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Category decomposition of $\mathrm{Rep}_k(\mathrm{SL_n}(F))$

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ABSTRACT

Let F be a non-archimedean local field with residual characteristic p, and k an algebraically closed field of characteristic $\ell \neq p$. We establish a category decomposition of $\mathrm{Rep}_k(\mathrm{SL}_n(F))$ with respect to the $\mathrm{GL}_n(F)$-inertially equivalent supercuspidal classes of $\mathrm{SL}_n(F)$, and we establish a block decomposition of the supercuspidal subcategory of $\mathrm{Rep}_k(\mathrm{SL}_n(F))$. Finally we give an example to show that in general a block of $\mathrm{SL}_n(F)$ is not defined with respect to a unique inertially equivalent supercuspidal class of $\mathrm{SL}_n(F)$, which is different from the case when $\ell = 0$.

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1. Introduction

Let F be a non-archimedean local field with residual characteristic p, and k an algebraically closed field with characteristic ℓ different from p. We say G is a p-adic group if it is the group of F-rational points of a connected reductive group $\mathbb{G}$ defined over F. Let $\mathrm{Rep}_k(\mathrm{G})$ be the category of smooth k-representations of G. In this article, we always denote by M' a Levi subgroup of $\mathrm{SL}_n(F)$, and we study the category $\mathrm{Rep}_k(\mathrm{M}')$.

For arbitrary p-adic group G, we say that $\mathrm{Rep}_k(\mathrm{G})$ has a **category decomposition** with respect to an index set $\mathcal{A}$, if there exists an equivalence:

$$\mathrm{Rep}_k(\mathrm{G}) \cong \prod_{\alpha\in\mathcal{A}} \mathrm{Rep}(\mathrm{G})_\alpha, \tag{1}$$

where $\mathrm{Rep}_k(\mathrm{G})_\alpha$ are full sub-categories of $\mathrm{Rep}_k(\mathrm{G})$. The equivalence implies that:

- Each object $\Pi \in \mathrm{Rep}_k(\mathrm{G})$ can be decomposed as a direct sum $\Pi \cong \oplus_{\alpha\in\mathcal{A}}\Pi_\alpha$, where $\Pi_\alpha \in \mathrm{Rep}_k(\mathrm{G})_\alpha$.
- For $i = 1, 2$ and $\alpha_i \in \mathcal{A}$, if $\alpha_1 \neq \alpha_2$, then $\mathrm{Hom}_\mathrm{G}(\Pi_1, \Pi_2) = 0$ for $\Pi_i \in \mathrm{Rep}_k(\mathrm{G})_{\alpha_i}$;

Furthermore, if

- for $\alpha \in \mathcal{A}$, there is no such decomposition for $\mathrm{Rep}_k(\mathrm{G})_\alpha$, we say that $\mathrm{Rep}_k(\mathrm{G})_\alpha$ is **non-split**.

If $\mathrm{Rep}_k(\mathrm{G})_\alpha$ is non-split for each $\alpha \in \mathcal{A}$, we call this category decomposition a **block decomposition** of $\mathrm{Rep}_k(\mathrm{G})$, which means the finest category decomposition of $\mathrm{Rep}_k(\mathrm{G})$, and we call each $\mathrm{Rep}_k(\mathrm{G})_\alpha$ a **block** of $\mathrm{Rep}_k(\mathrm{G})$.

When $\ell = 0$, a block decomposition of $\mathrm{Rep}_k(\mathrm{G})$ has been established with respect to $\mathcal{A} = \mathcal{SC}_\mathrm{G}$, where $\mathcal{SC}_\mathrm{G}$ is the set of G-inertially equivalent supercuspidal classes of G (see Section 2.1 for the definition). Let $[\mathrm{M}, \pi]_\mathrm{G} \in \mathcal{SC}_\mathrm{G}$, where (M, π) is a supercuspidal pair of G (see Section 2.1). The subcategory $\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{M},\pi]_\mathrm{G}}$ consists of the objects whose irreducible subquotients have supercuspidal supports (see Section 2.1) in $[\mathrm{M}, \pi]_\mathrm{G}$.

When ℓ is positive, a block decomposition has been established when $\mathbb{G}$ is GL_n ([8]) and its inner forms ([11]). For $\mathbb{G} = \mathrm{GL}_n$, the block decomposition is with respect to $\mathcal{SC}_\mathrm{G}$ as well, which is the same as the case when $\ell = 0$. It is worth noting that when we restrict the block decomposition in Equation (1) to the set of irreducible k-representations of G, the block decomposition with respect to $\mathcal{SC}_\mathrm{G}$ requires the supercuspidal support of which is an irreducible k-representation of G belongs to a unique G-inertially equivalent supercuspidal class, which can be deduced from the uniqueness of supercuspidal support proved in [13, §V.4] for $\mathrm{GL}_n(F)$. However the uniqueness of supercuspidal support is not true in general, in [6] an irreducible k-representation of $\mathrm{Sp}_8(F)$ that the supercuspidal support is not unique up to $\mathrm{Sp}_8(F)$-conjugation has been found. As for $\mathrm{SL}_n(F)$, the uniqueness of supercuspidal support holds true and has been proved in [5], hence the block decomposition with respect to $\mathrm{SL}_n(F)$-inertially equivalent supercuspidal classes was expected. However in this article, we show that this is **not** always true by providing a counter-example in Section 4.

1.1. Main results

Now we describe the work in this article with more details. Let G' be $\mathrm{SL}_n(F)$, and M' be a Levi subgroup of G'. Let M be a Levi subgroup of $\mathrm{GL}_n(F)$ such that $\mathrm{M}\cap\mathrm{G}'=\mathrm{M}'$. We establish a category decomposition of $\mathrm{Rep}_k(\mathrm{M}')$ with respect to M-inertially equivalent supercuspidal classes $\mathcal{SC}_{\mathrm{M}'}^{\mathrm{M}}$ (see Section 2.1 for the definitions), which is different from $\mathcal{SC}_{\mathrm{M}'}$, the set of M'-inertially equivalent supercuspidal classes. In fact, let L be a Levi subgroup of M and $\mathrm{L}'=\mathrm{L}\cap\mathrm{M}'$ a Levi subgroup of M', and let τ be an irreducible supercuspidal k-representation of L. Denote by $\mathcal{I}(\tau)$ the set of isomorphic classes of irreducible components of $\tau|_{\mathrm{L}'}$. Let $\tau'\in\mathcal{I}(\tau)$, denote by $[\mathrm{L}',\tau']_{\mathrm{M}'}$ the M'-inertially equivalent supercuspidal class defined by (M',τ'). The M-inertially equivalent supercuspidal class of (L',τ') is $\cup_{\gamma'\in\mathcal{I}(\tau)}[\mathrm{L}',\gamma']_{\mathrm{M}'}$, and we denote it by $[\mathrm{L}',\tau']_{\mathrm{M}}$.

Theorem 1.1 *(Theorem 3.15). Let $\mathcal{SC}_{\mathrm{M}'}^{\mathrm{M}}$ be the set of M-inertially equivalent supercuspidal classes of M'. There is a category decomposition of $\mathrm{Rep}_k(\mathrm{M}')$ with respect to $\mathcal{SC}_{\mathrm{M}'}^{\mathrm{M}}$.*

In particular, let $[\mathrm{L}',\tau']_{\mathrm{M}}\in\mathcal{SC}_{\mathrm{M}'}^{\mathrm{M}}$, a k-representation of M' belongs to the full subcategory $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{L}',\tau']_{\mathrm{M}}}$, if and only if the supercuspidal support of each of its irreducible subquotients is contained in $[\mathrm{L}',\tau']_{\mathrm{M}}$.

The above theorem gives a category decomposition

$$\mathrm{Rep}_k(\mathrm{M}')\cong\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}\times\mathrm{Rep}_k(\mathrm{M}')_{non-\mathcal{SC}},$$

where a k-representation Π of M' belongs to $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ (resp. $\mathrm{Rep}_k(\mathrm{M}')_{non-\mathcal{SC}}$) if each (resp. none) of its irreducible subquotients is supercuspidal. We call $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ **the supercuspidal subcategory** of $\mathrm{Rep}_k(\mathrm{M}')$. In Section 4, we establish a block decomposition of $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$.

Let π be an irreducible supercuspidal k-representation of M, and let $\mathcal{I}(\pi)$ be the set of isomorphic classes of irreducible components of $\pi|_{\mathrm{M}'}$. In Section 4, we introduce an equivalence relation $\sim$ on $\mathcal{I}(\pi)$. For $\pi'\in\mathcal{I}(\pi)$, an irreducible supercuspidal k-representation of M', let $(\pi',\sim)$ be the connected component of $\mathcal{I}(\pi)$ containing π' under this equivalence relation, which is the subset of $\mathcal{I}(\pi)$ consisting of the elements equivalent to π'. Let $[\pi',\sim]$ be the union of M'-inertially equivalent supercuspidal classes of $\pi_j'\in(\pi',\sim)$. In general, we have

$$[\mathrm{M}', \pi']_{\mathrm{M}'} \subsetneq [\pi', \sim] \subsetneq [\mathrm{M}', \pi']_{\mathrm{M}}.$$

Denote by $\mathcal{SC}_{\mathrm{M}',\sim}$ the set of pairs of the form $[\pi', \sim]$. We establish a block decomposition of $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$:

Theorem 1.2 *(Theorem 4.12). There is a block decomposition of* $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ *with respect to* $\mathcal{SC}_{\mathrm{M}',\sim}$. *In particular, let* $[\pi', \sim] \in \mathcal{SC}_{\mathrm{M}',\sim}$. *A k-representation* Π *of* M' *belongs to* $\mathrm{Rep}_k(\mathrm{M}')_{[\pi',\sim]}$ *if and only if each of the irreducible subquotients of* Π *belongs to* $[\pi', \sim]$.

This article ends with Example 4.13 of a k-representation in the supercuspidal subcategory of $\mathrm{Rep}_k(\mathrm{SL}_2(F))$ when $\ell = 3$. In this example, we construct a finite length projective k-representation of $\mathrm{SL}_2(F)$ which is induced from a projective cover of a maximal simple supercuspidal k-type of depth zero. By using the theory of k-representations of finite SL_2 group, we compute the irreducible subquotients of this projective cover, and we show that there exist two different supercuspidal k-representations of $\mathrm{SL}_2(F)$, which are not inertially equivalent, such that they belong to a same block. Or equivalently, this example shows that there exists an irreducible supercuspidal k-representation π' of $\mathrm{SL}_2(F)$, such that $[\pi', \sim]$ is not a unique $\mathrm{SL}_2(F)$-inertially equivalent supercuspidal class, hence the equivalence relation defined on $\mathcal{I}(\pi)$ is non-trivial in general. This example shows that a block decomposition of $\mathrm{Rep}_k(\mathrm{G}')$ (resp. $\mathrm{Rep}_k(\mathrm{M}')$) with respect to G'-inertially equivalent supercuspidal classes $\mathcal{SC}_{\mathrm{G}'}$ (resp. $\mathrm{Rep}_k(\mathrm{M}')$-inertially equivalent supercuspidal classes $\mathcal{SC}_{\mathrm{M}'}$) is not always possible in general.

1.2. Structure of this paper

The author is inspired by the method in [8]. We use the theory of maximal simple k-types, which has been firstly established for $\mathbb{C}$-representations of $\mathrm{GL}_n(F)$ in [2] and generalised by the author to the cuspidal k-representations of M' a Levi subgroup of $\mathrm{SL}_n(F)$ in [4]. In this article, we construct a family of projective objects defined from the projective cover of maximal simple k-types. In Section 3.1, we show that the projective cover of a maximal simple k-type of M' is an indecomposable direct summand of the restriction of the projective cover of a maximal simple k-type of M. We apply the compact induction functor $\mathrm{ind}_{\mathrm{M}'}^{\mathrm{M}}$ to these projective objects and describe their decomposition under the block decomposition of $\mathrm{Rep}_k(\mathrm{M})$. The above two parts lead to a family of injective objects verifying the conditions stated in Proposition 2.1, which gives the category decomposition in Theorem 1.1.

Section 4 concentrates on the supercuspidal subcategory of $\mathrm{Rep}_k(\mathrm{M}')$, where M' is a Levi subgroup of $\mathrm{SL}_n(F)$. We introduce an equivalence relation generated by putting all the irreducible subquotients of the projective cover of a maximal simple supercuspidal k-type of M' into a same equivalent class. Let π be an irreducible supercuspidal k-representation of M. The above equivalence relation on maximal simple supercuspidal k-types induces an equivalence relation on $\mathcal{I}(\pi)$, which is the equivalence relation $\sim$ needed in Theorem 1.2.

It is natural to expect that a block decomposition of $\mathrm{Rep}_k(\mathrm{G}')$ can be given with respect to the set of G'-conjugacy classes of elements in $\mathcal{SC}_{\mathrm{M}',\sim}$ for all Levi subgroup M', which involves the study of projective cover of maximal simple k-types (non-supercuspidal) of M' and leads to a study of semisimple k-types of G'.

2. Preliminary

2.1. Notations

Let F be a non-archimedean local field with residual characteristic equal to p.

- $\mathfrak{o}_F$: the ring of integers of F, and $\mathfrak{p}_F$: the unique maximal ideal of $\mathfrak{o}_F$.
- k: an algebraically closed field with characteristic $\ell \neq p$.
- Let K be a closed subgroup of a p-adic group G, then $\mathrm{ind}_K^{\mathrm{G}}$: compact induction, $\mathrm{Ind}_K^{\mathrm{G}}$: induction, $\mathrm{res}_K^{\mathrm{G}}$: restriction.
- Fix a split maximal torus of G, and M be a Levi subgroup, then $i_{\mathrm{M}}^{\mathrm{G}}, r_{\mathrm{M}}^{\mathrm{G}}$: normalised parabolic induction and normalised parabolic restriction.
- Denote by δ_{G} the module character of G.

