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Rational curves on lattice-polarised K3 surfaces

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Abstract

Fix a K3 lattice Λ of rank 2 and a big and nef divisor $L \in \Lambda$ that is suitably positive. We prove that the generic Λ-polarised K3 surface has an integral nodal rational curve in the linear system $|L|$, in particular strengthening previous work of the first-named author. The technique is by degeneration and also works for many lattices of higher rank.

1. Introduction

In [Che99], the first-named author proved the existence of integral nodal rational curves in $|nL|$ on a generic K3 surface with Picard group generated by L for all $n > 0$ and $L^2 \geqslant 4$. In this paper, we follow a similar strategy and prove the following (see Section 3 for a more precise statement).

Theorem A. *Let $a, b \in \mathbb{Z}$ and $d \in \mathbb{Z}_{>0}$ satisfy $4bd - a^2 < 0$, and let Λ be a lattice of rank 2 with intersection matrix*

$$\begin{bmatrix} 2d & a \\ a & 2b \end{bmatrix}.$$

Fixing $L \in \Lambda$ with $L^2 > 0$, let M_Λ be the moduli space of Λ-polarised K3 surfaces such that L is ample on a general $X \in M_\Lambda$. Then there exists a Zariski-open dense subset U_L of M_Λ such that there is an integral nodal rational curve in $|L|$ on $X \in U_L$ if L is the sum of three ample divisors on X.

The method of proof of Theorem A is a sequence of two degenerations. One first proceeds by degenerating to a smooth K3 surface of higher Picard rank that contains several (-2)-curves. The main technical difficulty is that one must distinguish between rank 2 lattices of even and odd discriminant, and each case requires a different degeneration, which significantly adds to the length of the argument. In particular, we prove that any rank 2 K3 lattice embeds primitively into one of the lattices

$$\begin{bmatrix} 2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & & & \\ 1 & -2 & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & -2 \end{bmatrix} \tag{1.0.1}$$

of size $r_1 \times r_1$ and $r_2 \times r_2$ for some $r_1 \leqslant 7$ and $r_2 \leqslant 5$, respectively.

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To prove the existence of integral nodal rational curves in $|L|$ for L a big and nef class in one of the above two matrices, we degenerate further, like in [Che99], to unions of smooth rational surfaces à la Ciliberto–Lopez–Miranda. One now constructs reducible, *limiting rational curves* with prescribed singularities in such log K3 surfaces, in the sense of Iitaka [Iit79], which now deform out to integral and nodal rational curves on the general K3 with lattice as above.

The technique in fact works to produce many nodal rational curves for a general K3 with Picard lattice which embeds into one of the lattices in (1.0.1). At the end of this paper, we show how this can work for the two special rank 4 lattices of Nikulin [Nik87], for which the K3 in question has finite automorphism group and is not elliptic.

The above results will be used in a follow-up paper [CGL19], whose main result completes the project (initiated by Bogomolov–Mumford) of showing that every complex projective K3 surface contains infinitely many rational curves. More specifically, Theorem A will be used in two ways in [CGL19]. First, it will be used in the proof of *regeneration theorem*: Given a family of K3 surface X/B, if there is an integral rational curve C on a fibre X_b, then there is a curve C' on X_b such that the union $C \cup C'$ can be deformed to an integral rational curve on a general fibre of X/B. Second, combined with a technique called *marked point trick*, Theorem A will be generalised to *every* K3 surface of Picard rank 2 as long as L is indivisible in Λ.

Notation. A *K3 surface* X will be a geometrically integral, smooth, proper and separated scheme of relative dimension 2 over the complex numbers, so that $\omega_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Let S be a connected base scheme. Then a morphism

$$f \colon \mathscr{X} \longrightarrow S$$

is a *smooth family of surfaces* if f is a smooth and proper morphism of algebraic spaces of relative dimension 2 whose geometric fibres are irreducible. In particular, a *family of K3 surfaces* is a family of surfaces where every fibre is a K3 surface as above. We say that a property holds for a *general point* in a set if it holds for all points of a Zariski-open subset, whereas a *very general point* will be one in the complement of countably many Zariski-closed subsets.

2. Degenerations of type II

In this section, we discuss degenerations of K3 surfaces that are of type II in the sense of Kulikov [Kul77]; see also [Per77, PP81]. We will need these degenerations in order to produce nodal rational curves in the next section.

Definition 2.1. A *degeneration of type II* (in the sense of Kulikov) of K3 surfaces is a flat and proper family $\pi \colon \mathscr{X} \to S$, where S is the spectrum of a discrete valuation ring with residue field $\mathbb{C}$ and where the geometric generic fibre $\overline{\mathscr{X}}_\eta$ is a K3 surface and the special fibre $\mathscr{X}_0$ is a union $Y_1 \cup Y_2 \cup \cdots \cup Y_m$ such that

(1) each Y_i is a smooth surface for all i;
(2) $Y_i \cap Y_j = \emptyset$ for $|i-j| \neq 0, 1$ and Y_i and Y_{i+1} meet transversally along a smooth elliptic curve D_i, where all D_i are isomorphic to a fixed smooth elliptic curve D;
(3) we have anti-canonical divisors E_1 and E_{m-1} in Y_1 and Y_m;
(4) each Y_i for $1 < i < m$ is a ruled surface over D.

It follows from item (3) that Y_1 and Y_m are rational surfaces. We note that the chain $Y_2 \cup \cdots \cup Y_{m-1}$ of ruled surfaces can be contracted to a family $\mathscr{X}' \to S$ that has the same generic fibre but whose special fibre has the form $\mathscr{X}'_0 := Y'_1 \cup Y'_2$, where Y'_1 and Y'_2 are smooth rational surfaces meeting transversally along a smooth anti-canonical curve D. In this case, (Y'_i, D), for $i = 1, 2$ are two genuine log K3 surfaces.

Conversely, given a union of $Y_1 \cup Y_2$ of two smooth rational surfaces meeting transversely along a smooth anti-canonical curve D, one may ask whether it can be deformed to a K3 surface. We refer the interested reader to [CLM93], where this question is studied.

More generally, let $Y = Y_1 \cup Y_2$ be the union of two smooth projective varieties meeting transversely along a smooth hypersurface D in each Y_i. That is, étale locally around D, the union is given by $xy = 0$. In particular, Y is reducible. The Picard group $\operatorname{Pic}(Y)$ is given by the exact sequence

$$0 \longrightarrow \operatorname{Pic}(D) \xrightarrow{\imath_1^* - \imath_2^*} \operatorname{Pic}(Y_1) \oplus \operatorname{Pic}(Y_2) \longrightarrow \operatorname{Pic}(Y) \longrightarrow 0\,,$$

where $\imath_i \colon D \hookrightarrow Y_i$ denotes the embedding of D into Y_i for $i = 1, 2$. In other words, an invertible sheaf $\mathscr{L}$ on Y is given by a pair of invertible sheaves $\mathscr{L}_i \in \operatorname{Pic}(Y_i)$ satisfying

$$\imath_1^* \mathscr{L}_1 \cong \imath_2^* \mathscr{L}_2\,. \tag{2.0.1}$$

(For the reader that wants to avoid this descent construction, we have an alternative realisation of Y given below.) In particular, Y is projective if and only if there exists an ample invertible sheaf $\mathscr{L} \in \operatorname{Pic}(Y)$ if and only if there exists a pair of ample invertible sheaves $\mathscr{L}_i$ on Y_i that satisfies (2.0.1). If such an $\mathscr{L}$ exists, then we can embed Y into some $\mathbb{P}^n$ via $|\mathscr{L}^{\otimes m}|$ for m sufficiently large.

Alternatively, we may start with two smooth projective varieties Y_1 and Y_2, embeddings $\imath_i \colon D \hookrightarrow Y_i$ of a smooth hypersurface D in Y_i and two ample line bundles $\mathscr{L}_i \in \operatorname{Pic}(Y_i)$ satisfying (2.0.1) for $i = 1, 2$. Let us choose m sufficiently large such that the $\mathscr{L}_i^{\otimes m}$ are very ample and the maps $\mathrm{H}^0\big(Y_i, \mathscr{L}_i^{\otimes m}\big) \to \mathrm{H}^0\big(D, \mathscr{L}_i^{\otimes m}\big)$ are surjective for $i = 1, 2$. We choose a basis $\{(s_{1j}, s_{2j}) \colon j = 0, 1, \ldots, n\}$ for the kernel of the map

$$\mathrm{H}^0(Y_1, \mathscr{L}_1) \oplus \mathrm{H}^0(Y_2, \mathscr{L}_2) \xrightarrow{\imath_1^* - \imath_2^*} \mathrm{H}^0(D, \imath_1^* \mathscr{L}_1) = \mathrm{H}^0(D, \imath_2^* \mathscr{L}_2)$$

and define the maps $\phi_i \colon Y_i \to \mathbb{P}^n$ to be $(s_{i0}, s_{i1}, \ldots, s_{in})$ for $i = 1, 2$. Then we see that $Y = \phi_1(Y_1) \cup \phi_2(Y_2)$ is the union of two smooth projective varieties meeting along $\phi_1(D) = \phi_2(D)$ such that $\phi_{i,*} T_{D,p} = \phi_{1,*} T_{Y_1,p} \cap \phi_{2,*} T_{Y_2,p}$ for the tangent spaces of Y_i and D at $p \in D$, as subspaces of $T_{\mathbb{P}^n, \phi(p)}$.

The first-order embedded deformations of $Y \subset P := \mathbb{P}^n$ are classified by $\mathrm{H}^0(Y, \mathscr{N}_Y)$, where $\mathscr{N}_Y$ denotes the normal sheaf of $Y \subset P$. We note that these deformations of Y preserve the line bundle $\mathscr{L}^{\otimes m}$. We want to deform Y to a smooth variety, that is, to "smooth" out D, which is the singular locus of Y. This is governed by the map

$$\mathrm{H}^0(Y, \mathscr{N}_Y) \longrightarrow \mathrm{H}^0\big(Y, T_Y^1\big)\,, \tag{2.0.2}$$

where T_Y^1 is the sheaf of $\mathcal{O}_Y$-modules

$$T_Y^1 := \mathscr{E}\mathrm{xt}(\Omega_Y, \mathcal{O}_Y) \cong \mathscr{N}_{D/Y_1} \otimes \mathscr{N}_{D/Y_2}\,,$$

where $\mathscr{N}_{D/Y_i}$ denotes the normal bundle of D in Y_i. A general deformation of $Y \subset P$ smooths out D if the embedded deformations of $Y \subset P$ are unobstructed and if the image of the map (2.0.2) is base-point-free on D.

Let $\mathscr{X} \subset P \times \Delta$ be a deformation of Y given by some $\xi \in \mathrm{H}^0(Y, \mathscr{N}_Y)$ with $\mathscr{X}_0 = Y$, where $\Delta = \{|t| < 1\}$ is the unit disk. Then $\mathscr{X}$ is singular along the vanishing locus $z(\rho(\xi))$ of $\rho(\xi)$, where ρ is the map (2.0.2). If $z(\rho(\xi))$ is smooth as a closed subscheme of D, then $\mathscr{X}$ has singularities of type

$$\mathbb{C}[[x_1, x_2, \ldots, x_n, t]]/(x_1 x_2 - t x_3)$$

at $z(\rho(\xi))$ and hence the generic fibre $\mathscr{X}_\eta$ is smooth. So a general deformation of $Y \subset P$ smooths out D under the above hypotheses.

A general deformation of $Y \subset P$, a priori, only preserves $\mathscr{L}^{\otimes m}$. For the above family $\mathscr{X}$, the restriction of $\mathscr{H} = \mathcal{O}_{\mathscr{X}}(1)$ to Y is obviously $\mathscr{L}^{\otimes m}$. On the other hand, in our application, $\mathrm{H}^2(Y, \mathbb{Z})$ is always torsion-free; by the Mayer–Vietoris sequence, this is guaranteed if $\mathrm{H}^1(Y_i) = 0$ and $\mathrm{H}^1(D, \mathbb{Z})$ and $\mathrm{H}^2(Y_i, \mathbb{Z})$ are torsion-free. By deformation retraction, $\mathrm{H}^2(\mathscr{X}, \mathbb{Z}) \cong \mathrm{H}^2(Y, \mathbb{Z})$ is torsion-free. Consequently, $(1/m)c_1(\mathscr{H}) \in \mathrm{H}^2(\mathscr{X}, \mathbb{Z})$, and $\mathscr{L}$ extends to a line bundle on $\mathscr{X}$. In conclusion, a general deformation of Y preserves $\mathscr{L}$ if $\mathrm{H}^2(Y, \mathbb{Z})$ is torsion-free.

Moreover, if we construct Y with arbitrary Picard rank $r := \rho(Y)$, we can deform Y to preserve $\mathrm{Pic}(Y)$ as follows. We choose very ample line bundles $\mathscr{H}_1, \ldots, \mathscr{H}_r$ on Y which generate $\mathrm{Pic}_{\mathbb{Q}}(Y)$ and embed Y into $P = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ via the complete linear systems $|\mathscr{H}_i|$. If the embedded deformations of $Y \subset P$ smooth out D, we can deform Y to a smooth variety while preserving $\mathrm{Pic}_{\mathbb{Q}}(Y)$. In addition, as commented above, if $\mathrm{H}^2(Y, \mathbb{Z})$ is torsion-free, then $\mathrm{Pic}(Y)$ is preserved when Y deforms in P. Using the techniques in [CLM93, Section 1], we can prove the following theorem on the deformation of $Y \subset P$.

Theorem 2.2 (Ciliberto–Lopez–Miranda + ε). *Let $Y = Y_1 \cup Y_2$ be a union of two smooth projective varieties meeting transversely along a smooth hypersurface D in Y_i. Suppose that Y is embedded into $P = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ by very ample line bundles $\mathscr{H}_1, \ldots, \mathscr{H}_r \in \mathrm{Pic}(Y)$ satisfying that $\mathscr{H}_1, \ldots, \mathscr{H}_r$ are linearly independent in $\mathrm{H}^{1,1}(Y_1)$.*

(1) *The sheaf T^1_Y is isomorphic to the cokernel of the inclusion $\mathscr{N}_{Y_i} \to \mathscr{N}_Y \otimes \mathcal{O}_{Y_i}$ for $i = 1, 2$; that is, the sequence*

$$0 \longrightarrow \mathscr{N}_{Y_i} \longrightarrow \mathscr{N}_Y\big|_{Y_i} \longrightarrow T^1_Y \longrightarrow 0 \tag{2.0.3}$$

is exact for $i = 1, 2$.

(2) *We have $\mathrm{H}^1(Y, \mathscr{N}_Y) = 0$, and the map (2.0.2) is surjective if*

$$\begin{aligned} &\mathrm{H}^1(Y_i, \mathscr{H}_j) = \mathrm{H}^1(Y_1, \mathscr{H}_j(-D)) = 0\,,\\ &\mathrm{H}^1(\mathscr{N}_{D/Y_2}) = \mathrm{H}^1(\mathscr{N}_{D/Y_1} \otimes \mathscr{N}_{D/Y_2}) = 0\,,\\ &\mathrm{H}^2(\mathcal{O}_{Y_i}) = \mathrm{H}^2(T_{Y_i}) = \mathrm{H}^2(T_{Y_1}(-D)) = 0\,, \end{aligned} \tag{2.0.4}$$

for $i = 1, 2$ and $j = 1, 2, \ldots, r$ and, moreover,

$$\text{either} \quad \mathrm{H}^2(\mathcal{O}_{Y_1}(-D)) = 0\,, \quad \text{or} \quad K_{Y_1} + D = 0 \quad \text{and} \quad \dim Y = 2\,. \tag{2.0.5}$$

Proof. In [CLM93], this was proved for $r = 1$ and $\dim Y = 2$.

We basically follow the same argument as in [CLM93]: the exactness of (2.0.3) is a consequence of the commutative diagram

$$\begin{array}{ccccccc} T_P\big|_{Y_i} & \longrightarrow & \mathscr{N}_{Y_i} & \longrightarrow & 0 & & \\ \| & & \downarrow & & & & \\ T_P\big|_{Y_i} & \longrightarrow & \mathscr{N}_Y\big|_{Y_i} & \longrightarrow & T^1_Y & \longrightarrow & 0\,, \end{array}$$

which shows that $\operatorname{coker}(\mathscr{N}_{Y_i} \to \mathscr{N}_Y \otimes \mathcal{O}_{Y_i})$ surjects onto T^1_Y. And since $\operatorname{coker}(\mathscr{N}_{Y_i} \to \mathscr{N}_Y \otimes \mathcal{O}_{Y_i})$ and T^1_Y are line bundles supported on D, the surjection must be an isomorphism and we obtain (2.0.3).

