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Transcendental Tools in Quantum Information Theory

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Prindërve, motrave,
dhe gjyshërve të mi.

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"*We are all agreed that your theory is crazy. The question which divides us is whether it is crazy enough to have a chance of being correct.* "

——— Niels Bohr

Zusammenfassung

In dieser Dissertation werden Probleme der Quanteninformationstheorie durch die Brille der transzendentalen Zahlentheorie und der algebraischen Geometrie betrachtet, wobei ein besonderer Schwerpunkt auf der Unterscheidung zwischen semialgebraischen Strukturen und transzendentalen Strukturen liegt. Insbesondere untersuchen wir die Beschaffenheit von Mengen, die für die Untersuchung spezifischer Aufgaben in der Quanteninformationsverarbeitung von zentraler Bedeutung sind, wie z. B. Niveaumengen entropischer Größen und Zustandsmengen, die für die Optimierung von Grundzustandseigenschaften in Quanten-Vielkörpersystemen relevant sind. Darüber hinaus analysieren wir die Auswirkungen der transzendentalen Natur dieser Größen und beweisen No-Go-Resultate für verschiedene entropische Größen. Dieser Rahmen gibt Aufschluss darüber, wann asymptotische Methoden, unendliche Ressourcen oder transzendentale Methoden für Aufgaben der Quanteninformationsverarbeitung unerlässlich sind und wann 'single-letter'-Formeln, endliche Ressourcen oder algebraische Techniken ausreichen.

Abstract

This dissertation addresses problems in quantum information theory through the lens of transcendental number theory and algebraic geometry, with a particular focus on distinguishing semialgebraic structures from transcendental ones. Specifically, it examines the nature of sets that are central to studying specific tasks in quantum information processing, such as level sets of entropic quantities and sets relevant to optimizing ground-state properties in quantum many-body systems. Furthermore, we analyze the implications of the transcendental nature of these sets and establish no-go results for several entropic quantities. This framework provides insights into when asymptotic settings, infinite resources, or transcendental methods are indispensable for quantum information processing tasks, and when single-letter formulas, finite resources, or algebraic techniques suffice.

Contents

Index of Publications

Core articles as principal author

I) Vjosa Blakaj and Michael M. Wolf
"Transcendental properties of entropy-constrained sets"
Annales Henri Poincaré 24, 349-362 (2022)
(see also article [1] in the bibliography, with core factor 1)

II) Vjosa Blakaj and Chokri Manai
"Transcendental properties of entropy-constrained sets II"
Journal of Mathematical Physics 65, 072203 (2024)
(see also article [2] in the bibliography, with core factor 1)

III) Vjosa Blakaj and Michael M. Wolf
"On the set of reduced states of translation invariant, infinite quantum systems"
Letters in Mathematical Physics 114, 28 (2024)
(see also article [3] in the bibliography, with core factor 1)

I, Vjosa Blakaj, am the principal author of the Core Articles I, II, and III, of this dissertation.

Chapter 1

Introduction

The integration of transcendental number theory and semialgebraic geometry opens up new perspectives for addressing problems in quantum information theory and beyond. The transcendental tools derived from these fields help explore the potential and limitations of various information-processing tasks, including the necessity of infinite resources, the feasibility of single-letter formulas, and the relevance of transcendental approaches in quantum information processing. This theme serves as the central focus of this thesis.

1.1 Outline

This section introduces the central questions in quantum information theory and quantum many-body physics addressed in this thesis, and provides an overview of its chapters. A wide range of tasks in quantum information theory, and beyond, depends on understanding the *relevant* foundational sets underlying specific processes. For example, investigating whether information-theoretic quantities with asymptotic operational characterizations can also admit single-shot characterizations, possibly through (correlated) catalysis, requires studying the level sets of functionals that govern such transformations. Similarly, characterizing the set of reduced quantum states of translation-invariant infinite quantum systems could provide significant insights into their ground-state properties.

The building elements of quantum information theory typically involve simple, low-dimensional objects, such as density matrices, quantum channels, and observables. The theory is often developed in finite-dimensional vector spaces, where standard constraints like operator positivity naturally lead to semialgebraic sets—those defined by a finite number of polynomial equalities and inequalities. Semialgebraic sets are particularly prolific due to their diverse closure properties, exemplified by the Tarski-Seidenberg theorem, which ensures that applying the existential or universal quantifiers over semialgebraic sets, preserves their semialgebraic nature. While semialgebraic sets provide a powerful framework for many problems in finite-dimensional quantum information theory and beyond, their applicability can be limited when dealing with non-algebraic functions or limits. Such situations often arise in contexts involving unbounded dimensions or an asymptotic number of copies in information-theoretic tasks, where the constraints of semialgebraic geometry are insufficient. These challenges are particularly evident in questions about

whether infinite resources—such as infinite entanglement or infinite-dimensional catalysts—are essential for certain state transformations. Addressing these scenarios often requires extending beyond the semialgebraic framework to incorporate transcendental methods or alternative approaches. As a result, quantitatively computing the properties of objects in quantum information theory or qualitatively characterizing them is often a significant challenge. These difficulties give rise to the central question of this thesis: Can solutions based on single-shot formulas, finite resources, or finite mean methods effectively address these type of problems?

In this thesis, we identify situations in which semialgebraic single-shot characterizations can be ruled out for various entropic quantities and explore their implications in the context of (correlated) catalysis within quantum resource theories. Additionally, from a transcendental perspective, we examine the structure of the set of reduced bipartite density matrices that arise in translation-invariant, infinite quantum systems, and discuss its relevance to the analysis of ground-state properties in quantum many-body systems.

The remainder of this thesis is organized as follows: Section 1.2, which completes this chapter, provides a summary of the articles included in this thesis. Chapter 2 introduces selected topics in quantum information theory, along with an overview of manifolds, and (von) Neumann algebras. Chapters 3 and 4 present the transcendental tools that underpin the framework used throughout this work: Chapter 3 discusses algebraic and transcendental numbers and their properties, while Chapter 4 focuses on semialgebraic and transcendental sets and functions. Both chapters conclude with a discussion of the relevance of these mathematical structures in quantum information theory and beyond. In Chapter 5, we review several entropic quantities and their key properties, emphasizing how the distinction between semialgebraic and transcendental structures shapes the separation between single-shot and asymptotic settings in quantum information theory. Chapter 6 provides an overview of quantum spin systems and related concepts. Chapter 7 provides one-page summaries of each Core Article, emphasizing individual contributions. Finally, all contributed Core Articles are included, along with the corresponding permissions for their use in this thesis.

1.2 Summary of Results

This thesis builds on a collection of published research articles, with Articles I, II, and III serving as its central contributions, where the author of this thesis is the main author. These Core Articles address key questions in quantum information theory and quantum many-body physics. Core Article I investigates the (im)possibility of single-letter formulas for asymptotically defined quantities by analyzing the (non)semialgebraic structure of the level sets of von Neumann entropy and their connection to catalytic state transformations. We further extend this analysis to level sets constrained by conditions beyond fixed von Neumann entropy. In a similar vein, in Core Article II we delve deeper into the question of ruling out single-shot characterizations for quantum information-theoretic quantities, focusing on the level sets of various entropic quantities, including relative entropy, Rényi entropies, and mutual information. Core Article III explores the existence of a finite-mean description for the set of bipartite reduced density matrices in

translation-invariant infinite quantum systems, offering insights into the ground-state properties of quantum many-body systems.

Core articles as principal author

- *Article I [1]*: *Transecendental properties of entropy-constrained sets*

  In this first Core Article, we propose a method for determining whether information-theoretic quantities, with an asymptotic operational characterization, have an alternative single-shot characterization. This could potentially involve optimization over an ancillary system, and we explore this by studying the (non-)semialgebraic structure of their level sets, i.e., the set of quantum states with a fixed information-theoretic value. If these level sets are non-semialgebraic or transcendental, then a (semi)algebraic single-shot characterization with a finite ancillary system does not exist for the corresponding quantity.

  If expressions in a single-shot characterization are algebraic, and the ancillary system is finite-dimensional, the resulting construction leads to semialgebraic sets. The main contribution of this article is a criterion for distinguishing semialgebraic from non-semialgebraic sets. We show that a set is non-semialgebraic based on the principle that analytic continuations of algebraic functions can only have a finite number of branches. Additionally, when sets are given as level sets of a differentiable function, the normal space of these sets can be particularly accessible via the gradient of the defining function. We apply this resulting theorem to the level sets of von Neumann entropy, showing that these sets are transcendental everywhere for dimensions $d \geq 3$ over the real numbers, while for qubits, they are semialgebraic. This result excludes a characterization of catalytic transformations with a universally bounded catalyst in terms of von Neumann entropy. We extend this analysis by introducing an additional constraint, which could be the distance of the quantum state to a specific target, its energy, or the expectation value of any observable. After establishing the transcendentality of this set, we apply it to the level sets of relative entropy, with a free first argument and an arbitrary but fixed second argument. We show that these sets are nowhere semialgebraic unless the set itself is empty. As an interlude, we provide a classification of rational, algebraic, and transcendental entropy values. A consequence of this result is that when using the natural logarithm, the level sets of von Neumann entropy cannot be semialgebraic over the algebraic numbers unless the entropy is zero.

- *Article II [2]*: *Transecendental properties of entropy-constrained sets II*

  In this second Core Article, we expand on the discussion of the (im)possibility of single-letter formulas, focusing on the transcendental structure of other key entropic quantities: relative entropy, mutual information, and α-Rényi entropies. We complete the proof for the level sets of relative entropy, building on the prior case in Core Article [1], which addressed a fixed second argument. Here, we extend this result to cases where the first argument is fixed and arbitrary while the second is free, as well as the general case where both arguments are free. To establish the transcendentality of these sets in dimensions $d \geq 3$, we apply techniques from algebraic geometry and complex analysis, leveraging the fact that non-algebraic functions possess more than algebraic singularities.

Following this approach, we use irrational poles to show that, for $\alpha \in (0,1) \cup (1,\infty)$, the sets constrained by α-Rényi entropy are everywhere transcendental when parametrized by an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. For rational values of α, by applying the Tarski-Seidenberg quantifier elimination theorem, we demonstrate that these constrained sets are semialgebraic. We also examine the geometry of the level sets of special cases, such as the Max entropy ($\alpha \to 0$) and Min entropy ($\alpha \to \infty$). Furthermore, we prove that the set of bipartite quantum states with fixed quantum mutual information is transcendental. A key technical ingredient here is the transcendentality of von Neumann entropy level sets. For extreme values of quantum mutual information, we reformulate mutual information in terms of conditional entropies and state purifications and use the Tarski-Seidenberg theorem to establish semialgebraicity.

Our findings rule out semialgebraic single-shot characterization of relative entropy, mutual information, and α-Rényi entropies with a bounded ancillary system, provided their level sets are transcendental. As specific applications of these results, we discuss catalytic LOCC transformations between pure entangled states and catalytic transformations between pairs of commutative quantum states.

- *Article III [3]*: *On the set of reduced states of translation invariant, infinite quantum systems*

  In this third Core Article, we study the geometry of the set of reduced bipartite density matrices of translation-invariant infinite quantum systems. Our work builds on an observation from [4], where certain quantum states were excluded from being C^*-finitely correlated by analyzing their ground-state energy, which must be an algebraic number if the respective quantum state is C^*-finitely correlated. However, their argument breaks down when transcendental parameters are allowed. To address this limitation, we utilize transcendental sets and functions to study the set of reduced bipartite density matrices.

  In translation-invariant infinite quantum spin chains, the ground-state energy density of a nearest-neighbor interaction Hamiltonian can, in principle, be obtained by optimizing over all admissible reduced states. However, the exact characterization of this set remains unknown, even for two-dimensional local Hilbert spaces. Inner and outer approximations corresponding to semialgebraic sets of reduced density matrices are known, but we demonstrate that the set itself is not semialgebraic. Consequently, any algebraic approximation to this set requires an infinite sequence of such approximations, as it cannot be fully described at any finite algebraic level. We establish this result by contradiction. Assuming the set of reduced bipartite density matrices is semialgebraic would imply that the ground-state energy of an algebraic family of Hamiltonians is also algebraic. To refute this, we construct a family of spin-1/2, two-body Hamiltonians—specifically, we use the Hamiltonian of the XY model—and show that the ground-state energy density is a transcendental function over $\mathbb{R}$. This contradiction validates our claim. We also explore the possibility of a finite, non-algebraic relaxation that could precisely characterize the set of reduced bipartite density matrices and present evidence against this possibility.

Chapter 2

Mathematical Elements of Quantum Information Theory

In this chapter, we establish the mathematical foundations of quantum information theory and other relevant topics necessary for the understanding of this thesis. The content presented here is primarily based on textbooks such as [5–8], as well as lecture notes like [9, 10]. Section 2.1 introduces the density operator formalism, which describes quantum states, along with the corresponding framework for quantum measurements. We also present quantum channels as means of describing transformations of quantum systems. In Section 2.2, we examine the class of quantum operations, which involve only local operations and classical communication (LOCC). Finally, we close this chapter with a brief discussion on key concepts from differential topology found in this thesis in Section 2.3 and (von Neumann) algebras in Section 2.4.

We begin by establishing some notation that will be useful for understanding this chapter and the remainder of the thesis. Throughout, we will work with separable complex Hilbert spaces $\mathcal{H}$, which are vector spaces over the complex numbers $\mathbb{C}$ with a countable orthonormal basis, and are equipped with an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that induces a norm $||\phi|| \coloneqq \langle \phi, \phi \rangle^{1/2}$ with respect to which the space is complete. As commonly used in mathematical physics, the inner product will be conjugate linear in the first argument, and linear in the second. Unless otherwise stated, we work with finite-dimensional Hilbert spaces $\mathbb{C}^d$, of dimension $d \in \mathbb{N} \coloneqq \{1, 2, 3, \ldots\}$. Two vectors $\phi, \psi \in \mathcal{H}$ are said to be orthogonal if $\langle \phi, \psi \rangle = 0$. For any subset $M \subseteq \mathcal{H}$, its *orthogonal complement* $M^\perp$ is defined as the subset of $\mathcal{H}$ consisting of all elements that are orthogonal to every element in M. The orthogonal complement is a closed linear subspace and every closed linear subspace $\mathcal{H}_1$ of the Hilbert space $\mathcal{H}$ decomposes $\mathcal{H}$ into an orthogonal direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. This decomposition provides a unique way to express any element $\phi \in \mathcal{H}$ as $\phi = \phi_1 + \phi_2$, where $\phi_1 \in \mathcal{H}_1$ and $\phi_2 \in \mathcal{H}_1^\perp$. The uniqueness of the vectors ϕ_i for $i = 1, 2$ enables us to define the two orthogonal projectors $P_i : \mathcal{H} \to \mathcal{H}_i$, $\phi \mapsto \phi_i$. These projectors are linear idempotent maps and are related through $P_2 = \mathbb{1} - P_1$, where $\mathbb{1}$ is the identity map on $\mathcal{H}$. By $\mathcal{H}'$ we denote the topological dual space of $\mathcal{H}$, that is, the space of all continuous linear functionals. The famous Riesz representation theorem says that every continuous linear map from $\mathcal{H} \to \mathbb{C}$ is of the form $\phi \mapsto \langle \psi, \phi \rangle$ for some $\psi \in \mathcal{H}$, and vice versa. This establishes

the standard notation used in quantum theory, known as the *Dirac bra-ket* notation. A vector $\phi \in \mathcal{H}$ is represented as $|\phi\rangle$ (a *ket*), and elements of the dual space $\mathcal{H}'$ are denoted as $\langle\psi|$ (a *bra*). The inner product is then denoted as $\langle\psi|\phi\rangle$ - creating a "bra(c)ket". This notation allows us to introduce the outer product of $|\phi\rangle$ and $\langle\psi|$, the *ket-bra* map $|\phi\rangle\langle\psi| : \mathcal{H} \to \mathcal{H}$, $|\varphi\rangle \mapsto |\phi\rangle\langle\psi|\varphi\rangle$.

Throughout this thesis, we use *analytic continuation* as a technique to extend the domain of a given complex analytic function (see, for instance, Chapter 8 of [11]). For complex functions $f(z)$ that are not uniquely determined by their analytic expressions, a point $z_* \in \mathbb{C}$ is called a *branch point* if $f(z)$ does not return to its initial value after tracing a curve that encloses z_* and is sufficiently close to it, starting from an arbitrary point on the curve. Along such a path, f varies continuously. This phenomenon, in which functions fail to remain uniquely determined when looping around a singularity, gives rise to *monodromy*. A detailed introduction to monodromy can be found in the textbook [12].

For any set L, we denote its cardinality by $|L|$. The set of real numbers is written as $\mathbb{R} \coloneqq (-\infty, \infty)$, and the set of non-negative real numbers as $\mathbb{R}^+$. The set of rational numbers is denoted by $\mathbb{Q}$, and the set of all algebraic numbers by $\overline{\mathbb{Q}}$; the latter is defined and discussed in detail in Chapter 3.

2.1 Operators in Hilbert Spaces

In this section, we discuss the fundamental concepts related to operators on Hilbert spaces, which are essential throughout this thesis. By *operator* we refer to a linear map defined between Hilbert spaces. For a Hilbert space $\mathcal{H}$, the operator norm of an operator T is given by

$$\|T\| \coloneqq \sup_{\varphi \neq 0} \frac{\|T\varphi\|}{\|\varphi\|}. \tag{2.1}$$

We denote by $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ the set of bounded linear operators between the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$:

$$\mathcal{B}(\mathcal{H}_A, \mathcal{H}_B) \coloneqq \{T \,|\, T\colon \mathcal{H}_A \to \mathcal{H}_B \text{ linear and } \exists\, C > 0 \text{ such that } \forall\, \varphi \in \mathcal{H}_A \ \|T\varphi\|_B \leq C\|\varphi\|_A\},$$

where $\|\cdot\|_A$ and $\|\cdot\|_B$ denote the norms on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For the case where $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}$ we use the shorthand $\mathcal{B}(\mathcal{H}) \coloneqq \mathcal{B}(\mathcal{H}, \mathcal{H})$. For an operator $V \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ we define its adjoint $V^* \in \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ by the relation $\langle\varphi, V\psi\rangle =: \langle V^*\varphi, \psi\rangle$ for all $\varphi \in \mathcal{H}_B$ and $\psi \in \mathcal{H}_A$. In this thesis, we use V^* and $V^\dagger$ interchangeably to denote the adjoint, or conjugate transpose, of V. If $V^*V = \mathbb{1}$ we call the operator V an isometry, and if in addition $VV^* = \mathbb{1}$ then V is said to be a unitary. A bounded operator $W \in \mathcal{B}(\mathcal{H})$ is called Hermitian if $W = W^*$. Such operators are also referred to as *observables* in the context of quantum mechanics. Furthermore, an operator $W \in \mathcal{B}(\mathcal{H})$ is said to be positive, denoted $W \geq 0$, if $\langle\varphi, W\varphi\rangle \geq 0$ for all $\varphi \in \mathcal{H}$. The notion of positivity induces a partial ordering on the set of Hermitian operators: for two operators $C, D \in \mathcal{B}(\mathcal{H})$, the relation $C \geq D$ is understood as $C - D$ being positive semidefinite.

