A Bäcklund transformation of a curve $f(\cdot, l_0)$ is determined by one initial quaternion $v(0, l_0) \in \mathbb{H}$. This initial freedom makes the construction of a discrete tangent or Hashimoto flow for an arbitrary curve difficult since the initial direction for these flows should not be arbitrary. However, we can consider curves with special properties, for instance, curves with periodicity. Recall Theorem 4.6.1, stating that there are in general two Bäcklund transformations that keep the periodicity of the curve for given Bäcklund parameters $r := \text{Re } v, \Delta = \|\vec{v}\|$. In [87, 55] it is shown that such Bäcklund transformations can approximate the tangent and Hashimoto flow. Specifically, it is argued that



Figure 8.1.: Discrete Hashimoto flow (smoke-ring flow) of an ellipse.

for a single Bäcklund transformation the tangent flow component is dominant while for a double-step consisting of two Bäcklund transformations with opposite parameters r, Δ and $-r, \Delta$ the tangent flow components cancel and the Hashimoto flow becomes dominant. This discrete Hashimoto flow of an ellipse can be seen in Figure 8.1.

In this chapter, we will not be concerned with such flows for general curves but with special curves that are invariant under discrete flows built from general sequences of Bäcklund transformations.

8.2. The hierarchy of n -invariant curves

We will now investigate quasi-periodic sequences of Bäcklund transformations by considering quasi-periodic parallelogram nets $p = (u, v)$ with a period $T = (0, n)$ (as in Definition 3.5.1). Compared to our considerations in Section 4.6 (where the period was $T = (n, 0)$) there is a shift in perspective. We consider different sets of parameter lines as discrete curves. In fact, now, the curves themselves do not necessarily have periodicity but instead the sequence of Bäcklund transformations has periodicity. If one interprets the sequence of Bäcklund transformations as a discrete flow, then the quasi-periodicity of this sequence means that the curve is an invariant curve of the flow. While the original perspective presented in Section 4.6 was known, this shift of perspective seems to be new and we will see that it yields a hierarchy of discrete curves related to elastica theory.

We will now apply the techniques developed in Section 4.6. In complete analogy to (4.67) we define the *monodromy* of a quasi-periodic sequence of Bäcklund transformation

at the vertex k as

$$M^\eta(k) := \begin{pmatrix} A(k) & B(k) \\ -B(k) & A(k) \end{pmatrix} := \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} T_{v(k,n-1)}^{0,\eta} \cdots T_{v(k,0)}^{0,\eta} \quad (8.7)$$

where E encodes the periodicity and $T^{0,\eta}$ is defined as in (4.58).

We will consider a quasi-periodic sequence of Bäcklund transformations of the curve f to be *regular* if every edge-length η (together with $r = 0$) is an admissible pair of Bäcklund parameters of M^η , i.e., if $\det M^\eta(k) \neq 0$ and $\det \vec{M}^\eta(k) \neq 0$ for all k and every edge-length η . Recall that $\det M^\eta(k) \neq 0$ and $\det \vec{M}^\eta(k) \neq 0$ are invariant along the curve and, thus, it suffices to check this condition at $k = 0$.

Now, we can define the hierarchy of curves.

Definition 8.2.1 *We call a curve $f : \mathbb{Z} \rightarrow \mathbb{R}^3$ n -invariant if there exists a sequence of Bäcklund transformations of f that is quasi-periodic with period $T = (0, n)$ and regular.*

Clearly, these are exactly the curves that arise as parameter lines in the second lattice direction by iterative application of Theorem 4.6.1 (with $r = 0$). Remarkably, regular arc-length parametrized n -invariant curves are exactly the curves arising from the Algorithm in Theorem 4.6.4 (with $r = 0$). In particular, note the nice implication of Proposition 4.6.6 for the construction of n -invariant curves in general parametrization: Permuting some edge-lengths does not affect succeeding curve points (if the correct fixed points are chosen).

A fundamental piece of the sequence of transformations consists of n Bäcklund transformations such that the n -th transform is related to the original curve by a Euclidean motion. The existence of n transformations with this property for a curve is analogous to Definition 8.2.1 and we will often use this formulation.

Let us discuss some properties of n -invariant curves.

First, note that while we always require neighboring curve points to be distinct, i.e., $\eta \neq 0$ we did not exclude the case $v = 0$ which means that a curve and its Bäcklund transform coincide. We call this the *identity transformation*. In Chapter 9 we will see that it is reasonable to allow this special case. Trivially, the existence of the identity transformation implies that an n -invariant curve is also $(n + 1)$ -invariant.

Next, we can consider the associated family f_t of an n -invariant curve f . Now, generally we have $\operatorname{Re} u_t \neq 0$ and, therefore, f_t is not a curve as considered in this chapter. However, if f is arc-length parametrized we have $\operatorname{Re} u_t \equiv \operatorname{const.}$ (see (4.20)) and, hence, using primary equivalence of the underlying parallelogram net we can still consider f_t to be a curve.

Proposition 8.2.2 *If an arc-length parametrized curve f is n -invariant then the curves f_t in its associated family are n -invariant as well.*

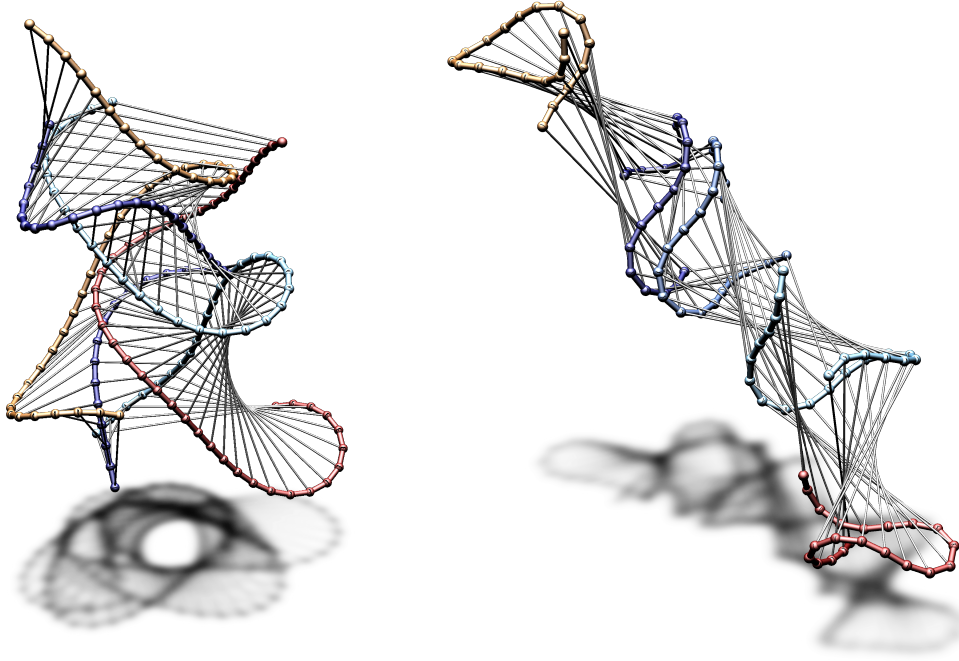


Figure 8.2.: Arc-length parametrized n -invariant curves with corresponding Bäcklund transformations for $n = 4, 5$. The first and last curve (orange and red) are related by a screw-motion.

Proof. This is a consequence of Proposition 3.5.2 since n -invariant curves are defined as parameter lines in quasi-periodic parallelogram nets.

We will show that regularity of the sequence of Bäcklund transformations is preserved as well. For this, recall that the monodromy (8.7) coincides with the polynomial $\mathcal{P}(x = (0, n)) = E\Phi(0, n)$ introduced in (3.51). Similarly, the monodromy $M_{t_0}^{\eta_{t_0}}$ of a member of the associated family f_{t_0} is given by $E_{t_0}\Psi(0, n) = E\Phi(0, n)\Psi(0, n)$ with E_{t_0} as in (3.47) and where Ψ is a moving frame of f_{t_0} . Now, in the proof of Proposition 3.4.2 we have shown that $\Phi(0, n)\Psi(0, n) = \tilde{\Phi}(0, n)$ for a frame $\tilde{\Phi}$ of f parametrized by functions as in (3.39). Thus, the monodromy of f_{t_0} is a reparametrization (in the spectral parameter) of the monodromy of f .

In particular, by plugging in $\operatorname{Re} u_{t_0}$ and η_{t_0} (as in (4.20) and (4.21)) we find $M_{t_0}^{\eta_{t_0}}(k) = M^\eta(k)$. It follows immediately that regularity of the sequence of Bäcklund transformations is preserved. $\square$

Remark 8.2.3 One can also consider n -invariant curves that are contained in the plane.

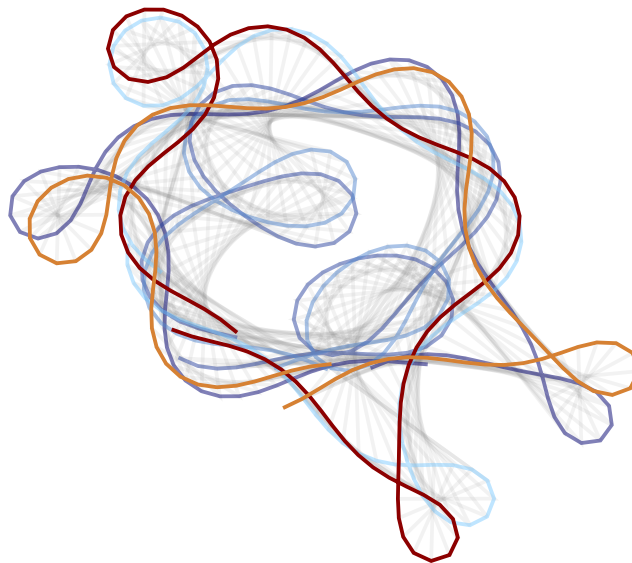


Figure 8.3.: A 5-invariant curve and its sequence of Bäcklund transformations contained in a plane (see Remark 8.2.3). The first and last curve (orange and red) are related by a rotation.

It is natural (but not necessary) to require their corresponding Bäcklund transformations to be contained in a plane as well. Such curves are then constructed by the reduction of our algorithm to the plane mentioned in Remark 4.6.7. It also turns out natural to make a distinction between the case where the axis of the screw motion describing this periodicity is in the plane and the case where it is orthogonal to the plane. We will exemplify this distinction in the next section.

8.3. Helices and 1-invariant curves

We start with our geometric investigation of the hierarchy of n -invariant curves by showing that 1-invariant curves are discrete helices. Curves this low in the hierarchy are still relatively simple to study and, thus, we will even consider the case of general parametrization (i.e., variable edge-length).

A discrete arc-length parametrized helix can be defined by sampling a smooth helix, for instance, by a map

$$\mathbb{Z} \rightarrow \mathbb{R}^3, \quad k \mapsto \begin{pmatrix} r \cos(\varphi k) \\ r \sin(\varphi k) \\ sk \end{pmatrix} \quad (8.8)$$

for some $\varphi, r, s \in \mathbb{R}$. In this case, the axis of the helix is the z -axis (or $\mathbf{k}$ -axis in $\text{Im } \mathbb{H} \cong \mathbb{R}^3$) and all points are contained in the cylinder around this axis with radius r . Note that for any two points f, g on this cylinder there exist unique $\alpha, h \in \mathbb{R}$ and a screw-motion

$$R : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad p \mapsto \left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \mathbf{k}\right) p \left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \mathbf{k}\right)^{-1} + h \mathbf{k} \quad (8.9)$$

such that $g = R(f)$. More generally, for two point on any cylinder there exists a unique corresponding screw-motion around the axis of the cylinder. One can characterize discrete arc-length parametrized helices as curves inscribed in a cylinder such that the screw-motion relating consecutive points is constant.

To define helices in general parametrization we introduce the *slope* of the screw motion R by

$$s(R) := \frac{h}{\tan \frac{\alpha}{2}}. \quad (8.10)$$

Definition 8.3.1 *We call a discrete curve $f : \mathbb{Z} \rightarrow \mathbb{R}^3$ a discrete helix if its points are contained in a common cylinder and if the slope $s(R)$ of the screw motion R relating consecutive points by $R(f) = f_1$ is constant.*

Note that the curve points generally do not lie on a common smooth helix anymore.

We can construct skew parallelograms on a cylinder by using the fact that two screw motions R_1, R_2 around the same axis commute (also compare Example 4.2.5). As discussed in Section 4.3, such quads are described by our algebraic framework if we find the correct real part r as (4.25).

Lemma 8.3.2 *Consider two screw motions R^1, R^2 with a common axis and a quad defined by an initial point f and $f_1 := R^1(f), f_2 := R^2(f)$ and $f_{12} := R^1(f_2) = R^2(f_1)$. Then, the real part r such that $u := f_1 - f, u_2 := f_{12} - f_2$ and $v := r + f_2 - f, v_1 := r + f_{12} - f_1$ fulfill the parallelogram equations (4.28) is given by*

$$r = s(R^2) - s(R^1). \quad (8.11)$$

Proof. Up to a common Euclidean motion (mapping the axis to $\mathbf{k}$ -axis) the points can be explicitly written as

$$f = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}, f_1 = \begin{pmatrix} r \cos(\alpha_1) \\ r \sin(\alpha_1) \\ h_1 \end{pmatrix}, f_2 = \begin{pmatrix} r \cos(\alpha_2) \\ r \sin(\alpha_2) \\ h_2 \end{pmatrix}, f_{12} = \begin{pmatrix} r \cos(\alpha_1 + \alpha_2) \\ r \sin(\alpha_1 + \alpha_2) \\ h_1 + h_2 \end{pmatrix}. \quad (8.12)$$

Then, the statement is shown directly by plugging these points into (4.30). $\square$

We can turn to our main observation.

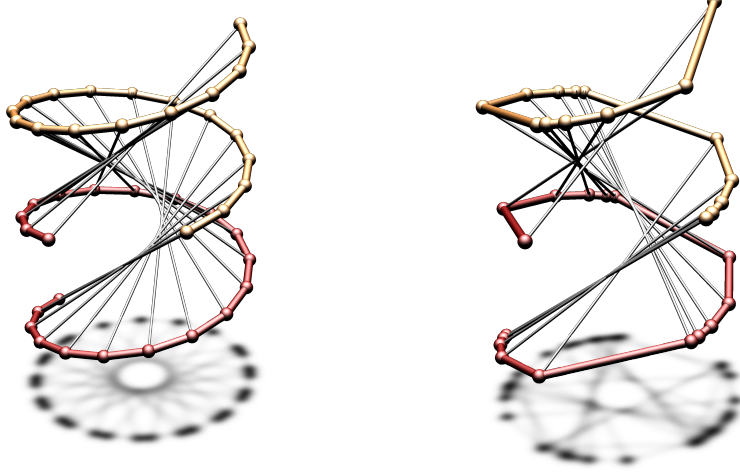


Figure 8.4.: A 1-invariant curve is a discrete helix. On the left the curve is arc-length parametrized and on the right it is not.

Proposition 8.3.3 *A discrete curve is a discrete helix if and only if it is 1-invariant.*

Proof. Consider a discrete helix f and denote the screw-motions relating consecutive points by R^1 . We can define a second curve f_2 by applying a screw motion R^2 around the axis of the helix to each curve point. This second curve is a Bäcklund transform of the helix since Lemma 8.3.2 implies that $p = (u, v)$ with $v = s(R^2) - s(R^1) + f_2 - f$ is a skew parallelogram net. Hence, the curve is 1-invariant.

Conversely, consider a 1-invariant curve and a corresponding parallelogram net $p = (u, v)$ with quasi-periodicity described by a quaternion E . We will first show that all points are contained in a cylinder. For this consider the points $X := f + \frac{1}{2}\vec{E}^{-1}Ev$. They lie on a common line with direction vector $\vec{E}$ since

$$\begin{aligned} X_1 - X &= u + \frac{1}{2}\vec{E}^{-1}E(v_1 - v) = u + \frac{1}{2}\vec{E}^{-1}(uE - Eu) \\ &= u + \frac{1}{2}\vec{E}^{-1}(2\operatorname{Re}(u\vec{E}) - 2\vec{E}u) = \operatorname{Re}(u\vec{E})\vec{E}^{-1} \end{aligned} \quad (8.13)$$

To compute the distance of the points to this line consider

$$\begin{aligned} X - f &= \frac{1}{2}\vec{E}^{-1}Ev \\ X - f_2 &= \frac{1}{2}\vec{E}^{-1}Ev - v = \frac{1}{2}\vec{E}^{-1}\bar{E}v \end{aligned} \quad (8.14)$$

These vectors have constant length since v has constant length. Now, due to the evolution $v_1 - v = E^{-1}uE - u$ the real part and the $\vec{E}$ -component of v are constant. Therefore, this holds for the two vectors $X - f, X - f_2$ as well. Hence, the length of the two vectors projected into the $\vec{E}^\perp$ -plane is constant and this is exactly the Euclidean distance of the points to the axis. Hence, the points of both curves lie on a common cylinder.

Now, the second curve is related to the first curve by a screw motion R^2 around the axis of the cylinder. Thus, every quad is a quad as considered in Lemma 8.3.2 and we have $\text{Re } v = s(R^2) - s(R^1)$. Hence, $s(R^1)$ is constant and the curve is a discrete helix. $\square$



Figure 8.5.: 1-invariant curves in a plane.

We finish this section by considering, again, the case of curves contained in a plane.

Remark 8.3.4 *Following up on Remark 8.2.3, observe that there are two distinct cases for which the discrete helix and its transform are contained in a plane: First, both curves can be inscribed in a circle, as visible in Figure 8.5 on the left. This case has a smooth analogue: smooth helices contained in a plane are smooth circles. In this case, the axis of the helix (and the screw-motion) is orthogonal to the plane and the motion relating the curves is a rotation. Thus, the restriction of the motion to the plane is orientation preserving.*

Second, it can be a 'zig-zag', as visible in Figure 8.5 on the right and this case has no smooth analogue. In this case the axis of the helix is contained in the plane and the motion relating both curves is a combination of a reflection and a translation and, thus, its restriction to the plane is orientation reversing.

This phenomenon extends to higher n : Alternating between orientation preserving (axis orthogonal to plane) and orientation reversing (axis in the plane) motions, only one case has a smooth analogue. In Chapter 9, we will discuss a hierarchy of n -invariant curves. We will be interested in curves with a smooth analogue and, thus, include this condition on the orientation of the motion in the definition.

This creates some ambiguity: For instance, curves belonging to the 'false' planar limit (like the 'zig-zag') are not considered to be n -invariant anymore. Instead, they are $(n+1)$ -invariant. Also, we have seen that n -invariant curves are $(n+1)$ -invariant since we can add in an identity transformation. In the planar version this is not the case since the motion relating first and last curve does not change its orientation. Instead, n -invariant curves will be $(n+2)$ -invariant.

8.4. Elastic rods and 2-invariant curves

In this section, we show that 2-invariant curves are exactly elastic rods as given in [21]. For this, we restrict ourselves to arc-length parametrized curves with $\eta = \|u\| = 1$ (thus, we have $u \in \mathbb{S}^2$).

First, for $n = 2$, the entries A and B of the monodromy (8.7) are

$$\begin{aligned} A &= Ev_2v - E, \\ B &= E(v + v_2). \end{aligned} \tag{8.15}$$

We can write the evolution for A and B along the curve as

$$\begin{aligned} A_1 &= E(v_{12}v_1 - 1) = E(\tilde{u}v_2vu^{-1} - 1) = E(E^{-1}uE(v_2v - 1 + 1)u^{-1} - 1) \\ &= uAu^{-1} + uEu^{-1} - E, \end{aligned} \tag{8.16}$$

$$\begin{aligned} B_1 &= E(v_1 + v_{12}) = E(-u + v + v_2 + \tilde{u}) = E(-u + v + v_2 + E^{-1}uE) \\ &= B + uE - Eu. \end{aligned} \tag{8.17}$$

We know that arc-length parametrized 2-invariant curves are constructed as in Theorem 4.6.4 (recall, that we swapped the lattice directions). In fact, each $u \in \mathbb{S}^2$ is a fixed point of the monodromy (8.7) and, hence, (4.69) reads $u = (\beta + \vec{B})^{-1}(\alpha + \vec{A})$ for α, β (which are constant due to Lemma 4.6.5). We define

$$\hat{A} := \alpha + \vec{A}, \quad \hat{B} := \beta + \vec{B} \tag{8.18}$$

which simplifies our formulas to

$$u = \hat{B}^{-1}\hat{A}, \quad u_{\bar{1}} = \hat{A}\hat{B}^{-1} \Rightarrow u = \hat{B}^{-1}u_{\bar{1}}\hat{B} = \hat{A}^{-1}u_{\bar{1}}\hat{A}. \tag{8.19}$$

Note, that we have $(\text{Re}(\hat{B}u), \text{Re}(\hat{B})) = (\alpha, \beta) \neq (0, 0)$ since the contrary would imply $\det \vec{M} = 0$.

We can now formulate a simplified algorithm.

Theorem 8.4.1 (Simplified algorithm for 2-invariant curves) *Let $E, \hat{B}(0) \in \mathbb{H} \setminus \{0\}$, $u(0) \in \mathbb{S}^2$, such that at $k = 0$, we have $(\text{Re}(\hat{B}u), \text{Re}(\hat{B})) \neq (0, 0)$. We can iterate the following steps, starting from $k = 0$:*

1. Calculate $\hat{B}(k+1) := \hat{B}(k) + u(k)E - Eu(k)$.
2. Calculate $u(k+1) := \hat{B}^{-1}(k)u\hat{B}(k)$.

Then, the curve f given (up to translation) by $u = f_1 - f$ is a 2-invariant curve. Any 2-invariant curve can be constructed this way.