To prove $\mathrm{H}^1(Y, \mathscr{N}_Y) = 0$ and the surjectivity of (2.0.2), we combine (2.0.3) with the exact sequence

$$0 \longrightarrow \mathscr{N}_Y \otimes \mathcal{O}_{Y_1}(-D) \longrightarrow \mathscr{N}_Y \longrightarrow \mathscr{N}_Y \otimes \mathcal{O}_{Y_2} \longrightarrow 0\,.$$

With these two exact sequences, it suffices to prove

$$\mathrm{H}^1(\mathscr{N}_{Y_1}(-D)) = \mathrm{H}^1(\mathscr{N}_{Y_1}) = \mathrm{H}^1(\mathscr{N}_{Y_2}) = \mathrm{H}^1(\mathscr{N}_{D/Y_2}) = \mathrm{H}^1\big(T^1_Y\big) = 0\,.$$

The vanishing of these cohomological groups mostly follows from (2.0.4). Let us say something about $\mathrm{H}^1(\mathscr{N}_{Y_1}(-D)) = 0$.

If $\mathrm{H}^2(\mathcal{O}_{Y_1}(-D)) = 0$, then the vanishing of $\mathrm{H}^1(\mathscr{N}_{Y_1}(-D))$ follows from (2.0.4) and the exact sequences

$$0 \longrightarrow T_{Y_1} \longrightarrow T_P \otimes \mathcal{O}_{Y_1} \longrightarrow \mathscr{N}_{Y_1} \longrightarrow 0\,,$$

$$0 \longrightarrow \mathcal{O}_{Y_1}^{\oplus r} \longrightarrow \sum_{i=1}^{r} \mathscr{H}_i^{\oplus(n_i+1)} \longrightarrow T_P \otimes \mathcal{O}_{Y_1} \longrightarrow 0\,.$$

If we instead have $K_{Y_1} + D = 0$ and $\dim Y = 2$ in (2.0.5), then the vanishing of $\mathrm{H}^1(\mathscr{N}_{Y_1}(-D))$ follows from (2.0.4) and the injectivity of the map

$$\begin{array}{ccc} \mathrm{H}^1(\Omega_P) & \hookrightarrow & \mathrm{H}^1(\Omega_{Y_1}) \\ \| & & \| \\ \mathrm{H}^1(T_P \otimes K_{Y_1})^\vee & \longrightarrow & \mathrm{H}^1(T_{Y_1} \otimes K_{Y_1})^\vee \\ \| & & \| \\ \mathrm{H}^1(T_P \otimes \mathcal{O}_{Y_1}(-D))^\vee & \longrightarrow & \mathrm{H}^1(T_{Y_1}(-D))^\vee\,, \end{array}$$

where the injectivity of $\mathrm{H}^1(\Omega_P) \to \mathrm{H}^1(\Omega_{Y_1})$ is a consequence of the hypothesis that $\mathscr{H}_1, \ldots, \mathscr{H}_r$ are linearly independent in $\mathrm{H}^{1,1}(Y_1)$. □

If $Y = Y_1 \cup Y_2$ is a degeneration of type II of K3 surfaces with Y_i Fano varieties (that is, del Pezzo surfaces) for $i = 1, 2$, then the hypotheses (2.0.4) and (2.0.5) are clearly satisfied. In this case, Y can be deformed to a smooth projective K3 surface of Picard rank $r = \rho(Y)$. Thus, we conclude the following.

THEOREM 2.3. *Let $Y = Y_1 \cup Y_2$ be the union of two smooth rational surfaces Y_i meeting transversely along a smooth anti-canonical curve D in Y_i for $i = 1, 2$ satisfying that*

$$\deg \mathscr{N}_{D/Y_1} + \deg \mathscr{N}_{D/Y_2} = K^2_{Y_1} + K^2_{Y_2} \geqslant 1 \quad \text{and}$$
$$\deg \mathscr{N}_{D/Y_2} = K^2_{Y_2} \geqslant 1\,.$$

Suppose that there exist $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_r \in \operatorname{Pic}(Y)$ such that the $\mathscr{L}_i$ are linearly independent in $\mathrm{H}^2(Y_1)$ and the subgroup of $\operatorname{Pic}(Y)$ generated by the $\mathscr{L}_i$ contains an ample line bundle on Y. Then Y can be deformed to a projective K3 surface of Picard rank r. More precisely, there exists a flat projective family $\pi\colon \mathscr{X} \to \operatorname{Spec} \mathbb{C}[[t]]$ such that its central fibre is $\mathscr{X}_0 = Y$, its generic *fibre $\mathscr{X}_\eta$ is a projective K3 surface of Picard rank r and the image of* $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(Y)$ *contains* $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_r$.

Proof. Let us choose sufficiently ample $\mathscr{H}_1, \mathscr{H}_2, \dots, \mathscr{H}_r \in \mathrm{Pic}(Y)$ such that

$$\mathrm{Span}_{\mathbb{Q}}\{\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_r\} = \mathrm{Span}_{\mathbb{Q}}\{\mathscr{H}_1, \mathscr{H}_2, \dots, \mathscr{H}_r\}$$

in $\mathrm{Pic}(Y)$. We embed Y into $P = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ by $|\mathscr{H}_i|$. By Theorem 2.2, the embedded deformations of $Y \subset P$ are unobstructed, and a general deformation $Y' \subset P$ of Y smooths out D; hence Y' is a projective K3 surface of Picard rank at least r. To see that $\rho(Y') = r$, we compute $h^0(\mathscr{N}_Y) = 20 - r + h^0(T_P)$ and conclude that the image of $\mathrm{H}^0(\mathscr{N}_Y) \to \mathrm{Ext}(\Omega_Y, \mathscr{O}_Y)$ has dimension at least $20 - r$.

Let $\mathscr{X} \subset P \times \mathbb{C}[[t]]$ be the family given by a general deformation of $Y \subset P$. Then for every $\mathscr{L}_i$, the line bundle $m_i\mathscr{L}_i$ lies in the image of $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(Y)$ for some integer $m_i \neq 0$; since $\mathrm{H}^2(Y, \mathbb{Z})$ is torsion-free, $\mathscr{L}_i$ lies in the image of $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(Y)$ for $i = 1, 2, \dots, r$. □

Let $\mathscr{X}$ be the family given by a general deformation of Y in Theorem 2.3. Then $\mathscr{X}$ has rational double points at $x_1, x_2, \dots, x_s \in D$ for $s = K_{Y_1}^2 + K_{Y_2}^2$, which are the vanishing locus of a section in $\mathrm{H}^0(T_Y^1)$. Clearly, the x_i satisfy

$$\mathscr{O}_D(x_1 + x_2 + \cdots + x_s) = \mathscr{N}_{D/Y_1} \otimes \mathscr{N}_{D/Y_2} = \mathscr{O}_D(-K_{Y_1} - K_{Y_2})\,. \tag{2.0.6}$$

For a general deformation of Y, the points $x_1, x_2, \dots, x_s$ are s general points with only the relation (2.0.6) on D.

Even if Y is not projective, we can still deform Y to a K3 surface, although the resulting family is obviously non-projective. The issue of projectivity is purely technical.

Theorem 2.4. *Keep the hypotheses of Theorem 2.3, except that instead of assuming that the subgroup of* $\mathrm{Pic}(Y)$ *generated by the $\mathscr{L}_i$ contains an ample line bundle and $K_{Y_1}^2 + K_{Y_2}^2 \geqslant 1$, we assume that $K_{Y_1}^2 + K_{Y_2}^2 \geqslant 2$. Then Y can be deformed to a projective K3 surface of Picard rank $r+1$. More precisely, there exists a flat proper (possibly non-projective) family $\pi\colon \mathscr{X} \to$* $\mathrm{Spec}\,\mathbb{C}[[t]]$ *such that its central fibre is $\mathscr{X}_0 = Y$, its generic fibre $\mathscr{X}_\eta$ is a projective K3 surface of Picard rank $r+1$ and the image of* $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(Y)$ *contains $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_r$.*

Proof. We choose an ample line bundle M_1 on Y_1 and an ample line bundle M_2 on Y_2. Let $m_i = M_i D$ on Y_i for $i = 1, 2$ and $m = \gcd(m_1, m_2)$. Let a_1 and a_2 be positive integers such that $a_1 m_1 - a_2 m_2 = m$, and let p be a point on D such that

$$\mathscr{O}_D(a_1 M_1) = \mathscr{O}_D(a_2 M_2) \otimes \mathscr{O}_D(mp)\,.$$

Let us choose M_i and p such that $\mathscr{O}_D(2p) \neq \mathscr{O}_D\big(-K_{Y_1}^2 - K_{Y_2}^2\big)$. Let $\widehat{Y}_1$ be the blowup of Y_1 at p. We can construct a union of $\widehat{Y} = \widehat{Y}_1 \cup Y_2$ meeting transversely along D such that there is a morphism $\varphi\colon \widehat{Y} \to Y$ such that $\varphi|_{\widehat{Y}_1}$ is the blowup map $\widehat{Y}_1 \to Y_1$ and $\varphi|_{Y_2} = \mathrm{id}$.

For a_1 sufficiently large, $a_1\varphi^* M_1 - mE$ is ample on $\widehat{Y}_1$, where E is the exceptional divisor of $\widehat{Y}_1 \to Y_1$. Therefore, there exists an ample line bundle $\widehat{M}$ on $\widehat{Y}$ whose restriction to $\widehat{Y}_1$ is $a_1\varphi^* M_1 - mE$ and whose restriction to Y_2 is $a_2\varphi^* M_2$.

Applying Theorem 2.3 to $\widehat{Y}$ with $\widehat{M}, \varphi^*\mathscr{L}_1, \varphi^*\mathscr{L}_2, \dots, \varphi^*\mathscr{L}_r$, we obtain a flat projective family $\mathscr{Y} \to \mathrm{Spec}\,\mathbb{C}[[t]]$ such that $\mathscr{Y}_0 = \widehat{Y}$, that $\mathscr{Y}_\eta$ is a K3 surface of Picard rank $r+1$ and that the image of $\mathrm{Pic}\big(\mathscr{Y}\big) \to \mathrm{Pic}\big(\widehat{Y}\big)$ contains $\widehat{M}, \varphi^*\mathscr{L}_1, \varphi^*\mathscr{L}_2, \dots, \varphi^*\mathscr{L}_r$.

For a general choice of $\mathscr{Y}$, it has at worst rational double points on D satisfying (2.0.6). We may assume that $\mathscr{Y}$ is smooth along E. As a complex manifold, $\mathscr{Y}$ admits a small contraction of E. Let us still use φ to denote this map:

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\varphi} & \mathscr{X} \\ \downarrow & \swarrow & \\ \operatorname{Spec}\mathbb{C}[[t]] \,. & & \end{array}$$

Clearly, $\mathscr{X}_0 = Y$ and $\mathscr{X}_\eta$ is a projective K3 surface of Picard rank $r+1$, while $\mathscr{X}$ is flat and proper but possibly non-projective over $\operatorname{Spec}\mathbb{C}[[t]]$. The image of $\operatorname{Pic}(\mathscr{X}) \to \operatorname{Pic}(Y)$ contains $\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_r$. □

In this paper, we mainly use this degeneration to construct rational curves on generic K3 surfaces. Let $\pi\colon \mathscr{X} \to \operatorname{Spec}\mathbb{C}[[t]]$ be the family constructed in Theorems 2.3 and 2.4. Suppose that $\mathscr{X}$ has rational double points $x_1, x_2, \dots, x_s$ on D satisfying (2.0.6). We are going to find integral (nodal) rational curves in $|\mathscr{L}|$ on $\mathscr{X}_\eta$ for some $\mathscr{L} \in \operatorname{Pic}(\mathscr{X})$. In order to do that, we construct some $\Gamma \in |\mathscr{L}|$ on $\mathscr{X}_0 = Y$, which we call "limiting rational curves" and show that Γ can be deformed to an integral (nodal) rational curve on $\mathscr{X}_\eta$.

We consider $\Gamma = f_*C$ as the image of a stable map $f\colon C \to Y$. Instead of deforming Γ, we try to deform the map f. Actually, we can construct "deformable" stable maps $f\colon C \to Y$ of arbitrary genus g, up to the arithmetic genus of $\mathscr{L}$, as follows:

- Let $h^0(\mathscr{X}_\eta, \mathscr{L}) = h^0(Y, \mathscr{L})$, where $\mathrm{H}^0(Y, \mathscr{L})$ is the kernel of the map

$$\begin{array}{ccc} \mathrm{H}^0(Y_1, \mathscr{L}_1) \oplus \mathrm{H}^0(Y_2, \mathscr{L}_2) & \xrightarrow{(\gamma_1,\gamma_2)\to\gamma_1-\gamma_2} & \mathrm{H}^0(D, \mathscr{L}_1) \\ & & \| \\ & & \mathrm{H}^0(D, \mathscr{L}_2) \end{array}$$

  with $\mathscr{L}_i$ the restriction of $\mathscr{L}$ on Y_i for $i = 1, 2$.
- Take $D \not\subset \Gamma \in \mathbb{P}\mathrm{H}^0(Y, \mathscr{L})$.
- The map f sends each irreducible component $G \subset C$ birationally onto its image:

$$f_*G = f(G) \quad \text{for all irreducible components } G \subset C\,. \tag{2.0.7}$$

- Let $C_\times$ be the set of points of C lying on two distinct components of C. Then

$$\begin{aligned} &\text{for each } p \in G_1 \cap G_2 \subset C_\times\,, \quad f(p) \in D\backslash\{x_1, x_2, \dots, x_s\}\,, \\ &\quad f(G_1) \subset Y_1\,, \quad f(G_2) \subset Y_2\,, \\ &\quad f_{G_1} = f|_{G_1}\colon G_1 \to Y_1\,, \quad f_{G_2} = f|_{G_2}\colon\ G_2 \to Y_2 \\ &\quad \text{and } v_p(f_{G_1}^*D) = v_p(f_{G_2}^*D)\,, \end{aligned} \tag{2.0.8}$$

  where G_1 and G_2 are two irreducible components of C meeting at p and $v_p(f_{G_i}^*D)$ is the multiplicity of p in $f_{G_i}^*D$.
- Outside of $f(C_\times)$, the curves Γ and D only meet at $x_1, x_2, \dots, x_s$. More precisely,

$$\begin{aligned} &\text{for each } q \in f^{-1}(D)\backslash C_\times\,, \quad f(q) \in \{x_1, x_2, \dots, x_s\}\,, \\ &\quad f_G = f|_G\colon G \to Y_i \quad \text{and} \quad v_q(f_G^*D) = 1\,, \end{aligned} \tag{2.0.9}$$

  where G is the irreducible component of C containing q.

Using the deformation theory of curves on $\mathscr{X}$ as explained in [Che99], we can deform f to the generic fibre $\mathscr{X}_\eta$. The above statement includes several improvements over [Che99]. For example, we do not assume that f_*C_i has simple tangency with D in (2.0.8). The difficulties caused by loosening these restrictions on Γ can be overcome by studying the deformation of the stable map $f\colon C \to \mathscr{X}$ instead of the deformation of $\Gamma \subset \mathscr{X}$, which is carried out in the same way as in the case that $\mathscr{X}$ is a smooth family of K3 surfaces. On the other hand, these assumptions do not guarantee that Γ can be deformed to a nodal curve on $\mathscr{X}_\eta$; for that to happen, we do need the same restrictions on Γ as in [Che99].

We will make one more improvement over [Che99]. Instead of only considering $|\mathscr{L}|$, we will also consider the "twisted" linear series $|\mathscr{L} + mY_1|$ on $\mathscr{X}$. Note that $\mathscr{X}$ is smooth outside of x_j so Y_1 is a Cartier divisor on $\mathscr{X}^\circ = \mathscr{X} \backslash \{x_j\}$; the linear system $|\mathscr{L} + mY_1|$ is interpreted as $\mathbb{P}\mathrm{H}^0(\mathscr{X}^\circ, \mathscr{L} + mY_1)$. The restrictions of $\mathscr{L} + mY_1$ to the Y_i are

$$(\mathscr{L} + mY_1)|_{Y_1} = \mathscr{L}_1 - mD \quad \text{and} \quad (\mathscr{L} + mY_1)|_{Y_2} = \mathscr{L}_2 + mD\,,$$

respectively, where the $\mathscr{L}_i$ are the restrictions of $\mathscr{L}$ to Y_i for $i = 1, 2$. Although m can be chosen to be an arbitrary integer, we take $m \geqslant 0$ for simplicity.