Finally, an operator $W \in \mathcal{B}(\mathcal{H})$ is called a projection if $W^2 = W$. If, additionally, $W = W^*$, then we say that W is an orthogonal projection. For a linear operator $T : D(T) \to \mathcal{H}$ with domain $D(T) \subseteq \mathcal{H}$ we write:

$$\ker(T) \coloneqq \{\varphi \in D(T) \,|\, T\varphi = 0\}, \tag{2.2}$$

$$\mathrm{supp(T)} \coloneqq \{\varphi \in \mathrm{D(T)} \,|\, \varphi \perp \psi \text{ for all } \psi \in \ker(\mathrm{T})\}, \tag{2.3}$$

$$\mathrm{ran(T)} \coloneqq \{\psi \in \mathcal{H} \,|\, \psi = \mathrm{T}\varphi,\ \varphi \in \mathrm{D(T)}\}, \tag{2.4}$$

$$\Gamma(T) \coloneqq \{(\varphi, T\varphi) \,|\, \varphi \in D(T)\} \subseteq \mathcal{H} \times \mathcal{H} \tag{2.5}$$

to denote its kernel, support, range (or image), and graph, respectively. The rank of T is defined as the dimension of its range. Given an orthonormal basis $\{|e_n\rangle\}_n$ of $\mathcal{H}$, the trace of a positive operator $W \in \mathcal{B}(\mathcal{H})$ is defined as

$$\mathrm{tr}[W] \coloneqq \sum_n \langle e_n, W e_n\rangle \in [0, \infty], \tag{2.6}$$

and it is independent of the choice of the orthonormal basis. The set of all operators $W \in \mathcal{B}(\mathcal{H})$ for which $\mathrm{tr}[|W|] < \infty$ is called the *trace-class operators*, denoted by $\mathcal{B}_1(\mathcal{H})$, where $|W| \coloneqq \sqrt{W^*W} \in \mathcal{B}(\mathcal{H})$ denotes the positive absolute value of W. Equipped with the trace norm $\|W\| \coloneqq \mathrm{tr}[|W|]$ for $W \in \mathcal{B}_1(\mathcal{H})$, the set of trace-class operators forms a Banach space.

Throughout, we often work with matrix representations of operators, upon fixing a basis. We will implicitly identify $\mathcal{B}(\mathbb{C}^d)$ with the space of $d \times d$ complex matrices, $\mathcal{M}_{d\times d}(\mathbb{C})$, and using the trace we can equip it with the Hilbert-Schmidt inner product $\langle V, W\rangle_{HS} \coloneqq \mathrm{tr}[V^\dagger W]$. By $H_d \subseteq \mathbb{C}^{d\times d}$ we denote the space of Hermitian $d \times d$ matrices, by $P_d \subseteq \mathbb{C}^{d\times d}$ the set of positive definite $d \times d$ matrices ($V \in P_d$ whenever $V > 0$) with a non-degenerate spectrum, and by $\mathcal{U}_d \subseteq \mathbb{C}^{d\times d}$ the set of $d \times d$ unitary matrices. The *commutator* of two operators X and Y is defined as $[X, Y] \coloneqq XY - YX$. When $[X, Y] = 0$, we say that the operators X and Y commute.

Pauli matrices. Among the most important 2×2 matrices extensively used in quantum information theory and quantum computation are the *Pauli matrices*. These are unitary, Hermitian, traceless matrices, which are defined as follows:

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7}$$

Together with the identity matrix, the Pauli matrices form a basis for the vector space of complex 2×2 matrices, $\mathcal{M}_{2\times 2}(\mathbb{C})$.

2.1.1 Density Operators and Measurements

In quantum mechanics, it is generally not possible to predict the exact outcome of a measurement, even if we have complete knowledge of the preparation of the system. Instead, we can only predict the probabilities of different outcomes across statistical experiments. The mathematical framework for describing physical experiments in quantum mechanics is divided into two stages:

preparation and *measurement*. The preparation of a quantum system is formally described using *density operators*.

Definition 2.1.1 (Density operators)**.** *The set of density operators or quantum states is defined as*

$$\mathcal{S}(\mathcal{H}) \coloneqq \{\rho \in \mathcal{B}_1(\mathcal{H}) \mid \rho \geq 0,\ \mathrm{tr}[\rho] = 1\}. \tag{2.8}$$

A density operator ρ is said to be *pure* if there exists a unit vector $|\varphi\rangle \in \mathcal{H}$ such that $\rho = |\varphi\rangle\langle\varphi|$, and *mixed* otherwise. Pure density operators are uniquely defined by their corresponding unit vectors $|\varphi\rangle$, which in turn are determined by the density operator up to a phase factor (a complex scalar of modulus one). We characterize pure states by their purity in the following way:

Proposition 2.1.1 (Purity)**.** *For $\rho \in \mathcal{S}(\mathcal{H})$, it holds that $0 < \mathrm{tr}[\rho^2] \leq 1$ with equality for the latter inequality if and only if ρ describes a pure state.*

If $\mathcal{H}$ is a finite-dimensional Hilbert space, i.e., $\mathcal{H} = \mathbb{C}^d$, then the set of $d-$dimensional density operators is denoted by $\mathcal{S}(\mathbb{C}^d)$. When $d = 2$ we speak of *qubit* states. In this thesis, the terms "density operators", "density matrices", and "quantum states" are used interchangeably. The set of density operators forms a convex set, with the extreme points being the pure states, which are rank-one projections. The set $\mathcal{S}(\mathbb{C}^d)$ generalizes the set of $d-$dimensional probability vectors, i.e., non-negative, normalized $d-$dimensional vectors.

One way to measure the closeness of two quantum states is through the *trace distance*, which is a metric in the mathematical sense. For quantum states ρ and σ, the trace distance is defined as

$$T(\rho, \sigma) \coloneqq \frac{\|\rho - \sigma\|_1}{2}. \tag{2.9}$$

Although there are other 'distance measures' for this purpose, not all of them qualify as metrics in the rigorous mathematical sense. For example, the *quantum relative entropy* is neither symmetric nor does it satisfy the triangle inequality. Given two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ the quantum relative entropy is defined as $S(\rho\|\sigma) \coloneqq \mathrm{tr}[\rho \log \rho] - \mathrm{tr}[\rho \log \sigma]$ whenever $\mathrm{supp}(\rho) \cap \ker(\sigma) = \emptyset$ and it is ∞ otherwise. For more details on quantum relative entropy and other 'distance measures', we refer to Chapter 5.

While Proposition 2.1.1 provides a method for determining the purity or mixedness of a quantum state, this choice is somewhat arbitrary. For example, instead of using the square of the density operator to define purity, one could instead use the function $\rho \mapsto \rho^3$ to order quantum states from pure to mixed. However, in dimensions beyond qubits, these two orderings can be inconsistent. This inconsistency highlights the need for a more reasonable ordering of states, which can be established through a preorder based on the concept of *majorization*.

Definition 2.1.2 (Majorization)**.** *For two sequences x and y of non-negative real numbers, either finite with equal length or infinite with $\|x\|_1 = \|y\|_1$, we say that x is majorized by y, and write $x \prec y$, if*

$$\sum_{i=1}^{n} x_i^{\downarrow} \leq \sum_{i=1}^{n} y_i^{\downarrow} \quad \forall n, \tag{2.10}$$

where $x^{\downarrow}$ and $y^{\downarrow}$ denote the sequences x and y arranged in non-increasing order.

In other words, majorization establishes a preorder for vectors, providing a meaningful way to describe one vector as being more or less disordered, or not compatible, compared to another vector [13]. For a pair of quantum states $\rho, \sigma \in \mathcal{B}_1(\mathcal{H})$, we say that ρ is majorized by σ if the sequence of eigenvalues of ρ (counted with multiplicity) is majorized by the sequence of eigenvalues of σ. In quantum information theory, majorization plays a crucial role in the theory of bipartite pure state entanglement, due to Nielsen's theorem [14], which is discussed in Section 2.2.

Measurements. In quantum theory, quantum states cannot be directly accessed in experiments. Instead, we can make observations by performing measurements on them. Mathematically, when $\mathcal{H} = \mathbb{C}^d$, we describe measurements as follows:

Definition 2.1.3 (Positive Operator-Valued Measures - POVMs)**.** *An m-outcome positive operator-valued measure, shortly POVM, is a family $\{E_k \in \mathcal{B}(\mathbb{C}^d)\}_{k=1}^m$ of $m \in \mathbb{N}$ operators such that $0 \leq E_k \leq \mathbb{1}$ and*

$$\sum_{k=1}^{m} E_k = \mathbb{1}_d.$$

The operators E_k are commonly known as effect operators.

We use the terms "POVM" and "measurement" interchangeably in this thesis. By combining POVMs with density operators, *Born's rule* instructs us on how to compute the predicted measurement probabilities in quantum theory. Specifically, the probability of obtaining the outcome $k \in \{1, \ldots, m\}$ when performing the measurement $\{E_k\}_{k=1}^m$ on a quantum system in the state $\rho \in \mathcal{S}(\mathcal{H})$ is given by $p_k \coloneqq \mathrm{tr}[\rho E_k]$. The term "measurement" is often used synonymously with "*observable*", although traditionally, an observable is associated with self-adjoint operators (which is the convention we will adhere to in this thesis). In this context, measurable quantities such as the energy or momentum of a quantum system are mathematically represented by observables. The expectation value, also called the *mean*, of an observable X in the state ρ is given by $\mathrm{tr}[\rho X]$.

2.1.2 Composite Systems

In the previous section, we defined states and measurements for individual systems. Now, we extend the discussion to composite systems. Two common methods for constructing new Hilbert spaces from two or more existing ones are through direct sums and tensor products.

Direct Sum. For the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, their *direct sum* $\mathcal{H}_1 \oplus \mathcal{H}_2$ is defined as the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2 \coloneqq \{(\varphi, \psi) \in \mathcal{H}_1 \times \mathcal{H}_2\}$ with the inner product

$$\langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle \coloneqq \langle \varphi_1, \varphi_2 \rangle + \langle \psi_1, \psi_2 \rangle.$$

The elements of the space $\mathcal{H}_1 \oplus \mathcal{H}_2$ are denoted by $\varphi \oplus \psi$, where $\varphi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$. The above construction results in a Hilbert space with dimension $\dim(\mathcal{H}_1 \oplus \mathcal{H}_2) = \dim(\mathcal{H}_1) + \dim(\mathcal{H}_2)$. For

a finite sequence of Hilbert spaces $(\mathcal{H}_i)_{i=1}^m$, the definition of $\bigoplus_{i=1}^m \mathcal{H}_i$ extends trivially, and the direct sum corresponding to an infinite sequence of Hilbert spaces $(\mathcal{H}_i)_{i\in\mathbb{N}}$ is defined as follows

$$\bigoplus_{i\in\mathbb{N}} \mathcal{H}_i \coloneqq \Big\{(\psi_i)_{i\in\mathbb{N}} \,|\, \psi_i \in \mathcal{H}_i, \sum_{i\in\mathbb{N}} ||\psi_i||^2 < \infty\Big\}, \tag{2.11}$$

with the inner product $\big\langle(\psi_i)_{i\in\mathbb{N}}, (\varphi_i)_{i\in\mathbb{N}}\big\rangle \coloneqq \sum_{i\in\mathbb{N}} \big\langle\psi_i, \varphi_i\big\rangle$.

Tensor Product. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote the Hilbert spaces corresponding to systems A and B, respectively. The composite system AB is then described by the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ which is defined as the completion of all finite linear combinations of maps of the form $\psi \otimes \varphi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$, given by $(\alpha, \beta) \mapsto \langle\alpha, \psi\rangle\langle\beta, \varphi\rangle$. This is done with respect to the inner product $\langle\psi_1 \otimes \psi_2, \varphi_1 \otimes \varphi_2\rangle \coloneqq \langle\psi_1, \varphi_1\rangle \langle\psi_2, \varphi_2\rangle$ and extended to the entire space through linearity and continuity. If $\{e_n\}_n$ is an orthonormal basis for $\mathcal{H}_1$ and $\{f_k\}_k$ an orthonormal basis for $\mathcal{H}_2$ then $\{e_n \otimes f_k\}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. The resulting Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ has dimension $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$. Similarly to vectors, for two operators $T \in \mathcal{B}(\mathcal{H}_1)$ and $V \in \mathcal{B}(\mathcal{H}_2)$ we define the tensor product $T \otimes V$ as an operator acting in the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. For $\varphi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$ we have

$$(T \otimes V)(\varphi \otimes \psi) \coloneqq T\varphi \otimes V\psi, \tag{2.12}$$

which is extended by linearity to all vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$. We are interested in the following properties: Hermiticity, positivity, unitarity, boundedness, and trace-class. These properties are preserved by tensor products, i.e., if T and V have one of these properties, then so does $T \otimes V$. Furthermore, if $T_1, T_2 \in \mathcal{B}_1(\mathcal{H})$, then $\mathrm{tr}[T_1 \otimes T_2] = \mathrm{tr}[T_1] \cdot \mathrm{tr}[T_2]$. For positive semi-definite operators $T \in \mathcal{B}(\mathcal{H}_1)$ and $V \in \mathcal{B}(\mathcal{H}_2)$, the following identity holds:

$$\log(T \otimes V) = \log T \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \log V. \tag{2.13}$$

Quantum states of the composite system AB, described by the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, are elements of $\mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. A state in $\mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called a *product state* if it can be expressed as $\rho_A \otimes \rho_B$, where $\rho_A \in \mathcal{S}(\mathcal{H}_1)$ and $\rho_B \in \mathcal{S}(\mathcal{H}_2)$. *Separable states* are those that can be expressed as convex combinations of product states. In contrast, *entangled states* are those that cannot be represented as such convex combinations.

Every $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be expressed in the following form:

$$\Psi = \sum_{i=1}^{d} \sqrt{\lambda_i} e_i \otimes f_i, \tag{2.14}$$

where $d \leq \min\{\dim(\mathcal{H}_1), \dim(\mathcal{H}_2)\}$, and $\{e_i\}_i$ and $\{f_j\}_j$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. This representation is known as the *Schmidt decomposition* [5]. The non-negative numbers $\{\lambda_i\}_{i=1}^d$, decreasingly ordered, are called the *Schmidt coefficients* and they form the *Schmidt vector* $\vec{\lambda}_\Psi$. The *Schmidt rank* of Ψ, denoted $r_s(\Psi)$, is the number of non-zero elements in the vector $\vec{\lambda}_\Psi$. A state Ψ is entangled if and only if its Schmidt rank satisfies $r_s(\Psi) \geq 2$.

Similarly to quantum states, measurements for composite systems are defined as in Definition 2.1.3, with the operators now taken from the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Partial trace. An essential concept for studying joint systems is the *partial trace*. This is the quantum analog of the marginalizing map in classical probability theory.

Definition 2.1.4 (Partial trace)**.** *The partial trace is the linear mapping* $\mathrm{tr}_1 : \mathcal{B}_1(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}_1(\mathcal{H}_2)$ *such that*

$$\mathrm{tr}[X(\mathbb{1} \otimes Y)] = \mathrm{tr}[\,\mathrm{tr}_1[X]\,Y\,] \;\; \forall Y \in \mathcal{B}(\mathcal{H}_2), \tag{2.15}$$

for all $X \in \mathcal{B}_1(\mathcal{H}_1 \otimes \mathcal{H}_2)$*. The partial trace over the second subsystem,* tr_2*, is defined similarly.*

Proposition 2.1.2 (Properties of the partial trace)**.** *The partial trace is trace-norm continuous and it satisfies the following properties for all* $X \in \mathcal{B}_1(\mathcal{H}_1 \otimes \mathcal{H}_2)$*:*

1) $X \geq 0 \Rightarrow \mathrm{tr}_1[X] \geq 0$ *and* $\mathrm{tr}_2[X] \geq 0$*,*

2) $X = X_1 \otimes X_2 \Rightarrow \mathrm{tr}_1[X] = \mathrm{tr}[X_1]X_2$ *and* $\mathrm{tr}_2[X] = X_1\,\mathrm{tr}[X_2]$*,*

3) $\mathrm{tr}[\,\mathrm{tr}_1[X]\,] = \mathrm{tr}[X] = \mathrm{tr}[\,\mathrm{tr}_2[X]\,]$*.*

For proof of these properties, we refer the reader to, for example, Theorem 1.38 in [9].

Having introduced the partial trace, we can now discuss *quantum marginals*. Let $\rho \in \mathcal{B}_1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be a density operator describing the preparation of the composite system AB with the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Disregarding the first system, the corresponding density operator is given by $\rho_2 \coloneqq \mathrm{tr}_1[\rho]$ and it is called the *reduced density operator* or a *quantum marginal* of ρ. Similarly, disregarding the second system, the reduced density operator describing the remaining part of the system is given by $\rho_1 \coloneqq \mathrm{tr}_2[\rho]$. A *quantum-classical (qc)* state is a state that combines both quantum and classical information. It has the form $\sum_{x \in \mathcal{X}} p_X(x) \rho_A^x \otimes |x\rangle\langle x|$, where ρ_A^x is a quantum state that describes the quantum part of the system, conditioned on x, $p_X(x)$ represents the probability of each classical outcome x, and $\{x\}_{x \in \mathcal{X}}$ is an orthonormal basis for the classical system X. In this form, X is the classical system, while A represents the quantum subsystem.

2.1.3 Quantum Channels and Instruments

In a statistical experiment, various events can occur between preparation and measurement. These can involve time evolution, interactions among different parts of the system or with the environment, as well as active interventions performed by the experimentalists. These intermediate operations can be described in two ways: by updating the quantum states in the preparation framework, known as the *Schrödinger picture*, or by updating the POVMs in the measurement framework, known as the *Heisenberg picture*. Mathematically, the characterization of these operations in both frameworks must be consistent with the probabilistic interpretation of quantum theory, therefore these maps are taken to be linear.

We call a linear map $T : \mathcal{L} \subseteq \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ *trace-preserving* if $T(X)$ is trace-class and $\mathrm{tr}[T(X)] = \mathrm{tr}[X]$ for any $X \in \mathcal{L} \cap \mathcal{B}_1(\mathcal{H}_1)$, *unital* if $T(\mathbb{1}) = \mathbb{1}$ (under the assumption that

$\mathbb{1} \in \mathcal{L}$), and *positive* if $T(X) \geq 0$ for all positive $X \in \mathcal{L}$. However, since quantum states might describe a part of a larger system with other parts remaining untouched by the mapping T, it is essential to ensure that also $T \otimes \mathbb{1}$ maps quantum states to quantum states to accurately describe the entire physical process. This is captured by the notion of *complete positivity*: the linear map T is said to be completely positive if $(T \otimes \mathbb{1}_d)$ is positive $\forall d \in \mathbb{N}$, with $\mathbb{1}_d$ being the identity map on $\mathcal{B}(\mathbb{C}^d)$. An example of a map that is positive but not completely positive is the matrix transposition map $\Theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, defined by $\Theta(X) \coloneqq X^T$, where the transpose is taken with respect to a fixed basis $\{|l\rangle\}_l \subset \mathcal{H}$.

Definition 2.1.5 (Quantum channels)**.** *A linear map $T : \mathcal{B}_1(\mathcal{H}_1) \to \mathcal{B}_1(\mathcal{H}_2)$ is called a* quantum channel *if it is both* Completely Positive and Trace-Preserving (CPTP)*.*

The definition given above describes quantum channels in the Schrödinger picture, where quantum states evolve over time while measurements remain fixed. Alternatively, we can use the Heisenberg picture, in which states are kept fixed and measurements evolve over time. Mathematically, moving from the Schrödinger picture to the Heisenberg picture involves taking the adjoint with respect to the Hilbert-Schmidt inner product. For a bounded linear map $T : \mathcal{B}_1(\mathcal{H}_1) \to \mathcal{B}_1(\mathcal{H}_2)$ describing state evolution in the Schrödinger picture, there exists a unique linear map $T^* : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ in the Heisenberg picture. This map is called the *dual map* and is connected to T via the relation

$$\mathrm{tr}[T(\rho)\, E] = \mathrm{tr}[\rho\, T^*(E)] \text{ for all } \rho \in \mathcal{B}_1(\mathcal{H}_1),\, E \in \mathcal{B}(\mathcal{H}_2). \tag{2.16}$$

Examples of quantum channels include unitary evolutions defined by $A \mapsto UAU^*$, where U is a unitary operator; the partial trace operation in a composite system, $A \mapsto \mathrm{tr}_1[A]$; the addition of an ancillary quantum state ν via $A \mapsto A \otimes \nu$; and others. We conclude this section with a brief remark on classical channels. For random variables X and Y, a classical channel is characterized by the conditional probabilities $P(Y|X)$, which specify the probability of receiving output Y given that input X was transmitted.