Remarkably, as we will see in the proof, recovering the Bäcklund transformations turns out to be an application of factorization.

Proof. If we have a 2-invariant curve, we know it fulfills the evolutions for $\hat{B}$ and u . Therefore, it can be constructed by this algorithm.

For the converse, if we have a curve constructed by this algorithm, we can define $\hat{A} := \hat{B}u$. $\hat{A}$ evolves as in (8.16) by

$$\hat{A}_1 = \hat{B}_1 u_1 = u \hat{B}_1 = u \hat{B} + u^2 E - u E u = u \hat{A} u^{-1} - E + u E u^{-1}. \quad (8.20)$$

Unfortunately, for $\hat{M} := \begin{pmatrix} \hat{A} & \hat{B} \\ -\hat{B} & \hat{A} \end{pmatrix}$ we have $\det \hat{M} = 0$, and, hence, this is not a valid monodromy map. But we can alter the real parts, i.e., we can define $A := a + \hat{A}$, $B := b + \hat{B}$ with $a, b \in \mathbb{R}$, such that $\det M \neq 0$ for $M := \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. The new A and B still fulfill (8.16) and (8.17). Now, we can apply the factorization theorem 6.2.1 to the polynomial

$$\mathcal{P} := t^2 E + tB(0) + (A(0) + E) \quad (8.21)$$

to obtain a quaternionic factorization

$$\mathcal{P} = E(t + v(0, 1))(t + v(0, 0)) \quad (8.22)$$

The factors v are not in $\mathbb{S}^2$, since this would have implied $\det M(0) = 0$. Also, since $0 = \det \hat{M}(0) = (\alpha + i\beta)^2 + \det \vec{M}(0)$, we have $\det \vec{M}(0) \neq 0$.

We can apply algorithm in Theorem 4.6.4 to the quasi-periodic curve (in $\mathbb{H}$) in transformation direction given by $E, v(0, 0), v(0, 1), u(0)$ to obtain a 2-invariant curve. The quantities u and A, B must agree with the corresponding values of the new curve, since their evolution agrees and their initial values agree. Hence, our curve u is 2-invariant. $\square$

The freedom in choosing a, b means that there are many Bäcklund transformations that leave the curve invariant. There is the third freedom to also choose $\operatorname{Re} E$ (which does not affect the curve) which corresponds to choosing the angle of rotation of the Euclidean motion between f and $\hat{f}$.

We will now show that 2-invariant curves are precisely elastic rods as described in [21]. Elastica are usually described as curves minimizing bending energy under certain constraints. However, one can equivalently characterize them as curves that stay invariant under certain flows. In [21] it is shown that discrete elastic rods can be characterized as discrete curves that stay invariant under a combination of the tangent flow (8.6) and the Hashimoto flow (8.6). A curve stays invariant under a flow if $\dot{f}$ is the vector field of a Euclidean motion which in this case means that there exist constant $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $e, x \in \mathbb{R}^3$, such that

$$\dot{f} = \alpha F^T + \beta F^H = e \times f + x. \quad (8.23)$$

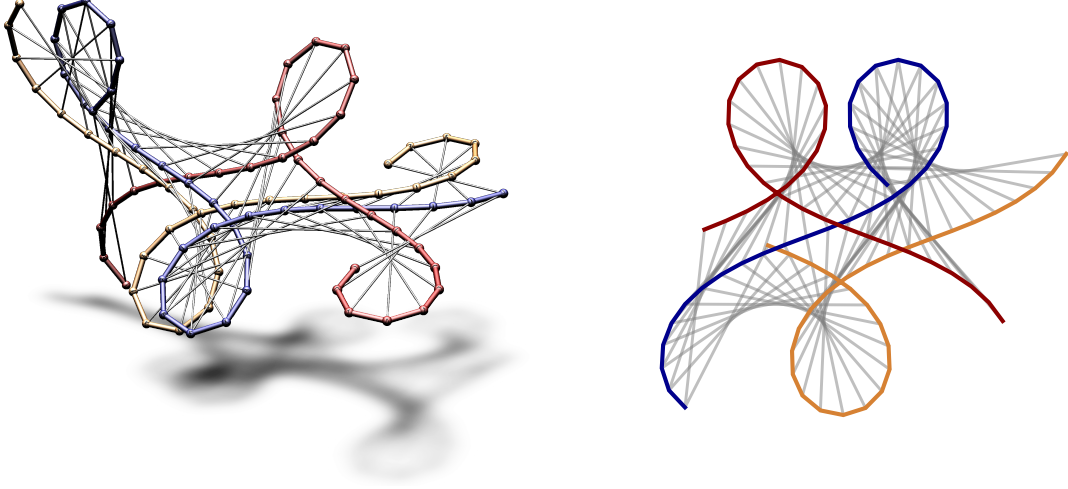


Figure 8.6.: Elastic rods and their corresponding Bäcklund transformations. On the right all curves are in a common plane and the first and last curve (orange and red) are related by reflection and translation.

Theorem 8.4.2 *A regular arc-length parametrized discrete curve is a 2-invariant curve if and only if stays invariant under a combination of tangent and Hashimoto flow, i.e., if it is an elastic rod as in [21].*

Since Bäcklund transformations can be seen as discrete flows this just means that curves invariant under the smooth flow are also invariant under the discrete flow and vice versa.

In [55] it is already shown that curves that stay invariant under two Bäcklund transformations are elastic rods. Here, we also prove the converse and, in particular, know from Theorem 8.4.1 how one can reconstruct the Bäcklund transformations for a given elastic rod.

Proof. First, we express the two flows in terms of quaternions:

$$\begin{aligned} F^T &= -2(u_{\bar{1}} + u)^{-1}, \\ F^H - 1 &= 2 \frac{u_{\bar{1}} \times u - (1 + \langle u_{\bar{1}}, u \rangle)}{|u_{\bar{1}} + u|^2} = 2 \frac{u_{\bar{1}} u + u^2}{|u_{\bar{1}} + u|^2} = -2(u_{\bar{1}} + u)^{-1} u. \end{aligned} \quad (8.24)$$

Now, assume f is 2-invariant. Then, there exists $\hat{A}, \hat{B}$ fulfilling (8.16), (8.17) and (8.19). Now, we can calculate $\hat{B}$ from u :

$$(u_{\bar{1}} + u)\hat{B} = (\hat{B}u\hat{B}^{-1} + u)\hat{B} = \hat{B}u + u\hat{B} = 2\alpha + 2\beta u, \quad (8.25)$$

where in the last step, we use $u\vec{B} + \vec{B}u = 2\text{Re}(\hat{B}u) = 2\alpha$. Then,

$$\hat{B} = 2(u_{\bar{1}} + u)^{-1}(\alpha + \beta u) = \beta - \alpha F^T - \beta F^H. \quad (8.26)$$

The evolution for $\vec{B}$ is

$$\vec{B}_1 - \vec{B} = uE - Eu = -2\vec{E} \times u = -2\vec{E} \times (f_1 - f), \quad (8.27)$$

which implies that there exist $x \in \text{Im}(\mathbb{H})$, such that for $e = 2\vec{E}$

$$-\frac{1}{2}(2\alpha F^T + \beta F^H) = \vec{B} = -\frac{1}{2}(e \times f + x). \quad (8.28)$$

This implies (8.23) and the curve is an elastic rod.

For the converse, assume there exist $\alpha, \beta \in \mathbb{R}$ and $e, x \in \mathbb{R}^3$ fulfilling (8.23). Define $\hat{B} := \beta - \alpha F^T - \beta F^H$ and $\hat{A} := \hat{B}u$. By assumption, we have $\hat{B}_1 - \hat{B} = uE - Eu$ for $E \in \mathbb{H}$ with $\vec{E} = \frac{1}{2}e$. Also,

$$\hat{B} = 2(u_{\bar{1}} + u)^{-1}(\alpha + \beta u). \quad (8.29)$$

implies $u_{\bar{1}}\hat{B} = \hat{B}u$ and, hence, the curve can be obtained from Theorem 8.4.1 and is, thus, a 2-invariant curve. $\square$

If $\alpha = 0$ the curve is known to be an elastic curve. By (8.19), $u_{\bar{1}}, u, A$ are all orthogonal to $\vec{B}$. A and $\vec{B}$ become tangent and binormal vector:

$$A \parallel F^T, \quad \vec{B} \parallel F^H. \quad (8.30)$$

If, additionally, $\vec{B}$ is also orthogonal to $\vec{E}$, the curve is a planar elastic curve.

We conclude this section by remarking that we believe 2-invariant curves in generaleal parametrization to be discrete elastic rods in general parametrization. However, finding geometric characterizations such as Theorem 8.4.2 is difficult in this general case.

9. Curve theory in space forms and n -invariance

This chapter contains a major part of the submitted preprint [64] which has been joint work with Gudrun Szewieczek and Tim Hoffmann. In the paper we also prove results using a light-cone model. The corresponding sections have been written by Gudrun Szewieczek and are, thus, only summarized (as in the case of Section 9.3) or not contained at all in this thesis. Still, we use some results and we point to the preprint [64] for proof.

The paper is self-contained and some language and notation differs from the rest of this thesis. Therefore, let us point out some important connections between this paper and the rest of this thesis.

In this chapter we consider curves and Bäcklund transformations in the plane. Due to the restriction to the plane we are able to introduce Bäcklund transformations in a more geometric manner via anti-parallelograms or *$\mathcal{Q}$ -Darboux butterflies* (Definition 9.4.3). The curve points live on vertices and we consider corresponding edge-based quantities which we call *transport matrices*. In the language of this thesis, the transport matrices form a skew parallelogram net and the curves arise as primitive maps. Specifically, analogous to other parts of this thesis (e.g., Section 5.2), the curves arise as additive primitive maps in Euclidean geometry and as multiplicative primitive maps in non-Euclidean geometry.

The associated family for curves (Proposition 9.2.1) and their Bäcklund transformations (Proposition 9.6.2) coincides with the special case of our general associated family (3.23) described in Theorem 5.2.1.

Clearly, our Definition 9.4.6 for n -invariant curves is a planar version of Definition 8.2.1. We have pointed out the motivation behind restricting to orientation preserving or reversing isometries (and some ambiguities created by this) in Remark 8.3.4. Once more, we will introduce some concepts such as the polynomial (9.42) (which corresponds to the polynomial (3.51) and the monodromy (8.7)). Also, we have seen in Section 8.2 that n -invariant curves are constructed by the algorithm for closed Bäcklund transformations in Section 4.6. Here, this construction is hidden in Lemma 9.5.1 and we explain the connection in Remark 9.5.4.

9.1. Space forms

We study discrete curves in 2-dimensional connected, geodesically complete Riemannian manifolds $\mathcal{Q}$ of constant sectional curvature $\kappa_{\mathcal{Q}}$, so-called *space forms*. Depending on the sign of the sectional curvature, we obtain three types which can be represented by the following standard models:

- *Euclidean space forms* ($\kappa_{\mathcal{Q}} = 0$) modeled as the Euclidean plane $\mathbb{E}^2$;
- *hyperbolic space forms* ($\kappa_{\mathcal{Q}} < 0$) represented either by the Poincaré half-plane, the disc model or the unit hyperboloid

$$\mathbb{H}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1\}$$

together with the Minkowski bilinear form induced by $(x, y, z) \mapsto -x^2 - y^2 + z^2$.

- *spherical space forms* ($\kappa_{\mathcal{Q}} > 0$) canonically considered as the unit sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

equipped with the standard scalar product.

The aim of this section is to provide a framework for discrete curve theory in 2-dimensional space forms, which extends established concepts for discrete curves in Euclidean space. Throughout the paper we will restrict to discrete curves parametrized by arc-length.

Our intention is to define the basic concepts, such as tangent line congruences and curvature circles, in the most geometric manner possible. These will be followed by a detailed description of the geometric objects in the light cone model as well as in a matrix model.

Matrix model for space forms

Here, we consider standard models for each space form as subsets of the complex matrix algebra $\mathbb{C}^{2 \times 2}$. We represent points in the Euclidean plane $\mathbb{E}^2$ and in spherical space (modelled by the unit sphere $\mathbb{S}^2$) by quaternions while we represent points in hyperbolic space (modelled by the unit hyperboloid $\mathbb{H}^2$) by split-quaternions. Thus, we only consider spherical space with $\varepsilon := \kappa_{\mathcal{Q}} = 1$ and hyperbolic space with $\varepsilon := \kappa_{\mathcal{Q}} = -1$. Other curvatures can be obtained by rescaling the metric. Both the quaternions and split-quaternions arise as real sub-algebras of $\mathbb{C}^{2 \times 2}$. For this, consider the basis of $\mathbb{C}^{2 \times 2}$ given by the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{C}^{2 \times 2}. \quad (9.1)$$

The real sub-algebra generated by $\mathbf{1} = \sigma_0, \mathbf{i} = -i\sigma_1, \mathbf{j} = -i\sigma_2, \mathbf{k} = -i\sigma_3$ is isomorphic to the quaternions. We identify the Euclidean plane with a plane in the quaternions via

$$\mathbb{E}^2 \ni (x, y) \leftrightarrow x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} 0 & -y - ix \\ y - ix & 0 \end{pmatrix} \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\} \quad (9.2)$$

and the unit sphere with imaginary unit quaternions via

$$\mathbb{S}^2 \ni (x, y, z) \leftrightarrow x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} -iz & -y - ix \\ y - ix & iz \end{pmatrix} \in \{F \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \mid \det F = 1\}. \quad (9.3)$$

Now, the real sub-algebra generated by $\mathbf{1} = \sigma_0, \sigma_1, \sigma_2, \mathbf{k} = -i\sigma_3$ is isomorphic to the split-quaternions and we identify the unit hyperboloid with imaginary unit split-quaternions via

$$\mathbb{H}^2 \ni (x, y, z) \leftrightarrow x\sigma_1 + y\sigma_2 + z\mathbf{k} = \begin{pmatrix} -iz & x - iy \\ x + iy & iz \end{pmatrix} \in \{F \in \text{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \mathbf{k}\} \mid \det F = 1\}. \quad (9.4)$$

To each matrix $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2 \times 2}$ we denote the adjugate matrix by

$$F^* = (\det F)F^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \quad (9.5)$$

The complex symmetric bilinearform $\langle F, G \rangle := \frac{1}{2} \text{tr} FG^*$ induces the usual Euclidean scalar product with signature $(4, 0)$ on the quaternions while it induces a real inner product with signature $(2, 2)$ on the split-quaternions. Also, the metric in each space form arises from this scalar product:

Space form	Points f, g	Signature of $\langle \cdot, \cdot \rangle$	distance of f and g
Euclidean	$F, G \in \mathbb{E}^2$	$(2, 0)$	$\sqrt{\langle G - F, G - F \rangle}$
Spherical	$F, G \in \mathbb{S}^2$	$(3, 0)$	$\arccos \langle F, G \rangle$
Hyperbolic	$F, G \in \mathbb{H}^2$	$(1, 2)$	$\text{arcosh} \langle F, G \rangle$

Note that the hyperboloid $\mathbb{H}^2$ has two sheets which can be distinguished by the sign of the $\mathbf{k}$ -component of any $F \in \mathbb{H}^2$. The distance between two points F, G is only well-defined if both points lie on the same sheet and, hence, fulfill $\langle F, G \rangle \geq 1$.

A geodesic in Euclidean space $\mathbb{E}^2$ is the solution set of the equation $\langle N, F \rangle = d$ for a unit imaginary quaternion N and normal distance d . The geodesic through two distinct

points F and G is given by

$$N = \frac{\mathbf{k}(G - F)}{\sqrt{\det(G - F)}} \quad \text{and} \quad d = \frac{\langle \mathbf{k}G, F \rangle}{\sqrt{\det(G - F)}}. \quad (9.6)$$

Similarly, a geodesic in a non-Euclidean space form is the solution set of the equation $\langle N, F \rangle = 0$ for a trace-free normalized (split-)quaternion N . The geodesic through two distinct points F, G is given by

$$N = \frac{\overrightarrow{FG}}{\sqrt{\varepsilon \det \overrightarrow{FG}}}, \quad (9.7)$$

where $\varepsilon := \kappa_Q = \pm 1$ and where $\overrightarrow{M} = M - \frac{1}{2} \text{tr } M \mathbf{1}$ denotes the trace-free component of a matrix. Note that in hyperbolic case we have $|\langle F, G \rangle| > 1$ and $\det \overrightarrow{FG} < 0$ and, thus, we normalize to $\det N = -1$.

Isometries in Euclidean space can be expressed by maps of the form

$$F \mapsto EFE^{-1} + E_T, \quad (9.8)$$

where either $E, E_T \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ (orientation reversing) or $E \in \text{span}\{\mathbf{1}, \mathbf{k}\}$ and $E_T \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ (orientation preserving). Isometries in non-Euclidean space can be expressed by maps of the form

$$F \mapsto -EFE^{-1} \text{ (orientation reversing),} \quad (9.9)$$

or

$$F \mapsto EFE^{-1} \text{ (orientation preserving),} \quad (9.10)$$

where E is a quaternion in the spherical case and an invertible split-quaternion in the hyperbolic case.

9.2. Discrete curve theory in space forms

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a 1-dimensional graph with vertices $\mathcal{V}$ and directed combinatorial edges $\mathcal{E}$. A *discrete curve* is a map $f : \mathcal{V} \rightarrow \mathcal{Q}$ that assigns to each vertex of the graph a point in some space form $\mathcal{Q}$. As illustrated in Figure 9.1, we will use the shift notation $f_{\bar{1}}, f_0, f_1$ for three consecutive curve points that are then connected by the two combinatorial edges $e_{\bar{1}0}$ and e_{01} .

When we need to distinguish initial vertex, we specify one vertex $t_0 \in \mathcal{V}$ and write $f(t_0)$ for the initial point of the curve.

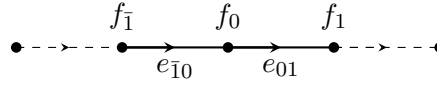


Figure 9.1.: Notation convention used for discrete curves.

Definition 9.2.1 A discrete curve $f : \mathcal{V} \rightarrow \mathcal{Q}$ is said to be parametrized by constant arc-length, if the distance (with respect to the metric of $\mathcal{Q}$) between any two consecutive points of f is constant.

Throughout the paper, we consider *regularly* arc-length parametrized discrete curves in a distinguished space form $\mathcal{Q}$: that is, any three consecutive curve points are pairwise distinct and, additionally,

- if $\mathcal{Q}$ is a hyperbolic space form, all curve points lie inside or outside the hyperbolic boundary;
- if $\mathcal{Q}$ is a spherical space form, then two consecutive points are not antipodal.

Definition 9.2.2 Suppose that $f : \mathcal{V} \rightarrow \mathcal{Q}$ is a discrete regular curve in the space form $\mathcal{Q}$. The unique distance-minimizing geodesic g_{01} in $\mathcal{Q}$ that connects two consecutive curve points f_0 and f_1 is called the geodesic edge between f_0 and f_1 .

Moreover, we say that an orientation for the geodesic edges is consistent if, for any two neighbouring oriented geodesic edges $g_{\bar{1}0}$ and g_{01} , there exists a circle in oriented contact with them at the curve points $f_{\bar{1}}$ and f_1 .

Note that a choice of orientation for one geodesic edge uniquely determines a global orientation for the discrete curve. As we will see below, those oriented geodesic edges encode first order information about the geometry of the curve (see Figure 9.2).

Thus, in what follows, we will consider discrete curves together with their consistently oriented geodesic edges.

Definition 9.2.3 The space of all regular and consistently oriented discrete curves that have constant arc-length in the space form $\mathcal{Q}$ is denoted by $\mathcal{C}_{\mathcal{Q}}$.

In some models it is more convenient to work with the complete geodesics rather than with the geodesic edges. Thus, in analogy with the smooth case, we define a tangent line congruence for discrete curves:

Definition 9.2.4 For a discrete curve $f \in \mathcal{C}_{\mathcal{Q}}$, the map

$$t : \mathcal{E} \rightarrow E_{\mathcal{Q}} := \{\text{complete oriented geodesics in } \mathcal{Q}\}$$

is called the tangent line congruence of f if it extends the oriented geodesic edges.

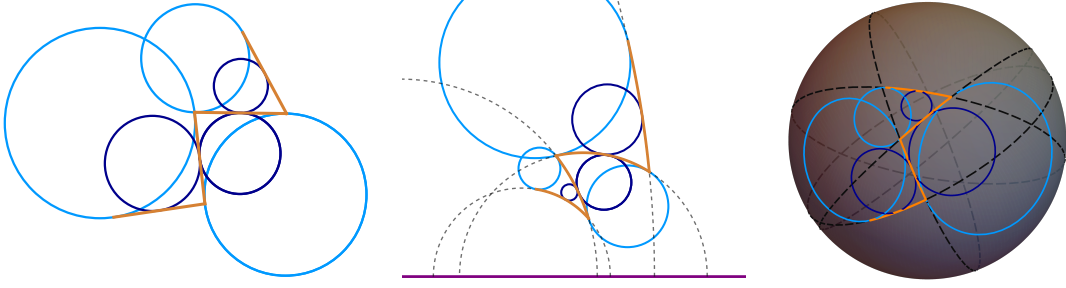


Figure 9.2.: Arc-length parametrized discrete curves in Euclidean, hyperbolic and elliptic space with their geodesic edges (orange), curvature circles (dark blue) and double-curvature circles (light blue).