The restriction of $\gamma \in \mathrm{H}^0(\mathscr{L} + mY_1)$ to Y consists of $\gamma_1 \in \mathrm{H}^0(\mathscr{L}_1 - mD)$ on Y_1 and $\gamma_2 \in \mathrm{H}^0(\mathscr{L}_2 + mD)$ on Y_2. Furthermore, the image of the restriction

$$\mathrm{H}^0(\mathscr{X}^\circ, \mathscr{L} + mY_1) \longrightarrow \mathrm{H}^0(Y_2\backslash\{x_i\}, \mathscr{L}_2 + mD) = \mathrm{H}^0(Y_2, \mathscr{L}_2 + mD)$$

is actually contained in the subspace

$$\mathrm{H}^0(\mathcal{O}_{Y_2}(\mathscr{L}_2 + mD) \otimes \mathcal{O}_{Y_2}(-mx_1 - mx_2 - \cdots - mx_s))\,,$$

where $\mathcal{O}_{Y_2}(-x_j)$ is the ideal sheaf of the point x_j and $\mathcal{O}_{Y_2}(-mx_j)$ is the mth symmetric product of $\mathcal{O}_{Y_2}(-x_j)$ for $j = 1, 2, \ldots, s$. That is,

$$\gamma_2 \in \mathrm{H}^0\Big(\mathcal{O}_{Y_2}\Big(\mathscr{L}_2 + mD - m\sum x_j\Big)\Big).$$

This is easy to see after we resolve the double points of $\mathscr{X}$ by blowing it up along Y_2. In summary, the restriction of $\mathrm{H}^0(\mathscr{L} + mY_1)$ to Y lies in the kernel, denoted by $\mathrm{H}^0(Y, \mathscr{L} + mY_1)$, of the map

$$\begin{array}{c} \mathrm{H}^0(\mathcal{O}_{Y_1}(\mathscr{L}_1 - mD)) \oplus \mathrm{H}^0(\mathcal{O}_{Y_2}(\mathscr{L}_2 + mD - m\sum x_j)) \\ \downarrow \\ \mathrm{H}^0(\mathcal{O}_D(\mathscr{L}_2 + mD - m\sum x_j)) = \mathrm{H}^0(\mathcal{O}_D(\mathscr{L}_1 - mD)) \end{array}$$

sending (γ_1, γ_2) to $\gamma_1 - \gamma_2$. We summarise the above discussion in the following theorem.

Theorem 2.5. *Let $\pi\colon \mathscr{X} \to B = \operatorname{Spec}\mathbb{C}[[t]]$ be a flat proper family of surfaces whose generic fibre $\mathscr{X}_\eta$ is a K3 surface and whose central fibre $\mathscr{X}_0 = Y = Y_1 \cup Y_2$ is the union of two smooth rational surfaces Y_i meeting transversely along a smooth anti-canonical curve D in Y_i for $i = 1, 2$. Suppose that $\mathscr{X}$ is smooth outside of the s distinct points $x_1, x_2, \ldots, x_s \in D$ satisfying (2.0.6). Let $\mathscr{L} \in \operatorname{Pic}(\mathscr{X})$, let $f\colon C \to Y$ be a stable map of genus g, and let m be a non-negative integer satisfying*

$$h^0(\mathscr{X}_\eta, \mathscr{L}) = h^0(Y, \mathscr{L} + mY_1)\,,$$
$$D \not\subset \Gamma = f_*C \in \mathbb{P}\mathrm{H}^0(Y, \mathscr{L} + mY_1)$$

and (2.0.7)–(2.0.9). Then after a finite base change, there exists a family of stable maps $\phi\colon \mathscr{C}/B \to \mathscr{X}/B$ such that $\phi_0 = f$ and $\phi_*\mathscr{C}_\eta$ is an integral curve of geometric genus g in $|\mathscr{L}|$ on $\mathscr{X}_\eta$.

If $f_G \circ \nu\colon \widehat{G} \to Y_i$ is an immersion for the normalisation $\nu\colon \widehat{G} \to G$ of every component $G \subset C$, that is,

$$\begin{aligned} &T_{\widehat{G}} \xhookrightarrow{(f_G\circ\nu)_*} f_G^* T_{Y_i} \quad \text{is injective for every irreducible component } G \subset C\,, \\ &\qquad f_G = f|_G\colon G \to Y_i \text{and normalisation } \nu\colon \widehat{G} \to G\,, \end{aligned} \tag{2.0.10}$$

then $\phi_\eta\colon \mathscr{C}_\eta \to \mathscr{X}_\eta$ is an immersion.

If in addition to (2.0.7)–(2.0.10), we assume that $f_* C_i$ has normal crossings on $Y_i \backslash D$ for $C_i = f^{-1}(Y_i)$ and $i = 1, 2$, $f(p_1) \neq f(p_2)$ for all $p_1 \neq p_2 \in C_\times$ and $f(G_1)$ and $f(G_2)$ meet transversely at $x_1, x_2, \ldots, x_s$ on Y_i for all pairs of distinct components G_j with $f(G_j) \subset Y_i$, then $\phi_* \mathscr{C}_\eta$ is nodal.

3. Nodal rational curves on generic K3 surfaces

In this section, we use degenerations of type II of K3 surfaces as considered in the previous section to construct nodal rational curves on *generic* surfaces inside moduli spaces of Λ-polarised K3 surfaces, generalising a result of the first-named author [Che99] to the higher-rank case (cf. [KLV21] for similar type II degenerations from higher-rank lattices).

Theorem 3.1. *Let Λ be a lattice of rank 2 with intersection matrix*

$$\begin{bmatrix} 2d & a \\ a & 2b \end{bmatrix} \tag{3.0.1}$$

for some $a, b \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$ satisfying $4bd - a^2 < 0$. Let M_Λ be the moduli space of Λ-polarised complex K3 surfaces and $L \in \Lambda$ such that L is big and nef on a general K3 surface $X \in M_\Lambda$. Then there exists an open and dense subset $U \subseteq M_\Lambda$ (with respect to the Zariski topology), depending on L, such that on every K3 surface $X \in U$, the complete linear series $|L|$ contains an integral nodal rational curve if one of the following holds:

A1. The determinant $\det(\Lambda)$ is even.

A2. We have $L = L_1 + L_2 + L_3$ for some $L_i \in \Lambda$ satisfying $LL_i > 0$ and $L_i^2 > 0$ for $i = 1, 2, 3$.

A3. We have $L = L_1 + L_2$ for some $L_i \in \Lambda$ satisfying $LL_i > 0$ for $i = 1, 2$, $L_1^2 > 0$, $L_2^2 = -2$, $L_1 \notin 2\Lambda$, $L_1 - L_2 \notin n\Lambda$ for all $n \in \mathbb{Z}$ and $n \geqslant 2$, and $L_1^2 + 2L_1L_2 \geqslant 18$.

Remark 3.2. Every even lattice of signature $(1, 1)$ (the signature is dictated by the Hodge index theorem) is of the form (3.0.1). For some special lattices of rank 2, namely where a is even and $b = 0$, the above is due to Lewis and the first-named author [CL13].

Of course, the existence of rational curves on general K3 surfaces of Picard rank 2 implies the same on general K3 surfaces of Picard rank 1. More precisely, Theorem 3.1 implies the existence of nodal rational curves in $|nL|$ on a general K3 surface with Picard lattice $\begin{bmatrix}2d\end{bmatrix}$ for all $n \in \mathbb{Z}^+$. This is due to the first-named author when $d \geqslant 2$; see [Che99]. Theorem 3.1 also resolves the case $d = 1$ in Picard rank 1.

Case A3 is a technical extension required for the main theorem of [CGL19] so can be ignored by the casual reader.

3.1 The proof of Theorem 3.1

We prove the theorem in three steps:

(1) First, we embed a rank 2 K3 lattice (3.0.1) into that of a K3 surface with many (-2)-curves: When $\det(\Lambda)$ is even, that is, a is even in (3.0.1), we embed Λ into a lattice with intersection matrix

$$\begin{bmatrix} 2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & -2 \end{bmatrix}_{(r+1)\times(r+1)} \tag{3.1.1}$$

for some $r \leqslant 6$. When $\det(\Lambda)$ is odd, that is, a is odd in (3.0.1), we embed Λ into a lattice with intersection matrix

$$\begin{bmatrix} 0 & 1 & & & \\ 1 & -2 & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & -2 \end{bmatrix}_{(r+1)\times(r+1)} \tag{3.1.2}$$

for some $r \leqslant 4$. The existence of the embedding is itself a purely arithmetic problem. However, for our purposes, we also require the embedding to have the additional property that the image of a "designated" ample divisor L remains (at least) big and nef, that is, preserves a given polarisation. This introduces some extra complexity.

(2) Second, we use the degeneration of K3 surfaces in Section 2 to show the existence of nodal rational curves in almost all big and nef linear systems on a general K3 surface with Picard lattice (3.1.1) or (3.1.2).

(3) Third and finally, we deform a K3 surface X_0 with Picard lattice (3.1.1) or (3.1.2) to K3 surfaces X_η with Picard lattice (3.0.1) such that a nodal rational curve on X_0 deforms to a nodal rational curve on X_η.

In summary, our argument involves two degenerations: the degeneration of K3 surfaces of Picard rank 2 to K3 surfaces with Picard lattices (3.1.1) or (3.1.2) and the degeneration of the latter to unions of rational surfaces.

In order to embed the lattice (3.0.1) to (3.1.1) or (3.1.2), we will make use of the classical result of Lagrange that every non-negative integer can be written as the square sum of four integers and that of Legendre that every non-negative integer not of the form $4^a(8b+7)$ can be written as the square sum of three integers. However, in order to obtain a primitive embedding, as we will see, we require these integers to be coprime. This is not possible in general, but we can choose these integers such that their greatest common divisor is a power of 2, a fact not in the standard formulation of these two theorems but implied by Dirichlet's proof of Legendre's theorem. So let us restate their theorems as follows.

THEOREM (Lagrange–Legendre–Dirichlet). *Every positive integer n not of the form $4^a(8b+7)$ for any $a, b \in \mathbb{N}$*[1] *can be written as*

$$n = m_1^2 + m_2^2 + m_3^2 \tag{3.1.3}$$

for some $m_1, m_2, m_3 \in \mathbb{N}$ with $\gcd(m_1, m_2, m_3) = 2^l$, where $l \in \mathbb{N}$ satisfies $4^l \mid n$ and $4^{l+1} \nmid n$. As a consequence, every positive integer n can be written as

$$n = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

for some $m_1, m_2, m_3, m_4 \in \mathbb{N}$ with $\gcd(m_1, m_2, m_3, m_4) = 2^l$, where $l \in \mathbb{N}$ satisfies $2^{2l+1} \mid n$ and $2^{2l+3} \nmid n$. Furthermore, every positive integer can be written as the square sum of five coprime integers.

Dirichlet's proof of Legendre's 3-square. Let us outline Dirichlet's proof. It is enough to prove (3.1.3) for $n \equiv 1, 2, 3, 5, 6 \pmod 8$. The key is to find a *ternary quadratic form*

$$F(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

such that A is a 3×3 positive definite symmetric integral matrix with $\det(A) = 1$ and $F(x,y,z) = n$ has an integral solution (x_0, y_0, z_0). If we can find such an A, then there exists a matrix $P \in \mathrm{SL}_3(\mathbb{Z})$ such that $A = P^T P$, and hence

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = P \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \tag{3.1.4}$$

is a solution of (3.1.3) with $\gcd(m_1, m_2, m_3) = \gcd(x_0, y_0, z_0)$. It turns out that we can choose

$$A = \begin{bmatrix} a_{11} & a_{12} & 1 \\ a_{12} & a_{22} & 0 \\ 1 & 0 & n \end{bmatrix}$$

with integers a_{ij} satisfying

$$a_{11} > 0\,, \quad d = a_{11}a_{22} - a_{12}^2 > 0 \quad \text{and} \quad a_{22} = dn - 1\,. \tag{3.1.5}$$

The corresponding $F(x,y,z) = n$ has an obvious solution $(x,y,z) = (0,0,1)$. So the m_i given by (3.1.4) are coprime, as required.

To find a_{ij} satisfying (3.1.5), we use the quadratic reciprocity law and Dirichlet's theorem on arithmetic progressions. We will skip this part of the proof. □

We start with the embedding of the lattice (3.0.1) into (3.1.1) when $\det(\Lambda)$ is even. In the following, by a primitive lattice embedding, we mean an injective lattice homomorphism with torsion-free cokernel.

LEMMA 3.3. *For every even lattice Λ of rank 2, even determinant and signature $(1,1)$, there exists a positive integer $r \leqslant 6$ such that Λ can be primitively embedded into a lattice Σ_r with intersection matrix* (3.1.1).

[1] We use $\mathbb{N}$ for the set of non-negative integers.

Proof. Such a lattice Λ has intersection matrix

$$\begin{bmatrix} 2a & 2m \\ 2m & 2b \end{bmatrix} \tag{3.1.6}$$

for some integers a, b, m satisfying $m^2 > ab$. We claim that there exists a basis of Λ such that $a \geqslant 0$, $m \geqslant 0$ and $b < 0$ in (3.1.6). It is easy to make $a \geqslant 0$ and $m \geqslant 0$. If $b < 0$, we are done. Otherwise, let us consider all bases of Λ whose intersection matrix (3.1.6) satisfies $a \geqslant 0$, $m \geqslant 0$ and $b \geqslant 0$. Let us choose a basis $\{F_1, F_2\}$ among these bases that minimises the trace $2(a+b)$ of the matrix (3.1.6). Without loss of generality, let us assume that $b \geqslant a \geqslant 0$. Since $m^2 > ab$, we have $m > a = F_1^2$. Then the intersection matrix of $\{F_1, F_2 - F_1\}$ is

$$\begin{bmatrix} 2a & 2(m-a) \\ 2(m-a) & 2a+2b-4m \end{bmatrix}$$

with $b > a + b - 2m$. By our choice of $\{F_1, F_2\}$, we must have $2a + 2b - 4m < 0$, which proves our claim.

Let us assume that Λ is generated by F_1 and F_2 such that

$$F_1^2 = 2a \geqslant 0\,, \quad F_1F_2 = 2m \geqslant 0 \quad \text{and} \quad F_2^2 = 2b < 0\,.$$

When $2 \nmid a$, we let

$$\sigma(F_1) = \frac{a+1}{2}(A - E_6) + E_6\,, \quad \sigma(F_2) = m(A - E_6) - \sum_{i=1}^{5} m_i E_i$$

with $-b = \sum_{i=1}^5 m_i^2$ and $\gcd(m_1, m_2, m_3, m_4, m_5) = 1$ be the embedding $\sigma\colon \Lambda \hookrightarrow \Sigma_6$, where $A, E_1, E_2, \ldots, E_r$ are the generators of Σ_r with intersection matrix (3.1.1).

When $2 \mid a$, we let

$$\sigma(F_1) = \frac{a+2}{2}(A - E_6) - E_1 + E_6\,, \quad \sigma(F_2) = (m + m_1)(A - E_6) - \sum_{i=1}^{5} m_i E_i$$

with $-b = \sum_{i=1}^5 m_i^2$ and $\gcd(m_1, m_2, m_3, m_4, m_5) = 1$. □

Remark 3.4. A K3 surface X with Picard lattice (3.1.1) can be realised as a double cover $\varphi\colon X \to S$, where S is a del Pezzo surface of degree $9-r$ and φ is ramified along a general curve in $|-2K_S|$.

We have the pullback map $\varphi^*\colon \operatorname{Pic}(S) \xrightarrow{\sim} \operatorname{Pic}(X)$ on the Picard groups such that $(\varphi^* L)^2 = 2L^2$ for all $L \in \operatorname{Pic}(S)$. Therefore, φ^* induces an isomorphism of nef cones of S and X. Recall that the effective cone of curves on S is generated by (-1)-curves for $2 \leqslant r \leqslant 8$. Correspondingly, the effective cone of curves on X is generated by (-2)-curves. It is also useful to us that there are lattice automorphisms $\sigma_{i_1 i_2 i_3}$ of $\operatorname{Pic}(X)$ given by

$$\begin{aligned} \sigma_{i_1 i_2 i_3}(A) &= 2A - E_{i_1} - E_{i_2} - E_{i_3}\,, \\ \sigma_{i_1 i_2 i_3}(E_{i_1}) &= A - E_{i_2} - E_{i_3}\,, \\ \sigma_{i_1 i_2 i_3}(E_{i_2}) &= A - E_{i_3} - E_{i_1}\,, \\ \sigma_{i_1 i_2 i_3}(E_{i_3}) &= A - E_{i_1} - E_{i_2}\,, \\ \sigma_{i_1 i_2 i_3}(E_i) &= E_i \quad \text{when } i \neq i_1, i_2, i_3 \end{aligned}$$

for $1 \leqslant i_1 < i_2 < i_3 \leqslant r$. These are induced by the Cremona (or quadratic) transformations of $\operatorname{Pic}(S)$. Together with the symmetric group acting on $\{E_i\}$, the $\sigma_{i_1 i_2 i_3}$ generate the subgroup

$$\operatorname{Aut}(\operatorname{Pic}(X))^+ \subset \operatorname{Aut}(\operatorname{Pic}(X))$$

that preserves the nef cone of X. The action of $\mathrm{Aut}(\mathrm{Pic}(X))^+$ on the set of (-2)-curves on X is transitive for $2 \leqslant r \leqslant 8$.

Although it is possible to construct many rational curves on X using this double cover [BHT11], it is very hard to construct rational curves of every ample class in this way. Namely, we cannot prove Theorem 3.1 in its full generality along this line of argument.

Remark 3.5. The bound $r \leqslant 6$ in Lemma 3.3 is optimal. For example, in the case $8 \mid a$, $b = 0$ and $4 \mid m$ in (3.1.6), a primitive embedding $\sigma\colon \Lambda \hookrightarrow \Sigma_r$ must be of the form

$$\sigma(F_1) = m_r(A - E_r) + mE_r - \sum_{i=1}^{r-1} m_i E_i\,, \quad \sigma(F_2) = A - E_r$$

with $2mm_r - m^2 - a = \sum_{i=1}^{r-1} m_i^2$ and $\gcd(m_1, m_2, \dots, m_{r-1}) = 1$ after composing σ with an action of $\mathrm{Aut}(\Sigma_r)$. Since $8 \mid \big(m^2 + 2mm_r - a\big)$, we have the inequality $\gcd(m_1, m_2, \dots, m_{r-1}) \neq 1$ if $r \leqslant 5$. So we need $r = 6$.