Quantum Instruments. At a general level, instruments unify quantum channels and positive operator-valued measures (POVMs), offering insights into how to describe the state of a quantum system after the measurement. Assume $\mathcal{X}$ is a finite set of possible measurement outcomes. Then, a quantum instrument is defined as follows:

Definition 2.1.6. *A tuple $\{I_x : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})\}_{x \in \mathcal{X}}$ is a* quantum instrument *if*

i) every I_x is a completely positive map, and

ii) $\sum_{x \in \mathcal{X}} I_x$ is trace-preserving.

A general measurement scheme described by a quantum instrument can be outlined as follows: if a measurement on a quantum system in the state $\rho \in \mathcal{S}(\mathcal{H})$ yields the outcome x with probability $p_x \coloneqq \mathrm{tr}[I_x(\rho)]$ (this represents the classical part of the procedure) then the post-measurement state of the system, conditioned on the outcome x, is given by $\rho_x = \frac{I_x(\rho)}{p_x}$ (this constitutes the

quantum part of the procedure). If we disregard the information about measurement outcome, we derive a quantum channel T as follows:

$$\rho \mapsto \sum_x p_x \rho_x = \underbrace{\sum_x I_x(\rho)}_{=: T}. \tag{2.17}$$

Additionally, if we ignore the quantum system and focus solely on the measurement outcomes, we identify the associated POVM $M = \{M_x \coloneqq I_x^*(\mathbb{1})\}_{x \in \mathcal{X}}$. The probability of observing the outcome x is given by:

$$p_x = \mathrm{tr}[I_x(\rho)] = \mathrm{tr}[\underbrace{I_x^*(\mathbb{1})}_{=: M_x} \rho], \tag{2.18}$$

where $I_x^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the dual map corresponding to I_x. Since $\sum_{x \in \mathcal{X}} I_x$ is trace-preserving, we have $\sum_{x \in \mathcal{X}} p_x = \mathrm{tr}[\rho] = 1$. Instruments can be viewed as a special case of quantum channels by associating them with channels of the form $\rho \mapsto \sum_{x \in \mathcal{X}} I_x(\rho) \otimes |x\rangle\langle x|$, where $\{|x\rangle\}_x$ is an orthonormal basis for the underlying space.

2.2 Local Operations and Classical Communication (LOCC)

Having introduced quantum channels in the previous section, we now delve into a particular class of multipartite quantum operations: *Local Operations and Classical Communication* (LOCC). In quantum information theory, the distant-lab paradigm is fundamental to both theoretical advancements and experimental applications. These paradigms are such that various parties share a multipartite quantum system, and each party is allowed to only act locally on their respective subsystems, either by performing measurements or applying more general quantum operations. Therefore, the physical meaning of 'local' is that actions are performed independently on the subsystems. To optimize their measurement strategies, the parties can communicate any classical data, including, among other things, previous measurement outcomes. Any quantum operation that can be performed in this manner is known as *LOCC*.

We provide a detailed mathematical description of LOCC operations, focusing on the scenario relevant to this thesis where a quantum state is shared between two parties. Suppose that an entangled state is shared between two distant parties, commonly referred to as Alice and Bob. In the first round of the protocol, Alice performs a measurement on her part of the system, described by an instrument defined on a finite outcome space Λ_1, where the operations are represented by $\mathcal{F}_{i_1}^A$. The sum of these operations $\sum_{i_1 \in \Lambda_1} \mathcal{F}_{i_1}^A =: T_1^A$, forms a quantum channel. Alice then communicates the outcome of her measurement, i_1, to Bob. In the second round of the protocol, Bob performs a measurement on his portion of the system. This measurement is described by an instrument over a finite outcome space Λ_2, with operations denoted by $\mathcal{F}_{i_2|i_1}^B$. Bob's choice of measurement can depend on the information i_1 that Alice shared with him. It holds that $\sum_{i_1 \in \Lambda_2} \mathcal{F}_{i_2|i_1}^B =: T_{2|i_1}^B$, creating a quantum channel conditioned on Alice's result. After performing his measurement, Bob communicates his outcome i_2 back to Alice, who continues her actions accordingly, and this process repeats. Assume this protocol involves n rounds

of communication. After the final exchange of information, both Alice and Bob apply a local channel $T^A_{n+1|i_n,\ldots,i_2,i_1} \otimes T^B_{n+1|i_n,\ldots,i_2,i_1}$. Thus, the LOCC protocol is composed of n rounds of communication (though LOCC protocols can, in principle, involve an infinite number of communication rounds), resulting in Alice and Bob jointly applying an LOCC operation described by the following expression:

$$T_{\mathrm{LOCC}} \coloneqq \sum_{i_n,\ldots,i_2,i_1} (T^A_{n+1|i_n,\ldots,i_2,i_1} \otimes T^B_{n+1|i_n,\ldots,i_2,i_1}) \cdots (\mathcal{F}^A_{i_1} \otimes \mathcal{F}^B_{i_2|i_1}), \tag{2.19}$$

with $\sum_{i_1} \mathcal{F}^A_{i_1} = T^A_1$, $\sum_{i_2} \mathcal{F}^B_{i_2|i_1} = T^B_{2|i_1}$, and similarly for subsequent rounds, ensuring that all operations represent quantum channels. Therefore, LOCC operations are considered a specific subset of all physically realizable operations on the global system. These operations are motivated by the technological challenges associated with communicating quantum data and serve as a means to explore not only quantum correlations and other nonlocal quantum effects but also resource transformations, including channel capacities [15].

Entanglement is a resource that cannot be increased through LOCC operations. This forms the basis of entanglement theory as a resource theory, where LOCC operations are treated as "free operations" and separable states make up the "free states". For clarity, a *quantum resource theory* can be described as a framework for understanding and quantifying resources in quantum systems. It is defined by a tuple $\mathcal{R} = (\mathcal{F}, \mathcal{O})$, where:

- $\mathcal{F}$ is the set of *free states*, which are states considered to lack any resource.
- $\mathcal{O}$ is the set of *free operations*, a restricted subset of all quantum channels.

The restriction on $\mathcal{O}$ usually reflects the physical limitations of the system. The free states in $\mathcal{F}$ are those that can be prepared using only the operations in $\mathcal{O}$. States that lie outside the free set are called *resource states*. A key requirement of any resource theory is that free operations cannot transform a free state into a resource state, thereby preserving the free set. For more information on resource theories we refer to [14–17].

Quantifying entanglement in a quantum state is a central goal in entanglement theory, leading to the development of various entanglement measures. Here, we define only specific bipartite entanglement measures; for a broader discussion, see [14, 17]. A bipartite entanglement measure is a function that maps density matrices of bipartite quantum systems to the positive real numbers. It satisfies the following key properties: It assigns a value of zero to all separable states, it does not increase under local operations and classical communication, and for pure states, it reduces to the entanglement entropy, defined as the von Neumann entropy of either marginal of the bipartite quantum state. Additional conditions, such as continuity and additivity, may also be required for certain entanglement measures [14].

An important subset of LOCC operations is the class of *one-way* LOCC operations. In this scenario, communication is restricted so that only one party—say, Alice—is allowed to send information to Bob, with no communication permitted in the reverse direction. This restriction limits the protocol to a single round of communication from Alice to Bob and proceeds as follows:

i) Alice performs a measurement described by an instrument $i \mapsto \mathcal{F}_i^A$.

ii) After receiving the measurement outcome i from Alice, Bob applies a corresponding channel T_i^B.

This interaction defines the one-way LOCC operation, which can be expressed as:

$$T_{1-LOCC} = \sum_i \mathcal{F}_i^A \otimes T_i^B. \tag{2.20}$$

In contrast, allowing for an infinite number of communication rounds between distant parties significantly complicates the characterization of LOCC protocols [15]. Although transformations with a finite number of classical communication rounds exist and provide a simple characterization of states that can be converted into others through such protocols, it has been shown that certain protocols can only be achieved if an infinite number of communication rounds between the parties is allowed [18].

In [19], Lo and Popescu study the transformation of entanglement between pure bipartite quantum states, focusing on scenarios where only a single copy of the initial state is considered. They demonstrate that one-way LOCC is equally effective as two-way communication for such transformations. As a result, one-way communication is shown to be both a necessary and sufficient approach for manipulating entanglement in bipartite pure states. We apply this finding in Core Article [2] in the discussion of *catalytic LOCC* transformations.

Determining whether an LOCC operation can transform one quantum state—say, ρ_1—into another state ρ_2 is, in general, a challenging problem. A significant advancement in this direction was achieved by Nielsen, who showed that the transformation of pure bipartite states under LOCC operations is determined by the concept of majorization.

Theorem 2.2.1 (Nielsen's majorization [14])**.** *Let $|\phi\rangle$ and $|\psi\rangle$ be two unit vectors in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Denote by $\vec{\lambda}_\phi$ and $\vec{\lambda}_\psi$ the Schmidt vectors associated with the states $|\phi\rangle\langle\phi|$ and $|\psi\rangle\langle\psi|$, respectively. Then, $|\phi\rangle\langle\phi|$ can be transformed into $|\psi\rangle\langle\psi|$ via LOCC if and only if $\vec{\lambda}_\psi$ majorizes $\vec{\lambda}_\phi$.*

There exist pairs of pure states for which Nielsen's criterion cannot be applied, as neither state majorizes the other. Consequently, these states cannot be transformed into one another using LOCC. Such pairs of quantum states are said to be *incomparable*. This raises the question of whether *additional resources* could enable these otherwise impossible LOCC transformations. In certain cases, transformations that cannot be achieved through LOCC alone become possible with the assistance of an additional quantum system, through *quantum catalysis* (refer to Section 5.5 of this thesis). For instance, consider two pure bipartite entangled states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, where no LOCC protocol can directly transform $|\varphi_1\rangle$ into $|\varphi_2\rangle$. By introducing a catalyst—a third quantum system in the state $|\nu\rangle_C$—the transformation

$$|\varphi_1\rangle \otimes |\nu\rangle_C \to |\varphi_2\rangle \otimes |\nu\rangle_C \tag{2.21}$$

becomes possible using LOCC operations. The catalyst $|\nu\rangle_C$ is 'borrowed' to make the transformation possible and is 'returned' unchanged at the conclusion of the process. This phenomenon is commonly known as *entanglement catalysis* [20].

Further results in this area show that entanglement entropy fully characterizes state transformations in the context of bipartite quantum entangled states [21]. Before detailing this result, we elaborate on the definition they give about catalytic LOCC transformations: For a bipartite quantum system AB described by the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, a catalytic LOCC transformation from $\rho^{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ to $\sigma^{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as:

$$\rho^{AB} \to \lim_{m\to\infty} \mathrm{tr}_C[\Lambda_m(\rho^{AB} \otimes \eta_m^C)], \tag{2.22}$$

where $C = A'B'$ denotes the bipartite system of the catalyst, $\{\eta_m^C\}_m$ is a sequence of catalyst states, and $\{\Lambda_m\}_m$ is a sequence of LOCC protocols. For all m, the state of the catalyst remains unchanged in the procedure:

$$\mathrm{tr}_{AB}[\Lambda_m(\rho^{AB} \otimes \eta_m^C)] = \eta_m^C, \tag{2.23}$$

and, as $m \to \infty$, the system decouples from the catalyst, i.e.:

$$\lim_{m\to\infty} \|\Lambda_m(\rho^{AB} \otimes \eta_m^C) - \sigma^{AB} \otimes \eta_m^C\|_1 = 0. \tag{2.24}$$

Theorem 2.2.2 (Catalytic LOCC transformations for pure bipartite states [21])**.** *A pure bipartite entangled quantum state $\varphi \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be transformed into another pure bipartite entangled quantum state $\psi \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ via catalytic LOCC if and only if the entanglement entropy of φ is greater or equal to that of ψ.*

For a detailed proof of this result, see [21]. When exploring catalytic transformations of this kind, relatively little is known about the catalyst itself. In Core Articles [1, 2], we focus on a specific question regarding the dimension of the catalyst: Specifically, we investigate whether a universal bound exists on the dimension of the catalyst in the aforementioned catalytic LOCC transformations, characterized solely by the entanglement entropy.

2.3 Manifolds and Related Concepts

In this section, we introduce some fundamental concepts from differential topology, following the approach presented in [22]. For arbitrary sets, $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, a mapping $f : A \to B$ is called a *diffeomorphism* if it is a homomorphism—meaning that f is continuous, bijective, and has a continuous inverse—and both f and its inverse f^{-1} are smooth. A subset $M \subset \mathbb{R}^m$ is said to be a *(smooth) manifold* of dimension n if, for every point $x \in M$, there exists a neighborhood $U \cap M$ that is (diffeomorphic) homeomorphic to an open set $V \subset \mathbb{R}^n$. A specific diffeomorphism $g : V \to U \cap M$ is called a *parametrization* of the region $U \cap M$. By definition, a manifold M of dimension 0 is one in which each point $x \in M$ has a neighborhood $U \cap M$ that consists solely of the point x itself.

For any open set $V \subset \mathbb{R}^n$, the *tangent space* T_xV at a point x is defined as the vector space $\mathbb{R}^n$ itself, with its elements referred to as *tangent vectors*. The derivative of a smooth map $f : V \to W$ between open sets $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ at a point $x \in V$ is defined as the map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ given by:

$$df_x(u) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t}, \tag{2.25}$$

for $u \in \mathbb{R}^n$. This derivative is a linear function of u, corresponding to the $m \times n$ matrix of partial derivatives $(df_x)_{i,j} = \left(\frac{\partial f_i}{\partial x_j}\right)_x$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, evaluated at the point x. This matrix is known as the *Jacobi-matrix*. The tangent space for an arbitrary smooth manifold $M \subset \mathbb{R}^m$ at a point x is constructed as follows. Start by choosing a parametrization $g : V \to M \subset \mathbb{R}^m$ of a neighborhood $g(V)$ of $x \in M$ such that $g(v) = x$, with $V \subset \mathbb{R}^n$ an open set. The derivative $dg_v : \mathbb{R}^n \to \mathbb{R}^m$ is then defined, and T_xM is set to be equal to $dg_v(\mathbb{R}^n)$. This construction is independent of the particular choice of parametrization g. Consider now a smooth map $f : M \to N$ between smooth manifolds $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^l$, with $f(x) = y$. We briefly outline how to construct the derivative $df_x : T_xM \to T_yN$. Since f is smooth, there exists an open set $V \ni x$ and a smooth map $F : V \to \mathbb{R}^l$ that coincides with f on $V \cap M$. For any vector $w \in T_xM$, $df_x(w)$ is defined to be equal to $dF_x(w)$. To justify this definition, it can be shown that $dF_x(w)$ lies in T_yN and does not depend on the specific choice of F. For further details, see §1 in [22].

Let $f : M \to N$ be a smooth map between smooth manifolds M and N of the same dimension. A point $x \in M$ is called a *regular point* of f if the derivative map df_x is nonsingular. In this case, f maps a neighborhood of $x \in M$ diffeomorphically onto an open subset of the manifold N. A point $y \in N$ is referred to as a *regular value* if the preimage $f^{-1}(y)$ consists solely of regular points. Conversely, if df_x is singular, then x is known as a *critical point* of f, and the corresponding image $f(x)$ is called a *critical value*. Brown's theorem states that the set of regular values of a smooth map $f : M \to N$ is everywhere dense in N (see §2 of [22]). For a smooth map $f : M \to N$ between smooth manifolds M and N of dimensions n and k, respectively, where $n \geq k$, if $y \in N$ is a regular value of f, then the preimage $f^{-1}(y) \subset M$ is a smooth manifold of $n - k$ (see §2 Lemma 1 in [22]).

Two submanifolds M_1 and M_2 of a finite-dimensional smooth manifold M intersect *transversally* if, at each intersection point, their tangent spaces together span the tangent space of M. In other words M_1 and M_2 intersect transversely if for every point $x \in M_1 \cap M_2$, we have $T_xM_1 + T_xM_2 = T_xM$. If this transversality condition is not satisfied, the intersection of M_1 and M_2 could have singularities, and thus not be a smooth submanifold.

We conclude this section by discussing the concept of a *normal space*. Let $\tilde{M}$ be a manifold contained in M. At a point $x \in \tilde{M} \subset M$, the tangent space $T_x\tilde{M}$ is a subspace of the tangent space T_xM. The *normal space*, also known as the space of normal vectors to $\tilde{M}$ in M at x, is a vector space of dimension $n - \tilde{m}$, n being the dimension of M and $\tilde{m}$ the dimension of $\tilde{M}$, respectively. It is defined as the orthogonal complement of $T_x\tilde{M}$ in T_xM. Finally, a *Gauss map*

is a map that assigns to each point on a manifold a unit vector that is orthogonal to the tangent space at that point.

2.4 (von Neumann) Algebras

Let $\mathcal{A}$ be a vector space over the complex numbers $\mathbb{C}$. The vector space $\mathcal{A}$ is said to be an *algebra* if it is endowed with a multiplication operation that assigns to each pair of elements $X, Y \in \mathcal{A}$ a product XY, where this operation is both associative and distributive. The algebra $\mathcal{A}$ is called *commutative*, or *abelian*, if the multiplication is commutative, i.e., $XY = YX$. A subspace $\mathcal{M}$ of $\mathcal{A}$ is called a *subalgebra* if $\mathcal{M}$ is itself an algebra, with respect to the operations inherited from $\mathcal{A}$.

Definition 2.4.1 (Involution [7]). *A mapping $\mathcal{A} \to \mathcal{A}$, defined by $X \mapsto X^*$, is called an* involution *or* adjoint operation *of the algebra $\mathcal{A}$, if it satisfies the following properties:*

i) $X^{**} = X$,

ii) $(XY)^* = Y^* X^*$,

iii) $(aX + bY)^* = \overline{a} X^* + \overline{b} Y^*$,

for all $X, Y \in \mathcal{A}$ and scalars $a, b \in \mathbb{C}$.

An algebra $\mathcal{A}$ equipped with an involution is called a **-algebra*. A subset $\mathcal{M} \subset \mathcal{A}$ is called *selfadjoint* if $X \in \mathcal{M} \Rightarrow X^* \in \mathcal{M}$. Furthermore, an algebra $\mathcal{A}$ is called a *normed algebra* if there exists an associated real number $\|X\|$, the norm of X, for each element $X \in \mathcal{A}$. In addition to its usual properties, this norm satisfies the product inequality$\|XY\| \leq \|X\|\|Y\|$, for all $X, Y \in \mathcal{A}$. The norm induces the uniform topology on $\mathcal{A}$, where the neighborhood of an element $X \in \mathcal{A}$ is defined by $\mathcal{V}_{X,\varepsilon} \coloneqq \{Y \,|\, Y \in \mathcal{A}, \;\; \|Y - X\| < \varepsilon\}$, for $\varepsilon > 0$. An algebra $\mathcal{A}$ is called a *Banach algebra* if it is complete with respect to the uniform topology. A normed algebra with involution that is complete and meets the condition $\|X\| = \|X^*\|$ is known as a *Banach* **-algebra*. Finally, a C^**-algebra* is a Banach *-algebra $\mathcal{A}$ such that $\|X^*X\| = \|X\|^2$ for all elements $X \in \mathcal{A}$. An identity (or multiplicative identity) element $\mathbb{1}$ in a (C^*-)algebra $\mathcal{A}$ is an element of $\mathcal{A}$ that satisfies $X\mathbb{1} = X = \mathbb{1}X$, for every $X \in \mathcal{A}$. If a (C^*-)algebra has an identity, then the identity is unique, and the (C^*-)algebra is called *unital*. For an algebra $\mathcal{A}$ with identity $\mathbb{1}$, an element $X \in \mathcal{A}$ is invertible if there exists an element $X^{-1} \in \mathcal{A}$, called the *inverse* of X, such that $XX^{-1} = \mathbb{1} = X^{-1}X$. If the inverse exists, it is unique and also invertible. Furthermore, when working with *-algebras, if the element X is invertible, then its adjoint X^* is also invertible.