We are now prepared to introduce a notion of curvature for planar discrete curves in space forms. Our definition naturally generalizes the notion of curvature circle as discussed in [56] for discrete arc-length parametrized curves in Euclidean space: suppose that $f \in \mathcal{C}_{\mathbb{E}^2}$ has constant Euclidean arc-length given by $|f_0 - f_1|^2 \equiv \eta^2 \in \mathbb{R}^\times$. Then the Euclidean curvature at the curve point f_0 is defined as $\kappa_0 := \frac{1}{r_0}$, where r_0 is the radius of the circle c_0 tangent to the oriented edges $g_{\bar{1}0}$ and g_{01} at the midpoints of these two edges (see Figure 9.2 Left).

We observe that the Euclidean curvature of the circle c_0 is exactly twice the Euclidean curvature of the circle tangent to the oriented edges $g_{\bar{1}0}$ and g_{01} at the curve points $f_{\bar{1}}$ and f_1 . As illustrated in Figure 9.2, this fact inspires our curvature notion for discrete curves in space forms:

Definition 9.2.5 *Suppose that $f \in \mathcal{C}_{\mathcal{Q}}$ is a discrete curve in the space form $\mathcal{Q}$. Its geodesic curvature κ_0 at the curve point f_0 is defined as two times the geodesic curvature of the double-curvature circle which is the circle tangent to the geodesic edges $g_{\bar{1}0}$ and g_{01} at the curve points $f_{\bar{1}}$ and f_1 .*

In what follows, we point out how those basic geometric objects for discrete curves in space forms are represented in the two models under consideration.

Discrete curves in the matrix model

Here, we consider curves in $\mathcal{C}_{\mathcal{Q}}(\eta)$ for the standard models $\mathcal{Q} = \mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2$ and identify every curve point f with a complex matrix F as explained above.

The first order information is encoded in an edge based *transport matrix* $u : \mathcal{E} \rightarrow \mathbb{C}^{2 \times 2}$ which we have to define separately for Euclidean and non-Euclidean space forms: In the Euclidean plane we define

$$u_{01} := F_1 - F_0. \quad (9.11)$$

Since the curve has constant arc-length we have that $\sqrt{\det u_{01}} = \eta > 0$ and $\text{tr } u_{01} = 0$ are constant. The geodesic connecting neighboring points is given by the equation $\langle N_{01}, G \rangle = d_{01}$ for $N_{01} = \frac{1}{\eta} \mathbf{k} u_{01}$ and a corresponding normal distance d_{01} .

For a spherical or hyperbolic space form we define

$$u_{01} := F_1 F_0^{-1} = -F_1 F_0. \quad (9.12)$$

Since the curve has constant arc-length we have that

$$\frac{1}{2} \text{tr } u_{01} = \langle F_0, F_1 \rangle = 1 + \varepsilon \langle \mathbf{f}_0, \mathbf{f}_1 \rangle = 1 - \varepsilon \frac{\eta^2}{2} \begin{cases} \in (-1, 1) & \text{for } \mathcal{Q} = \mathbb{S}^2 \\ \in (1, \infty) & \text{for } \mathcal{Q} = \mathbb{H}^2 \end{cases} \quad (9.13)$$

is constant. We introduce the conserved quantity

$$\zeta := \sqrt{\varepsilon \det \vec{u}_{01}} = \sqrt{\varepsilon(1 - \langle F_0, F_1 \rangle^2)} = \eta \sqrt{1 - \varepsilon \frac{\eta^2}{4}} \quad (9.14)$$

which is the sin (or sinh) of the distance between consecutive points. Then, the geodesic connecting neighboring points is given by the equation $\langle N_{01}, G \rangle = 0$ where $N_{01} = \frac{1}{\zeta} \vec{u}_{01}$.

Both quantities η and ζ encode the arc-length. In the Euclidean case we set $\zeta = \eta$ and their relation $\zeta^2 = \eta^2(1 - \varepsilon \frac{\eta^2}{4})$ still holds (with $\varepsilon = \kappa_{\mathcal{Q}} = 0$).

Note that in this model the curve direction induces a consistent orientation on the geodesic edges. Thus, the tangent line congruence is determined by $N_{01} = \frac{1}{\zeta} \vec{u}_{01}$ (and d_{01} in the Euclidean case).

As motivated in [56] for the Euclidean case, we will introduce the harmonic mean

$$T_0 := 2(\vec{u}_{10}^{-1} + \vec{u}_{01}^{-1})^{-1} \quad (9.15)$$

as a vertex tangent vector $T : \mathcal{V} \rightarrow \mathbb{C}^{2 \times 2}$. A quick computation shows that

$$\vec{u}_{01}^{-1} T_0 = 2(\vec{u}_{10}^{-1} \vec{u}_{01} + 1)^{-1} = 2(-\frac{1}{\zeta^2} \vec{u}_{10} \vec{u}_{01} + 1)^{-1} = 2(\vec{u}_{10} \vec{u}_{01}^{-1} + 1)^{-1} = T_0 \vec{u}_{10}^{-1}. \quad (9.16)$$

Thus, we can define a second vertex based map $H : \mathcal{V} \rightarrow \mathbb{C}^{2 \times 2}$ by

$$H_0 := \vec{u}_{01}^{-1} T_0 = T_0 \vec{u}_{10}^{-1}. \quad (9.17)$$

Both T and H transport u by

$$u_{01} = T_0 u_{10} T_0^{-1} = H_0 u_{10} H_0^{-1}. \quad (9.18)$$

The vector H_0 encodes the curvature κ_0 at each curve point F_0 :

Proposition 9.2.6 *In Euclidean space we have $H_0 = \mathbf{1} + \frac{\eta}{2} \kappa_0 \mathbf{k}$. In non-Euclidean space we have $H_0 = \mathbf{1} + \frac{\zeta}{2} \kappa_0 F_0$.*

Proof. We obtain the curvature from the equation $N_{\bar{1}0} - N_{01} = \frac{\kappa_0}{2}(F_1 - F_{\bar{1}})$. Here, again, N denotes the normalized matrix describing the geodesic edge. For details on this, see the preprint [64].

In Euclidean geometry we have $N_{01} = \frac{\mathbf{k}u_{01}}{\eta}$ and, therefore,

$$\begin{aligned} \mathbf{1} + \eta \mathbf{k} \frac{\kappa_0}{2} &= \mathbf{1} + \eta \mathbf{k} (N_{\bar{1}0} - N_{01})(F_1 - F_{\bar{1}})^{-1} = \mathbf{1} - (u_{\bar{1}0} - u_{01})(u_{\bar{1}0} + u_{01})^{-1} \\ &= 2u_{01}(u_{\bar{1}0} + u_{01})^{-1} = H_0. \end{aligned} \quad (9.19)$$

In non-Euclidean geometry we have $N_{01} = \frac{\vec{u}_{01}}{\zeta}$ and, therefore,

$$\begin{aligned} \mathbf{1} + \zeta \frac{\kappa_0}{2} F_0 &= \mathbf{1} + \zeta (N_{\bar{1}0} - N_{01})(F_1 - F_{\bar{1}})^{-1} F_0 = \mathbf{1} + (u_{\bar{1}0} - u_{01})(F_0^* F_1 - F_0^* F_{\bar{1}})^{-1} \\ &= (u_{01}^* - u_{\bar{1}0} + u_{\bar{1}0} - u_{01})(u_{01}^* - u_{\bar{1}0})^{-1} = -2\vec{u}_{01}(-\vec{u}_{01} - \vec{u}_{\bar{1}0})^{-1} = H_0. \end{aligned} \quad (9.20)$$

□

Associated family of curves

As in the smooth case, a discrete curve with constant arc-length in a space form is, up to space form motions, uniquely described by its curvature function and we can formulate a fundamental theorem.

Theorem 9.2.7 *To every space form Q , an admissible arc-length given by η and each map $\kappa : \mathcal{V} \rightarrow \mathbb{R}$ there exists a curve $f \in \mathbb{C}(\eta)$ that has κ as curvature function. This discrete curve f is unique up to motions in Q .*

Geometrically, it is clear that the curve can be iteratively constructed. We exemplify this by explaining the construction of the curve in the matrix model. In Euclidean space $H_0 = \mathbf{1} + \frac{\eta}{2}\kappa_0 \mathbf{k}$ is already determined from the curvature and after prescribing two initial points with distance η we construct the curve by iterating $u_{01} = H_0 u_{\bar{1}0} H_0^{-1}$ and $F_1 = F_0 + u_{01}$. Similarly, in spherical and hyperbolic space we also obtain the curve from two initial points with distance given by η by iterating $H_0 = \mathbf{1} + \frac{\zeta}{2}\kappa_0 F_0$, $u_{01} = H_0 u_{\bar{1}0} H_0^{-1}$ and $F_1 = u_{01} F_0$.

To a given curve we will now compute associated curves with the same curvature function (up to global scaling) in other space forms. This is an application of the associated family presented in [61] and, therefore, closely related to other associated families of curves and surfaces [17]. The associated family provides another way of integrating the curvature function to a non-Euclidean space form by first integrating it into Euclidean space and then applying the associated family. We will also show that

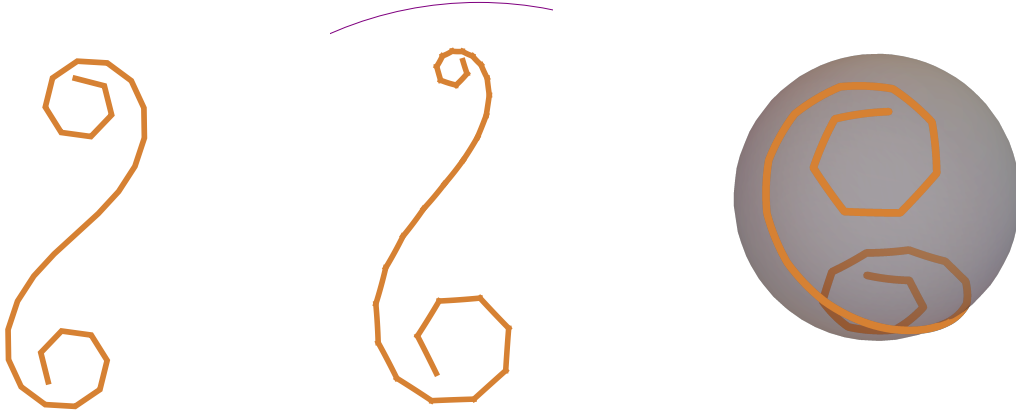


Figure 9.3.: Curves obtained from $\kappa(t_i) = ai$ for some constant $a \in \mathbb{R}$ form discrete Euler spirals/clothoids in Euclidean space $\mathbb{E}^2$, hyperbolic space $\mathbb{H}^2$ (the Poincaré disc) and spherical space $\mathbb{S}^2$.

applying the associated family is reversible. Thus, we can go back and forth between non-Euclidean and Euclidean space. In Section 9.4, we will apply this to generalize some Euclidean statements to space forms.

Here, we consider the curves $f \in \mathbb{C}$ for $\mathcal{Q} = \mathbb{E}^2, \mathbb{S}^2, \mathbb{H}^2$ in the matrix model and use that all space forms are contained in the common complex algebra $\mathbb{C}^{2 \times 2}$. We will also fix the initial point to $F(t_0) = \mathbf{0}$ for curves in Euclidean space and to $F(t_0) = \mathbf{k}$ for curves in non-Euclidean space. This is always achievable by applying an isometry. Now, the curve F is completely determined by its transport matrix u .

To a curve $F \in \mathbb{C}$ and each $\lambda \in \mathbb{C}$ (such that $\mathbf{1} \pm \lambda u_{01}$ is invertible) we assign a map $\Phi^\lambda : \mathcal{V} \rightarrow \mathbb{C}^{2 \times 2}$ defined by $\Phi^\lambda(t_0) = \mathbf{1}$ and $\Phi_1^\lambda = (\mathbf{1} + \lambda u_{01})\Phi_0^\lambda$. Also, we define an edge based map $u^\lambda : \mathcal{E} \rightarrow \mathbb{C}^{2 \times 2}$ by

$$u_{01}^\lambda := (\Phi_1^\lambda)^{-1}(\mathbf{1} - \lambda u_{01})\Phi_0^\lambda. \quad (9.21)$$

Proposition and Definition 9.2.1 *For $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$ the transport matrix u^λ defines a curve*

1. in $\mathcal{C}_{\mathbb{S}^2}$ for each $\lambda \in \mathbb{R} \setminus \{0\}$,
2. in $\mathcal{C}_{\mathbb{H}^2}$ for each $\lambda \in i\mathbb{R}$ with $|\lambda| \in (0, \frac{1}{\eta})$

which we denote by $\mathcal{T}^\lambda f$. Its points can be computed by $\mathcal{T}^\lambda F = (\Phi^\lambda)^{-1} \mathbf{k} \Phi^\lambda$. Similarly, the transport matrix

1. u^1 for a curve $f \in \mathcal{C}_{\mathbb{S}^2}$, or

2. iu^1 for a curve $f \in \mathcal{C}_{\mathbb{H}^2}$

defines a curve in $\mathcal{C}_{\mathbb{E}^2}$ which we denote by $\mathcal{T}^1 f$. We call the set of curves obtained by these operations from a curve $f \in \mathbb{C}$ the associated family of f .

Proof. First, let us state the general equation

$$u_{01}^\lambda = (\Phi_0^\lambda)^{-1}(\mathbf{1} + \lambda u_{01})^{-1}(\mathbf{1} - \lambda u_{01})\Phi_0^\lambda = (\Phi_1^\lambda)^{-1}(\mathbf{1} + \lambda u_{01})^{-1}(\mathbf{1} - \lambda u_{01})\Phi_1^\lambda. \quad (9.22)$$

Now, let $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$ be a Euclidean curve. First, we prove the equation $\mathcal{T}^\lambda F = (\Phi^\lambda)^{-1} \mathbf{k} \Phi^\lambda$ iteratively: If it holds at some point F_0 then it also holds at the next point F_1 since

$$\begin{aligned} \mathcal{T}^\lambda F_1 &= \mathcal{T}^\lambda u_{01} \mathcal{T}^\lambda F_0 = (\Phi_1^\lambda)^{-1}(\mathbf{1} - \lambda u_{01})\Phi_0^\lambda (\Phi_0^\lambda)^{-1} \mathbf{k} \Phi_0^\lambda \\ &= (\Phi_1^\lambda)^{-1} \mathbf{k} (\mathbf{1} + \lambda u_{01})\Phi_0^\lambda = (\Phi_1^\lambda)^{-1} \mathbf{k} \Phi_1^\lambda. \end{aligned} \quad (9.23)$$

Analogously, one can iterate backwards. The equation holds at t_0 and, hence, it holds everywhere. For $\lambda \in \mathbb{R}$ each Φ^λ is a quaternion and, therefore, $\mathcal{T}^\lambda f \in \mathbb{S}^2$ is a spherical curve. Similarly, for $\lambda \in i\mathbb{R}$ each Φ^λ is a split-quaternion and $\mathcal{T}^\lambda f \in \mathbb{H}^2$ is a hyperbolic curve. From (9.22) and

$$(\mathbf{1} + \lambda u_{01})^{-1}(\mathbf{1} - \lambda u_{01}) = \frac{(\mathbf{1} - \lambda u_{01})^2}{1 + \lambda^2 \eta^2} = \frac{(1 - \lambda^2 \eta^2)\mathbf{1} - 2\lambda u_{01}}{1 + \lambda^2 \eta^2} \quad (9.24)$$

we conclude that the arc-length of $\mathcal{T}^\lambda f$ is given (with $\varepsilon = \pm 1$ depending on the space form of $\mathcal{T}^\lambda f$) by

$$\mathcal{T}^\lambda \zeta = \sqrt{\varepsilon \det \vec{u}_{01}^\lambda} = \frac{2|\lambda|\eta}{1 + \lambda^2 \eta^2} \quad \text{and} \quad \mathcal{T}^\lambda \eta^2 = 2\varepsilon \left(1 - \frac{1}{2} \text{tr } u_{01}^\lambda\right) = \frac{4\varepsilon \lambda^2 \eta^2}{1 + \lambda^2 \eta^2} \quad (9.25)$$

which is constant along the curve. Regularity of $\mathcal{T}^\lambda f$ follows from regularity of f and the stated restrictions on λ . For instance, for imaginary λ the condition $|\lambda| \in (0, \frac{1}{\eta})$ ensures that neighboring points are distinct and lie on the same sheet of the hyperboloid. We conclude $\mathcal{T}^\lambda f \in \mathbb{C}$ for $\mathcal{Q} = \mathbb{S}^2$ and $\mathcal{Q} = \mathbb{H}^2$ respectively.

Now, let $f \in \mathbb{C}(\eta)$ for $\mathcal{Q} = \mathbb{S}^2$ or $\mathcal{Q} = \mathbb{H}^2$. We will show that the transport matrix is a quaternion in $\text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$. We have $(\mathbf{1} \pm u_{01})F_0 = F_0 \pm F_1 = \pm F_1(\mathbf{1} \pm u_{01})$ and, therefore, $\Phi^1 F(t_0) = F\Phi^1$. With $F(t_0) = \mathbf{k}$ we have

$$u_{01}^1 \mathbf{k} = (\Phi_1^1)^{-1}(\mathbf{1} - u_{01})F_0\Phi_0^1 = -(\Phi_1^1)^{-1}F_1(\mathbf{1} - u_{01})\Phi_0^1 = -\mathbf{k}u_{01}^1. \quad (9.26)$$

If $\mathcal{Q} = \mathbb{S}^2$ then u_{01}^1 is a quaternion anti-commuting with $\mathbf{k}$ and, thus, we have $u_{01}^1 \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$. If $\mathcal{Q} = \mathbb{H}^2$ then u_{01}^1 is a split-quaternion anti-commuting with $\mathbf{k}$ and, thus,

we have $iu_{01}^1 \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$. In both cases $\mathcal{T}^1 f$ is a well-defined Euclidean curve and from (9.22) and (with $\varepsilon = \pm 1$ depending on the space form of f)

$$(\mathbf{1} + u_{01})^{-1}(\mathbf{1} - u_{01}) = \frac{(\mathbf{1} + u_{01}^*)(\mathbf{1} - u_{01})}{(1 + \frac{1}{2} \text{tr } u_{01})^2 + \det \vec{u}_{01}} = \frac{u_{01}^* - u_{01}}{2 + \text{tr } u_{01}} = \frac{-2\vec{u}_{01}}{4 - \varepsilon\eta^2} \quad (9.27)$$

we conclude that the arc-length of $\mathcal{T}^1 f$ is given by

$$\mathcal{T}^1 \eta = \sqrt{\varepsilon \det u_{01}^1} = \frac{2\zeta}{4 - \varepsilon\eta^2} = \frac{\eta^2}{2\zeta} \quad (9.28)$$

which is constant. Regularity follows from regularity of f and we conclude that $\mathcal{T}^1 f \in \mathcal{C}_{\mathbb{E}^2}$. $\square$

The associated family preserves the curvature (up to a constant):

Proposition 9.2.8 *The curvature of $\mathcal{T}^\lambda f$ is given by $\mathcal{T}^\lambda \kappa = c\kappa$ for the constant $c = \frac{\zeta}{\mathcal{T}^\lambda \zeta}$ where*

1. $c = \frac{1+\lambda^2\eta^2}{2|\lambda|}$ for $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$ and
2. $c = 2(1 + \varepsilon\frac{\eta^2}{4})$ for $f \in \mathcal{C}_{\mathbb{S}^2}(\eta)$ or $f \in \mathcal{C}_{\mathbb{H}^2}(\eta)$ with $\lambda = 1$.

Proof. From (9.22), (9.24) and (9.27) we see that $\mathcal{T}^\lambda u_{01} = (\Phi_0^\lambda)^{-1}(x + y\vec{u}_{01})\Phi_0^\lambda = (\Phi_1^\lambda)^{-1}(x + y\vec{u}_{01})\Phi_1^\lambda$ holds in all cases for some constant scalar quantities x, y . From this, we obtain $\mathcal{T}^\lambda H = (\Phi^\lambda)^{-1}H\Phi^\lambda$. We use the identities in Proposition 9.2.6 for H and $\mathcal{T}^\lambda H$ and in the Euclidean case we write $\zeta = \eta$: For $f \in \mathcal{C}_{\mathbb{E}^2}$ this yields

$$\mathbf{1} + \frac{\mathcal{T}^\lambda \zeta}{2} \mathcal{T}^\lambda \kappa \mathcal{T}^\lambda F = \mathcal{T}^\lambda H = (\Phi^\lambda)^{-1}H\Phi^\lambda = \mathbf{1} + \frac{\zeta}{2} \kappa \mathcal{T}^\lambda F \quad (9.29)$$

while for $f \in \mathcal{C}_{\mathbb{S}^2}$ or $f \in \mathcal{C}_{\mathbb{H}^2}$ this yields (using $F\Phi^1 = \Phi^1 \mathbf{k}$)

$$\mathbf{1} + \frac{\mathcal{T}^1 \zeta}{2} \mathcal{T}^1 \kappa \mathbf{k} = \mathcal{T}^1 H = (\Phi^1)^{-1}H\Phi^1 = \mathbf{1} + \frac{\zeta}{2} \kappa \mathbf{k}. \quad (9.30)$$

In both cases we have $\mathcal{T}^\lambda \kappa = \frac{\zeta}{\mathcal{T}^\lambda \zeta} \kappa$ and with (9.25), (9.28) the claim follows. $\square$

By rescaling the metric of a space form by a factor one scales the curvature with the inverse factor. Therefore, if we scale the metric of the space form containing $\mathcal{T}^\lambda f$ by the factor c we find that f and $\mathcal{T}^\lambda f$ have the same curvature.

For suitable λ_1, λ_2 applying the associated family to a curve f twice yields a curve $\mathcal{T}^{\lambda_2} \mathcal{T}^{\lambda_1} f$ in the original space form of f .

