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Review Article

$B \longrightarrow P, V$ Form Factors with the B-Meson Light-Cone Sum Rules

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In this review, we discuss the calculation of the $B \longrightarrow P, V$ form factors within the framework of the light-cone sum rules with the light-cone distribution amplitudes of the B-meson. A detailed introduction to the definition, scale evolution, and phenomenological models of the B-meson distribution amplitudes is presented. We show two equivalent approaches of calculating the next-to-leading order QCD corrections to the sum rules for the form factors, i.e., the method of regions and the step-by-step matching in the soft-collinear effective theory. The power suppressed corrections to the $B \longrightarrow P, V$ form factors especially the contributions from the higher-twist B-meson distribution amplitudes are displayed. We also present numerical results of the form factors including both the QCD and the power corrections, and phenomenological applications of the predicted form factors such as the determination of the CKM matrix element $|V_{ub}|$.

1. Introduction

The decays of B_q mesons $(q = d, s)$ have been playing a crucial role in the determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements as well as the understanding of the QCD dynamics in the heavy-light meson system. The heavy-to-light transition form factors are essential ingredients in the semileptonic decays $B \longrightarrow M\ell\nu$ (M stands for vector (V) or pseudoscalar (P) meson), in the flavor-changing-neutral-current (FCNC) processes $B \longrightarrow M\ell^+\ell^-(\nu\bar{\nu})$ and $B \longrightarrow V\gamma$, and in the nonleptonic B-meson decays. In the small recoil region, the heavy-to-light form factors can be determined from the experiments or calculated by the nonperturbative approach, among which the lattice QCD simulation which is based on the first principle of QCD is regarded to give the most reliable predictions. At small hadronic recoil, the lattice QCD calculations of $B \longrightarrow \pi, K$, $B_s \longrightarrow K$ form factors have been performed [1–3] using the gauge-field ensembles with $(2+1)$-flavor lattice configurations. In addition, the Flavor Lattice Average Group (FLAG) has given the results of these form factors with an extrapolation to the whole kinematic region from the small hadronic recoil region of the light meson [4]. The unquenched lattice QCD calculations of $B \longrightarrow K^*$ form factors have been performed [5, 6] by employing the gauge-field ensembles with an improved staggered quark action from the MILC Collaboration [7].

Due to the limited computing capability, the lattice simulation cannot be applied to the large recoil region directly. In the framework of the QCD factorization, the heavy-to-light form factors at large recoil contain both the soft contribution satisfying the large-recoil symmetry relations and the hard spectator scattering effect violating the symmetry relations at leading power in Λ/m_Q [8]. The soft-collinear effective theory (SCET) provides a more transparent insight on the factorization property of heavy-to-light form factors by integrating out the hard and hard-collinear fluctuations step by step. Implementing the first-step matching procedure for the QCD current $\bar{\psi}\,\Gamma_i Q$ will give rise to the so-called $A0$-type and $B1$-type $\mathrm{SCET_I}$ operators [9–11], both of which can contribute to heavy-to-light form factors at leading power in Λ/m_Q. The matrix elements of the $A0$-type operator are nonfactorizable due to the emergence of endpoint divergences in the convolution integrals of the jet functions from the matching between $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$ and the light-cone distribution amplitudes (LCDAs). By contrast, the matrix elements of the $B1$-type $\mathrm{SCET_I}$ operators can be further factorized into convolutions of the jet functions and the LCDAs [11]. Since the latter one is suppressed by the strong coupling constant, the heavy-to-light form factors are dominated by the soft form factor in the QCD factorization. An alternative approach to compute the heavy-to-light form factors is based upon the transverse-momentum-dependent (TMD) factorization for hard processes, where the on-shell Sudakov form factor [12] arises from the resummation of large logarithmic terms that can effectively suppress the region with small transverse momentum [13]. In the TMD factorization approach (also called the PQCD approach), the endpoint singularity will be regularized by the transverse momentum; then, the form factors are perturbative calculable. The $B \longrightarrow \pi$ form factors within the PQCD approach have been pushed to $O(\alpha_s)$ for twist-2 [14, 15] and twist-3 [16] contributions of pion LCDAs. However, the infrared subtractions beyond the leading order in α_s [17] are much more complex than that in the QCD factorization and a complete understanding of the TMD factorization for exclusive processes with large momentum transfer has not been achieved to date on the conceptual side.

The lattice QCD simulation for the heavy-to-light form factors is valid in the small recoil region; its predictions need to be extrapolated to the whole kinematic region from the small hadronic recoil of the light meson through special models. The extrapolation-model dependence will produce unavoidable uncertainties in the determination of observations, especially at large recoil. Therefore, to obtain the q^2 (invariant mass of lepton pair in the semileptonic decays) dependence of the form factors with high accuracy, it is important to compute the form factors in the large hadronic recoil region directly. The light-cone sum rule (LCSR) approach, which is a combination of the SVZ sum rules with the QCD theory of hard exclusive processes, provides an appropriate method to evaluate the form factors at large recoil. According to the correlation functions employed in the calculation, two different frameworks of LCSR, i.e., LCSR with the light-meson LCDAs [18–25] and LCSR with the B-meson LCDAs (we will also call this the B-meson LCSR) [26–29], have been established. The advantage of LCSR with light-meson LCDAs is that it can be applied to a larger region of the square of the momentum transfer and also the LCDAs of the light meson are better determined than that of the B-meson. Meanwhile, the B-meson LCSR has its unique advantage that the input LCDAs are universal for all the $B \longrightarrow M$ form factors, and the theoretical uncertainty can be sizeably reduced in the calculation of ratios of the form factors. The LCSR for the $\mathrm{SCET_I}$ matrix elements entering the QCD factorization formulae of heavy-to-light B-meson decay form factors, which is also called the SCET improved sum rules, has been achieved [30, 31] employing the vacuum-to-B-meson correlation functions. To improve the accuracy of the $B \longrightarrow P(P = \pi, K, D)$ form factors, the next-to-leading order (NLO) corrections to the correlation functions in the strong coupling α_s have been performed in [32–35], where the factorization formulae for the correlation functions were established with the diagrammatic approach and the strategy of regions [36, 37]. With a similar method, the baryonic transition $\Lambda_b \longrightarrow \Lambda$ form factors have also been investigated [38]. The $B \longrightarrow V$ form factors have also been evaluated with the SCET improved sum rules, and the NLO correction to the hard-collinear function is performed [39]. Except for the QCD corrections, the power suppressed contributions also play an important role in the precise calculation of the heavy-to-light form factors. Since it is a notoriously difficult task to perform a complete power expansion on a physical amplitude, one has to deal with some special kinds of power corrections with respect to the leading power contribution so far. The power suppressed contributions from the high-twist LCDAs of B-meson to the $B \longrightarrow P$ and $B \longrightarrow V$ form factors were calculated in [35, 39, 40].

The fundamental nonperturbative inputs in the B-meson LCSR are the LCDAs of the B-meson. Since they are nonperturbative in nature, the hard effect must be integrated out in the definition of the LCDAs. Therefore, one has to employ the effective bottom quark field in the heavy quark effective theory (HQET) to construct the matrix elements of the nonlocal operators [41]. As a soft objective, the B-meson LCDAs do not have a definite twist in principle, while when entering a process with large momentum transfer and thus only one of the light-cone components of the momentum of the soft light quark would be picked up, the twist of the B-meson LCDAs makes sense. The scale dependence of the LCDAs of the B-meson can be obtained by the renormalization group (RG) equation approach, and the RG equation for the leading-twist LCDA has been derived at two-loop level [42, 43], and the one-loop level evolution equations for higher-twist LCDAs are also known [44]. The evolution behavior of the LCDAs can provide constraints on the model of the LCDAs; in addition, it also plays an important role in proving the factorization-scale independence of the factorization formula.

The plan of this review is as follows. In Section 2, we will discuss the LCDAs of the B-meson, including the definition, the evolution behavior, and the phenomenological models of the leading-twist and higher-twist LCDAs. In Sections 3 and 4, we will introduce the LCSR with B-meson LCDAs, emphasizing the two equivalent methods to evaluate the QCD corrections to the correlation functions, i.e., the method of regions and the matching between the $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$. In Section 5, we will discuss the power suppressed contributions to the heavy-to-light form factors with LCSR, concentrating on the contributions from high-twist B-meson LCDAs. In Section 6, we will present some numerical results of the form factors and the phenomenological applications. We summarize in the last section.

2. The Light-Cone Distribution Amplitudes of the B-Meson

2.1. Definition of the Leading-Twist B-Meson LCDAs. The B-meson LCDAs are among the most important ingredients of the QCD factorization formula for exclusive B decays. Before talking about the heavy-to-light form factors, we would like to introduce the LCDAs of the B-meson. The two-particle LCDAs of the B-meson in the HQET can be obtained from the coordinate-space matrix elements [41]:

$$\left\langle 0\left|\bar{q}^{\beta}(x)[x,0]h_v^{\alpha}(0)\right|\bar{B}(v)\right\rangle = -\frac{i\tilde{f}_B(\mu)m_B}{4}\left[\frac{1+v}{2}\left\{2\Phi_+\left(t,x^2\right)+\frac{\Phi_-\left(t,x^2\right)-\Phi_+\left(t,x^2\right)}{t}x\right\}\right]^{\alpha\beta}, \quad (1)$$

where $t=v\cdot x$. The LCDAs $\Phi_\pm(t,x^2)$ can be expanded around $x^2=0$. In the limit $x^2\longrightarrow 0$, we assume $t\longrightarrow z=n\cdot x/2$; then, the B-meson LCDAs in the momentum space are defined through the Fourier transformation:

$$\phi_\pm(\omega)=\int\frac{dz}{2\pi}e^{i\omega z}\Phi_\pm(z), \quad \Phi_\pm(z)\equiv\Phi_\pm(z,0). \quad (2)$$

For convenience, we introduce two light-like vectors n and $\bar{n}$ satisfying $n\cdot\bar{n}=2, n^2=\bar{n}^2=0$. Any four-vector k^μ can be expressed as $k^\mu=n\cdot k\,\bar{n}^\mu/2+\bar{n}\cdot k\,n^\mu/2+k_\perp^\mu$.

When the LCDAs of the B-meson are applied in the calculation of B-meson decay processes, the momentum space projector is usually required. To obtain the projector, we adopt the reference frame satisfying $v=(n+\bar{n})/2$ and assume $\bar{n}\cdot x\ll x_\perp\ll n\cdot x$, thus $x\simeq z\bar{n}+x_\perp$, then

$$\Phi_+\left(t,x^2\right)+\frac{\Phi_-\left(t,x^2\right)-\Phi_+\left(t,x^2\right)}{t}x\simeq\Phi_+(z)n+\Phi_-(z)\bar{n}+\frac{\Phi_-(z)-\Phi_+(z)}{z}x_\perp. \quad (3)$$

After the Fourier transform, the projector of B-meson LCDAs is obtained as [8]

$$M_B(\omega)=-\frac{i\tilde{f}_B(\mu)m_B}{4}\left[\frac{1+v}{2}\left\{\phi_+(\omega)n+\phi_-(\omega)\bar{n}-\int_0^\omega d\eta[\phi_-(\eta)-\phi_+(\eta)]\gamma_\perp^\mu\frac{\partial}{\partial k_\perp^\mu}\right\}\right]\gamma_5, \quad (4)$$

with k is the momentum of the antilight quark in the B-meson.

The LCDAs $\Phi_\pm$ defined above do not contribute at the same power in the QCD factorizations of the heavy hadron decay processes, since they do not have the same collinear twist (in the following, we call "twist" for short). The twist t and conformal spin j of the light quark and gluon fields are given by the usual expressions [45]:

$$t=d-s, \quad j=\frac{1}{2}(d+s), \quad (5)$$

where d is the canonical dimension and s is the spin projection on the light cone. The twist and conformal spin are closely related to the collinear subgroup of the conformal group. The collinear subgroup of the conformal group is locally equivalent to the $SL(2,R)$ group. The subgroup contains four generators, namely, P_+, M_{-+}, D, and K_-, where P_+ and M_{-+} are the projections of the generator of the Poincare group $p_\mu, M_{\nu\mu}$ on the light cone, D is the generator of dilatation, and K_- is the generator of the special conformal transformation along the light cone [45]. Specifically, if we write the light-like vector $x=\alpha n$, then the dilatation indicates the transform $\alpha\longrightarrow\alpha'=\lambda\alpha$, and for the special conformal transformation,

$$\alpha\longrightarrow\alpha'=\frac{a\alpha+b}{c\alpha+d}, \quad ad-bc=1. \quad (6)$$

The four generators of the collinear subgroup of the conformal group can be rearranged to form the algebra of $SL(2,R)$, i.e.,

$$\begin{aligned} J_+&=J_1+iJ_2=-iP_+,\\ J_-&=J_1-iJ_2=\left(\frac{i}{2}\right)K_-,\\ J_0&=\left(\frac{i}{2}\right)(D+M_\mp),\\ E&=\left(\frac{i}{2}\right)(D-M_{-+}), \end{aligned} \quad (7)$$

with

$$[J_0,J_\mp]=\mp J_\mp, \quad [J_-,J_+]=-2J_0. \quad (8)$$

For the quantized field $\Phi(\alpha)$, we have

$$\begin{aligned} [J_+,\Phi(\alpha)]&=-\partial_\alpha\Phi(\alpha),\\ [J_-,\Phi(\alpha)]&=\left(\alpha^2\partial_\alpha+2j\alpha\right)\Phi(\alpha),\\ [J_0,\Phi(\alpha)]&=(\alpha\partial_\alpha+j)\Phi(\alpha),\\ [E,\Phi(\alpha)]&=\frac{1}{2}(\ell-s)\Phi(\alpha). \end{aligned} \quad (9)$$

For a spinor field Ψ, the projectors $\Psi_+=(n\bar{n}/4)\Psi$ and $\Psi_-=(\bar{n}n/4)\Psi$ have definite twist, namely, $t[\Psi_\pm]=\pm1$. In an appropriate reference frame, $\Psi_\pm$ only have two nonzero components; thus, it is more convenient to define these fields with two-component spinors. Any light-like vector can be represented by a product of two spinors. One can write

$$\begin{aligned} n_{\alpha\dot\alpha} &= n_\mu \sigma^\mu_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}, \\ \bar n_{\alpha\dot\alpha} &= \bar n_\mu \sigma^\mu_{\alpha\dot\alpha} = \mu_\alpha \bar\mu_{\dot\alpha}, \end{aligned} \tag{10}$$

where for the auxiliary λ and μ spinors, $\bar\lambda = \lambda^\dagger$, $\bar\mu = \mu^\dagger$ which satisfy $(\lambda\,\mu) = \lambda^\alpha \mu_\alpha = 2$, $(\bar\mu\,\bar\lambda) = \bar\mu_{\dot\alpha}\bar\lambda^{\dot\alpha} = 2$. The "+" and "−" fields are defined as

$$\begin{aligned} \chi_+ &= \lambda^\alpha \psi_\alpha, \\ \bar\psi_+ &= \bar\lambda^{\dot\alpha} \psi_{\dot\alpha}, \\ f_{++} &= \lambda^\alpha \lambda^\beta f_{\alpha\beta}, \\ f_\pm &= \lambda^\alpha \mu^\beta f_{\alpha\beta}, \\ \bar f_{++} &= \bar\lambda^{\dot\alpha} \bar\lambda^{\dot\beta} \bar f_{\dot\alpha\dot\beta}, \end{aligned} \tag{11}$$

etc. The Dirac (antiquark) spinors

$$\begin{aligned} q &= \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix}, \\ \bar q &= \left(\chi^\beta, \bar\psi_{\dot\alpha}\right) \end{aligned} \tag{12}$$

are written in terms of the following two component fields:

$$\begin{aligned} (\lambda\mu)\chi^\alpha &= \mu^\alpha \chi_+ - \lambda^\alpha \chi_-, \\ (\bar\mu\bar\lambda)\bar\psi_{\dot\alpha} &= \bar\mu_{\dot\alpha}\bar\psi_+ - \bar\lambda_{\dot\alpha}\bar\psi_-. \end{aligned} \tag{13}$$

The large component of the heavy quark field in the HQET satisfies $v h_v = h_v$; then,

$$\begin{aligned} h_+ &= -\bar h_-, \\ h_- &= \bar h_+. \end{aligned} \tag{14}$$

The gluon strength tensor $F_{\mu\nu}$ can be decomposed as

$$\begin{aligned} F_{\alpha\beta,\dot\alpha\dot\beta} &= \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} - \epsilon_{\alpha\beta} \bar f_{\dot\alpha\dot\beta}\right), \\ i\tilde F_{\alpha\beta,\dot\alpha\dot\beta} &= \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} i\tilde F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} + \epsilon_{\alpha\beta} \bar f_{\dot\alpha\dot\beta}\right). \end{aligned} \tag{15}$$

Here, $f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ are chiral and antichiral symmetric tensors, $f^* = \bar f$, which belong to $(1, 0)$ and $(0, 1)$ representations of the Lorentz group, respectively. It is easy to see that the twist of the field ψ_+ and χ_+ is 1, the twist of ψ_- and χ_- is 2, and we assign the twist of $h_\pm$ to be 1. Employing the relevant operators with definite twist in spinor notation, one obtains

$$\begin{aligned} \tilde f_B(\mu) m_B \Phi_+(z) &= i\left\langle 0 \middle| \bar\psi_+(z) h_+(0) - \chi_+(z)\bar h_+(0) \middle| \bar B(v)\right\rangle, \\ \tilde f_B(\mu) m_B \Phi_-(z) &= i\left\langle 0 \middle| \bar\psi_-(z) h_-(0) - \chi_-(z)\bar h_-(0) \middle| \bar B(v)\right\rangle. \end{aligned} \tag{16}$$

2.2. Evolution of the Leading-Twist B-Meson LCDA. At leading power, only $\phi_+(\omega)$ is relevant in the factorization formula of various B-meson decay processes, and the RG equation of $\phi_+(\omega)$ up to the leading-logarithmic (LL) accuracy is the well-known Lange-Neubert (LN) equation [42], which reads

$$\frac{d}{d\ln\mu}\phi_+(\omega,\mu) = -\int_0^\infty d\omega' \Gamma_+\left(\omega,\omega',\mu\right)\phi_+\left(\omega',\mu\right), \tag{17}$$

where μ is the renormalization scale. The anomalous dimensions are

$$\begin{aligned} \Gamma_+\left(\omega,\omega',\mu\right) &= \left(\Gamma_{\text{cusp}} \ln\frac{\mu}{\omega} + \gamma_+\right)\delta\left(\omega-\omega'\right) + \omega\,\Gamma_{\text{cusp}}\Gamma\left(\omega,\omega'\right), \\ \Gamma^{(0)}_{\text{cusp}} &= 4C_F, \\ \Gamma^{(1)}_{\text{cusp}} &= 4C_F\left[\frac{67}{3} - \pi^2 - \frac{10}{9}n_f\right], \\ \gamma^{(0)}_+ &= -2C_F, \\ \Gamma\left(\omega,\omega'\right) &= -\left[\frac{\theta\left(\omega'-\omega\right)}{\omega'\left(\omega'-\omega\right)} + \frac{\theta\left(\omega-\omega'\right)}{\omega\left(\omega-\omega'\right)}\right]_+, \\ \int_0^\infty dy[f(x,y)]_+ g(y) &= \int_0^\infty dy\, f(x,y)[g(y)-g(x)], \end{aligned} \tag{18}$$

where the anomalous dimensions are expanded in the same way as

$$\gamma_+ = \sum_{n=0}\gamma_+^{(n)} a^{n+1}, \quad \text{with} \quad a = \frac{\alpha_s}{4\pi}. \tag{19}$$

Since the evolution equation of the leading-twist B-meson LCDA is the integrodifferential equation, it is difficult to obtain the solution directly. A commonly used method is to simplify the evolution equation by an integral transformation. There exist several kinds of integral transformations which are helpful to work out the solution of the evolution equation. It was found that the evolution kernel is diagonalized when it is transformed into the so-called "dual" space [46]. The leading-twist LCDA in the dual space can be obtained by

$$\rho_+\left(\omega',\mu\right) = \int_0^\infty \frac{d\omega}{\omega}\sqrt{\frac{\omega}{\omega'}} J_1\left(2\sqrt{\frac{\omega}{\omega'}}\right)\phi_+(\omega,\mu), \tag{20}$$

which satisfies an ordinary differential equation:

$$\frac{d}{d\ln\mu}\rho_+\left(\omega',\mu\right)=-\left[\Gamma_{\text{cusp}}\ln\frac{\mu}{e^{-2\gamma_E}\omega'}+\gamma_+\right]\rho_+\left(\omega',\mu\right). \tag{21}$$

The evolution equation can also be calculated in the position space, where it takes the form [47]

$$\begin{aligned}\frac{d}{d\ln\mu}\Phi_+(z,\mu)=&-\left[\Gamma_{\text{cusp}}\ln(iz\tilde{\mu})+\tilde{\gamma}_+\right]\Phi_+(z,\mu)\\&-\Gamma_{\text{cusp}}\int_0^1 du\frac{\bar{u}}{u}\left[\Phi_+(z,\mu)-\Phi_+(\bar{u}z,\mu)\right],\end{aligned} \tag{22}$$

where $\tilde{\mu}=\mu e^{\gamma_E}$, $\bar{u}=1-u$, and $\tilde{\gamma}_+^{(0)}=2C_F$. This equation can be related to the LN equation by a Fourier transform. Performing the Mellin transformation to the evolution equation in the position space [47, 48]:

$$\varphi_+(j,\mu)=\frac{1}{2\pi i}\int_{-i0}^{-i\infty}\frac{dz}{z}(iz\tilde{\mu})^{-j}\Phi_+(z,\mu), \tag{23}$$

the evolution equation can also be diagonalized and easily solved. The different kinds of integral transform mentioned above are equivalent, and the LCDAs $\phi_+(\omega)$, $\Phi_+(z)$, $\varphi_+(j)$, and $\rho_+(\omega')$ are different expressions of an identical objective. Because the momentum space and the position space are related through a standard Fourier transformation, we are able to derive

$$\begin{aligned}\varphi_+(j,\mu)&=\frac{\Gamma(-j)}{2\pi i}\int_0^\infty d\omega\left(\frac{\omega}{\tilde{\mu}}\right)^j\phi_+(\omega,\mu),\\\varphi_+(j,\mu)&=\frac{\tilde{\mu}}{2\pi i}\Gamma(2+j)\int_0^\infty\frac{d\omega'}{\omega'}\rho_+\left(\omega',\mu\right)\left(\frac{\tilde{\mu}}{\omega'}\right)^{-1-j}.\end{aligned} \tag{24}$$

At the one-loop level, the most convenient method is to work in the dual space since the Bessel function is the eigenfunction of the LN kernel, which is confirmed in [49, 50]. The LN kernel can be expressed as a logarithm of the generator of special conformal transformations along the light cone. When the eigenfunction of the generator is transformed to the momentum space, it is simply the Bessel function in (20).