On the other hand, there are situations when we can embed Λ into Σ_5. For example, when $8 \nmid b$, we can always write $-b$ as the square sum of four coprime integers. So the construction in the proof works for $r = 5$ when $8 \nmid b$.

Next, let us embed the lattice (3.0.1) into (3.1.2) when $\det(\Lambda)$ is odd.

LEMMA 3.6. *Every even lattice Λ of rank 2, odd determinant and signature $(1,1)$ can be primitively embedded into a lattice Σ_r with intersection matrix (3.1.2) for some $r \leqslant 4$.*

Proof. As in the proof of Lemma 3.3, we can find a basis $\{F_1, F_2\}$ of Λ such that

$$F_1^2 = 2a \geqslant 0\,, \quad F_1F_2 = m > 0 \quad \text{and} \quad F_2^2 = 2b < 0$$

for some integers a, b, m with $2 \nmid m$. We can always find an $m_1 \in \mathbb{Z}^+$ such that $(m - bm_1)m_1 > a$ and $(m - bm_1)m_1 - a$ is not in the form of $4^\alpha(8\beta + 7)$ for any $\alpha, \beta \in \mathbb{N}$. Then we let

$$\sigma(F_1) = (m - bm_1)A + m_1(A + E_1) - \sum_{i=2}^{4} m_i E_i\,, \quad \sigma(F_2) = bA + (A + E_1)$$

with $(m - bm_1)m_1 - a = \sum_{i=2}^{4} m_i^2$ and $\gcd(m_2, m_3, m_4) = 2^l$, where $A, E_1, E_2, \dots, E_r$ are the generators of Σ_r with intersection matrix (3.1.2). □

Remark 3.7. Let X be a K3 surface with Picard lattice (3.1.2). We claim that the effective cone of X is generated by the (-2)-curves

$$E_i \text{ and } P_j = A - E_j \quad \text{for } 1 \leqslant i \leqslant r \text{ and } 2 \leqslant j \leqslant r \tag{3.1.7}$$

and $L = dA + m_1E_1 - \sum_{i=2}^{r} m_iE_i$ is nef if and only if $LE_i \geqslant 0$ and $LP_j \geqslant 0$ or, equivalently,

$$d \geqslant 2m_1 \geqslant 4m_j \geqslant 0 \quad \text{for } 2 \leqslant j \leqslant r \tag{3.1.8}$$

when $2 \leqslant r \leqslant 5$.

Clearly, all curves in (3.1.7) are (-2)-curves, and the inequalities in (3.1.8) are necessary for $L = dA + m_1E_1 - \sum m_iE_i$ to be nef. On the other hand, (3.1.8) guarantees that $L^2 \geqslant 0$. Therefore, X does not contain (-2)-curves other than those in (3.1.7), and the inequalities in (3.1.8) are also sufficient for L to be nef.

Neither of Lemmas 3.3 and 3.6 produces an embedding σ preserving a given polarisation L. It turns out that we can always compose an existing $\sigma\colon \Lambda \hookrightarrow \Sigma$ with a lattice automorphism $\alpha \in \mathrm{Aut}(\Sigma)$ such that $\alpha \circ \sigma(L)$ is big and nef.

Lemma 3.8. *Suppose that there exists a primitive embedding $\Lambda \hookrightarrow \Sigma$ between K3 Picard lattices Λ and Σ. Then for each $L \in \Lambda$ with $L^2 > 0$, there is a primitive embedding $\sigma\colon \Lambda \hookrightarrow \Sigma$ such that $\sigma(L)$ is big and nef on X, when Σ is identified with the Picard lattice of a projective K3 surface X. Moreover, fixing another class $C \in \Lambda$, we can choose σ such that $\sigma(NL - C)$ is big and nef on X for N sufficiently large.*

Proof. Fixing an ample divisor D on X, we consider all primitive embeddings $\sigma\colon \Lambda \hookrightarrow \Sigma$ satisfying $\sigma(L).D > 0$. We choose σ among these embeddings such that $\sigma(L).D$ achieves the minimum. So $\sigma(L)$ is pseudo-effective. By the Zariski decomposition, we can write

$$\sigma(L) = P + N\,,$$

where P is a nef $\mathbb{Q}$-divisor, N is a $\mathbb{Q}$-effective divisor whose components have negative self-intersection matrix and $PN = 0$. If $\sigma(L)$ is nef, that is, $\sigma(L) = P$, we are done. Otherwise, there exists an integral curve $R \subset X$ such that $\sigma(L).R < 0$. Then $R \subset \operatorname{supp}(N)$ is a (-2)-curve on X.

We let $\alpha\colon \Sigma \to \Sigma$ be the group homomorphism given by

$$\alpha(F) = F + (F.R)R$$

for $F \in \Sigma$. Note that $\alpha^2 = \mathrm{id}$ and $(\alpha(F))^2 = F^2$ for all $F \in \Sigma$. So α is a lattice automorphism.

Let $\widehat{\sigma} = \alpha \circ \sigma$. Since $\widehat{\sigma}(L).P = P^2 > 0$ and $L^2 > 0$, the divisor $\widehat{\sigma}(L)$ is big and hence $\widehat{\sigma}(L).D > 0$. On the other hand, since $\sigma(L).R < 0$,

$$\widehat{\sigma}(L).D = \sigma(L).D + (\sigma(L).R)RD < \sigma(L).D\,,$$

which contradicts our hypothesis that σ minimises $\sigma(L).D$. So $\sigma(L) = P$ is big and nef.

Let $E = \sigma(C)$, and let $R_1, R_2, \dots, R_l$ be the integral curves on X such that $PR_i = 0$ for $i = 1, 2, \dots, l$. Let us consider the set

$$\Pi = \big\{\xi\colon \Lambda \hookrightarrow \Sigma \text{ primitive embedding} \;\big|\; \xi(L) = P \text{ and } \xi(C) = E + m_1R_1 + m_2R_2 + \cdots + m_lR_l \text{ for some } m_i \in \mathbb{Z}\big\}\,.$$

Since $R_1, R_2, \dots, R_l$ have negative definite self-intersection, there are only finitely many $(m_1, m_2, \dots, m_l) \in \mathbb{Z}^l$ satisfying

$$(E + m_1R_1 + m_2R_2 + \cdots + m_lR_l)^2 = E^2\,.$$

Therefore, there exists a $\xi \in \Pi$ maximising $\xi(C).D$. We claim that $\xi(C).R_i \leqslant 0$. Otherwise, suppose that $\xi(C).R > 0$ for some $R = R_i$. Then $\widehat{\xi} = \alpha \circ \xi \in \Pi$ and

$$\widehat{\xi}(C).D = \xi(C).D + (\xi(C).R)RD > \xi(C).D\,,$$

which contradicts our choice of ξ. Therefore, $\xi(C).R > 0$ for all integral curves R with $PR = 0$. Replacing σ with ξ, we see that $\sigma(NL - C)$ is big and nef for N sufficiently large. $\square$

Now we have embedded the lattice (3.0.1) into (3.1.1) or (3.1.2). Next, we want to prove the existence of nodal rational curves on K3 surfaces with Picard lattices (3.1.1) and (3.1.2). Here we use the type II degeneration of K3 surfaces introduced in Section 2. It turns out that in order to produce rational curves on a general K3 surface, we need to construct rational curves on a log K3 surface with some tangency conditions. More precisely, we want to find rational curves on a del Pezzo surface satisfying some tangency conditions with a fixed anti-canonical curve.

Definition 3.9. For a Cartier divisor A on a projective surface X, we use the notation $V_{A,g}$ to denote the Severi variety of integral curves of geometric genus g in $|A|$. For a curve $D \subset X$ and a zero cycle $\alpha = m_1p_1 + m_2p_2 + \cdots + m_lp_l \in Z_0(D)$, we use the notation $V_{A,g,D,\alpha}$ to denote the subvariety of $V_{A,g}$ consisting of integral curves $C \in |A|$ of genus g with the properties that C meets D properly and that there exist $q_i \in \nu^{-1}(p_i)$ and $n_i \geqslant m_i$ such that $q_1, q_2, \ldots, q_l$ are distinct and $\nu^*D = n_iq_i$ when ν is restricted to the open neighbourhoods of p_i and q_i for $i = 1, 2, \ldots, l$, where $\nu\colon \widehat{C} \to X$ is the normalisation of C, $m_1, m_2, \ldots, m_l \in \mathbb{N}$ and $p_1, p_2, \ldots, p_l$ are points on D such that D is locally Cartier at each p_i.

The variety $V_{A,g,D,m_1p_1+m_2p_2+\cdots+m_lp_l}$ parametrises the curves of fixed tangencies with D. We can also define the subvariety of $V_{A,g}$ of curves of moving tangencies with D by letting some of the p_i move. For example, with $p_1, p_2, \ldots, p_s$ moving, these curves are parametrised by

$$\bigcup_{(p_1,p_2,\ldots,p_s)\in(D^s)^*} V_{A,g,D,m_1p_1+m_2p_2+\cdots+m_lp_l}\,,$$

where $(D^s)^*$ is the open set of $D^s = D^{\times s}$ of points $p_i \neq p_j$ for $1 \leqslant i < j \leqslant l$. In the following, we write $A \geqslant B$ or $B \leqslant A$ if $A - B$ is effective.

Theorem 3.10. *Let X be a smooth projective complex rational surface containing a smooth anti-canonical curve $D \in |-K_X|$. Let $A_1, A_2, \ldots, A_n$ be divisors on X such that*

- *$A_iD \geqslant 1$ and $A_1D \geqslant 2$,*
- *$(A_1 + A_2 + \cdots + A_j)A_{j+1} \geqslant 1$ and*
- *$V_{A_i,0} \neq \emptyset$*

for $i = 1, 2, \ldots, n$ and $j = 1, \ldots, n-1$. Then for $A = A_1 + A_2 + \cdots + A_n$, all distinct points $p_1, p_2, \ldots, p_l$ on D satisfying

$$\text{there does not exist an integral curve } B \subset X \text{ such that } A \geqslant B \text{ and } B \cap D \subset \{p_1, p_2, \ldots, p_l\} \tag{3.1.9}$$

and all $m_1, m_2, \ldots, m_l \in \mathbb{N}$ satisfying $m = \sum m_i \leqslant AD - 1$, there exists an effective divisor G on X such that $DG = 0$ and

$$V_{A-G,0,D,m_1p_1+m_2p_2+\cdots+m_lp_l} \neq \emptyset\,.$$

Moreover, for $V = V_{A-G,0,D,m_1p_1+m_2p_2+\cdots+m_lp_l}$ and a general member C in V, the following hold:

1. *We have $\dim V = AD - m - 1$.*
2. *If $m \leqslant AD - 2$, the normalisation $\nu\colon \widehat{C} \to X$ of C induces an injection $\nu_*\colon T_{\widehat{C}} \to \nu^*T_X$, and $(CD)_{p_i} = m_i$ for $i = 1, 2, \ldots, l$.*
3. *If $m = AD - 1$, item (2) holds for $p_1 \in D$ general.*
4. *If $m \leqslant AD - 3$, the curve C meets a fixed reduced curve $F \subset X$ transversely outside of $\{p_1, p_2, \ldots, p_l\}$.*
5. *If $m = AD - 3 + s$ for some $s \in \mathbb{N}$, item (4) holds for $p_1, p_2, \ldots, p_s \in D$ general and $m_1, m_2, \ldots, m_s \in \mathbb{Z}^+$.*
6. *If $m \leqslant AD - 4$, all singularities of C are of type $\mathbb{C}[[x, y]]/(x^a - y^a)$, that is, ordinary.*
7. *If $m = AD - 4 + s$ for some $s \in \mathbb{N}$, item (6) holds for $p_1, p_2, \ldots, p_s \in D$ general and $m_1, m_2, \ldots, m_s \in \mathbb{Z}^+$.*
8. *If $m \leqslant AD - 5$, the curve C is nodal.*

(9) *If* $m = AD - 5 + s$ *for some* $s \in \mathbb{N}$, *item* (8) *holds for* $p_1, p_2, \dots, p_s \in D$ *general and* $m_1, m_2, \dots, m_s \in \mathbb{Z}^+$.

Proof. By the standard deformation theory of curves on surfaces [HM98, § 3.B], the variety $V_{A_i,0}$ has the expected dimension $A_i D - 1$. Since $\dim V_{A_1,0} = A_1 D - 1 \geqslant 1$, a general member $C_1 \in V_{A_1,0}$ meets a fixed reduced curve $F \subset X$ transversely (see, for example, [Ded20, Theorem 1.4.1]). In particular, for a fixed $C_2 \in V_{A_2,0}$, the curve C_2 meets the normalisation of C_1 transversely. Let us consider $C = C_1 \cup C_2$ and the stable map $\nu\colon C^\nu \to X$ that normalises all singularities of C except one point among $C_1 \cap C_2$. The deformation space of the genus 0 stable map ν has dimension at least

$$-K_X(C_1 + C_2) + \dim X - 3 = D(A_1 + A_2) - 1\,,$$

which is simply the virtual dimension of the moduli space of rational stable maps to X. We give a quick proof of this fact in this case for lack of a precise reference. Suppose that $C^\nu = C_1^\nu \cup C_2^\nu$, where the C_i^ν are the normalisations of the C_i. Let $p = C_1^\nu \cap C_2^\nu$, and let $\phi\colon C^\nu \to X \times \mathbb{P}^1$ be the morphism whose restrictions to C_i^ν are the maps $\nu \times \phi_i$, where the $\phi_i\colon C_i^\nu \to \mathbb{P}^1$ are isomorphisms satisfying $\phi_1(p) = \phi_2(p)$. For a general choice of ϕ_1 and ϕ_2, the morphism ϕ is a closed embedding. So the deformation space of ϕ is the same as the embedded deformation space of $\Gamma = \phi(C^\nu)$ in $X \times \mathbb{P}^1$. Since Γ is a local complete intersection in $X \times \mathbb{P}^1$, its normal sheaf is locally free and its deformation space in $X \times \mathbb{P}^1$ has dimension at least

$$h^0\big((I_\Gamma/I_\Gamma^2)^\vee\big) - h^1\big((I_\Gamma/I_\Gamma^2)^\vee\big) = -K_{X\times\mathbb{P}^1}.\Gamma = -K_X(C_1 + C_2) + 4\,,$$

where I_Γ is the ideal sheaf of Γ in $X \times \mathbb{P}^1$ and the first equality comes from adjunction and the Riemann–Roch theorem. Among these deformations, those with $\pi_X \circ \phi$ constant have dimension 5, where $\pi_X\colon X \times \mathbb{P}^1 \to X$ is the projection to X. So we arrive at the above lower bound for $\dim \mathrm{Def}(\nu)$.

On the other hand, we have

$$\dim V_{A_1,0} + \dim V_{A_2,0} = (A_1 + A_2)D - 2\,.$$

Hence C deforms to an integral rational curve in $|A_1 + A_2|$. Consequently, $V_{A_1+A_2,0}$ is non-empty of the expected dimension $\dim V_{A_1+A_2,0} = (A_1 + A_2)D - 1$. We may continue to apply the same argument to $C_{12} \cup C_3$, where C_{12} is a general member of $V_{A_1+A_2,0}$ and $C_3 \in V_{A_3,0}$. Eventually, we conclude that $V_{A,0}$ is non-empty of the expected dimension $AD - 1$.

Next let us prove $V_{A-G,0,D,m_1p_1+m_2p_2+\cdots+m_lp_l} \neq \emptyset$ by induction on m. There is nothing to do when $m = 0$. Suppose that

$$V = V_{A-G,0,D,m_1p_1+m_2p_2+\cdots+m_lp_l} \neq \emptyset$$

for some $m \leqslant AD - 2$. It suffices to show that

$$V_{A-G-G',0,D,(m_1+1)p_1+m_2p_2+\cdots+m_lp_l} \neq \emptyset \tag{3.1.10}$$

for some $G' \geqslant 0$ and $DG' = 0$.

Note that V has the expected dimension $AD - m - 1$. Let $\overline{V}$ be the closure of V in $|A - G|$. Let $g\colon \Gamma \hookrightarrow \overline{V}$ be an integral projective curve passing through a general point of $V_{A-G,0,D,mp}$. After a finite base change, there exists a family $f\colon \mathscr{C} \to X$ of stable maps of genus 0 over Γ such that $f_*\mathscr{C}_b = g(b)$ for every point $b \in \Gamma$. We may also choose f such that $f^{-1}(D)$ is a union of sections over Γ and some "vertical" components. That is,

$$f^*D = m_1P_1 + m_2P_2 + \cdots + m_lP_l + n_1Q_1 + n_2Q_2 + \cdots + n_aQ_a + W\,,$$

where the P_i and Q_j are sections of $\pi\colon \mathscr{C} \to \Gamma$, $f(P_i) = p_i$ and $\pi_* W = 0$. Since all components of $\mathscr{C}_b$ are rational and D is a smooth elliptic curve, $f_* W = 0$.