Proposition 9.2.9 *The associated family is reversible in the following sense:*

1. $\mathcal{T}^1 \circ \mathcal{T}^1 = \text{id}$ on $\mathcal{C}_{\mathbb{S}^2}$ and
2. $\mathcal{T}^{-i} \circ \mathcal{T}^1 = \text{id}$ on $\mathcal{C}_{\mathbb{H}^2}$.

Proof. Let $f \in \mathbb{C}$ be a curve in a non-Euclidean space form and set $\lambda = 1$ in the spherical case and $\lambda = -i$ in the hyperbolic case. Then, in both cases $\lambda \mathcal{T}^1 u_{01} = u_{01}^1$. Therefore,

$$\begin{aligned} \mathbf{1} + \lambda \mathcal{T}^1 u_{01} &= \mathbf{1} + u_{01}^1 = (\Phi_1^1)^{-1}(\mathbf{1} + u_{01} + \mathbf{1} - u_{01})\Phi_0^1 = 2(\Phi_1^1)^{-1}\Phi_0^1, \\ \mathbf{1} - \lambda \mathcal{T}^1 u_{01} &= \mathbf{1} - u_{01}^1 = (\Phi_1^1)^{-1}(\mathbf{1} + u_{01} - \mathbf{1} + u_{01})\Phi_0^1 = 2(\Phi_1^1)^{-1}u_{01}\Phi_0^1. \end{aligned} \quad (9.31)$$

The frame $\Psi = \mathcal{T}^1 \Phi$ of $\mathcal{T}^1 f$ fulfils $\Psi_1^\lambda = (\mathbf{1} + \lambda \mathcal{T}^1 u_{01})\Psi_0^\lambda$ and, hence, $\Psi^\lambda(t_k) = 2^k(\Phi^1(t_k))^{-1}$ where $t_k \in \mathcal{V}$ denotes the k -th vertex. The transport matrix $\mathcal{T}^\lambda \mathcal{T}^1 u$ of the curve $\mathcal{T}^\lambda \mathcal{T}^1 f$ is given by (9.21) applied to $\mathcal{T}^1 u$ and we compute

$$\mathcal{T}^\lambda \mathcal{T}^1 = (\Psi_1^\lambda)^{-1}(\mathbf{1} - \lambda \mathcal{T}^1 u_{01})\Psi_0^\lambda = \frac{1}{2}\Phi_1^1(\mathbf{1} - \lambda \mathcal{T}^1 u_{01})(\Phi_0^1)^{-1} = u_{01}. \quad (9.32)$$

Hence, $\mathcal{T}^\lambda \circ \mathcal{T}^1 f = f$. □

Remark 9.2.10 *Using similar techniques one can also show that $\mathcal{T}^1 \circ \mathcal{T}^\lambda = c \text{id}$ on $\mathcal{C}_{\mathbb{E}^2}$ for some constant $c \in \mathbb{R}$.*

9.3. Discrete constrained elastic curves in space forms

In this section we introduce discrete elastic and area-constrained elastic curves in various space forms via a curvature equation. This definition generalizes the known discrete notion of (area-constrained) elastic curves in Euclidean space as discussed in [21, 55, 59, 108].

Based on this definition, we show that these curves satisfy discrete versions of several properties well-known from smooth theory such as the existence of a straight or circular directrix and a certain proportionality condition for the curvature. These observations will then be key to prove in Section 9.4 that discrete constrained elastic curves in space forms are invariant curves of discrete flows built from Bäcklund transformations.

Throughout this section we consider regular discrete curves, $f \in \mathcal{C}_{\mathcal{Q}}$, which are parametrized by arc-length in the space form $\mathcal{Q}$ of constant sectional curvature $\varepsilon = \kappa_{\mathcal{Q}}$.

Definition 9.3.1 *A discrete curve $f \in \mathcal{C}_{\mathcal{Q}}(\eta)$ is said to be constrained elastic in the space form $\mathcal{Q}$ if it satisfies the following curvature equation*

$$\kappa_{\bar{1}} + \kappa_1 = \frac{\xi \kappa_0 + \delta}{1 + \frac{\xi^2}{4} \kappa_0^2}, \quad (9.33)$$

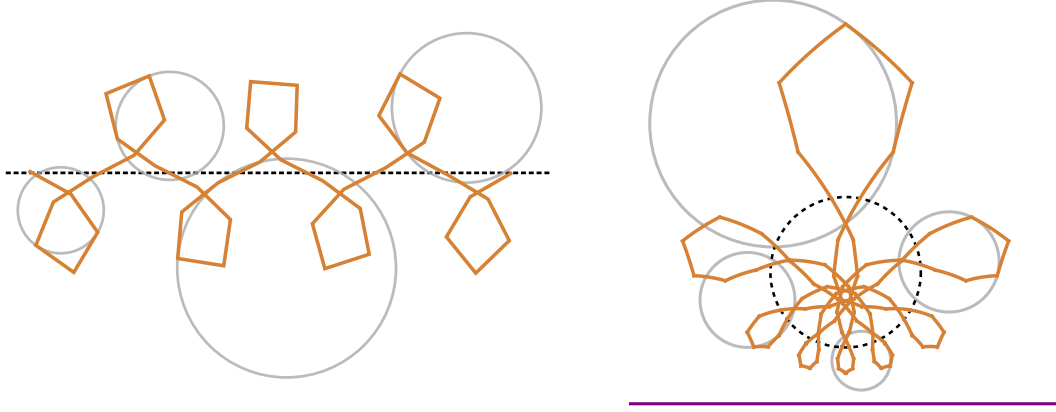


Figure 9.4.: A discrete elastic curve in Euclidean space and a discrete area-constrained elastic curve in hyperbolic space with their directrices (dotted) and some double-curvature circles (gray). The latter intersect the directrix at a constant angle.

where $\zeta^2 := \eta^2(1 - \varepsilon \frac{\eta^2}{4})$ and $\xi, \delta \in \mathbb{R}$ are suitable constants.

More specifically, the curve is called elastic if $\delta = 0$ and area-constrained elastic, otherwise.

As a first simple example we see that discrete circles, that is, curves with non-vanishing constant curvature, satisfy the curvature equation. In particular, in this case the choice of the constants ξ and δ is not unique. To avoid case studies, in what follows we will consider discrete circles as area-constrained elastic curves.

Smooth constrained elastic curves in space forms can be characterized by the existence of a special tangential circle congruence, which gives rise to a distinguished directrix for each such curve [37]. In our preprint [64] we prove the existence of a similar circle congruence in the discrete case, the double-curvature circle congruence. As illustrated in Figure 9.2, this double-curvature circle associated to the curve point f_0 is in oriented contact to the two adjacent tangent lines $t_{\bar{1}0}$ and t_{01} and goes through the curve points $f_{\bar{1}}$ and f_1 .

For smooth elastic curves in Euclidean space, it is well-known that they may be characterized as curves whose curvature is proportional to the distance of a line, its directrix. As discussed in [5], also smooth area-constrained elastic curves can be described in a similar way: their curvature is proportional to the squared tangential distance to a fixed circle.

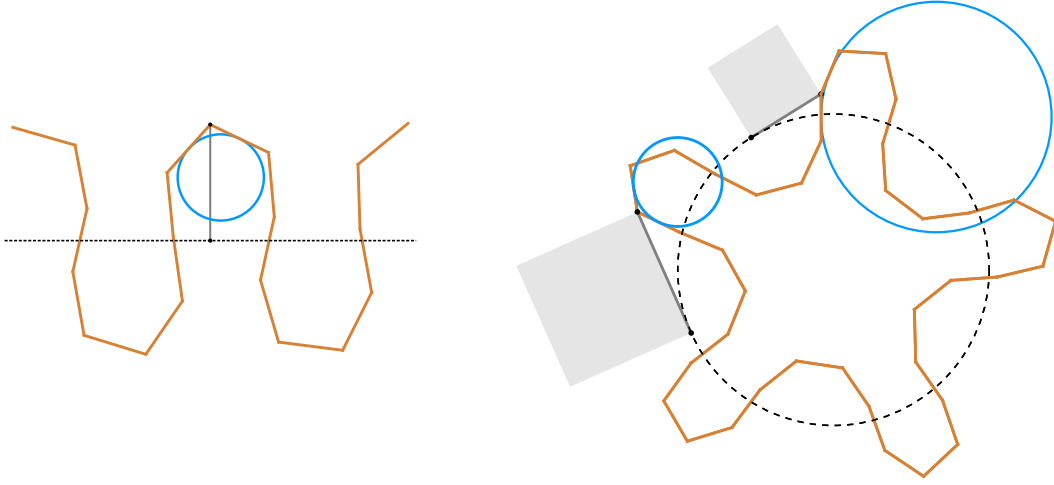


Figure 9.5.: A discrete (area-constrained) elastic curve in Euclidean space: its curvature and orthogonal (squared tangential) distance to the corresponding directrix are proportional (see Corollary 9.3.2).

The *tangential distance* d in Euclidean space of a point $f \in \mathbb{R}^2$ to a circle with center $c \in \mathbb{R}^2$ and radius r is given by

$$d := \sqrt{\|f - c\|^2 - r^2}. \quad (9.34)$$

For $r \in \mathbb{R}$ and the point f outside the circle this is the distance of f to a point on the circle such that the line through f and this point is tangent to the circle. However, we extend this definition to include imaginary radii and points inside the circle and, thus, d can be imaginary.

As a consequence of [64, Proposition 18] in our preprint, we obtain these characterizations also in the discrete case (cf. Figure 9.5):

Corollary 9.3.2 (i) *A curve $f \in \mathcal{C}_{\mathbb{E}^2}$ is elastic in $\mathbb{E}^2$ if and only if there is a line such that the signed orthogonal distance d of each curve point to the line is proportional to the curvature at this vertex: $d = c\kappa$ for a constant $c \in \mathbb{R}$.*

(ii) *A curve $f \in \mathcal{C}_{\mathbb{E}^2}$ is area-constrained elastic in $\mathbb{E}^2$ if and only if there exists a circle such that the squared tangential distance d of each curve point to this circle is proportional to the curvature, that is, $d^2 = c\kappa$ for a constant $c \in \mathbb{R}$.*

9.4. Bäcklund transformations and n -invariant curves

We will now apply an integer number of Bäcklund transformations to a discrete curve which we can interpret as a step of a discrete flow of the curve. The invariant curves of each flow give rise to a hierarchy of curves and, as we will see in Theorem 9.4.11, this hierarchy contains elastic and constrained elastic curves.

In this section we investigate transformations of curves built from elementary quads. For each quad (a, b, c, d) we will assume the regularity condition of non-vanishing diagonals $a \neq c$ and $b \neq d$. However, we will encounter some special cases where neighboring points can coincide.

Definition 9.4.1 We call a quad (a, b, c, d) a $\mathcal{Q}$ -Darboux butterfly if there is a reflection in a geodesic of the space form $\mathcal{Q}$ interchanging the pairs $a \leftrightarrow c$ and $b \leftrightarrow d$.

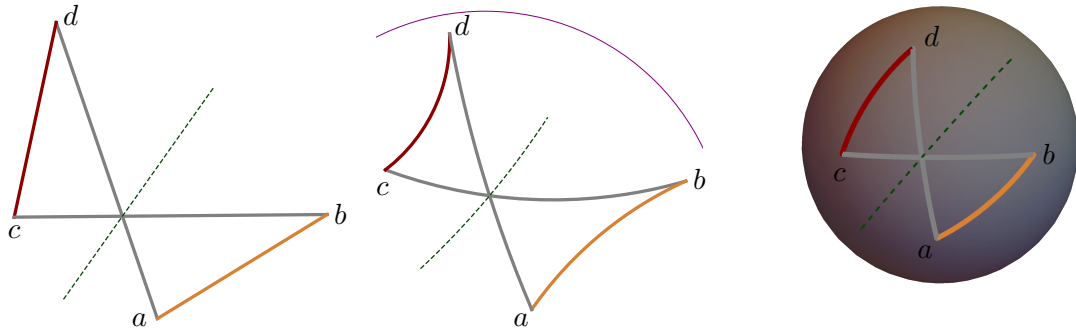


Figure 9.6.: An $\mathbb{E}^2$ -, $\mathbb{H}^2$ - and $\mathbb{S}^2$ -Darboux butterfly with its axis of reflection (green).

The name *Darboux butterfly* is adopted from [105]. In Euclidean geometry these quads are *anti-parallelograms*. Any $\mathcal{Q}$ -Darboux butterfly is circular and opposite edges have the same length in $\mathcal{Q}$. On the other hand, adjacent edges of the quad do not have the same length: If a would have the same distance to b and d then the reflection interchanging $b \leftrightarrow d$ would map a to itself violating the regularity assumption. A $\mathcal{Q}$ -Darboux butterfly is uniquely determined by any three of its points. We study these quads in the matrix model. For $\mathcal{Q} = \mathbb{S}^2$ and $\mathcal{Q} = \mathbb{H}^2$ we assume neighboring points not to be antipodal. Many equations can be translated to the light cone model using a Moutard equation (see [24, Section 2.3]).

Proposition 9.4.2 A quad (a, b, c, d) is a $\mathcal{Q}$ -Darboux butterfly if and only if its matrix representations A, B, C, D fulfil

1. $(C - B)(B - A) = (C - D)(D - A)$ for $\mathcal{Q} = \mathbb{E}$ or
2. $CB + BA = CD + DA$ for $\mathcal{Q} = \mathbb{S}$ and $\mathcal{Q} = \mathbb{H}$.

Proof. In Euclidean geometry the statement is translation invariant and we can assume that the fixed line of the reflection interchanging $B \leftrightarrow D$ intersects the origin. This implies $\det B = \det D$ and the equation reads

$$0 = (C - B)(B - A) - (C - D)(D - A) = C(B - D) + (B - D)A. \quad (9.35)$$

This is equivalent to

$$C = -(B - D)A(B - D)^{-1} = (\mathbf{k}(B - D)) A (\mathbf{k}(B - D))^{-1} \quad (9.36)$$

which is exactly the condition that the pair A, C gets interchanged by the same reflection as the pair B, D . Similarly, in non-Euclidean geometry the equation is equivalent to

$$C = -(B - D)A(B - D)^{-1} \quad (9.37)$$

which, again, is the condition that the pair A, C gets interchanged by the same reflection as the pair B, D . $\square$

Definition 9.4.3 Two curves $f, \tilde{f} \in \mathbb{C}$ are said to be related by a Bäcklund transformation if each quad $(f_0, f_1, \tilde{f}_1, \tilde{f}_0)$ forms a $\mathcal{Q}$ -Darboux butterfly.

This transformation is also a special *Darboux transformation* since all quads have the same cross-ratio (which is positive) and, in the Euclidean case, it is known as the *discrete bicycle transformation* [105].

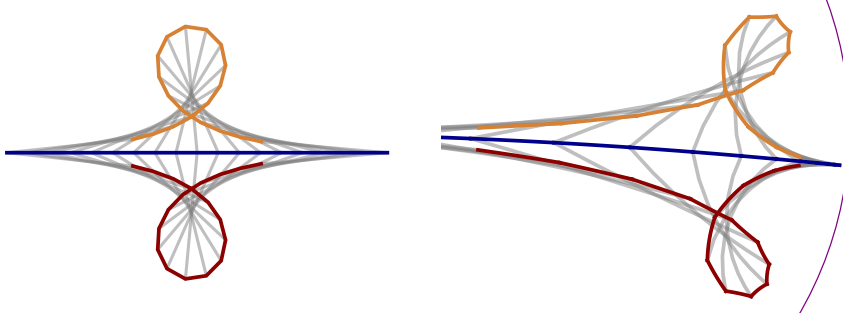


Figure 9.7.: A Bäcklund transformation of an equally sampled geodesic in a space form (blue) yields a discrete Euler loop (orange). Since its reflection (red) along the geodesic is also a Bäcklund transform of the geodesic the Euler loop is 2-invariant.

One initial point $\tilde{f}(t_0)$ already determines the Bäcklund transformation of a curve $f \in \mathbb{C}$. Here, the point $f(t_0)$ can be any point (distinct from the antipodal $f(t_0)$ point

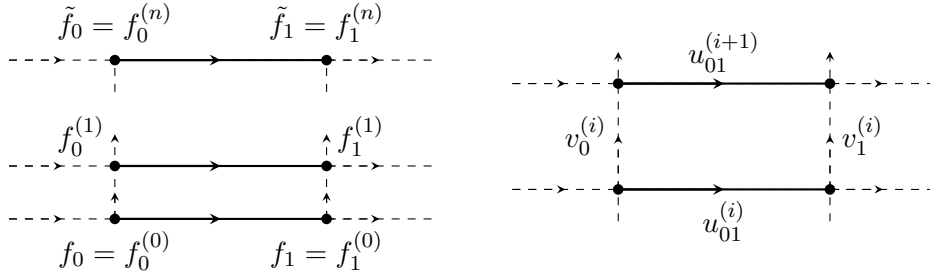


Figure 9.8.: Notation convention for a sequence of curves (*left*) and, in particular, the transport matrices at one quad (*right*).

in the non-Euclidean cases) that does not have the arc-length of the curve as distance to $f(t_0)$.

In particular, if $\tilde{f}(t_0) = f(t_0)$ we obtain the trivial case where $\tilde{f} = f$ holds for all vertices and we call this the *identity transformation*. This is the only case for which different points of a quad can coincide.

Now, we consider a sequence of curves $f^{(0)}, \dots, f^{(n)} \in \mathbb{C}$. We consider this sequence f to live on the directed graph $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ with vertices $\mathcal{V}_n = \mathcal{V} \times \{0, \dots, n\}$ and $\mathbb{Z}^2$ combinatorics (see Figure 9.8). The edge set $\mathcal{E}_n$ consists of edges directed in curve direction and edges directed in transformation direction. Analogous to the transport matrix u in curve direction we define the transport matrix v in transformation direction by $v_0^{(i)} := F_0^{(i+1)} - F_0^{(i)}$ in Euclidean space and by $v_0^{(i)} := F_0^{(i+1)}(F_0^{(i)})^{-1}$ in non-Euclidean space.

We are interested in sequences of Bäcklund transformations, i.e., in the case where consecutive curves $f^{(i)}$ and $f^{(i+1)}$ are related by a Bäcklund transformation. As a direct consequence of Proposition 9.4.2 we have

Corollary 9.4.4 *A sequence of curves $f^{(0)}, \dots, f^{(n)}$ is a sequence of Bäcklund transformations if and only if the corresponding map $p = (u, v) : \mathcal{E}_n \rightarrow \mathbb{C}^{2 \times 2}$ defined by the transport matrices fulfills*

$$u_{01}^{(i)} + v_1^{(i)} = v_0^{(i)} + u_{01}^{(i+1)} \quad (9.38)$$

and

$$v_1^{(i)} u_{01}^{(i)} = u_{01}^{(i+1)} v_0^{(i)} \quad (9.39)$$

at every quad.

One can also consider general maps $p = (u, v) : \mathcal{E}_n \rightarrow \mathbb{C}^{2 \times 2}$ (not necessarily coming from transport matrices).

Definition 9.4.5 An edge based map $p := (u, v) : \mathcal{E}_n \rightarrow \mathbb{C}^{2 \times 2}$ is called a skew parallelogram net if it fulfills (9.38) and (9.39) at every quad.

Thus, the transport matrices of a sequence of Bäcklund transformations form a skew parallelogram net. Skew parallelogram nets are studied in [61] as lattice equation system encompassing many integrable systems studied in discrete differential geometry.

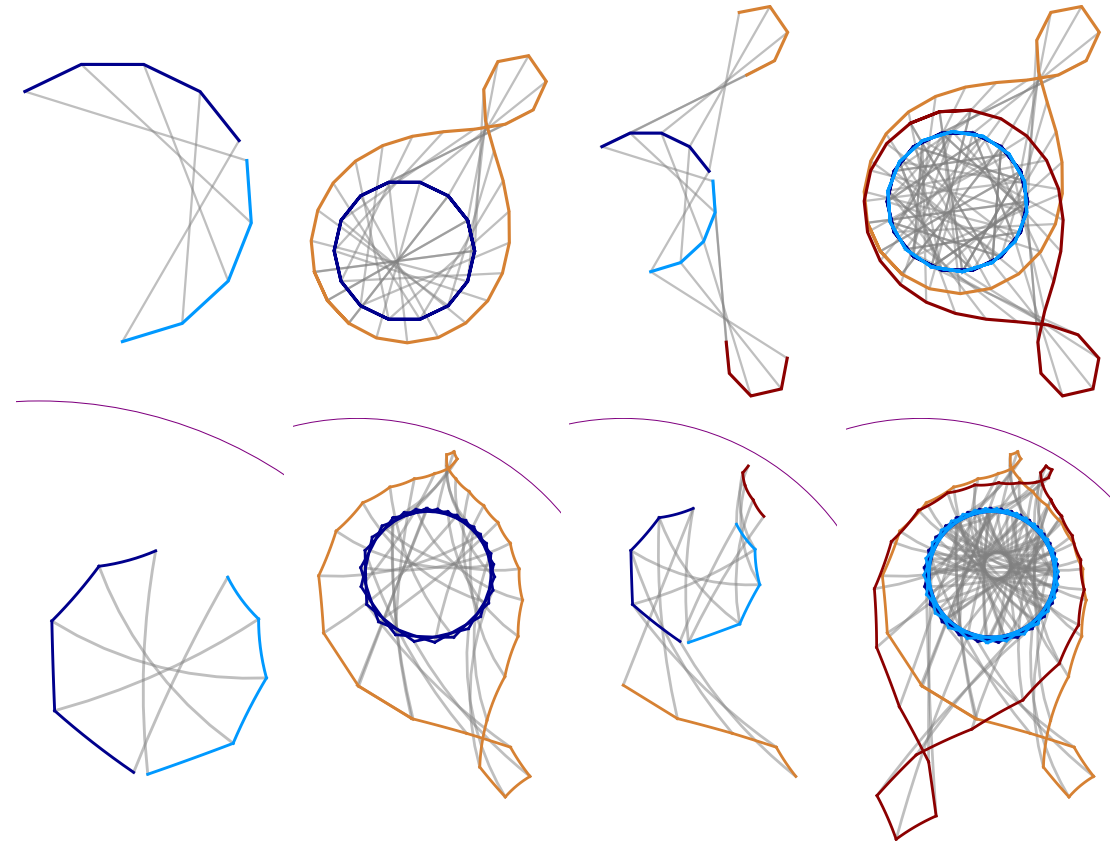


Figure 9.9.: Bäcklund transformations of an equally sampled circle in Euclidean space (top row) and hyperbolic space (bottom row). The figures on the right show a sequence of the transformations seen on the left which causes the orange curve to be 3-invariant.