In this article, without specified we always denote by G the group of F-rational points of GL_n and by G' the group of F-rational points of SL_n. Let ι be the canonical embedding from G' to G, which induces an isomorphism between the Weyl group of G' and G, hence gives a bijection from the set of Levi subgroups of G' to those of G. In particular, if M is a Levi subgroup of G, we always denote by M' the Levi subgroup $\mathrm{M} \cap \mathrm{G}'$ of G'. We say an irreducible k-representation π of a p-adic group G is **cuspidal**, if $r_{\mathrm{M}}^{\mathrm{G}}\pi$ is zero for every proper Levi subgroup M; we say π is **supercuspidal** if it does not appear as a subquotient of $i_{\mathrm{M}}^{\mathrm{G}}\tau$ for each proper Levi subgroup M and its irreducible representation τ.

Let π be an irreducible k-representation of G. Its restriction $\pi|_{\mathrm{G}'}$ is semisimple with finite length, and each irreducible k-representation π' of G' appears as a direct component of $\pi|_{\mathrm{G}'}$. A pair (M, τ) is called a **cuspidal** (resp. **supercuspidal**) **pair** if M is a Levi subgroup and τ is an irreducible cuspidal (resp. supercuspidal) of M. Let $(\mathrm{M}_1', \tau_1'), (\mathrm{M}_2', \tau_2')$ be two cuspidal pairs of G' and K be a subgroup of G. We say they are K**-inertially equivalent**, if there exists an element $g \in K$ such that $g(\mathrm{M}_1') = \mathrm{M}_2'$ and there exists an unramified k-quasicharacter θ of $F^\times$ such that $g(\tau_1') \cong \tau_2' \otimes \theta$. We denote by $[\mathrm{M}', \tau']_K$ the K-inertially equivalent class defined by (M', τ'), and we call it a K-inertially equivalent supercuspidal (resp. cuspidal) class if (M', τ') is a supercuspidal (resp. cuspidal) pair. A same definition of $[\mathrm{M}, \tau]_{\mathrm{G}}$ is applied for cuspidal pairs of G. We always abbreviate $[\mathrm{M}', \tau']_{\mathrm{G}'}$ as $[\mathrm{M}', \tau']$, and abbreviate $[\mathrm{M}, \tau]_{\mathrm{G}}$ as $[\mathrm{M}, \tau]$.

We say that a cuspidal (resp. supercuspidal) pair (M, τ) belongs to the **cuspidal** (resp. **supercuspidal**) **support** of π, if π appears as a subrepresentation or a quotient-representation (resp. subquotient representation) of $i_{\mathrm{M}}^{\mathrm{G}}\tau$. When π is an irreducible k-representation of G (resp. G'), its supercuspidal support as well as its cuspidal support is **unique** up to G (resp. G′)-conjugation (see Theorem 4.16 of [5] and §V.4 of [13] for the uniqueness of supercuspidal support, and see §III 2.4 of [12] for the uniqueness of cuspidal support).

To decompose $\mathrm{Rep}_k(\mathrm{G}')$ as a direct product of a family of full-subcategories, we construct a family of injective objects and follow the method as below, which is the same strategy as in [8, Proposition 2.4]. We state it here for convenient reason.

Proposition 2.1. *Let $\mathcal{I}_1, \mathcal{I}_2$ be two injective objects in $\mathrm{Rep}_k(\mathrm{G}')$, and denote by $\mathcal{S}_1, \mathcal{S}_2$ the sets of irreducible k-representations of G' which appear as subquotients of $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. Suppose the following conditions are verified:*

- *an object in $\mathcal{S}_1$ can be embedded into $\mathcal{I}_1$;*
- *an object in $\mathcal{S}_1$ does not belong to $\mathcal{S}_2$ up to isomorphism;*
- *an irreducible k-representation of G', which does not belong to $\mathcal{S}_1$ up to isomorphism, can be embedded into $\mathcal{I}_2$.*

Then $\mathrm{Rep}_k(\mathrm{G}')$ can be decomposed as a direct product of two full subcategories R_1 and R_2, such that

- *every object $\Pi \in \mathrm{Rep}_k(\mathrm{G}')$ is isomorphic to a direct sum $\pi_1 \oplus \pi_2$, where each irreducible subquotient of π_1 belongs to $\mathcal{S}_1$ and each irreducible subquotient of π_2 belongs to $\mathcal{S}_2$;*
- *every object in R_1 has an injective resolution by direct sums of copies of $\mathcal{I}_1$, and every object in R_2 has an injective resolution by direct sums of copies of $\mathcal{I}_2$ (copies means direct product by itself).*

Remark 2.2 *(Projective version).* Let $\mathcal{P}_1, \mathcal{P}_2$ be two projective objects in $\mathrm{Rep}_k(\mathrm{G}')$, and denote by $\mathcal{S}_1, \mathcal{S}_2$ the sets of irreducible k-representations of G' which appears as a subquotient of $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Suppose the following conditions are verified:

- an object in $\mathcal{S}_1$ is a quotient of $\mathcal{P}_1$;
- an object in $\mathcal{S}_1$ does not belong to $\mathcal{S}_2$ up to isomorphism;
- an irreducible k-representation of G', which does not belong to $\mathcal{S}_1$ up to isomorphism, can be realised as a quotient of $\mathcal{P}_2$.

Then $\mathrm{Rep}_k(\mathrm{G}')$ can be decomposed as a direct product of two full subcategories R_1 and R_2, such that every object $\Pi \in \mathrm{Rep}_k(\mathrm{G}')$ is isomorphic to a direct sum $\pi_1 \oplus \pi_2$, where each irreducible subquotient of π_1 belongs to $\mathcal{S}$ and each irreducible subquotient of π_2 belongs to $\mathcal{S}_2$.

The proof of Remark 2.2 is done in the same manner as in Proposition 2.4 of [8] by changing injective objects to projective objects as suggested in Remark 2.5 of [8].

2.2. Maximal simple k-types of M′

In this section, we recall notations and definitions in the theory of maximal simple k-types of Levi subgroups M' of G' which has been studies in [4]. It requires the theory of maximal simple k-types of G which has been established in [3] for complex case. The later is related to modulo ℓ maximal simple types in [12, §III] by considering the reduction modulo ℓ, while [9] gives a more intrinsic description. We state some useful properties which will be needed for the further use.

A **maximal simple k-type** of G is a pair (J, λ), where J is an open compact subgroup of G and λ is an irreducible k-representation of J. We have a groups inclusion:

$$H^1 \subset J^1 \subset J,$$

where J^1 is a normal pro-p open subgroup J^1 of J, such that the quotient J/J^1 is isomorphic to $\mathrm{GL}_m(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a field extension of the residue field of F, and H^1 is open. The k-representation λ is a tensor product $\kappa \otimes \sigma$, where κ is irreducible whose restriction to H^1 is a multiple of a k-character, and σ is inflated from a cuspidal k-representation of J/J^1. By [12, §III 4.25] or [9, Proposition 3.1], for an irreducible k-cuspidal representation π of G, there exists a maximal simple k-type (J, λ), a compact modulo centre subgroup K and an irreducible representation Λ of K, where J is the unique largest compact open subgroup of K and Λ is an extension of λ, such that $\pi \cong \mathrm{ind}_K^{\mathrm{G}} \Lambda$. Since a Levi subgroup of G is a tensor product of GL-groups of lower rank, so we can define maximal simple k-types $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ and obtain the same property for cuspidal k-representations of M as above.

For the reason that a Levi subgroup M' of G' is not a product of SL-groups of lower rank, so it is not sufficient to consider only the maximal simple k-types of G'. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple k-type of M. The group of projective normaliser $\tilde{J}_{\mathrm{M}}$ contains J_{M} as a normal subgroup, which is defined in [4, 2.15] and [3, 2.2]. In particular, for any $g \in \tilde{J}_{\mathrm{M}}$, we have $g(\lambda_{\mathrm{M}}) \cong \lambda_{\mathrm{M}} \otimes \chi$, where χ is a k-quasicharacter of $F^{\times}$. As in [4, 2.48], a **maximal simple k-type** of M' is a pair of the form $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$, where $\tilde{\lambda}'_{\mathrm{M}}$ is an irreducible direct component of $(\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \lambda_{\mathrm{M}})|_{\tilde{J}'_{\mathrm{M}}}$, and we set $\tilde{\lambda}_{\mathrm{M}} := \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \lambda_{\mathrm{M}}$, which is irreducible as proved in [4, Theorem 2.47]. For any irreducible cuspidal k-representation π' of M', there exists an irreducible cuspidal k-representation π of M such that π' is a direct component of $\pi|_{\mathrm{M}'}$. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple k-types contained in π, then there exists a maximal simple k-type $(\tilde{J}_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ as well as an open compact modulo centre subgroup $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})$, the normaliser group of $\tilde{\lambda}'_{\mathrm{M}}$ in M', containing $\tilde{J}'_{\mathrm{M}}$ as its largest open compact subgroup, as well as an extension $\Lambda_{\mathrm{M}'}$ of $\tilde{\lambda}'_{\mathrm{M}}$ to $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})$, such that $\pi' \cong \mathrm{ind}_{N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})}^{\mathrm{M}'} \Lambda_{\mathrm{M}'}$. We call $(N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}}), \Lambda'_{\mathrm{M}})$ an **extended maximal simple k-type**.

Proposition 2.3 *(Proposition 2.29 and Lemma 4.2 of [5]). Let π' be an irreducible cuspidal k-representation of* M'*. There exists a cuspidal k-representation π of* M*, such that π' is a direct component of $\pi|_{\mathrm{M}'}$. Then π' is supercuspidal if and only if π is supercuspidal.*

When π is supercuspidal, we call a k-type $(J_{\mathrm{M}},\lambda_{\mathrm{M}})$ (resp. $(\tilde{J}'_{\mathrm{M}},\tilde{\lambda}'_{\mathrm{M}})$) contained in π (resp. π') a ***maximal simple supercuspidal k-type****.*

Let K_1, K_2 be two open subgroups of M' and ρ_1,ρ_2 two irreducible k-representations of K_1, K_2 respectively. We say that ρ_1 is **weakly intertwined** with ρ_2 in M', if there exists an element $m\in\mathrm{M}'$ such that ρ_1 is isomorphic to a subquotient of $\mathrm{ind}_{K_1\cap m(K_2)}^{K_1}\mathrm{res}_{K_1\cap m(K_2)}^{m(K_2)}m(\rho_2)$.

Proposition 2.4 *(Theorem 3.19 and Theorem 3.25 of [4]).*

1. *We have $\tilde{J}_{\mathrm{M}}=\tilde{J}'_{\mathrm{M}}J_{\mathrm{M}}$.*
2. *Let $(\tilde{J}'_{\mathrm{M},1},\tilde{\lambda}'_{\mathrm{M},1})$ and $(\tilde{J}'_{\mathrm{M},2},\tilde{\lambda}'_{\mathrm{M},2})$ be two maximal simple k-types of M'. They are weakly intertwined in M' if and only if they are M'-conjugate.*

3. Category decomposition

In this section, to simplify the notations, we denote by G a Levi subgroup of $\mathrm{GL}_n(F)$ and $\mathrm{G}'=\mathrm{G}\cap\mathrm{SL}_n(F)$, which is a Levi subgroup of $\mathrm{SL}_n(F)$. Let M be a Levi subgroup of G. We denote by $\mathrm{M}'=\mathrm{M}\cap\mathrm{G}'$ a Levi subgroup of G', and let K be an open subgroup of G. We always denote by $K'=K\cap\mathrm{G}'$. If π is an irreducible k-representation of K, then π' is one of the irreducible summand of $\pi|_{K'}$.

3.1. Projective objects

In this section, we will follow the strategy of [8] to construct some projective objects of $\mathrm{Rep}_k(\mathrm{G}')$. We study first the projective cover of maximal simple k-types of Levi subgroups M', then we consider their induced representations. Proposition 3.6 and Corollary 3.7 give the relation between these projective objects and irreducible k-representations whose cuspidal support is given by the corresponding maximal simple k-type. The later properties will be used in Section 3.2.

Let $(J_{\mathrm{M}},\lambda_{\mathrm{M}})$ be a maximal simple k-type of M, and $\tilde{J}_{\mathrm{M}}$ be the group of projective normaliser of $(J_{\mathrm{M}},\lambda_{\mathrm{M}})$ (see Section 2.2). Write λ_{M} as $\kappa_{\mathrm{M}}\otimes\sigma_{\mathrm{M}}$. Let $\mathcal{P}_{\lambda_{\mathrm{M}}}$ be the projective cover of λ_{M}. From [8, Lemma 4.8] we know that $\mathcal{P}_{\lambda_{\mathrm{M}}}$ is isomorphic to $\mathcal{P}_{\sigma_{\mathrm{M}}}\otimes\kappa_{\mathrm{M}}$, where $\mathcal{P}_{\sigma_{\mathrm{M}}}$ is the projective cover of σ_{M}. Denote by $\tilde{\lambda}_{\mathrm{M}}$ the irreducible k-representation $\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}}\lambda_{\mathrm{M}}$. Let $(\tilde{J}'_{\mathrm{M}},\tilde{\lambda}'_{\mathrm{M}})$ be a maximal simple k-type of M' defined from $(J_{\mathrm{M}},\lambda_{\mathrm{M}})$ as in Section 2.2. Since $\mathcal{P}_{\lambda_{\mathrm{M}}}$ has finite length, we have $\mathcal{P}_{\lambda_{\mathrm{M}}}|_{J'_{\mathrm{M}}}=\oplus_{i=1}^{s}\mathcal{P}_i$, where $\mathcal{P}_i$ is an indecomposable projective k-representation of J'_{M} for each i.