The two-loop level anomalous dimension of the B-meson LCDA was first calculated in the coordinate space in [43]

$$\begin{aligned}\frac{d}{d\ln\mu}\Phi_+(z,\mu)=&-\left[\Gamma_{\text{cusp}}\ln(iz\tilde{\mu})+\tilde{\gamma}_+\right]\Phi_+(z,\mu)\\&-\Gamma_{\text{cusp}}\int_0^1 du\frac{\bar{u}}{u}\left[1+ah(u)\right]\left[\Phi_+(z,\mu)-\Phi_+(\bar{u}z,\mu)\right],\end{aligned} \tag{25}$$

where

$$\begin{aligned}\tilde{\gamma}_+^{(1)}&=C_F\left[4C_F\left(2+\pi^2-6\zeta_3\right)+\frac{662}{9}-\frac{35}{6}\pi^2-18\zeta_3-n_f\left(\frac{80}{27}-\frac{1}{9}\pi^2\right)\right],\\\Gamma_{\text{cusp}}^{(2)}&=C_F\left[1470-\frac{536\pi^2}{3}+\frac{44\pi^4}{5}+264\zeta_3+n_f\left(-\frac{1276}{9}+\frac{80\pi^2}{9}-\frac{208}{3}\zeta_3\right)-\frac{16}{27}n_f^2\right],\\h(u)&=\ln\bar{u}\left[11-\frac{2}{3}n_f+2C_F\left(\ln\bar{u}-\frac{1+\bar{u}}{\bar{u}}\ln u-\frac{3}{2}\right)\right].\end{aligned} \tag{26}$$

This equation has also been transformed into the momentum space in [51], resulting in the two-loop level LN equation (17) with the integral kernel modified as

$$\begin{aligned}\Gamma_+\left(\omega,\omega',\mu\right)=&\left(\Gamma_{\text{cusp}}\ln\frac{\mu}{\omega}+\gamma_+\right)\delta\left(\omega-\omega'\right)\\&+\omega\Gamma_{\text{cusp}}\Gamma\left(\omega,\omega'\right)+\hat{\gamma}_+\left(\omega,\omega'\right).\end{aligned} \tag{27}$$

The anomalous dimensions up to two loops are

$$\begin{aligned}\gamma_+&=\tilde{\gamma}_+-\Gamma_{\text{cusp}}\left[1-a\left(\beta_0\left(1-\frac{\pi^2}{6}\right)-C_F\left(3-\frac{\pi^2}{6}\right)\right)\right],\\\hat{\gamma}_+\left(\omega,\omega'\right)&=-a\,4C_F\frac{\omega\,\theta\left(\omega'-\omega\right)}{\omega'\left(\omega'-\omega\right)}h\left(\frac{\omega}{\omega'}\right).\end{aligned} \tag{28}$$

The advantage of solving the evolution equation at the two-loop level in the dual space does not hold since the two-loop evolution kernel is not diagonal in this space. On the contrary, the elegant form of the RG equation in the Mellin space is maintained:

$$\left[\frac{d}{d\ln\mu}+V_+(j,\alpha_s)\right]\varphi_+(j,\mu)=0, \tag{29}$$

with

$$V_+(j,\alpha_s)=j-\tilde{\gamma}_+ +\Gamma_{\mathrm{cusp}}\left[\psi(j+2)-\psi(2)+\vartheta(j)\right], \tag{30}$$

where

$$\vartheta(j)=a\left\{(\beta_0-3C_F)\left(\psi'(j+2)-\psi'(2)\right)+2C_F\left(\frac{1}{(j+1)^3}+\psi'(j+2)(\psi(j+2)-\psi(1))+\psi'(j+1)(\psi(j+1)-\psi(1))-\frac{\pi^2}{6}\right)\right\}. \tag{31}$$

The solution in the Mellin space is then obtained directly [48]

$$\varphi_+(j(\mu),\alpha_s(\mu),\mu)=\varphi_+(j(\mu_0),\alpha_s(\mu_0),\mu_0)\exp\cdot\left\{-\int_{\mu_0}^{\mu}\frac{ds}{s}V_+[j(s),\alpha_s(s)]\right\}. \tag{32}$$

In a recent paper [52], an alternative approach to solving the evolution equation at two-loop level was proposed. The essential idea of this approach is to perform a Laplace transformation to the B-meson LCDA:

$$\widetilde{\phi}_+(\eta,\mu)=\int_0^\infty\frac{d\omega}{\omega}\left(\frac{\omega}{\bar{\omega}}\right)^{-\eta}\phi_+(\omega,\mu), \tag{33}$$

where $\bar{\omega}$ is a fixed reference scale, which can be used to eliminate the logarithmic moment σ_1 in the factorization formula of $B\longrightarrow\gamma\ell\bar{\nu}_\ell$. We note that the LCDA $\varphi_+(j)$ is related to $\widetilde{\phi}_+(\eta)$ through [53]

$$\varphi_+(j,\mu)=\frac{\Gamma(-j)}{2\pi i}e^{\gamma_E}\mu\left(\frac{\bar{\omega}}{e^{\gamma_E}\mu}\right)^{j+1}\widetilde{\phi}_+(-j-1,\mu). \tag{34}$$

Then, one could derive the RG equation for $\widetilde{\phi}_+$ and solve the evolution equation directly [52, 53].

2.3. Higher-Twist B-Meson LCDAs. Power corrections to the B-meson decay processes are of great importance, and higher-twist B-meson LCDAs provide one kind of important power suppressed contributions. The LCDA ϕ_- defined in the previous subsection is of twist-3, and it is suppressed due to different components of the quark field in the definition of LCDAs. Besides, the additional gluon or quark fields will also give rise to higher-twist LCDAs. Compared with two-particle LCDAs, the three-particle quark-gluon LCDAs are more numerous. There exist eight independent Lorentz structures [54], and they can be defined as

$$\begin{aligned}&\langle 0|\bar{q}(\bar{n}z_1)[\bar{n}z_1,\bar{n}z_2]gG_{\mu\nu}(\bar{n}z_2)[\bar{n}z_2,0]\Gamma h_v(0)|\bar{B}(v)\rangle\\ &==\frac{1}{2}\tilde{f}_B(\mu)m_B Tr\Big\{\gamma_5\Gamma P_+\Big[\left(v_\mu\gamma_\nu-v_\nu\gamma_\mu\right)[\Psi_A-\Psi_V]\\ &-i\sigma_{\mu\nu}\Psi_V-(\bar{n}_\mu v_\nu-\bar{n}_\nu v_\mu)X_A+\left(\bar{n}_\mu\gamma_\nu-\bar{n}_\nu\gamma_\mu\right)[W+Y_A]\\ &-i\epsilon_{\mu\nu\alpha\beta}\bar{n}^\alpha v^\beta\gamma_5\tilde{X}_A+i\epsilon_{\mu\nu\alpha\beta}\bar{n}^\alpha\gamma^\beta\gamma_5\tilde{Y}_A-(\bar{n}_\mu v_\nu-\bar{n}_\nu v_\mu)\bar{n}\,W\\ &+\left(\bar{n}_\mu\gamma_\nu-\bar{n}_\nu\gamma_\mu\right)\bar{n}\,Z\Big]\Big\}(z_1,z_2;\mu),\end{aligned} \tag{35}$$

where the totally antisymmetric tensor $\varepsilon_{0123}=1$, the covariant derivative is defined as $D_\mu=\partial_\mu-igA_\mu$, and the dual gluon strength tensor is $\tilde{G}_{\mu\nu}=(1/2)\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$. The momentum space distributions are defined through Fourier transformations:

$$\Psi_A(z_1,z_2)=\int_0^\infty d\omega_1\int_0^\infty d\omega_2\,e^{-i\omega_1 z_1-i\omega_2 z_2}\,\psi_A(\omega_1,\omega_2), \tag{36}$$

and similarly for the other functions. The LCDAs defined above do not have definite twist. In order to construct the LCDAs with definite twist, one can take advantage of the two-component spinors as follows:

$$\begin{aligned}
&2\tilde{f}_B(\mu)m_B\Phi_3(z_1,z_2;\mu)\\
&\quad=-\langle 0|\chi_+(z_1)\bar{f}_{++}(z_2)h_+(0)+\bar{\psi}_+(z_1)f_{++}(z_2)\bar{h}_+(0)|\bar{B}(v)\rangle,\\
&2\tilde{f}_B(\mu)m_B\Phi_4(z_1,z_2;\mu)\\
&\quad=\langle 0|\chi_-(z_1)f_{++}(z_2)h_-(0)+\bar{\psi}_-(z_1)\bar{f}_{++}(z_2)\bar{h}_-(0)|\bar{B}(v)\rangle,\\
&\tilde{f}_B(\mu)m_B\left[\Psi_4+\tilde{\Psi}_4\right](z_1,z_2;\mu)\\
&\quad=-\langle 0|\chi_+(z_1)f_{+-}(z_2)h_-(0)+\bar{\psi}_+(z_1)\bar{f}_{+-}(z_2)\bar{h}_-(0)|\bar{B}(v)\rangle,\\
&\tilde{f}_B(\mu)m_B\left[\Psi_4-\tilde{\Psi}_4\right](z_1,z_2;\mu)\\
&\quad=-\langle 0|\chi_+(z_1)\bar{f}_{+-}(z_2)h_-(0)+\bar{\psi}_+(z_1)f_{+-}(z_2)\bar{h}_-(0)|\bar{B}(v)\rangle,\\
&2\tilde{f}_B(\mu)m_B\Phi_5(z_1,z_2;\mu)\\
&\quad=\langle 0|\chi_+(z_1)f_{--}(z_2)h_+(0)+\bar{\psi}_+(z_1)\bar{f}_{--}(z_2)\bar{h}_+(0)|\bar{B}(v)\rangle,\\
&\tilde{f}_B(\mu)m_B\left[\Psi_5+\tilde{\Psi}_5\right](z_1,z_2;\mu)\\
&\quad=\langle 0|\chi_-(z_1)f_{+-}(z_2)h_+(0)+\bar{\psi}_-(z_1)\bar{f}_{+-}(z_2)\bar{h}_+(0)|\bar{B}(v)\rangle,\\
&\tilde{f}_B(\mu)m_B\left[\Psi_5-\tilde{\Psi}_5\right](z_1,z_2;\mu)\\
&\quad=\langle 0|\chi_-(z_1)\bar{f}_{+-}(z_2)h_+(0)+\bar{\psi}_-(z_1)f_{+-}(z_2)\bar{h}_+(0)|\bar{B}(v)\rangle,\\
&2\tilde{f}_B(\mu)m_B\Phi_6(z_1,z_2;\mu)\\
&\quad=\langle 0|\chi_-(z_1)\bar{f}_{--}(z_2)h_-(0)+\bar{\psi}_-(z_1)f_{--}(z_2)\bar{h}_-(0)|\bar{B}(v)\rangle,
\end{aligned} \tag{37}$$

where $f_{\alpha\beta}$ and $\bar{f}_{\dot{\alpha}\dot{\beta}}$ are chiral and antichiral symmetric tensors, and the twist of their light-cone components satisfies: $t[f_{++}]=t[\bar{f}_{++}]=1$, $t[f_{+-}]=t[\bar{f}_{+-}]=2$, and $t[f_{--}]=t[\bar{f}_{--}]=3$. This eight invariant function is related to the B-meson higher-twist LCDAs:

$$\begin{aligned}
\Phi_3&=\Psi_A-\Psi_V,\\
\Phi_4&=\Psi_A+\Psi_V,\\
\Psi_4&=\Psi_A+X_V,\\
\tilde{\Psi}_4&=\Phi_A-\tilde{X}_A,\\
\Psi_5&=-\Psi_A+X_A-2Y_A,\\
\Phi_5&=\Psi_A+\Psi_V+2Y_A-2\tilde{Y}_A+2W,\\
\tilde{\Psi}_5&=-\Psi_V-\tilde{X}_A+2\tilde{Y}_A,\\
\Phi_6&=\Psi_A-\Psi_V+2Y_A+2\tilde{Y}_A+2W-4Z.
\end{aligned} \tag{38}$$

Except for the higher Fock state, the higher twist also arises from the nonvanishing parton transverse momenta (or virtuality). The twist-4 and twist-5 two-particle B-meson LCDAs can be defined as

$$\begin{aligned}
&\langle 0|\bar{q}(x)\Gamma[x,0]h_v(0)|\bar{B}(v)\rangle\\
&\quad=-\frac{i}{2}\tilde{f}_B(\mu)m_B Tr[\gamma_5\Gamma P_+]\int_0^\infty d\omega e^{-i\omega(vx)}\left\{\phi_+(\omega)+x^2 g_+(\omega)\right\}\\
&\qquad+\frac{i}{4}\tilde{f}_B(\mu)m_B Tr[\gamma_5\Gamma P_+ x]\frac{1}{vx}\\
&\qquad\times\int_0^\infty d\omega e^{-i\omega(vx)}\left\{[\phi_+-\phi_-](\omega)+x^2[g_+-g_-](\omega)\right\},
\end{aligned} \tag{39}$$

where we have to assume $|x^2|\ll 1/\Lambda_{\mathrm{QCD}}^2$; thus, (39) can be understood as a light-cone expansion to the tree-level accuracy. This definition contains the constraints

$$\int_0^\infty d\omega[\phi_+(\omega)-\phi_-(\omega)]=0,\ \int_0^\infty d\omega[g_+(\omega)-g_-(\omega)]=0. \tag{40}$$

From QCD equation of motion (EOM), one can derive the following relations among the LCDAs [55] (the last two relations in (41a) follow from the expressions given in [55] by simple algebra).

$$\begin{aligned}
&\left[z\frac{d}{dz}+1\right]\Phi_-(z)=\Phi_+(z)+2z^2\int_0^1 udu\,\Phi_3(z,uz),\\
&2z^2\mathrm{G}_+(z)=-\left[z\frac{d}{dz}-\frac{1}{2}+iz\bar{\Lambda}\right]\Phi_+(z)-\frac{1}{2}\Phi_-(z)\\
&\qquad-z^2\int_0^1\bar{u}du\Psi_4(z,uz)\equiv 2z^2\widehat{\mathrm{G}}_+(z)-z^2\int_0^1\bar{u}du\Psi_4(z,uz),\\
&2z^2\mathrm{G}_-(z)=-\left[z\frac{d}{dz}-\frac{1}{2}+iz\bar{\Lambda}\right]\Phi_-(z)-\frac{1}{2}\Phi_+(z)\\
&\qquad-z^2\int_0^1\bar{u}du\Psi_5(z,uz)\equiv 2z^2\widehat{\mathrm{G}}_-(z)-z^2\int_0^1\bar{u}du\Psi_5(z,uz),\\
&\Phi_-(z)=\left(z\frac{d}{dz}+1+2iz\bar{\Lambda}\right)\Phi_+(z)+2z^2\int_0^1 du[u\Phi_4(z,uz)+\Psi_4(z,uz)],
\end{aligned} \tag{41a}$$

where

$$\begin{aligned}
\mathrm{G}_\pm(z,\mu)&=\int_0^\infty d\omega\, e^{-i\omega z}g_\pm(\omega,\mu),\\
\bar{\Lambda}&=m_B-m_b.
\end{aligned} \tag{42}$$

With the above relations, one can calculate the LCDAs $G_\pm$ with the leading-twist and three-particle higher-twist LCDAs.

2.4. The Phenomenological Models. Different from the LCDAs of light mesons, which can be expanded in terms of Gagenbauer polynomials and the corresponding Gagenbauer moments can be calculated by Lattice or QCD sum rules since they are determined by the matrix elements of local operators, the LCDAs of the B meson are more difficult to be modeled. The evolution of the B-meson can provide some model-independent constraints to the behavior of the leading-twist LCDA of the B-meson. In the large ω region, the operator product expansion (OPE) can be employed to explore the model-independent properties of the LCDA. The method is to calculate the first several moments of the distribution amplitude, derive its asymptotic behavior, and study its properties under RG evolution; then, the constraints on the LCDAs can be found. The result indicates that at large ω, the LCDA falls off faster than $1/\omega$. In the low ω region, the behavior of the LCDA cannot be constrained by the perturbative QCD and only can be modeled with a nonperturbative method. In practice, the B-meson LCDA is usually applied to the decay processes of the B meson; therefore, the spectator is regarded as a soft quark, and low ω region behavior is more important. Hereafter, we introduce some commonly used models.