Example 2.4.1 (Examples of algebras). *Some examples of (C^*-)algebras are the following:*

i) The set of polynomials $\mathbb{F}[X]$ in one variable with coefficients taken from a field $\mathbb{F}$ is an abelian algebra with identity. The identity is the constant polynomial $p(x) = 1$. The algebra operations are the standard ones with polynomials, and the only invertible elements are the non-zero constant polynomials.

ii) The set $\mathcal{M}_{n\times n}(\mathbb{F})$ is an algebra with the usual matrix addition and multiplication. It is called the (full) matrix algebra *and for $n > 1$ is non-abelian as matrix multiplication is non-commutative. The invertible elements in this algebra are the matrices with a non-zero determinant.*

iii) The algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ endowed with the operator norm is a C^*-algebra. The involution in $\mathcal{B}(\mathcal{H})$ corresponds to the adjoint operation on the Hilbert space, and the C^*-norm condition is satisfied, as $\|X^*X\| = \|X\|^2$ holds for all $X \in \mathcal{B}(\mathcal{H})$.

Let $\mathcal{L}$ be a non-empty subset of an algebra $\mathcal{A}$. The subalgebra generated by $\mathcal{L}$ is the smallest subalgebra of $\mathcal{A}$ that contains $\mathcal{L}$:

$$\mathrm{Alg}(\mathcal{L}) = \mathrm{span}\{l_1, \ldots, l_k \mid k \in \mathbb{Z}^+,\ l_1, \ldots, l_k \in \mathcal{L}\}.$$

The *center* or *commutant* of an algebra $\mathcal{A}$ is $Z(\mathcal{A}) = \{X \in \mathcal{A} \mid XY = YX \text{ for all } Y \in \mathcal{A}\}$. A *homomorphism* between two algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear transformation $\phi : \mathcal{A} \to \mathcal{B}$ that satisfies the property $\phi(XY) = \phi(X)\phi(Y)$ for all $X, Y \in \mathcal{A}$. If in addition, the mapping ϕ is bijective, then ϕ is called an *isomorphism*.

Definition 2.4.2 (von Neumann algebra [8])**.** *In a finite-dimensional Hilbert space $\mathcal{H}$, a* von Neumann algebra *is defined as a subalgebra $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ satisfying the two following conditions:*

1) $\mathcal{U}$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$, and

2) for every $X \in \mathcal{U}$, its adjoint X^* also belongs to $\mathcal{U}$.

A von Neumann algebra $\mathcal{U}$ is called a *factor* if it has a trivial center, which means that $Z(\mathcal{U}) = \mathbb{C}\mathbb{1} \coloneqq \{\lambda \cdot 1 \mid \lambda \in \mathbb{C}\}$. Furthermore, if there exist subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_n \subseteq \mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, and if each subspace $\mathcal{H}_i$ corresponds to a von Neumann algebra acting on $\mathcal{H}_i$ for all $i \in \{1, \ldots, n\}$, then the algebra

$$\mathcal{U} = \{T_1 \oplus \cdots \oplus T_n \mid T_i \in \mathcal{U}_i,\ \forall i\} \tag{2.26}$$

is called a *direct sum of von Neumann algebras* and is denoted by $\mathcal{U} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_n$. Every von Neumann algebra $\mathcal{U}$ can be expressed as a direct sum of n factors, where n corresponds to the dimension of the center of $\mathcal{U}$ (for a proof see e.g., Proposition 5.3 of [8]). An example of a factor is the algebra $\mathcal{M}_{n\times n}(\mathbb{C})$, viewed as a von Neumann algebra acting on the complex inner product space $\mathbb{C}^d$. Moreover, it can be shown that all factors acting on finite-dimensional spaces have the structure of matrix algebras (see Theorem 5.4 of [8]). The structure of von Neumann algebras acting on $\mathbb{C}^d$ is described by the following theorem.

Theorem 2.4.1 (Theorem 5.6 of [8])**.** *Assume $\mathcal{U}$ is a von Neumann algebra acting on $\mathbb{C}^d$ and $n = \dim Z(\mathcal{U})$. Then, there exist positive integers $p_1, \ldots, p_n, l_1, \ldots, l_n$ and a unitary operator U acting on $\mathbb{C}^d$ such that*

1) $l_1 p_1 + \ldots + l_n p_n = d$, and

2) $\mathcal{U}\mathcal{U}U^* = (\mathcal{M}_{p_1\times p_1}(\mathbb{C})\otimes \mathbb{1}_{l_1}) \oplus \ldots \oplus (\mathcal{M}_{p_n\times p_n}(\mathbb{C})\otimes \mathbb{1}_{l_n})$,

where the term $\mathcal{M}_{p\times p}(\mathbb{C})\otimes \mathbb{1}_k$ *refers to the following factor acting on* $\mathbb{C}^{p_n l_n}$*:*

$$\mathcal{M}_{p\times p}(\mathbb{C})\otimes \mathbb{1}_k = \Big\{ \underbrace{T\oplus \ldots \oplus T}_{l_n \ times} \mid T\in \mathcal{M}_{p\times p}(\mathbb{C})\Big\}.$$

Definition 2.4.3. *A* state *over a* C^*-*algebra is defined as a linear functional* $\omega : \mathcal{A}\to\mathbb{C}$ *that meets the following conditions:*

i) Positivity: $\omega(X^*X)\geq 0$ *for all* $X\in\mathcal{A}$*.*

ii) Normalization: $\|\omega\| = 1$*.*

A state ω on a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ is characterized by the existence of a density matrix $\rho\in\mathcal{H}$ such that $\omega(X) = \mathrm{tr}[\rho X],\ \ \forall X\in\mathcal{A}$ (see Theorem 2.4.21 of [7]).

Chapter 3

Transcendental Number Theory

This chapter introduces the fundamental concepts of algebraic and transcendental numbers, which are crucial for understanding the algebraic structures explored in the next chapter and the methods used to address the research questions in Core Articles [1–3]. We examine their properties, provide examples, and explore their classifications, while also addressing the challenges remaining open to this day. For additional insights, we recommend and follow textbooks such as [23–25]. We also refer to the summaries in [26–28], particularly in Section 3.1 when discussing Baker's results and Schanuel's conjecture. Section 3.2 provides an overview of the role of transcendental numbers in tackling problems in quantum information theory and beyond.
The study of transcendental numbers dates back to 1844 when Liouville identified numbers that do not satisfy any algebraic equation with integer coefficients. However, research into closely related irrational numbers began at least a century earlier. By 1744, Euler had proved the irrationality of e, and by 1761, Lambert had established the irrationality of π.

Definition 3.0.1 (Algebraic numbers)**.** *A complex number $\alpha \in \mathbb{C}$ is called algebraic if it is the root of a non-zero polynomial $p(x) \in \mathbb{Z}[X]$.*

For any algebraic number $\alpha \in \mathbb{C}$, there exists a unique irreducible monic polynomial $p(x) \in \mathbb{Q}[X]$ (a polynomial whose leading coefficient is 1) such that $p(\alpha) = 0$. This polynomial is known as the *minimal polynomial* of α. If the minimal polynomial has degree d, then α is said to have *degree d*. We denote the field of all complex algebraic numbers by $\overline{\mathbb{Q}}$.
Examples of algebraic numbers include:

- *All rational numbers*: For any $k, l \in \mathbb{Z}$ with $l \neq 0$, every rational number $\frac{k}{l}$ is a root of the linear polynomial $p_1(x) \coloneqq k - lx$.
- *Quadratic irrational numbers*: These are roots of quadratic polynomials of the form $p_2(x) \coloneqq ax^2 + bx + c$, where $a, b, c \in \mathbb{Z}$.
- *The golden ratio* $\varphi = \frac{1+\sqrt{5}}{2}$: It is the root of the polynomial equation $p_3(x) \coloneqq x^2 - x - 1$.

A complex number $a + ib$ is algebraic if and only if both $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are (real) algebraic. Furthermore, any expression formed from algebraic numbers using basic arithmetic operations (addition, subtraction, multiplication, and division, where defined) and extracting nth roots is

also algebraic. Since the usual algebraic rules (commutativity and associativity) are inherited from $\mathbb{C}$, the set of algebraic numbers forms a subfield of the complex numbers. For a detailed proof, see Section 2.3 in [25].

If a number $\alpha \in \mathbb{C}$ satisfies a monic polynomial equation with integer coefficients, then α is called an *algebraic integer*. The set of all algebraic integers is denoted by $\overline{\mathbb{Z}}$, and it forms a ring with the inclusions $\mathbb{Z} \subset \overline{\mathbb{Z}} \subset \overline{\mathbb{Q}}$ (see, for instance, Corollary 2.28 in [25]). In addition, every algebraic number is computable—meaning it can be calculated to any desired precision by a finite, terminating algorithm—and therefore, each is definable.

Definition 3.0.2 (Algebraic independence)**.** *The complex numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ are said to be* algebraically independent *over $\mathbb{Q}$, if there exists no non-zero polynomial $p(x_1, \ldots, x_k) \in \mathbb{Q}[X_1, \ldots, X_k]$ such that $p(\alpha_1, \ldots, \alpha_k) = 0$. Conversely, if such a polynomial exists, the numbers are called* algebraically dependent*.*

The concept of algebraic independence leads to the following definition in transcendental number theory:

Definition 3.0.3 (Transcendence degree)**.** *Let $A \subseteq \mathbb{C}$ be a set of complex numbers. The* transcendence degree *$\gamma(A) \in \mathbb{N} \cup \{0, \infty\}$ of A is defined as the maximal number of algebraically independent elements of A.*

Definition 3.0.4 (Transcendental numbers)**.** *A complex number is called* transcendental *if it is not algebraic.*

Algebraic numbers are countable, and as a subset of the complex numbers, the set of all algebraic numbers $\overline{\mathbb{Q}}$ has Lebesgue measure zero (see, for instance, Cantor's Theorem 2.2 in [25]). As a result, algebraic numbers occupy no 'space' in the complex plane, meaning that almost all complex numbers are transcendental. In 1853, Liouville established a fundamental theorem on approximating algebraic numbers by rational numbers, which allowed him to construct specific transcendental numbers explicitly.

Theorem 3.0.1 (Liouville's Theorem [23])**.** *Let α be an algebraic number of degree $d > 1$. Then, there exists a positive constant $c = c(\alpha)$ such that for all integers k and $l > 0$ with $(k, l) = 1$, the following inequality holds:*

$$\left|\alpha - \frac{k}{l}\right| > \frac{c(\alpha)}{l^d}.$$

The proof of this theorem uses the mean value theorem as a key step. As a consequence, Liouville showed that the series

$$\sum_{j=0}^{\infty} \frac{1}{10^{j!}}$$

is transcendental.

In 1873, Hermite made a groundbreaking contribution to mathematics by proving the transcendence of e [23]. Building on Hermite's methods, Lindemann demonstrated within a decade that π is also transcendental [29]. This result solved the ancient Greek problem of *squaring the circle*, which involves constructing, using only a ruler and compass, a square with the same area as a given circle.

Lindemann further proposed a general result, special cases of which confirm the transcendence of both e and π. This generalization was rigorously proved by Weierstrass in 1885 and is formalized in the following theorem:

Theorem 3.0.2 (Lindemann-Weierstrass I, Chapter 4 of [23]). *Let $\alpha_1, \ldots, \alpha_k$ be distinct algebraic numbers. Then, for any algebraic numbers $\gamma_1, \ldots, \gamma_k$, the equation*

$$\gamma_1 e^{\alpha_1} + \ldots + \gamma_k e^{\alpha_k} = 0$$

holds only if $\gamma_1 = \ldots = \gamma_k = 0$. In other words, the numbers $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are linearly independent over $\overline{\mathbb{Q}}$.

An equivalent formulation of the Lindemann-Weierstrass theorem is as follows:

Theorem 3.0.3 (Lindemann-Weierstrass II, Theorem 4.4 of [23]). *Let $\alpha_1, \ldots, \alpha_k$ be algebraic numbers that are linearly independent over $\mathbb{Q}$. Then the numbers $e^{\alpha_1}, \ldots, e^{\alpha_k}$ are algebraically independent.*

We omit the proof of this theorem in this thesis but explain how the transcendence of the numbers e and π follows from the Lindemann-Weierstrass theorem:

Consider the case where $\alpha_1 = 0$ and $\alpha_2 = \alpha \in \overline{\mathbb{Q}}$. The set $\{0, \alpha\}$ consists of two distinct algebraic numbers. By the Lindemann-Weierstrass theorem, the set $\{e^0 = 1, e^\alpha\}$ is linearly independent over $\overline{\mathbb{Q}}$. In particular, this implies that e^α can not be algebraic. Setting $\alpha = 1$, we conclude that e is transcendental.

To show that π is transcendental, we begin by assuming the contrary: suppose π is algebraic. Then $i\pi$ would also be algebraic. However, by the Lindemann-Weierstrass theorem, $e^{\pi i} = -1$ would then have to be transcendental, which leads to a contradiction. Thus, π must be transcendental.

Proving that a number is transcendental can be extremely challenging. For example, it remains unknown whether $e + \pi$, or π^e are transcendental, even though both e and π individually are known to be transcendental. One result established through the work of Hermite and Lindemann is that for any non-zero algebraic number α, the number e^α is transcendental. Consequently, if α is a non-zero algebraic number not equal to 1, it follows that $\ln \alpha$ is transcendental.

3.1 Baker's Results and Schanuel's Conjecture

In 1934, Gelfond, and independently Schneider in 1935, proved a landmark result concerning the logarithm of algebraic numbers. Specifically, they showed that the logarithm of an algebraic number with an algebraic base—other than 0 or 1—is either transcendental or rational [26]. This result resolved Hilbert's seventh problem, which explored the irrationality and transcendence of certain numbers. Hilbert's seventh problem asked whether, for an algebraic number $\alpha \neq 0, 1$ and an irrational algebraic number β, the expression α^β is always transcendental. The affirmative answer, known as the Gelfond-Schneider theorem, represents a major milestone in the study of transcendental numbers. A detailed proof of this result can be found, for example, in Chapter

9 of [23]. The requirement that β is irrational is essential because α^β is algebraic when α is algebraic and β is rational.

Gelfond subsequently refined its proof and established a positive lower bound for the expression

$$|\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2|,$$

where $\beta_1, \beta_2 \in \overline{\mathbb{Q}}$ are not both 0, and $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ with $\frac{\log \alpha_1}{\log \alpha_2} \in \mathbb{R} \setminus \mathbb{Q}$. He also highlighted that it would be of great interest in number theory to obtain an analogous result for linear forms in an arbitrary number of logarithms of algebraic numbers. This extension was achieved by Baker, who provided a general result in [26]. He established a lower bound for the linear combination

$$|\beta_1 \log \alpha_1 + \ldots + \beta_k \log \alpha_k| \tag{3.1}$$

where $\alpha_1, \ldots, \alpha_k$ are algebraic numbers distinct from 0 or 1 such that $\log \alpha_1, \ldots, \log \alpha_k$ and $2\pi i$ are linearly independent over $\mathbb{Q}$. The coefficients $\beta_1, \ldots, \beta_k$ are algebraic numbers, not all 0. A consequence of this result is that the expression $\beta_1 \log \alpha_1 + \ldots + \beta_k \log \alpha_k$ is different from 0. The complete theorem, its proof, and further refinements are detailed in [26] and [27]. However, in this thesis, we are interested in the implications presented in [26].

Corollary 3.1.1 (Linear independendce I)**.** *Let $\alpha_1, \ldots, \alpha_k$ be non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_k$ and $2\pi i$ are linearly independent over $\mathbb{Q}$. Then, $\log \alpha_1, \ldots, \log \alpha_k$ are linearly independent over $\overline{\mathbb{Q}}$.*

Corollary 3.1.2. *Let $\alpha_1, \ldots, \alpha_k$ be positive real algebraic numbers distinct from 1, and let $\beta_1, \ldots, \beta_k$ be real algebraic numbers such that $1, \beta_1, \ldots, \beta_k$ are linearely independent over $\mathbb{Q}$. Then, the number $\alpha_1^{\beta_1} \ldots \alpha_k^{\beta_k}$ is transcendental.*

Baker extended his results further in [27] and showed that the term $2\pi i$ can be omitted from Corollary 3.1.1 without affecting the validity of its conclusion.

Theorem 3.1.1 (Linear independence II)**.** *Let $\alpha_1, \ldots, \alpha_k \in \overline{\mathbb{Q}} \setminus \{0\}$. Then, $\log \alpha_1, \ldots, \log \alpha_k$ are linearly independent over $\mathbb{Q}$ if and only if they are linearly independent over $\overline{\mathbb{Q}}$.*

He further extended Corollary 3.1.2 to a more general case that includes both real and complex algebraic numbers (see Theorem 2 of [27]). Furthermore, Baker extended the homogeneous linear combination of logarithms of non-zero algebraic numbers (Equation3.1) to an inhomogeneous form by including a constant term $\beta_0 \neq 0$. Specifically, he considered the expression

$$|\beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_k \log \beta_k|,$$

as presented in [28]. As a consequence of this generalization, the following result holds.

Theorem 3.1.2 (Baker's Theorem [23, 28])**.** *Let $\alpha_1, \ldots, \alpha_k$ be non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_k$ are linearly independent over $\mathbb{Q}$. Then, the $k+1$ numbers*

$$1, \log \alpha_1, \ldots, \log \alpha_k$$

are linearly independent over $\overline{\mathbb{Q}}$.

The case $k = 1$ is a consequence of the Lindemann-Weierstrass theorem, while $k = 2$ yields the Gelfond-Schneider theorem. Due to the complexity of the proof, we omit it here and direct the reader to [28] for the full details and further refinements of Baker's work. We apply Baker's result to identify the conditions under which the von Neumann entropy of a quantum state ρ, defined as $S(\rho) = -\operatorname{tr}[\rho \log \rho]$, is transcendental, algebraic, or rational (see Theorem 2 in the Core Article [1]). With this understanding, we can also draw conclusions about the level sets of the von Neumann entropy, particularly whether they are semialgebraic or transcendental. In the following chapter, we introduce semialgebraic sets, after which we revisit the implications of having transcendental entropy values.

In the remainder of this section, before addressing open problems in the field, we present two applications of Baker's theorem, along with their proofs, to provide insight into working with transcendental and algebraic numbers. The first application establishes that any non-zero algebraic linear combination of logarithms of algebraic numbers is either zero or transcendental.

Corollary 3.1.3 (Linear combination of logarithms of algebraic numbers, Corollary 20 of [23])**.** *Let $\alpha_1, \ldots, \alpha_k$ be non-zero algebraic numbers, and $\beta_1, \ldots, \beta_k$ be algebraic numbers. Then, the expression*

$$\beta_1 \log \alpha_1 + \ldots \beta_k \log \alpha_k \tag{3.2}$$

is either zero or transcendental.