Remark 3.1. The projective cover $\mathcal{P}_{\sigma_{\mathrm{M}}}$ is given by the theory of k-representations of finite general linear groups. When σ_{M} is inflated from a supercuspidal k-representation of M, which means $(J_{\mathrm{M}},\lambda_{\mathrm{M}})$ is a maximal simple supercuspidal k-type of M, according to the construction of $\mathcal{P}_{\sigma_{\mathrm{M}}}$ (see Lemma 5.11 of [7] or see Corollary 3.5 of [5]) as well as Deligne-

Lusztig theory, we conclude that the irreducible subquotients of $\mathcal{P}_{\sigma_{\mathrm{M}}}$ are isomorphic to σ_{M}.

Let π be an irreducible cuspidal k-representation of M which contains $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$, and π' be an irreducible cuspidal k-representation of M' which contains $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ such that $\pi' \hookrightarrow \pi|_{\mathrm{M}'}$. We denote by

$$\mathcal{P}_{[\mathrm{M},\pi]} = i_{\mathrm{M}}^{\mathrm{G}} \mathrm{ind}_{J_{\mathrm{M}}}^{\mathrm{M}} \mathcal{P}_{\lambda_{\mathrm{M}}},$$

and by

$$\mathcal{P}_{[\mathrm{M}',\pi']} = i_{\mathrm{M}'}^{\mathrm{G}'} \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}.$$

Lemma 3.2. $\mathcal{P}_{[\mathrm{M}',\pi']}$ *is an direct summand of* $\mathcal{P}_{[\mathrm{M},\pi]}|_{\mathrm{M}'}$.

Proof. We have

$$(i_{\mathrm{M}}^{\mathrm{G}} \mathrm{ind}_{J_{\mathrm{M}}}^{\mathrm{M}} \mathcal{P}_{\lambda_{\mathrm{M}}})|_{\mathrm{G}'} \cong i_{\mathrm{M}'}^{\mathrm{G}'} (\mathrm{ind}_{J_{\mathrm{M}}}^{\mathrm{M}} \mathcal{P}_{\lambda_{\mathrm{M}}})|_{\mathrm{M}'}.$$

Since $\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda}|_{\tilde{J}'_{\mathrm{M}}}$ is projective and has a surjection to $\tilde{\lambda}'_{\mathrm{M}}$, we obtain that $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ is a direct summand of $\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda}|_{\tilde{J}'_{\mathrm{M}}}$. Hence $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ is a direct summand of $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}}|_{\mathrm{M}'}$ where $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \cong \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda_{\mathrm{M}}}$, and $\mathcal{P}_{[\mathrm{M}',\pi']}$ is a direct summand of $\mathcal{P}_{[\mathrm{M},\pi]}|_{\mathrm{G}'}$. □

Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple supercuspidal k-type of M, and $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ be a maximal simple supercuspidal k-type of M' defined from $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ as in Section 2.2.

Lemma 3.3. *Let π be an irreducible supercuspidal k-representation of* M *which contains $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$, and τ' be an irreducible subquotient of the projective cover $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ of $\tilde{\lambda}'_{\mathrm{M}}$. Then $(\tilde{J}'_{\mathrm{M}}, \tau')$ is also a maximal simple supercuspidal k-type defined by $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$, and there exists an irreducible direct component π'_0 of $\pi|_{\mathrm{M}'}$ which contains $(\tilde{J}'_{\mathrm{M}}, \tau')$. In particular, when $\mathrm{M}' = \mathrm{G}' = \mathrm{SL}_n(F)$, if τ' is different from $\tilde{\lambda}'_{\mathrm{M}}$, and suppose π' is an irreducible direct component of $\pi|_{\mathrm{M}'}$ containing $\tilde{\lambda}'_{\mathrm{M}}$, then π'_0 is different from π'.*

Proof. Recall that $\mathcal{P}_{\lambda_{\mathrm{M}}}$ is the projective k-cover of λ_{M}, as explained in Remark 3.1, its irreducible subquotients are isomorphic to λ_{M}. As in the proof of Lemma 3.2, we know that the projective representation $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ is an indecomposable direct component of $\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda_{\mathrm{M}}}|_{\tilde{J}'_{\mathrm{M}}}$. As in Section 2.2, the induced representation $\tilde{\lambda}_{\mathrm{M}} := \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \lambda_{\mathrm{M}}$ is irreducible. By the exactness of induction functor, we know that the irreducible subquotients of $\mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda_{\mathrm{M}}}$ are isomorphic to $\tilde{\lambda}_{\mathrm{M}}$, which implies that an irreducible subquotient of $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ is isomorphic to an irreducible direct component of $\tilde{\lambda}_{\mathrm{M}}|_{\tilde{J}'_{\mathrm{M}}}$. Since π contains $\tilde{\lambda}_{\mathrm{M}}$ after restricting to $\tilde{J}_{\mathrm{M}}$, by the Mackey's theory, any irreducible direct component of $\tilde{\lambda}_{\mathrm{M}}|_{\tilde{J}'_{\mathrm{M}}}$ must be contained in an irreducible direct component of $\pi|_{\mathrm{M}'}$.

When $\mathrm{M}' = \mathrm{G}' = \mathrm{SL}_n(F)$, by Mackey's theory the induction $\mathrm{ind}_{\tilde{J}'_{\mathrm{G}}}^{\mathrm{G}'} \mathrm{res}_{\tilde{J}'_{\mathrm{G}}}^{\tilde{J}_{\mathrm{G}}} \tilde{\lambda}_{\mathrm{G}}$ is a subrepresentation of $\pi|_{\mathrm{G}'}$, and each irreducible component of $\tilde{\lambda}_{\mathrm{G}}|_{\tilde{J}'_{\mathrm{G}}}$ is irreducibly induced to G'. The second statement is directly from the fact that $\pi|_{\mathrm{M}'}$ is multiplicity-free as proved in Proposition 2.35 of [4]. □

Remark 3.4. When M' is a proper Levi subgroup of G', it is possible that two different maximal simple supercuspidal k-types $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ and $(\tilde{J}'_{\mathrm{M}}, \tau')$, which are defined from a same maximal simple supercuspidal k-type, are M'-conjugate to each other, which implies that they may be contained in a same irreducible supercuspidal k-representation of M'.

Lemma 3.5.

1. *Let* $\alpha \in \tilde{J}_{\mathrm{M}}$*, then* $\alpha(\mathcal{P}_{\lambda_{\mathrm{M}}}) \cong \mathcal{P}_{\alpha(\lambda_{\mathrm{M}})} \cong \mathcal{P}_{\lambda_{\mathrm{M}}} \otimes \theta$*, where* θ *is a* k*-quasicharacter of* $\det(J_{\mathrm{M}})$ *and* $\alpha(\lambda_{\mathrm{M}}) \cong \lambda_{\mathrm{M}} \otimes \theta$*.*
2. *Let* $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_1)$ *and* $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_2)$ *be two different maximal simple* k*-types defined from* $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$*. Let* $\alpha \in \tilde{J}_{\mathrm{M}}$ *such that* $\alpha(\tilde{\lambda}'_1) \cong \tilde{\lambda}'_2$*, then for the projective covers we have* $\alpha(\mathcal{P}_{\tilde{\lambda}'_1}) \cong \mathcal{P}_{\tilde{\lambda}'_2}$*.*

Proof. For the first part, there is a surjective morphism from $\mathcal{P}_{\lambda_{\mathrm{M}}} \otimes \theta$ to $\lambda_{\mathrm{M}} \otimes \theta$ and is indecomposable. Moreover, the projectivity can be easily deduced directly from the definition. Since $\alpha(\mathcal{P}_{\lambda_{\mathrm{M}}})$ is the projective cover of $\alpha(\lambda_{\mathrm{M}})$, we obtain the expected equality. The second part can be deduced in a similar way. □

Proposition 3.6. *Recall that* G' *is a Levi subgroup of* $\mathrm{SL}_n(F)$*. Let* ρ' *be an irreducible* k*-representation of* G' *and* (M', π') *be a cuspidal pair of* G' *inside the cuspidal support of* ρ'*, then there is a surjective morphism*

$$\mathcal{P}_{[\mathrm{M}',\pi']} \to \rho'.$$

Proof. Let $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ be a maximal simple k-type contained in π', hence there is an injection $\tilde{\lambda}'_{\mathrm{M}} \to \mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \pi'$. By Frobenius reciprocity, it gives a surjection $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}} \to \pi'$, which induces a surjection $\mathcal{P}_{[\mathrm{M}',\pi']} \to i_{\mathrm{M}'}^{\mathrm{G}'} \pi'$, hence a surjection from $\mathcal{P}_{[\mathrm{M}',\pi']}$ to ρ' by [12, §II, 2.20]. □

Corollary 3.7. *Let* $\mathcal{I}_{[\mathrm{M}',\pi']}$ *be the contragredient of* $\mathcal{P}_{[\mathrm{M}',\pi'^{\vee}]}$*, where* $\pi'^{\vee}$ *is the contragredient of* π'*. Suppose that the cuspidal support of* τ' *is* $[\mathrm{M}', \pi']$*, then* τ' *is embedding to* $\mathcal{I}_{[\mathrm{M}',\pi']}$*.*

3.2. Category decomposition

Recall that in this section G' is a Levi subgroup of $\mathrm{SL}_n(F)$ and G is a Levi subgroup of $\mathrm{GL}_n(F)$ such that $\mathrm{G}' = \mathrm{G} \cap \mathrm{SL}_n(F)$. A decomposition of $\mathrm{Rep}_k(\mathrm{G}')$ by its full sub-categories will be given in Theorem 3.15 with respect to the G-twist equivalent supercuspidal classes of G' (see the paragraph below Proposition 3.12 for G-twist equivalent equivalence). This will not be a block decomposition in general, which means it does not always verify the last condition of Equation (1), however we will see in Section 4 that it is not always possible to decompose $\mathrm{Rep}_k(\mathrm{G}')$ with respect to the G'-inertially equivalent supercuspidal classes as for $\mathrm{Rep}_k(\mathrm{G})$ in Equation (1).

Let $\mathcal{A}$ be a family of G-inertially equivalent supercuspidal classes of G, and denote by $\mathrm{Rep}_k(\mathrm{G})_{\mathcal{A}}$ the union of blocks $\bigcup_{[\mathrm{M},\pi]_{\mathrm{G}}\in\mathcal{A}}\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{M},\pi]_{\mathrm{G}}}$. Let $\mathcal{A}'$ be a family of G'-inertially equivalent supercuspidal classes of G', verifying that $[\mathrm{M}',\pi']_{\mathrm{G}'}\in\mathcal{A}'$ if and only if there exists $[\mathrm{M},\pi]_{\mathrm{G}}\in\mathcal{A}$ such that $\mathrm{M}'=\mathrm{M}\cap\mathrm{G}'$ and $\pi'\to\pi|_{\mathrm{M}'}$. Let L be a Levi subgroup of G which contains M. Denote by $\mathcal{A}_{\mathrm{L}}$ the family of L-inertially equivalent supercuspidal classes of the form $[w(\mathrm{M}),w(\pi)]_{\mathrm{M}}$, where $[\mathrm{M},\pi]_{\mathrm{G}}\in\mathcal{A}$, and recall that $[\cdot,\cdot]_{\mathrm{L}}$ is the L-inertially equivalent class, and $w\in\mathrm{G}$ such that $w(\mathrm{M})\subset\mathrm{L}$. We define $\mathcal{A}'_{\mathrm{L}}$ in the same manner of $\mathcal{A}'$ by replacing G by L.