The asymptotic behavior of the LCDAs at the small quark and glue momenta is relative to the conformal spins of the quark and gluon field:

$$\phi(\omega_1,\omega_2) \sim \omega_1^{2j_1-1}\omega_2^{2j_2-1}, \quad \phi \in \{\phi_3, \phi_4, \psi_4 \cdots\}. \tag{43}$$

This relation can be obtained from the correlation function of the light-ray operators and suitable local current [56]. Several models for the two-particle and three-particle LCDAs to the twist-four accuracy have been given with a more general ansatz in [57], such as the exponential model, the free parton model, and the local duality model. Similarly, we can obtain all the LCDA models in accord with the correct low-momentum behavior [56] and EOM constrains (tree level):

$$\begin{aligned}
\phi_+(\omega) &= \omega f(\omega),\\
\phi_-(\omega) &= \int_\omega^\infty d\rho f(\rho) + \frac{1}{6}\varkappa\left(\lambda_E^2 - \lambda_H^2\right)\\
&\quad \cdot \left[\omega^2 f'(\omega) + 4\omega f(\omega) - 2\int_\omega^\infty d\rho f(\rho)\right],\\
g_-(\omega) &= \frac{1}{4}\int_\omega^\infty dx(x-\omega)[\phi_+(\omega) - \phi_-(\omega)] - 2\left(\bar{\Lambda} - x\right)\phi_-(\omega)\\
&\quad - \frac{1}{2}\int_0^\omega d\omega_1 \int_{\omega-\omega_1}^\infty d\omega_2 \frac{1}{\omega_2}\left(1 - \frac{\omega-\omega_1}{\omega_2}\right)\psi_5(\omega_1,\omega_2),\\
\phi_3(\omega_1,\omega_2) &= -\frac{1}{2}\varkappa\left(\lambda_E^2 - \lambda_H^2\right)\omega_1\omega_2^2 f'(\omega_1+\omega_2),\\
\phi_4(\omega_1,\omega_2) &= \frac{1}{2}\varkappa\left(\lambda_E^2 + \lambda_H^2\right)\omega_2^2 f(\omega_1+\omega_2),\\
\psi_4(\omega_1,\omega_2) &= \varkappa\lambda_E^2\omega_1\omega_2 f(\omega_1+\omega_2),\\
\phi_5(\omega_1,\omega_2) &= \varkappa\left(\lambda_E^2 + \lambda_H^2\right)\omega_1\int_{\omega_1+\omega_2}^\infty d\omega f(\omega),\\
\psi_5(\omega_1,\omega_2) &= \varkappa\lambda_E^2\omega_2\int_{\omega_1+\omega_2}^\infty d\omega f(\omega),\\
\tilde{\psi}_5(\omega_1,\omega_2) &= \varkappa\lambda_H^2\omega_2\int_{\omega_1+\omega_2}^\infty d\omega f(\omega),\\
\phi_6(\omega_1,\omega_2) &= \varkappa\left(\lambda_E^2 - \lambda_H^2\right)\int_{\omega_1+\omega_2}^\infty d\omega \int_\omega^\infty d\omega' f\left(\omega'\right),
\end{aligned} \tag{44}$$

where the normalization constant

$$\varkappa^{-1} = \frac{1}{2}\int_0^\infty \omega^3 f(\omega) d\omega. \tag{45}$$

It is clear that this model is in agreement with the local duality model and the exponential model in [35]. The parameters λ_E and λ_H in HQET are defined as the hadronic matrix element for both the light and the heavy quarks with EOM constraints:

$$\begin{aligned}
&\left\langle 0\left|\bar{q} g_s G_{\mu\nu}\Gamma h_v\right|\bar{B}(v)\right\rangle\\
&= -\frac{\tilde{f}_B m_B}{6}\mathrm{Tr}\left\{\frac{1+v}{2}\left[i\sigma_{\mu\nu}\lambda_H^2 + \left(v_\mu\gamma_\nu - v_\nu\gamma_\mu\right)\left(\lambda_H^2 - \lambda_E^2\right)\right]\gamma_5\Gamma\right\}.
\end{aligned} \tag{46}$$

The renormalization scale dependence of λ_E and λ_H can be obtained by solving the RG equation at the one-loop order:

$$\begin{pmatrix}\lambda_E^2(\mu)\\ \lambda_H^2(\mu)\end{pmatrix} = \hat{V}\left[\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_i^{(0)}/(2\beta_0)}\right]_{\mathrm{diag}} V\wedge^{-1}\begin{pmatrix}\lambda_E^2(\mu_0)\\ \lambda_H^2(\mu_0)\end{pmatrix}. \tag{47}$$

Here, we employ the three-parameter model of $\phi_+(\omega,\mu_0) = \omega f_{3p}(\omega,\mu_0)$ where

$$f_{3p}(\omega,\mu_0) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{1}{\omega_0^2}e^{-\omega/\omega_0}\mathrm{U}\left(\beta-\alpha, 3-\alpha, \frac{\omega}{\omega_0}\right). \tag{48}$$

The three parameters α, β, and ω_0 determine the model of B-meson LCDAs at the initial scale $\mu = 1.0$ GeV. The values of the logarithmic moments λ_B, $\hat{\sigma}_1$, and $\hat{\sigma}_2$ are needed, which are defined as [57]

$$\hat{\sigma}_n(\mu) = \int_0^\infty d\omega \frac{\lambda_B}{\omega}\ln^n\frac{\lambda_B e^{-\gamma_E}}{\omega}\phi_+(\omega,\mu), \tag{49}$$

where λ_B is defined by $\hat{\sigma}_0 = 1$. In the terms of the three parameters ω_0, α, and β, we obtain with a short calculation

$$\begin{aligned}
\lambda_B &= \alpha - 1\beta - 1\omega_0,\\
\hat{\sigma}_1 &= \psi(\beta-1) - \psi(\alpha-1) + \ln\left(\frac{\alpha-1}{\beta-1}\right),\\
\hat{\sigma}_2 &= \pi^2 6 + \left[\psi(\beta-1) - \psi(\alpha-1) + \ln\left(\frac{\alpha-1}{\beta-1}\right)\right]^2\\
&\quad - \left[\psi'(\beta-1) - \psi'(\alpha-1)\right].
\end{aligned} \tag{50}$$

3. LCSR with B-Meson LCDAs

The LCSR with B-meson LCDAs is widely employed to calculate the heavy-to-light form factors which receive the well-known end-point divergence in the QCD factorization. Starting from the QCD two-point correlation functions at the Euclidean space, the B-meson LCSR can avoid the endpoint divergence with the help of the dispersion relation, the Borel transformation, and the quark-hadron duality. The heavy-to-light form factors can also be calculated with the SCET improved LCSR. In the following, we will calculate the $B \longrightarrow P$ form factors and the $B \longrightarrow V$ form factors with the B-meson LCSR and the SCET improved LCSR, respectively.

3.1. The Heavy-to-Light Form Factors. The form factors for the B-meson decays into a pseudoscalar meson are defined by the matrix elements of the weak transition current sandwiched between the B-meson and a pseudoscalar meson states, i.e.,

$$\begin{aligned}\langle P(p)|\bar{q}\,\gamma^\mu b|\bar{B}(p_B)\rangle &= f^+_{B\longrightarrow P}(q^2)\left[p_B^\mu + p^\mu - \frac{m_B^2 - m_P^2}{q^2}q^\mu\right] \\ &\quad + f^0_{B\longrightarrow P}(q^2)\,\frac{m_B^2 - m_P^2}{q^2}q^\mu, \\ \langle P(p)|\bar{q}\,\sigma^{\mu\nu}q_\nu b|\bar{B}(p_B)\rangle \\ &= \frac{if^T_{B\longrightarrow P}(q^2)}{m_B + m_P}\left[q^2\left(p_B^\mu + p^\mu\right) - \left(m_B^2 - m_P^2\right)q^\mu\right],\end{aligned} \tag{51}$$

where m_B and m_P are, respectively, the masses of the B meson and the pseudoscalar meson and the momentum transfer $q = p_B - p$. The relevant form factors for B decays into vector mesons are defined as

$$\begin{aligned}\langle V(p,\varepsilon^*)|\bar{q}\gamma^\mu b|\bar{B}(p_B)\rangle &= \frac{2iV(q^2)}{m_B + m_V}\,\epsilon^{\mu\nu\rho\sigma}\varepsilon^*_\nu\,p_\rho p_{B,\sigma}, \\ \langle V(p,\varepsilon^*)|\bar{q}\gamma^\mu\gamma_5 b|\bar{B}(p_B)\rangle &= 2m_V A_0(q^2)\,\frac{\varepsilon^*\cdot q}{q^2}\,q^\mu \\ &\quad + (m_B + m_V)A_1(q^2)\left[\varepsilon^{*\mu} - \frac{\varepsilon^*\cdot q}{q^2}\,q^\mu\right] \\ &\quad - A_2(q^2)\,\frac{\varepsilon^*\cdot q}{m_B + m_V}\left[p_B^\mu + p^\mu - \frac{m_B^2 - m_V^2}{q^2}\,q^\mu\right], \\ \langle V(p,\varepsilon^*)\,|\,\bar{q}\sigma^{\mu\nu}q_\nu b\,|\,\bar{B}(p_B)\rangle &= 2\,T_1(q^2)\,\epsilon^{\mu\nu\rho\sigma}\varepsilon^*_\nu\,p_{B,\rho}p_\sigma, \\ \langle V(p,\varepsilon^*)\,|\,\bar{q}\sigma^{\mu\nu}\gamma_5 q_\nu b\,|\,\bar{B}(p_B)\rangle \\ &= (-i)T_2(q^2)\left[\left(m_B^2 - m_V^2\right)\varepsilon^{*\mu} - (\varepsilon^*\cdot q)\left(p_B^\mu + p^\mu\right)\right] \\ &\quad + (-i)T_3(q^2)(\varepsilon^*\cdot q)\left[q^\mu - \frac{q^2}{m_B^2 - m_V^2}\left(p_B^\mu + p^\mu\right)\right],\end{aligned} \tag{52}$$

where m_V (ε) is the mass (polarisation vector) of the vector meson and we use the sign convention $\varepsilon_{0123} = 1$. At maximal hadronic recoil $q^2 = 0$, there exist three relations for the above-mentioned $B \longrightarrow M$ form factors in QCD:

$$\begin{aligned} f^+_{B\longrightarrow P}(0) &= f^0_{B\longrightarrow P}(0), \\ \frac{m_B + m_V}{2m_V}A_1(0) - \frac{m_B - m_V}{2m_V}A_2(0) &= A_0(0), \\ T_1(0) &= T_2(0),\end{aligned} \tag{53}$$

which are free of both radiative and power corrections.

3.2. The $B \longrightarrow P$ Form Factors with LCSR. In this subsection, we intend to obtain the light-cone sum rules for the $B \longrightarrow P$ form factors at the leading power in $1/m_b$ and leading order in α_s. In order to obtain the light-cone sum rules for $B \longrightarrow P$ form factors, we use the following vacuum-to-B-meson correlation function:

$$\begin{aligned}\Pi_\mu(n\cdot p, \bar{n}\cdot p) &= \int d^4x\, e^{ip\cdot x}\langle 0|T\{\bar{d}(x)n\gamma_5\, q(x), \bar{q}(0)\Gamma_\mu\, b(0)\}|\bar{B}(p_B)\rangle \\ &= \begin{cases}\Pi(n\cdot p, \bar{n}\cdot p)\, n_\mu + \tilde{\Pi}(n\cdot p, \bar{n}\cdot p)\bar{n}_\mu, & \Gamma_\mu = \gamma_\mu, \\ \Pi_T(n\cdot p, \bar{n}\cdot p)\,[\bar{n}\cdot q\, n_\mu - n\cdot q\,\bar{n}_\mu], & \Gamma_\mu = \sigma_{\mu\nu}q^\nu,\end{cases}\end{aligned} \tag{54}$$

where the Lorentz structure Γ_μ stands for the two different $b \longrightarrow q$ weak currents in QCD. In the center-of-mass frame, the B-meson momentum $p_B = p + q = m_B v$, where p and q stand for the momentum of the light-meson and the weak current. At large recoil $n\cdot p \sim m_b$, we work with $p^2 < 0$ in order that the correlation function can be calculated perturbatively. We take $\bar{n}$ standing for the direction of p and use the following power counting with the representation of $q \sim (n\cdot q, \bar{n}\cdot q, q_\perp)$:

$$p_B \sim m_b\,(1,1,1), \quad p \sim m_b\left(1,\lambda^2,\lambda\right), \quad k \sim m_b\left(\lambda^2,\lambda^2,\lambda^2\right), \tag{55}$$

where the power counting parameter $\lambda = \sqrt{\Lambda_{\rm QCD}/m_b}$, k is the momentum of the light-quark in the B meson, and m_b is the b-quark mass. For the interpolation current, we employ $\bar{d}(x)\,n\gamma_5\,q(x)$ so that we can obtain the leading power contribution to the form factors directly.

To guarantee the validity of the light-cone OPE, one must prove the light-cone dominance of the correlation functions. In the correlation functions, we have assumed that the four-momentum p is spacelike, $p^2 < 0$, and sufficiently large: $P^2 = -p^2 \gg \Lambda^2_{\rm QCD}$; in addition, the ratio $\xi = 2p\cdot k/P^2 \sim O(1)$; then, $x^2 \sim 1/P^2$ must be satisfied under the condition that the exponent e^{ipx} does not oscillate strongly. This leads to a constraint on region of the q^2 which is accessible to OPE on the light-cone in the B

$\longrightarrow P$ transitions

$$0 \le q^2 < m_b^2 - \frac{m_b P^2}{\Lambda_{\rm QCD}}. \quad (56)$$

The light-cone dominance of the correlation function allows one to contract the q and $\bar{q}$ fields and use the free-quark propagator $S_q(x) = -i\langle 0 \mid q(x)\bar{q}(0) \mid 0\rangle$ as a leading-order approximation. Then, the correlation function can be written by

$$\Pi_\mu(n\cdot p, \bar{n}\cdot p) = \int d^4x e^{ip\cdot x}\left[n\gamma_5 S_q(x)\Gamma_\mu\right]_{\alpha\beta}\left\langle 0\middle|\bar{d}(x)_\alpha h_v(0)_\beta\middle|\bar{B}(v)\right\rangle, \quad (57)$$

where the heavy quark field has been expanded using HQET and only the large component has been kept. If one expands the bilocal operator $\bar{d}(x)_\alpha h_v(0)_\beta$ in the small x region, an infinite series of matrix elements of local operators are required for the vacuum-B-meson amplitude. Instead, one has to retain the matrix element of the nonlocal operator, which introduces the LCDAs of the B-meson. The detailed discussion on the B-meson LCDAs has been done in the previous section.

We will use the calculation of the form factors $f^+_{B\longrightarrow P}$ and $f^0_{B\longrightarrow P}$ as an illustration of the LCSR approach. With $\Gamma_\mu = \gamma_\mu$, the correlation function is then expressed in terms of the convolution of the hard scattering kernel and the LCDAs of the B meson, and at tree level, the result reads

$$\begin{aligned}\tilde{\Pi}(n\cdot p, \bar{n}\cdot p) &= \tilde{f}_B(\mu)m_B\int_0^\infty d\omega\frac{\phi_-(\omega)}{\omega - \bar{n}\cdot p - i0},\\ \Pi(n\cdot p, \bar{n}\cdot p) &= 0.\end{aligned} \quad (58)$$

In order to arrive at the sum rules, one has to express this result in terms of the dispersion integral with respect to $\bar{n}\cdot p$:

$$\tilde{\Pi}_{QCD}(n\cdot p, \bar{n}\cdot p) = \int_0^\infty \frac{d\omega'}{\omega' - \bar{n}\cdot p}{\rm Im}_{\omega'}\tilde{\Pi}\left(n\cdot p, \omega'\right), \quad (59)$$

and $\Pi(n\cdot p, \omega')$ should also be expressed in the dispersion form. The relative hadronic representation of the vacuum-to-B-meson correlation function and the decay constant of the pseudoscalar meson are given by

$$\begin{aligned}\Pi^{\rm had}_\mu(P_B, p) &= \frac{\langle 0|\bar{d}n\gamma_5 q|P(p)\rangle\left\langle P(p)\middle|\bar{q}\gamma_\mu b\middle|\bar{B}(p_B)\right\rangle}{m_P^2 - p^2} + \text{continuum}\\ &= \frac{f_P m_B}{2\left(m_P^2/n\cdot p - \bar{n}\cdot p\right)}\left\{\bar{n}_\mu\left[\frac{n\cdot p}{m_B}f^+_{B\longrightarrow P}\left(q^2\right) + f^0_{B\longrightarrow P}\left(q^2\right)\right]\right.\\ &\quad\left. + n_\mu\frac{m_B}{n\cdot p - m_B}\left[\frac{n\cdot p}{m_B}f^+_{B\longrightarrow P}\left(q^2\right) - f^0_{B\longrightarrow P}\left(q^2\right)\right]\right\}\\ &\quad + \int_{\omega_s^P}^\infty \frac{d\omega'}{\omega' - \bar{n}\cdot p - i0}\left[\rho_n^{\rm had}\left(n\cdot p, \omega'\right)n_\mu + \rho_{\bar{n}}^{\rm had}\left(n\cdot p, \omega'\right)\bar{n}_\mu\right],\end{aligned} \quad (60), (61)$$

where we have used the definitions of the form factors (51) and the light-meson decay constant

$$\left\langle 0\middle|\bar{d}n\gamma_5 q\middle|P(p)\right\rangle = {\rm i}n\cdot p f_P. \quad (62)$$

The parameter $\omega_s^P = s_0^P/n\cdot p \sim \lambda^2$ corresponds to the hadronic threshold of the light-meson channel. Taking advantage of the parton-hadron duality assumption, one can relate the result from the parton level calculation to the parametrization in the hadronic level:

$$\int_{\omega_s^P}^\infty \frac{d\omega'}{\omega' - \bar{n}\cdot p}\rho_{\bar{n}}^{\rm had}\left(n\cdot p, \omega'\right) = \int_{\omega_s^P}^\infty \frac{d\omega'}{\omega' - \bar{n}\cdot p}{\rm Im}_{\omega'}\tilde{\Pi}\left(n\cdot p, \omega'\right). \quad (63)$$

A similar relation also holds for $\rho_n^{\rm had}(n\cdot p, \omega')$ and $\Pi(n\cdot p, \omega')$. Then, we should perform the Borel transformation with respect to the variable $\bar{n}\cdot p$ to both the hadronic and partonic representation of the correlation [58]:

$$\Pi_B(\omega_M) \equiv B_{\omega_M}\Pi(\bar{n}\cdot p) = \lim_{\substack{-\bar{n}\cdot p/r = \omega_M\\ -\bar{n}\cdot p, r\longrightarrow\infty}}\frac{(-\bar{n}\cdot p)^{r+1}}{r!}\left(\frac{d}{d\bar{n}\cdot p}\right)^r\Pi(\bar{n}\cdot p). \quad (64)$$

With the above procedures, one can obtain the sum rules for the form factors

$$\begin{aligned}f^+_{B\longrightarrow P}\left(q^2\right) &= \frac{\tilde{f}_B(\mu)m_B}{f_P\, n\cdot p}\exp\left[m_P^2 n\cdot p\,\omega_M^P\right]\int_0^{\omega_s^P} d\omega' e^{-\omega'/\omega_M^P}\phi_-\left(\omega'\right),\\ f^0_{B\longrightarrow P}\left(q^2\right) &= \frac{n\cdot p}{m_B}f^+_{B\longrightarrow P}\left(q^2\right),\end{aligned} \quad (65)$$

where the Borel mass $\omega_M^P = M_P^2/n\cdot p \sim \lambda^2$. At tree level as well as leading power, the scalar correlation functions $\Pi(q^2)$ vanishes, which exhibits the large recoil symmetry of the form factors. The tensor form factor can also be calculated in the same method by using the correlation function with $\Gamma_\mu = \sigma_{\mu\nu}q^\nu$.