Proof. We proceed by induction on $k \in \mathbb{N}$. For $k = 1$, the result is a consequence of the Lindemann-Weierstrass theorem. Assume that the results hold true for all $k < n$. We now prove it for $k = n$. Suppose, to the contrary, that there exists a non-zero algebraic number β_0 such that

$$\beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n = \beta_0. \tag{3.3}$$

By Baker's theorem (Theorem 3.1.2), the logarithms $\log \alpha_1, \ldots, \log \alpha_n$ must be linearly dependent over the rationals. Therefore, there exist rational numbers $a_1, \ldots, a_n$, not all zero, such that

$$a_1 \log \alpha_1 + \ldots + a_n \log \alpha_n = 0. \tag{3.4}$$

Without loss of generality, assume $a_n \neq 0$. Multiply equation (3.3) by a_n and equation (3.4) by β_n, then subtract the resulting expressions. This yields:

$$(a_n\beta_1 - a_1\beta_n) \log \alpha_1 + \ldots + (a_n\beta_{n-1} - a_{n-1}\beta_n) \log \alpha_{n-1} = \beta_0 a_n \tag{3.5}$$

$$\Longleftrightarrow$$

$$\tilde{\beta}_1 \log \alpha_1 + \ldots \tilde{\beta}_{n-1} \log \alpha_{n-1} = \beta_0 a_n \neq 0. \tag{3.6}$$

By the inductive hypothesis, the expression in (3.6) cannot hold because $\tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1}$ are non-zero algebraic numbers, and $\beta_0 a_n \neq 0$. This contradiction completes the proof. □

The following result is a vast generalization of the Gelfond-Schneider theorem.

Corollary 3.1.4 (Generalization of Gelfond-Schneider Theorem, Corollary 20.2 [23]). *Let $\alpha_1, \ldots, a_k$ and $\beta_0, \beta_1, \ldots, \beta_k$ be non-zero algebraic numbers. Then,*

$$e^{\beta_0} \alpha_1^{\beta_1} \ldots \alpha_k^{\beta_k}$$

is a transcendental number.

Proof. Suppose, for contradiction, that $e^{\beta_0} \alpha_1^{\beta_1} \ldots \alpha_k^{\beta_k}$ is algebraic, i.e.,

$$e^{\beta_0} \alpha_1^{\beta_1} \ldots \alpha_k^{\beta_k} = \alpha_{k+1} \in \overline{\mathbb{Q}}. \tag{3.7}$$

Taking the logarithm of both sides gives:

$$\beta_1 \log \alpha_1 + \ldots \beta_k \log \alpha_k - \log \alpha_{n+1} = -\beta_0 \neq 0. \tag{3.8}$$

This equation represents a non-zero linear combination of logarithms of algebraic numbers. By Corollary 3.1.3, such a combination must either be zero or transcendental. However, the right-hand side is non-zero and algebraic, leading to a contradiction. Therefore, $e^{\beta_0} \alpha_1^{\beta_1} \ldots \alpha_k^{\beta_k}$ must be transcendental. □

We conclude this chapter with a discussion of what is widely regarded as the central open problem in transcendental number theory.

Conjecture 3.1.1 (Schanuel's conjecture, Chapter 21 [23]). *Let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ be linearly independent numbers over $\mathbb{Q}$. Then, the set $\{\alpha_1, \ldots \ldots, \alpha_k, e^{\alpha_1}, \ldots, e^{\alpha_k}\}$ contains at least k algebraically independent numbers.*

A particularly notable special case of Schanuel's conjecture occurs when all the complex numbers α_j are algebraic. In this scenario, the conjecture reduces to the Lindemann-Weierstrass theorem. A special case of Schanuel's conjecture, known as the Weak Schanuel's conjecture, serves as a generalization of Baker's theorem. It states that if $\alpha_1, \ldots, \alpha_k$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_k$ are linearly independent over $\mathbb{Q}$, then these logarithms are also algebraically independent.

3.2 Algebraic and Transcendental Numbers in Quantum Information Theory and Beyond

The use of transcendentality as a criterion for excluding certain representations—both in quantum information theory and beyond—has a long and somewhat elusive history. Our investigation into the (im)possibility of single-letter formulas (Core Articles [1, 2]) and the search for finite algebraic descriptions of marginal sets in quantum many-body systems (Core Article [3]) is inspired by foundational work by Fannes, Werner, and Nachtergaele [4]. In their work, they use the distinction between algebraic and transcendental numbers to demonstrate that the ground state of the antiferromagnetic spin-1/2 Heisenberg model cannot be a C^*-finitely correlated state because its ground state energy is transcendental. If the ground state were a C^*-finitely correlated state,

the ground state energy would necessarily be an algebraic number, as algebraic methods produce only algebraic values.
Their general result, using C^*-finitely correlated states as trial states in the variational problem for ground states of a given interaction, is presented in Proposition 7.3 of [4]. A concrete application of this method to the antiferromagnetic spin-1/2 Heisenberg model appears in Corollary 7.4 of the same work. For readers unfamiliar with the antiferromagnetic spin-1/2 Heisenberg chain and C^*-finitely correlated states, we refer to Chapter 6, where these concepts are introduced and explained.

Schanuels's conjecture is particularly interesting in Core Article [3]. When analyzing the set of reduced bipartite states in translation-invariant infinite quantum systems, we pose the question of whether incorporating transcendental ingredients would allow for a finitary, explicit description of this set. Assuming Schanuel's conjecture to be true, we show that this set is not definable within the first-order language of the ordered field of real numbers with exponentiation, denoted by:

$$\mathbb{R}_{\exp} \coloneqq (\mathbb{R}, +, \cdot, \exp, <, 0, 1) \tag{3.9}$$

A subset $S \subseteq \mathbb{R}^n$ is said to be definable in the first-order language of $\mathbb{R}_{\exp}$ if there exists a first-order formula $\Phi(s)$ such that $s \in S \iff \Phi(s)$. For a more detailed discussion, we refer the reader to Section 5 of Core Article [3].

Another transcendental tool that has proven useful for studying problems in quantum information theory is the transcendence degree. In [30], Jia and Wolf provide explicit examples of quantum states and unitaries with exponential quantum circuit complexity. Their approach is based on the idea that small quantum circuits are characterized by a small number of parameters, which in turn can only produce matrices with a small number of "independent entries". This notion of independence is rigorously defined using the concept of transcendence degree and using the Lindemann-Weierstrass theorem as a tool to lift linear independence to algebraic independence. Consequently, their problem reduces to identifying sets with a large transcendence degree, as a high transcendence degree directly implies high circuit complexity.

Chapter 4

Semialgebraic and Transcendental Structures

Having introduced algebraic and transcendental numbers in Chapter 3, we now extend our discussion to semialgebraic and transcendental structures within the framework of *real algebraic geometry*. Our focus will be on aspects most relevant to our research, specifically the study of semialgebraic and transcendental sets, as well as semialgebraic and transcendental functions. For additional details on these topics, we recommend and follow the textbook [31], the overview on semialgebraic sets [32], and the lecture notes [33]. The structure of this chapter is as follows: in Section 4.1 we introduce semialgebraic sets and give some examples. Section 4.2 covers one of the fundamental results in semialgebraic theory, the Tarski-Seidenberg Theorem, and discusses its implications regarding the properties of semialgebraic sets. In Section 4.3 we explore semialgebraic and transcendental functions. Section 4.4 offers an overview of the decomposition of semialgebraic sets. Finally, Section 4.5 examines the role of distinguishing between semialgebraic and transcendental structures in addressing fundamental problems in quantum information theory and beyond. This chapter, along with the previous one, lays the groundwork necessary to address the research questions examined in Core Articles [1–3].

4.1 Semialgebraic and Transcendental Sets

Algebraic geometry, in simplest terms, studies the set of solutions of a system of polynomial equations, whereas the primary focus of real algebraic geometry is on subsets of $\mathbb{R}^n$ defined by polynomial equations. However, our interest here extends beyond polynomial equations, allowing in addition polynomial inequalities, leading us to the study of *semialgebraic sets*. These sets, along with semialgebraic functions (functions with semialgebraic graphs), are distinctive features of real algebraic geometry, known for their remarkable stability properties—most notably, stability under projections. Practically, almost all useful constructions involving semialgebraic sets produce other semialgebraic sets, and they also exhibit favorable topological properties, including well-behaved stratifications. We examine these polynomials over *real closed fields*, which in this thesis are denoted by R, due to the special behaviors they exhibit in these fields. Real closed fields are fields that exhibit a unique ordering, such that each positive element has a square root,

and each polynomial of odd degree has a root. Another characterization of real closed fields is given in terms of the intermediate value theorem. Specifically, for a polynomial $p(x) \in R[X]$ and elements $x, y \in R$ with $x < y$ and $p(x) < 0 < p(y)$, then there $\exists z \in R$ such that $x < z < y$ and $p(z) = 0$ (see, for instance, Proposition 1.2.4 in [31]). Examples of real closed fields we will use include the field of real numbers and the field of real algebraic numbers. In contrast, the field of rational numbers is not a real closed field.
To formally define a semialgebraic set, let R be a real closed field and let $n \geq 1$.

Definition 4.1.1 (Semialgebraic sets)**.** *Let A be a subset of R^n. We say that A is a* semialgebraic set *if it can be expressed in the following form:*

$$A \coloneqq \bigcup_{i=1}^{m} \bigcap_{j=1}^{k_i} \{x \in R^n \,|\, p_{i,j} \ *_{i,j} \ 0\}, \tag{4.1}$$

where the polynomials $p_{i,j} \in R[X_1, X_2, \ldots, X_n]$ and $*_{i,j} \in \{<, >, =\}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k_i$. In other words, a subset $A \subseteq R^n$ is called semialgebraic if it can be defined as the set of points $(x_1, ..., x_n)$ in R^n that satisfy a boolean combination (using conjunctions, disjunctions, and negations) of a finite number of polynomial equations and inequalities with coefficients in R.

Semialgebraic sets form the smallest class of sets in R^n containing all sets of the form:

$$\{x \in R^n : p(x) = 0\}, \ \{x \in R^n : p(x) > 0\}, \text{ and } \{x \in R^n : p(x) < 0\},$$

where $p(x)$ is a polynomial in $R[X_1, X_2, \ldots, X_n]$. Moreover, semialgebraic sets are closed under finite unions $(A \cup B)$, finite intersection $(A \cap B)$, and complements $(R^n \setminus A, \ R^n \setminus B)$, where $A, B \subseteq R^n$ are semialgebraic sets. If we restrict the definition to sets defined solely by polynomial equations (without inequalities), the resulting set is called an *algebraic set.*
A semialgebraic set $S \subseteq R^n$ is called *basic open* if it has the form:

$$\{x \in R^n \mid p_1(x) > 0, \, p_2(x), \ldots, p_k(x) > 0\},$$

where $p_1, p_2, \ldots, p_k \in R[X_1, X_2, \ldots, X_n]$. Similarly, S is a *basic closed* semialgebraic set if it is has the form $\{x \in R^n \mid p_1(x) \geq 0, \, p_2(x) \geq, \ldots, p_k(x) \geq 0\}$.

Example 4.1.1 (Examples of (non)semilgebraic sets)**.**

a) Any polyhedron is a semialgebraic set.

b) An algebraic set is also semialgebraic.

c) For $X \in \mathcal{M}_n(\mathbb{C})$, the set of unitary matrices, defined as the solution of the quadratic equation $X^*X = \mathbb{1}_n$, is semialgebraic.

d) The graph $\{(a, b) \in \mathbb{R} \times \mathbb{R} \,|\, b = e^a\}$ is not semialgebraic.

Why the set in d) is not semialgebraic will become clear as we delve deeper into the structure and properties of (semi)algebraic sets and functions. Note that a finite collection of polynomial equations $p_i(x)$ defining an algebraic set can be replaced by a single polynomial $P(x) \coloneqq \sum_i p_i^2(x)$.

Proposition 4.1.1 (Semialgebraic subsets of R^n)**.** *Any semialgebraic subset of R^n can be expressed as a finite union of semialgebraic sets of the form*

$$\{x \in R^n : P(x) = 0,\, q_1(x) > 0,\, q_2(x) > 0, \ldots, q_k(x) > 0\} \tag{4.2}$$

where $P, q_1, q_2, ..., q_k$ are polynomials in $R[X_1, X_2, ..., X_n]$.

The proof follows directly from the definition of semialgebraic sets (see Proposition 2.1.8 [31]).

4.2 A Tool from Tarski and Seidenberg

Semialgebraic sets possess several important properties by definition: they are closed under finite intersections, finite unions, and complements. A key feature distinguishing semialgebraic sets from algebraic sets is their stability under projections, a result ensured by the Tarski-Seidenberg theorem. Originally announced by Tarski in 1931 [34] and later generalized by Seidenberg in 1954 [35], this theorem states that applying quantifiers $(\forall, \exists)$ to semialgebraic sets does not take us outside the semialgebraic realm. This preservation under logical operations greatly expands their utility and enables a wide range of applications, which we will explore in this section.

To present the Tarski-Seidenberg theorem, we first introduce the concepts of *first-order formulas* and *quantifier-free formulas.*

Definition 4.2.1 (First-order formula and quantifier-free formula [32, 33])**.** *In the language of real closed fields, a first-order formula is constructed according to the following rules:*

i) For $f \in \mathbb{R}[X_1, \ldots, X_n]$, the expression $f * 0$ where $*$ denotes any of the relations $\{=, <, >\}$, is a first-order formula.

ii) If Φ and Ψ are formulas, then their disjunction $\Phi \wedge \Psi$, their conjunction $\Phi \vee \Psi$, and negation $\neg\Psi$ are also first-order formulas.

iii) For any first-order formula Ψ and variable y that ranges over R, the expressions $\exists y\, \Psi$ and $\forall y\, \Psi$ are first-order formulas.

Formulas constructed solely from rules i) and ii) are called quantifier-free formulas.

Equipped with these definitions, we can clarify the relationship between Boolean algebras and logical notation in the following remark.

Remark 4.2.1 (Connection between Boolean algebras and logical notation [36])**.** *For U and V non-empty sets, consider the first-order formulas $\Psi(u, v)$ and $\Phi(u, v)$ defined on $(u, v) \in U \times V$, which represent the sets*

$$\Psi = \{(u, v) \in U \times V \,:\, \psi(u, v)\} \;\; \textit{and} \;\; \Phi = \{(u, v) \in U \times V \,:\, \phi(u, v)\}.$$

The following relations hold:

- $\phi(u,v) \wedge \psi(u,v)$ *defines the intersection* $\Phi \cap \Psi$,
- $\phi(u,v) \vee \psi(u,v)$ *defines the union* $\Phi \cup \Psi$,
- $\neg\phi(u,v)$ *defines the complement* $U \times V \setminus \Phi$,
- $\forall u\, \psi(u,v)$ *defines* $V \setminus \Pi_V(U \times V \setminus \Psi)$, *where* $\Pi_V : U \times V \to V$ *denotes the natural projection onto* V,
- $\exists u\, \psi(u,v)$ *defines the projection* $\Pi_V(\Psi)$.

We are now prepared to present the Tarski-Seidenberg theorem.

Theorem 4.2.1 (Tarski-Seidenberg, quantifier elimination)**.** *Let R be a real closed field. Consider a finite set of polynomial equations and inequalities $\{p_i(x,a)\}_{i=1}^{l}$ with variables $(x,a) \in R^n \times R^m$ and coefficients in $\mathbb{R}$. Define $\psi(x,a)$ as a Boolean combination of the p_i's constructed using the logical operators $\vee$ (or), $\wedge$ (and), and $\neg$ (not). We then define*

$$\Phi(a) \coloneqq \big(Q_1 x_1 \ldots Q_n x_n : \psi(x,a)\big), \text{ where } Q_j \in \{\exists, \forall\}. \tag{4.3}$$

Under these conditions, there exists a formula $\phi(a)$ such that:

i) $\phi(a)$ is a quantifier-free Boolean combination of finitely many polynomial equations and inequalities with coefficients in $\mathbb{R}$, and

ii) $\phi(a)$ is equivalent to the formula (4.3) in the sense that

$$\forall a : \ \big(\phi(a) \iff \Phi(a)\big). \tag{4.4}$$

Even more, there exists an effective algorithm that constructs the quantifier-free formula ϕ that is equivalent to any such formula Φ.

For a proof of this theorem, see, for instance, [37]. In essence, the Tarski-Seidenberg theorem states that in the language of real closed fields, every first-order formula is equivalent to a quantifier-free formula. It is important to emphasize that quantifiers are only permitted on variables that range over real closed fields R.

Example 4.2.1. *Let $a, b, c \in \mathbb{R}$ such that $ax^2 + bx + c = 0$. This equation has a real solution if and only if*

$$(a \neq 0 \text{ and } b^2 - 4ac \geq 0) \text{ or } (a = 0 \text{ and } b \neq 0) \text{ or } (a = b = c = 0). \tag{4.5}$$

The expression in Equation (4.5) is a quantifier-free formula with the same true value as the original quantifier-containing statement:

$$\exists x \ ax^2 + bx + c = 0. \tag{4.6}$$

Before exploring the key implications of the Tarski-Seidenberg theorem, we introduce an alternative formulation often referred to in the literature as the "Tarski-Seidenberg Second Form" or the "Geometric Version of the Tarski-Seidenberg theorem". This version states that the projection of a semialgebraic set is also semialgebraic.

Theorem 4.2.2 (Projection of semialgebraic sets, Theorem 2.2.1 [31])**.** *Let A be a semialgebraic subset of R^{n+1}, and $\Pi : R^{n+1} \to R^n$ the projection onto the first n-coordinates. Then, $\Pi(A) \subseteq R^n$ is a semialgebraic set.*

Proof. Assume the semialgebraic set $A \subseteq R^{n+1}$ is defined as

$$A = \{(b, a) \in R^n \times R \mid p(b, a) = 0,\ q_j(b, a) > 0,\ j = 1, \ldots, l\},$$

where $p(b, a)$ and $q_j(b, a)$ are polynomials in the variables $b \in R^n$ and $a \in R$, with coefficients in $\mathbb{R}$. The projection $\Pi(A)$ onto the n-dimensional space is defined as:

$$\Pi(A) = \{b \in R^n \mid \exists a \in R \text{ such that } (b, a) \in A\}.$$

By the Tarski-Seidenberg theorem, the projection $\Pi(A)$ is semialgebraic because it is defined by quantifying the variable a from the description of A. This quantification ensures that $\Pi(A)$ can be expressed as a Boolean combination of polynomial equations and inequalities in b, with coefficients in $\mathbb{R}$. Thus, $\Pi(A)$ is a semialgebraic subset of R^n, completing the proof. □

The closure of semialgebraic sets under projections enhances their mathematical significance by highlighting an important relationship between algebraic and semialgebraic sets: the projection of an algebraic set is a semialgebraic set. Furthermore, any semialgebraic set in R^n can be expressed as the projection of an algebraic set in R^{n+l}, for some positive integer l. For instance, consider the following semialgebraic set:

$$\{x \in R^n \mid P(x) = 0,\ q_1(x) > 0, \ldots, q_l(x) > 0\}.$$

This set can be seen as the projection of the algebraic set

$$\{(x, y) \in R^{n+l} \mid P(x) = 0,\ y_1^2 q_1(x) = 1, \ldots, y_l^2 q_l(x) = 1\}.$$

Furthermore, every semialgebraic subset of R^n can be expressed as the projection of an algebraic subset of R^{n+1} (see [38]). Assuming that R is equipped with the Euclidean topology, the Tarski-Seidenberg theorem also implies that the interior, closure, and boundary of a semialgebraic set are themselves semialgebraic.