Lemma 3.8. *Let $P\in\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{M},\pi]_{\mathrm{G}}}$, and L be a Levi subgroup of G. Then $r_{\mathrm{L}}^{\mathrm{G}}P\in\prod_{w\in\mathrm{G},w(\mathrm{M})\subset\mathrm{L}}\mathrm{Rep}_k(\mathrm{G})_{[w(\mathrm{M}),w(\pi)]_{\mathrm{L}}}$.*

Proof. Suppose Π is an irreducible subquotient of $r_{\mathrm{L}}^{\mathrm{G}}P$, whose cuspidal support is (N,τ), where N is a Levi subgroup of L and τ is a cuspidal representation of N. Let $\mathcal{P}_{[\mathrm{N},\tau]_{\mathrm{L}}}$ be the projective object defined from the maximal simple k-type of τ, then there is a non-trivial morphism $\mathcal{P}_{[\mathrm{N},\tau]_{\mathrm{L}}}\to r_{\mathrm{L}}^{\mathrm{G}}P$. By the second adjunction of Bernstein, we have a non-trivial morphism from $\overline{i_{\mathrm{L}}^{\mathrm{G}}}\mathcal{P}_{[\mathrm{N},\tau]_{\mathrm{L}}}$ to P, where $\overline{i_{\mathrm{L}}^{\mathrm{G}}}$ is the opposite normalised parabolic induction from L to G. Since the module character δ_{L} is an unramified character on L, the k-representation $\overline{i_{\mathrm{L}}^{\mathrm{G}}}\mathcal{P}_{[\mathrm{N},\tau]_{\mathrm{L}}}$ belongs to the same block as $i_{\mathrm{L}}^{\mathrm{G}}\mathcal{P}_{[\mathrm{N},\tau]_{\mathrm{L}}}$, which implies that the supercuspidal support of τ belongs to the union $\cup_{w\in\mathrm{G},w(\mathrm{M})\subset\mathrm{L}}(w(\mathrm{M}),w(\pi))$. We finish the proof. □

Lemma 3.9. *Let $P\in\mathrm{Rep}_k(\mathrm{G})_{\mathcal{A}}$, and τ' be an irreducible subquotient of $P|_{\mathrm{G}'}$, then the supercuspidal support of τ' belongs to $\mathcal{A}'$.*

Proof. Suppose firstly that τ' is cuspidal, then there exists a maximal simple k-type (J,λ) of G, such that an irreducible component (J',λ') of $\lambda|_{\mathrm{G}'}$ is contained in τ' as a subrepresentation. By [4, Lemma 2.14], up to twist a k-character of $F^{\times}$, we can assume that λ is a subquotient of $P|_J$. Hence there is a non-trivial morphism from the projective cover $\mathcal{P}_\lambda$ of λ to $P|_J$, which implies that for any irreducible cuspidal k-representation τ of G which contains (J,λ), its supercuspidal support must belong to $\mathcal{A}$. In particular, we can choose τ such that $\tau'\hookrightarrow\tau|_{\mathrm{G}'}$, hence by [5, Proposition 4.4] we know that the supercuspidal support of τ' must belong to $\mathcal{A}'$.

Now suppose τ' is not cuspidal. Let (L',ρ') belong to its cuspidal support. The ρ' appears as a subquotient of $r_{\mathrm{L}'}^{\mathrm{G}'}P|_{\mathrm{G}'}\cong(r_{\mathrm{L}}^{\mathrm{G}}P)|_{\mathrm{L}'}$. By Lemma 3.8, and the previous paragraph, we know that the supercuspidal support of ρ' must belong to $\mathcal{A}'_{\mathrm{L}}$, from which we deduce the desired property of supercuspidal support of τ. □

Lemma 3.10. *Let π and π' be cuspidal k-representations of M and M' respectively and $\pi' \hookrightarrow \pi|_{\mathrm{M}'}$. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple k-type of π and $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ be a maximal simple k-type of π' defined from $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$. Suppose $[\mathrm{L}, \tau]$ is the supercuspidal support of $[\mathrm{M}, \pi]$, then we have*

$$\mathrm{ind}_{\mathrm{G}'}^{\mathrm{G}} \mathcal{P}_{[\mathrm{M}',\pi']} \in \prod_{\chi \in (\mathcal{O}_F^\times)^\vee} \mathrm{Rep}_k(\mathrm{G})_{[\mathrm{L},\tau\otimes\chi]}.$$

Proof. We set $\mathcal{P}' := \mathcal{P}_{[\mathrm{M}',\pi']}$, $\mathcal{P}'_{\mathrm{M}'} = \mathrm{ind}_{\tilde{J}_{\mathrm{M}'}}^{\mathrm{M}'} \mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ and $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} = \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda_{\mathrm{M}}}$ in this proof. Recall that $\mathcal{P}' = i_{\mathrm{M}'}^{\mathrm{G}'} \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\mathcal{P}}_{\tilde{\lambda}'_{\mathrm{M}}}$. Since the module character $\delta_{\mathrm{M}'} = \delta_{\mathrm{M}}|_{\mathrm{M}'}$, we have

$$\mathrm{ind}_{\mathrm{G}'}^{\mathrm{G}} \mathcal{P}' \cong i_{\mathrm{M}}^{\mathrm{G}} \mathrm{ind}_{\mathrm{M}'}^{\mathrm{M}} \mathcal{P}'_{\mathrm{M}'} \hookrightarrow i_{\mathrm{M}}^{\mathrm{G}} \mathrm{ind}_{\tilde{J}_{\mathrm{M}}}^{\mathrm{M}} (\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathbb{1}),$$

where

$$\begin{aligned} \mathrm{res}_{J^1_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} (\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathbb{1}) &= \mathrm{res}_{J^1_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathrm{ind}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathcal{P}_{\lambda_{\mathrm{M}}} \otimes \mathrm{res}_{J^1_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathbb{1} \\ &= \oplus_{\alpha \in \tilde{J}_{\mathrm{M}} \backslash J_{\mathrm{M}}} \mathrm{res}_{J^1_{\mathrm{M}}}^{J_{\mathrm{M}}} \alpha(\mathcal{P}_{\lambda_{\mathrm{M}}}) \otimes \oplus_{\tilde{J}'_{\mathrm{M}} \backslash \tilde{J}_{\mathrm{M}} / J^1_{\mathrm{M}}} \mathrm{ind}_{J^1_{\mathrm{M}} \cap \tilde{J}'_{\mathrm{M}}}^{J^1_{\mathrm{M}}} \mathbb{1}. \end{aligned} \tag{2}$$

Since J^1_{M} is a pro-p group, and by the definition of $\tilde{J}_{\mathrm{M}}$ the above representation is semisimple whose direct components are of the form $\eta \otimes \theta$, where η is the Heisenberg representation of the simple character of λ_{M}, and $\theta \in (\det(J^1_{\mathrm{M}}))^\wedge$ which can be extended to a character of $F^\times$ and we fix one of such extension by denoting it as θ as well. Hence we have the decomposition

$$\mathrm{res}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} (\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathbb{1}) \cong \oplus_{\theta \in (\det(H^1_{\mathrm{M}}))^\wedge} P_\theta, \tag{3}$$

where P_θ is the $\eta \otimes \theta$-isogeny subrepresentation. Notice that we require θ is non-trivial on H^1_{M}, because otherwise $\eta \cong \eta \otimes \theta$. By a similar computation as in Equation (2), we have

$$\mathrm{res}_{J_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} (\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\tilde{J}_{\mathrm{M}}} \mathbb{1}) \cong \oplus_{\alpha \in \tilde{J}_{\mathrm{M}} / J_{\mathrm{M}}} \alpha(\mathcal{P}_{\lambda_{\mathrm{M}}}) \otimes \oplus_{\rho \in (\det(J_{\mathrm{M}}))^\wedge} \rho \otimes \mathrm{ind}_{J_{\mathrm{M},\ell'}}^{J_{\mathrm{M}}} \mathbb{1},$$

where $J_{\mathrm{M},\ell}$ is the subgroup of J_{M} consisting with the elements whose determinant belongs to the ℓ'-part of $F^\times$. By Lemma 3.5, the right hand side of the above equation is isomorphic to $\oplus_{\rho \in (\det(J_{\mathrm{M}}))^\wedge} (\mathcal{P}_{\lambda_{\mathrm{M}}} \otimes \rho)^{\tilde{q}} \otimes \mathrm{ind}_{J_{\mathrm{M},\ell'}}^{J_{\mathrm{M}}} \mathbb{1}$, where $\tilde{q}$ is the index $[\tilde{J}_{\mathrm{M}} : J_{\mathrm{M}}]$ and $(\cdot)^{\tilde{q}}$ is the $\tilde{q}$-multiple of $\cdot$. Hence P_θ in Equation (3) is isomorphic to $\oplus_\rho (\mathcal{P}_{\lambda_{\mathrm{M}}} \otimes \rho)^{\tilde{q}} \otimes \mathrm{ind}_{J_{\mathrm{M},\ell'}}^{J_{\mathrm{M}}} \mathbb{1}$, where $\rho \in (\det J_{\mathrm{M}})^\wedge, \rho|_{H^1_{\mathrm{M}}} = \theta$. Recall that $\lambda_{\mathrm{M}} \cong \kappa \otimes \sigma$, where σ is inflated from a supercuspidal k-representation of $J_{\mathrm{M}}/J^1_{\mathrm{M}}$, and $\mathcal{P}_\lambda \cong \mathcal{P}_\sigma \otimes \kappa$. Since an irreducible subquotient of P_θ is isomorphic to $\kappa\otimes\sigma_0\otimes\rho$, where σ_0 is inflated from $J_{\mathrm{M}}/J^1_{\mathrm{M}}$ and its supercuspidal support is the same as that of σ, and ρ is an character as above. Now we fix an extension of θ to J_{M} and denote it again by θ. We have $P_\theta\cong\Pi_\theta\otimes\kappa\otimes\theta$, where Π_θ is inflated from $J_{\mathrm{M}}/J^1_{\mathrm{M}}$, and the supercuspidal support of each of its irreducible subquotient is the same as $\sigma\otimes\overline{\rho}$, where $\overline{\rho}\in\det(J_{\mathrm{M}}/J^1_{\mathrm{M}})^\wedge$. By [11, Theorem 9.6], the induction $\mathrm{ind}_{J_{\mathrm{M}}}^{\mathrm{M}}P_\theta$ belongs to the subcategory $\prod_{\chi\in(\mathfrak{o}_F^\times)^\wedge}\mathrm{Rep}_k(\mathrm{M})_{[\mathrm{L},\tau\otimes\chi]}$, hence $i_{\mathrm{M}}^{\mathrm{G}}P_\theta\in\prod_{\chi\in(\mathfrak{o}_F^\times)^\wedge}\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{L},\tau\otimes\chi]}$. Since $\mathrm{ind}_{\mathrm{G}'}^{\mathrm{G}}\mathcal{P}'\cong\oplus_{\theta\in(\det(H^1_{\mathrm{M}}))^\wedge}i_{\mathrm{M}}^{\mathrm{G}}\mathrm{ind}_{J_{\mathrm{M}}}^{\mathrm{M}}P_\theta$, we deduce the desired property. □

Lemma 3.11. *Let $\mathcal{A}$ be as above, and $P\in\mathrm{Rep}_k(\mathrm{G})_{\mathcal{A}}$. Then $P^\vee\in\mathrm{Rep}_k(\mathrm{G})_{\mathcal{A}^\vee}$, where $\mathcal{A}^\vee$ consists of the* G*-inertially equivalent supercuspidal classes $[\mathrm{M},\pi^\vee]$ such that $[\mathrm{M},\pi]\in\mathcal{A}$.*

Proof. Suppose there exists an irreducible subquotient π of $P^\vee$. Denote by $[\mathrm{M}_0,\tau_0]$ its supercuspidal support and by $[\mathrm{L}_0,\pi_0]$ is cuspidal support. There is non-trivial morphism $\mathcal{P}_{[\mathrm{L}_0,\pi_0]}\to P^\vee$, which implies a non-trivial morphism $P\to\mathcal{P}^\vee_{[\mathrm{L}_0,\pi_0]}$. Since $\mathcal{P}^\vee_{[\mathrm{L}_0,\pi_0]}$ belongs to the block $\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{M}_0,\tau_0^\vee]}$ by [8, Corollary 11.7], we have $[\mathrm{M}_0,\tau_0^\vee]\in\mathcal{A}$ by the Bernstein decomposition [8, Theorem 11.8]. □

Proposition 3.12. *We keep the notations as in Lemma 3.10. Let ρ' be an irreducible subquotient of the contragredient $\mathcal{P}^\vee_{[\mathrm{M}',\pi']}$, then the supercuspidal support of ρ' is contained in union of* G*-conjugacy classes of $[\mathrm{L}',\tau'^\vee]$. In other words, let $\tau|_{\mathrm{L}'}=\oplus_{i\in I}\tau'_i$, then the supercuspidal support of ρ' is contained in $\bigcup_{i\in I}[\mathrm{L}',\tau_i'^\vee]$.*

Proof. Let $\mathcal{P}'$ be $\mathcal{P}_{[\mathrm{M}',\pi']}$ in this proof. Since there is no non-trivial character on G', we have $(\mathrm{ind}_{\mathrm{G}'}^{\mathrm{G}}\mathcal{P}')^\vee\cong\mathrm{Ind}_{\mathrm{G}'}^{\mathrm{G}}\mathcal{P}'^\vee$. By Lemma 3.10 and Lemma 3.11, we have

$$(\mathrm{Ind}_{\mathrm{G}'}^{\mathrm{G}}\mathcal{P}')^\vee\in\prod_{\chi\in(\mathfrak{o}_F^\times)^\vee}\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{L},\tau^\vee\otimes\chi]}.$$

By the surjective morphism $\mathrm{res}_{\mathrm{G}'}^{\mathrm{G}}\mathrm{Ind}_{\mathrm{G}'}^{\mathrm{G}}\mathcal{P}'^\vee\to\mathcal{P}'^\vee$ and Lemma 3.9, we conclude that the supercuspidal support of an arbitrary irreducible subquotient of $\mathcal{P}'^\vee$ belongs to the G-conjugation of $[\mathrm{L}',\tau']$. □

Definition 3.13. Let (L_1,τ_1) and (L_2,τ_2) be cuspidal pairs of G. We say they are **G-twist equivalent**, if there exists $g\in\mathrm{G}$ such that $g(\mathrm{L}_1)=\mathrm{L}_2$ and $g(\tau_1)$ is isomorphic to τ_2 up to a k-quasicharacter of $F^\times$, which is an equivalence relation and denote by $[\mathrm{L}_1,\tau_1]^{tw}$ the G-twist equivalent class defined by (L_1,τ_1).