3.3. The $B \longrightarrow V$ Form Factors with SCET Improved LCSR. In this subsection, we will calculate the $B \longrightarrow V$ form factors with the SCET improved LCSR. First, we match QCD onto SCET where the heavy-to-light form factors are given by [9, 11, 59]

$$(\bar{q}\,\Gamma_i\, b)(0) = \int d\hat{s}\sum_j \tilde{C}^{(A0)}_{ij}(\hat{s})O^{(A0)}_j(s\,;0) + \int d\hat{s}\sum_j \tilde{C}^{(A1)}_{ij\mu}(\hat{s})O^{(A1)\mu}_j(s\,;0) + \int d\hat{s}_1\int d\hat{s}_2\sum_j \tilde{C}^{(B1)}_{ij\mu}(\hat{s}_1,\hat{s}_2)O^{(B1)\mu}_j(s_1,s_2\,;0)+\cdots. \tag{66}$$

The hard matching coefficients for both the $A0$-type and $B1$-type SCET currents have been computed at one-loop accuracy [60–62]. The seven QCD $B \longrightarrow V$ form factors are expressed in terms of the four "effective" form factors in SCET at leading power in the heavy quark expansion [63]:

$$\begin{aligned}
&\left\langle V(p,\varepsilon^*)\middle|\left(\bar{\xi}W_c\right)\gamma_5 h_v\middle|\bar{B}(v)\right\rangle = -n\cdot p(\varepsilon^*\cdot v)\xi_\parallel(n\cdot p),\\
&\left\langle V(p,\varepsilon^*)\middle|\left(\bar{\xi}W_c\right)\gamma_5\gamma_{\mu\perp} h_v\middle|\bar{B}(v)\right\rangle = -n\cdot p\left(\varepsilon^*_\mu - \varepsilon^*\cdot v\bar{n}_\mu\right)\xi_\perp(n\cdot p),\\
&\left\langle V(p,\varepsilon^*)\middle|\left(\bar{\xi}W_c\right)\gamma_5\left(W_c^\dagger iD_{c\perp}W_c\right)(rn) h_v\middle|\bar{B}(v)\right\rangle = -n\cdot p m_b\varepsilon^*\cdot v\int_0^1 d\tau e^{i\tau\, n\cdot p\, r}\Xi_\parallel(\tau,n\cdot p),\\
&\left\langle V(p,\varepsilon^*)\middle|\left(\bar{\xi}W_c\right)\gamma_5\gamma_{\mu\perp}\left(W_c^\dagger iD_{c\perp}W_c\right)(rn) h_v\middle|\bar{B}(v)\right\rangle = -n\cdot p m_b\left(\varepsilon^*_\mu - \varepsilon^*\cdot v\bar{n}_\mu\right)\int_0^1 d\tau e^{i\tau n\cdot pr}\Xi_\perp(\tau,n\cdot p),
\end{aligned} \tag{67}$$

where the light-cone Wilson line is introduced to restore the collinear gauge invariance [11, 64]:

$$W_c(x) = P\,\exp\left[ig_s\int_{-\infty}^0 ds\, n\cdot A_c(x+s\,n)\right]. \tag{68}$$

The relation between the QCD form factors and the SCET form factors are [63]

$$f^i_{B\longrightarrow V}(n\cdot p) = C^{(A0)}_i(n\cdot p)\xi_a(n\cdot p) + \int d\tau\, C^{(B1)}_i(\tau,n\cdot p)\Xi_a(\tau,n\cdot p),\quad (a=\parallel,\perp), \tag{69}$$

where

$$\begin{aligned}
\frac{m_B}{m_B+m_V}V(n\cdot p) &= C^{(A0)}_V\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\perp(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_V\left(\frac{n\cdot p\bar{\tau}}{m_b},\frac{n\cdot p\tau}{m_b},\mu\right)\Xi_\perp(\tau,n\cdot p),\\
\frac{2m_V}{n\cdot p}A_0(n\cdot p) &= C^{(A0)}_{f_0}\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\parallel(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_{f_0}\left(\frac{n\cdot p\bar{\tau}}{m_b},\frac{n\cdot p\tau}{m_b},\mu\right)\Xi_\parallel(\tau,n\cdot p),\\
\frac{m_B+m_V}{n\cdot p}A_1(n\cdot p) &= C^{(A0)}_V\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\perp(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_V\left(\frac{n\cdot p\bar{\tau}}{m_b},\frac{n\cdot p\tau}{m_b},\mu\right)\Xi_\perp(\tau,n\cdot p),\\
\frac{m_B+m_V}{n\cdot p}A_1(n\cdot p) - \frac{m_B-m_V}{m_B}A_2(n\cdot p) &= C^{(A0)}_{f_+}\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\parallel(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_{f_+}\left(\frac{n\cdot p\bar{\tau}}{m_b},\frac{n\cdot p\tau}{m_b},\mu\right)\Xi_\parallel(\tau,n\cdot p),\\
T_1(n\cdot p) &= C^{(A0)}_{T_1}\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\perp(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_{T_1}\left(\frac{n\cdot p\,\bar{\tau}}{m_b},\frac{n\cdot p\,\tau}{m_b},\mu\right)\Xi_\perp(\tau,n\cdot p),\\
\frac{m_B}{n\cdot p}\,T_2(n\cdot p) &= C^{(A0)}_{T_1}\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\perp(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_{T_1}\left(\frac{n\cdot p\,\bar{\tau}}{m_b},\frac{n\cdot p\,\tau}{m_b},\mu\right)\Xi_\perp(\tau,n\cdot p),\\
\frac{m_B}{n\cdot p}\,T_2(n\cdot p) - T_3(n\cdot p) &= C^{(A0)}_{f_T}\left(\frac{n\cdot p}{m_b},\mu\right)\xi_\parallel(n\cdot p) + \int_0^1 d\tau\, C^{(B1)}_{f_T}\left(\frac{n\cdot p\,\bar{\tau}}{m_b},\frac{n\cdot p\,\tau}{m_b},\mu\right)\Xi_\parallel(\tau,n\cdot p).
\end{aligned} \tag{70}$$

The coefficient functions $C^{(A0)}_{ij}$ and $C^{(B1)}_{ij\mu}$ are obtained from the Fourier transformations of the position-space coefficient functions $\tilde{C}^{(A0)}_{ij}$ and $\tilde{C}^{(B1)}_{ij\mu}$ [11]. It is evident that only five independent combinations of $A0$- and $B1$-type SCET operators appear in the factorization formulae for the seven different $B \longrightarrow V$ form factors, implying the two additional relations [8, 65]:

$$\begin{aligned}
\frac{m_B}{m_B+m_V}V(n\cdot p) &= \frac{m_B+m_V}{n\cdot p}A_1(n\cdot p),\\
T_1(n\cdot p) &= \frac{m_B}{n\cdot p}T_2(n\cdot p),
\end{aligned} \tag{71}$$

which are fulfilled to all orders in perturbative expansion at leading power in Λ/m_b.

With the relations between the form factors at hand, we only need to calculate the SCET form factors ξ_i and Ξ_i. We will use the calculation of the SCET form factor $\xi_\parallel(n\cdot p)$ as an example to illustrate the application of SCET improved

LCSR. In the SCET sum rules, the correlation function is constructed with fields in the SCET. We start with the the vacuum-to-B-meson correlation function:

$$\Pi_{\nu,\parallel}(p,q) = \int d^4x\, e^{ip\cdot x} \left\langle 0 \middle| \mathrm{T}\left\{ j_\nu(x), \left(\bar{\xi}\, W_c\right)(0)\gamma_5\, h_v(0) \right\} \middle| \bar{B}(v) \right\rangle, \quad (72)$$

where the local QCD current j_ν interpolates current for the longitudinal polarization state of the collinear vector meson [10]:

$$j_\nu(x) = \bar{q}'(x)\gamma_\nu q(x) = j^{(0)}_{\xi\xi,\nu} + j^{(1)}_{\xi\xi,\perp\nu} + j^{(2)}_{\xi q_s,\parallel\nu} + j^{(2)}_{\xi q_s,\perp\nu} + \cdots, \quad (73)$$

where the explicit expressions of the effective currents are given by

$$\begin{aligned} j^{(0)}_{\xi\xi,\nu} &= \bar{\xi}\frac{n}{2}\xi\bar{n}_\nu, \\ j^{(1)}_{\xi\xi,\perp\nu} &= \bar{\xi}\gamma_{\nu\perp}\frac{1}{\mathrm{i}n\cdot D_c} iD_{c\perp}\frac{n}{2}\xi + \bar{\xi}iD_{c\perp}\frac{1}{\mathrm{i}n\cdot D_c}\gamma_{\nu\perp}\frac{n}{2}\xi, \\ j^{(2)}_{\xi q_s,\parallel\nu} &= \left(\bar{\xi}W_c\frac{n}{2}Y_s^\dagger q_s + \bar{q}_s Y_s \frac{n}{2} W_c^\dagger \xi\right)\bar{n}_\nu, \\ j^{(2)}_{\xi q_s,\perp\nu} &= \bar{\xi}W_c\gamma_{\perp\nu}Y_s^\dagger q_s + \bar{q}_s Y_s\gamma_{\perp\nu}W_c^\dagger\xi, \end{aligned} \quad (74)$$

where the collinear Wilson line defined in (68) and the following soft Wilson line

$$Y_s(x) = P\ \exp\left[ig_s\int_{-\infty}^{0} ds\, \bar{n}\cdot A_s(x+s\bar{n})\right], \quad (75)$$

is introduced to keep the current gauge invariance. It is then straightforward to write down the leading-power contribution to the correlation function at tree level:

$$\Pi_{\nu,\parallel}(p,q) = \frac{\tilde{f}_B(\mu)m_B}{2}\int_0^\infty d\omega\, \frac{1}{\bar{n}\cdot p - \omega + i0}\, \phi_-(\omega,\mu)\bar{n}_\nu \quad (76)$$

Now, we apply the standard method of sum rules and match the spectral representation of the factorization formula (76) with the corresponding hadronic dispersion relation:

$$\begin{aligned} \Pi^{\mathrm{had}}_{\nu,\parallel}(p,q) = \Bigg[&-\frac{f_{V,\parallel}\, m_V}{m_V^2/n\cdot p - \bar{n}\cdot p - i0}\,(n\cdot p2\, m_V)^2\, \xi_\parallel(n\cdot p) \\ &+ \int_{\omega_s^V}^{\infty} \frac{d\omega'}{\omega' - \bar{n}\cdot p - i0}\rho^{\mathrm{had}}_\parallel\left(n\cdot p,\omega'\right)\Bigg]\bar{n}_\nu, \end{aligned} \quad (77)$$

to obtain the SCET form factor $\xi_\parallel(n\cdot p)$

$$\xi_\parallel(n\cdot p) = 2\frac{\tilde{f}_B(\mu)}{f_{V,\parallel}}\frac{m_B\, m_V}{(n\cdot p)^2}\int_0^{\omega_s^V} d\omega'\ \exp\left[-\frac{n\cdot p\,\omega' - m_V^2}{n\cdot p\,\omega_M^V}\right]\phi_-\left(\omega',\mu\right). \quad (78)$$

The scale-independent longitudinal decay constant of the vector meson is defined as follows:

$$c_V\,\langle V(p,\varepsilon^*)\,|\,j_\nu\,|\,0\rangle = -if_{V,\parallel}\, m_V\, \varepsilon_\nu^*(p). \quad (79)$$

The other SCET form factors ξ_i and Ξ_i can be calculated with the same procedure from different SCET correlation functions (72).

4. QCD Corrections to the $B \longrightarrow M$ Form Factors

In this section, we will show the calculation of the QCD corrections to the $B \longrightarrow M$ form factors. First, we calculate the NLO correction to the $B \longrightarrow P$ form factors with traditional LCSR with B-meson LCDAs. Then, we provide the NLO computation of the $B \longrightarrow V$ form factors with SCET improved LCSR.

4.1. The $B \longrightarrow P$ Form Factors with B-Meson LCSR. The objective of this section is to establish the factorization formulae for $\Pi_\mu(n\cdot p, \bar{n}\cdot p)$ in (54) at the one-loop level. In the previous section, we have seen that the correlation function can be factorized into the hard scattering part and the LCDAs of the B meson. Since the B-meson LCDA is nonperturbative, what we need is to find the QCD correction to the hard scattering part. The correlator $\Pi_{\mu,b\bar{d}}$ with free partonic states can be expanded as

$$\begin{aligned} \Pi_{\mu,b\bar{d}} &= \Pi^{(0)}_\mu + \Pi^{(1)}_\mu + \cdots = \Phi_{b\bar{q}} \otimes T = \Phi^{(0)}_{b\bar{q}} \otimes T^{(0)} \\ &+ \left[\Phi^{(0)}_{b\bar{q}} \otimes T^{(1)} + \Phi^{(1)}_{b\bar{q}} \otimes T^{(0)}\right] + \cdots, \end{aligned} \quad (80)$$

where $\otimes$ denotes the convolution in the variable ω', and $T^{(1)}$ is the one-loop level hard scattering kernel; it is then determined by the matching condition:

$$\Phi^{(0)}_{b\bar{q}} \otimes T^{(1)} = \Pi^{(1)}_\mu - \Phi^{(1)}_{b\bar{q}} \otimes T^{(0)}, \quad (81)$$

where the second term serves as the infrared (soft) subtraction. We will demonstrate that the soft divergence will be completely absorbed into the B-meson LCDA and that there is no leading contribution to the correlation function from the collinear region (with the momentum scaling $l_\mu \sim (1, \lambda^4, \lambda^2)$), which confirms the factorization at one-loop level. As a result, the hard-scattering kernel T can be contributed only from hard and/or hard-collinear regions at leading power in Λ/m_b. We will evaluate the master formula of $T^{(1)}$ in (81) diagram by diagram. There exist two typical perturbative scales, namely, the hard scale m_b and the hard-collinear scale $\sqrt{m_b\Lambda}$; therefore, the hard scattering part can be further factorized into the hard function and the jet function. It is more convenient to apply the method of regions [36] to compute the loop integrals in order to obtain the hard coefficient function (C) and the jet function (J) simultaneously. C and J must be well defined in dimensional regularization. This guarantees that we can adopt dimensional regularization to evaluate the leading-power contributions of $\Pi_{\mu,b\bar{q}}$ without introducing an

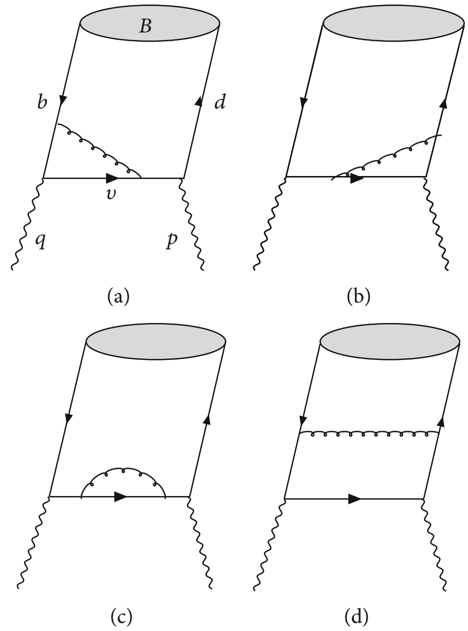

FIGURE 1: Diagrammatic representation of the correlation function $\Pi_\mu(n\cdot p, \bar n\cdot p)$ at $O(\alpha_s)$. Figure 1 is reproduced from [32].

additional "analytical" regulator. In the following, we will calculate the contribution in Figure 1(a) in detail and present the final results for the other diagram directly.

4.1.1. Weak Vertex Diagram. The contribution to $\Pi^{(1)}_\mu$ from the QCD correction to the weak vertex (Figure 1 (a)) is

$$\Pi^{(1)}_{\mu,\text{weak}} = \frac{g_s^2 C_F}{2(\bar n\cdot p-\omega)} \int \frac{d^D l}{(2\pi)^D} \frac{1}{\left[(p-k+l)^2+i0\right]\left[(m_b v+l)^2-m_b^2+i0\right]\left[l^2+i0\right]} \bar d(k) n\gamma_5 \bar n\gamma_\rho(\not p-k+l)\gamma_\mu(m_b v+l+m_b)\gamma^\rho h_v, \tag{82}$$

where $D=4-2\varepsilon$. For the following scalar integral:

$$I_1 = \int [d\,l] \, \frac{1}{\left[(p-k+l)^2+i0\right]\left[(m_b v+l)^2-m_b^2+i0\right]\left[l^2+i0\right]}, \tag{83}$$

the power counting of I_1 is $I_1 \sim \lambda^0$ for the three leading regions (the hard, hard-collinear, and soft regions); thus, only the leading-power contributions of the numerator in (82) need to be kept for a given region. We define the integration measure as

$$[d\,l] \equiv \frac{(4\pi)^2}{i}\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\varepsilon} \frac{d^D l}{(2\pi)^D}. \tag{84}$$

Inserting the partonic light-cone projector yields the hard contribution of $\Pi^{(1)}_{\mu,\text{weak}}$ at leading power

$$\begin{aligned}\Pi^{(1),h}_{\mu,\text{weak}} = {} & i g_s^2\, C_F \tilde f_B(\mu) m_B \frac{\phi^-_{b\bar q}(\omega)}{\bar n\cdot p-\omega} \int \frac{d^D l}{(2\pi)^D} \frac{1}{\left[l^2+n\cdot p\,\bar n\cdot l+i0\right]\left[l^2+2\,m_b\, v\cdot l+i0\right]\left[l^2+i0\right]} \\ & \times \left\{\bar n_\mu\left[2\,m_b n\cdot(p+l)+(D-2)l_\perp^2\right]-n_\mu(D-2)(\bar n\cdot l)^2\right\},\end{aligned} \tag{85}$$

where the superscript "h" denotes the hard contribution and we adopt the conventions

$$l_\perp^2 \equiv g_\perp^{\mu\nu}\, l_\mu\, l_\nu, \quad g_\perp^{\mu\nu} \equiv g^{\mu\nu} - \frac{n^\mu \bar{n}^\nu}{2} - \frac{n^\nu \bar{n}^\mu}{2}. \tag{86}$$

The loop integrals can be evaluated directly, and we obtain

$$\begin{aligned}\Pi_{\mu,\mathrm{weak}}^{(1),h} = {} & \frac{\alpha_s C_F}{4\pi} \tilde{f}_B(\mu) m_B \% \int_0^\infty d\omega \frac{\phi_{b\bar{q}}^-(\omega)}{\bar{n}\cdot p - \omega} \left\{ \bar{n}_\mu \left[\frac{1}{\int^2} + \frac{1}{\int}\left(2\ln\frac{\mu}{n\cdot p} + 1\right)\right.\right. \\ & + 2\ln^2\frac{\mu}{n\cdot p} + 2\ln\frac{\mu}{m_b} - \ln^2 r - 2Li_2\left(-\frac{\bar{r}}{r}\right) \\ & \left.\left. + \frac{2-r}{r-1}\ln r + \frac{\pi^2}{12} + 3\right] + n_\mu \left[\frac{1}{r-1}\left(1 + \frac{r}{\bar{r}}\ln r\right)\right]\right\},\end{aligned} \tag{87}$$

with $r = n\cdot p/m_b$ and $\bar{r} = 1 - r$.

Along the same vein, one can identify the hard-collinear contribution of $\Pi_{\mu,\mathrm{weak}}^{(1)}$ at leading power

$$\begin{aligned}\Pi_{\mu,\mathrm{weak}}^{(1),hc} = {} & i g_s^2 C_F \tilde{f}_B(\mu) m_B \frac{\phi_{b\bar{q}}^-(\omega)}{\bar{n}\cdot p - \omega} \\ & \cdot \int \frac{d^D l}{(2\pi)^D} \frac{2\, m_b\, n\cdot(p+l)}{\left[n\cdot(p+l)\bar{n}\cdot(p-k+l) + l_\perp^2 + i0\right]\left[m_b\, n\cdot l + i0\right]\left[l^2 + i0\right]},\end{aligned} \tag{88}$$

where the superscript "hc" indicates the hard-collinear contribution and the propagators have been expanded systematically in the hard-collinear region. Evaluating the integrals yields

$$\begin{aligned}\Pi_{\mu,\mathrm{weak}}^{(1),hc} = {} & \frac{\alpha_s C_F}{4\pi} \tilde{f}_B(\mu) m_B \int_0^\infty d\omega \frac{\phi_{b\bar{q}}^-(\omega)}{\omega - \bar{n}\cdot p} \bar{n}_\mu \\ & \cdot \left[\frac{2}{\int^2} + \frac{2}{\int}\left(\ln\frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)} + 1\right) + \ln^2\frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)}\right. \\ & \left. + 2\ln\frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)} - \frac{\pi^2}{6} + 4\right].\end{aligned} \tag{89}$$

Applying the method of regions we extract the soft contribution of $\Pi_{\mu,weak}^{(1)}$

$$\begin{aligned}\Pi_{\mu,weak}^{(1),s} = {} & \frac{g_s^2 C_F}{2(\bar{n}\cdot p - \omega)} \int_0^\infty \frac{d\omega}{\bar{n}\cdot p - \omega} \\ & \cdot \int \frac{d^D l}{(2\pi)^D} \frac{1}{[\bar{n}\cdot(p-k+l) + i0][v\cdot l + i0]\left[l^2 + i0\right]} \bar{d}(k) n\gamma_5 \bar{n}\gamma_\mu h_v \\ = {} & \frac{\alpha_s C_F}{4\pi} \tilde{f}_B(\mu) m_B \int_0^\infty d\omega \frac{\phi_{b\bar{d}}^-(\omega)}{\bar{n}\cdot p - \omega} \bar{n}_\mu \\ & \cdot \left[\frac{1}{\int^2} + \frac{2}{\int}\ln\frac{\mu}{\omega - \bar{n}\cdot p} + 2\ln^2\frac{\mu}{\omega - \bar{n}\cdot p} + \frac{3\pi^2}{4}\right],\end{aligned} \tag{90}$$

where the superscript "s" represents the soft contribution.

