Cut Set Coloring - Integer Linear Program Formulation

Shakthivelu Janardhanan, *Student Member, IEEE,* Ritanshi Agarwal, Wolfgang Kellerer, *Senior Member, IEEE,* Carmen Mas-Machuca, *Senior Member, IEEE.*

***Abstract*—A minimal cut set of a system refers to the smallest possible set of the system's components which on becoming unavailable, cause the system to fail. However, all the components in a minimal cut set must fail for the system to fail. The same concept also applies to communication networks. In a network, traffic flows between a source and a destination. Such a flow also has minimal cut sets of nodes in the network which when failed, cause the flow to fail. If all the nodes in a minimal cut set are purchased from the same manufacturer and if that manufacturer becomes unavailable due to any reason, then the minimal cut set fails, ultimately causing the flow to fail. Therefore, it is wise to purchase these different critical nodes from different manufacturers. In this work, we present a graph-coloring approach based on an Integer Linear Program (ILP) called the *Cut Set Coloring (CSC) - ILP*, to identify how to purchase these critical components in the minimal cut set of a system from different manufacturers, such that the impact of failure(s) of the manufacturer(s) is minimized.**

***Index Terms*—minimal cut sets, cut set coloring, ILP**

I. Cut Set Coloring - Integer Linear Program Formulation (*CSC-ILP*)

The research question addressed in this work is as follows.

(Q1) Given a topology and the number of manufacturers available, what is the best manufacturer assignment possible to minimized the impact of manufacturer failures?

To answer this question, we use the *CSC-ILP*.

A. CSC-ILP Constants

Let us consider a network topology graph G, with vertices V and edges E. Let us consider the set of flows R, where each flow r is defined by its source, destination, and weight (d_r). The weight of the flow is its priority- the operator's choice. Let the manufacturer list be M. Each flow r has its set of minimal cut sets denoted by K_r. κ_{rj} is the j^{th} minimal cut set of the r^{th} demand. We remove the minimal cut sets with only the src and dst, because they are irrelevant to a sovereignty study. We also remove all the flows with a one-hop path to the destination because there are no minimal cut sets other than the src and dst themselves. For a realistic analysis, we consider the normalized cost per component from each manufacturer $m \in M$ to be C_m. The cost of a component from the manufacturer varies linearly between 0.995 and 0.985, depending on $|M|$. The cost threshold for each operator varies as per their preferences. However, we use a cost threshold C_T in this study to restrict operators' choices. The cost threshold C_T is given by the product of $|V|$ and the average cost of components, i.e., 0.9.

$$C_T = |V| \times \frac{0.995 + 0.985}{2} \tag{1}$$

B. CSC-ILP variables

There are two major variables to consider. b_{mn} is a binary variable that is equal to 1 if node n is from manufacturer m. q_{mrj} is a binary variable that is equal to 1 if the j^{th} minimal cut set of the r^{th} demand uses a component from manufacturer m. The solution to the ILP problem lies in the variable b_{mn} because it gives the most sovereign manufacturer assignment.

C. CSC-ILP Constraints

The first constraint in Eq. 2 is to ensure one component can be purchased from one manufacturer only.

$$\sum_{m \in M} b_{mn} = 1, \forall n \in V \tag{2}$$

The second constraint in Eq. 3 is to identify which manufacturers are present in each minimal cut set.

$$q_{mrj} = \bigvee_{n \in \kappa_{rj}} (b_{mn}), \ \forall \kappa_{rj} \in K_r, \forall m \in M, \forall r \in R \tag{3}$$

This third constraint in Eq. 4 is the cost threshold constraint that ensures the total cost of the network is lower than the threshold C_T.

$$\sum_{n \in V} \left[\sum_{m \in M} (b_{mn} \times C_m) \right] \leq C_T \tag{4}$$

D. CSC-ILP Objective

CSC-ILP aims to solve the problem statement in (Q1) by choosing the most appropriate manufacturer assignment to minimize the impact of manufacturer failures. *CSC-ILP* achieves the aforementioned objective by maximizing manufacturer diversity inside the minimal cut set. That is, *CSC-ILP* maximizes the number of manufacturers in each minimal cut set of each flow. The score can be normalized with the weight variable w_r if the operator wishes to prioritize the survival of particular flows above others. For our study, we assume that

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all flows have equal weights. Hence, the objective formulation is presented in Eq. 5.

$$\text{maximize} \sum_{r \in R} \left\{ \sum_{j \in J} \left[\sum_{m \in M} q_{mrj} \right] \right\} \tag{5}$$

Solving this objective function, we obtain the best manufacturer assignment to maximize network sovereignty in the variable b_{mn}. Though the example considered in this work pertains to communication networks, the same formulation will work for any system.