Corollary 4.2.1 (Properties of semialgebraic sets, Proposition 2.2.2 [31, 33])**.** *The closure, interior, and boundary of a semialgebraic set are themselves semialgebraic sets.*

Proof. Let S be a semialgebraic subset of R^m. The closure of S is given by

$$\overline{S} = \{x \in R^m \mid \forall \varepsilon \in R,\ \exists y \in S\ (\|x - y\|^2 < \varepsilon\ \vee \varepsilon = 0)\}. \tag{4.7}$$

By the Tarksi-Seidenberg Theorem 4.2.1, $\overline{S}$ is also semialgebraic. The interior of S, defined as $\text{int}(S) = R^n \setminus (\overline{R^n \setminus S})$, is also semialgebraic because it is the complement of a semialgebraic set. Finally, the boundary of S, given by $\text{bd}(S) = \overline{S} \cap (\overline{R^n \setminus S})$, is semialgebraic as it is the intersection of two semialgebraic sets. □

In summary, using formulas is generally the more effective method for demonstrating that a set is semialgebraic, as opposed to relying on projections.

4.3 Semialgebraic and Transcendenal Functions

We begin this section by defining the concept of an *algebraic function.* A function $g : J \to \mathbb{R}$ defined on an interval $J \subseteq \mathbb{R}$ is referred to as *algebraic* over a subfield $\mathbb{F} \subseteq \mathbb{R}$ if there exists a polynomial $q \in \mathbb{F}[y, x]$ and an interval $I \subseteq \mathbb{R}$ such that the equation $y = g(x)$ is equivalent to $q(y, x) = 0$ for all pairs (x, y) in the product $J \times I$.

To define *semialgebraic functions*, let $U \subset R^n$ and $V \subset R^k$ be semialgebraic sets.

Definition 4.3.1 (Semialgebraic functions)**.** *A function $f : U \to V$ is called* semialgebraic *if its graph, $\Gamma_f = \{(u, v) \in U \times V \mid v = f(u)\}$, is a semialgebraic set in R^{n+k}.*

If f is also bijective and continuous, with a continuous inverse $f^{-1} : V \to U$, then f is said to be a *semialgebraic homeomorphism.* Additionally, if both f and its inverse f^{-1} are smooth, f is called a *semialgebraic diffeomorphism.*

Definition 4.3.2 (Transcendental Functions)**.** *Functions that are not piecewise algebraic over a subfield $\mathbb{F} \subseteq \mathbb{R}$, are called* transcendental functions*.*

In Core Articles [1–3] we work with $\mathbb{F} = \mathbb{R}$. If $f : \mathbb{R} \supseteq J \to \mathbb{R}$ is semialgebraic over $\mathbb{R}$, then it is piecewise algebraic over $\mathbb{R}$ (see § 2.5.2 in [39]).

Example 4.3.1 (Examples of semialgebraic functions)**.** *Let $U \subset R^n$ and $V \subset R^m$ are semialgebraic sets. Then,*

i) Any polynomial function $p : U \to V$, given by $(u_1, \ldots, u_n) \mapsto (p_1(u_1, \ldots, u_n), \ldots, p_m(u_1, \ldots, u_n))$ where each $p_j \in R[U_1, \ldots, U_n]$ for $j = 1, \ldots, m$ is semialgebraic.

This can be shown by looking at its graph:

$$\Gamma_p = \{(u, v) \in U \times V \mid \forall j = 1, \ldots, m,\ p_j(u_1, \ldots, u_n) = v_j\}.$$

Clearly, Γ_p is an algebraic set and, consequently, a semialgebraic subset of R^{m+n}.

ii) Any rational function $f : U \to V$ is semialgebraic, provided that each coordinate of f is a rational function whose denominator does not vanish on U.

Consider f in the form

$$(u_1, \ldots, u_n) \mapsto \Big(\frac{f_1(u_1, \ldots, u_n)}{g_1(u_1, \ldots, u_n)}, \ldots, \frac{f_m(u_1, \ldots, u_n)}{g_m(u_1, \ldots, u_n)}\Big).$$

Its graph is given by

$$\Gamma_f = \{(u, v) \in U \times V \mid \forall j = 1, \ldots, m,\ f_j(u_1, \ldots, u_n) = v_j g_j(u_1, \ldots, u_n)\},$$

where $f_j, g_j \in R[U_1, \ldots, U_N]$. Clearly, Γ_f is an algebraic subset of R^{n+m}, and therefore, it is also semialgebraic

Definition 4.3.3 (Nash functions)**.** *Let $V \subset R^n$ be an open semialgebraic set. A* Nash function *$f : V \to R$ is a function that is both smooth and semialgebraic.*

An example of a Nash function is $f(x) = \sqrt{1 + x^2}$. When the field of real numbers $\mathbb{R}$ is chosen as the real closed field, Nash functions exhibit a special characterization.

Proposition 4.3.1 (Nash functions on $\mathbb{R}$)**.** *Assume V is an open semialgebraic subset of $\mathbb{R}^m$. A function $f : V \to \mathbb{R}$ is a Nash function if and only if it is both analytic and algebraic on V.*

For a proof, see Proposition 8.1.8 of [31]. The concept of a *Nash diffeomorphism* between open semialgebraic subsets R^m is straightforward: it is a mapping that is both a Nash function and a diffeomorphism.

Proposition 4.3.2 (Image of semialgebraic sets under semialgebraic functions)**.** *Let $U \subset R^n$ and $V \subset R^m$ be semialgberaic sets, and $f : U \to V$ a semialgebraic map. If $S \subset U$ is a semialgebraic set, then its image $f(S)$ is also semialgebraic. Similarly, if $L \subset V$ is a semialgebraic set, its preimage $f^{-1}(L)$ is also semialgebraic.*

For a proof, see, for instance, Proposition 2.2.7 of [33].

An example of a semialgebraic function used in the study of level sets of the von Neumann entropy in the Core Article [1] is the distance function.

Proposition 4.3.3 (Distance function, Proposition 2.2.8 [31])**.** *Let S be a nonempty semialgebraic subset of R^m. For any point $u \in R^m$, the distance between u and S, defined as*

$$\mathit{dist}(u, S) = \inf\{\|u - v\| \mid v \in S\}, \tag{4.8}$$

is well-defined. Moreover, the function $R^m \ni u \mapsto \operatorname{dist}(u, S) \in R$ is continuous and semialgebraic.

Proof. The set $\{\|u - v\| \mid v \in S\}$ is defined as the image of S under the semialgebraic function $S \ni v \mapsto \|u - v\| \in R$, making it semialgebraic as well. Since this set is bounded from below, it has a lower bound. To verify that the function $R^m \ni u \mapsto \operatorname{dist}(u, S) \in R$ is semialgebraic, we examine its graph:

$$\begin{aligned}\Gamma_{dist} = \{(u, t) \in R^{m+1} \mid t \geq 0 \text{ and } \forall v \in S \quad & \|u - v\|^2 \geq t^2 \\ & \text{and } \forall \varepsilon \in R,\ \varepsilon > 0 \Rightarrow \exists v \in S \quad \|u - v\|^2 < t^2 + \varepsilon\}.\end{aligned}$$

By applying the Tarski-Seidenberg Theorem to the set Γ_{dist}, we conclude its semialgebraicity. The continuity of the function follows directly from its definition. □

We now give some examples of transcendental functions and explain why they are indeed transcendental.

Example 4.3.2 (Transcendental functions)**.** *The following functions are transcendental:*

i) The complex logarithmic function [40]: Consider a complex number $z = x + iy = re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$. The logarithm is defined by $\log(z) = \log(r) + i\theta$. Let $A \coloneqq e^{i\theta_0}$ be an arbitrary point on the complex plane. Starting from A and tracing a closed path counterclockwise around the origin $z = 0$, we return to A with the angle θ_0 increased by 2π. Thus, $\log(z)$ does not return to its original value after the path is traced, as $\log(A) \to \log(A) + 2\pi i$. Since A was chosen arbitrarily, this "identity crisis" occurs at every point in the complex plane, making $z = 0$ a *branch point* for $\log(z)$. To be more precise, this behavior must occur for all curves that enclose $z = 0$ and lie sufficiently close to it. This way any integer multiple of 2π is obtained, depending on how many times the path around the origin is enclosed. We can uniquely define $\log(z)$ in regions that do not enclose the origin by choosing a specific angle θ in each region, thus specifying a particular *branch* of $\log(z)$. This choice allows $\log(z)$ to be analytic in each region.

The complex logarithm is well-defined in any region that does not contain a closed path encircling the origin. Therefore, to make $\log(z)$ single-valued across the entire complex plane, we exclude an infinitesimally small cut along the positive real x-axis, thereby preventing paths from enclosing the origin. Here, a point A just above the $x-$axis differs in $\log(z)$ value from a point B infinitesimally below it, creating a discontinuity along this *branch cut* on the positive real $x-$axis.

One way to visualize this structure is by imagining the function $\log(z)$ defined not on a plane, but on a multi-level 'parking garage' with infinitely many floors. When we trace a path around the origin from A, we end up one 'level' above A in the 'garage'. Hence, although the value of $\log(z)$ changes by 2π, there are no 'identity crises'. The value of $\log(z)$ then depends on which level of the 'garage' we occupy. These 'garage levels' are known as *Riemann sheets*. Starting from the 'first' Riemann sheet and making k counterclockwise loops around the origin we reach the $(k+1)th$ Riemann sheet, and the function shifts by

$$\log(z) \to \log(z) + 2k\pi i, \quad \textit{where} \;\; k \in \mathbb{Z}. \tag{4.9}$$

Since $k \in \mathbb{Z}$, $\log(z)$ has infinitely many Riemann sheets, or infinitely many branches, forming a surface where $\log(z)$ becomes a single-valued and analytic function, known as the *Riemann surface* for $\log(z)$. In addition to $z = 0$ being a branch point, $z = \infty$ is also a branch point for $\log(z)$. A branch cut for $\log(z)$ can be any curve connecting these two points, with the objective of preventing curves from going around any of them. Since algebraic analytic functions have only finitely many branches (see Theorem 4 in [11]), $\log(z)$ is therefore a transcendental function. This fact is used in Core Article [1] to demonstrate the transcendentality of the level sets of von Neumann entropy.

ii) The exponential function: Since the inverse of an algebraic function is also algebraic if $\log(z)$ were algebraic, then e^z would be as well. However, because we have shown that the logarithm is not an algebraic function, the exponential function is likewise not algebraic.

iii) One specific form of an elliptic integral is the *complete elliptic integral of the second kind.* These integrals are called 'elliptic' because they initially arose in the context of determining the arc length of an ellipse. The complete elliptic integral of the second kind is defined as

$$E(k) = \int_0^{\pi/2} (1 - k^2 sin^2(\phi))^{1/2} d\phi, \tag{4.10}$$

where the elliptic modulus k satisfies $0 < k^2 < 1$. In Core Article [3], this integral is examined, and it is shown to be transcendental over $\mathbb{R}$.

Algebraic functions can also be characterized in terms of their *algebraic singularities* [41]. A singular point $z_* \in \mathbb{C}$ of a complex function $f(z)$ is called algebraic if, in a neighborhood of z_*, the function can be represented by a Puiseux series:

$$f(z) = \sum_{l=M}^{\infty} a_l (z - z_*)^{l/n}, \tag{4.11}$$

where $n > 0$ and M are integers, with $a_M \neq 0$. For a complex function $f(z)$ and a point $z_* \in \mathbb{C} \cup \{\infty\}$, the pair $(f(z), z_*)$ is called a *singular element* of an analytic function $F(z)$ if $f(z)$ consists of branches of $F(z)$ that satisfy the following conditions:

i) z_* is an isolated singular point shared by each branch of $f(z)$ in a small punctured disk $D_* = \{0 < \|z - z_*\| < \varepsilon\}$, and any two analytic branches of $f(z)$ can be analytically continued into each other along a path within D_*.

ii) No other analytic branch of $F(z)$ can be obtained by continuing any branch of $f(z)$ along a path in D_*.

If, in a neighborhood of z_*, $f(z)$ can be expressed by a Puiseux series as in Equation (4.11), then the singular element $(f(z), z_*)$ is called algebraic.

Theorem 4.3.1 (Algebraic functions and algebraic singularities, [41])**.** *A global analytic function $F(z)$, $z \in \mathbb{C} \cup \{\infty\}$, is algebraic if and only if it has a finite number of algebraic singularities, and all its singular points are isolated.*

The set of $R-$valued semialgebraic functions on a semialgebraic set forms a ring endowed with pointwise addition and pointwise multiplication (see e.g., Theorem 2.2.6 of [31]). Additionally, the absolute value of a semialgebraic map remains semialgebraic, and if the map is positive, its square root is also semialgebraic. As the usual notions of differentiation over $\mathbb{R}$ can be copied to an arbitrary real closed field, the derivative of a differentiable semialgebraic function $f : (a, b) \to R$ is also semialgebraic. This is shown by describing the graph of f' using a first-order formula in the language of ordered fields, with parameters in R, and then applying the Tarski-Seidenberg quantifier elimination theorem (see Proposition 2.9.1 of [31]). The differentiability of semialgebraic functions in multiple variables follows similar principles, as explained in [31].

Proposition 4.3.4 (Composition of semialgebraic maps, Proposition 2.2.6 [31])**.** *Let $f : U \to V$ and $g : V \to W$ be semialgebraic maps, where $U \subset R^n$, $V \subset R^m$, and $W \subset R^k$ are semialgebraic sets. Then, the composition $g \circ f : U \to W$ is also a semialgebraic map.*

Proof. Let $\Gamma_f \subset R^{m+n}$ denote the graph of the function f, and let $\Gamma_g \subset R^{m+k}$ denote the graph of the function g. The graph of the composition $g \circ f$ is the projection of the intersection $(\Gamma_f \times R^k) \cap (R^n \times \Gamma_g)$ onto R^{n+k}. Since the projection of a semialgebraic set remains semialgebraic, the proof is concluded. □

4.4 Decomposition of Semialgebraic Sets

Every semialgebraic subset of R can be represented as a finite union of points and open intervals, which may be either bounded or unbounded. In higher-dimensional spaces R^m, for some finite m, semialgebraic subsets can also be decomposed in a similar way. The goal is to break semialgebraic sets down into simpler, more 'manageable' semialgebraic subsets. Collins [42] pioneered research in this area, developing the cylindrical algebraic decomposition (CAD) algorithm, a major tool for studying such decompositions. The cylindrical algebraic decomposition algorithm shows that any semialgebraic set $S \subset R^n$ can be expressed as the disjoint union of a finite number of semialgebraic subsets $A_1, \ldots, A_q$, where each A_i is semialgebraically homeomorphic to an open hypercube $(0,1)^{\dim(C_i)} \subset R^m$. Here $(0,1)^0$ represents a point. The decomposition

$$S = \bigcup_{j=1}^{\cdot q} A_j \tag{4.12}$$

is called a *stratification* of S, and the components A_j for $j = 1, \ldots, q$ are called *strata* (see, for instance, Theorem 2.3.6 [31], or §2.3 of [32]). Before presenting the specific form of this decomposition used to prove the transcendence of the set of quantum states with fixed von Neumann entropy in Core Article [1], we introduce the concept of *Nash manifolds*.

Definition 4.4.1 (Nash manifolds)**.** *Let N be a semialgebraic subset of R^m. We say that N is a* Nash submanifold *of R^m if it is a submanifold that is both smooth and semialgebraic.*

Theorem 4.4.1 (Decomposition into Nash submanifolds)**.** *Let $S \subset \mathbb{R}^m$ be a semialgebraic set. Then S can be decomposed into a finite disjoint union of Nash submanifolds N_i, each Nash diffeomorphic to an open hypercube $(0,1)^{\dim(N_i)}$.*

This result is proved by induction on m, using the cylindrical algebraic decomposition (for details see Proposition 2.9.10 in [31]).

Another concept related to the decomposition of semialgebraic sets is connectedness. A semialgebraic set $S \subset R^m$ is called *semialgebraically connected* if, for any pair of semialgebraic subsets A_1 and A_2 that are closed in S (in the Euclidean topology) and satisfy $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = S$, it must hold that either $A_1 = S$ or $A_2 = S$. Since an open hypercube $(0,1)^m \subset R^m$ is semialgebraically connected (for a proof, see Proposition 2.4.3 in [31]), the following result holds:

Theorem 4.4.2 (Connected components of semialgebraic sets)**.** *Each semialgebraic subset S of R^m can be written as the finite disjoint union of semialgebraically connected semialgebraic sets $A_1, \ldots, A_q$ that are both open and closed in S. The sets A_i for $i = 1, \ldots, q$ are known as the* semialgebraically connected components *of S.*

The proof is established using cylindrical algebraic decomposition. For further details, see Theorem 2.4.4 in [31]. Since an open hypercube is connected when $R = \mathbb{R}$, it follows that a semialgebraic set $S \subset \mathbb{R}^m$ is semialgebraically connected if and only if it is connected. Consequently, every semialgebraic set—specifically, every semialgebraic subset of $\mathbb{R}^m$—possesses a finite number of connected components, that are semialgebraic (see Theorem 2.4.5 of [31]).

Upon introducing the cylindrical algebraic decomposition of a semialgebraic set $S \subset R^m$ as the disjoint union $S = \dot{\bigcup}_{j=1}^{q} A_j$, with each A_j semialgebraically homeomorphic to an open hypercube $(0,1)^{d_j}$, the dimension of S is defined as the maximum of the dimensions d_j: $\dim(S) = \max(d_1, \ldots, d_q)$ (see Corollary 2.8.9 of [31]). This dimension is independent of the chosen decomposition (see Section 3.3 of [32]). The general concept of dimension for semialgebraic sets is rooted in ring theory and prime ideals, but in this thesis, we will focus on specific cases where semialgebraic sets are either smooth manifolds or open subsets.

Proposition 4.4.1 (Dimension of open semialgebraic sets, Proposition 2.8.4 of [31])**.** *If Vm is a non-empty open semialgebraic subset of R^m, then the dimension of V is $dim(V) = m$.*

We conclude this section with a result regarding the dimension of semialgebraic sets over $\mathbb{R}$.

Proposition 4.4.2 (Dimension of Nash manifolds, Proposition 2.8.14 of [31])**.** *Let $S \subset \mathbb{R}^m$ be a semialgebraic set that is also a smooth submanifold of $\mathbb{R}^m$ with dimension d. Then, the dimension of S is d.*

4.5 Semialgebraic and Transcendenal Structures in Quantum Information Theory and Beyond

Building on the discussion in Section 3.2, which highlighted the role of transcendental number theory in addressing foundational problems in quantum information theory and beyond, this section turns to the contributions of semialgebraic geometry. This mathematical framework has likewise proven instrumental in tackling a variety of challenges in the field. Its relevance is demonstrated both in earlier work [43–45] and in our own contributions presented in the Core Articles [1–3].

Wolf, Cubitt, and Pérez-García [43] investigated the algorithmic decidability of problems arising in quantum information theory using tools from Tarski and Seidenberg. They show that many problems involving quantum channels and entanglement theory are decidable using the Tarski-Seidenberg quantifier elimination theorem, as long as these problems have a compact formulation and do not require quantification over integers. More broadly, problems expressed in terms of semialgebraic sets are decidable under this framework.

Many challenges, nevertheless, in quantum information theory arise from problems that require consideration of all possible protocols, decompositions, or models. Additional complications stem from asymptotically defined properties that must hold for all or some values of $n \in \mathbb{N}$. Such problems are often formulated mathematically using quantifiers over infinite domains, which is precisely what makes them difficult to resolve. This quantification over infinite sets makes effective procedures seem, at best elusive and, at worst, impossible. To address the decidability of some of these problems, one promising direction is to explore extensions of the Tarski-Seidenberg framework. For instance, quantifier elimination over the real field augmented by exponentiation and all restricted analytic functions [43,46], provides a potential pathway for proving decidability within specific classes of problems.