Now, we compute the corresponding infrared subtraction term $\Phi_{b\bar{d},a}^{(1)} \otimes T^{(0)}$ as displayed in Figure 2(a). With the Wilson-line Feynman rules, we obtain

$$\begin{aligned}& \Phi_{b\bar{q},a}^{\alpha\beta,(1)}\left(\omega, \omega'\right) \\ & = i\, g_s^2\, C_F \int \frac{d^D l}{(2\pi)^D} \frac{\delta\left(\omega' - \omega - \bar{n}\cdot l\right) - \delta\left(\omega' - \omega\right)}{[\bar{n}\cdot l + i0][v\cdot l + i0]\left[l^2 + i0\right]} \bar{d}(k)_\alpha\, h_{v\beta},\end{aligned} \tag{91}$$

from which we can derive the soft subtraction term

$$\Phi_{b\bar{q},a}^{(1)} \otimes T^{(0)} = \frac{g_s^2 C_F}{2(\bar{n}\cdot p - \omega)} \int \frac{d^D l}{(2\pi)^D} \frac{\bar{d}(k) n\gamma_5 \bar{n}\gamma_\mu h_v}{[\bar{n}\cdot(p-k+l) + i0][v\cdot l + i0]\left[l^2 + i0\right]}. \tag{92}$$

We then conclude that

$$\Pi_{\mu,weak}^{(1),s} = \Phi_{b\bar{q},a}^{(1)} \otimes T^{(0)}, \tag{93}$$

at leading power in Λ/m_b, which is an essential point to prove factorization of the correlation function Π_μ.

4.1.2. Interpolation Current Vertex Diagram. Now, we turn to compute the QCD correction to the interpolation current vertex (Figure 1(b)). The soft region contribution will generate a scaleless integral which vanishes in dimensional regularization, and the hard region induces a power-suppression factor λ from the spinor structure. Therefore, the contribution from the hard-collinear region will be the same as the full QCD result. It is then straightforward to obtain the contribution of the current vertex:

$$\begin{aligned}\Pi_{\mu,P}^{(1)} = \Pi_{\mu,P}^{(1),hc} = {} & \frac{\alpha_s C_F}{4\pi} \tilde{f}_B(\mu) m_B \% \\ & \cdot \int_0^\infty d\omega \frac{1}{\bar{n}\cdot p - \omega} \left\{ n_\mu \phi_{b\bar{q}}^+(\omega) \left[\frac{\bar{n}\cdot p - \omega}{\omega} \ln\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p}\right] + \bar{n}_\mu \phi_{b\bar{q}}^-(\omega) \right. \\ & \cdot \left[\left(\frac{1}{\int} + \ln\left(-\frac{\mu^2}{p^2}\right)\right)\left(\frac{2\bar{n}\cdot p}{\omega}\ln\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p} + 1\right) \frac{\bar{n}\cdot p}{\omega}\ln\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p}\right. \\ & \left.\left. \cdot \left(\ln\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p} + \frac{2\omega}{\bar{n}\cdot p} - 4\right) + 4\right]\right\}.\end{aligned} \tag{94}$$

The soft contribution of $\Pi_{\mu,P}^{(1)}$ vanishes in dimensional regularization; it can be proved that the precise cancelation of $\Pi_{\mu,P}^{(1),s}$ and $\Phi_{b\bar{q},b}^{(1)} \otimes T^{(0)}$ from Figure 2(b) is independent of regularization schemes.

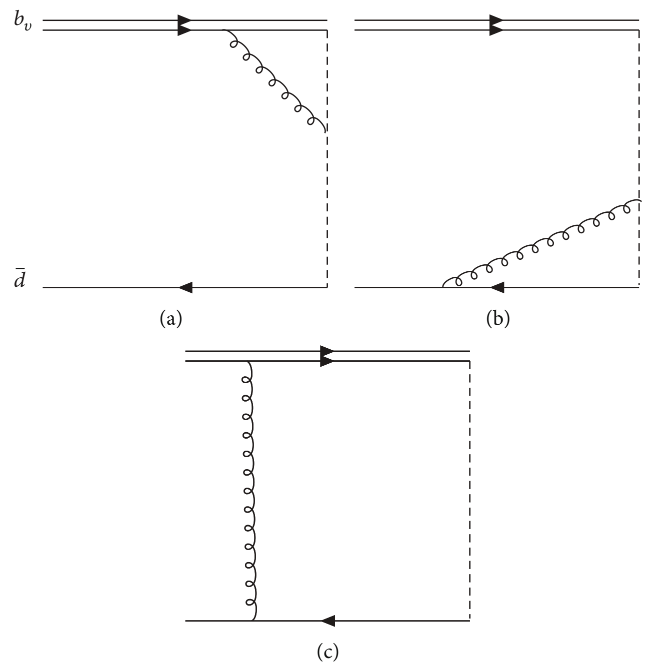

Figure 2: One-loop diagrams for the B-meson LCDA. The figure is reproduced from [32].

4.1.3. Wave Function Renormalization. The self-energy correction to the intermediate quark propagator (Figure 1(c)) is free of soft and collinear divergences and a straightforward calculation gives

$$\Pi^{(1)}_{\mu,wfc} = \frac{\alpha_s C_F}{4\pi} \tilde{f}_B(\mu) m_B \int_0^\infty d\omega \frac{\phi^-_{b\bar{q}}(\omega)}{\bar{n}\cdot p - \omega} \bar{n}_\mu \cdot \left[\frac{1}{\int} + \ln \frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)} + 1\right]. \quad (95)$$

For the external quark fields, the wave function renormalization of a massless quark does not contribute to the matching coefficient when dimensional regularization is applied to regularize both ultraviolet and infrared divergences, i.e.,

$$\Pi^{(1)}_{\mu,dwf} - \Phi^{(1)}_{b\bar{q},dwf} \otimes T^{(0)} = 0. \quad (96)$$

The wave function renormalization of the b-quark is actually the match coefficient between the full QCD and HQET:

$$\Pi^{(1)}_{\mu,bwf} - \Phi^{(1)}_{b\bar{q},bwf} \otimes T^{(0)} = -\frac{\alpha_s C_F}{8\pi}\left[\frac{3}{\int} + 3\ln \frac{\mu^2}{m_b^2} + 4\right]\Pi^{(0)}_\mu. \quad (97)$$

4.1.4. Box Diagram. The one-loop contribution to Π_μ from the box diagram (Figure 1(d)) receives leading-power contribution from both the hard-collinear and the soft regions. Evaluating the hard-collinear contribution of $\Pi^{(1)}_{\mu,\mathrm{box}}$ yields

$$\Pi^{(1),hc}_{\mu,\mathrm{box}} = \frac{\alpha_s C_F}{4\pi} \tilde{f}_B(\mu) \int_0^\infty d\omega \frac{m_B}{\omega} \bar{n}_\mu \cdot \left\{\phi^+_{b\bar{q}}(\omega)[r \ln(1+\eta)] - 2\phi^-_{b\bar{q}}(\omega)\ln(1+\eta) \times \left[\frac{1}{\int} + \ln \frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)} + \frac{1}{2}\ln(1+\eta) + 1\right]\right\}, \quad (98)$$

with $\eta = -\omega/\bar{n}\cdot p$.

Extracting the soft contribution of $\Pi^{(1)}_{\mu,box}$ with the method of regions gives

$$\Pi^{(1),s}_{\mu,box} = -\frac{g_s^2 C_F}{2}\int \frac{d^D l}{(2\pi)^D} \frac{\bar{d}(k)\nu(k-l) n\gamma_5 \bar{n}\gamma_\mu h_v}{[v\cdot l + i0][\bar{n}\cdot(p-k+l)+i0]\left[(k-l)^2+i0\right]\left[l^2+i0\right]}. \quad (99)$$

Now, we compute the corresponding NLO contribution to the partonic LCDA (Figure 2(c)):

$$\Phi^{\alpha\beta,(1)}_{b\bar{q},c}\left(\omega,\omega'\right) = -ig_s^2 C_F \cdot \int \frac{d^D l}{(2\pi)^D} \frac{\delta\left(\omega' - \omega + \bar{n}\cdot l\right)}{\left[(l-k)^2+i0\right][v\cdot l + i0]\left[l^2+i0\right]} \left[\bar{d}(k)\nu(l-k)\right]_\alpha [h_v]_\beta, \quad (100)$$

from which one can deduce the soft subtraction term

$$\Phi^{(1)}_{b\bar{q},c} \otimes T^{(0)} = \frac{g_s^2 C_F}{2} \int \frac{d^D l}{(2\pi)^D} \frac{\bar{d}(k) v(l-k) n\gamma_5 \bar{n}\gamma_\mu h_v}{[v\cdot l + i0][\bar{n}\cdot(p-k+l)+i0]\left[(l-k)^2+i0\right]\left[l^2+i0\right]}, \quad (101)$$

which cancels out the soft contribution of the correlation function $\Pi^{(1),s}_{\mu,\mathrm{box}}$ completely. The absence of such soft contribution to the perturbative matching coefficient is particularly important for the box diagram, since the relevant loop integrals in the soft region depend on *two* components of the soft spectator momentum $\bar{n}\cdot k$ and $v\cdot k$, and the light-cone OPE fails in the soft region.

After the results of all the diagrams obtained, the one-loop hard-scattering kernel of the correlation function Π_μ can be readily computed from the matching condition in (81) by collecting different pieces together

$$\begin{aligned}
\Phi^{(0)}_{b\bar{q}} \otimes T^{(1)} &= \left[\Pi^{(1)}_{\mu,\mathrm{weak}} + \Pi^{(1)}_{\mu,P} + \Pi^{(1)}_{\mu,wfc} + \Pi^{(1)}_{\mu,\mathrm{box}} + \Pi^{(1)}_{\mu,bwf} + \Pi^{(1)}_{\mu,dwf}\right] \\
&\quad - \left[\Phi^{(1)}_{b\bar{q},a} + \Phi^{(1)}_{b\bar{q},b} + \Phi^{(1)}_{b\bar{q},c} + \Phi^{(1)}_{b\bar{q},bwf} + \Phi^{(1)}_{b\bar{q},dwf}\right] \otimes T^{(0)} \\
&= \left[\Pi^{(1),h}_{\mu,\mathrm{weak}} + \left(\Pi^{(1)}_{\mu,bwf} - \Phi^{(1)}_{b\bar{q},bwf}\right)\right] \\
&\quad + \left[\Pi^{(1),hc}_{\mu,\mathrm{weak}} + \Pi^{(1),hc}_{\mu,P} + \Pi^{(1),hc}_{\mu,wfc} + \Pi^{(1),hc}_{\mu,\mathrm{box}}\right],
\end{aligned} \quad (102)$$

where the one-loop level hard function and the jet function can be extracted from the first and second square brackets of the second equality, respectively. Finally, the factorization formulae of Π and $\tilde{\Pi}$ are given by

$$\begin{aligned}
\Pi &= \tilde{f}_B(\mu) m_B \sum_{k=\pm} C^{(k)}(n\cdot p,\mu) \int_0^\infty \frac{d\omega}{\omega - \bar{n}\cdot p} J^{(k)}\left(\frac{\mu^2}{n\cdot p\omega}, \frac{\omega}{\bar{n}\cdot p}\right) \phi_k(\omega,\mu), \\
\tilde{\Pi} &= \tilde{f}_B(\mu) m_B \sum_{k=\pm} \tilde{C}^{(k)}(n\cdot p,\mu) \int_0^\infty \frac{d\omega}{\omega - \bar{n}\cdot p} \tilde{J}^{(k)}\left(\frac{\mu^2}{n\cdot p\omega}, \frac{\omega}{\bar{n}\cdot p}\right) \phi_k(\omega,\mu),
\end{aligned} \quad (103)$$

at leading power in Λ/m_b, where we keep the factorization-scale dependence explicitly, the hard coefficient functions are given by

$$\begin{aligned}
C^{(+)} &= \tilde{C}^{(+)} = 1, \\
C^{(-)} &= aC_F \frac{1}{\bar{r}}\left[\frac{r}{\bar{r}} \ln r + 1\right], \\
\tilde{C}^{(-)} &= 1 - aC_F \left[2\ln^2 \frac{\mu}{n\cdot p} + 5\ln\frac{\mu}{m_b} \right.\\
&\quad \left. - \ln^2 r - 2\mathrm{Li}_2\left(-\frac{\bar{r}}{r}\right) + \frac{2-r}{r-1}\ln r + \frac{\pi^2}{12} + 5\right],
\end{aligned} \quad (104)$$

and the jet functions are

$$\begin{aligned}
J^{(+)} &= \frac{1}{r}\tilde{J}^{(+)} = aC_F\left(1 - \frac{\bar{n}\cdot p}{\omega}\right)\ln\left(1 - \frac{\omega}{\bar{n}\cdot p}\right), \\
J^{(-)} &= 1, \\
\tilde{J}^{(-)} &= 1 + aC_F\left[\ln^2 \frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)} \right.\\
&\quad - 2\ln\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p}\ln\frac{\mu^2}{n\cdot p(\omega - \bar{n}\cdot p)} - \ln^2\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p} \\
&\quad \left. - \left(1 + \frac{2\bar{n}\cdot p}{\omega}\right)\ln\frac{\bar{n}\cdot p - \omega}{\bar{n}\cdot p} - \frac{\pi^2}{6} - 1\right].
\end{aligned} \quad (105)$$

From the one-loop calculation, one can obtain the anomalous dimensions of the hard function, the jet function, and the soft contribution; then, the factorization-scale independence of Π and $\tilde{\Pi}$ can be demonstrated explicitly. It is straightforward to write down the following evolution equations:

$$\frac{d}{d\ln\mu}\tilde{C}^{(-)}(n\cdot p,\mu) = -aC_F\left[\Gamma^{(0)}_{\mathrm{cusp}}\ln\frac{\mu}{n\cdot p} + 5\right]\tilde{C}^{(-)}(n\cdot p,\mu), \quad (106)$$

$$\begin{aligned}
\frac{d}{d\ln\mu}\tilde{J}^{(-)}\left(\frac{\mu^2}{n\cdot p\omega}, \frac{\omega}{\bar{n}\cdot p}\right) &= aC_F\left[\Gamma^{(0)}_{\mathrm{cusp}}\ln\frac{\mu^2}{n\cdot p\omega}\right]\tilde{J}^{(-)}\left(\frac{\mu^2}{n\cdot p\omega}, \frac{\omega}{\bar{n}\cdot p}\right) \\
&\quad + aC_F\int_0^\infty d\omega'\,\omega\Gamma\left(\omega,\omega',\mu\right)\tilde{J}^{(-)}\left(\frac{\mu^2}{n\cdot p\omega'}, \frac{\omega'}{\bar{n}\cdot p}\right),
\end{aligned} \quad (107)$$

$$\begin{aligned}
\frac{d}{d\ln\mu}\left[\tilde{f}_B(\mu)\phi_-(\omega,\mu)\right] &= -aC_F\left[\Gamma^{(0)}_{\mathrm{cusp}}\ln\frac{\mu}{\omega} - 5\right]\left[\tilde{f}_B(\mu)\phi_-(\omega,\mu)\right] \\
&\quad - aC_F\int_0^\infty d\omega'\,\omega\Gamma\left(\omega,\omega',\mu\right)\left[\tilde{f}_B(\mu)\phi_-\left(\omega',\mu\right)\right],
\end{aligned} \quad (108)$$

where the evolution kernel in the last equation is actually the LN kernel (except for the anomalous dimension of the decay constant $\tilde{f}_B$) for ϕ_-, and Γ has been given in the previous section. The renormalization kernel of $\phi_-(\omega,\mu)$ at one-loop level was first computed in [66] and then confirmed in [67]. With the evolution equations displayed above, it is evident that

$$\frac{d}{d\ln\mu}\left[\Pi(n\cdot p,\bar{n}\cdot p),\tilde{\Pi}(n\cdot p,\bar{n}\cdot p)\right]=O\left(\alpha_s^2\right). \tag{109}$$

One cannot avoid the large logarithms of order $\ln(m_b/\Lambda)$ in the hard functions, the jet functions, $\tilde{f}_B(\mu)$, and the B-meson LCDAs concurrently by choosing a common value of μ, and the large logarithms have to be resummed by solving the three RG equations shown above. The solution to the RG equation of the LCDA has been discussed detailed in the previous section; in addition, the hadronic scale entering the initial conditions of the B-meson LCDAs $\mu_0\approx 1.0\,\mathrm{GeV}$ is quite close to the hard-collinear scale $\mu_{hc}\simeq\sqrt{m_b\Lambda}\approx 1.5\,\mathrm{GeV}$; it is not important phenomenologically to sum logarithms of μ_{hc}/μ_0 [68]. To achieve NLL resummation of large logarithms in the hard coefficient $\tilde{C}^{(-)}$, we need to generalize the RG equation (106) to

$$\begin{aligned}\frac{d}{d\ln\mu}\tilde{C}^{(-)}(n\cdot p,\mu)&=\left[-\Gamma_{\mathrm{cusp}}(\alpha_s)\ln\frac{\mu}{n\cdot p}+\gamma(\alpha_s)\right]\tilde{C}^{(-)}(n\cdot p,\mu),\\ \frac{d}{d\ln\mu}\tilde{f}_B(\mu)&=\gamma_{hl}(\alpha_s)\tilde{f}_B(\mu).\end{aligned} \tag{110}$$

The cusp anomalous dimension at the three-loop order and the remaining anomalous dimension $\gamma(\alpha_s)$ determining renormalization of the SCET heavy-to-light current at two loops will enter $U_1(n\cdot p,\mu_{h1},\mu)$ at NLL accuracy. The manifest expressions of $\gamma^{(i)}$, $\gamma_{hl}^{(i)}$, and β_i can be found in [68] and references therein (note that there is a factor C_F difference of our conventions of $\Gamma_{cusp}^{(i)}$ and $\gamma^{(i)}$ compared with [68]); the evolution function $U_1(n\cdot p,\mu_{h1},\mu)$ can be read from (A.3) in [68] with the replacement rules $E_\gamma\longrightarrow n\cdot p/2$ and $\mu_h\longrightarrow\mu_{h1}$. Because the hard scale $\mu_{h1}\sim n\cdot p$ in the hard function $\tilde{C}^{(-)}(n\cdot p,\mu)$ differs from the one $\mu_{h2}\sim m_b$ in $\tilde{f}_B(\mu)$, the resulting evolution functions due to running of the renormalization scale from $\mu_{h1}(\mu_{h2})$ to μ_{hc} in $\tilde{C}^{(-)}(n\cdot p,\mu)$ $(\tilde{f}_B(\mu))$ are

$$\begin{aligned}\tilde{C}^{(-)}(n\cdot p,\mu)&=U_1(n\cdot p,\mu_{h1},\mu)\tilde{C}^{(-)}(n\cdot p,\mu_{h1}),\\ \tilde{f}_B(\mu)&=U_2(\mu_{h2},\mu)\tilde{f}_B(\mu_{h2}).\end{aligned} \tag{111}$$

The final factorization formulae of Π and $\tilde{\Pi}$ with RG improvement at NLL accuracy can be written as

$$\begin{aligned}\Pi&=m_B\left[U_2(\mu_{h2},\mu)\tilde{f}_B()\right](\mu_{h2})\int_0^\infty\frac{d\omega}{\omega-\bar{n}\cdot p}J^{(+)}\left(\frac{\mu^2}{n\cdot p\omega},\frac{\omega}{\bar{n}\cdot p}\right)\phi_+(\omega,\mu)\\ &\quad+m_B\left[U_2(\mu_{h2},\mu)\tilde{f}_B(\mu_{h2})\right]C^{(-)}(n\cdot p,\mu)\int_0^\infty\frac{d\omega}{\omega-\bar{n}\cdot p}\phi_-(\omega,\mu),\\ \tilde{\Pi}&=m_B\left[U_2(\mu_{h2},\mu)\tilde{f}_B(\mu_{h2})\right]\int_0^\infty\frac{d\omega}{\omega-\bar{n}\cdot p}\tilde{J}^{(+)}\left(\frac{\mu^2}{n\cdot p\omega},\frac{\omega}{\bar{n}\cdot p}\right)\phi_+(\omega,\mu)\\ &\quad+m_B[U_1(n\cdot p,\mu_{h1},\mu)U_2(\mu_{h2},\mu)]\left[\tilde{f}_B(\mu_{h2})\tilde{C}^{(-)}(n\cdot p,\mu_{h1})\right]\\ &\quad\times\int_0^\infty\frac{d\omega}{\omega-\bar{n}\cdot p}\tilde{J}^{(-)}\left(\frac{\mu^2}{n\cdot p\omega},\frac{\omega}{\bar{n}\cdot p}\right)\phi_-(\omega,\mu),\end{aligned} \tag{112}$$

where μ should be taken as a hard-collinear scale of order $\sqrt{m_b\Lambda}$.