We observe that in the above definition, we do not require the k-quasicharacter of $F^\times$ is unramified, which is different comparing to the relation of G-inertial equivalence. We define the depth of a G-twist equivalent class as the minimal depth among all the pairs inside this class. Denote by $\mathcal{C}_{[\mathrm{L},\tau]^{tw}}$ the set of G-twist equivalent cuspidal classes whose supercuspidal support belong to $[\mathrm{L},\tau]^{tw}$ up to an isomorphism, and denote by

$\mathcal{C}_{\overline{[\mathrm{L},\tau]^{tw}}}$ the set of G-twist equivalent cuspidal classes whose supercuspidal support does not belong to $[\mathrm{L},\tau]^{tw}$ up to an isomorphism. It is worth noticing that

1. $\mathcal{C}_{[\mathrm{L},\tau]^{tw}}$ is a finite set;
2. fix a positive number $n \in \mathbb{N}$, there are only finitely many object in $\mathcal{C}_{\overline{[\mathrm{L},\tau]^{tw}}}$ whose depth is smaller than n.

Define

$$\mathcal{I}_{(\mathrm{L},\tau)} = \bigoplus_{[\mathrm{M}',\pi']\in\mathcal{C}'_{[\mathrm{L},\tau]^{tw}}} \mathcal{P}^{\vee}_{[\mathrm{M}',\tau'^{\vee}]},$$

$$\mathcal{I}_{\overline{(\mathrm{L},\tau)}} = \bigoplus_{[\mathrm{M}',\pi']\in\mathcal{C}'_{\overline{[\mathrm{L},\tau]^{tw}}}} \mathcal{P}^{\vee}_{[\mathrm{M}',\tau'^{\vee}]},$$

where the relation between $\mathcal{C}_{[\mathrm{L},\tau]^{tw}}$ and $\mathcal{C}'_{[\mathrm{L},\tau]^{tw}}$ is as explained in the beginning of Section 3.2.

Lemma 3.14. *$\mathcal{I}_{\overline{(\mathrm{L},\tau)}}$ is injective.*

Proof. In fact $\mathcal{I}_{\overline{(\mathrm{L},\tau)}}$ is the smooth part of the contragredient $\prod_{[\mathrm{M}',\pi']\in\mathcal{C}'_{\overline{[\mathrm{L},\tau]^{tw}}}} \mathcal{P}^{*}_{[\mathrm{M}',\tau'^{\vee}]}$ of $\oplus_{[\mathrm{M}',\pi']\in\mathcal{C}'_{\overline{[\mathrm{L},\tau]^{tw}}}}\mathcal{P}_{[\mathrm{M}',\tau'^{\vee}]}$, where $\mathcal{P}^{*}_{[\mathrm{M}',\tau'^{\vee}]}$ is the contragredient (not necessarily smooth) of $\mathcal{P}_{[\mathrm{M}',\tau'^{\vee}]^{tw}}$. Fix an open compact subgroup K' of G', there exist finitely many $[\mathrm{M}',\pi'] \in \mathcal{C}'_{\overline{[\mathrm{L},\tau]^{tw}}}$ such that the K-invariant part of $\mathcal{P}_{[\mathrm{M}',\tau'^{\vee}]}$ is non-trivial, which implies the same property for the contragredient $\mathcal{P}^{*}_{[\mathrm{M}',\tau'^{\vee}]}$ by [12, §4.15]. Hence an K-invariant non-trivial linear form f in the smooth part of $\prod_{[\mathrm{M}',\pi']\in\mathcal{C}'_{\overline{[\mathrm{L},\tau]^{tw}}}} \mathcal{P}^{*}_{[\mathrm{M}',\tau'^{\vee}]}$ must belong to $\oplus_{[\mathrm{M}',\pi']\in\mathcal{C}'_{\overline{[\mathrm{L},\tau]^{tw}}}}\mathcal{P}^{\vee}_{[\mathrm{M}',\tau'^{\vee}]}$, which finishes the proof. □

Theorem 3.15. *Let G be a Levi subgroup of $\mathrm{GL}_n(F)$ and G' be a Levi subgroup of $\mathrm{SL}_n(F)$, such that $\mathrm{G}' = \mathrm{G} \cap \mathrm{SL}_n(F)$. We have a category decomposition*

$$\mathrm{Rep}_k(\mathrm{G}') \cong \prod_{[\mathrm{L},\tau]^{tw}\in\mathcal{SC}^{tw}_{\mathrm{G}}} \mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L},\tau]^{tw}},$$

where

1. *$\mathcal{SC}^{tw}_{\mathrm{G}}$ is the set of G-twist equivalent supercuspidal classes in G;*
2. *$\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L},\tau]^{tw}}$ is the full subcategory consisting with the objects whose irreducible subquotients have supercuspidal support belonging to $[\mathrm{L}',\tau']_{\mathrm{G}}$ and τ' is an irreducible direct component of $\tau|_{\mathrm{L}'}$.*

In particular, each object in $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L},\tau]^{tw}}$ has an injective resolution with direct sums of copies of $\mathcal{I}_{(\mathrm{L},\tau)}$.

Proof. By the definition of $\mathcal{I}_{(\mathrm{L},\tau)}$, Corollary 3.7 and Proposition 3.12, each irreducible subquotient of $\mathcal{I}_{(\mathrm{L},\tau)}$ is a subrepresentation of $\mathcal{I}_{(\mathrm{L},\tau)}$, and none of the irreducible subquotient of $\mathcal{I}_{\overline{(\mathrm{L},\tau)}}$ appears as a subquotient of $\mathcal{I}_{(\mathrm{L},\tau)}$. Furthermore, each irreducible k-representation is either a subrepresentation of $\mathcal{I}_{(\mathrm{L},\tau)}$ or a subrepresentation of $\mathcal{I}_{\overline{(\mathrm{L},\tau)}}$ by the unicity of cuspidal support as well as the unicity of supercuspidal support. Hence by Proposition 2.1, for any object $\Pi \in \mathrm{Rep}_k(\mathrm{G}')$ and any G-twist equivalent supercuspidal class $[\mathrm{L},\tau]^{tw}$ of G, define $\Pi_{[\mathrm{L},\tau]^{tw}}$ to be the largest subrepresentation of Π belonging to $\mathrm{Rep}_k(\mathrm{G}')_{[rL,\tau]^{tw}}$, we have $\Pi \cong \oplus_{[\mathrm{L},\tau]\in\mathcal{SC}_{\mathrm{G}}}\Pi_{[\mathrm{L},\tau]^{tw}}$, and by applying Proposition 2.1 we know that there is no morphism between objects of sub-categories defined from different G-twist equivalent supercuspidal classes, hence we finish the proof. □

Remark 3.16. Let (L,τ) be a supercuspidal pair of G, and $\tau|_{\mathrm{L}'} \cong \oplus_{j=1}^{s}\tau_j'$, where (L',τ_j') are supercuspidal pairs of G′. Denote by $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L}',\tau_j']}$ the full subcategory of $\mathrm{Rep}_k(\mathrm{G}')$, consisting of objects of which any irreducible subquotient has supercuspidal support belonging to the G′-inertially equivalent class $[\mathrm{L}',\tau_j']$. The subcategory $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L},\tau]^{tw}}$ is generated by sub-categories $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L}',\tau_j']}$, for all $1 \leq j \leq s$. In other words, let $\mathcal{SC}_{\mathrm{G}'}^{\mathrm{G}}$ be the set of G-inertially equivalent supercuspidal classes of G′. Theorem 3.15 establishes a category decomposition of $\mathrm{Rep}_k(\mathrm{G}')$ with respect to $\mathcal{SC}_{\mathrm{G}'}^{\mathrm{G}}$.

Corollary 3.17. *Let $[\mathrm{L},\tau]$ be a G-inertially equivalent class of G, where G is a Levi subgroup of $\mathrm{GL}_n(F)$. The functor $\mathrm{res}_{\mathrm{G}'}^{\mathrm{G}}$ gives functors from blocks $\mathrm{Rep}_k(\mathrm{G})_{[\mathrm{L},\tau\otimes\chi]}$ for any k-quasicharacter χ of $F^{\times}$ to the subcategory $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{L},\tau]^{tw}}$.*

Proof. It follows directly from Theorem 3.15 and Lemma 3.9. □

Corollary 3.18. *Let G′ be a Levi subgroup of $\mathrm{SL}_n(F)$. There is a category decomposition*

$$\mathrm{Rep}_k(\mathrm{G}') \cong \mathrm{Rep}_k(\mathrm{G}')_{\mathcal{SC}} \times \mathrm{Rep}_k(\mathrm{G}')_{non-\mathcal{SC}},$$

where

1. *an object belongs to $\mathrm{Rep}_k(\mathrm{G}')_{\mathcal{SC}}$, if and only if all its irreducible subquotients are supercuspidal;*
2. *an object belongs to $\mathrm{Rep}_k(\mathrm{G}')_{non-\mathcal{SC}}$, if and only if none of its irreducible subquotients is supercuspidal.*

Proof. Directly from Theorem 3.15. □

Definition 3.19. We call $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ the supercuspidal sub-category of $\mathrm{Rep}_k(\mathrm{M}')$, and the blocks of $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ are called supercuspidal blocks of $\mathrm{Rep}_k(\mathrm{M}')$.

4. Supercuspidal subcategory of $\mathbf{Rep}_k(\mathrm{M}')$

In this section, let G be $\mathrm{GL}_n(F)$ and G′ be $\mathrm{SL}_n(F)$. In the previous section, Theorem 3.15 gives a category decomposition of $\mathrm{Rep}_k(\mathrm{G}')$, according to which we define the supercuspidal subcategory $\mathrm{Rep}_k(\mathrm{G}')_{\mathcal{SC}}$. In this section, Theorem 4.12 gives a description of the blocks of the supercuspidal subcategory of $\mathrm{Rep}_k(\mathrm{G}')$ and $\mathrm{Rep}_k(\mathrm{M}')$, where M′ is a Levi subgroup of G′.

4.1. M′*-inertially equivalent supercuspidal classes*

In this section, we give a bijection between M′-conjugacy classes of maximal simple k-types of M′, and M′-inertially equivalent cuspidal classes of M′. The most complexity of this section comes from the fact that the Levi subgroup of G′ is not a special linear group in lower rank.

Let M be a Levi subgroup of G such that $\mathrm{M}' = \mathrm{M} \cap \mathrm{G}'$. Let $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ be a maximal simple k-type of M′ defined from a maximal simple k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ of M. As explained in Section 2.2, if π is an irreducible cuspidal k-representation of M containing $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$, then there exists a direct component π' of $\pi|_{\mathrm{M}'}$, such that π' contains $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$.

Lemma 4.1. *Let E be a field extension of F, such that there is an embedding $E^{\times} \hookrightarrow \mathrm{GL}_n(F)$. Let ϖ_E be a uniformiser of E, and Z_{ϖ_E} be a subgroup of $\mathrm{GL}_n(F)$ generated by the image of ϖ_E under the embedding. Then a k-character of Z_{ϖ_E} can be extended to a character of $\mathrm{GL}_n(F)$.*

Proof. A k-character of Z_{ϖ_E} factors through determinant of $\mathrm{GL}_n(F)$. □

Under the assumption on E as in Lemma 4.1, denote by $Z_{\mathcal{O}_E}$ the group generated by the image of $\mathcal{O}_E^{\times}$ under the embedding. For general Levi subgroup M of $\mathrm{G} = \mathrm{GL}_n(F)$. Suppose M is a direct product of m general linear groups, and there exist field extensions $E_i, 1 \leq i \leq m$ of F, such that $\prod_{i=1}^{m} E_i^{\times} \hookrightarrow \mathrm{M}$. Then after fixing a uniformiser ϖ_i for each E_i, we denote by $Z_{\varpi_{E_{\mathrm{M}}}}$ the group generated by the image of $\{1 \times \cdots \times \varpi_i \times \cdots \times 1, 1 \leq i \leq m\}$ under the embedding, and by $Z_{\mathcal{O}_{E_{\mathrm{M}}}}$ the group generated by the image of $\prod_{i=1}^{m} \mathcal{O}_i^{\times}$, where $\mathcal{O}_i$ is the ring of integers of E_i. It is obvious that the image of $\prod_{i=1}^{m} E_i^{\times}$ can be decomposed as a direct product $Z_{\varpi_{E_{\mathrm{M}}}} \times Z_{\mathcal{O}_{E_{\mathrm{M}}}}$. In particular, when $E_i = F$ for $1 \leq i \leq m$, we consider the canonical embedding, which is the equivalence between $(F^{\times})^m$ and the centre of M. Then the centre Z_{M} of M decomposes as $Z_{\varpi_{F_{\mathrm{M}}}} \times Z_{\mathcal{O}_{F_{\mathrm{M}}}}$. We denote by $Z'_{\varpi_{E_{\mathrm{M}}}}$ as $Z_{\varpi_{E_{\mathrm{M}}}} \cap \mathrm{M}'$ and $Z'_{\mathcal{O}_{E_{\mathrm{M}}}}$ as $Z_{\mathcal{O}_{E_{\mathrm{M}}}} \cap \mathrm{M}'$.

Remark 4.2. Lemma 4.1 implies that a k-character of $Z_{\varpi_{F_{\mathrm{M}}}}$ can be extended to a k-character of M. In particular, for two irreducible k-representations of M, if their central characters coincide to each other on $Z_{\mathcal{O}_{F_{\mathrm{M}}}}$, then up to modifying by an unramified k-character, they share the same central character.