Now, consider a sequence of Bäcklund transformations $f = f^{(0)}, f^{(1)}, \dots, f^{(n)} = \tilde{f}$ that preserves the shape of the curve f , meaning that the first and last curve f and $\tilde{f}$ are related by an isometry. We express the action of the isometry on the transport matrices by $\tilde{u}_{01} = E^{-1}u_{01}E$ for an invertible (split-)quaternion E .

Equations (9.38) and (9.39) yield

$$(\mathbf{1} + \lambda v_1^{(i)})(\mathbf{1} + \lambda u_{01}^{(i)}) = (\mathbf{1} + \lambda u_{01}^{(i+1)})(\mathbf{1} + \lambda v_0^{(i)}) \quad (9.40)$$

for any $\lambda \in \mathbb{C}$ and at every quad. Thus, multiplication of such linear factors built from the transport matrices is path-independent. In particular, we have

$$\begin{aligned} (\mathbf{1} + \lambda v_1^{(n-1)}) \cdots (\mathbf{1} + \lambda v_1^{(0)})(\mathbf{1} + \lambda u_{01}) &= (\mathbf{1} + \lambda \tilde{u}_{01})(\mathbf{1} + \lambda v_0^{(n-1)}) \cdots (\mathbf{1} + \lambda v_0^{(0)}) \\ &= E^{-1}(\mathbf{1} + \lambda u_{01})E(\mathbf{1} + \lambda v_0^{(n-1)}) \cdots (\mathbf{1} + \lambda v_0^{(0)}) \end{aligned} \quad (9.41)$$

and, therefore, the vertex based polynomial

$$\mathcal{P}_0(\lambda) := E(\mathbf{1} + \lambda v_0^{(n-1)}) \cdots (\mathbf{1} + \lambda v_0^{(0)}) =: C_0^0 + \lambda C_0^1 + \cdots + \lambda^n C_0^n \quad (9.42)$$

fulfills

$$\mathcal{P}_1(\mathbf{1} + \lambda u_{01}) = (\mathbf{1} + \lambda u_{01})\mathcal{P}_0. \quad (9.43)$$

Other examples can be obtained by going back and forth (e.g., by $f, f^{(1)}, f^{(2)}, f^{(1)}, f$) or by using permutability of Bäcklund transformations (see [87, 98]). All these special sequences can be identified by the fact that the polynomial $\mathcal{P}$ is scalar, i.e., by $\vec{\mathcal{P}} = 0$. On the other hand, we call a sequence of Bäcklund transformations *regular* if it fulfills $\vec{\mathcal{P}} \neq 0$.

Finding such a regular sequence of transformations is more complicated and, in fact, it only exists for a special class of curves:

Definition 9.4.6 *A curve $f \in \mathbb{C}$ is called n -invariant if there exists a regular sequence of n Bäcklund transformations $f = f^{(0)}, \dots, f^{(n)} = \tilde{f}$ such that the n -th transform $\tilde{f}$ is related to f by an isometry of $\mathcal{Q}$ which is*

1. *orientation preserving if n is odd and*
2. *orientation reversing if n is even.*

A corresponding version of this hierarchy for curves in Euclidean 3-space is given by [61, Definition 3.2] where the isometry is a screw motion. By restriction to the plane one needs to distinguish the two cases of the axis of the screw motion being contained in the plane or being orthogonal to the plane. For each n only one case yields interesting curves and this depends on the parity of n , thus, motivating our condition on the orientation of the isometry.

Note that by adding two identity transformations to a sequence of Bäcklund transformations we can observe that every n -invariant curve is also $(n+2)$ -invariant. Also,

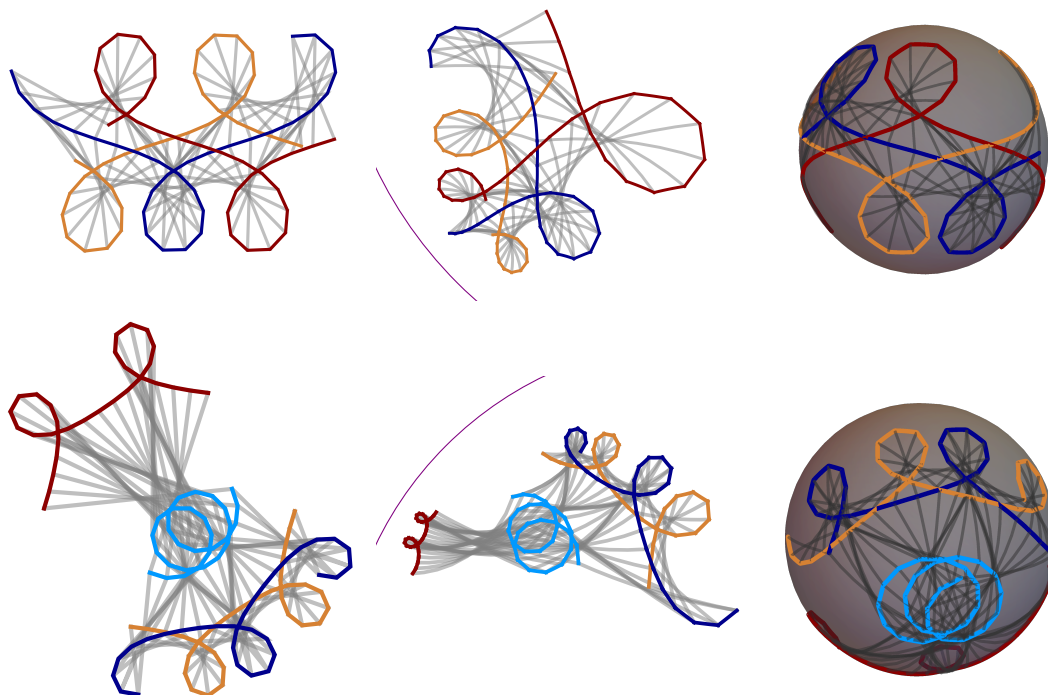


Figure 9.10.: Elastic and area-constrained elastic curves in each space form and their corresponding Bäcklund transformations. The first and last curve (orange and red) are isometric in the respective space form.

by adding one identity transformation, we can observe that curves invariant after a sequence of n Bäcklund transformations with the isometry having 'wrong' orientation (i.e., reversing for odd n , preserving for even n) are $(n + 1)$ -invariant.

One can interpret the n transformations as fundamental piece of a (bi-)infinite sequence of Bäcklund transformations with period n . In fact, one could analogously define n -invariant curves via the existence of such a periodic infinite sequence. This observation also shows that the intermediate curves $f^{(1)}, \dots, f^{(n-1)}$ are n -invariant as well. Note that in this way we can interpret the n Bäcklund transformations as step of a discrete flow (as in [55, 87] for $n = 2$) and the n -invariant curves as the invariant curves of this flow.

Example 9.4.7 A Bäcklund transform of an equally sampled circle with initial point on the circle (as seen in Figure 9.9 on the left) is a rotated version of the original curve and, as such, 1-invariant. Such discrete circles (and lines) are the only 1-invariant curves.

Example 9.4.8 A Bäcklund transform of an equally sampled straight line yields a discrete Euler loop as seen in Figure 9.7. Reflecting this construction along the line yields a

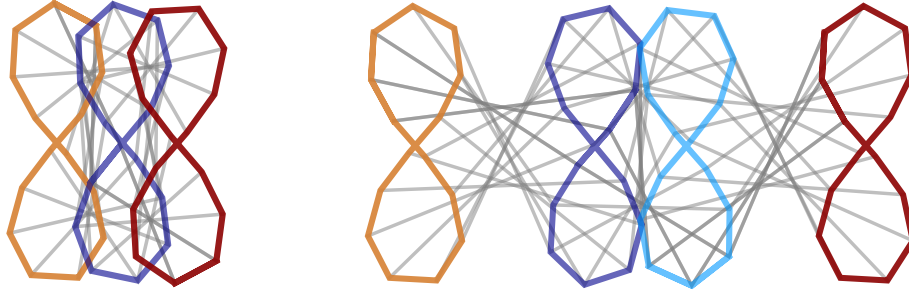


Figure 9.11.: An elastic curve is 2-invariant (left) and 3-invariant (right). Observe that the last curve of the sequence (red) is obtained from the first curve (orange) by reflection and translation on the left but only by translation on the right.

second Euler loop which is related to the first Euler loop by two Bäcklund transformations and by reflection. As such, the Euler loop is 2-invariant.

Proposition 9.4.9 *A Bäcklund transform $\hat{f}$ of an n -invariant curve f is $(n + 2)$ -invariant.*

Proof. There is a curve $\tilde{f}$ related to f by a regular sequence of n Bäcklund transformations and by an isometry φ (i.e., $\tilde{f} = \psi(f)$). Then, application of the isometry to the curve $\hat{f}$ yields a curve $\psi(\hat{f})$ which is related to $\tilde{f}$ by the regular sequence of $(n + 2)$ Bäcklund transformations $\hat{f}, f = f^{(0)}, \dots, f^{(n)} = \tilde{f} = \psi(f), \psi(\hat{f})$. $\square$

Example 9.4.10 *A Bäcklund transform of an equally sampled circle with an arbitrary initial point is 3-invariant as seen in Figure 9.9 on the right. This includes closed Darboux transforms of discrete circles as studied in [36].*

We are prepared to state the main theorem of this section:

Theorem 9.4.11 *A curve $f \in \mathbb{C}$ is 2-invariant if and only if it is a discrete elastic curve in $\mathcal{Q}$. It is 3-invariant if and only if it is a discrete constrained elastic curve in $\mathcal{Q}$.*

In particular, as demonstrated in Figure 9.11, elastic curves are both 2- and 3-invariant, however, the relation between the corresponding sequences of Bäcklund transformations is not obvious.

In Section 9.5, we will prove this theorem for curves in Euclidean space. In Section 9.6, we will extend the result to space forms using the associated family.

9.5. Euclidean n -invariant curves in the plane

We will now consider n -invariant curves $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$. First, we discuss how the coefficients C^j of the polynomial $\mathcal{P}$ given by (9.42) contain invariants along the curve and how they are related to the curvature and tangential vector of the curve.

Note that in the case of an identity transformation the transport matrix $v_0^{(i)}$ vanishes and the corresponding linear factor becomes trivial. Thus, the polynomial $\mathcal{P}$ can have degree smaller than n and some of the higher coefficients can vanish.

Now, each coefficient C^j is a product of E with j transport matrices $v^{(i)}$. Recall that $v \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ and $E \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ for even n and $E \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ for odd n and, therefore, we have

$$C^n, C^{n-2}, \dots \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\} \quad \text{and} \quad C^{n-1}, C^{n-3}, \dots \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}. \quad (9.44)$$

From (9.43) we see that the real polynomials $\text{tr } \mathcal{P}$ and $\det \mathcal{P}$ remain constant along the curve and their coefficients are invariants. The coefficients of $\text{tr } \mathcal{P}$ are given by $r_j := \frac{1}{2} \text{tr } C^j$ and vanish for $j = n, n-2, \dots$. Also, we have

$$\det \mathcal{P} = \det E (1 + \lambda^2 \det v^{(n-1)}) \cdots (1 + \lambda^2 \det v^{(0)}) \quad (9.45)$$

and, hence, all coefficients of $\det \mathcal{P}$ are given by $\det E$ and $\det v^{(i)}$ which we already knew to be constant. For later use, recall that non-vanishing diagonals of the Darboux butterflies implies $\det v^{(i)} \neq \eta^2$ and, therefore, we have $\det \mathcal{P}(\frac{i}{\eta}) \neq 0$.

Invariants contain information about the curve and the Bäcklund transformations. The coefficients of the invariant polynomial $\det \vec{\mathcal{P}} = \det \mathcal{P} - (\text{tr } \mathcal{P})^2 = -(\vec{\mathcal{P}})^2$ will turn out to only depend on the curve. The λ^j coefficient is given by

$$\theta_j := \sum_{k=\max\{0, j-n\}}^{\min\{j, n\}} \langle \vec{C}^k, \vec{C}^{j-k} \rangle. \quad (9.46)$$

We have that $\vec{C}^k \perp \vec{C}^l$ if k and l have different parity and, hence, $\theta_j = 0$ holds for odd j . For $n \leq 3$ the invariants θ_j are given by the following table:

θ_0	$n = 1$ $\det \vec{C}^0$	$n = 2$ $\det C^0$	$n = 3$ $\det \vec{C}^0$
θ_2	$\det C^1$	$\det \vec{C}^1 + 2\langle C^0, C^2 \rangle$	$\det C^1 + 2\langle \vec{C}^0, \vec{C}^2 \rangle$
θ_4		$\det C^2$	$\det \vec{C}^2 + 2\langle C^1, C^3 \rangle$
θ_6			$\det C^3$

The following lemma shows how the polynomial contains information about the curve.

Lemma 9.5.1 *Let $f \in \mathcal{C}_{\mathbb{R}^2}(\eta)$ be n -invariant with polynomial $\mathcal{P}$ given by (9.42). Then, the vertex based quaternions*

$$\begin{aligned} A &:= C^n - \eta^2 C^{n-2} + \eta^4 C^{n-4} - \dots \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\} \\ B &:= C^{n-1} - \eta^2 C^{n-3} + \eta^4 C^{n-5} - \dots \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\} \end{aligned} \quad (9.47)$$

and an invariant $\beta \in \mathbb{R}$ given by $\beta^2 = -(-\eta^2)^{n-1} \det \vec{\mathcal{P}}(\frac{i}{\eta}) \neq 0$ have the following relations to the curve:

1. A is related to the tangential vector via $A = \beta T$ and
2. $\vec{B}$ encodes the curvature via $\beta - \vec{B} = \beta H$ and, thus, $\vec{B} = -\kappa \frac{\beta \eta}{2} \mathbf{k}$.

Proof. Expanding (9.43) in powers of λ gives us equations for the coefficients C^i :

$$\begin{aligned} \lambda^0 : \quad C_1^0 &= C_0^0 (= E) \\ \lambda^i : \quad C_1^i + C_1^{i-1} u_{01} &= C_0^i + u_{01} C_0^{i-1}, \quad 1 \leq i \leq n \\ \lambda^{n+1} : \quad C_1^n u_{01} &= u_{01} C_0^n \end{aligned} \quad (9.48)$$

The equations for $0 \leq i \leq n$ already determine $\mathcal{P}_1$ from $\mathcal{P}_0$ and u_{01} . Plugging these into the last equation provides a condition relating $\mathcal{P}_0$ and u_{01} :

$$\begin{aligned} u_{01} C_0^n &= C_1^n u_{01} = C_0^n u_{01} + u_{01} C_0^{n-1} u_{01} - C_1^{n-1} u_{01}^2 = \dots \\ &= C_0^n u_{01} + u_{01} C_0^{n-1} u_{01} - C_0^{n-1} u_{01}^2 - u_{01} C_0^{n-2} u_{01}^2 + C_0^{n-2} u_{01}^3 + \dots \end{aligned} \quad (9.49)$$

With $u_{01}^2 = -\eta^2$ this can be rearranged to

$$u_{01}(C_0^n - C_0^{n-1} u_{01} - \eta^2 C_0^{n-2} + \dots) = (C_0^n - C_0^{n-1} u_{01} - \eta^2 C_0^{n-2} + \dots) u_{01} \quad (9.50)$$

which reads

$$u_{01}(A_0 - B_0 u_{01}) = (A_0 - B_0 u_{01}) u_{01}. \quad (9.51)$$

Two imaginary quaternions commute if and only if they are multiples of each other and, hence, there exists $\beta \in \mathbb{R}$ with $A_0 - B_0 u_{01} = (\beta - \frac{1}{2} \text{tr } B_0) u_{01}$ which is equivalent to $A_0 = (\beta + \vec{B}_0) u_{01}$. We can also obtain $A_0 = -(\hat{\beta} + \vec{B}_0) u_{\bar{1}0}$ for some $\hat{\beta} \in \mathbb{R}$ since $-u_{\bar{1}0}$ must fulfil the same equations as u_{01} due to the symmetry of the construction. Taking the determinant on both sides of these equations and using $(-i\eta)^n \mathcal{P}(\frac{i}{\eta}) = A - i\eta B$ yields

$$\eta^2 \hat{\beta}^2 = \eta^2 \beta^2 = \det A - \eta^2 \det \vec{B} = \det(A - i\eta \vec{B}) = (-\eta^2)^n \det \vec{\mathcal{P}}(\frac{i}{\eta}). \quad (9.52)$$

Since the curve is regular $-u_{\bar{1}0}$ and u_{01} are distinct and, thus, we have $\hat{\beta} = -\beta \neq 0$ and $A_0 = (\beta - \vec{B}_0) u_{\bar{1}0} = u_{\bar{1}0}(\beta + \vec{B}_0)$. Note that a quaternion $X \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ is

determined up to scaling by the equation $u_{01} = X^{-1}u_{\bar{1}0}X$. Both quaternions $\beta + \vec{B}_0$ and $H_0^* = \mathbf{1} - \vec{H}_0$ fulfill this equation and, hence, $\beta + \vec{B}_0 = \beta H_0^*$ and $A_0 = \beta H_0^* u_{01} = \beta T_0$ show both statements.

Finally, we prove that β is constant (not only up to sign). At neighboring vertices we write

$$\begin{aligned} A_0 + u_{01}B_0 &= (\beta_0 + \vec{B}_0)u_{01} + u_{01}B_0 = (\beta_0 + \frac{1}{2} \operatorname{tr} B_0)u_{01}, \\ A_1 + B_1u_{01} &= u_{01}(\beta_1 + \vec{B}_1) + B_1u_{01} = (\beta_1 + \frac{1}{2} \operatorname{tr} B_0)u_{01}. \end{aligned} \quad (9.53)$$

Both of these terms agree which follows from (9.43) with $\lambda = \frac{i}{\eta}$ which reads

$$(A_1 - i\eta B_1)(\mathbf{1} + \frac{i}{\eta}u_{01}) = (\mathbf{1} + \frac{i}{\eta}u_{01})(A_0 - i\eta B_0). \quad (9.54)$$

Thus, we have $\beta_1 = \beta_0$. □

Remark 9.5.2 *There is one special case which, for brevity, is not mentioned in the Lemma: If $A_0 = \vec{B}_0 = 0$ or, equivalently, $\vec{\mathcal{P}}_0(\frac{i}{\eta}) = 0$ at some point f_0 there is no condition on u . In this case we have $\det \vec{\mathcal{P}}(\frac{i}{\eta}) = 0$ and since this is an invariant along the curve we have $\vec{\mathcal{P}}(\frac{i}{\eta}) = 0$ at every point. Therefore, $1 + \lambda^2\eta^2$ is a factor of $\vec{\mathcal{P}}$ and the unique polynomial $\mathcal{Q}$ defined by $\vec{\mathcal{P}} = (1 + \lambda^2\eta^2)^k \mathcal{Q}$ with $\mathcal{Q}(\frac{i}{\eta}) \neq 0$ still fulfils (9.43). Thus, the statement of the Lemma still holds true if we replace $\vec{\mathcal{P}}$ with the reduced polynomial $\mathcal{Q}$. Factoring of such real polynomials until the polynomial has no divisors as in Proposition 9.5.5 (2b) also allows us to find the minimal n such that the curve is n -invariant.*

Example 9.5.3 *For a 1-invariant curve, we have that $B = C^0 = E$ is constant. Hence, its curvature is constant and the curve is a discrete circle or a line (cf. Example 9.4.7). Specifically, with normalized $\vec{E} = \mathbf{k}$ the center of the circle turns out to be the point $X = F - \frac{1}{2}\mathbf{k}C^1$ (which can be shown to be constant, see (9.62)) and its squared radius is given by*

$$\|F - X\|^2 = \det(F - X) = \frac{1}{4} \det C^1 = \frac{1}{4} \theta^2. \quad (9.55)$$

The evolution formula (9.48) allows us to make some initial observations about the coefficients C^i . Clearly, $C^0 = E$ is constant and contains information about the isometry. As exemplified in Figure 9.12, we can interpret the coefficients as flow vector fields. For example, the coefficient C^1 evolves by $C_1^1 - C_0^1 = u_{01}E - Eu_{01} = 2u_{01} \times \vec{E}$. For odd n it describes the Euclidean motion induced by the isometry. This interpretation

extends to even n if we consider our Euclidean plane as plane in Euclidean 3-space $\mathbb{R}^3 \cong \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$: We have $\vec{C}^1 \parallel \mathbf{k}$ and the flow leaves the plane. As a second example, observe that the vector field $\mathbf{k}C^3 \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ evolves by $\mathbf{k}C_1^3 = -u_{01}\mathbf{k}C_0^3u_{01}^{-1}$ which is a reflection along the perpendicular bisector of each edge. As such, it is a parallel normal field to the curve and, in particular, has constant length.