On the other hand, every connected component of $f^{-1}(D)$ must dominate D. Since $f_* P_1 = 0$, the section P_1 must lie on the same connected component as some Q_j. Therefore, there exists a $W' \subset W$ such that $P_1 \cup W' \cup Q_j$ is connected. So P_1 and Q_j are joined by a chain of components contained in $f^{-1}(p_1) \cap \mathscr{C}_b$ for some point $b \in \Gamma$. In an open neighbourhood $U \subset \mathscr{C}_b$ of the connected component of $f^{-1}(p_1) \cap \mathscr{C}_b$ containing $P_1 \cap \mathscr{C}_b$, we have

$$(f_* U.D)_{p_1} \geqslant m_1 + 1\,.$$

Let us write

$$f_* \mathscr{C}_b = \mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_r C_r + G'\,, \tag{3.1.11}$$

where G' is supported on the components of $f_* \mathscr{C}_b$ that are disjoint from D and the C_j are the components satisfying $C_j \in V_{C_j,0,D,\alpha_j}$ for effective 0-cycles α_j on D satisfying

$$\begin{aligned}
&\operatorname{supp} \alpha_j \subset \{p_1, p_2, \ldots, p_l\} \quad \text{and} \\
&\sum_{j=1}^{r} \mu_j \alpha_j \geqslant (m_1 + 1) p_1 + m_2 p_2 + \cdots + m_l p_l\,.
\end{aligned}$$

Due to our choice of $p_1, p_2, \ldots, p_l$ in (3.1.9), the points $p_1, p_2, \ldots, p_l$ cannot be the only intersections between C_j and D. Therefore, $\deg \alpha_j \leqslant C_j D - 1$.

Since Γ is an arbitrary curve in $\overline{V}$ passing through a general point of V, the above argument shows that there is a codimension 1 subvariety Z of $\overline{V}$ parametrising the curves (3.1.11):

$$\begin{aligned}
Z = \Big\{ &\mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_r C_r + G' \in \overline{V} \colon DG' = 0, \\
&C_j \in V_{C_j,0,D,\alpha_j},\ \deg \alpha_j \leqslant C_j D - 1 \text{ for } j = 1, 2, \ldots, r, \\
&\sum_{j=1}^{r} \mu_j \alpha_j \geqslant (m_1 + 1) p_1 + m_2 p_2 + \cdots + m_l p_l \Big\}, \\
\dim Z = \dim &V - 1 = AD - m - 2\,.
\end{aligned}$$

Since $\deg \alpha_j \leqslant C_j D - 1$, we obtain $\dim V_{C_j,0,D,\alpha_j} \leqslant C_j D - \deg \alpha_j - 1$. There are at most countably many rational curves that are disjoint from D, so G' is rigid. Therefore,

$$\begin{aligned}
AD - m - 2 = \dim Z &\leqslant \sum_{j=1}^{r} \dim V_{C_j,0,D,\alpha_j} \\
&\leqslant \sum_{j=1}^{r} \mu_j \dim V_{C_j,0,D,\alpha_j} = \sum_{j=1}^{r} \mu_j (C_j D - \deg \alpha_j - 1) \\
&= AD - \sum_{j=1}^{r} \mu_j \deg \alpha_j - \sum_{j=1}^{r} \mu_j \leqslant AD - m - 1 - \sum_{j=1}^{r} \mu_j\,.
\end{aligned}$$

Then we must have $\sum \mu_j = 1$. That is, $\sum \mu_j C_j$ contains only one component $C = C_1$ with multiplicity one. Clearly,

$$C \in V_{A-G-G',0,D,(m_1+1)p_1 + m_2 p_2 + \cdots + m_l p_l}\,,$$

and (3.1.10) follows. Once we have $V_{A-G,0,D,m_1 p_1 + m_2 p_2 + \cdots + m_l p_l} \neq \emptyset$, all the other statements follow from the standard deformation theory of curves on surfaces [HM98, § 3.B]. □

Corollary 3.11. *Let X be a complex del Pezzo surface and $D \in |-K_X|$ be a smooth anti-canonical curve on X. Then for all big and nef divisors A on X, all points $p_1, p_2, \ldots, p_l \in D$ satisfying (3.1.9) and all $m_1, m_2, \ldots, m_l \in \mathbb{N}$ satisfying $m = \sum m_i \leqslant AD - 1$,*

$$V_{A,0,D,m_1p_1+m_2p_2+\cdots+m_lp_l} \neq \emptyset\,.$$

Proof. By Theorem 3.10, it suffices to show that $V_{A,0} \neq \emptyset$. Note that D is ample and there does not exist a $G \geqslant 0$ with $G \neq 0$ and $DG = 0$.

It should be a well-known fact that every big and nef complete linear series on a del Pezzo surface contains an integral rational curve, but we include a simple argument proving this. It suffices to write $A = A_1 + A_2 + \cdots + A_n$ such that $A_iD \geqslant 1$, $A_1D \geqslant 2$ and $V_{A_i,0} \neq \emptyset$ as in Theorem 3.10.

We may assume that A is ample. Otherwise, we simply blow down the (-1)-curves disjoint from A to obtain $f\colon X \to Y$. Then f_*A is ample on Y and $A = f^*(f_*A)$.

Let $\Lambda, E_1, E_2, \ldots, E_r$ be the effective divisors generating $\operatorname{Pic}(X)$ with $\Lambda^2 = 1$, $\Lambda E_i = 0$ and $E_i^2 = -1$ for $i = 1, 2, \ldots, r$. When $r = 0$, we have $A = d\Lambda$ for some $d \in \mathbb{Z}^+$ with $\Lambda D = 3$ and $V_{\Lambda,0} \neq \emptyset$, obviously. So $V_{A,0} \neq \emptyset$. When $r = 1$, we have $A = d\Lambda + m(\Lambda - E_1)$ for some $d, m \in \mathbb{Z}^+$ with $V_{\Lambda,0} \neq \emptyset$ and $V_{\Lambda-E_1,0} \neq \emptyset$. So $V_{A,0} \neq \emptyset$. When $r \geqslant 2$, the effective cone of curves of X is generated by (-1)-curves on X. Therefore, there exists an $m \in \mathbb{Z}^+$ such that $A - mD = G$ is nef while $A - (m+1)D$ is not. It is not hard to see that $V_{D,0} \neq \emptyset$.

If $G^2 > 0$, then by our choice of m, we have $GR = 0$ for some (-1)-curve R. We can blow down the (-1)-curves disjoint from G to obtain an $f\colon X \to Y$ such that f_*G is ample and $G = f^*(f_*G)$. So by induction on $\operatorname{rank}\operatorname{Pic}(X)$, we conclude that $V_{G,0} \neq \emptyset$. Obviously, $GD \geqslant 2$ by the Hodge index theorem. Therefore, $V_{A,0} \neq \emptyset$.

If $G \neq 0$ and $G^2 = 0$, then $G = aF$ for some indivisible $F \in \operatorname{Pic}(X)$ with F base-point-free and $F^2 = 0$. It is not hard to see that $V_{F,0} \neq \emptyset$ and $FD = 2$. Therefore, $V_{A,0} \neq \emptyset$.

If $G = 0$ and $D^2 \geqslant 2$, we can again derive $V_{A,0} \neq \emptyset$ from $V_{D,0} \neq \emptyset$. So the only case left is that $A = mD$ and $D^2 = 1$, that is, $r = 8$ and $A = -mK_X$ for some $m \geqslant 2$. In this case, we can write

$$2D = \underbrace{\left(3\Lambda - 2E_1 - \sum_{i=2}^{7} E_i\right)}_{R_1} + \underbrace{\left(3\Lambda - \sum_{i=2}^{7} E_i - 2E_8\right)}_{R_2},$$

where R_1 and R_2 are (-1)-curves with $R_1R_2 = 3$. By deforming the union $R_1 \cup R_2$, we conclude that $V_{2D,0} \neq \emptyset$. Thus, $V_{A,0} \neq \emptyset$. □

The appearance of the effective divisor G in Theorem 3.10, which we will call a (-2)-*tail*, is quite inconvenient for us. In the case that X is a del Pezzo surface, we automatically have $G = 0$ due to the ampleness of $-K_X$. However, we also need to apply the theorem to the case that $-K_X$ is big and nef. Namely, we need to work with singular del Pezzo surfaces. In this case, every connected component of G, if non-zero, is supported on a tree of smooth (-2)-curves. Fortunately, the following proposition guarantees that (-2)-tails do not appear in the flat limits of integral rational curves.

Proposition 3.12. *Let $\mathscr{X}$ be a smooth proper family of surfaces over $B = \operatorname{Spec}\mathbb{C}[[t]]$ and $f\colon \mathscr{C}/B \to \mathscr{X}/B$ be a family of stable maps over B such that*

- *the geometric generic fibre $\overline{\mathscr{C}}_\eta$ of $\mathscr{C}/B$ is connected and smooth, and f maps $\mathscr{C}$ birationally onto its image;*

- *the image of the central fibre of $\mathscr{C}_0$ of $\mathscr{C}/B$ under f is*
$$f_*\mathscr{C}_0 = C_0 + m_1C_1 + m_2C_2 + \cdots + m_rC_r\,,$$
where $m_i \in \mathbb{N}$, $C_1, C_2, \ldots, C_r$ are smooth rational curves satisfying $C_i^2 \leqslant -2$ and $C_1 + C_2 + \cdots + C_r$ has simple normal crossings and C_0 is a (possibly reducible and non-reduced) curve meeting $C_1 + C_2 + \cdots + C_r$ transversely on $X = \mathscr{X}_0$;
- *each curve C_i deforms in the family $\mathscr{X}/B$ for $i = 0, 1, \ldots, r$;*
- *$\mathscr{C}_0 - f^{-1}(C_1 \cup C_2 \cup \cdots \cup C_r)$ and $\mathscr{C}_0$ have the same arithmetic genus.*

Then $m_1 = m_2 = \cdots = m_r = 0$.

Proof. Let $M = \mathscr{C}_0 - f^{-1}(C_1 \cup C_2 \cup \cdots \cup C_r)$. The fact that M and $\mathscr{C}_0$ have the same arithmetic genus is equivalent to the fact that every connected component T of $f^{-1}(C_1 \cup C_2 \cup \cdots \cup C_r)$ is a tree of smooth rational curves and $TM = 1$.

Suppose that at least one of the m_i is positive. We will construct a (possibly infinite) sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_n, \ldots$ such that

- $\Gamma_0 = M$ and each Γ_i is either M or an irreducible component of $\mathscr{C}_0$ dominating one of $C_1, C_2, \ldots, C_r$;
- for each $i \in \mathbb{N}$, we have $\Gamma_i \neq \Gamma_{i+1}$, and there exist a point $p \in C_1 \cup C_2 \cup \cdots \cup C_r$ and a connected component T_i of $f^{-1}(p)$ satisfying $T_i \cap \Gamma_i \neq \emptyset$ and $T_i \cap \Gamma_{i+1} \neq \emptyset$;
- $T_i \neq T_{i+1}$ for all $i \in \mathbb{N}$;
- the sequence terminates at $n \geqslant 2$ if and only if $\Gamma_n = M$.

Once we have such a sequence, we must have $\Gamma_i = \Gamma_j$ for some $j - i \geqslant 2$. Then it is easy to see that the dual graph of $\mathscr{C}_0$ contains a path $G_1G_2 \ldots G_m$ such that $G_1, G_2, \ldots, G_m$ are distinct components of $\mathscr{C}_0$ for some $m \geqslant 2$, $G_i \subset \mathscr{C}_0 - M$ for $1 < i < m$, $G_j \cap G_{j+1} \neq \emptyset$ for $j = 1, 2, \ldots, m-1$ and either $G_1G_2 \ldots G_m$ is a circuit, or G_1 and G_m are two distinct components of M. Either way, this contradicts the hypothesis that M and $\mathscr{C}_0$ have the same arithmetic genus. So it suffices to produce the sequence $\{\Gamma_i\}$ with the above properties.

Let $C = C_a$ for some $1 \leqslant a \leqslant r$. Let $\mathscr{X}^{[1]}$ be the blowup of $\mathscr{X}$ along C. The central fibre of $\mathscr{X}_0^{[1]}$ of $\mathscr{X}^{[1]}$ over B is the union of the proper transform of X, which we still denote by X, and the exceptional divisor R_1 meeting transversely along $X \cap R_1 = D_1$. One of the key hypotheses is that C deforms in the family $\mathscr{X}/B$. So the normal bundle of C in $\mathscr{X}$ splits as
$$\mathscr{N}_{C/\mathscr{X}} = \mathscr{N}_{X/\mathscr{X}}\Big|_C \oplus \mathscr{N}_{C/X} = \mathcal{O}_C \oplus \mathcal{O}_C(-n_a)\,,$$
where $C^2 = C_a^2 = -n_a \leqslant -2$. Consequently, $R_1 \cong \mathbb{F}_{n_a}$ for $n_a \geqslant 2$. And since $D_1^2 = n_a$ on R_1, the divisor R_1 contains a section D_2 over C with $D_2^2 = -n_a$ and $D_1 \cap D_2 = \emptyset$.

We continue to blow up $\mathscr{X}^{[1]}$ along D_2 to obtain $\mathscr{X}^{[2]}$. Then the central fibre $\mathscr{X}_0^{[2]}$ of $\mathscr{X}^{[2]}/B$ is the union $X \cup R_1 \cup R_2$, where X is the proper transform of $X \subset \mathscr{X}$, the divisor R_1 is the proper transform of $R_1 \subset \mathscr{X}^{[1]}$, and R_2 is the exceptional divisor, X and R_1 meet transversely along $D_1 = X \cap R_1$, the divisors R_1 and R_2 meet transversely along $D_2 = R_1 \cap R_2$ and $X \cap R_2 = \emptyset$. Here we again abuse the notation by using X, R_i, D_j for the subvarieties of all $\mathscr{X}^{[k]}$. Again, we have $R_2 \cong \mathbb{F}_{n_a}$ and a section D_3 of R_2/C with $D_3^2 = -n_a$ on R_2 and $D_2 \cap D_3 = \emptyset$. We may continue to blow up $\mathscr{X}^{[2]}$ along D_3 to obtain $\mathscr{X}^{[3]}$. So we have a sequence of blowups
$$\mathscr{X} = \mathscr{X}^{[0]} \longleftarrow \mathscr{X}^{[1]} \longleftarrow \mathscr{X}^{[2]} \longleftarrow \cdots \longleftarrow \mathscr{X}^{[l]}\,, \tag{3.1.12}$$

where $\mathscr{X}_0^{[l]} = X \cup R_1 \cup R_2 \cup \cdots \cup R_l$, such that

- X is the proper transform of $X = \mathscr{X}_0$;
- $R_i \cong \mathbb{F}_{n_a}$ for $i = 1, 2, \ldots, l$;
- $R_i \cap R_j = \emptyset$ for $0 \leqslant i < j - 1 \leqslant l - 1$ and $R_0 = X$;
- R_{i-1} and R_i meet transversely along $D_i = R_{i-1} \cap R_i$ and $D_i^2 = -n_a$ on R_{i-1} and $D_i^2 = n_a$ on R_i for $i = 1, 2, \ldots, l$.

Over a general point $q \in C$, the map $f\colon \mathscr{C}_0 \to X$ is finite and unramified onto its image if $C \subset f(\mathscr{C})$. Therefore, the proper transform of $f(\mathscr{C})$ under $\mathscr{X}^{[l]} \to \mathscr{X}$ does not contain D_i for $i = 1, 2, \ldots, l$. And since $f(\mathscr{C})$ is irreducible and $C_0 \neq C$, for l large enough, the proper transform of $f(\mathscr{C})$ does not contain D_{l+1} either, where D_{l+1} is the section of R_l/C with $D_{l+1}^2 = -n_a$. Let us choose l with this property and also lift $f\colon \mathscr{C} \to \mathscr{X}$ to a family $\widehat{f}\colon \mathscr{C} \to \mathscr{X}^{[l]}$ of stable maps with the diagram

$$\begin{array}{ccc} \widehat{\mathscr{C}} & \xrightarrow{\widehat{f}} & \mathscr{X}^{[l]} \\ {\scriptstyle\varphi}\downarrow & & \downarrow \\ \mathscr{C} & \xrightarrow{f} & \mathscr{X} \end{array}$$

after a base change.

For every component Γ of $\mathscr{C}_0$ that dominates C via f, our choice of l implies that $\widehat{f}\big(\widehat{\Gamma}\big)$ lies in R_i for some $1 \leqslant i \leqslant l$ and $D_i, D_{i+1} \not\subset \widehat{f}\big(\widehat{\Gamma}\big)$, where $\widehat{\Gamma} \subset \widehat{\mathscr{C}_0}$ is the proper transform of Γ under φ. Let us define two things using $\widehat{f}$:

(1) We define a partial order among the components of $\mathscr{C}_0$ that dominates C via f. Let Γ and Γ' be two components of $\mathscr{C}_0$ dominating C. Let $\widehat{\Gamma}$ and $\widehat{\Gamma}' \subset \widehat{\mathscr{C}_0}$ be their proper transforms under φ. Suppose that $\widehat{f}\big(\widehat{\Gamma}\big) \subset R_i$ and $\widehat{f}\big(\widehat{\Gamma}'\big) \subset R_j$ for some $1 \leqslant i, j \leqslant l$. We say that $\Gamma \prec \Gamma'$ or $\Gamma' \succ \Gamma$ if $i < j$ and $\Gamma \not\prec \Gamma'$ or $\Gamma' \not\succ \Gamma$ if $i \geqslant j$.