Proposition 4.3. *Let π_1, π_2 be two irreducible cuspidal k-representations of* M' *which contain* $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$. *Then there exists an unramified k-character χ of $F^\times$, such that $\pi_1 \cong \pi_2 \otimes \chi$.*

Proof. Let $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})$ be the normaliser of $\tilde{\lambda}'_{\mathrm{M}}$ in M', which contains the centre $Z_{\mathrm{M}'}$ of M' as mentioned in Section 2.2, then by Theorem 4.4 of [4] there exist extensions $\Lambda_{\mathrm{M}',1}, \Lambda_{\mathrm{M}',2}$ of $\tilde{\lambda}'_{\mathrm{M}}$ to $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})$, such that $\pi_1 \cong \mathrm{ind}_{N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})}^{\mathrm{M}'} \Lambda_{\mathrm{M}',1}$ and $\pi_2 \cong \mathrm{ind}_{N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})}^{\mathrm{M}'} \Lambda_{\mathrm{M}',2}$.

After modifying an unramified k-character of M', we can assume that $\Lambda_{\mathrm{M}',1}$ and $\Lambda_{\mathrm{M}',2}$ have the same central character on $Z_{\mathrm{M}'}$. In fact, we have $Z'_{\mathcal{O}_{F_{\mathrm{M}}}} \subset J'_{\mathrm{M}} \subset \tilde{J}'_{\mathrm{M}}$, hence the central characters of $\Lambda_{\mathrm{M}',1}$ and $\Lambda_{\mathrm{M}',2}$ coincide on $Z'_{\mathcal{O}_{F_{\mathrm{M}}}}$. On the other hand, since $Z_{\varpi_{F_{\mathrm{M}}}} \cong \mathbb{Z}^m$ for an integer m decided by M, a character of a sub-$\mathbb{Z}$-module of $Z_{\varpi_{F_{\mathrm{M}}}}$ can be extended to $Z_{\varpi_{F_{\mathrm{M}}}}$. In particular, we can extend a character of $Z'_{\varpi_{F_{\mathrm{M}}}}$ to $Z_{\varpi_{F_{\mathrm{M}}}}$, then to M by Lemma 4.1, finally restricting to M'. Hence we prove that a character of $Z'_{\varpi_{F_{\mathrm{M}}}}$ can be extended to M'. Combining with the above discussion, we conclude that there is an unramified k-character χ_1 of M', such that $\Lambda_{\mathrm{M}',1} \otimes \chi_1|_{Z_{\mathrm{M}'}\tilde{J}'_{\mathrm{M}}} \cong \Lambda_{\mathrm{M}',2}|_{Z_{\mathrm{M}'}\tilde{J}'_{\mathrm{M}}}$. By the Frobenius reciprocity, there is an injection

$$\Lambda_{\mathrm{M}',1} \otimes \chi_1 \hookrightarrow \Lambda_{\mathrm{M}',2} \otimes \mathrm{ind}_{Z_{\mathrm{M}'}\tilde{J}'_{\mathrm{M}}}^{N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})} \mathbb{1}. \tag{4}$$

As observed in Remark 2.42 of [4], the group $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})$ (see Section 2.2 for definition) is a subgroup with finite index of $E^\times_{\mathrm{M}} \tilde{J}_{\mathrm{M}} \cap \mathrm{M}'$, where $E^\times_{\mathrm{M}} \cong \prod_{i=1}^m E_i^\times$ and E_i is a field extension of F for each $1 \leq i \leq m$. Since the quotient group $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})/Z'_{\mathrm{M}}\tilde{J}'_{\mathrm{M}}$ is isomorphic to a subquotient group of $Z_{\varpi_{E_{\mathrm{M}}}}$, hence a character of $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})/Z_{\mathrm{M}'}\tilde{J}'_{\mathrm{M}}$ can be extended to a character of M by Lemma 4.1, hence a character of M'.

Now we look back to Equation (4). The k-representation $\mathrm{ind}_{Z_{\mathrm{M}'}\tilde{J}'_{\mathrm{M}}}^{N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})} \mathbb{1}$ has finite length and each of its irreducible subquotient is a character of $N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})/Z'_{\mathrm{M}}\tilde{J}'_{\mathrm{M}}$, hence can be viewed as a character of M'. By the unicity of Jordan-Hölder factors, there exists a character χ_2 of M', such that $\Lambda_{\mathrm{M}',1} \otimes \chi_1 \cong \Lambda_{\mathrm{M}',2} \otimes \chi_2$, since χ_1, χ_2 are k-characters of M', applying the induction functor $\mathrm{ind}_{N_{\mathrm{M}'}(\tilde{\lambda}'_{\mathrm{M}})}^{\mathrm{M}'}$ on both sides gives an equivalence that $\pi_1 \otimes \chi_1 \cong \pi_2 \otimes \chi_2$. Define χ to be $\chi_2 \chi_1^{-1}$, which is the required unramified k-character of M'. □

Proposition 4.4. *Let* $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ *be a maximal simple k-type of* M', *and π' an irreducible k-representation of* M' *containing* $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$. *Then any irreducible subquotient of* $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\lambda}'_{\mathrm{M}}$ *must belong to* $[\mathrm{M}', \pi']_{\mathrm{M}'}$, *or equivalently saying, must be* M'*-inertially equivalent to π'.*

Proof. By Proposition IV.1.6 of [13], we know that $\mathrm{ind}_{J_{\mathrm{M}}}^{\mathrm{M}} \lambda_{\mathrm{M}}$ is cuspidal, hence its subrepresentation $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\lambda}'_{\mathrm{M}}$ is cuspidal as well. Let π_0 be an irreducible subquotient of $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\lambda}'_{\mathrm{M}}$, and (J'_0, λ'_0) a maximal simple k-type contained in π_0. The latter is weakly intertwined with $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ by Mackey's theory. By the property of weakly intertwining implying conjugacy of maximal simple k-types of M' in Theorem 3.25 of [4], we conclude that a maximal simple k-type contained in π_0 must M'-conjugate to $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$, and hence π_0 contains $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$. By Proposition 4.3, we conclude that π_0 is M'-inertially equivalent to π'. □

Remark 4.5.

1. Lemma 4.3 and Proposition 4.4 give a bijection between the set of M'-conjugacy classes of maximal simple k-types and the set of M'-inertially equivalent cuspidal classes:

$$\nu : [\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}}]_{\mathrm{M}'} \mapsto [\mathrm{M}', \pi']_{\mathrm{M}'},$$

   where $[\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}}]_{\mathrm{M}'}$ is the M'-conjugacy class of $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$, and π' is an irreducible cuspidal k-representation that contains $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$.
2. Let $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_1)$ and $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_2)$ be two different maximal simple k-types defined by $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$. When $\mathrm{M}' = \mathrm{G}' = \mathrm{SL}_n(F)$ by Lemma 3.3, the associated G'-inertially equivalent cuspidal classes defined by $(\tilde{J}_{\mathrm{M}'}, \tilde{\lambda}_i), i = 1, 2$ are different. When M' is a proper Levi of $\mathrm{SL}_n(F)$ by Remark 3.4, the associated G'-inertially equivalent cuspidal classes may be the same.

4.2. Supercuspidal blocks of $\mathrm{Rep}_k(\mathrm{M}')$

In this Section, we give a block decomposition of the supercuspidal subcategory $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ of $\mathrm{Rep}_k(\mathrm{M}')$, of which the blocks are called supercuspidal blocks of $\mathrm{Rep}_k(\mathrm{M}')$ as defined in the end of Section 3.2. Let $[\mathrm{M}', \pi']_{\mathrm{M}'}$ be a M'-inertially equivalent supercuspidal class of M'. Denote by $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M}', \pi']}$ the full subcategory of $\mathrm{Rep}_k(\mathrm{M}')$, such that the irreducible subquotients of an object of $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M}', \pi']}$ belong to $[\mathrm{M}', \pi']_{\mathrm{M}'}$. As in Proposition of [13][§III], a subcategory $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M}', \pi']}$ is non-split, and a block of $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ is generated by a finitely number of subcategories of the form $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M}', \pi']}$.

Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple supercuspidal k-type of M, and $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M}})$ be a maximal simple supercuspidal k-type defined from $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ as explained in Section 2.2. Recall that $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ is the projective cover of $\tilde{\lambda}'_{\mathrm{M}}$. By Lemma 3.3, its irreducible subquotients are maximal simple supercuspidal k-types of M' as well, and we denote by $\mathcal{I}(\tilde{\lambda}'_{\mathrm{M}})$ the set of isomorphic classes of irreducible subquotients of $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$. We define a set of M'-inertially equivalent supercuspidal classes $\mathcal{SC}(\tilde{\lambda}'_{\mathrm{M}})$, such that there is a bijection

$$\nu : \mathcal{I}(\tilde{\lambda}'_{\mathrm{M}}) \to \mathcal{SC}(\tilde{\lambda}'_{\mathrm{M}}),$$

which is given as in Remark 4.5.

Proposition 4.6. *Suppose that the image $\mathcal{SC}(\tilde{\lambda}'_{\mathrm{M}})$ is not a singleton. For any non-trivial disjoint union $\mathcal{SC}(\tilde{\lambda}'_{\mathrm{M}}) = \mathcal{SC}_1 \sqcup \mathcal{SC}_2$, and let $\mathcal{I}(\tilde{\lambda}'_{\mathrm{M}}) = \mathcal{I}_1 \sqcup \mathcal{I}_2$ such that $\mathcal{SC}_1 = \nu(\mathcal{I}_1)$ and*

$\mathcal{SC}_2 = \nu(\mathcal{I}_2)$. *It is not possible to decompose* $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'}\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ *as* $P_1 \oplus P_2$*, where any irreducible subquotients of* P_1 *belongs to* $\mathcal{SC}_1$ *and any irreducible subquotients of* P_2 *belongs to* $\mathcal{SC}_2$*.*

Proof. We abbreviate $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'}\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ by $\mathcal{P}_{\mathrm{M}'}$ in this proof. By Theorem 3.15, the irreducible subquotients of $\mathcal{P}_{\mathrm{M}'}$ are supercuspidal. Suppose the contrary that, there exists a non-trivial disjoint union $\mathcal{SC}(\tilde{\lambda}'_{\mathrm{M}}) = \mathcal{SC}_1 \sqcup \mathcal{SC}_2$, such that $\mathcal{P}_{\mathrm{M}'} = P_1 \oplus P_2$ verifying the conditions in the statement of the proposition. Without loss of generality, we suppose $\tilde{\lambda}'_{\mathrm{M}} \in \mathcal{I}_1$. Let ι'_{M} be a maximal simple supercuspidal k-type in $\mathcal{I}_2$, and τ' be a supercuspidal k-representation of M' containing ι'_{M}. Hence τ' is a subrepresentation of $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'}\iota_{\mathrm{M}'}$, and the later is a subquotient of $\mathcal{P}_{\mathrm{M}'}$, hence P_2 is non-trivial (P_1 is also non-trivial since $\tilde{\lambda}'_{\mathrm{M}} \in \mathcal{I}_1$).

By Lemma 3.3, there exists a filtration of $\{0\} = W_0 \subset W_1 \cdots \subset W_s = \mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ for an $s \in \mathbb{N}$, such that each quotient $\tilde{\lambda}'_i := W_i/W_{i-1}, 1 \leq i \leq s$ is irreducible and $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_i)$ is a maximal simple supercuspidal k-type of M' defined also from $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$. In particular, $\tilde{\lambda}'_{\mathrm{M}}$ as well as ι'_{M} are isomorphic to $\tilde{\lambda}'_i$ for some $0 \leq i \leq s$ respectively. Now define $\tilde{\lambda}'_0$ to be null, and denote by $V_i = \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} W_i$, then $\{V_i\}_{0 \leq i \leq s}$ is a filtration of $\mathcal{P}_{\mathrm{M}'}$ and $V_i/V_{i-1} \cong \mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\lambda}'_i, 1 \leq i \leq s$. Denote by $V_{i,1}$ the image of V_i in P_1 under the canonical projection, and $V_{i,2}$ the image of V_i in P_2 under the canonical projection. Hence $\{V_{i,1}\}_{0 \leq i \leq s}$ (resp. $\{V_{i,2}\}_{0 \leq i \leq s}$) forms a filtration of P_1 (resp. P_2). By Proposition 4.4, the quotient $V_{i,1}/V_{i-1,1}$ (resp. $V_{i,2}/V_{i-1,2}$) is non-trivial if and only if $\tilde{\lambda}'_i \in \mathcal{I}_1$ (resp. $\tilde{\lambda}'_i \in \mathcal{I}_2$).

Now we consider the canonical injective morphism

$$\alpha : \mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}} \hookrightarrow \mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \mathcal{P}_{\mathrm{M}'}.$$

Under the above assumption, we have $\mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \mathcal{P}_{\mathrm{M}'} \cong \mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} P_1 \oplus \mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} P_2$. Since we consider a representation of infinite length, the unicity of Jordan-Hölder factors is not sufficient, and we need a simple but practical lemma as below to continue the proof: □

Lemma 4.7. *Let* G *be a locally pro-finite group, and* π *a* k*-representation of* G*. Let* π_1 *be a subrepresentation of* π*. Suppose* τ *is an irreducible subquotient of* π*, then* τ *is either isomorphic to an irreducible subquotient of* π_1 *or to an irreducible subquotient of* π/π_1*.*

Proof. Easy to check. □

Continue the proof of Proposition 4.6. Suppose $\alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}) \subset \mathrm{res}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} P_1$. Let $\iota'_{\mathrm{M}} \in \mathcal{I}_2$ be an irreducible subquotient of $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$. By Lemma 4.7 there exists $1 \leq i \leq s$, such that ι'_{M} is an irreducible subquotient of $V_{i,1}/V_{i-1,1}$, and the later is a subquotient of $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\lambda}'_i$. In other words, ι'_{M} is an irreducible subquotient of $\mathrm{ind}_{\tilde{J}'_{\mathrm{M}}}^{\mathrm{M}'} \tilde{\lambda}'_i$. Applying Mackey's theorem, it is equivalent to say that ι'_{M} is weakly intertwined with $\tilde{\lambda}'_i$ in M' (see Section 2.2 for weakly intertwining), hence by Theorem 3.25 of [4] they are M'-conjugate to each other, hence they define the same M'-inertially equivalent class as in Remark 4.5. Meanwhile, by the above analysis, we know that $\nu(\tilde{\lambda}_i) \in \mathcal{SC}_1$ and $\nu(\iota'_{\mathrm{M}}) \in \mathcal{SC}_2$, which is a contradiction. Hence $\alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}) \cap \mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}} P_1 \neq \alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}})$.