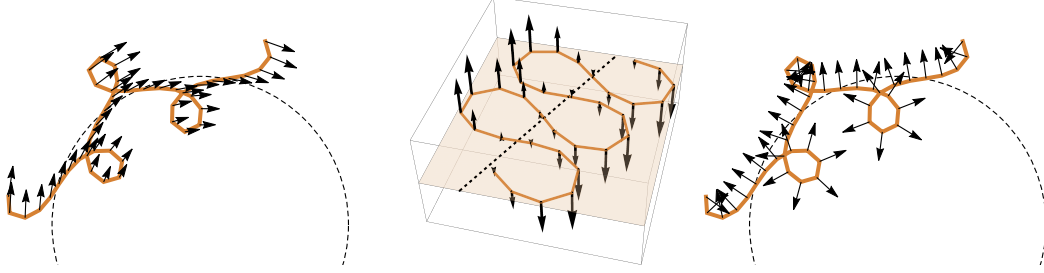


Figure 9.12.: Flow vector fields for n -invariant curves: The vector field $\vec{C}^1$ induces a rotation which stays inside the plane for odd n (left) and leaves the plane for even n (middle). The vector field $\mathbf{k}C^3$ constitutes an edge-parallel normal map (right).

Remark 9.5.4 Using Lemma 9.5.1 we can construct an n -invariant curve iteratively from $E \in \mathbb{H} \setminus \{0\}$ and initial points $f(t_0) = f^{(0)}(t_0), \dots, f^{(n)}(t_0)$ in transformation direction. Indeed, this initial data determines an initial polynomial $\mathcal{P}(t_0)$ which determines $T(t_0), H(t_0)$ as stated in the lemma and, due to (9.17), it determines $u(t_0, t_1)$ for a neighboring vertex t_1 as well. Then, the points $f^{(i)}(t_1)$ are determined and we can iterate the process to obtain an n -invariant curve. By interpreting $f^{(i)}(t_0)$ as an initial polygon with a monodromy given by E we see that our construction is analogous to the construction of periodic Bäcklund transformations [55, 87]. In our planar case, this is known as discrete bicycle transformation [105] and constitutes a special case of cross-ratio dynamics [3, 4, 52] on the twisted polygon $f^{(i)}(t_0)$. Note that if the curve itself and its transformations are periodic we find this dynamic in both lattice directions.

We have investigated the polynomial (9.42) determined by the Bäcklund transformations corresponding to an n -invariant curve. For the converse direction, we will now reconstruct the Bäcklund transformations from a given polynomial. We will need some results about the factorization of *quaternionic polynomials* which are polynomials $\sum_{i=0}^n \lambda^i C^i$ in a scalar variable λ with quaternions C^i as coefficients.

Proposition 9.5.5 1. Any quaternionic polynomial can be factorized into linear factors.

2. Let $1 + \lambda q$ be right (or left) divisor of a quaternionic polynomial and denote the set of quaternions with the same trace and determinant as q by $[q]$. Then, either

- a) any right (or left) divisor $\mathbf{1} + \lambda \tilde{q}$ with $\tilde{q} \in [q]$ agrees with $\mathbf{1} + \lambda q$ or
 - b) $\mathbf{1} + \lambda \tilde{q}$ is a right/left divisor for all $\tilde{q} \in [q]$. This is the case if and only if $1 + \lambda \operatorname{tr} q + \lambda^2 \det q$ divides the polynomial.
3. A quaternionic polynomial is called special if $C^i \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ for odd i and $C^i \in \operatorname{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ for even i . A special quaternionic polynomial can be factorized into special linear factors if and only if all roots of the determinant polynomial are imaginary.

Proof and more context can be found in [68, Section 4.1]. Such factorization results are also currently studied for other algebras in the context of kinematics (see, e.g., [76]). Note, that for an n -invariant curve $E^{-1}\mathcal{P}$ is a special polynomial.

Lemma 9.5.6 *A curve $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$ is n -invariant if and only if at every vertex there exists a quaternionic polynomial of the form $\mathcal{P} = C^0 + \lambda C^1 + \dots + \lambda^n C^n$ fulfilling*

- 1. $C^n, C^{n-2}, \dots \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ and $C^{n-1}, C^{n-3}, \dots \in \operatorname{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ and $C^0 \neq 0$,
- 2. equation (9.43) and
- 3. $\det \mathcal{P}$ has only imaginary roots distinct from $\frac{i}{\eta}$.

Proof. If the curve is n -invariant the polynomial $\mathcal{P}$ given in (9.42) fulfills (1)-(3). Conversely, consider a polynomial $\mathcal{P}$ fulfilling (1)-(3). We will iteratively construct a factorization at every point. Initially, with $E := C^0$ consider $E^{-1}\mathcal{P}(t_0)$ which is a special polynomial with the same roots as $\mathcal{P}(t_0)$ and, therefore, Proposition 9.5.5 (3) implies the existence of a factorization

$$E^{-1}\mathcal{P}(t_0) = (\mathbf{1} + \lambda v^{(n-1)}(t_0)) \cdots (\mathbf{1} + \lambda v^{(0)}(t_0)) \quad (9.56)$$

with $v^{(i)}(t_0) \in \operatorname{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$. These transport matrices are initial data for a sequence of Bäcklund transformations which we denote by $f = f^{(0)}, \dots, f^{(n)} = \tilde{f}$. Note that if $\deg \mathcal{P} < n$ the sequence is completed by $n - \deg \mathcal{P}$ identity transformations (for which $v^{(i)} = 0$).

Now, we show that the equality $\mathcal{P} = E(\mathbf{1} + \lambda v^{(n-1)}) \cdots (\mathbf{1} + \lambda v^{(0)})$ (which holds at t_0) gets preserved along the curve. If it holds at some point f_0 we have

$$\begin{aligned} (\mathbf{1} + \lambda \tilde{u}_{\bar{1}0})(\mathbf{1} + \lambda v_{\bar{1}}^{(n-1)}) \cdots (\mathbf{1} + \lambda v_{\bar{1}}^{(0)}) &= (\mathbf{1} + \lambda v_0^{(n-1)}) \cdots (\mathbf{1} + \lambda v_0^{(0)})(\mathbf{1} + \lambda u_{\bar{1}0}) \\ &= E^{-1}\mathcal{P}_0(\mathbf{1} + \lambda u_{\bar{1}0}) = E^{-1}(\mathbf{1} + \lambda u_{\bar{1}0})\mathcal{P}_{\bar{1}} = (\mathbf{1} + \lambda E^{-1}u_{\bar{1}0}E)E^{-1}\mathcal{P}_{\bar{1}} =: \mathcal{Q} \end{aligned} \quad (9.57)$$

The determinant of $\mathcal{Q}$ is not divided by $(1 + \lambda^2 \eta^2)^2$ since, by (3), $1 + \lambda^2 \eta^2$ does not divide $\det \mathcal{P}$. Therefore, $1 + \lambda^2 \eta^2$ does not divide $\mathcal{Q}$ and by Proposition 9.5.5 (2) the left divisor is unique implying $\tilde{u}_{\bar{1}0} = E^{-1}u_{\bar{1}0}E$ and $\mathcal{P}_{\bar{1}} = E(\mathbf{1} + \lambda v_{\bar{1}}^{(n-1)}) \cdots (\mathbf{1} + \lambda v_{\bar{1}}^{(0)})$. By

symmetry of the construction, the same argument works in the other curve direction. Now, $\tilde{u}_{01} = E^{-1}u_{01}E$ implies that the curve is n -invariant with isometry given by E . $\square$

Note that addition of a real polynomial to $\mathcal{P}$ does not affect (9.43). Thus, by changing the real part of $\mathcal{P}$ (in a manner that preserves condition (1) and (3) as well) we will be able to find different sequences of Bäcklund transformations that make the curve n -invariant.

Now, we have the tools to characterize n -invariant curves geometrically.

Example 9.5.7 *One can check that for an equally sampled circle with center X the polynomial given by the coefficients*

$$\begin{aligned} C_0^0 &:= E := r_0 + \mathbf{k}, \\ C_0^1 &:= 2(F_0 - X)\mathbf{k} \end{aligned} \tag{9.58}$$

fulfills the conditions of Lemma 9.5.6 for almost all $r_0 \in \mathbb{R}$.

The following statement is a version for plane curves of a known result for elastic rods [55, 61].

Proposition 9.5.8 *A curve $f \in \mathcal{C}_{\mathbb{E}^2}$ is 2-invariant if and only if it is an elastic curve.*

Proof. Let $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$ be 2-invariant and let $f = f^{(0)}, f^{(1)}, f^{(2)} = \tilde{f} \in \mathcal{C}_{\mathbb{E}^2}$ be a corresponding sequence of Bäcklund transformations with $\tilde{u}_{01} = E^{-1}u_{01}E$ for some $E \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ with $\det E = 1$. We will find the directrix as the line of reflection belonging to the orientation-reversing isometry: All points $\frac{1}{2}(F + \tilde{F})$ lie on a line with direction vector E since

$$\begin{aligned} \frac{1}{2}(F_1 + \tilde{F}_1) - \frac{1}{2}(F_0 + \tilde{F}_0) &= \frac{1}{2}(\tilde{u}_{01} + u_{01}) = \frac{E^{-1}}{2}(u_{01}E + Eu_{01}) \\ &= -\langle E, u_{01} \rangle E^{-1} = \langle E, u_{01} \rangle E. \end{aligned} \tag{9.59}$$

Now, the signed distance d_0 of a point F_0 to this line is given by

$$\begin{aligned} d_0 &= \langle \mathbf{k}E, F_0 - \frac{1}{2}(F_0 + \tilde{F}_0) \rangle = -\frac{1}{2}\langle \mathbf{k}E, v_0^{(0)} + v_0^{(1)} \rangle \\ &= \frac{1}{4} \left(\text{tr } \mathbf{k}E(v_0^{(0)} + v_0^{(1)}) \right) = \frac{1}{4} \text{tr}(\mathbf{k}B_0) = \frac{\beta\eta}{4}\kappa_0 \end{aligned} \tag{9.60}$$

where we used Lemma 9.5.1 and, in particular, $B = C^1 = E(v^{(0)} + v^{(1)})$. We conclude that f is elastic by Corollary 9.3.2.

Conversely, consider an elastic curve $f \in \mathcal{C}_{\mathbb{E}^2}(\eta)$ and its directrix with normalized direction vector E such that the signed distance of each curve point to the directrix

fulfills $d_0 = \frac{\beta\eta}{4}\kappa_0$ for some $\beta \neq 0$. We define a polynomial $\mathcal{P}_0$ of degree 2 at each vertex by the coefficients

$$\begin{aligned} C_0^0 &:= E, \\ C_0^1 &:= r_1 - \beta\vec{H}_0, \\ C_0^2 &:= \beta T_0 + \eta^2 E \end{aligned} \tag{9.61}$$

for some constant $r_1 \in \mathbb{R}$. We need to check the conditions (1)-(3) of Lemma 9.5.6. Condition (1) is clearly fulfilled. A calculation (see Appendix) shows that condition (2) holds as well and that condition (3) holds for some values r_1 . Hence, by Lemma 9.5.6, the curve is 2-invariant. $\square$

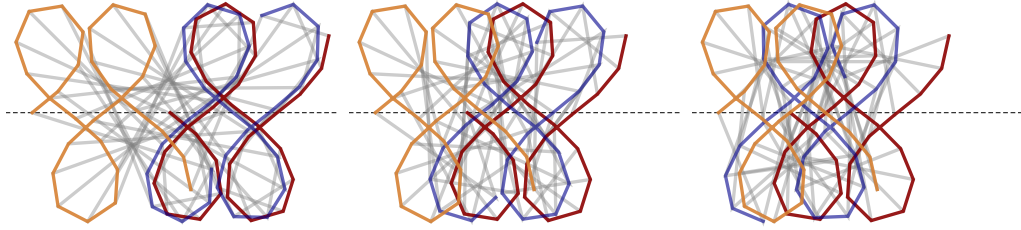


Figure 9.13.: Different sequences of Bäcklund transformations for an elastic curve (orange). The last curve (red) can be obtained from the first curve (orange) by reflection in (and translation along) the directrix.

As visible in Figure 9.12 in the middle, the flow vector field $\vec{C}^1 = \vec{B} = -\beta\vec{H}$ for a 2-invariant curve induces a motion in Euclidean 3-space. In fact $\vec{H}$ is the semi-discrete Hashimoto flow defined in [21] and, hence, (as discussed in [21, 55, 61]) discrete elastica are the invariant curves of the Hashimoto flow.

Proposition 9.5.9 *A curve $f \in \mathcal{C}_{\mathbb{E}^2}$ is 3-invariant if and only if it is a constrained elastic curve.*

Proof. We will distinguish the case where there isometry is a rotation ($\vec{E} \neq 0$) from the special case where it is a translation ($\vec{E} = 0$). We will show that the case of a rotation corresponds to an area-constrained elastic curve while the case of a translation corresponds to an elastic curve or a circle.

First, if the curve is 3-invariant with a rotation as isometry we can assume normalized $\vec{E} = \mathbf{k}$. Consider the point $X = F - \frac{1}{2}\mathbf{k}C^1$. This point is in fact constant and will turn out to be the center of the directrix:

$$X_1 - X_0 = u_{01} - \frac{1}{2}\mathbf{k}(C_1^1 - C_0^1) = u_{01} - \frac{1}{2}\mathbf{k}(u_{01}\mathbf{k} - \mathbf{k}u_{01}) = 0. \tag{9.62}$$

We calculate

$$\begin{aligned}\|F - X\|^2 &= \det(F - X) = \frac{1}{4} \det C^1 = \frac{1}{4}(\theta_2 - 2\langle \vec{C}^0, \vec{C}^2 \rangle) = \frac{1}{4}(\theta_2 - 2\langle \mathbf{k}, \vec{B} + \eta^2 \mathbf{k} \rangle) \\ &= \frac{1}{4}(\theta_2 - 2\eta^2 + \beta\eta\kappa).\end{aligned}\tag{9.63}$$

Now, with $r^2 := \frac{1}{4}(\theta_2 - 2\eta^2)$, the squared tangential distance of F to the circle with center X and radius r is

$$x^2 = \|F - X\|^2 - r^2 = \frac{\beta\eta}{4}\kappa.\tag{9.64}$$

Note, that x^2 and r^2 can be negative. The curve is area-constrained elastic by Corollary 9.3.2.

For the converse, let $f \in \mathcal{C}_{\mathbb{E}^2}$ be an area-constrained elastic curve fulfilling (9.64) for some circle with center X , radius r and some constant $\beta \in \mathbb{R}$. We will define a polynomial $\mathcal{P}$ by

$$\begin{aligned}C_0^0 &:= E := r_0 + \mathbf{k}, \\ C_0^1 &:= 2(F_0 - X)\mathbf{k}, \\ C_0^2 &:= r_2 - \beta\vec{H}_0 + \eta^2\mathbf{k}, \\ C_0^3 &:= \beta T_0 + \eta^2 C_0^1.\end{aligned}\tag{9.65}$$

for some constants $r_0, r_2 \in \mathbb{R}$. We need to check the conditions (1)-(3) of Lemma 9.5.6. Analogous to the 2-invariant case, condition (1) is clearly fulfilled and a calculation (see Appendix) shows that condition (2) holds as well and that condition (3) holds for some r_0, r_2 . Hence, the curve is 3-invariant.

Now, consider an elastic curve. We know it is 2-invariant with corresponding polynomial $\mathcal{P}$ given by (9.61). To show that the curve is 3-invariant consider the polynomial $\hat{\mathcal{P}} := r_0 + r_2\lambda^2 + \lambda\vec{\mathcal{P}}$ which fulfills conditions (1) and (2) of Lemma 9.5.6 for any $r_0, r_2 \in \mathbb{R}$ with $r_0 \neq 0$. Applying Lemma A.1 (2) (in the Appendix) to the constant real polynomial $\det \hat{\mathcal{P}}$ yields that $\hat{\mathcal{P}}$ also fulfills condition (3) for some r_0, r_2 . Hence, the elastic curve is 3-invariant as well.

Conversely, consider a curve which is 3-invariant with a corresponding polynomial $\mathcal{P}$, isometry given by $E \in \mathbb{R}$ (i.e., the isometry is a translation) and $C^1 \neq 0$. To show that the curve is 2-invariant consider the polynomial $\hat{\mathcal{P}} := r_1\lambda + \frac{1}{\lambda}\vec{\mathcal{P}}$ which fulfills conditions (1) and (2) of Lemma 9.5.6 for all $r_1 \in \mathbb{R}$. Again, applying Lemma A.1 (1) (in the Appendix) to the constant real polynomial $\det \hat{\mathcal{P}}$ yields that $\hat{\mathcal{P}}$ also fulfills condition (3) for some r_1 . Hence, the curve is 2-invariant and, thus, elastic.

Finally, in the special case of a 3-invariant curve with $E \in \mathbb{R}$ and $C^1 = 0$ analogous argumentation to above for the polynomial $\hat{\mathcal{P}} := r_0 + \frac{1}{\lambda^2}\vec{\mathcal{P}}$ shows that the curve is 1-invariant and, thus, a circle. $\square$

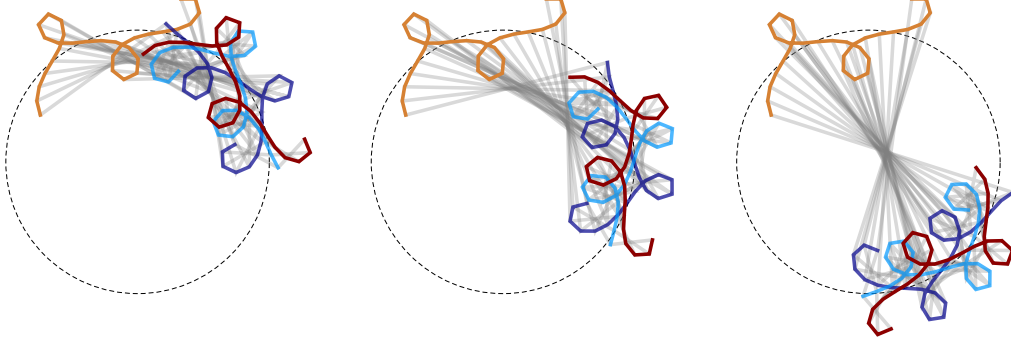


Figure 9.14.: Different sequences of Bäcklund transformations for an area-constrained elastic curve (orange). The last curve (red) can be obtained from the first curve (orange) by rotation around the center of the directrix.

In [59], constrained elastic curves are introduced as invariant curves of a semi-discrete mKdV flow. Indeed, the vector field C^1 induces a Euclidean motion on the curve and we will relate it to the mKdV flow given in [59]. For this consider the discrete Schwarzian derivative

$$S_{01} := H_1 H_0^* = \left(1 + \frac{\eta}{2} \kappa_1 \mathbf{k}\right) \left(1 - \frac{\eta}{2} \kappa_0 \mathbf{k}\right) = \left(1 + \frac{\eta^2}{4} \kappa_1 \kappa_0\right) \mathbf{1} + \frac{\eta}{2} (\kappa_1 - \kappa_0) \mathbf{k}. \quad (9.66)$$

The semi-discrete tangent flow and mKdV flow are then given by

$$F_0^T := T_0, \quad F_0^{mKdV} := \frac{1}{2} (S_{10} + S_{01}) T_0. \quad (9.67)$$

Proposition 9.5.10 *An area-constrained elastic curve $f \in \mathcal{C}_{\mathbb{E}^2}$ is invariant under a linear combination of mKdV flow and tangent flow.*

Proof. The curve is 3-invariant and the coefficient C^1 of the corresponding polynomial is the flow vector field which induces a rotation around the center of the directrix (see Figure 9.12 on the left). A long calculation (see the Appendix) shows that

$$C_0^1 = \frac{-\beta}{2\eta^2} F_0^{mKdV} + \frac{1}{2\beta\eta^4} (\theta_6 - \eta^4 \theta_2 + 2\eta^6 \theta_0) F_0^T \quad (9.68)$$

which is a linear combination of mKdV and tangent flow with constant coefficients. $\square$

9.6. Associated family and n -invariant curves

We will now show that the associated family preserves the property of a curve of being n -invariant allowing us to complete the proof of Theorem 9.4.11. For this, we extend

the associated family to sequences of Bäcklund transformations $f = f^{(0)}, \dots, f^{(n)}$. As in Section 9.2, we will assume a fixed initial point $F^{(0)}(t_0) = \mathbf{0}$ for curves in Euclidean space and $F^{(0)}(t_0) = \mathbf{k}$ for curves in non-Euclidean space. Thus, the curve f and its transformations $f^{(i)}$ are completely determined by the transport matrices u and v .

Recall (9.40), stating that multiplication of linear factors is path-independent for $\lambda \in \mathbb{C}$. We consider the value λ to be *admissible* if it fulfills the assumptions of Proposition 9.2.1 and if $\mathbf{1} \pm \lambda v^{(i)}$ is invertible. For admissible $\lambda \in \mathbb{C}$ the map Φ^λ given by

$$\Phi^{(0),\lambda}(t_0) = \mathbf{1}, \quad \Phi_1^{(i),\lambda} = (\mathbf{1} + \lambda u_{01}^{(i)})\Phi_0^{(i),\lambda}, \quad \Phi_0^{(i+1),\lambda} = (\mathbf{1} + \lambda v_0^{(i)})\Phi_0^{(i),\lambda} \quad (9.69)$$

is well-defined. It extends the map $\Phi^\lambda : \mathcal{V}_n \rightarrow \mathbb{C}^{2 \times 2}$ given in Section 9.2 in transformation direction. Analogous to (9.21), we also define

$$v_0^{(i),\lambda} := (\Phi_0^{(i+1),\lambda})^{-1}(\mathbf{1} - \lambda v_0^{(i)})\Phi_0^{(i),\lambda}. \quad (9.70)$$

The pair $p^\lambda = (u^\lambda, v^\lambda)$ defines a map $\mathcal{E}_n \rightarrow \mathbb{C}^{2 \times 2}$.