Beyond questions of decidability, the distinction between transcendental and semialgebraic structures has also proven useful for distinguishing between different quantum states. Fanizza, Lumbreras, and Winter [45] prove the existence of finitely correlated states (FCS) that are not C^*-finitely correlated states (C^*-FCS) (for readers not familiar with FCS and C^*-FCS, please refer to Section 6.3 of this thesis for a brief introduction). Their work builds on ideas introduced in [47] and the framework of transcendental sets, addressing a question initially posed by Fannes, Nachtergaele, and Werner in [4]: If a process has a finite-dimensional linear realization (see Definition 6.3.2), can it always be explained by quantum mechanics? In other words, is every FCS state a C^*-FCS? Finitely correlated states are states generated by finite-dimensional real vector spaces, also known as "memory systems" within a general probabilistic framework. They can be described by linear realizations, which use linear maps to evolve memory states (comparing to Definition 6.3.2, these states are elements of a real vector space W). The positivity of the probabilities defined by the linear realizations (as given in the expression (6.17) in Section 6.3) is shown to be ensured by the existence of convex cones $C \subset W$ and $\tilde{C} \subset C' \subset W^*$ such that $e \in C$ and $\rho \in \tilde{C}$. Moreover, these cones are preserved under the transformations $\mathbb{E}_M$, where e, ρ, and $\mathbb{E}_M$ are as defined in Definition 6.3.2. Conversely, if the expression (6.17) is non-negative, such cones exist, and they are not unique. For C^*-FCS, these convex cones are semidefinite representable (SDR), meaning they satisfy a given set of semidefinite constraints. The SDR cones are semialgebraic, i.e., they can be described by a finite system of polynomial equations and inequalities.
The regular representation of a C^*-FCS (refer to Definition 6.3.3) is unique and can be obtained from the C^*-representation via a quotient construction, as shown in [45] and [47]. This regular representation admits an SDR cone. In [45], the authors construct processes with finite-dimensional linear realizations that are valid FCS but whose defining convex cone is transcendental. Since SDR cones are semialgebraic, these processes cannot be described as C^*-FCS. This provides a definitive answer to the question posed in [4]: there exist processes with finite-dimensional linear realizations that can be explained by general probabilistic theories (GPTs) but cannot be represented using quantum mechanics. Formulated differently, there exist finitely correlated states that are not C^*-finitely correlated states.

Chapter 5

Quantum Entropies and Information-Theoretic Quantities

Entropies are fundamental to quantum information theory, serving as key tools for characterizing the optimal rates of information-theoretic tasks and quantifying the uncertainty in the state of a physical system. This chapter provides an overview of the definitions and fundamental properties of entropic quantities, which are essential for understanding some of the core topics of this thesis. For further reading, we recommend textbooks such as [6, 48] and relevant research articles cited throughout this chapter.

This chapter is organized as follows: In Section 5.1, we introduce von Neumann entropy, covering its properties, and its continuity bounds. Section 5.2 discusses quantum relative entropy as a measure of distinguishability between quantum states. We then explore mutual information in Section 5.3 and $\alpha-$Rényi entropies in Section 5.4. Finally, Section 5.5 concludes with a discussion on the distinction between semialgebraic and transcendental structures, highlighting their implications for the separation between single-shot and asymptotic settings. Throughout this discussion, we assume a base 2 logarithm when defining the following entropic quantities.

5.1 von Neumann Entropy

The von Neumann entropy stands out as one of the most extensively studied and arguably the most significant entropy functional in quantum information theory. Before entering the quantum domain, we first review the concept of entropy in the classical setting. Consider a discrete random variable X that takes values in a finite set $\mathcal{X}$, with a corresponding probability density distribution $p = (p_x)_{x\in\mathcal{X}}$. The *Shannon entropy*, introduced by Claude Shannon in 1948 [49], quantifies (in bits) the average information gained upon learning the value of X. It is defined as the concave function:

$$H(X) \coloneqq -\sum_x p_x \log p_x, \tag{5.1}$$

with the convention $0 \log 0 \equiv 0$. Equivalently, the Shannon entropy can be viewed as a function of the (discrete) probability distribution p, denoted $H(p)$. Shannon entropy has an operational meaning in the information-processing task of source coding: it represents the minimum number

of bits needed to encode messages generated by a statistical source. From its definition, it follows that Shannon entropy is always non-negative and equals 0 if and only if the outcome of X is certain, i.e., the probability distribution is concentrated on a single event. The maximum possible value of $H(X)$ is $\log|\mathcal{X}|$, achieved when all outcomes in $\mathcal{X}$ are equally likely. A key property of Shannon entropy is subadditivity: for random variables X and Y, the inequality $H(X,Y) \leq H(X) + H(Y)$ holds, with equality if and only if X and Y are independent. Here, $H(X,Y)$, known as the *joint entropy* of X and Y, is defined analogously:

$$H(X,Y) \coloneqq -\sum_{x,y} p(x,y) \log p(x,y), \tag{5.2}$$

and it quantifies the uncertainty about the pair (X,Y). This definition extends naturally to the joint entropy of n-random variables. Another important concept in information theory that captures the relationship between the information content of X and Y is *conditional entropy*. It quantifies the average uncertainty about X given that the value of Y is known. When Y is known, it provides $H(Y)$ bits of information about (X,Y), leaving the remaining uncertainty attributable to X. The conditional entropy, therefore, measures the residual uncertainty and is defined as:

$$H(X|Y) \coloneqq H(X,Y) - H(Y). \tag{5.3}$$

Conditioning does not increase Shannon entropy. In other words, having access to a side variable Y can only reduce our uncertainty about another variable X, i.e., $H(X|Y) \leq H(X)$.

When a random variable has only two possible outcomes with probabilities p and $1-p$, the Shannon entropy reduces to a simpler form:

$$h(p) \coloneqq -p \log p - (1-p) \log(1-p), \tag{5.4}$$

which is known as the *binary entropy*.

Shannon entropy quantifies uncertainty in classical probability distributions. In the quantum setting, this concept is generalized by replacing probability distributions with density operators, leading to the notion of von Neumann entropy, introduced by John von Neumann in 1927 [50]. Its operational interpretation emerged much later, in 1995, when Schumacher demonstrated that it represents the number of qubits needed to faithfully transmit quantum states produced by a statistical source [51].

Definition 5.1.1 (von Neumann entropy)**.** *The von Neumann entropy of a quantum state $\rho \in \mathcal{S}(\mathcal{H})$ is defined as:*

$$S(\rho) \coloneqq -\operatorname{tr}[\rho \log \rho], \tag{5.5}$$

with the standard convention $0 \log 0 \equiv 0$.

If $\{\lambda_i\}_i$ represents the spectrum of ρ, the von Neumann entropy of ρ can be expressed as the Shannon entropy of its spectrum (considered as a multiset):

$$S(\rho) = H(\{\lambda_i\}_i) = -\sum_i \lambda_i \log \lambda_i. \tag{5.6}$$

Similar to Shannon entropy, the von Neumann entropy is always non-negative. It equals zero if and only if the quantum state is pure, indicating that the quantum system is fully described by this pure state. In a d-dimensional Hilbert space, the von Neumann entropy is at most $\log d$, with this maximum achieved if and only if the state is maximally mixed. Several additional properties of von Neumann entropy, some derived directly from its definition and others requiring specific calculations, are summarized in the following theorem.

Theorem 5.1.1 (Properties of the von Neumann entropy)**.** *The von Neumann entropy*

i) is *continuous* for finite-dimensional Hilbert spaces, and *lower semicontinuous* for infinite-dimensional Hilbert spaces*, i.e.,* $\liminf_{\rho\to\rho_0} S(\rho) \geq S(\rho_0)$.

ii) is *subadditive, i.e.,* for a quantum state ρ_{AB} describing the composite quantum system AB we have

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B), \tag{5.7}$$

where ρ_A and ρ_B are the reduced states of systems A and B, respectively. This inequality was proven by Araki and Lieb in 1970 [52]. Equality holds in (5.7) if and only if the systems A and B are uncorrelated, meaning $\rho_{AB} = \rho_A \otimes \rho_B$. In such cases, the von Neumann entropy is said to be *additive.*

iii) is a *concave* function of its inputs. Specifically, for a set of probabilities p_i ($\forall i\ p_i \geq 0$ and $\sum_i p_1 = 1$) and the corresponding quantum states ρ_i, the following inequality holds:

$$S\left(\sum_i p_i\rho_i\right) \geq \sum_i p_i S(\rho_i). \tag{5.8}$$

This property reflects that the entropy of a mixture of quantum states is at least as large as the weighted sum of the entropies of the individual states. It reflects the idea that mixing quantum states introduces additional uncertainty.

iv) is *isometric-invariant, i.e.,* for any isometry $U : \mathcal{H} \to \mathcal{H}'$, the entropy satisfies the following equality:

$$S(U\rho U^*) = S(\rho). \tag{5.9}$$

v) satisfies the *triangle inequality, i.e.,* for two distinct quantum systems A and B, the entropy of the composite system AB satisfies the inequality:

$$|S(A)_\rho - S(B)_\rho| \leq S(AB)_\rho. \tag{5.10}$$

For a quantum system A, we denote the von Neumann entropy of the quantum state ρ describing the system by any of the following notations: $S(A)$, $S(A)_\rho$, or $S(\rho_A)$. Throughout this thesis, we use the terms 'von Neumann entropy' and 'entropy' interchangeably.

The concepts of subadditivity and the triangle inequality for von Neumann entropy generalize to tripartite systems via the notion of *strong subadditivity*. This fundamental inequality was proven by Lieb and Ruskai in 1973 [53], and holds for any three quantum systems A, B, and C:

$$S(ABC)_\rho + S(B)_\rho \leq S(AB)_\rho + S(BC)_\rho. \tag{5.11}$$

For rigorous proofs and further discussion of von Neumann entropy's properties, see Sections 11.3 and 11.4 of [6].

Similar to its classical counterpart, the *conditional quantum entropy* is defined as follows:

Definition 5.1.2. *For a quantum state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the conditional quantum entropy $S(A|B)_\rho$ is defined as:*

$$S(A|B)_\rho \coloneqq S(AB)_\rho - S(B)_\rho. \tag{5.12}$$

Unlike its classical counterpart, conditional quantum entropy can take negative values, e.g., if ρ_{AB} is pure. However, it remains non-negative if at least one of the systems, A or B, is classical (see, for instance, 11.4.2 in [48]), and if ρ_{AB} is separable it is always non-negative. An important property of conditional quantum entropy is that conditioning does not increase the von Neumann entropy, i.e., $S(A)_\rho \geq S(A|B)_\rho$. For a proof of this property, see Theorem 11.4.1 in [48].

As mentioned in Theorem 5.1.1, von Neumann entropy is continuous in finite-dimensional spaces. Its practical utility, however, is enhanced when explicit continuity bounds are established. For a general entropic function f, continuity bounds quantify how the values of f for two quantum states ρ and σ relate when these states are close in trace distance. Specifically, if $T(\rho, \sigma) \leq \varepsilon$ for $0 < \varepsilon < 1$, a continuity bound provides a measure of the difference between $f(\rho)$ and $f(\sigma)$. One of the earliest and most fundamental continuity bounds for entropy was derived by Fannes in 1973 [54]. For any two density operators $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ satisfying $T(\rho, \sigma) \leq 1/e$, the von Neumann entropies of ρ and σ satisfy the following inequality:

$$|S(\rho) - S(\sigma)| \leq T(\rho, \sigma) \log d - T(\rho, \sigma) \log(T(\rho, \sigma)), \tag{5.13}$$

where $d = |\mathcal{H}|$ denotes the dimension of the Hilbert space. An enhanced form of this bound, known as the Audenaert-Fannes inequality, was later introduced in [55] and independently proven in [56] using a different approach.

Theorem 5.1.2 (Audenaert-Fannes [55])**.** *Let $\rho, \sigma \in \mathcal{S}(\mathbb{C}^d)$ be density operators in a d-dimensional Hilbert space, and let $\varepsilon > 0$. If $T(\rho, \sigma) \leq \varepsilon$, then the following inequality holds:*

$$|S(\rho) - S(\sigma)| \leq \begin{cases} \varepsilon \log(d-1) + h(\varepsilon) & \text{if } \varepsilon < 1 - \frac{1}{d} \\ \log d & \text{if } \varepsilon \geq 1 - \frac{1}{d}, \end{cases} \tag{5.14}$$

where $h(\varepsilon)$ is the binary entropy function.

Theorem 5.1.3 (Continuity of conditional entropy)**.** *Let $\rho, \sigma \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be density operators, and let $0 < \varepsilon \leq 1$. If $T(\rho, \sigma) \leq \varepsilon$, then*

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq 2\varepsilon \log |\mathcal{H}_A| + (1+\varepsilon) h\Big(\frac{\varepsilon}{1+\varepsilon}\Big). \tag{5.15}$$

*If B is classical (i.e., ρ and σ are qc-*states*), the bound tightens to*

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq \varepsilon \log |\mathcal{H}_A| + (1+\varepsilon) h\Big(\frac{\varepsilon}{1+\varepsilon}\Big). \tag{5.16}$$

For a proof of this theorem see Lemma 2 in [57].

5.2 Relative Entropy

Classical relative entropy, also known as the *Kullback-Leibler* (KL) divergence, is another fundamental measure in information theory used to quantify the "distance" between two probability distributions. For two distributions $p = (p(x))_{x \in \mathcal{X}}$ and $q = (q(x))_{x \in \mathcal{X}}$ defined over the same alphabet $\mathcal{X}$, the KL divergence is given by:

$$H(p\|q) \coloneqq \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}, \tag{5.17}$$

where we adopt the standard conventions $0 \log 0 \equiv 0$ and $-p(x) \log 0 \equiv \infty$ if $p(x) > 0$.

Similar to Shannon entropy, classical relative entropy exhibits several important properties. One key property is non-negativity: it is always ≥ 0 and equals zero if and only if $p(x) = q(x)$ for all $x \in \mathcal{X}$. This makes classical relative entropy a natural measure of similarity between distributions (see Theorem 10.7.1 in [48] for a proof). An important consequence of this non-negativity is the *Data Processing Inequality* (DPI), also known as the *monotonicity of relative entropy*. This result formalizes the principle that information content cannot increase through any processing, and any lost information cannot be recovered. This result is stated rigorously in the following theorem, with a proof available in Corollary 10.7.2 of [48].

Theorem 5.2.1 (Data processing inequality (DPI))**.** *Let p and q be probability distributions over an alphabet $\mathcal{X}$, and $N(y|x)$ a classical channel. The relative entropy between p and q does not increase when passed through the channel $N(y|x)$:*

$$H(Np\|Nq) \leq H(p\|q). \tag{5.18}$$

Quantum relative entropy. Quantum relative entropy extends the classical concept of relative entropy to quantum systems, serving as a measure of how distinguishable one quantum state is from another. Although it is not a 'distance' in the strict mathematical sense, it serves as an important tool for comparing quantum states and is widely used in quantum information theory. It forms the basis for several fundamental entropic quantities in quantum information theory, including von Neumann entropy, quantum conditional entropy, and mutual information (see Section 5.3).

Definition 5.2.1 (Quantum relative entropy)**.** *Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ be positive semi-definite operators. The quantum relative entropy between ρ and σ is defined as:*

$$S(\rho\|\sigma) \coloneqq \mathrm{tr}[\rho(\log\rho - \log\sigma)], \tag{5.19}$$

whenever $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$*; otherwise, it is defined to be* ∞*.*

From this point forward, we will refer to the quantum relative entropy between ρ and σ simply as "relative entropy".

Theorem 5.2.2 (Klein's inequality)**.** *Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ be density operators. Then, the quantum relative entropy of ρ to σ is always non-negative:*

$$S(\rho\|\sigma) \geq 0, \tag{5.20}$$

with equality if and only if $\rho = \sigma$.

The proof follows from the definition of relative entropy by decomposing ρ and σ into their orthonormal eigenbases and applying the strict concavity of the logarithm. For a detailed explanation, refer to Theorem 11.7 in [6]. This non-negativity property, can be further understood through the following proposition, which establishes a connection between quantum relative entropy and the trace distance.

Proposition 5.2.1 (Pinsker's inequality)**.** *For density operators $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the relative entropy and trace distance are related as follows:*

$$\frac{1}{2}\|\rho - \sigma\|_1^2 \leq S(\rho\|\sigma). \tag{5.21}$$

Inequality (5.21) confirms that $S(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$, reinforcing Klein's inequality. For a detailed proof, see Proposition 1.46 in [9].

Theorem 5.2.3 (Joint convexity of relative entropy)**.** *Let $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathcal{B}(\mathcal{H})$ be positive semi-definite operators. For any $\lambda \in (0, 1)$, the relative entropy is jointly convex in its arguments:*

$$S(\lambda\rho_1 + (1-\lambda)\rho_2\|\lambda\sigma_1 + (1-\lambda)\sigma_2) \leq \lambda S(\rho_1\|\sigma_1) + (1-\lambda)S(\rho_2\|\sigma_2). \tag{5.22}$$

Joint convexity is a crucial property that highlights the stability of relative entropy under probabilistic mixture of states. For a proof, refer to Theorem 11.12 in [6].

Like its classical counterpart, the quantum relative entropy satisfies the data processing inequality, which states that the distinguishability between quantum states ρ and σ cannot increase when they are subjected to noise or processed through a quantum channel.

Theorem 5.2.4 (Data processing inequality - DPI)**.** *Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ be two density operators, and $T : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ a quantum channel. Then:*

$$S(T(\rho)\|T(\sigma)) \leq S(\rho\|\sigma). \tag{5.23}$$

If the Hilbert space $\mathcal{H}$ represents a quantum system composed of subsystems A and B, and the quantum channel T corresponds to taking the partial trace over one of the subsystems, say B, the monotonicity of relative entropy becomes:

$$S(\rho_A \| \sigma_A) = S(\mathrm{tr}_B[\rho_{AB}] \| \mathrm{tr}_B[\sigma_{AB}]) \leq S(\rho_{AB} \| \sigma_{AB}), \tag{5.24}$$

where $\rho_{AB}, \sigma_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$. This inequality illustrates that ignoring parts of the system (in this case, subsystem B) reduces the ability to distinguish between the two quantum states. For a proof of DPI, see Theorem 11.9.2 in [48].

Additional properties of relative entropy include *additivity for tensor product states*:

$$S(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = S(\rho_1 \| \sigma_1) + S(\rho_2 \| \sigma_2), \tag{5.25}$$

where $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$ are density operators and $\sigma_1, \sigma_2 \in \mathcal{B}(\mathcal{H})$ are positive semi-definite operators. Relative entropy is also invariant under isometric transformations:

$$S(\rho \| \sigma) = S(U \rho U^* \| U \sigma U^*) \tag{5.26}$$

where $U : \mathcal{H} \to \mathcal{H}_1$ is an isometry.

We conclude this section with the following identity:

Lemma 5.2.1. *Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ be positive semi-definite operators, and let $a, b > 0$ be positive constants. Then,*

$$S(a\rho \| b\sigma) = a\, S(\rho \| \sigma) + a \log \frac{a}{b} \mathrm{tr}[\rho]. \tag{5.27}$$

These properties make quantum relative entropy a versatile and fundamental tool in quantum information theory.