Now we consider $\alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}})/(\alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}) \cap \mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}} P_1)$, which is non-null as above, and is a subrepresentation of $\mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}} \mathcal{P}_{\mathrm{M}'}/\mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}} P_1 \cong \mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}} P_2$. By the same manner as above, we conclude that each irreducible subquotient of $\alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}})/(\alpha(\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}) \cap \mathrm{res}^{\mathrm{M}'}_{\tilde{J}'_{\mathrm{M}}} P_1)$ belongs to $\mathcal{I}_2$, which implies that there exists $\tilde{\lambda}'_{i_0} \in \mathcal{I}_2$ such that $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}} \to \tilde{\lambda}'_{i_0}$. Since $\tilde{\lambda}'_{i_0}$ is different from $\tilde{\lambda}'_{\mathrm{M}}$, the maximal semisimple quotient of $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ contains $\tilde{\lambda}'_{i_0} \oplus \tilde{\lambda}'_{\mathrm{M}}$, which contradicts to the fact that $\mathcal{P}_{\tilde{\lambda}'_{\mathrm{M}}}$ is the projective cover of $\tilde{\lambda}'_{\mathrm{M}}$ by Proposition 41 *c*) [10]. Hence we finish the proof. □

Lemma 4.8. *Let $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_1)$ and $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_2)$ be two maximal simple supercuspidal k-types. Suppose $\tilde{\lambda}'_2 \in \mathcal{I}(\tilde{\lambda}'_1)$, then $\tilde{\lambda}'_1 \in \mathcal{I}(\tilde{\lambda}'_2)$ (see the beginning of this section for the definition of $\mathcal{I}(\cdot)$).*

Proof. Let $W(k)$ be the ring of Witt vectors of k, and $\mathcal{K}$ be the fractional field of $W(k)$. Let $\tilde{\mathcal{K}}$ be a finite field extension of $\mathcal{K}$, such that $\tilde{\mathcal{K}}$ contains the $|\tilde{J}_{\mathrm{M}}/N|$-th roots, where N is the kernel of $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}}$, and let $\tilde{\mathcal{O}}$ be its ring of integers. Consider the ℓ-modular system $(\tilde{\mathcal{K}}, \tilde{\mathcal{O}}, k)$, we have that $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{K}}$ is semisimple, whose direct components are absolutely irreducible. By Proposition 42 of [10], the projective cover $\mathcal{P}_{\tilde{\lambda}'_1}$ can be lifted over $\tilde{\mathcal{O}}$, and we denote the lifting to $\tilde{\mathcal{O}}$ by $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}}$ as well. Now we consider $\mathcal{P}_{\tilde{\lambda}'_1} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{K}}$, which is semisimple with finite length. Suppose P is an irreducible component of $\mathcal{P}_{\tilde{\lambda}'_1} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{K}}$, then the semisimplification of its reduction modulo ℓ must contain $\tilde{\lambda}'_1$, otherwise it will induce a surjection from $\mathcal{P}_{\tilde{\lambda}'_1}$ to an irreducible k-representation different from $\tilde{\lambda}'_1$, which contradicts with the fact the $\mathcal{P}_{\tilde{\lambda}'_1}$ is the projective cover of $\tilde{\lambda}'_1$ by Proposition 41 of [10]. Since $\tilde{\lambda}'_2$ is a subquotient of $\mathcal{P}_{\tilde{\lambda}'_1}$, their exists an irreducible direct component P'_2 of $\mathcal{P}_{\tilde{\lambda}'_1} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{K}}$, of which the semisimplification of reduction modulo-ℓ contains $\tilde{\lambda}'_1$ as well as $\tilde{\lambda}'_2$. Let $\alpha \in \tilde{J}_{\mathrm{M}}$, such that $\alpha(\tilde{\lambda}'_1) \cong \tilde{\lambda}'_2$. By the second part of Lemma 3.5, we have $\alpha(\mathcal{P}_{\tilde{\lambda}'_1}) \cong \mathcal{P}_{\tilde{\lambda}'_2}$, which implies that $\alpha(P'_2)$ is a direct component of $\mathcal{P}_{\tilde{\lambda}'_2}$. We state that α stabilises P'_2. In fact, by the proof of Lemma 3.2, we have $\mathcal{P}_{\tilde{\lambda}'_1}$ is an indecomposable direct factor of $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}}$. In particular, the reduction modulo-ℓ of each irreducible components of $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{K}}$ is isomorphic to $\tilde{\lambda}_{\mathrm{M}}$. By the unicity of Jordan-Holdar factors, there exists an irreducible component P_2 of $\mathcal{P}_{\tilde{\lambda}_{\mathrm{M}}} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{K}}$, such that P'_2 is an irreducible component of $P_2|_{\tilde{J}'_{\mathrm{M}}}$. Since $\alpha(\tilde{\lambda}'_1) \cong \tilde{\lambda}'_2$, the semisimplification of the reduction modulo-ℓ of $\alpha(P'_2)$ contains $\tilde{\lambda}'_2$. Since $\alpha(P'_2)$ is isomorphic to an irreducible component of $P_2|_{\tilde{J}'_{\mathrm{M}}}$, and the reduction modulo ℓ of P_2 is isomorphic to $\tilde{\lambda}_{\mathrm{M}}$, combining with the fact that $\tilde{\lambda}_{\mathrm{M}}|_{\tilde{J}'_{\mathrm{M}}}$ is multiplicity-free, we conclude that $\alpha(P'_2) \cong P'_2$. Hence $\tilde{\lambda}'_1 \in \mathcal{I}(\tilde{\lambda}'_2)$. □

Definition 4.9. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple supercuspidal k-type of M, and denote by $\mathcal{I}(\lambda_{\mathrm{M}})$ the set of isomorphic classes of maximal simple supercuspidal k-types of M' defined by $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$. Let $(\tilde{J}'_{\mathrm{M}}, \gamma')$ and $(\tilde{J}'_{\mathrm{M}}, \tau')$ be two elements in $\mathcal{I}(\lambda_{\mathrm{M}})$, we say

1. γ' is related to τ', if $\gamma' \in \mathcal{I}(\tau')$ (or equivalently $\tau' \in \mathcal{I}(\gamma')$ by Lemma 4.8) and we denote by $\gamma' \leftrightarrow \tau'$;
2. $\gamma' \sim \tau'$ if there exists a series $(\tilde{J}'_{\mathrm{M}}, \tilde{\lambda}'_i), 1 \leq i \leq t$ for an integer t, such that

$$\gamma' \leftrightarrow \tilde{\lambda}'_1 \leftrightarrow \cdots \leftrightarrow \tilde{\lambda}'_t \leftrightarrow \tau',$$

   and we call the series $\{\tilde{\lambda}'_i, 1 \leq i \leq t\}$ a connected relation of γ' and τ'. The relation "$\sim$" defines an equivalence relation on $\mathcal{I}(\lambda_{\mathrm{M}})$ (By Proposition 2.6 of [4], the relations $\leftrightarrow$ and $\sim$ on $\mathcal{I}(\lambda_{\mathrm{M}})$ do not depend on the choice of λ_{M}).
3. Denote by $[\tilde{\lambda}'_{\mathrm{M}}, \sim]$ the subset of $\mathcal{I}(\lambda_{\mathrm{M}})$ consisting of all τ' such that $\tau' \sim \tilde{\lambda}'_{\mathrm{M}}$, or equivalently the connected component containing $\tilde{\lambda}'_{\mathrm{M}}$ defined by $\sim$.

Let π be an irreducible supercuspidal k-representation of M, and denote by $\mathcal{I}(\pi)$ the isomorphy classes of the irreducible direct components of $\pi|_{\mathrm{M}'}$. Let $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ be a maximal simple supercuspidal k-type contained in π. The above equivalence relation "$\sim$" on $\mathcal{I}(\lambda_{\mathrm{M}})$ induces an equivalence relation on $\mathcal{I}(\pi)$.

Definition 4.10. Let $\pi'_1, \pi'_2 \in \mathcal{I}(\pi)$, and we say $\pi'_1 \sim \pi'_2$ if there exists a maximal simple supercuspidal k-type $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$ contained in π, and two maximal simple supercuspidal k-types $(\tilde{J}_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M},1})$ and $(\tilde{J}_{\mathrm{M}}, \tilde{\lambda}'_{\mathrm{M},2})$ defined from $(J_{\mathrm{M}}, \lambda_{\mathrm{M}})$, such that π'_i contains $\tilde{\lambda}'_{\mathrm{M},i}$ for $i = 1, 2$, and $\tilde{\lambda}'_{\mathrm{M},1} \sim \tilde{\lambda}'_{\mathrm{M},2}$. By the unicity property that two maximal simple supercuspidal k-types of M$'$, which are contained in a same irreducible supercuspidal k-representation, are M$'$-conjugate to each other (Theorem 3.25 of [4]), we have that "$\sim$" defines an equivalence relation on $\mathcal{I}(\pi)$.

Remark 4.11. Let $\pi' \in \mathcal{I}(\pi)$, and define $[\pi', \sim]$ to be a subset of $\mathcal{I}(\pi)$, consisting of the elements that are equivalent to π'. In other words, $(\pi', \sim)$ is the connected component containing π' under the equivalence relation "$\sim$" on $\mathcal{I}(\pi)$. In particular, there exists a subset $\{\pi'_j, 1 \leq j \leq s\}$ of $\mathcal{I}(\pi)$ for an integer s, such that $(\pi'_j, \sim)$ are two-two disjoint, and $\cup_{j=1}^s (\pi'_j, \sim) = \mathcal{I}(\pi)$. Denote by $[\pi'_j, \sim]$ the family of M$'$-inertially equivalent classes of $\pi' \in (\pi'_j, \sim)$, and we call $[\pi'_j, \sim]$ **a connected M$'$-inertially equivalent class of** π'_j.

By Theorem 3.15, giving a block decomposition of $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$ is equivalent to giving a block decomposition of $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M},\pi]^{tw}}$ for each irreducible supercuspidal k-representation π of M.

Theorem 4.12 *(Block decomposition of* $\mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}}$*). Let* π *be an irreducible supercuspidal k-representation of* M*, and we keep the notations in Remark 4.11. For each* $1 \leq j \leq s$*, define the full subcategory* $\mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j,\sim]}$*, consisting of the objects, of which each irreducible subquotient belongs to* $[\pi'_j, \sim]$*. Then* $\mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j,\sim]}$ *is a block, and the subcategory* $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M},\pi]^{tw}} \cong \prod_{j=1}^s \mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j,\sim]}$.

Proof. First we prove that $\mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j,\sim]}$ is non-split. By Proposition of [13][§III], we only need to prove that for any non-trivial disjoint union $[\pi'_j,\sim] = I_1 \sqcup I_2$, where I_1, I_2 are two non-trivial families of M'-inertially equivalence classes, then there exists an object $P \in \mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j,\sim]}$, such that P cannot be decomposed as $P_1 \oplus P_2$, where $P_1 \in \mathrm{Rep}_k(\mathrm{M}')_{I_1}$ and $P_2 \in \mathrm{Rep}_k(\mathrm{M}')_{I_2}$. Without loss of generality, we assume that $\pi'_j \in I_1$ and let $\pi'_{j_0} \in \mathcal{I}(\pi)$ such that $\pi'_{j_0} \in I_2$. Since $\pi'_j \sim \pi'_{j_0}$, there exists a maximal simple supercuspidal k-type $(J_\mathrm{M}, \lambda_\mathrm{M})$ of π and two maximal simple supercuspidal k-types $(\tilde{J}'_\mathrm{M}, \tilde{\lambda}'_\mathrm{M})$ of π'_j and $(\tilde{J}'_\mathrm{M}, \tau'_\mathrm{M})$ of π'_{j_0}, such that $\tilde{\lambda}'_\mathrm{M} \sim \tau'_\mathrm{M}$ in $\mathcal{I}(\lambda_\mathrm{M})$. By the second part of Definition 4.9, let $\{\tilde{\lambda}'_i, 1 \leq i \leq t\}$ be a series of a connected relation of $\tilde{\lambda}'_\mathrm{M}$ and τ'_M. Define a new series $\{\tilde{\lambda}'_i, 0 \leq i \leq t+1\}$, by putting $\tilde{\lambda}'_0 = \tilde{\lambda}'_\mathrm{M}$ and $\tilde{\lambda}'_{t+1} = \tau'_\mathrm{M}$. There exists $0 \leq i \leq t$, such that $\nu(\tilde{\lambda}'_i) \in I_1$ but $\nu(\tilde{\lambda}'_{i+1}) \in I_2$, where ν is defined as in Remark 4.5. Now we consider $\mathcal{P}_{\mathrm{M}'} := \mathrm{ind}_{\tilde{J}'_\mathrm{M}}^{\mathrm{M}'} \mathcal{P}_{\tilde{\lambda}'_i} \in \mathrm{Rep}_k(\mathrm{M}')_{\mathcal{SC}(\tilde{\lambda}'_i)}$ (see the beginning of Section 4.2 for the definition of $\mathcal{SC}(\tilde{\lambda}'_i)$), hence $\mathcal{P}_{\mathrm{M}'} \in \mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j\sim]}$. Assume contrarily that $\mathcal{P}_{\mathrm{M}'} \cong P_1 \oplus P_2$, where $P_1 \in \mathrm{Rep}_k(\mathrm{M}')_{I_1}$ and $P_2 \in \mathrm{Rep}_k(\mathrm{M}')_{I_2}$. Then $P_1 \in \mathrm{Rep}_k(\mathrm{M}')_{I_1 \cap \mathcal{SC}(\tilde{\lambda}'_i)}$ and $P_2 \in \mathrm{Rep}_k(\mathrm{M}')_{I_2 \cap \mathcal{SC}(\tilde{\lambda}'_i)}$. Since the union of $I_1 \cap \mathcal{SC}(\tilde{\lambda}'_i)$ and $I_2 \cap \mathcal{SC}(\tilde{\lambda}'_i)$ is a non-trivial disjoint union of $\mathcal{SC}(\tilde{\lambda}'_i)$, the decomposition $\mathcal{P}_{\mathrm{M}'} \cong P_1 \oplus P_2$ is contradicted with Proposition 4.6.