(2) Let Γ be a component of $\mathscr{C}_0$ that dominates C via f and $\widehat{\Gamma} \subset \widehat{\mathscr{C}_0}$ be its proper transform. Suppose that $\widehat{f}\big(\widehat{\Gamma}\big) \subset R_i$ for some $1 \leqslant i \leqslant l$. We define ξ_Γ to be the effective 0-cycle on Γ given by

$$\xi_\Gamma = \varphi_*\big(\big(\widehat{f}^* R_{i-1}\big).\widehat{\Gamma}\big)\,.$$

Note that $\widehat{f}\big(\widehat{\Gamma}\big)$ is an integral curve on $R_i \cong \mathbb{F}_{n_a}$ meeting D_{i-1} and D_i properly. Therefore, we have

$$\deg \xi_\Gamma = \big(\widehat{f}_*\widehat{\Gamma}\big).R_{i-1} \geqslant n_a \deg_\Gamma(f) \geqslant 2 \deg_\Gamma(f)\,, \tag{3.1.13}$$

where $\deg_\Gamma(f)$ is the degree of the map $f\colon \Gamma \to C$.

One of our basic tools is the following observation:

($*$) Let $V \subset \mathscr{X}^{[l]}$ be an étale/analytic/formal open neighbourhood of a point $p \in D_i = R_{i-1} \cap R_i$ for some $1 \leqslant i \leqslant l$ such that

$$V \cong \mathbb{C}[[x, y, z, t]]/\big(xy - t^m\big)\,.$$

Let $U \subset \widehat{\mathscr{C}}$ be a connected component of $\widehat{f}^{-1}(V)$. We write

$$U_0 = W_{i-1} + W_i$$

with $\widehat{f}(W_{i-1}) \subset R_{i-1}$ and $\widehat{f}(W_i) \subset R_i$. Then

$$\widehat{f}_* W_{i-1}.R_i = \widehat{f}_* W_i.R_{i-1}\,.$$

We will construct the sequence $\{\Gamma_i\}$ inductively such that for each $i \in \mathbb{Z}^+$, either $\Gamma_i = M$, or Γ_i dominates some C_a via f and

$$\operatorname{supp}(\xi_{\Gamma_i}) \not\subset T_{i-1}\,. \tag{3.1.14}$$

We have $\Gamma_0 = M$. Let us first find Γ_1. Since $\mathscr{C}_0$ is connected, there exist a point $p \in C_0 \cap C_a$ for some $1 \leqslant a \leqslant r$, a connected component T_0 of $f^{-1}(p)$ and a component Γ_1 of $\mathscr{C}_0$ dominating C_a such that $T_0 \cap \Gamma_0 \neq \emptyset$ and $T_0 \cap \Gamma_1 \neq \emptyset$. Since C_0 meets C_a transversely at p, we must have

$$v_q(\xi_{\Gamma_1}) = 1$$

by $(*)$, where $q = T_0 \cap \Gamma_1$ and $v_q(\xi_{\Gamma_1})$ is the multiplicity of q in the 0-cycle ξ_{Γ_1}. By (3.1.13), the support $\operatorname{supp}(\xi_{\Gamma_1})$ contains at least another point $q' \neq q$. So (3.1.14) holds for $i = 1$. We have found Γ_1 with the required property.

Suppose that we have found Γ_i. If $\Gamma_i = M$, the sequence terminates and we are done. Suppose that Γ_i dominates C_a for some $1 \leqslant a \leqslant r$. By (3.1.14), there is a point $q \in \operatorname{supp}(\xi_{\Gamma_i})$ such that $q \notin T_{i-1}$. Let T_i be the connected component of $f^{-1}(f(q))$ such that $q = T_i \cap \Gamma_i$. There are three cases:

(1) We have $M \cap T_i \neq \emptyset$. In this case, we simply let $\Gamma_{i+1} = M$.

(2) There is a component Γ of $\mathscr{C}_0$ dominating C_a such that $\Gamma \cap T_i \neq \emptyset$ and $\Gamma \prec \Gamma_i$. Then we let $\Gamma_{i+1} = \Gamma$ since we have $\operatorname{supp}(\xi_\Gamma) \not\subset T_i$ by $(*)$.

(3) Cases (1) and (2) both fail. By $(*)$, there must be a component G of $\mathscr{C}_0$ dominating C_b for some $1 \leqslant b \neq a \leqslant r$ such that $G \cap T_i \neq \emptyset$. This case requires more effort.

Now let us deal with case (3). Since cases (1) and (2) both fail, $M \cap T_i = \emptyset$ and for all components $\Gamma \neq \Gamma_i$ of $\mathscr{C}_0$ dominating C_a and satisfying $\Gamma \cap T_i \neq \emptyset$, we have $\Gamma \not\prec \Gamma_i$.

Let P be the union of the components Γ of $\mathscr{C}_0$ dominating C_a and satisfying $\Gamma \cap T_i \neq \emptyset$, and let Q be the union of the components G of $\mathscr{C}_0$ dominating C_b and satisfying $G \cap T_i \neq \emptyset$. We let U be an étale open neighbourhood of T_i in $\mathscr{C}$ and let f_U be the restriction of f to U. Then by $(*)$, we have

$$\deg_P(f_U) \leqslant \deg_Q(f_U)\,, \tag{3.1.15}$$

where $\deg_P(f_U)$ and $\deg_Q(f_U)$ are the degrees of the maps

$$f\colon P \cap U \to C_a \cap f(U) \quad \text{and} \quad f\colon\; Q \cap U \to C_b \cap f(U)\,,$$

respectively. We claim that there exists at least one component $G \subset Q$ such that

$$\operatorname{supp}(\xi_G) \not\subset T_i\,. \tag{3.1.16}$$

Otherwise, suppose that $\operatorname{supp}(\xi_G) \subset T_i$ for all components $G \subset Q$. And since G and T_i meet at a unique point s, this implies that $\operatorname{supp}(\xi_G)$ consists of the single point s, the map $f\colon G \to C_b$ is totally ramified at s and

$$v_s(\xi_G) = \deg \xi_G \geqslant 2 \deg_G(f) \tag{3.1.17}$$

by (3.1.13).

Then by (3.1.17) and by applying $(*)$ to the blowup sequence (3.1.12) over $C = C_b$, we conclude that

$$\deg_P(f_U) = \sum_{\substack{G \subset Q\\ s = G \cap T_i}} v_s(\xi_G) \geqslant 2 \deg_Q(f) = 2 \deg_Q(f_U)\,, \tag{3.1.18}$$

where $\deg_Q(f) = \deg_Q(f_U)$ since the map $f\colon G \to C_b$ is totally ramified at $G \cap T_i$ for all components $G \subset Q$. Clearly, (3.1.15) and (3.1.18) contradict each other. This proves (3.1.16) for some component $G \subset Q$. So it suffices to take $\Gamma_{i+1} = G$. □

Corollary 3.13. *Under the hypotheses of Theorem 3.10, we further assume that $DP > 0$ for all nef and effective divisors $P \not\supset D$. Then Theorem 3.10 holds for $G = 0$.*

Proof. Let Σ be the union of all rational curves $R \subset X$ such that $DR = 0$. We claim that Σ is a union of smooth rational curves with negative definite self-intersection matrix.

Suppose that $R_1, R_2, \ldots, R_n \subset \Sigma$ are rational curves whose self-intersection matrix is not negative definite. We may choose $\{R_1, R_2, \ldots, R_n\}$ such that every proper subset of $\{R_1, R_2, \ldots, R_n\}$ has negative definite self-intersection matrix. Since the self-intersection matrix of $\{R_1, R_2, \ldots, R_n\}$ is not negative definite, we can find $c_1, c_2, \ldots, c_n \in \mathbb{Z}$ not all zero such that $(c_1R_1 + c_2R_2 + \cdots + c_nR_n)^2 \geqslant 0$. We may choose c_i such that at least one of the c_i is positive. Let us write

$$c_1R_1 + c_2R_2 + \cdots + c_nR_n = \underbrace{\sum_{c_i>0} c_iR_i}_{A} - \underbrace{\sum_{c_i\leqslant 0}(-c_i)R_i}_{B} .$$

We claim that $B = 0$; otherwise, $A^2 < 0$, $B^2 < 0$ and $AB \geqslant 0$ by our hypothesis on R_i and hence $(A - B)^2 < 0$. Therefore, $B = 0$ and $c_1, c_2, \ldots, c_n > 0$. In other words, there exists an effective divisor $A = \sum c_iR_i$ supported on $R_1 + R_2 + \cdots + R_n$ such that $A^2 \geqslant 0$. Let us choose A such that $B^2 < 0$ for all $0 < B < A$. Clearly, A is nef; otherwise, $AR_i \leqslant -1$ for some i and then

$$(A - R_i)^2 = A^2 - 2AR_i + R_i^2 \geqslant A^2 + 2 - 2 = A^2 \geqslant 0 .$$

So A is nef and hence $DA > 0$, which gives a contradiction.

In conclusion, all subsets $\{R_1, R_2, \ldots, R_n\} \subset \Sigma$ have negative definite self-intersection matrices. This actually implies that Σ is a union of finitely many smooth rational (-2)-curves with simple normal crossings.

In the proof of Theorem 3.10, the support of G' in (3.1.11) is contained in Σ. Hence $G' = 0$ by Proposition 3.12. Thus, Theorem 3.10 holds for $G = 0$. □

Theorem 3.14. *For a general complex K3 surface X with Picard lattice (3.1.1), $r \leqslant 8$ and a big and nef divisor L on X, there exists an integral rational curve $C \in |L|$ such that the normalisation $\nu\colon \widehat{C} \to X$ of C induces an injection $\nu_*\colon T_{\widehat{C}} \to \nu^*T_X$. In addition, if $r \leqslant 6$, then C can be chosen to be nodal.*

Proof. We consider a type II degeneration $Y = Y_1 \cup Y_2$, where the Y_i are two del Pezzo surfaces whose Picard groups are generated by effective divisors $A_i, E_{i1}, E_{i2}, \ldots, E_{ir}$ with intersection matrix

$$\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}_{(r+1)\times(r+1)}$$

for $i = 1, 2$ and Y_1 and Y_2 meet transversely along a smooth anti-canonical curve $D = Y_1 \cap Y_2$. Let $\imath_i\colon D \hookrightarrow Y_i$ be the inclusion. We further require

$$\imath_1^*(A_1) = \imath_2^*(A_2) \quad \text{and} \quad \imath_1^*E_{1j} = \imath_2^*E_{2j} \tag{3.1.19}$$

in $\operatorname{Pic}(D)$ for $j = 1, 2, \ldots, r$.

For a general choice of such a Y, the relations from (3.1.19) are the only relations among $\imath_i^* A_i$ and $\imath_i^* E_{ij}$ in $\operatorname{Pic}(D)$. If these are satisfied, then $\operatorname{Pic}(Y)$ is freely generated by A and E_j, for $j = 1, 2, \ldots, r$, whose restrictions to Y_i are A_i and E_{ij}, respectively. By Theorem (2.3), the surface Y can be deformed to a K3 surface with Picard lattice (3.1.1). Clearly, the E_j deform to disjoint (-2)-curves, and A deforms to a big and nef divisor orthogonal to E_j correspondingly.

Let $\pi\colon \mathscr{X} \to \operatorname{Spec}\mathbb{C}[[t]]$ be such a family with $\mathscr{X}_0 = Y$. Now we use $A, E_1, E_2, \ldots, E_r$ to denote the effective divisors on $\mathscr{X}$ whose restrictions to Y_i are $A_i, E_{i1}, E_{i2}, \ldots, E_{ir}$, respectively, for $i = 1, 2$. Meanwhile, the big and nef divisor L on the generic fibre $\mathscr{X}_\eta$ extends to a divisor, which we still denote by L, on $\mathscr{X}$. We let L_i be the restriction of L to Y_i for $i = 1, 2$.

Clearly, the 3-fold $\mathscr{X}$ has $18 - 2r$ rational double points x_1, x_2, $\ldots$, x_{18-2r} on D satisfying

$$\begin{aligned}\mathscr{O}_D(x_1 + x_2 + \cdots + x_{18-2r}) &= \mathscr{N}_{D/Y_1} \otimes \mathscr{N}_{D/Y_2} \\ &= \mathscr{O}_D(-K_{Y_1}) \otimes \mathscr{O}_D(-K_{Y_2}) \\ &= \mathscr{O}_D(6A - 2E_1 - 2E_2 - \cdots - 2E_r)\,,\end{aligned} \tag{3.1.20}$$

which is the only relation among $x_1, x_2, \ldots, x_{18-2r}$ for a general choice of $\mathscr{X}$.

To find a rational curve in $|L|$ on the generic fibre $\mathscr{X}_\eta$ of $\mathscr{X}$, it suffices to locate a "limiting rational curve" Γ in $|L|$ on $\mathscr{X}_0$.

Suppose that $LD \geqslant 2$. By Corollary 3.11, we have $V_{L_i,0,D,mp} \neq \emptyset$ for $p \in D$ general and $m = LD - 1$. And since x_1 is a general point on D, we can find rational curves $\Gamma_i \in |L_i|$ for $i = 1, 2$ such that $\Gamma_i.D = x_1 + mp$ on Y_i, that Γ_i is smooth at p and that the normalisation $\nu\colon \widehat{\Gamma}_i \to Y_i$ of Γ_i induces an injection $\nu_*\colon T_{\widehat{\Gamma}_i} \to \nu^* T_{Y_i}$; that is, ν is an immersion.

By Theorem 2.5, the union $\Gamma = \Gamma_1 \cup \Gamma_2$ can be deformed to a rational curve $\mathscr{C}_\eta$ on the generic fibre $\mathscr{X}_\eta$ of $\mathscr{X}$ after a finite base change. The normalisation $\nu\colon \widehat{\mathscr{C}_\eta} \to \mathscr{X}_\eta$ of $\mathscr{C}_\eta$ is an immersion since the same holds for Γ_i, the point $x_1 \in \Gamma_1 \cap \Gamma_2$ deforms to a node, and the point $p \in \Gamma_1 \cap \Gamma_2$ deforms to $m - 1$ nodes of $\mathscr{C}_\eta$.

If $LD = 1$, this only happens when $r = 8$ and $L_i = -K_{Y_i} = D$. For a general del Pezzo surface Y_i of degree 1, there exists a nodal rational curve Γ_i in $|-K_{Y_i}|$ that meets D transversely at a unique point p. Then it is easy to see that $\Gamma = \Gamma_1 \cup \Gamma_2$ can be deformed to a nodal rational curve $\mathscr{C}_\eta$ on the generic fibre $\mathscr{X}_\eta$.

Suppose that $r \leqslant 6$. By the Hodge index theorem, $LD \geqslant 3$.

If $LD \leqslant 4$, it is easy to see that $(K_{Y_i} + L_i)L_i \leqslant 0$. Namely, the arithmetic genus of L_i is at most 1. Then a general member of $V_{L_i,0}$ must be nodal since its normalisation is an immersion. So there is a nodal rational curve in $|L_i|$ passing through the $a = LD - 1$ general points $x_1, x_2, \ldots, x_a$ on D. Thus, we may find $\Gamma = \Gamma_1 \cup \Gamma_2$ such that Γ_i are nodal rational curves in $|L_i|$ satisfying

$$\Gamma_i.D = x_1 + x_2 + \cdots + x_a + p$$

on Y_i for $i = 1, 2$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ can be deformed to a nodal rational curve $\mathscr{C}_\eta$ on the generic fibre $\mathscr{X}_\eta$.

If $LD \geqslant 5$, then by Theorem 3.10 and Corollary 3.11, we have $V_{L_i,0,D,mp} \neq \emptyset$, and a general member of $V_{L_i,0,D,mp}$ is nodal for $m = LD - 4$ and a general point $p \in D$. And since x_1, x_2, x_3, x_4 are four general points on D by (3.1.20), we can find nodal rational curves $\Gamma_i \in V_{L_i,0,D,mp}$ such that

$$\Gamma_i.D = x_1 + x_2 + x_3 + x_4 + mp$$

on Y_i for $i = 1, 2$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ can be deformed to a nodal rational curve $\mathscr{C}_\eta$ on the generic fibre $\mathscr{X}_\eta$. □

Theorem 3.15. *Let X be a general complex K3 surface with Picard lattice (3.1.2) generated by effective divisors $A, E_1, E_2, \dots, E_r$ for $r \leqslant 5$, and let L be a big and nef divisor on X satisfying*

$$\begin{cases} LA \geqslant 3\,, \\ LE_5 \leqslant 2 \quad \text{if } r = 5\,. \end{cases} \tag{3.1.21}$$

Then there exists an integral nodal rational curve $\Gamma \in |L|$. Moreover, there exist integral nodal rational curves P and Q in $|A|$ and $|4A + 2E_1 - E_2 - \cdots - E_r|$, respectively, such that $\Gamma + P + Q$ has normal crossings on X.