Lemma 9.6.1 *Let $f^{(0)}, \dots, f^{(n)}$ be a sequence of Bäcklund transformations. Then, the pair $p^\lambda = (u^\lambda, v^\lambda)$ is a skew parallelogram net for admissible $\lambda \in \mathbb{C}$.*

Proof. We will use that $p = (u, v)$ is a skew parallelogram net. The multiplicative condition holds since

$$\begin{aligned} v_1^{(i),\lambda} u_{01}^{(i),\lambda} &= (\Phi_1^{(i+1),\lambda})^{-1}(\mathbf{1} - \lambda v_1^{(i)})(\mathbf{1} - \lambda u_{01}^{(i)})\Phi_0^{(i),\lambda} \\ &= (\Phi_1^{(i+1),\lambda})^{-1}(\mathbf{1} - \lambda u_{01}^{(i+1)})(\mathbf{1} - \lambda v_0^{(i)})\Phi_0^{(i),\lambda} = u_{01}^{(i+1),\lambda} v_0^{(i),\lambda} \end{aligned} \quad (9.71)$$

and the additive condition holds since

$$\begin{aligned} u_{01}^{(i),\lambda} + v_1^{(i),\lambda} &= (\Phi_1^{(i+1),\lambda})^{-1}((\mathbf{1} + \lambda v_1^{(i)})(\mathbf{1} - \lambda u_{01}^{(i)}) + (\mathbf{1} - \lambda v_1^{(i)})(\mathbf{1} + \lambda u_{01}^{(i)}))\Phi_0^{(i),\lambda} \\ &= 2(\Phi_1^{(i+1),\lambda})^{-1}(\mathbf{1} - \lambda^2 v_1^{(i)} u_{01}^{(i)})\Phi_0^{(i),\lambda} \end{aligned} \quad (9.72)$$

coincides with

$$\begin{aligned} v_0^{(i),\lambda} + u_{01}^{(i+1),\lambda} &= (\Phi_1^{(i+1),\lambda})^{-1}((\mathbf{1} + \lambda u_{01}^{(i+1)})(\mathbf{1} - \lambda v_0^{(i)}) + (\mathbf{1} - \lambda u_{01}^{(i+1)})(\mathbf{1} + \lambda v_0^{(i)}))\Phi_0^{(i),\lambda} \\ &= 2(\Phi_1^{(i+1),\lambda})^{-1}(\mathbf{1} - \lambda^2 u_{01}^{(i+1)} v_0^{(i)})\Phi_0^{(i),\lambda}. \end{aligned} \quad (9.73)$$

□

Now, we extend the associated family to Bäcklund transformations.

Proposition 9.6.2 *Let $f^{(0)}, \dots, f^{(n)} \in \mathbb{C}$ be a sequence of Bäcklund transformations of the curve $f = f^{(0)}$. The transport matrices*

1. u^λ, v^λ for $\mathcal{Q} = \mathbb{E}^2$ and admissible λ ,
2. u^1, v^1 for $\mathcal{Q} = \mathbb{S}^2$,
3. iu^1, iv^1 for $\mathcal{Q} = \mathbb{H}^2$

define a sequence of curves $\mathcal{T}^\lambda f^{(0)}, \dots, \mathcal{T}^\lambda f^{(n)} \in \mathcal{C}_{\mathcal{Q}_2}$ (with $\mathcal{C}_{\mathcal{Q}_2}$ as in Proposition 9.2.1) which is again a sequence of Bäcklund transformations of $\mathcal{T}^\lambda f = \mathcal{T}^\lambda f^{(0)}$.

Proof. The transport is closed around every quad since $p^\lambda = (u^\lambda, v^\lambda)$ is a skew parallelogram net (Lemma 9.6.1). Therefore, $\mathcal{T}^\lambda f^{(i)}$ is well-defined. The same arguments as in the proof of Proposition 9.2.1 show that $\mathcal{T}^\lambda f^{(i)} \in \mathcal{C}_{\mathcal{Q}_2}$. Now, the property of p^λ being a skew parallelogram net also implies that the curves constitute a sequence of Bäcklund transformations. One can easily check that our regularity assumptions stay preserved as well. $\square$

Proposition 9.6.3 *The associated family preserves n -invariance.*

Proof. Let $f \in \mathbb{C}$ be n -invariant with a sequence of Bäcklund transformations $f = f^{(1)}, \dots, f^{(n)} = \tilde{f}$ and a (split-)quaternion E fulfilling $\tilde{u}_{01} = E^{-1}u_{01}E$. We will show that the sequence of Bäcklund transformations $\mathcal{T}^\lambda f^{(1)}, \dots, \mathcal{T}^\lambda f^{(n)}$ preserves the shape of $\mathcal{T}^\lambda f$ and, thus, causes $\mathcal{T}^\lambda f$ to be n -invariant. For this, observe that the matrix $E^\lambda := (\Phi^\lambda)^{-1}E\tilde{\Phi}^\lambda$ (with $\Phi^\lambda = \Phi^{(0),\lambda}$ and $\tilde{\Phi}^\lambda = \Phi^{(n),\lambda}$) is constant along the curve since

$$(\Phi_1^\lambda)^{-1}E\tilde{\Phi}_1^\lambda = (\Phi_0^\lambda)^{-1}(\mathbf{1} + \lambda u_{01})^{-1}E(\mathbf{1} + \lambda \tilde{u}_{01})\tilde{\Phi}_0^\lambda = (\Phi_0^\lambda)^{-1}E\tilde{\Phi}_0^\lambda. \quad (9.74)$$

First, if $\mathcal{Q} = \mathbb{E}$ we have

$$\mathcal{T}^\lambda \tilde{F} = (\tilde{\Phi}^\lambda)^{-1}\mathbf{k}\tilde{\Phi}^\lambda = \pm(\tilde{\Phi}^\lambda)^{-1}E^{-1}\mathbf{k}E\tilde{\Phi}^\lambda = \pm(E^\lambda)^{-1}\mathcal{T}^\lambda FE^\lambda. \quad (9.75)$$

The sign depends on whether E commutes with $\mathbf{k}$ (orientation preserving $E \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$) or anti-commutes with $\mathbf{k}$ (orientation reversing $E \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$). Now, $\mathcal{T}^\lambda f \in \mathcal{C}_{\mathcal{Q}_2}$ is n -invariant and the new isometry is given by

1. the quaternion E^λ if $\lambda \in \mathbb{R}$ (and, hence, $\mathcal{Q}_2 = \mathbb{S}^2$).
2. the split-quaternion E^λ if $\lambda \in i\mathbb{R}$ (and, hence, $\mathcal{Q}_2 = \mathbb{H}^2$) and $E \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$.
3. the split-quaternion iE^λ if $\lambda \in i\mathbb{R}$ (and, hence, $\mathcal{Q}_2 = \mathbb{H}^2$) and $E \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$.

On the other hand, if $\mathcal{Q} = \mathbb{S}$ or $\mathcal{Q} = \mathbb{H}$ we have $\mathcal{T}^1 f \in \mathcal{C}_{\mathbb{E}^2}$ and

$$\tilde{u}_{01}^1 = (\tilde{\Phi}_1^1)^{-1}(1 - \tilde{u}_{01})\tilde{\Phi}_0^1 = (\tilde{\Phi}_1^1)^{-1}E^{-1}(1 - u_{01})E\tilde{\Phi}_0^1 = (E^1)^{-1}u_{01}^1E^1 \quad (9.76)$$

and it only remains to show that E^1 (or a scalar multiple) is a quaternion in $\text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ or $\text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$. Indeed, using $\Phi^{(i),1}\mathbf{k} = F^{(i)}\Phi^{(i),1}$ and $\tilde{F} = \pm E^{-1}FE$, we have

$$E^\lambda \mathbf{k} = (\Phi^1)^{-1} E \tilde{F} \Phi^1 = \pm (\Phi^1)^{-1} F E \Phi^1 = \pm \mathbf{k} E^\lambda \quad (9.77)$$

and, hence, $\mathcal{T}^1 f \in \mathcal{C}_{\mathbb{E}^2}$ is n -invariant with the isometry given by

1. the quaternion $E^1 \in \text{span}_{\mathbb{R}}\{\mathbf{1}, \mathbf{k}\}$ for orientation preserving E and $\mathcal{Q} = \mathbb{S}$ or $\mathcal{Q} = \mathbb{H}$.
2. the quaternion $E^1 \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ for orientation reversing E and $\mathcal{Q} = \mathbb{S}$.
3. the quaternion $iE^1 \in \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}\}$ for orientation reversing E and $\mathcal{Q} = \mathbb{H}$.

In all cases, one can also check that regularity of the sequence of transformations is preserved. $\square$

Thus, n -invariant curves and their Bäcklund transformations in space forms can be obtained from corresponding n -invariant curves in Euclidean space. For example, the n -invariant curves in different space forms visible in Figure 9.10 are related by the associated family.

Finally, we combine our observations to prove our main Theorem.

Proof. [Proof of Theorem 9.4.11.] In Proposition 9.5.8 and 9.5.9 we have shown the statement for curves in Euclidean space. Note that the associated family preserves the curvature equation 9.33 of constrained elastic curves since $\mathcal{T}^\lambda \zeta \mathcal{T}^\lambda \kappa = \zeta \kappa$ (Proposition 9.2.8).

Now, if a curve f in a non-Euclidean space form is 2-invariant then the associated Euclidean curve $\mathcal{T}^1 f$ is also 2-invariant (Proposition 9.6.3) and, therefore, elastic (Proposition 9.5.8). Then, every curve $\mathcal{T}^\lambda \circ \mathcal{T}^1 f$ is elastic and, since the associated family is reversible (Proposition 9.2.9), this includes f .

For the converse, we can use the same results in the other direction: If f is elastic the associated Euclidean curve $\mathcal{T}^1 f$ must be elastic and, therefore, it is 2-invariant. All curves $\mathcal{T}^\lambda \circ \mathcal{T}^1 f$ are 2-invariant and this includes the original curve f .

Analogously, using Proposition 9.5.9, we can show that 3-invariant curves are exactly constrained elastic curves. $\square$

A. Polynomials and Euclidean elastica

We provide some calculations for elastic and area-constrained elastic curves in $\mathbb{E}^2$. We start with two statements that we will use to show condition (3) of Lemma 9.5.6.

Lemma A.1 1. For any $\theta_0, \theta_2, \theta_4 \in \mathbb{R}$ with $\theta_0, \theta_4 > 0$ there exist infinitely many $r_1 \in \mathbb{R}$ such that the real polynomial

$$\theta_0 + \lambda^2(r_1^2 + \theta_2) + \lambda^4\theta_4$$

has only imaginary roots.

2. For any $\theta_0, \theta_2, \theta_4, \theta_6 \in \mathbb{R}$ with $\theta_6 > 0$ there exist infinitely many combinations $r_0, r_2 \in \mathbb{R}$ such that the real polynomial

$$r_0^2 + \theta_0 + \lambda^2(2r_0r_2 + \theta_2) + \lambda^4(r_2^2 + \theta_4) + \lambda^6\theta_6$$

has only imaginary roots.

Proof. For the first statement, consider the quadratic polynomial $\mathcal{Q} := \theta_0 + \mu(r_1^2 + \theta_2) + \mu^2\theta_4$. We will show that for sufficiently large r_1^2 it has only real negative roots, which proves the claim of the lemma. In fact, the quadratic discriminant

$$(r_1^2 + \theta_2)^2 - 4\theta_0\theta_4$$

is positive for large r_1^2 and, hence, the roots of $\mathcal{Q}$ are real. Now, since the coefficients of $\mathcal{Q}$ are positive its roots are necessarily negative.

For the second statement, consider the cubic polynomial $\mathcal{Q} := r_0^2 + \theta_0 + \mu(2r_0r_2 + \theta_2) + \mu^2(r_2^2 + \theta_4) + \mu^3\theta_6$ and its cubic discriminant

$$\begin{aligned} & 18\theta_6(r_2^2 + \theta_4)(2r_0r_2 + \theta_2)(r_0^2 + \theta_0) - 4(r_2^2 + \theta_4)^3(r_0^2 + \theta_0) \\ & + (r_2^2 + \theta_4)^2(2r_0r_2 + \theta_2)^2 - 4\theta_6(2r_0r_2 + \theta_2)^3 - 27\theta_6^2(r_0^2 + \theta_0)^2. \end{aligned}$$

We write $r_0 = Rr_2$ and will show that for sufficiently large R and r_2 the polynomial $\mathcal{Q}$ has only real negative roots which, again, proves the claim of the lemma. With $r_0 = Rr_2$ the discriminant becomes a polynomial of degree six in r_2 with leading coefficient

$$-4\theta_0 + 4\theta_2R - 4\theta_4R^2 + 4\theta_6R^3$$

which is positive for sufficiently large R . Thus, if r_2 is sufficiently large as well, the discriminant is positive and $\mathcal{Q}$ has only real roots. Now, since the coefficients of $\mathcal{Q}$ are positive its roots are necessarily negative. $\square$

Proof. [Calculations for polynomial (9.61) for an elastic curve.] Consider an elastic curve and the polynomial $\mathcal{P}$ with coefficients given by (9.61). We will check (9.43) using its expanded version (9.48). Clearly, C^0 is constant. Also, we have

$$\begin{aligned} C_1^1 - C_0^1 &= -\beta(\vec{H}_1 - \vec{H}_0) = -\frac{\beta\eta}{2}(\kappa_1 - \kappa_0)\mathbf{k} \\ &= -2(d_1 - d_0)\mathbf{k} = -2\langle u_{01}, \mathbf{k}E \rangle \mathbf{k} = (u_{01}\mathbf{k}E + \mathbf{k}Eu_{01})\mathbf{k} = u_{01}E - Eu_{01} \end{aligned}$$

and, using (9.17), we have

$$C_1^2 - C_0^2 = \beta(T_1 - T_0) = \beta(H_1u_{01} - u_{01}H_0) = u_{01}C_0^1 - C_1^1u_{01}$$

and

$$C_1^2u_{01} = \eta^2(-\beta H_1 + Eu_{01}) = \eta^2(-\beta H_0 + u_{01}E) = u_{01}C_0^2.$$

Hence, the polynomial $\mathcal{P}$ fulfills (9.43) and, therefore, condition (2) of Lemma 9.5.6. Then, $\det \vec{P}$ is constant along the curve and by denoting (as in (9.46)) the coefficients of $\det \vec{P}$ by θ_i it is a polynomial as discussed in Lemma A.1 (1). Thus, it has only imaginary roots (also, $\det \mathcal{P}(\frac{i}{\eta}) \neq 0$ is true for almost all r_1). We conclude that there exists r_1 such that for our polynomial $\mathcal{P}$ all conditions of Lemma 9.5.6 are met. $\square$

Proof. [Calculations for polynomial (9.65) for an area-constrained elastic curve.] Consider an area-constrained elastic curve and the polynomial $\mathcal{P}$ with coefficients given by (9.65). Again, we will check (9.43) using its expanded version (9.48). Clearly, C^0 is constant and we have $C_1^1 - C_0^1 = 2u_{01}\mathbf{k} = u_{01}E - Eu_{01}$. From this, by writing $C_0^1u_{01} =: a_0 + b_0\mathbf{k}$ for $a_0, b_0 \in \mathbb{R}$ we obtain $C_1^1u_{01} = a_0 + (b_0 + 2\eta^2)\mathbf{k}$ and, thus, we can compute

$$\begin{aligned} C_1^2 - C_0^2 &= -\beta(\vec{H}_1 - \vec{H}_0) = -\frac{\beta\eta}{2}(\kappa_1 - \kappa_0)\mathbf{k} = -2(x_1^2 - x_0^2)\mathbf{k} \\ &= -2(\det(F_1 - X) - \det(F_0 - X))\mathbf{k} = -\frac{1}{2}(\det C_1^1 - \det C_0^1)\mathbf{k} \\ &= -\frac{1}{2\eta^2}(a_0^2 + (b_0 + 2\eta^2)^2 - a_0^2 - b_0^2)\mathbf{k} = -2(b_0 + \eta^2)\mathbf{k} \\ &= a_0 - b_0\mathbf{k} - a_0 - (b_0 + 2\eta^2)\mathbf{k} = u_{01}C_0^1 - C_1^1u_{01}. \end{aligned}$$

Next, using (9.17), we calculate

$$\begin{aligned} C_1^3 - C_0^3 &= \beta(T_1 - T_0) + \eta^2(C_1^1 - C_0^1) = \beta\vec{H}_1u_{01} - \beta u_{01}\vec{H}_0 + 2\eta^2u_{01}\mathbf{k} \\ &= (\beta\vec{H}_1 - \eta^2\mathbf{k})u_{01} - u_{01}(\beta\vec{H}_0 - \eta^2\mathbf{k}) = u_{01}C_0^2 - C_1^2u_{01} \end{aligned}$$

and

$$\begin{aligned} C_1^3 u_{01} &= \eta^2(-\beta H_1 + C_1^1 u_{01}) = \eta^2(C_1^2 + C_1^1 u_{01} - r_2 - \beta - \eta^2 \mathbf{k}) \\ &= \eta^2(C_0^2 + u_{01} C_0^1 - r_2 - \beta - \eta^2 \mathbf{k}) = \eta^2(-\beta H_0 + u_{01} C_0^1) = u_{01} C_0^3. \end{aligned}$$

Hence, the polynomial $\mathcal{P}$ fulfills (9.43) and, therefore, condition (2) of Lemma 9.5.6. Analogous to the 2-invariant case, $\det \vec{P}$ is constant along the curve and by denoting (as in (9.46)) the coefficients of $\det \vec{P}$ by θ_i it is a polynomial as discussed in Lemma A.1 (2). Thus, it has only imaginary roots (also, again, $\det \mathcal{P}(\frac{i}{\eta}) \neq 0$ is true for almost all r_0, r_2). We conclude that there exists r_0, r_2 such that for our polynomial $\mathcal{P}$ all conditions of Lemma 9.5.6 are met. $\square$

Proof. [Calculation of the coefficients of the flow C^1 written as linear combination of F^{mKdV} and F^T .]

In this calculation we will use (9.17), Proposition 9.2.6 and the relations in Lemma 9.5.1 for 3-invariant curves multiple times.