Since the one-loop partonic level result of the correlation function has been obtained, we are ready to construct the sum rules of $f_{B\longrightarrow P}^{+}(q^2)$ and $f_{B\longrightarrow P}^{0}(q^2)$ including the radiative corrections at $O(\alpha_s)$. To achieve this target, one must express the one-loop correlation function in terms of the dispersion integral. Using the quark-hadron duality assumption and the Borel transform, we obtain the form factors for the $B\longrightarrow\pi$ transition:

$$\begin{aligned}&f_\pi e^{-m_\pi^2/(n\cdot p\omega_M^\pi)}\left\{\frac{n\cdot p}{m_B}f_{B\longrightarrow\pi}^{+}(q^2),f_{B\longrightarrow\pi}^{0}(q^2)\right\}\\ &=\left[U_2(\mu_{h2},\mu)\tilde{f}_B(\mu_{h2})\right]\int_0^{\omega_s^\pi}d\omega' e^{-\omega'/\omega_M^\pi}\\ &\quad\times\left[r\phi_{B,\mathrm{eff}}^{+}\left(\omega',\mu\right)+\left[U_1(n\cdot p,\mu_{h1},\mu)\tilde{C}^{(-)}(n\cdot p,\mu_{h1})\right]\phi_{B,\mathrm{eff}}^{-}\right.\\ &\quad\left.\times\left(\omega',\mu\right)\pm\frac{n\cdot p-m_B}{m_B}\left(\phi_{B,\mathrm{eff}}^{+}\left(\omega',\mu\right)+C^{(-)}(n\cdot p,\mu)\phi_-\left(\omega',\mu\right)\right)\right],\end{aligned} \tag{113}$$

where the functions $\phi_{B,eff}^{\pm}(\omega',\mu)$ are defined as

$$\begin{aligned}\phi_{B,\mathrm{eff}}^{+}\left(\omega',\mu\right)&=a\,C_F\int_{\omega'}^{\infty}\frac{d\omega}{\omega}\phi_+(\omega,\mu),\\ \phi_{B,\mathrm{eff}}^{-}\left(\omega',\mu\right)&=\phi_-\left(\omega',\mu\right)\\ &\quad+a\,C_F\left\{\int_0^{\omega'}d\omega\left[\frac{2}{\omega-\omega'}\left(\ln\frac{\mu^2}{n\cdot p\,\omega'}-2\ln\frac{\omega'-\omega}{\omega'}\right)\right]_{\oplus}\phi_-(\omega,\mu)\right.\\ &\quad-\int_{\omega'}^{\infty}d\omega\left[\ln^2\frac{\mu^2}{n\cdot p\,\omega'}-\left(2\ln\frac{\mu^2}{n\cdot p\,\omega'}+3\right)\ln\frac{\omega-\omega'}{\omega'}\right.\\ &\quad\left.\left.+2\ln\frac{\omega}{\omega'}+\frac{\pi^2}{6}-1\right]\frac{d\phi_-(\omega,\mu)}{d\omega}\right\}.\end{aligned} \tag{114}$$

The other $B\longrightarrow P$ form factors could be calculated in the same way.

4.2. The $B \longrightarrow V$ Form Factors with SCET Improved LCSR. In this subsection, we will calculate the NLO corrections to the SCET form factors ξ_i and Ξ_i. The computation of the form factor $\xi_\parallel$ will be presented in detail; the calculation of the other form factors, which can be derived in a similar procedure, will be neglected.

We start with the SCET vacuum-to-B-meson correlation function (72):

$$\Pi_{v,\parallel}(p,q) = \int d^4x e^{ip\cdot x}\left\langle 0\left|\mathrm{T}\left\{j_v(x), \left(\bar{\xi}\, W_c\right)(0)\gamma_5 h_v(0)\right\}\right|\bar{B}(v)\right\rangle. \tag{115}$$

It is straightforward to write down the leading-power contribution to the correlation function:

$$\begin{aligned}\Pi_{v,\parallel}(p,q) &= \int d^4x e^{ip\cdot x}\left\langle 0\left|\mathrm{T}\left\{j^{(2)}_{\xi q_s,\parallel v}(x), \left(\bar{\xi} W_c\right)(0)\gamma_5 h_v(0)\right\}\right|\bar{B}(v)\right\rangle \\ &+ \int d^4x e^{ip\cdot x}\int d^4y\left\langle 0\left|\mathrm{T}\left\{j^{(0)}_{\xi\xi,v}(x), iL^{(2)}_{\xi q_s}(y),\right.\right.\right. \\ &\cdot \left(\bar{\xi} W_c\right)(0)\gamma_5 h_v(0)\Big\}|\bar{B}(v)\rangle + \int d^4x e^{ip\cdot x}\int d^4y \\ &\cdot \int d^4z\Big\langle 0\Big|\mathrm{T}\Big\{j^{(0)}_{\xi\xi,v}(x), iL^{(1)}_{\xi q_s}(y), iL^{(1)}_{\xi m}(z), \\ &\cdot \left(\bar{\xi} W_c\right)(0)\gamma_5 h_v(0)\Big\}|\bar{B}(v)\rangle \equiv \Pi^A_{v,\parallel}(p,q) \\ &+ \Pi^B_{v,\parallel}(p,q) + \Pi^C_{v,\parallel}(p,q),\end{aligned} \tag{116}$$

where the third term $\Pi^C_{v,\parallel}$ takes into account the light-quark mass effect. The multipole expanded SCET Lagrangian up to the $O(\lambda^2)$ accuracy [64] has been derived with the position-space formalism [10]:

$$\begin{aligned}\mathscr{L}^{(0)}_{\xi} &= \bar{\xi}(i\bar{n}\cdot D + iD_{\perp c})\frac{n}{2}\xi, \\ \mathscr{L}^{(1)}_{\xi m} &= m\bar{\xi}\left[iD_{\perp c}, \frac{1}{\mathrm{in}\cdot D_c}\right]\frac{n}{2}\xi, \\ \mathscr{L}^{(2)}_{\xi m} &= -m^2\bar{\xi}\frac{1}{\mathrm{in}\cdot D_c}\frac{n}{2}\xi, \\ \mathscr{L}^{(1)}_{\xi q_s} &= \bar{q}_s W_c^\dagger iD_{\perp c}, \xi - \bar{\xi}i\overleftarrow{D}_{\perp c}W_c q_s, \\ \mathscr{L}^{(2)}_{\xi q_s} &= \bar{q}_s W_c^\dagger\left(i\bar{n}\cdot D + iD_{\perp c}, \frac{1}{\mathrm{in}\cdot D_c}iD_{\perp c}\right)\frac{n}{2}\xi - \bar{\xi}\frac{n}{2} \\ &\cdot\left(i\bar{n}\cdot\overleftarrow{D} + i\overleftarrow{D}_{\perp c}, \frac{1}{\mathrm{in}\cdot\overleftarrow{D}_c}i\overleftarrow{D}_{\perp c}\right)W_c q_s \\ &+ \bar{q}_s\overleftarrow{D}^\mu_s x_{\perp\mu}W_c^\dagger iD_{\perp c}\xi - \bar{\xi}i\overleftarrow{D}_{\perp c}W_c x_{\perp\mu}D^\mu_s qs.\end{aligned} \tag{117}$$

In the following, we aim at calculating the jet function in the following factorization formula $\Pi^i_{v,\parallel}$ as defined in (116) onto $\mathrm{SCET_{II}}$:

$$\begin{aligned}\Pi^i_{v,\parallel}(p,q) &= \frac{\tilde{f}_B(\mu)\, m_B}{2}\sum_{m=\pm}\int_0^\infty d\omega\, J^i_{\parallel,m} \\ &\cdot\left(\frac{\mu^2}{n\cdot p\,\omega}, \frac{\omega}{\bar{n}\cdot p}\right)\phi_m(\omega,\mu)\bar{n}_v, (i = A, B, C),\end{aligned} \tag{118}$$

at the one-loop accuracy.

Firstly, we calculate the jet function $J^A_{\parallel,m}$ entering the SCET factorization formula (118) by investigating the following matrix element:

$$F^A_\parallel = \int d^4x e^{ip\cdot x}\left\langle 0\left|T\left\{\bar{q}_s(x)Y_s\frac{n}{2}W_c^\dagger\xi(x), \left(\bar{\xi}W_c\right)(0)\gamma_5 h_v(0)\right\}\right|\bar{q}_s(k)h_v\right\rangle. \tag{119}$$

At tree level, we have

$$F^A_{\parallel,\mathrm{LO}} = -\frac{i}{\bar{n}\cdot p - \omega' + i0} * \left\langle O_{\parallel,-}\left(\omega,\omega'\right)\right\rangle^{(0)}, \tag{120}$$

where the light-cone matrix element $\langle O_{\parallel,-}(\omega,\omega')\rangle$ is defined as

$$\begin{aligned}\left\langle O_{\parallel\parallel,-}\left(\omega,\omega'\right)\right\rangle &= \left\langle 0\left|O_{\parallel\parallel,-}\left(\omega'\right)\right|\bar{q}_s(k)h_v\right\rangle \\ &= \bar{q}_s(k)\frac{n}{2}\gamma_5 h_v\delta\left(\omega-\omega'\right) + O(\alpha_s),\end{aligned} \tag{121}$$

and the HQET operator $O_{\parallel,\mp}(\omega')$ in the momentum space reads

$$O_{\parallel,\mp}\left(\omega'\right) = \frac{1}{2\pi}\int dt e^{it\omega'}(\bar{q}_s Y_s)(t\bar{n})\gamma_5\left\{\frac{n}{2}, \frac{\bar{n}}{2}\right\}\left(Y_s^\dagger h_v\right)(0). \tag{122}$$

Remembering the definition of the LCDA ϕ_-, we can obtain the jet functions at tree level as follows:

$$\begin{aligned}J^{A,(0)}_{\parallel,-} &= \frac{1}{\bar{n}\cdot p - \omega' + i0}, \\ J^{A,(0)}_{\parallel,+} &= 0.\end{aligned} \tag{123}$$

We proceed to determine the NLO contribution to the jet function $J^A_{\parallel,\pm}$; the corresponding one-loop $\mathrm{SCET_I}$ diagrams are presented in Figure 3 with the subleading power SCET Feynman rules collected in [69]. The self-energy correction to the hard-collinear quark propagator displayed in Figure 3(a) can be readily written as [38]

$$F^{A,(a)}_{\parallel,\mathrm{NLO}} = -\alpha_s C_F 4\pi\left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{n\cdot p\,(\omega - \bar{n}\cdot p) - i0} + 1\right]F^A_{\parallel,\mathrm{LO}}. \tag{124}$$

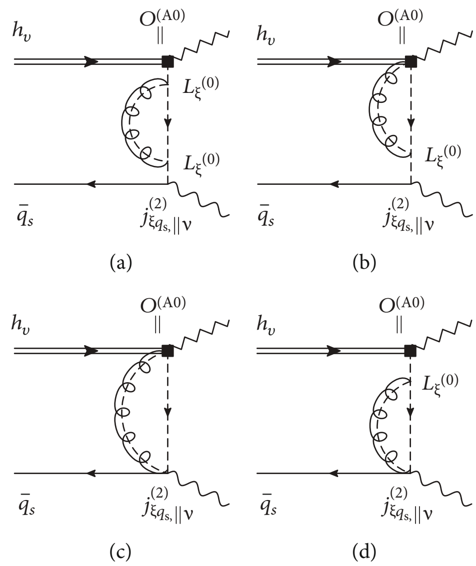

Figure 3: Diagrammatic representation of the vacuum-to-B-meson correlation function $\Pi^{A}_{\nu,\parallel}(p,q)$ defined with the $A0$-type SCET operator $O_{\parallel}^{(A0)} = (\bar{\xi}\, W_c)\gamma_5 h_v$ and the power suppressed interpolating current $j^{(2)}_{\xi q_s,\parallel\nu}$ at one loop.

Diagram (c) of Figure 3 yields vanishing contribution due to $n^2 = 0$ where n is from the Wilson line. One can further verify that the hard-collinear corrections displayed in the diagrams (b) and (d) of Figure 3 give rise to the identical results:

$$F^{A,(b)}_{\parallel,\mathrm{NLO}} = F^{A,(d)}_{\parallel,\mathrm{NLO}} = -\frac{2g_s^2 C_F}{\bar{n}\cdot p-\omega}\bar{q}_s(k)\frac{n}{2}\gamma_5 h_v \times \int\frac{d^D l}{(2\pi)^D}\frac{n\cdot(p+l)}{\left[n\cdot(p+l)\bar{n}.(p-k+l)+l_\perp^2+i0\right]\left[n\cdot l+i0\right]\left[l^2+i0\right]}, \tag{125}$$

which can be evaluated straightforwardly with dimensional regularization scheme

$$F^{A,(b)}_{\parallel,\mathrm{NLO}} = F^{A,(d)}_{\parallel,\mathrm{NLO}} = \frac{\alpha_s C_F}{2\pi}\left\{\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left[\ln\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}+1\right] + \frac{1}{2}\ln^2\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}+\ln\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}-\frac{\pi^2}{12}+2\right\}F^{A}_{\parallel,\mathrm{LO}}. \tag{126}$$

Adding up different pieces together leads to the jet functions at the one-loop accuracy:

$$J^{A}_{\parallel,-} = J^{A,(0)}_{\parallel,-}\left\{1+aC_F\left[\frac{4}{\varepsilon^2}+\frac{1}{\varepsilon}\left(4\ln\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}+3\right) + 2\ln^2\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}+3\ln\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)} - \frac{\pi^2}{3}+7\right]\right\}, J^{A}_{\parallel,+}=0, \tag{127}$$

which are in precise agreement with the results presented in [31].

The jet function $J^{B}_{\parallel,\pm}$ and $J^{C}_{\parallel,\pm}$ appear only at loop level, and they can be determined with a similar method with $J^{A}_{\parallel,\pm}$. $J^{B}_{\parallel,\pm}$ and $J^{C}_{\parallel,\pm}$ can be obtained through diagrams in Figure 4 and the diagram in Figure 5 with the SCET Feynman rules, respectively. We collect the corresponding jet functions here:

$$\begin{aligned} J^{B}_{\parallel,-} &= a\,C_F\,J^{A,(0)}_{\parallel,-} \cdot\left\{-\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}\left[-2\ln\left(\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}\right)-2\ln(1+\eta)-3\right] - \ln^2\left(\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}\right)+\ln \right. \\ &\quad \cdot\left(\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}\right)\left[-2\ln(1+\eta)-3\right]-\ln^2(1+\eta) \\ &\quad \left. +\left(\frac{2}{\eta}-1\right)\ln(1+\eta)+\frac{\pi^2}{6}-8\right\}, J^{B}_{\parallel,+}=0, J^{C}_{\parallel,+} \\ &= -a\,C_F J^{A,(0)}_{\parallel,-}\frac{m}{\omega}\ln\left(\frac{\bar{n}\cdot p-\omega}{\bar{n}\cdot p}\right), J^{C}_{\parallel,-}=0. \end{aligned} \tag{128}$$

Plugging the obtained jet functions into the factorization formula (118) and employing the decomposition of $\Pi_{\nu,\parallel}$ defined in (116) yields

$$\Pi_{\nu,\parallel}(p,q) = \frac{\tilde{f}_B(\mu)\,m_B}{2}\int_0^\infty\frac{d\omega}{\bar{n}\cdot p-\omega+i0} \cdot\left\{\left[1+aC_F\,\widehat{J}^{(A0)}_{\parallel,-}\left(\frac{\mu^2}{n\cdot p\,\omega},\frac{\omega}{\bar{n}\cdot p}\right)\right]\phi_-(\omega,\mu) + \left[aC_F\,\widehat{J}^{(m)}_{\parallel,+}\left(\frac{\mu^2}{n\cdot p\,\omega},\frac{\omega}{\bar{n}\cdot p}\right)\right]\phi_+(\omega,\mu)\right\}\bar{n}_\nu, \tag{129}$$

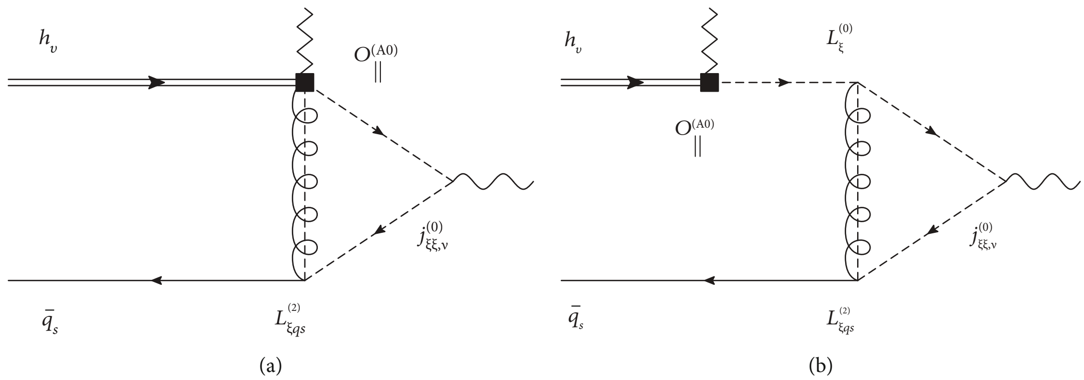

FIGURE 4: Diagrammatic representation of the vacuum-to-B-meson correlation function $\Pi_{v,\|}^{B}(p,q)$ defined with the $A0$-type SCET operator $O_{\|}^{(A0)}=(\bar{\xi}\,W_c)\gamma_5 h_v$, the leading power interpolating current $j_{\xi\xi,v}^{(0)}$, and the subleading power SCET Lagrangian $L_{\xi q_s}^{(2)}$.

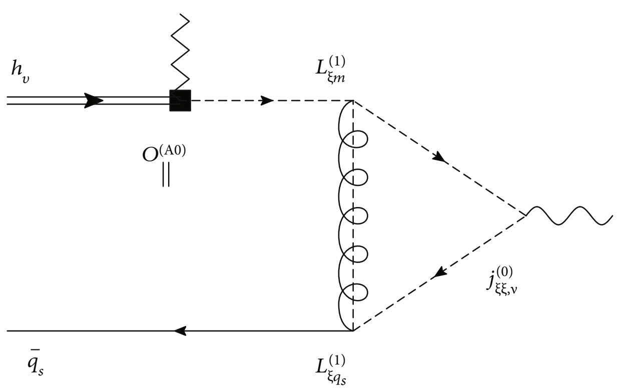

FIGURE 5: Diagrammatic representation of the vacuum-to-B-meson correlation function $\Pi_{v,\|}^{B}(p,q)$ defined with the $A0$-type SCET operator $O_{\|}^{(A0)}=(\bar{\xi}\,W_c)\gamma_5 h_v$, the leading power interpolating current $j_{\xi\xi,v}^{(0)}$ and the subleading power SCET Lagrangians $L_{\xi q_s}^{(1)}$ and $L_{\xi m}^{(1)}$.

where the normalized one-loop jet functions $\widehat{J}_{\|,-}^{(A0)}$ and $\widehat{J}_{\|,+}^{(m)}$ read

$$\begin{aligned}\widehat{J}_{\|,-}^{(A0)}&=\ln^2\left(\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}\right)-2\ \ln\left(\frac{\mu^2}{n\cdot p\,(\omega-\bar{n}\cdot p)}\right)\ln(1+\eta)\\&\quad-\ln^2(1+\eta)+\left(\frac{2}{\eta}-1\right)\ln(1+\eta)-\frac{\pi^2}{6}-1,\\ \widehat{J}_{\|,+}^{(m)}&=-\frac{m}{\omega}\ \ln\left(\frac{\bar{n}\cdot p-\omega}{\bar{n}\cdot p}\right).\end{aligned}\tag{130}$$

To construct the SCET sum rules for the effective form factor $\xi_{\|}(n\cdot p)$, we write the correlation function as a dispersion integral:

$$\begin{aligned}\Pi_{v,\|}(p,q)&=-\frac{\tilde{f}_B(\mu)\,m_B}{2}\\&\quad\cdot\int_0^\infty\frac{d\omega'}{\omega'-\bar{n}\cdot p-i0}\left[\phi_{B,\mathrm{eff}}^{-}\left(\omega',\mu\right)+\phi_{B,m}^{+}\left(\omega',\mu\right)\right]\bar{n}_v,\end{aligned}\tag{131}$$

where the effective B-meson "distribution amplitudes" are identical to that used in the $B\longrightarrow P$ form factors. After applying the NLL resummation to the decay constants $\tilde{f}_B$, we derive

$$\begin{aligned}\xi_{\|}(n\cdot p)&=2\,\frac{U_2(\mu_{h2},\mu)\,\tilde{f}_B(\mu_{h2})f_{V,\|}\,m_B\,m_V}{(n\cdot p)^2}\\&\quad\times\int_0^{\omega_s^V}d\omega'\ \exp\left[-\frac{n\cdot p\,\omega'-m_V^2}{n\cdot p\,\omega_M^V}\right]\\&\quad\cdot\left[\phi_{B,\mathrm{eff}}^{-}\left(\omega',\mu\right)+\phi_{B,m}^{+}\left(\omega',\mu\right)\right].\end{aligned}\tag{132}$$

The calculations of the other $\xi_{\perp}$, $\Xi_{\|}$, and $\Xi_{\perp}$ are similar but lengthy, see [39] for the details. We will not show the results of these three form factors here.