Secondly, we prove that $\mathrm{Rep}_k(\mathrm{M}')_{[\mathrm{M},\pi]^{tw}} \cong \prod_{j=1}^{s} \mathrm{Rep}_k(\mathrm{M}')_{[\pi'_j,\sim]}$. We use the projective version in Remark 2.2. Now fix j_0, and let $(\tilde{J}'_\mathrm{M}, \tilde{\lambda}'_{j_0})$ be a maximal simple supercuspidal k-type contained in π'_{j_0}, defined from a maximal simple supercuspidal k-type $(J_\mathrm{M}, \lambda_\mathrm{M})$ of M. By Definition 4.10 and Remark 4.11, we fix a maximal simple supercuspidal k-type for each M'-inertially equivalent supercuspidal class contained in $[\pi'_{j_0},\sim]$, and denote by $\mathcal{I}_{j_0}$ the finite set of these maximal simple supercuspidal k-types. Define $\mathcal{P}_{[\pi'_{j_0},\sim]} := \oplus_{\tau' \in \mathcal{I}_{j_0}} \mathrm{ind}_{\tilde{J}'_\mathrm{M}}^{\mathrm{M}'} \mathcal{P}_{\tau'}$ where $\mathcal{P}_{\tau'}$ is the projective cover of τ'. For each $1 \leq j \leq s$ different from j_0, and let $[\pi'_j,\sim] = \sqcup_{i=1}^{t} [\mathrm{M}', \pi'_{j,i}]_{\mathrm{M}'}$ where $\pi'_{j,i}$ are irreducible supercuspidal and $t \in \mathbb{N}$. Fix a maximal simple supercuspidal k-type $(\tilde{J}'_{j,i}, \tilde{\lambda}'_{j,i})$ contained in $\pi'_{j,i}$. Define $[\pi_{j_0},\sim]^\perp$ to be the union $\cup_{j \neq j_0} [\pi_j,\sim]$ and $\mathcal{P}_{[\pi_{j_0},\sim]^\perp} := \oplus_{j \neq j_0} \oplus_{i=1}^{t} \mathrm{ind}_{\tilde{J}'_{j,i}}^{\mathrm{M}'} \mathcal{P}_{\tilde{\lambda}'_{j,i}}$. We show that $\mathcal{P}_{[\pi_{j_0},\sim]}$ and $\mathcal{P}_{[\pi_{j_0},\sim]^\perp}$ verify the conditions in Remark 2.2. By Proposition 4.4 and Lemma 4.7, we know that an irreducible subquotient of $\mathcal{P}_{[\pi_{j_0},\sim]}$ belong to $[\pi_{j_0},\sim]$. Meanwhile an irreducible subquotient of $\mathcal{P}_{[\pi_{j_0},\sim]^\perp}$ belong to $[\pi_{j_0},\sim]^\perp := \cup_{j \neq j_0} [\pi_j,\sim]$. Condition 1 and 3 of Remark 2.2 can be deduced from Proposition 3.6. Condition 2 of Remark 2.2 is verified from Remark 4.11 that "$\sim$" defines an equivalent relation, and $[\pi'_{j_0},\sim]$ is disjoint with $[\pi_{j_0},\sim]^\perp$. Hence by repeating the same operation on $\mathrm{Rep}_k(\mathrm{M}')_{[\pi_{j_0},\sim]^\perp}$, and after finite times we obtain the desired decomposition. □

Example 4.13. For $\mathrm{G}' = \mathrm{SL}_n(F)$, when ℓ is positive,

- it is not always true that the reduction modulo ℓ of an irreducible ℓ-adic supercuspidal representation of G' is irreducible;
- it is not always true that $\mathrm{Rep}_k(\mathrm{G}')$ can be decomposed with respect to the G'-inertially equivalent supercuspidal classes as in Equation (1) in the case where $\ell = 0$.

Proof. Let $p = 5, n = 2, \ell = 3$, and denote by $\overline{\mathrm{G}} = \mathrm{GL}_2(\mathbb{F}_5)$ and by $\overline{\mathrm{G}}' = \mathrm{SL}_2(\mathbb{F}_5)$. From [1, §11.3.2] we know that there exist two irreducible supercuspidal $\overline{\mathbb{Q}}_\ell$-representations π_1, π_2 of $\overline{\mathrm{G}}$ (π_1 corresponding to $-j^\wedge$ and π_2 corresponding to θ_0 as in [1, §11.3.2]), such that the reduction modulo ℓ of π_1 and π_2 are irreducible and coincide to each other. Meanwhile, the restriction $\pi_1|_{\overline{\mathrm{G}}'}$ is irreducible but $\pi_2|_{\overline{\mathrm{G}}'}$ is semisimple with length 2. We denote by $\bar{\pi}_2$ the reduction modulo ℓ of $\bar{\pi}_2$. By [1, §11.3.2] the length of $\bar{\pi}|_{\overline{\mathrm{G}}'}$ is two, and denote by $\bar{\pi}_{2,1}, \bar{\pi}_{2,2}$ the two irreducible direct components of $\bar{\pi}_2|_{\overline{\mathrm{G}}'}$ (in the notation of [1, §11.3.2], $\bar{\pi}_{2,1}$ and $\bar{\pi}_{2,2}$ correspond to the reduction modulo ℓ of $R'_-(\theta_0)$ and $R'_+(\theta_0)$ respectively). In other words, the reduction modulo ℓ of the irreducible supercuspidal $\overline{\mathbb{Q}}_\ell$-representation $\pi_1|_{\overline{\mathrm{G}}'}$ is reducible, and its Jordan-Hölder components consist of $\bar{\pi}_{2,1}$ and $\bar{\pi}_{2,2}$. Both of $\bar{\pi}_{2,1}$ and $\bar{\pi}_{2,2}$ are supercuspidal by [5, §3.2], since their projective covers are cuspidal.

We consider the $\overline{\mathbb{Z}}_\ell$-projective cover $\mathcal{Y}_{\bar{\pi}_{2,1}}$ of $\bar{\pi}_{2,1}$. The strategy is to prove that the irreducible $\overline{\mathbb{Q}}_\ell$-representation $\pi_1|_{\mathrm{G}'}$ is a subquotient of $\mathcal{Y}_{\bar{\pi}_{2,1}} \otimes \overline{\mathbb{Q}}_\ell$, from which we deduce that $\bar{\pi}_{2,2}$ is a subquotient of $\mathcal{Y}_{\bar{\pi}_{2,1}} \otimes k$, then we apply Proposition 4.6.

Let U be the subgroup of upper triangular matrices of $\overline{\mathrm{G}}$, then the reduction modulo ℓ gives a bijection between non-degenerate $\overline{\mathbb{Q}}_\ell$-characters of U and non-degenerate k-characters of U. Let $\theta_{\overline{\mathbb{Q}}_\ell}$ be a non-degenerate $\overline{\mathbb{Q}}_\ell$-character of U, and θ_ℓ be the reduction modulo ℓ of $\theta_{\overline{\mathbb{Q}}_\ell}$, which is a non-degenerate k-character of U, such that $\bar{\pi}_{2,1}$ is generic according to θ_ℓ. By the unicity of Whittaker model, it follows that $\bar{\pi}_{2,2}$ is not generic according to θ_ℓ. By [5], $\mathcal{Y}_{\bar{\pi}_{2,1}} \otimes \overline{\mathbb{Q}}_\ell$ is semisimple, and can be written as $\oplus_{s\in\mathcal{S}_{s_0}} \pi_{s,\theta_{\overline{\mathbb{Q}}_\ell}}$. Here s_0 is the ℓ'-semisimple conjugacy class in $\overline{\mathrm{G}}$ corresponding to π_2 by the theory of Deligne-Lusztig (or equivalently s_0 corresponds to θ_0 under the notations of [1, §11.3.2]), and $\mathcal{S}_{s_0}$ is the set of semisimple conjugacy classes in $\overline{\mathrm{G}}$ whose ℓ'-part is equal to s_0. Denote by π_s the irreducible supercuspidal $\overline{\mathbb{Q}}_\ell$-representation corresponding to s, and by $\pi_{s,\theta}$ the unique irreducible component of $\pi_s|_{\overline{\mathrm{G}}'}$ which is generic according to $\theta_{\overline{\mathbb{Q}}_\ell}$. Hence π_1 is a subrepresentation of $\mathcal{Y}_{\bar{\pi}_{2,1}} \otimes \overline{\mathbb{Q}}_\ell$, which implies that $\bar{\pi}_{2,2}$ is a subquotient of $\mathcal{Y}_{\bar{\pi}_{2,1}} \otimes k$, which is the k-projective cover of $\bar{\pi}_{2,1}$.

To go further to the p-adic groups, we conclude that the semisimplification of $\mathcal{Y}_{\pi_{2,1}} \otimes k$ consists with a non-trivial multiple of $\pi_{2,1}$ and a non-trivial multiple of $\pi_{2,2}$.

Now we consider the p-adic groups $\mathrm{G} = \mathrm{GL}_2(F)$ and $\mathrm{G}' = \mathrm{SL}_2(F)$, suppose that $F = \mathbb{Q}_5$, and $k = \overline{\mathbb{F}}_3$. Let $J = \mathrm{GL}_2(\mathbb{Z}_5)$, $J^1 = 1 + \mathrm{M}_2(5\mathbb{Z}_5)$ and $J' = J \cap \mathrm{G}'$, $J^{1'} = J^1 \cap \mathrm{G}'$. We have $J/J^1 \cong \overline{\mathrm{G}}$, and $J'/J^{1'} \cong \overline{\mathrm{G}}'$. We still denote by $\pi_i, \bar{\pi}_i, \bar{\pi}_{2,i}, i = 1, 2$ the corresponding inflation to J' respectively. Hence $(J, \bar{\pi}_i), i = 1, 2$ are maximal simple supercuspidal k-types of G. According to [4, 3.18] and the fact that there are 4 G'-conjugacy classes of non-degenerate characters on U, we deduce from the unicity of Whittaker models that for an irreducible cuspidal k-representation π of G, the length of $\pi|_{\mathrm{G}'}$ is a divisor of 4, hence is prime to 5. By Theorem 3.18 of [4], the index $|\tilde{J} : J|$ is a p-power and a divisor of the length $\pi|_{\mathrm{G}'}$, which implies that $\tilde{J} = J$. We deduce firstly that $(J', \pi_1|_{J'})$ is a maximal simple supercuspidal $\bar{\mathbb{Q}}_\ell$-type of G', and $(J', \bar{\pi}_{2,i}), i = 1, 2$ are maximal simple supercuspidal k-types of G'. Hence $\mathrm{ind}_{J'}^{\mathrm{G}'} \pi_1|_{J'}$ is irreducible, but its reduction modulo ℓ has length two, with two factors $\mathrm{ind}_{J'}^{\mathrm{G}'} \bar{\pi}_{2,i}, i = 1, 2$, which is the first of this example. Secondly, we have that $(J', \bar{\pi}_{2,1})$ and $(J', \bar{\pi}_{2,2})$ are non G'-conjugate by the second part of Remark 4.5. By [4, Proposition 2.35, Theorem 3.30], $\Pi_1 := \mathrm{ind}_{J'}^{\mathrm{G}'} \bar{\pi}_{2,1}$ and $\Pi_2 := \mathrm{ind}_{J'}^{\mathrm{G}'} \bar{\pi}_{2,2}$ are different irreducible supercuspidal k-representations, and they define different G'-inertially equivalent classes since there is no non-trivial k-character on G'. The inflation of $\mathcal{Y}_{\bar{\pi}_{2,1}}$ to J' is the $\overline{\mathbb{Z}}_3$-projective cover of $\bar{\pi}_{2,1}$. By applying the previous paragraphs, $\bar{\pi}_{2,2}$ appears as a subquotient of $\mathcal{Y}_{\bar{\pi}_{2,1}}$. Apply Theorem 4.12, we conclude that both the full subcategories $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{G}',\Pi_1]}$ and $\mathrm{Rep}_k(\mathrm{G}')_{[\mathrm{G}',\Pi_2]}$ belong to the same block. □

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