Proof. By the description of the nef cone of X (see Remark 3.7) and (3.1.21), we have

$$\begin{aligned} &L = dA + m_1E_1 - m_2E_2 - \cdots - m_rE_r \quad \text{for } d, m, m_i \in \mathbb{N}, \\ &d \geqslant 2m_1 \geqslant 4 \max_{2\leqslant i\leqslant r} m_i, \text{ and } m_1 \geqslant 3\,, \\ &m_5 \leqslant 1 \text{ if } r = 5\,. \end{aligned} \tag{3.1.22}$$

We let Y_1 be a smooth projective rational surface with Picard lattice

$$\begin{bmatrix} 0 & 1 & & & & \\ 1 & -2 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}_{2r\times 2r}$$

generated by effective divisors $A_1, B_1, G_1, G_2, \dots, G_{2r-2}$ and let D be a smooth anti-canonical curve on Y_1. We further require

$$\mathscr{O}_D(A_1) = \mathscr{O}_D(G_1 + G_2) = \mathscr{O}_D(G_3 + G_4) = \cdots = \mathscr{O}_D(G_{2r-3} + G_{2r-2})\,.$$

Such a Y_1 can be realised as the blowup of $\mathbb{F}_2$ at $2r - 2$ points $p_1, p_2, \dots, p_{2r-2}$ such that p_{2i-1} and p_{2i} lie on the same fibre of $\mathbb{F}_2$ over $\mathbb{P}^1$ for $i = 1, 2, \dots, r - 1$.

We let $Y_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $\operatorname{Pic}(Y_2)$ generated by two rulings A_2 and B_2 and let D be a smooth anti-canonical curve on Y_2.

Let $Y = Y_1 \cup Y_2$ be the union of Y_1 and Y_2 glued transversely along D satisfying

$$\mathscr{O}_D(\imath_1^* A_1) = \mathscr{O}_D(\imath_2^* A_2)\,,$$

where $\imath_i\colon D \hookrightarrow Y_i$ is the inclusion for $i = 1, 2$.

Note that such a $Y = Y_1 \cup Y_2$ is not projective. But we can deform it to a projective K3 surface whose Picard lattice has rank $r + 2$ and contains the lattice (3.1.2) as a primitive sublattice. That is, there exists a flat and proper (but non-projective) family $\pi\colon \mathscr{X} \to \operatorname{Spec} \mathbb{C}[[t]]$ of surfaces such that $\mathscr{X}_0 = Y$ and the generic fibre $\mathscr{X}_\eta$ of $\mathscr{X}$ is a K3 surface whose Picard lattice has rank $r + 2$ and contains (3.1.2) as a primitive sublattice. This follows from Theorem 2.4.

There are effective divisors $A, E_1, E_2, \dots, E_r$ on $\mathscr{X}$ such that

$$\begin{aligned} \mathscr{O}_{Y_1}(A) &= \mathscr{O}_{Y_1}(A_1)\,, \quad \mathscr{O}_{Y_2}(A) = \mathscr{O}_{Y_2}(A_2)\,, \\ \mathscr{O}_{Y_1}(E_1) &= \mathscr{O}_{Y_1}(B_1)\,, \quad \mathscr{O}_{Y_2}(E_1) = \mathscr{O}_{Y_2}\,, \end{aligned}$$

$$\mathcal{O}_{Y_1}(E_i) = \mathcal{O}_{Y_1}(G_{2i-3} + G_{2i-2}), \quad \mathcal{O}_{Y_2}(E_i) = \mathcal{O}_{Y_2}(A_2) \text{ for } i = 2, \ldots, r.$$

The 3-fold $\mathscr{X}$ has $18 - 2r$ rational double points $x_1, x_2, \ldots, x_{18-2r}$ on D satisfying

$$\begin{aligned}\mathcal{O}_D(x_1 + x_2 + \cdots + x_{18-2r}) &= \mathscr{N}_{D/Y_1} \otimes \mathscr{N}_{D/Y_2} \\ &= \mathcal{O}_D(-K_{Y_1}) \otimes \mathcal{O}_D(-K_{Y_2}) \\ &= \mathcal{O}_D((7-r)A) \otimes \mathcal{O}_D(2B_2),\end{aligned}$$

which is the only relation among $x_1, x_2, \ldots, x_{18-2r}$ for a general choice of $\mathscr{X}$.

Let L be the divisor on $\mathscr{X}$ defined by (3.1.22). As before, to prove the existence of rational curves in $|L|$ on the generic fibre $\mathscr{X}_\eta$, it suffices to find a limiting rational curve in $|L|$ on $\mathscr{X}_0 = Y$. However, due to the fact that L is not big when restricted to Y_2, we cannot construct such a curve in $|L|$ on Y. To overcome this, we need to work with the "twisted" linear series $|L + Y_1|$ on $\mathscr{X}$.

As explained in Section 2, the group $\mathrm{H}^0(Y, L + Y_1)$ is the kernel of the map

$$\begin{array}{c} \mathrm{H}^0(\mathcal{O}_{Y_1}(L_1 - D)) \oplus \mathrm{H}^0(\mathcal{O}_{Y_2}(L_2 + D - \sum x_j)) \\ \downarrow \\ \mathrm{H}^0(\mathcal{O}_D(L_2 + D - \sum x_j)) = \!= \!= \mathrm{H}^0(\mathcal{O}_D(L_1 - D)) \end{array}$$

sending (γ_1, γ_2) to $\gamma_1 - \gamma_2$, where the L_i are the restrictions of L to Y_i for $i = 1, 2$ and are given explicitly by

$$\begin{aligned} L_1 &= L|_{Y_1} = dA_1 + m_1 B_1 - \sum_{j=2}^{r} m_j (G_{2j-3} + G_{2j-2}), \\ L_2 &= L|_{Y_2} = \left(d - \sum_{j=2}^{r} m_j\right) A_2 . \end{aligned}$$

By a direct computation, we see that $h^0(\mathscr{X}_\eta, L) = h^0(Y, L+Y_1)$. So every (γ_1, γ_2) in $\mathrm{H}^0(Y, L+Y_1)$ can be deformed to a section in $\mathrm{H}^0(L)$ on the generic fibre $\mathscr{X}_\eta$. It suffices to find a limiting rational curve $\Gamma \subset Y$ cut out by such γ_i.

Without loss of generality, let us assume that $m_2 \geqslant \cdots \geqslant m_r$. Suppose that $m_i = 0$ for $i > a$ and $m_i > 0$ for $i \leqslant a$. We have

$$\begin{aligned} L_1 - D &= (d-4)A_1 + (m_1 - 2)B_1 - \sum_{j=2}^{r} (m_j - 1)(G_{2j-3} + G_{2j-2}) \\ &= M + \sum_{j=a+1}^{r} (G_{2j-3} + G_{2j-2}), \\ L_2 + D &= (d + 2 - \sum_{j=2}^{r} m_j) A_2 + 2B_2 \end{aligned}$$

$$\text{for } M = (d-4)A_1 + (m_1 - 2)B_1 - \sum_{j=2}^{a} (m_j - 1)(G_{2j-3} + G_{2j-2}).$$

Since $d-4 \geqslant 2(m_1-2) \geqslant 4(m_2-1) \geqslant \cdots \geqslant 4(m_a-1)$ by (3.1.22), we can write

$$\begin{aligned} M = {} & (m_a-1)\Big(2A_1+B_1-\sum_{j=2}^{a} G_{2j-3}\Big) + (m_a-1)\Big(2A_1+B_1-\sum_{j=2}^{a} G_{2j-2}\Big) \\ & + \sum_{i=2}^{a-1}(m_i-m_{i+1})\Big(2A_1+B_1-\sum_{j=2}^{i} G_{2j-3}\Big) + \sum_{i=2}^{a-1}(m_i-m_{i+1})\Big(2A_1+B_1-\sum_{j=2}^{i} G_{2j-2}\Big) \\ & + (m_1-2m_2)(2A_1+B_1)+(d-2m_1)A_1 \end{aligned} \tag{3.1.23}$$

and conclude that $V_{M,0} \neq \emptyset$. Similarly, $V_{L_2+D-cA_2,0} \neq \emptyset$ for all $c \leqslant 4$. We let

$$\lambda = \min(4, MD-1) \quad \text{and} \quad m = MD-\lambda\,.$$

If $MD \leqslant 4$, it is easy to see by (3.1.23) that the arithmetic genus of M is at most 1. So a general member of $V_{M,0}$ is nodal, and there exists a nodal rational curve in $|M|$ passing through λ general points on D. If $MD \geqslant 5$, by Corollary 3.13,

$$V_{M,0,D,mp} \neq \emptyset \text{ and } V_{L_2+D-4A_2,0,D,mp} \neq \emptyset$$

and general members of $V_{M,0,D,mp}$ and $V_{L_2+D-4A_2,0,D,mp}$ are nodal for $p \in D$ general. So we may find a $\Gamma \subset Y$ such that

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_\lambda \cup \Gamma_{\lambda+1} \cup \Gamma_{\lambda+2} \cup \Gamma_{\lambda+3}\,,$$

where

$$\begin{aligned} & \Gamma_1, \Gamma_2, \ldots, \Gamma_\lambda, \Gamma_{\lambda+2} \subset Y_2\,, \quad \Gamma_{\lambda+1}, \Gamma_{\lambda+3} \subset Y_1\,, \\ & \Gamma_{\lambda+1} \in V_{M,0,D,mp}\,, \quad \Gamma_1, \Gamma_2, \ldots, \Gamma_\lambda \in |A_2|\,, \\ & \Gamma_{\lambda+2} \in V_{L_2+D-\lambda A_2,0,D,mp}\,, \\ & \Gamma_{\lambda+3} = G_{2a-1} \cup G_{2a-2} \cup \cdots \cup G_{2r-3} \cup G_{2r-2}\,, \\ & \Gamma_1.D = x_1+y_1\,,\ \Gamma_2.D = x_2+y_2\,,\ \ldots\,,\ \Gamma_\lambda.D = x_\lambda+y_\lambda\,, \\ & \Gamma_{\lambda+1}.D = y_1+y_2+\cdots+y_\lambda+mp\,, \\ & \Gamma_{\lambda+2}.D = mp+w_1+w_2+\cdots+w_{2r-2a}+x_{\lambda+1}+x_{\lambda+2}+\cdots+x_{18-2r}\,, \\ & \Gamma_{\lambda+3}.D = w_1+w_2+\cdots+w_{2r-2a}\,. \end{aligned}$$

Here we choose $\Gamma_{\lambda+1}$ and $\Gamma_{\lambda+2}$ to be the general members of $V_{M,0,D,mp}$ and $V_{L_2+D-\lambda A_2,0,D,mp}$, respectively. So they are nodal, as explained above, in both cases $MD \leqslant 4$ and $MD \geqslant 5$. Therefore,

- $\Gamma_{\lambda+1}+\Gamma_{\lambda+3}$ and $\Gamma_1+\Gamma_2+\cdots+\Gamma_\lambda+\Gamma_{\lambda+2}$ have normal crossings on Y_i,
- $\Gamma_{\lambda+1}+\Gamma_{\lambda+3}$ and $\Gamma_1+\Gamma_2+\cdots+\Gamma_\lambda+\Gamma_{\lambda+2}$ meet D transversely outside of p on Y_i, and
- $\Gamma_{\lambda+1}+\Gamma_{\lambda+3}$ and $\Gamma_1+\Gamma_2+\cdots+\Gamma_\lambda+\Gamma_{\lambda+2}$ have simple tangencies with D at p on Y_i for $i=1,2$.

By Theorem 2.5, we see that Γ can be deformed to a nodal rational curve in $|L|$ on $\mathscr{X}_\eta$. To construct a nodal rational curve in $|A|$, we let

$$P = P_1 \cup P_2\,,$$

where

$$P_1 \subset Y_1\,,\ P_2 \subset Y_2\,,\ P_1 \in |A_1|\,,\ P_2 \in |A_2|\,, P_1.D = x_{\lambda+1}+q\,,\ P_2.D = x_{\lambda+1}+q\,.$$

Again by Theorem 2.5, we see that P can be deformed to a nodal rational curve in $|A|$ on $\mathscr{X}_\eta$. To construct a nodal rational curve in $|4A+2E_1-E_2-\cdots-E_r|$, we let

$$Q = Q_1 \cup Q_2 \cup \cdots \cup Q_{6-r}$$

where

$$\begin{aligned}
&Q_1 \subset Y_1\,, \quad Q_2, \ldots, Q_{6-r} \subset Y_2\,,\\
&Q_1 \in |4A_1 + 2B_1 - G_1 - \cdots - G_{2r-2}|\,, \quad Q_2 \in |A_2|\,,\\
&Q_1.D = s_2 + s_3 + \cdots + s_{6-r}\,,\\
&Q_2.D = x_{\lambda+2} + s_2\,,\ Q_3.D = x_{\lambda+3} + s_3\,,\ \ldots\,,\ Q_{6-r}.D = x_{\lambda+6-r} + s_{6-r}\,,
\end{aligned}$$

where Q_1 is a nodal rational curve in $|4A_1+2B_1-G_1-\cdots-G_{2r-2}|$ passing through the general points $s_2, \ldots, s_{6-r}$. Again by Theorem 2.5, we see that Q can be deformed to a nodal rational curve in $|4A+2E_1-E_2-\cdots-E_r|$ on $\mathscr{X}_\eta$.

Also it is easy to check that $\Gamma + P + Q$ has normal crossings on Y_1 and Y_2, and $p \notin P \cup Q$. So its deformation on $\mathscr{X}_\eta$ has normal crossings as well. □

Now we have produced nodal rational curves on K3 surfaces with Picard lattices (3.1.1) and (3.1.2). Theorem 3.1 follows more or less easily.

Proof of Theorem 3.1 when $\det(\Lambda)$ *is even.* Let Y be a general K3 surface with Picard lattice (3.1.1) for $r=6$. By Lemma 3.3 and 3.8, we can find a primitive lattice embedding $\sigma\colon \Lambda \hookrightarrow \mathrm{Pic}(Y)$ such that $\sigma(L)$ is big and nef on Y. Then there is a nodal rational curve $C \in |\sigma(L)|$ by Theorem 3.14.

There is a smooth proper family $\pi\colon \mathscr{X} \to \operatorname{Spec}\mathbb{C}[[t]]$ of K3 surfaces such that $\mathscr{X}_0 = Y$, the fibre $\mathscr{X}_\eta$ has Picard lattice Λ and L extends to a divisor $\mathscr{L}$ on $\mathscr{X}$ with $\mathscr{L}_0 = \sigma(L)$. Then C can be deformed to a nodal rational curve in $|\mathscr{L}|$ on the generic fibre $\mathscr{X}_\eta$ of $\mathscr{X}$. □

Proof of Theorem 3.1 when $\det(\Lambda)$ *is odd.* We are going to prove the theorem under hypothesis A2 or A3. Let Y be a general K3 surface with Picard lattice (3.1.2) for some $r \leqslant 5$. In both cases A2 and A3, it suffices to find a primitive lattice embedding $\sigma\colon \Lambda \hookrightarrow \mathrm{Pic}(Y)$ such that $\sigma(L)$ is big and nef on Y and

$$\begin{cases} \sigma(L).A \geqslant 3\,, \\ \sigma(L).E_5 \leqslant 2 \quad \text{if } r = 5\,, \end{cases} \tag{3.1.24}$$

where $A, E_1, E_2, \ldots, E_r$ are the effective generators of $\mathrm{Pic}(Y)$ with intersection matrix (3.1.2).

Suppose that L satisfies condition A2. By Lemmas 3.6 and 3.8, there is a primitive lattice embedding $\sigma\colon \Lambda \hookrightarrow \mathrm{Pic}(Y)$ for $r=4$ such that $\sigma(L)$ is big and nef on Y. In this case, we have $L = L_1 + L_2 + L_3$ such that $LL_i > 0$ and $L_i^2 > 0$ for $i = 1,2,3$. Let us write

$$\sigma(L) = \sigma(L_1) + \sigma(L_2) + \sigma(L_3) = M_1 + M_2 + M_3 \tag{3.1.25}$$

for $M_i = \sigma(L_i)$. We claim that $M_i A \geqslant 1$ for all nef divisors $A \neq 0$ on Y.

Since $\sigma(L)$ is nef and $\sigma(L).M_i = LL_i > 0$, we have $h^2(M_i) = h^0(-M_i) = 0$. Therefore, by the Riemann–Roch theorem,

$$h^0(M_i) = h^1(M_i) + \frac{M_i^2}{2} + 2 \geqslant \frac{L_i^2}{2} + 2 > 2\,.$$

Hence the linear system $|M_i|$ has a non-zero moving part. Let Γ be an irreducible component of the moving part of $|M_i|$. Then

$$M_i A \geqslant \Gamma A \geqslant 0\,.$$

If $\Gamma A > 0$, then $M_i A > 0$ follows. Otherwise, $\Gamma A = 0$. And since both Γ and A are nef, $A^2 = 0$ and Γ is numerically equivalent to aA for some $a \in \mathbb{Q}^+$ by the Hodge index theorem. This holds for all components Γ of the moving part of $|M_i|$. So we have

$$M_i \equiv aA + F\,,$$

where F is the fixed part of $|M_i|$. If $FA > 0$, we again have $M_i A > 0$. Otherwise, $FA = 0$; then

$$F^2 = a^2A^2 + 2aAF + F^2 = (aA + F)^2 = M_i^2 > 0\,.$$

Since F is effective and $F^2 > 0$, we again have $h^0(F) > 2$ by the Riemann–Roch theorem. This contradicts the fact that F is the fixed part of $|M_i|$.