5. Power Corrections to the $B\longrightarrow M$ Form Factors

In the previous sections, the form factors are calculated at the leading power. The power suppressed contributions are expected to be sizeable due to the finite bottom quark mass. So far, the high twist contribution has been considered in various works [35, 39, 40]. In this section, we will take the scalar and vector $B\longrightarrow P$ form factors as an example and investigate the contribution from both the two-particle and three-particle B-meson LCDAs employing a complete parametrization of the corresponding three-particle light-cone matrix element and the EOM constraints of the higher-twist LCDAs presented in [56].

To obtain the result from three-particle LCDAs, we make use of the light-cone expansion of the quark propagator in the background gluon field [70]:

$$\begin{aligned}\langle 0\,|\,\mathrm{T}\,\{\bar{q}(x),q(0)\}\,|\,0\rangle&\supset i\,g_s\int_0^\infty d^4k(2\pi)^4\,e^{-i\,k\cdot x}\int_0^1 du\\&\quad\cdot\left[\frac{u\,x_\mu\,\gamma_\nu}{k^2-m_q^2}-\frac{(k+m_q)\,\sigma_{\mu\nu}}{2\left(k^2-m_q^2\right)^2}\right]G^{\mu\nu}(u\,x),\end{aligned}\tag{133}$$

where we only keep the one-gluon part without the covariant derivative of the $G_{\mu\nu}$ terms. The tree-level diagram is displayed in Figure 6, and three-particle higher-twist corrections to the vacuum-to-B-meson correlation function (54) can be derived directly:

$$\Pi_\mu^{(3P)}(n\cdot p,\bar{n}\cdot p) = -\frac{\tilde{f}_B(\mu)\, m_B}{n\cdot p}\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \int_0^1 du\, \frac{1}{[\bar{n}\cdot p-\omega_1-u\,\omega_2]^2} \times\left\{\bar{n}_\mu\left[\rho_{\bar{n},\mathrm{LP}}^{(3P)}+\frac{m_q}{n\cdot p}\rho_{\bar{n},\mathrm{NLP}}^{(3P)}\right](u,\omega_1,\omega_2,\mu) + n_\mu\left[\rho_{n,\mathrm{LP}}^{(3P)}+\frac{m_q}{n\cdot p}\rho_{n,\mathrm{NLP}}^{(3P)}\right](u,\omega_1,\omega_2,\mu)\right\}, \tag{134}$$

where the light-quark mass-dependent term of the three-particle corrections have been taken into account, which is suppressed by one power of Λ/m_b. The explicit expressions of $\rho_{i,\mathrm{LP}}^{(3P)}$ and $\rho_{i,\mathrm{NLP}}^{(3P)}$ $(i=n,\bar{n})$ can be found in [35]. The definition of three particle LCDAs in [56] is essential in evaluating these results. Due to nonvanishing quark transverse momentum, the higher-twist two-particle B-meson LCDAs are related expressed in terms of the three-particle configurations with the exact EOM, and they must be taken into account simultaneously for consistency. The calculation of the higher-twist two-particle B-meson LCDAs is quite similar to the leading power calculation.

Adding up the two-particle and three-particle higher-twist corrections at tree level together and implementing the standard strategy to construct the sum rules for heavy-to-light form factors give rise to the following expressions:

$$f_P\, n\cdot p2\, \exp\left[-\frac{m_P^2}{n\cdot p}\omega_M^P\right]\left[f_{B\longrightarrow P}^{+,\mathrm{HT}}(q^2)+\frac{m_B}{n\cdot p}f_{B\longrightarrow P}^{0,\mathrm{HT}}(q^2)\right] = -\frac{\tilde{f}_B(\mu)\, m_B}{n\cdot p}\left\{e^{-\omega_s^P/\omega_M^P} H_{\bar{n},\mathrm{LP}}^{2\mathrm{PHT}}(\omega_s^P,\mu) + \int_0^{\omega_s^P} d\omega'\, \frac{1}{\omega_M^P}\, e^{-\omega'/\omega_M} H_{\bar{n},\mathrm{LP}}^{2\mathrm{PHT}}(\omega',\mu) + \int_0^{\omega_s^P} d\omega_1 \int_{\omega_s^P-\omega_1}^\infty \frac{d\omega_2}{\omega_2}\, e^{-\omega_s^P/\omega_M^P}\left[H_{\bar{n},\mathrm{LP}}^{3\mathrm{PHT}}+m_q n\cdot p\, H_{\bar{n},\mathrm{NLP}}^{3\mathrm{PHT}}\right] \times\left(\frac{\omega_s^P-\omega_1}{\omega_2},\omega_1,\omega_2,\mu\right) + \int_0^{\omega_s^P} d\omega' \int_0^{\omega'} d\omega_1 \times\int_{\omega'-\omega_1}^\infty \frac{d\omega_2}{\omega_2}\,\frac{1}{\omega_M^P}\, e^{-\omega'/\omega_M^P}\left[H_{\bar{n},\mathrm{LP}}^{3\mathrm{PHT}}+\frac{m_q}{n\cdot p}H_{\bar{n},\mathrm{NLP}}^{3\mathrm{PHT}}\right] \times\left(\frac{\omega'-\omega_1}{\omega_2},\omega_1,\omega_2,\mu\right)\right\}, \tag{135}$$

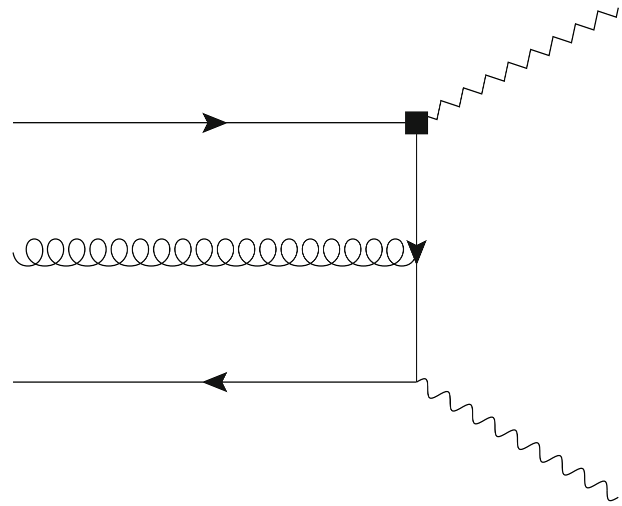

Figure 6: Diagrammatical representation of the three-particle higher-twist corrections to the vacuum-to-B-meson correlation function. The square box indicates the insertion of the weak vertex $\bar{q}\,\Gamma_\mu\, b$, and the waveline represents the interpolating current $\bar{d}n\,\gamma_5\, q$ for the light-pseudoscalar meson.

$$f_P\, n\cdot p2\, \exp\left[-\frac{m_P^2}{n\cdot p}\omega_M^P\right]\frac{m_B}{n\cdot p-m_B}\left[f_{B\longrightarrow P}^{+,\mathrm{HT}}(q^2)-\frac{m_B}{n\cdot p}f_{B\longrightarrow P}^{0,\mathrm{HT}}(q^2)\right] = -\frac{\tilde{f}_B(\mu)\, m_B}{n\cdot p}\left\{\int_0^{\omega_s^P} d\omega_1 \int_{\omega_s^P-\omega_1}^\infty \frac{d\omega_2}{\omega_2} e^{-\omega_s^P/\omega_M^P}\left[H_{n,\mathrm{LP}}^{3\mathrm{PHT}}+\frac{m_q}{n\cdot p}H_{n,\mathrm{NLP}}^{3\mathrm{PHT}}\right] \times\left(\frac{\omega_s^P-\omega_1}{\omega_2},\omega_1,\omega_2,\mu\right) + \int_0^{\omega_s^P} d\omega' \int_0^{\omega'} d\omega_1 \int_{\omega'-\omega_1}^\infty \frac{d\omega_2}{\omega_2}\frac{1}{\omega_M}e^{-\omega'/\omega_M^P} \times\left[H_{n,\mathrm{LP}}^{3\mathrm{PHT}}+\frac{m_q}{n\cdot p}H_{n,\mathrm{NLP}}^{3\mathrm{PHT}}\right]\left(\frac{\omega'-\omega_1}{\omega_2},\omega_1,\omega_2,\mu\right)\right\}, \tag{136}$$

where the nonvanishing spectral functions $H_{i,\mathrm{LP}}^{2\mathrm{PHT}}$ and $H_{i,(\mathrm{N})\mathrm{LP}}^{3\mathrm{PHT}}$ $(i=n,\bar{n})$ are given by

$$\begin{aligned} H_{\bar{n},\mathrm{LP}}^{2\mathrm{PHT}}(\omega,\mu) &= 4\,\hat{g}_-(\omega,\mu),\\ H_{n,\mathrm{LP}}^{3\mathrm{PHT}}(u,\omega_1,\omega_2,\mu) &= 2\,(u-1)\,\phi_4(\omega_1,\omega_2,\mu),\\ H_{n,\mathrm{NLP}}^{3\mathrm{PHT}}(u,\omega_1,\omega_2,\mu) &= \tilde{\psi}_5(\omega_1,\omega_2,\mu)-\psi_5(\omega_1,\omega_2,\mu),\\ H_{\bar{n},\mathrm{LP}}^{3\mathrm{PHT}}(u,\omega_1,\omega_2,\mu) &= \tilde{\psi}_5(\omega_1,\omega_2,\mu)-\psi_5(\omega_1,\omega_2,\mu),\\ H_{\bar{n},\mathrm{NLP}}^{3\mathrm{PHT}}(u,\omega_1,\omega_2,\mu) &= 2\,\phi_6(\omega_1,\omega_2,\mu). \end{aligned} \tag{137}$$

It is evident that the two-particle higher-twist corrections preserve the large-recoil symmetry relations of the $B\longrightarrow P$ form factors and the three-particle higher-twist contributions violate such relations already at tree level.

Employing the power counting scheme for the Borel mass ω_M^P and the threshold parameter ω_s^P [32]:

$$\omega_s^P \sim \omega_M^P \sim O(\Lambda^2/m_b), \tag{138}$$

we can identify the scaling behaviors of the higher-twist corrections to $B \longrightarrow P$ form factors

$$f_{B\longrightarrow P}^{+,\mathrm{HT}}(q^2) \sim f_{B\longrightarrow P}^{0,\mathrm{HT}}(q^2) \sim O(\lambda^5), \tag{139}$$

in the heavy quark limit, which is suppressed by one power of Λ/m_b compared with the leading-twist contribution. Collecting different pieces together, the final expressions for the LCSR of $B \longrightarrow P$ form factors at large hadronic recoil can be written as

$$\begin{aligned} f_{B\longrightarrow P}^{+}(q^2) &= f_{B\longrightarrow P}^{+,2\mathrm{PNLL}}(q^2) + f_{B\longrightarrow P}^{+,2\mathrm{PHT}}(q^2) + f_{B\longrightarrow P}^{+,3\mathrm{PHT}}(q^2), \\ f_{B\longrightarrow P}^{0}(q^2) &= f_{B\longrightarrow P}^{0,2\mathrm{PNLL}}(q^2) + f_{B\longrightarrow P}^{0,2\mathrm{PHT}}(q^2) + f_{B\longrightarrow P}^{0,3\mathrm{PHT}}(q^2), \end{aligned} \tag{140}$$

where the manifest expressions of $f_{B\longrightarrow P}^{i,2\mathrm{PNLL}}(q^2)$ $(i=+,0)$ include the light-quark mass effect, and the higher-twist corrections $f_{B\longrightarrow P}^{i,2\mathrm{PHT}}(q^2)$ and $f_{B\longrightarrow P}^{i,3\mathrm{PHT}}(q^2)$ can be extracted from (135) and (136).

The power suppressed contribution is from much more sources than the high twist contribution considered here. Various kinds of power suppressed contribution to the $B \longrightarrow \gamma\nu\ell$ [57, 71–73] and $B_s \longrightarrow \gamma\gamma$ [53] have been comprehensively studied, among which some kinds of sources can also contribute to the heavy-to-light form factor. One example is from heavy quark expansion, i.e.,

$$\bar{q}\Gamma_\mu b = e^{-im_b v\cdot x}\bar{q}\Gamma_\mu \left[1 + \frac{i\overrightarrow{D}}{2m_b}\right] h_v + \cdots, \tag{141}$$

where $\overrightarrow{D}^\mu = D^\mu - (v\cdot D)\, v^\mu$, and the b-quark mass m_b is defined in the pole-mass scheme in HQET. It is clear replacing the first term in the square bracket with the second term yields power suppressed contributions. In addition, the power suppressed local and nonlocal contribution in hard-collinear propagators might also play an important role. A specific calculation of these contributions will be published elsewhere.

6. Numerical Results

Having the sum rules for heavy-to-light form factors at hand, we will proceed to perform the numerical analysis. First, we will show the predictions of the form factors with LCSR at large recoil. Then, we will explore some phenomenological implications of the form factors by extrapolating the form factors to the entire physical region with the z-series parametrization.

6.1. LCSR Predictions of the Form Factors. Prior to presenting the predictions of the semileptonic $B \longrightarrow M$ decay form factors, we need to determine the input parameter used in the numerical analysis. Most importantly, the threshold and the Borel parameters:

$$\begin{aligned} s_0^{P,V} &\equiv n\cdot p\, \omega_s^{P,V}, \\ M_{P,V}^2 &\equiv n\cdot p\, \omega_M^{P,V}, \end{aligned} \tag{142}$$

should be extracted. The threshold parameter, which also enters the QCD sum rules for the light-meson decay constant, could be fixed by the sum rule calculation of the light-meson decay constant [58]. The Borel mass is introduced to suppress the continuum-state contributions; at the same time, the LCSR requires a plain window of the Borel mass inside which the form factor is stable. Then, two requirements are employed to obtain the interval of the Borel parameter [32]:

(i) The dependence of the form factor on the Borel mass should be small

$$\frac{\partial f_{B\longrightarrow P,V}}{\partial \omega_M^{P,V}} \le 0.35 \tag{143}$$

(ii) The continuum contribution in the dispersion integral of the correlation function should be smaller than the light-meson channel contribution

$$\begin{aligned} &\int_{\omega_s^{P,V}}^{\infty} d\omega' e^{-\omega'/\omega_M^{P,V}} \mathrm{Im}_{\omega'}\{\Pi, \tilde{\Pi}\}(n\cdot p, \omega') \\ &< \int_0^{\omega_s^{P,V}} d\omega' e^{-\omega'/\omega_M^{P,V}} \mathrm{Im}_{\omega'}\{\Pi, \tilde{\Pi}\}(n\cdot p, \omega') \end{aligned} \tag{144}$$

In Figure 7, we show the Borel mass and the threshold parameter dependence of the form factor $f_{B\longrightarrow\pi}^{+}$. From the left figure, we see that the form factor has a stable window with respect to the Borel mass. For the later convenience, we collect the dominant inputs in Table 1 (see [81–83] for a discussion about the flavor symmetry breaking effects of the decay constants of pseudoscalar mesons).

Except for these parameters, we still need to determine the inverse moment of the B meson $\lambda_B(\mu_0)$. The parameter $\lambda_B(\mu_0)$ could be calculated with nonperturbative HQET sum rules [84] and could also be extracted indirectly from $B \longrightarrow \gamma\ell\nu$ process [57, 71, 72, 85, 86]. While at present, we are still lack of satisfied constraints of $\lambda_B(\mu_0)$. Following the strategy displayed in [32], we will employ a fit approach to determine $\lambda_B(\mu_0)$. For the $B \longrightarrow P$ form factors, matching the LCSR calculation for the vector $B \longrightarrow \pi$ form factor at $q^2=0$ with the predictions from LCSR with pion LCDAs $f_{B\longrightarrow\pi}^{+}(q^2=0) = 0.28 \pm 0.03$ [18, 20, 87] will provide one with a suitable value of $\lambda_B(\mu_0)$. While for the $B \longrightarrow V$ form factors, $\lambda_B(\mu_0)$ could be determined by matching the LCSR prediction of the form factor $V_{B\longrightarrow\rho}(q^2=0)$ to that of the improved NLO LCSR with the ρ-meson LCDAs [25].

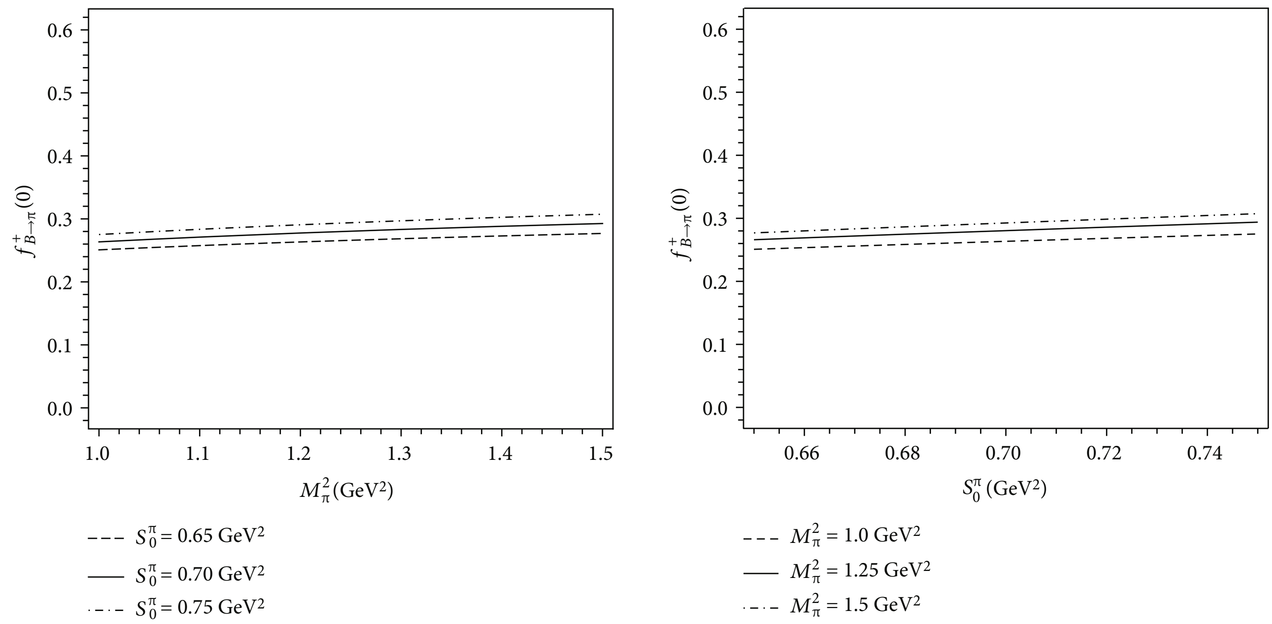

FIGURE 7: The Borel mass and threshold dependence of the form factor $f^+_{B\longrightarrow\pi}(0)$.

TABLE 1: The numerical values of the various input parameters employed in the theory predictions of the $B\longrightarrow M$ form factors with B -meson LCSR.