First, we have

$$F_0^{mKdV} = \frac{1}{2}(H_1^* u_{01}^{-1} + H_1 u_{10}^{-1}) T_0^2 = \frac{\det T_0}{2\eta^2}(u_{01} H_1 + H_1 u_{10}).$$

Now, the curve is 3-invariant and we consider a corresponding polynomial $\mathcal{P}$. Using (9.48), we get

$$\begin{aligned} -\beta \vec{H}_1 &= \vec{B}_1 = \vec{B}_0 + u_{01} C_0^1 - C_1^1 u_{01} = -\beta \vec{H}_0 + u_{01} C_0^1 - C_0^1 u_{01} - 2\eta^2 \vec{E} \\ -\beta \vec{H}_{\bar{1}} &= \vec{B}_{\bar{1}} = \vec{B}_0 - u_{10} C_0^1 + C_1^1 u_{10} = -\beta \vec{H}_0 - u_{10} C_0^1 + C_0^1 u_{10} - 2\eta^2 \vec{E} \end{aligned}$$

We combine

$$\begin{aligned} \beta F_0^{mKdV} &= -\frac{\det T_0}{2\eta^2}(u_{01}(-\beta \vec{H}_{\bar{1}}) + (-\beta \vec{H}_1)u_{10} - \beta(u_{01} + u_{10})) \\ &= -\frac{\det T_0}{2\eta^2}(-\beta u_{01} H_0 - \beta H_0 u_{10} + 2u_{01} C_0^1 u_{10} - (u_{01} u_{10} + u_{10} u_{01}) C_0^1 + 2\eta^2 \vec{E}(u_{01} - u_{10})) \\ &= -\frac{\det T_0}{\eta^2}(-\beta T_0 + (u_{01} C_0^1 u_{10} - \eta^2 C_0^1) + (\eta^2 + \langle u_{01}, u_{10} \rangle) C_0^1 + \eta^2 \vec{E}(u_{01} - u_{10})). \end{aligned}$$

Each term is a multiple of C^1 or T since

$$\begin{aligned} u_{01} C_0^1 u_{10} - \eta^2 C_0^1 &= (T_0 H_0^{-1} C_0^1 + C_0^1 H_0^{-1} T_0) u_{10} \\ &= (H_0^{-1})^*(T_0 C_0^1 + C_0^1 T_0) u_{10} = \frac{-2\langle T_0, C_0^1 \rangle}{\det H_0} T_0 \end{aligned}$$

and, with $\vec{E} =: e\mathbf{k}$,

$$\vec{E}(u_{01} - u_{10}) = e\mathbf{k}(T_0 H_0^{-1} - H_0^{-1} T_0) = \frac{e}{\det H_0} \mathbf{k}(H_0 - H_0^*) T_0 = -\frac{e\eta\kappa}{\det H_0} T_0.$$

We have $\eta^2 \det H_0 = \det T_0 = \frac{2\eta^4}{\eta^2 + \langle u_{01}, u_{10} \rangle}$ and, hence, we can combine

$$\beta F_0^{mKdV} = \frac{\beta \det T_0}{\eta^2} T_0 + 2\langle T_0, C_0^1 \rangle T_0 - 2\eta^2 C_0^1 + e\eta^3 \kappa T_0 = -2\eta^2 C_0^1 + \frac{a_0}{\beta\eta^2} F_0^T$$

with coefficient $a_0 := \beta^2 \det T_0 + 2\beta\eta^2 \langle T_0, C_0^1 \rangle + e\beta\eta^5 \kappa$. We need to show that a is constant. For this, we express it in terms of our invariants θ_j as

$$\begin{aligned} a_0 &= \det A_0 + 2\langle A_0, \eta^2 C_0^1 \rangle - 2\langle \eta^4 \vec{E}, \vec{B}_0 \rangle \\ &= \langle C_0^3 - \eta^2 C_0^1, C_0^3 + \eta^2 C_0^1 \rangle - 2\eta^4 \langle \vec{E}, \vec{C}_0^2 \rangle + 2\eta^6 \langle \vec{E}, \vec{E} \rangle \\ &= \det C_0^3 - \eta^4 \det C_0^1 - 2\eta^4 \langle \vec{C}^0, \vec{C}^2 \rangle + 2\eta^6 \det \vec{C}^0 = \theta_6 - \eta^4 \theta_2 + 2\eta^6 \theta_0. \end{aligned}$$

Thus, we showed that the flow vector field C^1 (which only induces a Euclidean motion) can be written as constant linear combination of mKdV- and tangent flow. $\square$

Bibliography

- [1] Mark J. Ablowitz and John F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *Journal of Mathematical Physics*, 17(6):1011–1018, 1976.
- [2] Vsevolod E. Adler. Recuttings of polygons. *Functional Analysis and Its Applications*, 27(2):141–143, 1993.
- [3] Niklas Affolter, Terrence George, and Sanjay Ramassamy. Integrable dynamics in projective geometry via dimers and triple crossing diagram maps on the cylinder. 2023.
- [4] Maxim Arnold, Dmitry Fuchs, Ivan Izmistiev, and Serge Tabachnikov. Cross-ratio dynamics on ideal polygons. *International Mathematics Research Notices*, 2022(9):6770–6853, 2020.
- [5] Guillermo Arreaga, Riccardo Capovilla, Chrysomalis Chrysomalakos, and Jemal Guven. Area-constrained planar elastica. *Phys. Rev. E*, 65, 2002.
- [6] Eduardo Bayro-Corrochano. *Geometric Algebra Applications Vol. III: Integral Transforms, Machine Learning, and Quantum Computing*. Springer Nature, 2024.
- [7] Miklós Bergou, Max Wardetzky, Stephen Robinson, Basile Audoly, and Eitan Grinspun. Discrete elastic rods. *ACM SIGGRAPH Papers*, 2008.
- [8] Wilhelm Blaschke. Kinematik und Quaternionen. *Deutscher Verlag der Wissenschaften*, 1960.
- [9] Alexander I. Bobenko. *Surfaces in terms of 2 by 2 matrices. Old and new integrable cases*. Springer, 1994.
- [10] Alexander I. Bobenko, Udo Hertrich-Jeromin, and Inna Lukyanenko. Discrete constant mean curvature nets in space forms: Steiner’s formula and Christoffel duality. *Discrete & Computational Geometry*, 52(4):612–629, 2014.
- [11] Alexander I. Bobenko, Tim Hoffmann, and Andrew O. Sageman-Furnas. Isothermic tori with one family of planar curvature lines and area constrained hyperbolic elastica. 2023.
- [12] Alexander I. Bobenko, Tim Hoffmann, and Nina Smeenk. Constant mean curvature surfaces from ring patterns: Geometry from combinatorics. 2024.
- [13] Alexander I. Bobenko, Tim Hoffmann, and Yuri B. Suris. Hexagonal circle patterns and integrable systems: Patterns with the multi-ratio property and Lax equations on the regular triangular lattice. *International Mathematics Research Notices*, 2002(3):111–164, 2002.
- [14] Alexander I. Bobenko, Christian Mercat, and Yuri B. Suris. Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green’s function. *Journal für die reine und angewandte Mathematik*, 2005(583):117–161, 2005.

- [15] Alexander I. Bobenko and Ulrich Pinkall. Discrete isothermic surfaces. *Journal für die reine und angewandte Mathematik*, 1996(475):187–208, 1996.
- [16] Alexander I. Bobenko and Ulrich Pinkall. Discrete surfaces with constant negative Gaussian curvature and the Hirota equation. *Journal of Differential Geometry*, 43(3):527 – 611, 1996.
- [17] Alexander I. Bobenko and Ulrich Pinkall. Discretization of surfaces and integrable systems. *Discrete Integrable Geometry and Physics*, 16:3–58, 1999.
- [18] Alexander I. Bobenko, Helmut Pottmann, and Johannes Wallner. A curvature theory for discrete surfaces based on mesh parallelity. *Mathematische Annalen*, 348:1–24, 2010.
- [19] Alexander I. Bobenko and Pascal Romon. Discrete CMC surfaces in $\mathbb{R}^3$ and discrete minimal surfaces in $\mathbb{S}^3$: a discrete Lawson correspondence. *Journal of Integrable Systems*, 2(1):1–18, 2017.
- [20] Alexander I. Bobenko and Wolfgang K. Schief. Affine Spheres: Discretization via Duality Relations. *Experimental Mathematics*, 8(3):261–280, 1999.
- [21] Alexander I. Bobenko and Yuri B. Suris. Discrete time Lagrangian mechanics on Lie groups, with an application to the Lagrange top. *Comm. Math. Phys.*, 204(1):147–188, 1999.
- [22] Alexander I. Bobenko and Yuri B. Suris. Integrable systems on quad-graphs. *International Mathematics Research Notices*, 2002(11):573–611, 2002.
- [23] Alexander I. Bobenko and Yuri B. Suris. Isothermic surfaces in sphere geometries as Moutard nets. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 463(2088):3171–3193, 2007.
- [24] Alexander I. Bobenko and Yuri B. Suris. *Discrete differential geometry: Integrable structure*, volume 98. AMS Graduate Studies in Mathematics, 2008.
- [25] Alexander I. Bobenko and Yuri B. Suris. Discrete Koenigs nets and discrete isothermic surfaces. *International Mathematics Research Notices*, 2009(11):1976–2012, 2009.
- [26] Christoph Bohle, Paul Peters, and Ulrich Pinkall. Constrained Willmore surfaces. *Calc. Var. Partial Differential Equations*, 32(2):263–277, 2008.
- [27] David Brander, Andreas Bærentzen, Kenn Clausen, Ann-Sofie Fisker, Jens Gravesen, Morten N. Lund, Toke B. Nørkjær, Kasper Steenstrup, and Asbjørn Søndergaard. Designing for hot-blade cutting: Geometric approaches for high-speed manufacturing of doubly-curved architectural surfaces. In *Advances in Architectural Geometry 2016*, pages 306–327, 2016.
- [28] David Brander, Jens Gravesen, and Toke B. Nørkjær. Approximation by planar elastic curves. *Advances in Computational Mathematics*, 43:25–43, 2015.
- [29] Raoul Bricard. Mémoire sur la théorie de l’octaèdre articulé. *Journal de Mathématiques pures et appliquées*, 3:113–148, 1897.
- [30] Francis Burstall, Udo Hertrich-Jeromin, Wayne Rossman, and Susanna Santos. Discrete special isothermic surfaces. *Geometriae Dedicata*, 174(1):1–11, 2015.

-
- [31] Francis E. Burstall, Joseph Cho, Udo Hertrich-Jeromin, Mason Pember, and Wayne Rossman. Discrete Ω -nets and Guichard nets via discrete Koenigs nets. *Proceedings of the London Mathematical Society*, 126(2):790–836, 2023.
 - [32] Francis E. Burstall, Udo Hertrich-Jeromin, and Wayne Rossman. Discrete linear Weingarten surfaces. *Nagoya Mathematical Journal*, 231:55–88, 2018.
 - [33] Francis E. Burstall, Udo Hertrich-Jeromin, Wayne Rossman, and Susana D. Santos. Discrete surfaces of constant mean curvature. In *Development in Differential Geometry of Submanifolds*, volume 1880, pages 133–179. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.
 - [34] Annalisa Calini, Thomas Ivey, and Gloria Mari Beffa. Integrable flows for starlike curves in centroaffine space. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 9, 2013.
 - [35] Albert Chern, Felix Knöppel, Franz Pedit, and Ulrich Pinkall. *Commuting Hamiltonian flows of curves in real space forms*, page 291–328. LMS Lecture Note Series. Cambridge University Press, 2020.
 - [36] Joseph Cho, Katrin Leschke, and Yuta Ogata. Periodic discrete Darboux transforms. *Differential Geometry and its Applications*, 91, 2023.
 - [37] Joseph Cho, Mason Pember, and Gudrun Szewieczek. Constrained elastic curves and surfaces with spherical curvature lines. *Indiana University Mathematics Journal*, 72:2059–2099, 2023.
 - [38] Joseph Cho, Wayne Rossman, and Tomoya Seno. Discrete mKdV equation via Darboux transformation. *Mathematical Physics, Analysis and Geometry*, 24, 2021.
 - [39] Adam Doliwa. Generalized isothermic lattices. *Journal of Physics A: Mathematical and Theoretical*, 40(42), 2007.
 - [40] Josef Dorfmeister, Franz Pedit, and Hongyong Wu. Weierstrass type representation of harmonic maps into symmetric spaces. *Communications in analysis and geometry*, 6(4):633–668, 1998.
 - [41] Heinz-Dieter Ebbinghaus, Hans Hermes, Friedrich Hirzebruch, Max Koecher, Klaus Mainzer, Jürgen Neukirch, Alexander Prestel, and Reinhold Remmert. *Zahlen*, volume 1. Springer-Verlag, 2013.
 - [42] Leonhard Euler. Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti. *Marcum-Michaelem Bousquet*, 1744.
 - [43] Matteo Gallet, Georg Grasegger, Jan Legerský, and Josef Schicho. Combinatorics of Bricard’s octahedra. *Comptes Rendus. Mathématique*, 359(1):7–38, 2021.
 - [44] Raymond E. Goldstein and Dean M. Petrich. The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane. *Phys. Rev. Lett.*, 67:3203–3206, Dec 1991.
 - [45] Basil Gordon and Theodore S. Motzkin. On the zeros of polynomials over division rings. *Transactions of the American Mathematical Society*, 116:218–226, 1965.

- [46] Ewain Gwynne and Matvei Libine. On a quaternionic analogue of the cross-ratio. *Advances in Applied Clifford Algebras*, 22(4):1041–1053, 2012.
- [47] Christian Hafner and Bernd Bickel. The design space of plane elastic curves. *ACM Transactions on Graphics*, 40(4), 2021.
- [48] Gábor Hegedüs, Josef Schicho, and Hans-Peter Schröcker. Factorization of rational curves in the study quadric. *Mechanism and Machine Theory*, 69:142–152, 2013.
- [49] Udo Hertrich-Jeromin. Transformations of discrete isothermic nets and discrete cmc-1 surfaces in hyperbolic space. *manuscripta mathematica*, 102(4):465–486, 2000.
- [50] Udo Hertrich-Jeromin. *Introduction to Möbius differential geometry*. Number 300 in London Mathematical Society Lecture Note Series. Cambridge University Press, 2003.
- [51] Udo Hertrich-Jeromin, Tim Hoffmann, and Ulrich Pinkall. *A discrete version of the Darboux transform for isothermic surfaces*, pages 59–81. Clarendon Press, Oxford, 1999.
- [52] Udo Hertrich-Jeromin, Ian McIntosh, Peter Norman, and Franz Pedit. Periodic discrete conformal maps. *J. Reine Angew. Math.*, 534, 1999.
- [53] Tim Hoffmann. Discrete cmc surfaces and discrete holomorphic maps. *Discrete integrable geometry and physics*, pages 97–112, 1999.
- [54] Tim Hoffmann. On the equivalence of the discrete nonlinear Schrödinger equation and the discrete isotropic Heisenberg magnet. *Physics Letters A*, 265(1-2):62–67, 2000.
- [55] Tim Hoffmann. Discrete Hashimoto surfaces and a doubly discrete smoke-ring flow. In *Discrete differential geometry*, pages 95–115. Springer, 2008.
- [56] Tim Hoffmann. Discrete differential geometry of curves and surfaces. *Kyushu University Lecture Note Series Vol.18*, 2009.
- [57] Tim Hoffmann. A Darboux transformation for discrete s-isothermic surfaces. *Journal of Math-for-Industry*, 2, 2010.
- [58] Tim Hoffmann, Shimpei Kobayashi, and Zi Ye. Discrete constant mean curvature surfaces on general graphs. *Geometriae Dedicata*, 216(71), 2022.
- [59] Tim Hoffmann and Nadja Kutz. Discrete curves in $\mathbb{CP}^1$ and the Toda lattice. *Studies in Applied Mathematics*, 113(1):31–55, 2004.
- [60] Tim Hoffmann and Andrew O. Sageman-Furnas. A 2×2 Lax Representation, Associated Family, and Bäcklund Transformation for Circular K-Nets. *Discrete & Computational Geometry*, 56(2):472–501, 2016.
- [61] Tim Hoffmann, Andrew O. Sageman-Furnas, and Jannik Steinmeier. Skew parallelogram nets and universal factorization. 2024.
- [62] Tim Hoffmann, Andrew O. Sageman-Furnas, and Max Wardetzky. A Discrete Parametrized Surface Theory in $\mathbb{R}^3$. *International Mathematics Research Notices*, 2017(14):4217–4258, 2016.
- [63] Tim Hoffmann, Andrew O. Sageman-Furnas, and Max Wardetzky. Discrete Bonnet pairs and conformal nets. *in preparation*, 2025.

-
- [64] Tim Hoffmann, Jannik Steinmeier, and Gudrun Szewieczek. Discrete curve theory in space forms: planar elastic and area-constrained elastic curves. 2025.
 - [65] Tim Hoffmann and Gudrun Szewieczek. Isothermic nets with spherical parameter lines from discrete holomorphic maps. 2024.
 - [66] Jun-ichi Inoguchi, Kenji Kajiwara, Nozomu Matsuura, and Yasuhiro Ohta. Discrete mKdV and discrete sine-Gordon flows on discrete space curves. *Journal of Physics A: Mathematical and Theoretical*, 47(23), 2014.
 - [67] Ivan Izmistiev. Classification of flexible Kokotsakis polyhedra with quadrangular base. *International Mathematics Research Notices*, 2017(3):715–808, 2017.
 - [68] Anton Izosimov. Polygon recutting as a cluster integrable system. *Selecta Mathematica*, 29(21), 2023.
 - [69] Martin Kilian, Georg Nawratil, Matteo Raffaelli, Arvin Rasoulzadeh, and Kiumars Sharifmoghaddam. Interactive design of discrete Voss nets and simulation of their rigid foldings. *Computer Aided Geometric Design*, 111, 2024.
 - [70] Antonios Kokotsakis. Über bewegliche Polyeder. *Mathematische Annalen*, 107(1):627–647, 1933.
 - [71] Benno König. *A Geometric Construction for the Associated Family of S-Isothermic CMC Surfaces*. PhD thesis, Technische Universität München, 2018.
 - [72] Joel Langer and Ron Perline. Poisson geometry of the filament equation. *Journal of Nonlinear Science*, 1:71–93, 1991.
 - [73] Joel Langer and David A. Singer. The total squared curvature of closed curves. *J. Differential Geom.*, 20(1):1–22, 1984.
 - [74] H. Blaine Lawson Jr. Complete minimal surfaces in $\mathbb{S}^3$. *Annals of Mathematics*, pages 335–374, 1970.
 - [75] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. In *Selected Papers Volume I*, pages 366–389. Springer, 2005.
 - [76] Zijia Li, Daniel F. Scharler, and Hans-Peter Schröcker. Factorization results for left polynomials in some associative real algebras: state of the art, applications, and open questions. *Journal of Computational and Applied Mathematics*, 349:508–522, 2019.
 - [77] Zijia Li, Hans-Peter Schröcker, and Johannes Siegle. A Geometric Algorithm for the Factorization of Spinor Polynomials. 2023.
 - [78] Shigeki Matsutani and Emma Previato. From Euler’s elastica to the mKdV hierarchy, through the Faber polynomials. *Journal of Mathematical Physics*, 57(8), 2016.
 - [79] J. Michael McCarthy and Gim Song Soh. *Geometric design of linkages*, volume 11. Springer Science & Business Media, 2010.
 - [80] Jürgen Moser and Alexander P. Veselov. Discrete Versions of Some Classical Integrable Systems and Factorization of Matrix Polynomials. *Communications in Mathematical Physics*, 139:217–243, 1991.

- [81] Frank Nijhoff. On some “Schwarzian” equations and their discrete analogues. In *Algebraic aspects of integrable systems: in memory of Irene Dorfman*, pages 237–260. Springer, 1997.
- [82] Frank Nijhoff and Hans Capel. The discrete Korteweg-de Vries equation. *Acta Applicandae Mathematica*, 39:133–158, 1995.
- [83] Ivan Niven. Equations in quaternions. *The American Mathematical Monthly*, 48(10):654–661, 1941.
- [84] Yuta Ogata and Masashi Yasumoto. Construction of discrete constant mean curvature surfaces in Riemannian spaceforms and applications. *Differential Geometry and its Applications*, 54:264–281, 2017.
- [85] Mason Pember, Denis Polly, and Masashi Yasumoto. Discrete Weierstrass-type representations. *Discrete & Computational Geometry*, 70(3):816–844, 2023.
- [86] Ulrich Pinkall and Oliver Gross. *Differential geometry - from elastic curves to Willmore surfaces*. Compact Textbooks in Mathematics. Birkhäuser/Springer, 2024.
- [87] Ulrich Pinkall, Boris Springborn, and Steffen Weißmann. A new doubly discrete analogue of smoke ring flow and the real time simulation of fluid flow. *Journal of Physics A: Mathematical and Theoretical*, 40(42), 2007.
- [88] Helmut Pottmann and Johannes Wallner. *Computational line geometry*, volume 6. Springer, 2001.
- [89] Colin Rogers and Wolfgang K. Schief. *Bäcklund and Darboux transformations: geometry and modern applications in soliton theory*, volume 30. Cambridge University Press, 2002.
- [90] Robert Sauer. Parallelogrammgitter als modelle pseudosphärischer Flächen. *Mathematische Zeitschrift*, 52(1):611–622, 1950.
- [91] Robert Sauer. *Differenzengeometrie*, volume 7. Springer, 1970.
- [92] Robert Sauer and Heinrich Graf. Über Flächenverbiegung in Analogie zur Verknickung offener Facettenfläche. *Mathematische Annalen*, 105:499–535, 1931.
- [93] Daniel F. Scharler and Hans-Peter Schröcker. An Algorithm for the Factorization of Split Quaternion Polynomials. *Advances in Applied Clifford Algebras*, 31:1–23, 2021.
- [94] Daniel F. Scharler, Johannes Siegele, and Hans-Peter Schröcker. Quadratic split quaternion polynomials: factorization and geometry. *Advances in Applied Clifford Algebras*, 30:1–23, 2020.
- [95] Wolfgang K. Schief. Isothermic surfaces in spaces of arbitrary dimension: integrability, discretization, and Bäcklund transformations—a discrete Calapso equation. *Studies in Applied Mathematics*, 106(1):85–137, 2001.
- [96] Wolfgang K. Schief. On the unification of classical and novel integrable surfaces. II. Difference geometry. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 459(2030):373–391, 2003.
- [97] Wolfgang K. Schief. On a maximum principle for minimal surfaces and their integrable discrete counterparts. *Journal of Geometry and Physics*, 56(9):1484–1495, 2006.

-
- [98] Wolfgang K. Schief. Discrete Chebyshev nets and a universal permutability theorem. *Journal of Physics A: Mathematical and Theoretical*, 40(18):4775–4801, 2007.
 - [99] Wolfgang K. Schief, Alexander I. Bobenko, and Tim Hoffmann. On the integrability of infinitesimal and finite deformations of polyhedral surfaces. *Discrete differential geometry*, pages 67–93, 2008.
 - [100] Oded Schramm. Circle patterns with the combinatorics of the square grid. *Duke Mathematical Journal*, 86(2):347 – 389, 1997.
 - [101] Hellmuth Stachel. Remarks on Bricard’s flexible octahedra of type 3. In *Proc. 10th International Conference on Geometry and Graphics*, pages 8–12, Kiev, 2002.
 - [102] Yuri B. Suris. A note on an integrable discretization of the nonlinear Schrödinger equation. *Inverse problems*, 13(4):1121–1136, 1997.
 - [103] Yuri B Suris. *The problem of integrable discretization: Hamiltonian approach*, volume 219. Birkhäuser, 2012.
 - [104] Antoni Sym. Soliton surfaces and their applications (soliton geometry from spectral problems). In *Geometric Aspects of the Einstein Equations and Integrable Systems: Proceedings of the Sixth Scheveningen Conference, Scheveningen, The Netherlands, August 26–31, 1984*, pages 154–231. Springer, 2005.
 - [105] Serge Tabachnikov and Emmanuel Tsukerman. On the discrete bicycle transformation. *Publ. Mat. Urug. 14*, pages 201–219, 2013.
 - [106] Ekkehard-Heinrich Tjaden. Einfache elastische Kurven. *Dissertation, TU Berlin*, 1991.
 - [107] Walter Wunderlich. Zur Differenzengeometrie der Flächen konstanter negativer Krümmung. *Osterreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsher. II*, 160:39–77, 1951.
 - [108] Sebastian Zurita and Kenji Kajiwar. Fairing of discrete planar curves by discrete Euler’s elasticae. *JSIAM Letters*, 11:73–76, 2019.