In conclusion, $M_i A \geqslant 1$ for all nef divisors $A \neq 0$ on X and $i = 1, 2, 3$. By (3.1.25), we have $\sigma(L).A \geqslant 3$. This proves (3.1.24) for case A2.

Suppose that L satisfies condition A3. In this case, $L = L_1 + L_2$ which $LL_i > 0$, $L_1^2 > 0$, $L_2^2 = -2$, $L_1 \notin 2\Lambda$, $L_1 - L_2 \notin n\Lambda$ for all $n \in \mathbb{Z}$ and $n \geqslant 2$, and

$$L_1^2 + 2L_1L_2 \geqslant 18 \Leftrightarrow a + b \geqslant 9\,, \tag{3.1.26}$$

where we let $L_1^2 = 2a$ and $L_1L_2 = b$.

Let us first assume that $\Lambda = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$. In this case, we will use the numerical condition (3.1.26) to explicitly construct a primitive embedding $\sigma\colon \Lambda \hookrightarrow \mathrm{Pic}(Y)$ for $r = 5$ such that $\sigma(L)$ is a big and nef divisor on Y satisfying (3.1.24).

When $b \equiv 0 \pmod 3$, we let

$$\begin{cases} \sigma(L_1) = \dfrac{a+9+\delta}{3}A + 3E_1 - \displaystyle\sum_{i=3}^{\delta+2} E_i \quad \text{for } \delta = 3 + 3\left\lfloor \dfrac{a}{3} \right\rfloor - a\,, \\ \sigma(L_2) = \dfrac{b}{3}A - E_2\,. \end{cases}$$

When $b \equiv 1 \pmod 3$, we let

$$\begin{cases} \sigma(L_1) = \dfrac{a+13+\delta}{3}A + 3E_1 - 2E_2 - \displaystyle\sum_{i=3}^{\delta+2} E_i \quad \text{for } \delta = 2 + 3\left\lfloor \dfrac{a+1}{3} \right\rfloor - a\,, \\ \sigma(L_2) = \dfrac{b-4}{3}A + E_2\,. \end{cases}$$

When $b \equiv 2 \pmod 3$, we let

$$\begin{cases} \sigma(L_1) = \dfrac{a+10+\delta}{3}A + 3E_1 - E_2 - \displaystyle\sum_{i=3}^{\delta+2} E_i \quad \text{for } \delta = 2 + 3\left\lfloor \dfrac{a+1}{3} \right\rfloor - a\,, \\ \sigma(L_2) = \dfrac{b-2}{3}A + E_2\,. \end{cases}$$

It is easy to check that $\sigma(L) = \sigma(L_1 + L_2)$ is big and nef divisor on Y satisfying (3.1.24). This settles the case $\Lambda = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$.

Now assume that

$$\Lambda \neq \mathbb{Z}L_1 \oplus \mathbb{Z}L_2\,. \tag{3.1.27}$$

Let $\sigma\colon \Lambda \hookrightarrow \operatorname{Pic}(Y)$ be a primitive lattice embedding for $r = 4$ such that $\sigma(L)$ is big and nef on Y. It suffices to prove that $\sigma(L).A \geqslant 3$. We write

$$\sigma(L) = \sigma(L_1 + L_2) = M_1 + M_2$$

for $M_i = \sigma(L_i)$. Since $LL_1 > 0$ and $L_1^2 > 0$, we have $M_1A \geqslant 1$ by the same argument as before. Since $LL_2 > 0$ and $L_2^2 = -2$, the divisor M_2 is effective by the Riemann–Roch theorem. So $M_2A \geqslant 0$. If $M_1A + M_2A \geqslant 3$, then $\sigma(L).A \geqslant 3$ and we are done. Otherwise, we have three cases:

- $M_1A = 1$ and $M_2A = 0$;
- $M_1A = 2$ and $M_2A = 0$;
- $M_1A = M_2A = 1$.

We will show that none of these cases are possible.

Suppose that $M_1A = 1$ and $M_2A = 0$. Since M_2 is effective, $M_2^2 = -2$ and $M_2A = 0$, we necessarily have $M_2 = mA \pm E_j$ for some $2 \leqslant j \leqslant 4$. And since $M_1A = 1$, it easy to see that M_1 and M_2 generate a primitive sublattice of $\operatorname{Pic}(Y)$. Then L_1 and L_2 generate Λ, contradicting (3.1.27).

Suppose that $M_1A = 2$ and $M_2A = 0$. Again, we have $M_2 = mA \pm E_j$. Then one of following must hold:

(1) M_1 and M_2 generate a primitive sublattice of $\operatorname{Pic}(Y)$;

(2) $M_1 = 2D$ for some $D \in \operatorname{Pic}(Y)$; or

(3) $M_1 - M_2 = 2D$ for some $D \in \operatorname{Pic}(Y)$.

As pointed out above, the first case is equivalent to $\Lambda = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2$, contradicting (3.1.27). The second and third cases are equivalent to $L_1 \in 2\Lambda$ and $L_1 - L_2 \in 2\Lambda$, respectively, both contradicting our hypotheses on L_i.

Suppose that $M_1A = M_2A = 1$. Then

(1) either M_1 and M_2 generate a primitive sublattice of $\operatorname{Pic}(Y)$, or

(2) $M_1 - M_2 = nD$ for some $D \in \operatorname{Pic}(Y)$, $n \in \mathbb{Z}$ and $n \geqslant 2$.

Again, the former contradicts (3.1.27) and the latter is equivalent to $L_1 - L_2 \in n\Lambda$, contradicting our hypotheses on L_i. This finishes the argument for case A3.

In conclusion, we can find a primitive embedding $\sigma\colon \Lambda \hookrightarrow \operatorname{Pic}(Y)$ such that $\sigma(L)$ satisfies the hypotheses of Theorem 3.15. So there is a nodal rational curve $C \in |\sigma(L)|$ on Y.

There is a smooth proper family $\pi\colon \mathscr{X} \to \operatorname{Spec}\mathbb{C}[[t]]$ of K3 surfaces such that $\mathscr{X}_0 = Y$, that $\mathscr{X}_\eta$ has Picard lattice Λ and that L extends to a divisor $\mathscr{L}$ on $\mathscr{X}$ with $\mathscr{L}_0 = \sigma(L)$. Then C can be deformed to a nodal rational curve in $|\mathscr{L}|$ on the generic fibre $\mathscr{X}_\eta$ of $\mathscr{X}$. □

3.2 Higher-rank lattices

It is natural to expect the above techniques to apply to various lattice of higher rank; for the purposes of [CGL19], however, we will carry this out for the following two specific rank 4 lattices.

Theorem 3.16. *Let Λ be one of the following lattices of rank 4:*

$$\begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}, \tag{3.2.1}$$

$$\begin{bmatrix} 12 & -2 & 0 & 0 \\ -2 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}. \tag{3.2.2}$$

Then for a general K3 surface X with $\mathrm{Pic}(X) = \Lambda$, there is an integral rational (respectively, geometric genus 1) curve in $|L|$ if L is a big and nef divisor L on X with the property that

$$L = L_1 + L_2 + L_3 \quad \text{for some } L_i \in \Lambda \text{ satisfying that } LL_i > 0 \text{ and } L_i^2 > 0 \text{ for } i = 1, 2, 3\,. \tag{3.2.3}$$

Proof. We claim that there is a primitive embedding $\sigma\colon \Lambda \hookrightarrow \Sigma_r$ for Σ_r given by (3.1.2) and $r \leqslant 5$.

When Λ is (3.2.1), we let $r = 4$ and

$$\sigma(B) = 2A + E_1\,,$$
$$\sigma(C_1) = -A + E_2\,, \quad \sigma(C_2) = -A + E_3\,, \quad \sigma(C_3) = -A + E_4\,,$$

where $\{B, C_1, C_2, C_3\}$ and $\{A, E_1, \ldots, E_r\}$ are the bases of Λ and Σ_r, respectively, with the corresponding intersection matrices.

When Λ is (3.2.2), we let $r = 5$ and

$$\sigma(B) = 12A + 6E_1 - 4E_2 - 3E_3 - 2E_4 - E_5\,,$$
$$\sigma(C_1) = A - E_2\,, \quad \sigma(C_2) = -E_1\,, \quad \sigma(C_3) = A - E_3\,.$$

This proves our claim. So there exists a flat proper family $\pi\colon \mathscr{X} \to \mathrm{Spec}\,\mathbb{C}[[t]]$ of K3 surfaces such that $\mathscr{X}_0$ is a general K3 surface with Picard lattice Σ_r, that $\mathscr{X}_\eta$ is a K3 surface with Picard lattice Λ and that there is a divisor $\mathscr{L}$ on $\mathscr{X}$ with $\mathscr{L}_0 = \sigma(L)$.

We may choose σ such that $\mathscr{L}_0 = \sigma(L)$ is big and nef on $\mathscr{X}_0$. By the same argument as in the proof of Theorem 3.1, we can show that

$$\sigma(L_i).A \geqslant 1 \Rightarrow \sigma(L).A \geqslant 3 \Leftrightarrow \mathscr{L}_0 A \geqslant 3$$

on $\mathscr{X}_0$ by (3.2.3).

When Λ is (3.2.1), we have $r = 4$. Then the existence of nodal rational curves in $|\mathscr{L}_0|$ on $\mathscr{X}_0$ is directly given by Theorem 3.15. Therefore, there are integral rational (respectively, geometric genus 1) curves in $|\mathscr{L}|$ on $\mathscr{X}_\eta$.

When Λ is (3.2.2), we have $r = 5$. Theorem 3.15 only gives the existence of nodal rational curves in $|\mathscr{L}_0|$ if $\mathscr{L}_0$ additionally satisfies $\min_{2\leqslant i\leqslant 5} \mathscr{L}_0 E_i \leqslant 2$. So some extra work is needed. Suppose that

$$\mathscr{L}_0 = dA + m_1 E_1 - \sum_{i=2}^{5} m_i E_i$$

for some $d, m_i \in \mathbb{Z}$. Without loss of generality, let us assume that $m_2 \geqslant m_3 \geqslant m_4 \geqslant m_5$. Since $\mathscr{L}_0$ is nef and $\mathscr{L}_0 A \geqslant 3$, we have

$$d \geqslant 2m_1 \geqslant 4m_2 \geqslant 4m_3 \geqslant 4m_4 \geqslant 4m_5 \geqslant 0 \quad \text{and} \quad m_1 \geqslant 3\,.$$

If $m_5 \leqslant 1$, then there is a nodal rational curve in $|\mathscr{L}_0|$ by Theorem 3.15, and we are done. Let us assume that $m_5 \geqslant 2$.

We write

$$\begin{aligned}\mathscr{L}_0 &= (d-4m_5+4)A + (m_1-2m_5+2)E_1 - \sum_{i=2}^{5}(m_i-m_5+1)E_i \\ &\quad + (m_5-1)\left(4A+2E_1-\sum_{i=2}^{5}E_i\right) = P+(m_5-1)F\end{aligned} \tag{3.2.4}$$

if $m_1 - 2m_5 \geqslant 1$ and

$$\mathscr{L}_0 = (d-4m_5)A + m_5\left(4A+2E_1-\sum_{i=2}^{5}E_i\right) = (d-4m_5)A + m_5F \tag{3.2.5}$$

if $m_1 = 2m_5$, which implies $m_1 = 2m_2 = 2m_3 = 2m_4 = 2m_5$.

Suppose that $m_1 - 2m_5 \geqslant 1$. That is, we have (3.2.4). Since P is big and nef and $PA \geqslant 3$, there exists an integral rational (respectively, geometric genus 1) curve $\Gamma \in |P|$ by Theorem 3.15. There is also an integral nodal rational curve $R \in |F|$ such that Γ and R meet transversely. As $F^2 = 0$, the curve R has a unique node q. Let

$$\widehat{\Gamma} \cup R_1 \cup R_2 \cup \cdots \cup R_{m_5-1} \xrightarrow{\ f\ } \mathscr{X}_0$$

be a stable map given as follows:

- $f\colon \widehat{\Gamma} \to \Gamma$ and $f\colon R_i \to R$ are the normalisations of Γ and R, respectively, for $i = 1,2,\dots,m_5-1$;
- $\widehat{\Gamma}$ and R_1 meet at one point, R_i and R_{i+1} meet at one point for $i = 1,2,\dots,m_5-2$, and there are no other intersections among Γ and R_i;
- f maps the point $\widehat{\Gamma} \cap R_1$ to one of the intersections in $\Gamma \cap R$, and it is a local isomorphism at $\widehat{\Gamma} \cap R_1$ onto its image;
- f maps the point $R_i \cap R_{i+1}$ to the node q of R, and it is a local isomorphism at $R_i \cap R_{i+1}$ onto its image for $i = 1,2,\dots,m_5-2$.

By a local isomorphism at $\widehat{\Gamma} \cap R_1$ and $R_i \cap R_{i+1}$, we mean that f maps an étale/analytic/formal neighbourhood of the point on the curve isomorphically onto its image.

It is clear that f deforms in the expected dimension on $\mathscr{X}_0$. So it deforms to $\mathscr{X}_\eta$. On the other hand, the divisor class F does not deform in the family $\mathscr{X}$ over $\operatorname{Spec}\mathbb{C}[[t]]$ since $\mathscr{X}_\eta$ is not elliptic. Therefore, f extends to a family of stable maps to $\mathscr{X}$ over $\operatorname{Spec}\mathbb{C}[[t]]$, still denoted by $f\colon \mathscr{C} \to \mathscr{X}$, such that $\mathscr{C}_\eta$ is smooth and $f_*\mathscr{C}_\eta$ is an integral rational (respectively, geometric genus 1) curve on $\mathscr{X}_\eta$. We are done.

Suppose that $m_1 = 2m_5$. That is, we have (3.2.5). There is a nodal rational curve $D \in |A|$ such that D and R meet transversely at two points. Clearly, D has a unique node p. Let

$$D_1 \cup D_2 \cup \cdots \cup D_{d-4m_5} \cup R_1 \cup R_2 \cup \cdots \cup R_{m_5} \xrightarrow{\ f\ } \mathscr{X}_0$$

be a stable map given as follows:

- $f\colon D_i \to D$ and $f\colon R_j \to R$ are the normalisations of D and R, respectively, for $i = 1,2,\dots,d-4m_5$ and $j = 1,2,\dots,m_5$;
- D_i and D_{i+1} meet at one point, D_{d-4m_5} and R_1 meet at one point, R_j and R_{j+1} meet at one point for $1 \leqslant i \leqslant d-4m_5-1$ and $1 \leqslant j \leqslant m_5-1$, and there are no other intersections among D_i and R_j;

- f maps the point $D_{d-4m_5} \cap R_1$ to one of the intersections in $D \cap R$, and it is a local isomorphism at $D_{d-4m_5} \cap R_1$ onto its image;
- f maps the point $D_i \cap D_{i+1}$ to the node p of D, and it is a local isomorphism at $D_i \cap D_{i+1}$ onto its image for $i = 1, 2, \ldots, d - 4m_5 - 1$;
- f maps the point $R_j \cap R_{j+1}$ to the node q of R, and it is a local isomorphism at $R_j \cap R_{j+1}$ onto its image for $j = 1, 2, \ldots, m_5 - 1$.

Again f deforms in the expected dimension on $\mathscr{X}_0$. So it deforms to $\mathscr{X}_\eta$. On the other hand, neither A nor F deforms in the family $\mathscr{X}$ over $\operatorname{Spec}\mathbb{C}[[t]]$ since $\mathscr{X}_\eta$ is not elliptic. Therefore, f extends to a family of stable maps to $\mathscr{X}$ over $\operatorname{Spec}\mathbb{C}[[t]]$, still denoted by $f\colon \mathscr{C} \to \mathscr{X}$, such that $\mathscr{C}_\eta$ is smooth and $f_*\mathscr{C}_\eta$ is an integral rational curve on $\mathscr{X}_\eta$.

To see that there is also an integral geometric genus 1 curve in $|\mathscr{L}|$ on $\mathscr{X}_\eta$, we let $s \in D \cap R$ be the intersection such that $s \neq f(D_{d-4m_5} \cap R_1)$. Obviously, there are points $s' \in D_{d-4m_5}$ and $s'' \in R_1$ such that $f(s') = f(s'') = s$. Therefore, $f(\mathscr{C}_\eta)$ has a singularity where it has two branches. Then it is well known that $f(\mathscr{C}_\eta)$ can be deformed to an integral genus 1 curve on $\mathscr{X}_\eta$ (see, for example, [CGL19, Lemma 6.5]) so we are done. □

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