Parameter	Value	Ref.	Parameter	Value	Ref.
$m_s(2\,\mathrm{GeV})$	$93.8\pm1.5\pm1.9\,\mathrm{MeV}$	[74]	$m_b(m_b)$	$4.193^{+0.022}_{-0.035}\,\mathrm{GeV}$	[75, 76]
μ	$1.5\pm0.5\,\mathrm{GeV}$		μ_{h1},μ_{h2},ν_h	$m_{b-m_b/2}^{+m_b}$	
f_B	$192.0\pm4.3\,\mathrm{MeV}$	[77]			
f_π	$130.2\pm1.7\,\mathrm{MeV}$	[74]	f_K	$155.6\pm0.4\,\mathrm{MeV}$	[74]
$f_{\rho,\parallel}$	$213\pm5\,\mathrm{MeV}$	[25]	$f_{\rho,\perp}(1\,\mathrm{GeV})$	$160\pm7\,\mathrm{MeV}$	[25]
$f_{\omega,\parallel}$	$197\pm8\,\mathrm{MeV}$	[25]	$f_{\omega,\perp}(1\,\mathrm{GeV})$	$148\pm13\,MeV$	[25]
$f_{K^*,\parallel}$	$204\pm7\,\mathrm{MeV}$	[25]	$f_{K^*,\perp}(1\,\mathrm{GeV})$	$159\pm6\,\mathrm{MeV}$	[25]
s_0^π	$0.70\pm0.05\,\mathrm{GeV}^2$	[32, 78]	s_0^K	$1.05\pm0.05\,\mathrm{GeV}^2$	[32, 78]
$s_0^{\rho,\parallel}$	$1.5\pm0.1\,\mathrm{GeV}^2$	[27, 79]	$s_0^{\rho,\perp}$	$1.2\pm0.1\,\mathrm{GeV}^2$	[80]
$s_0^{\omega,\parallel}$	$s_0^{\rho,\parallel}+m_\omega^2-m_\rho^2$	[27, 79]	$s_0^{\omega,\perp}$	$s_0^{\rho,\perp}+m_\omega^2-m_\rho^2$	[80]
$s_0^{K^*,\parallel}$	$s_0^{\rho,\parallel}+m_{K^*}^2-m_\rho^2$	[27, 79]	$s_0^{K^*,\perp}$	$s_0^{\rho,\perp}+m_{K^*}^2-m_\rho^2$	[80]
$M^2_{\pi,K}$	$1.25\pm0.25\,\mathrm{GeV}^2$	[32, 78]	M^2_ρ	$1.5\pm0.5\,\mathrm{GeV}^2$	[79]
M^2_ω	$M^2_\rho+m^2_\omega-m^2_\rho$	[79]	$M^2_{K^*}$	$M^2_\rho+m^2_{K^*}-m^2_\rho$	[79]

Performing such matching procedure, we obtain

$$\lambda_B(\mu_0)=\begin{cases}285^{+27}_{-23}\,\mathrm{MeV},\ (B\longrightarrow P),\\ 343^{+22}_{-20}\,\mathrm{MeV},\ (B\longrightarrow V).\end{cases}\tag{145}$$

It is interesting to notice that the extracted values of $\lambda_B(\mu_0)$ for the $B\longrightarrow V$ form factors are in a nice agreement with that of the $B\longrightarrow P$ form factors and are also consistent with the implications of experimental data for the two-body charmless hadronic B-meson decays from the QCD factorization approach [88]. The determined values of λ_B are also compatible with the recent extracted values from the $B\longrightarrow\gamma\ell\bar{\nu}_\ell$ process [89].

Employing the above-determined parameters, one can explore the LCSR predictions of the $B\longrightarrow M$ form factors in the large-recoil region. As an illustration, we show the breakdown of different contributions to the form factor $f^+_{B\longrightarrow\pi}(q^2)$ in Figure 8. One can see from the figure that higher-twist corrections to the form factor $f^+_{B\longrightarrow\pi}(q^2)$ are dominated by the two-particle twist-five B-meson LCDA $\hat{g}_-(\omega,\mu)$, which can shift the leading-power prediction by

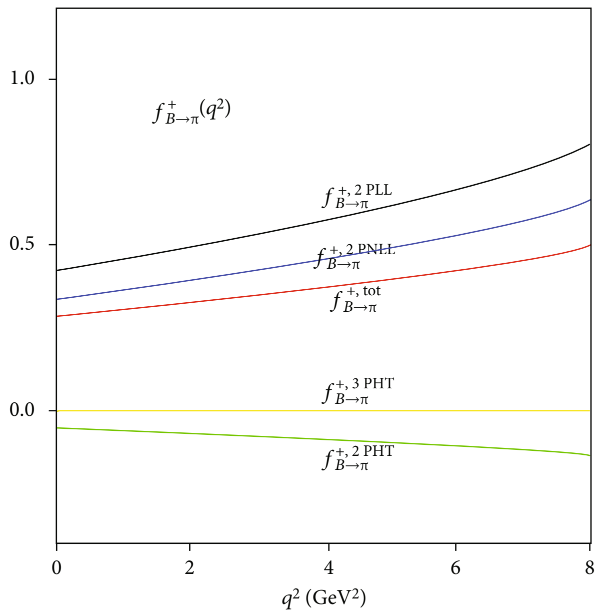

FIGURE 8: The q^2 dependence of the form factor $f^{+}_{B\longrightarrow\pi}(q^2)$. The red line represents the total contribution. The black, blue, green, and yellow correspond to the LP contribution at LL, the LP contribution at NLL, the two-particle higher-twist correction, and the three-particle higher-twist correction, respectively.

an amount of approximately $(20\sim30)\%$. Compared to the two-particle higher-twist contribution, the three-particle higher-twist contribution only generates a tiny contribution on the form factor. We further find that the radiative correction can introduce about a $O(20\%)$ reduction of the corresponding LL prediction. We have also verified that such observations also hold true for the other $B\longrightarrow M$ form factors at large hadronic recoil.

6.2. Phenomenological Applications. It is well-known that the light-cone operator-product expansion of the vacuum-to-B-meson correlation function is only valid in the large-recoil region [27, 32]. One then needs to extrapolate the LCSR predictions of $B\longrightarrow M$ form factors at $q^2\le 8\,\mathrm{GeV}^2$ to the full kinematic region with the z-series expansion. The parameter z corresponds to a map of the entire cut q^2-plane onto a unit disk $|z(q^2,t_0)|\le 1$:

$$z(q^2,t_0)=\frac{\sqrt{t_+-q^2}-\sqrt{t_+-t_0}}{\sqrt{t_+-q^2}+\sqrt{t_+-t_0}}. \tag{146}$$

Here, $t_+=(m_B+m_M)^2$ is determined by the threshold of the lowest continuum state which can be generated by the weak transition currents in QCD. The auxiliary parameter t_0, which determines the q^2 point to be mapped onto the origin of the complex z plane, will be further chosen as [18, 35]

$$\begin{aligned} t_0&=(m_B+m_P)\left(\sqrt{m_B}+\sqrt{m_P}\right)^2, B\longrightarrow P,\\ t_0&=(m_B-m_V)^2, B\longrightarrow V. \end{aligned} \tag{147}$$

We will use the simplified Bourrely-Caprini-Lellouch (BCL) series expansion for the $B\longrightarrow M$ form factors [90] (see [91, 92] for an alternative parametrization and [20] for more discussions for the $B\longrightarrow\pi$ form factors):

$$f^{+,T}_{B\longrightarrow P}(q^2)=f^{+,T}_{B\longrightarrow P}(0)1-\frac{q^2}{m^2_{B^*_{(s)}}}\left\{1+\sum_{k=1}^{N-1}b^{+,T}_{k,P}\left(z(q^2,t_0)^k-z(0,t_0)^k-(-1)^{N-k}kN\left[z(q^2,t_0)^N-z(0,t_0)^N\right]\right)\right\}, \tag{148}$$

$$f^{0}_{B\longrightarrow P}(q^2)=f^{0}_{B\longrightarrow P}(0)\left\{1+\sum_{k=1}^{N}b^{0}_{k,P}\left(z(q^2,t_0)^k-z(0,t_0)^k\right)\right\}, \tag{149}$$

$$f^{i}_{B\longrightarrow V}(q^2)=f^{i}_{B\longrightarrow V}(0)1-\frac{q^2}{m^2_{i,\mathrm{pole}}}\left\{1+\sum_{k=1}^{N}b^{i}_{k,V}\left[z(q^2,t_0)^k-z(0,t_0)^K\right]\right\}. \tag{150}$$

The expansion parameter is small in the entire region $|z(q^2,t_0)|^2\le 0.04$; the expansion could be truncated at certain N. We will truncate the z-series at $N=2$ for the form factors $f^{+,T}_{B\longrightarrow P}$ and at $N=1$ for all the other form factors (see [93] for further discussions on the systematic truncation uncertainties). The adopted values of the various resonance masses from the Particle Data Group (PDG) [74] and from the heavy-hadron chiral perturbation theory [94] are summarized in Table 2.

Table 3 shows the numerical predictions of the form factors $f^{+,0}_{B\longrightarrow\pi}$ at $q^2=0\,\mathrm{GeV}^2$. In this table, we display the significant uncertainties from different input parameters. One concludes from the table that the parameter λ_B introduces an about 10% uncertainty to the form factors. The fitted z-expansion parameters are also included in the table.

Having at our disposal the theory predictions for $B\longrightarrow\pi$ form factors, we proceed to explore

Table 2: The resonance masses with different quantum numbers entering the z-series expansions of the $B \longrightarrow V$ form factors (149) where $A_{12} = ((m_B + m_V)/(n\cdot p))A_1 - ((m_B - m_V)/m_B)A_2$ and $T_{23} = (m_B/(n\cdot p))T_2 - T_3$.

$f^i_{B\longrightarrow V}(q^2)$	J^P	$b \longrightarrow d$ (in GeV)	$b \longrightarrow s$ (in GeV)
$V(q^2)$, $T_1(q^2)$	1^-	5.325	5.415
$A_0(q^2)$	0^-	5.279	5.366
$A_1(q^2)$, $A_{12}(q^2)$, $T_2(q^2)$, $T_{23}(q^2)$	1^+	5.724	5.829

Table 3: Theory predictions for the form factors $f^{+,0}_{B\longrightarrow\pi}(0)$ and the corresponding z-expansion shape parameters $b^{+,0}_{1,\pi}$ with the dominant uncertainties from variations of different input parameters.

Parameters	Central value	λ_B	$\widehat{\sigma}_1$	μ	M^2_π	s^π_0
$f^{+,0}_{B\longrightarrow\pi}(0)$	0.280	$^{-0.030}_{+0.031}$	$^{-0.012}_{+0.013}$	$^{+0.000}_{-0.032}$	$^{+0.012}_{-0.017}$	$^{+0.014}_{-0.014}$
$b^+_{1,\pi}$	−2.77	$^{+0.05}_{-0.02}$	$^{+0.02}_{-0.01}$	$^{+0.09}_{-0.16}$	$^{+0.02}_{-0.03}$	$^{+0.07}_{-0.07}$
$b^0_{1,\pi}$	−4.88	$^{-0.10}_{+0.11}$	$^{-0.04}_{+0.04}$	$^{+0.17}_{-0.61}$	$^{+0.04}_{-0.06}$	$^{+0.11}_{-0.11}$

phenomenological aspects of the semileptonic $B \longrightarrow \pi\ell\nu$ decays, which serves as the golden channel for the determination of CKM matrix element $|V_{ub}|$ exclusively (see [95] for the future advances of precision measurements of Belle II). It is straightforward to write down the differential decay rate for $B \longrightarrow \pi\ell\bar{\nu}_\ell$:

$$\frac{d\Gamma(B\longrightarrow \pi\ell\bar{\nu}_\ell)}{dq^2} = \frac{G_F^2|V_{ub}|^2}{192\pi^3 m_B^3}\lambda^{\frac{3}{2}}\left(m_B^2, m_\pi^2, q^2\right)\left(1-\frac{m_l^2}{q^2}\right)^2\left(1+\frac{m_l^2}{2q^2}\right)\left[\left|f^+_{B\longrightarrow\pi}(q^2)\right|^2 + \frac{3m_l^2\left(m_B^2-m_\pi^2\right)^2}{\lambda\left(m_B^2,m_\pi^2,q^2\right)\left(m_l^2+2q^2\right)}\left|f^0_{B\longrightarrow\pi}(q^2)\right|^2\right],$$
$$\frac{d\Gamma(B\longrightarrow V\ell\bar{\nu}_\ell)}{dq^2} = \frac{G_F^2|V_{ub}|^2 q^2}{192\pi^3 m_B^3 c_V^2}\lambda^{1/2}\left(m_B^2, m_V^2, q^2\right)\left\{\left|H_0(q^2)\right|^2 + \left|H_+(q^2)\right|^2 + \left|H_-(q^2)\right|^2\right\}, \tag{151}$$

where $\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc$ and the three helicity amplitudes $H_i(q^2)$ $(i=\pm,0)$ can be expressed in terms of the semileptonic $B\longrightarrow V$ form factors:

$$H_\pm(q^2) = (m_B+m_V)\left[A_1(q^2) \mp \frac{2m_B|\vec{p}_V|}{(m_B+m_V)^2}V(q^2)\right],$$
$$H_0(q^2) = \frac{m_B+m_V}{2m_V\sqrt{q^2}}\left[\left(m_B^2-m_V^2-q^2\right)A_1(q^2) - \frac{4m_B^2\left|\vec{p}_V\right|^2}{(m_B+m_V)^2}A_2(q^2)\right], \tag{152}$$

with the momentum $|\vec{p}_V|$ of the light-vector meson in the B-meson rest frame given by $|\vec{p}_V| = 12\, m_B\lambda^{1/2}(m_B^2, m_V^2, q^2)$.

Employing the experimental measurements of $B\longrightarrow\pi\ell\bar{\nu}_\ell$ [96, 97], and taking advantage of the measurements of the partial branching fractions for $B\longrightarrow\rho\ell\bar{\nu}_\ell$ [97, 98] and $B\longrightarrow\omega\ell\bar{\nu}_\ell$ [96, 97], we can derive the following intervals for exclusive $|V_{ub}|$:

$$|V_{ub}|_{\rm exc.} = \left(3.23^{+0.66}_{-0.48}\big|_{\rm th.}{}^{+0.11}_{-0.11}\big|_{\rm exp.}\right)\times 10^{-3}, [\text{from } B\longrightarrow\pi\ell\nu_\ell],$$
$$|V_{ub}|_{\rm exc.} = \left(3.05^{+0.67}_{-0.52}\big|_{th.}{}^{+0.19}_{-0.20}\big|_{\rm exp.}\right)\times 10^{-3}, [\text{from } B\longrightarrow\rho\ell\nu_\ell],$$
$$|V_{ub}|_{\rm exc.} = \left(2.54^{+0.56}_{-0.40}\big|_{th.}{}^{+0.18}_{-0.19}\big|_{exp.}\right)\times 10^{-3}, [\text{from } B\longrightarrow\omega\ell\nu_\ell]. \tag{153}$$

The total uncertainties are obtained by adding in quadrature the separate uncertainties from individual variations of all input parameters. Apparently, the extracted values of $|V_{ub}|$ from $B\longrightarrow V\ell\bar{\nu}_\ell$ decay are lower than that from the $B\longrightarrow\pi\ell\bar{\nu}_\ell$ channel. We also note that the obtained values from the $B\longrightarrow\pi\ell\bar{\nu}_\ell$ decay are in good agreements with the results from $B\longrightarrow V\ell^+\ell^-$ decays [25]. The values of $|V_{ub}|$ from $B\longrightarrow\pi\ell\bar{\nu}_\ell$ decay are in agreement with the averaged exclusive determinations presented in PDG [74], while the central values of both determinations of $|V_{ub}|$ from $B\longrightarrow V$

$\ell \bar{\nu}_\ell$ are somewhat smaller than the corresponding result in PDG [74]. We also observe that the obtained $|V_{ub}|$ from the $B \longrightarrow \omega \ell \bar{\nu}_\ell$ process are significantly smaller than that from the exclusive channel $B \longrightarrow \rho \ell \bar{\nu}_\ell$ as already observed in [97]. All the three exclusively extracted values of $|V_{ub}|$ from the $B \longrightarrow M \ell \bar{\nu}_\ell$ decays are significantly smaller than the averaged inclusive determinations reported in [74]

$$|V_{ub}|_{\text{inc.}} = \left(4.49 \pm 0.15^{+0.16}_{-0.17} \pm 0.17\right) \times 10^{-3}. \tag{154}$$

7. Summary

In this review, we have discussed the LCSR with B-meson LCDAs and its application to calculating the $B \longrightarrow P$ and $B \longrightarrow V$ form factors. The fundamental nonperturbative inputs in this approach are the LCDAs of the B meson, which are defined in terms of nonlocal operators including the effective b-quark field and the soft light parton fields sandwiched between the vacuum and the B-meson state. At the one-loop level, the leading-twist B-meson LCDA satisfies the LN equation, which can be simplified by some kinds of integral transforms. The two-loop level evolution equation of the leading-twist B-meson LCDA has also been obtained. The evolution function is useful in the determination of the models of the LCDAs, especially the behavior at the endpoint. The higher-twist B-meson LCDAs can be introduced by adjusting the spinor structure, or adding an additional gluon field, or taking the higher power contribution from the light-cone expansion. They can be studied using a similar method as the leading-twist one.

We introduced the investigation of the $B \longrightarrow M$ form factors with the B-meson LCSR in detail. To obtain the sum rules, one must start from the correlation function which is defined as the vacuum-to-B-meson matrix element of the time-ordered product of the weak current and the interpolation current of the light meson. At the partonic level, the correlation function can be factorized into the convolution of the hard function, the jet function, and the LCDAs of the B meson. The short-distance hard function and jet function can be calculated by choosing free partonic external states. To evaluate the NLO corrections to the hard function and the jet function, one might employ the method of regions, i.e., the loop momentum was assigned to be hard, collinear, and soft, respectively. The loop integral was calculated in different regions, and the hard function and jet function can be extracted directly. An alternative approach to obtain the NLO hard function and jet function is to perform the match from QCD first to SCET_{I} and then to SCET_{II}, respectively. The most subtle place in the loop calculation is the infrared subtraction, especially under the condition that the evanescent operators took the place. The convolution of the hard function, the jet function, and the LCDAs is independent of the factorization scale. However, the large logarithmic terms need to be resummed with the RG equation approach. Having the hard function and the jet function at hand, the correlation function can be expressed in terms of the dispersion integral. After this, we employed the quark-hadron duality assumption and the Borel transformation and then obtained the sum rules for the form factors.

We have discussed the power suppressed contributions to the form factors. The higher-twist corrections to the form factors from both the two-particle and three-particle B-meson LCDAs were calculated at tree level. For the contribution from three-particle B-meson LCDAs, we employed the quark propagator in the background field. The quark-mass-dependent term in the sum rules with three-particle B-meson LCDAs is power suppressed, which is different from the leading-twist contribution. Except for the high-twist contribution, there exist more sources of power suppressed contributions, such as the power suppressed contribution from heavy quark expansion, the power suppressed contribution from hard-collinear propagators, and the local terms. The investigation of power suppressed contributions is one of the main goals of future studies.

After employing appropriate input parameters, we evaluated the $B \longrightarrow P$ and $B \longrightarrow V$ form factors numerically within the momentum region $0 < q^2 < 8\,\text{GeV}^2$ where the LCSR with B-meson LCSR is applicable. Then, we extrapolated the result to the whole physical region by the z-series expansion. Applying the experimental measurements of the $B \longrightarrow \pi\ell\bar{\nu}_\ell$ process and taking advantage of the measurements of the partial branching fractions for $B \longrightarrow \rho \ell \bar{\nu}_\ell$ and $B \longrightarrow \omega \ell \bar{\nu}_\ell$ decays, we obtained intervals for exclusive $|V_{ub}|$. The extracted values of $|V_{ub}|$ from $B \longrightarrow V \ell \bar{\nu}_\ell$ decay are lower than that from the $B \longrightarrow \pi \ell \bar{\nu}_\ell$ channel. The values of $|V_{ub}|$ from $B \longrightarrow \pi \ell \bar{\nu}_\ell$ decay are in agreement with the averaged exclusive determinations presented in PDG. All of the three exclusively extracted values of $|V_{ub}|$ from the $B \longrightarrow M \ell \bar{\nu}_\ell$ decays are significantly smaller than the averaged inclusive determinations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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