5.3 Mutual Information

Another important information-theoretic measure that quantifies the amount of information shared between two parties or random variables is *mutual information*. When summing the information content of two random variables X and Y, represented as $H(X)$ and $H(Y)$, the common information is counted twice, while the unique information from each variable is counted once. To correct for this, mutual information subtracts the joint information, given by $H(X, Y)$:

$$I(X : Y) \coloneqq H(X) + H(Y) - H(X, Y). \tag{5.28}$$

In essence, mutual information measures the correlation between X and Y. It quantifies how much knowing one variable reduces the uncertainty about the other:

$$I(X : Y) = H(X) - H(X|Y). \tag{5.29}$$

Mutual information is symmetric with respect to its arguments, i.e., $I(X : Y) = I(Y : X)$. Furthermore, it is always non-negative and equals zero if and only if X and Y are independent.

Quantum mutual information. In the quantum setting, mutual information extends its classical counterpart to quantum states, capturing the total amount of correlations—both classical and quantum—between subsystems A and B. A higher value of quantum mutual information indicates stronger correlations between the subsystems.

Definition 5.3.1 (Quantum mutual information)**.** *The quantum mutual information for a bipartite state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as:*

$$I(A : B)_\rho = I(\rho_{AB}) \coloneqq S(\rho_{AB} \| \rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{5.30}$$

where $\rho_A = \mathrm{tr}_B[\rho_{AB}] \in \mathcal{S}(\mathcal{H}_A)$ and $\rho_B = \mathrm{tr}_A[\rho_{AB}] \in \mathcal{S}(\mathcal{H}_B)$ are the marginal states of ρ_{AB}.

As expected, the quantum mutual information for any bipartite state $\rho_{AB} \in \mathcal{S}(\mathcal{H})$ is always non-negative, and vanishes for $\rho_{AB} = \rho_A \otimes \rho_B$. Additionally, it is upper-bounded by:

$$I(\rho_{AB}) \leq 2 \log[\min\{\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\}]. \tag{5.31}$$

Quantum mutual information can also be expressed in terms of the conditional von Neumann entropy as follows:

$$I(A : B)_\rho = S(A)_\rho - S(A|B)_\rho \tag{5.32}$$

$$= S(B)_\rho - S(B|A)_\rho. \tag{5.33}$$

5.4 Rényi Entropy

In this section, we introduce *Rényi entropy*, a generalization of Shannon entropy that retains many of its key properties. In his 1961 work [58], Alfréd Rényi used an axiomatic approach to derive Shannon entropy, identifying five natural requirements for information measures: continuity, unitary invariance, normalization, additivity, and the arithmetic mean property. By replacing the arithmetic mean with a generalized mean, these axioms lead to what is now called Rényi entropy.

Definition 5.4.1 (Classical Rényi entropy)**.** *For a discrete probability distribution $p = \{p_i\}_{i=1}^n$ and $\alpha \in \mathbb{R}_{>0} \setminus \{1\}$, the Rényi entropy of order α is defined as*

$$H_\alpha(p) \coloneqq \frac{1}{1-\alpha} \log\Big(\sum_{i=1}^n p_i^\alpha\Big). \tag{5.34}$$

This measure is often referred to as the *entropy of order α*. In the special case where $\alpha \to 1$, Rényi entropy reduces to Shannon entropy:

$$\lim_{\alpha \to 1} H_\alpha(p) = H_1(p) \equiv H(p).$$

Quantum Rényi entropy Extending the classical definition, Rényi entropy is adapted to the quantum domain by replacing probability distributions with quantum states.

Definition 5.4.2 (Quantum Rényi entropy)**.** *For a density operator $\rho \in \mathcal{S}(\mathcal{H})$, the quantum Rényi entropy of order $\alpha \in (0, \infty) \setminus \{1\}$ is defined as:*

$$S_\alpha(\rho) \coloneqq \frac{1}{1-\alpha} \log \operatorname{tr}[\rho^\alpha]. \tag{5.35}$$

If $\{\lambda_k\}_k$ are the eigenvalues of ρ, then the quantum Rényi entropy aligns with its classical counterpart:

$$S_\alpha(\rho) = H_\alpha(\{\lambda_k\}_k).$$

Similar to the classical case, the von Neumann entropy is recovered as the limit of the quantum Rényi entropy when $\alpha \to 1$. Many interesting and useful properties of the α-Rényi entropies are known [58, 59]. Below, we summarize some of these properties.

For density operators $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{S}(\mathcal{H}')$, the following hold:

i) *Non-negativity*: $S_\alpha(\rho) \geq 0$, with equality if and only if the state ρ is pure.

ii) *Maximal value*: Let d be the dimension of the Hilbert space $\mathcal{H}$. Then,

$$S_\alpha(\rho) \leq \log d, \tag{5.36}$$

with equality if and only if ρ is the maximally mixed state.

iii) *Isometric invariance*: Let $U : \mathcal{H} \to \mathcal{H}'$ be an isometry. Then,

$$S_\alpha(\rho) = S_\alpha(U\rho U^*). \tag{5.37}$$

iv) *Additivity*: Rényi entropy is additive under tensor products:

$$S_\alpha(\rho \otimes \sigma) = S_\alpha(\rho) + S_\alpha(\sigma). \tag{5.38}$$

v) *Concavity*: Let $\{\rho_i\}_{i=1}^n \in \mathcal{S}(\mathcal{H})$ be density operators and $\{p_i\}_{i=1}^n$ a probability distribution. For $0 < \alpha \leq 1$, the Rényi entropy satisfies:

$$S_\alpha\Big(\sum_{i=1}^n p_i \rho_i\Big) \geq \sum_{i=1}^n p_i S_\alpha(\rho_i). \tag{5.39}$$

vi) *Monotonicity*: Rényi entropy decreases monotonically as α increases:

for $0 \leq \alpha_1 \leq \alpha_2$

$$S_{\alpha_2}(\rho) \leq S_{\alpha_1}(\rho). \tag{5.40}$$

Apart from the limiting case as $\alpha \to 1$, for $\alpha = 0$ and $\alpha = \infty$, the function in (5.35) is also defined by taking a limit (see the definitions in (5.41) and (5.43)). Rényi entropies find extensive applications in quantum information theory, including studies on strong converse problems for the classical capacity of quantum channels [60], entanglement-assisted LOCC conversion [61], as well as strong converse problems in quantum hypothesis testing [62].

Max-entropy (Hartley entropy). For a density operator $\rho \in \mathcal{S}(\mathcal{H})$, the max-entropy is defined as the limit of the α-Rényi entropy as $\alpha \to 0$:

$$S_0(\rho) \coloneqq \lim_{\alpha \to 0} S_\alpha(\rho) = \log \operatorname{rank}(\rho). \tag{5.41}$$

Max-entropy satisfies properties $(i) - (iv)$ listed above but does not exhibit concavity. However, it is Schur concave:

$$\sigma \prec \rho \Longrightarrow S_0(\rho) \leq S_0(\sigma). \tag{5.42}$$

For a bipartite state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, with marginals $\rho_A = \operatorname{tr}_B[\rho_{AB}] \in \mathcal{S}(\mathcal{H}_A)$ and $\rho_B = \operatorname{tr}_A[\rho_{AB}] \in \mathcal{S}(\mathcal{H}_B)$, max-entropy satisfies subadditivity:

$$S_0(\rho_{AB}) \leq S_0(\rho_A) + S_0(\rho_B).$$

Min-entropy. For a density operator $\rho \in \mathcal{S}(\mathcal{H})$, the min-entropy is defined as the limit of the $\alpha-$Rényi entropy as $\alpha \to \infty$:

$$S_\infty(\rho) \coloneqq \lim_{\alpha \to \infty} S_\alpha(\rho) = -\log \|\rho\|, \tag{5.43}$$

where $\|\rho\|$ is the largest eigenvalue of ρ. Min-entropy satisfies properties (i) through (iv) listed above but does not satisfy property (v), concavity. However, like max-entropy, min-entropy is also Schur concave:

$$\sigma \prec \rho \Longrightarrow S_\infty(\rho) \leq S_\infty(\sigma). \tag{5.44}$$

Unlike max-entropy, however, min-entropy is not subadditive.

5.5 Transcendental and Semialgebraic Perspectives: From Asymptotic to Single-Shot Settings

Key quantities that determine the fundamental limits of information processing tasks are often derived from entropic quantities, both in the asymptotic setting–where many independent and identically distributed quantum systems are considered–and in the single-shot setting, where single quantum systems are involved. Schumacher compression [51] is an example of a task that falls within the asymptotic setting. Characteristic quantities in this setting include von Neumann entropy and related quantities such as quantum relative entropy and mutual information. In contrast, examples of tasks studied in the single-shot setting range from decoupling of quantum systems and hypothesis testing to state transitions in quantum thermodynamics. Relevant quantities in this context include the Rényi divergence [63–65] and its smooth variants [66, 67].

Asymptotic results often lack practical significance because they rely on assumptions, such as infinite or unrestricted resources. Therefore, it is important to study quantum information processing tasks under finite resource constraints. The belief that the asymptotic and single-shot settings are characterized by different quantities has been challenged in the context of *quantum catalysis*. Quantum catalysis enables transformations that would otherwise be impossible by utilizing auxiliary systems without degrading them. In this context, physical transformations become feasible by leveraging a quantum degree of freedom that remains unchanged after the process.

In any physical setting, catalysis requires at least two systems: the system of interest S and an ancillary system C. Given S, we introduce a family of quantum channels $\mathcal{O}$ that describes all the physical operations allowed under the constraints of the given physical setting. These constraints could arise from conservation laws, resource limitations, or other physical restrictions.

Suppose the quantum system S is prepared in the state ρ, and the goal is to transform it into a target state σ, i.e.:

$$\rho \xrightarrow{\mathcal{O}} \sigma. \tag{5.45}$$

In many cases, no quantum channel $\tilde{\mathcal{T}} \in \mathcal{O}$ can achieve this transformation, i.e., $\nexists \tilde{\mathcal{T}} \in \mathcal{O}$ such that $\tilde{\mathcal{T}}(\rho) = \sigma$. However, the transformation in (5.45) may become possible by acting on additional degrees of freedom, represented by a quantum ancillary system C.

Definition 5.5.1 (Quantum catalysis)**.** *Given a set of operations $\mathcal{O}$, we say that ρ is* catalytically transformed *to σ if there exists a state ω_C on an ancillary system C, called the* catalyst*, and a quantum channel $\mathcal{T} \in \mathcal{O}$ such that $\mathcal{T}(\rho \otimes \omega_C) = \sigma \otimes \omega_C$.*

Since the catalyst is returned unchanged at the end of the transformation, it can be reused in subsequent operations. If correlations are established between the system and the catalyst during the process, the transformation is referred to as *correlated-catalytic*. More precisely, we say that there exists a correlated-catalytic transformation from ρ and σ if there exists a channel $\mathcal{T} \in \mathcal{O}$ and a catalyst state ω_C such that:

$$\mathcal{T}(\rho \otimes \omega_C) =: \mu_{SC}, \tag{5.46}$$

where μ_{SC} is a state on the composite system $S \otimes C$ satisfying:

$$\mathrm{tr}_S[\mu_{SC}] = \omega_C \text{ and } \mathrm{tr}_C[\mu_{SC}] = \sigma.$$

If the final state of the system is only approximately equal to σ, the process is referred to as *(correlated-)approximate catalysis*. Specifically, we say that such a transformation is possible if, for every $\varepsilon > 0$, there exists a catalyst state ω_C and a quantum channel $\mathcal{T} \in \mathcal{O}$ such that:

$$\mathcal{T}(\rho \otimes \omega_C) = \xi_{SC}, \ \mathrm{tr}_S[\xi_{SC}] = \omega_C, \tag{5.47}$$

and the output on the system is $\varepsilon-$close to σ in trace distance:

$$\frac{1}{2}\| \mathrm{tr}_C[\xi_{SC}] - \sigma\|_1 \leq \varepsilon. \tag{5.48}$$

Initial studies of quantum catalysis primarily focused on entanglement manipulation [20, 68–72]. Over time, the scope of quantum catalysis has expanded to other areas of quantum theory, including [73–76]. For a comprehensive overview of quantum catalysis and its applications across various physical contexts, we follow and refer readers to the review in [77].

For information-theoretic quantities that have an asymptotic operational characterization, a natural question to ask is whether they also admit a single-shot characterization, potentially involving an optimization over an ancillary system. In Core Articles [1, 2], we explore this question by analyzing the level sets of various entropic quantities to determine the (im)possibility of single-letter formulas. Certain catalytic state transformations have been formally characterized by the monotonic behavior of entropic quantities, such as the von Neumann entropy [73, 74, 76]. However, the dimension of the catalyst required for these transformations has remained largely unexplored. In Core Article [1], we show that the level sets of the von Neumann entropy are everywhere transcendental for dimensions $d \geq 3$. This result excludes the possibility of a single-shot semialgebraic characterization of von Neumann entropy using universally bounded ancillary systems, such as those involved in catalytic transformations [73, 74, 76]: For an initial quantum state $\sigma \in \mathcal{S}(\mathbb{C}^d)$, let $\mathcal{C}_n$ denote the set of all states in $\mathcal{S}(\mathbb{C}^d)$ reachable from σ through correlated catalytic transformations, with the help of an n-dimensional catalyst. Specifically:

$$\mathcal{C}_n \coloneqq \{\rho \in \mathcal{S}(\mathbb{C}^d) | \forall \varepsilon > 0, \exists \mu \in \mathcal{S}(\mathbb{C}^n), \exists U \in \mathbb{C}^d \otimes \mathbb{C}^n : \mathrm{tr}_1[\sigma_{12}] = \mu \wedge \| \mathrm{tr}_2[\sigma_{12}] - \sigma\|_1 \leq \varepsilon\}, \tag{5.49}$$

where $\sigma_{12} \coloneqq U(\sigma \otimes \mu)U^*$. When the dimension of the catalyst is unrestricted, the above catalytic transformations are possible if and only if $S(\sigma) \leq S(\rho)$ [74, 76]. Therefore, we define the set of all states satisfying this entropy condition as:

$$\mathcal{C} \coloneqq \{\rho \in \mathcal{S}(\mathbb{C}^d) \mid S(\sigma) \leq S(\rho)\}. \tag{5.50}$$

A natural question to ask is whether there exists a finite $n = n(d)$ such that $\mathcal{C}_n = \mathcal{C}$. In Core Article [1], we prove that the answer to this is negative for dimensions $d \geq 3$. A direct application of the Tarski-Seidenberg theorem demonstrates that the set $\mathcal{C}_n$ is semialgebraic. On the other

hand, by conducting a detailed analysis involving complex analysis and the characterization of algebraic and transcendental functions, we establish that the set $\mathcal{C}$ is everywhere transcendental in dimensions 3 or higher. This distinction highlights the fundamental difference between the two sets and implies that there is no universal bound on the catalyst dimension for such transformations. Moreover, this result shows that no semialgebraic single-shot characterization of von Neumann entropy exists, unless infinite dimensions for the catalyst are considered.

A similar analysis applies to catalytic LOCC transformations between pure bipartite entangled states, as studied in [21, 78]. Here too, we show that there is no universal bound on the dimension of the catalyst required for these transformations. Detailed proofs can be found in Core Article [2]. Furthermore, in Core Articles [1, 2] we explore the (non)semialgebraic nature of the level sets of relative entropy, α-Rényi entropies ($\alpha \in (0,1) \cup (1,\infty)$) and mutual information. This includes extremal and limiting cases, further emphasizing the distinction between the semialgebraic and transcendental regimes.

To conclude, the distinction between semialgebraic and transcendental frameworks sheds light on the separation between single-shot and asymptotic settings in quantum information theory. Finite-resource scenarios are often described by polynomial equalities and inequalities, as they involve mathematically tractable structures within finite-dimensional vector spaces. By contrast, asymptotic or infinite-resource regimes exceed the scope of the semialgebraic framework and instead (often) fall within the transcendental domain. The complexity of these regimes arises from the need to quantify over infinite-dimensional spaces, making the analysis substantially more challenging.

Chapter 6

Quantum Many-Body Systems

This chapter explores the theoretical framework underpinning our final application of the transcendental tools introduced in Chapters 3 and 4, focusing on their use in quantum many-body systems. The problem we address here falls under the domain of the quantum marginal problems [79, 80]. Our primary interest is in the geometric properties of the set of reduced-density matrices for translation-invariant, infinite quantum systems. Specifically, we investigate the possibility of a *simple*, i.e., *semialgebraic*, characterization of the set of nearest-neighbor density operators in translation-invariant spin chains in the thermodynamic limit.

We organize this chapter as follows: Section 6.1 introduces the fundamental definitions and concepts underlying quantum spin systems, establishing the basis for later discussions. Once these key ideas are presented, we explore a specific example—the XY model—in Section 6.2 pursuing [81]. Finally, in Section 6.3, we provide an introduction to matrix product states, as discussed in [82], and (C^*-)finitely correlated states, building on insights from [4, 83]. We follow, and recommend to readers seeking a deeper understanding of quantum spin systems, the textbooks [84–86] as well as the lecture notes [87]. For a concise overview, see [88, 89].

6.1 Mathematical Framework of Quantum Spin Systems

Quantum spin systems are mathematical models commonly used to describe physical systems in quantum information theory and other fields. These models represent systems of particles arranged on a lattice, interacting according to the principles of quantum mechanics [84]. The primary objective of studying these systems is to achieve a balance between mathematical rigor and the adaptability required to represent a broad spectrum of physically relevant systems and phenomena.

In the physical interpretation relevant to this work, each lattice site is occupied by a particle with internal degrees of freedom, such as intrinsic spin, with various possible orientations. These particles interact through couplings between their internal degrees of freedom, leading to dynamic changes in the system, including the continuous evolution of spin orientations over time. Conceptually, spins can be thought of as atomic-sized magnets [85]. Mathematically, in this thesis, the term "spin" refers to any quantum system with a finite-dimensional complex Hilbert space of states. This space may represent the spin states of an atom, molecule, or particle [87].

Additionally, "spin" can refer to spin angular momentum [86], a quantum mechanical property described by the *spin operator* $\mathbf{S} = (S_x, S_y, S_z)$, which will be introduced in detail later in this section (see Equation (6.2)).

To establish the mathematical foundations of quantum spin systems, we consider a lattice denoted by $\Gamma = \mathbb{Z}^\nu$, where $\nu \geq 1$ is an integer, and the lattice is equipped with a metric. Let $\mathcal{F}$ represent the collection of all finite subsets of Γ. To each point $j \in \Gamma$, referred to as a *site*, we associate a finite-dimensional Hilbert space $\mathcal{H}_{\{j\}} \cong \mathbb{C}^{d_j}$ of dimension $d_j \geq 2$. These individual Hilbert spaces correspond to the spin at each site $j \in \Gamma$, and together they form the spin system. For any finite subset $\Lambda \in \mathcal{F}$, called a finite *volume*, the combined Hilbert space is given by the tensor product of the Hilbert spaces associated with each site in Λ:

$$\mathcal{H}_\Lambda = \bigotimes_{j \in \Lambda} \mathbb{C}^{d_j}.$$

The algebra of observables for the spin at site $j \in \Gamma$ consists of $d_j \times d_j$ matrices. Consequently, the algebra of observables for a finite subset $\Lambda \in \mathcal{F}$ is defined as:

$$\mathcal{A}_\Lambda = \bigotimes_{j \in \Lambda} \mathcal{A}_{d_j}, \text{ where } \mathcal{A}_{d_j} = \mathcal{B}(\mathbb{C}^{d_j}) = \mathcal{M}_{d_j \times d_j}(\mathbb{C}). \tag{6.1}$$

With respect to the standard matrix operations-addition, multiplication, Hermitian conjugation, and the operator norm-the algebra $\mathcal{A}_\Lambda$ forms a finite-dimensional C^*-algebra, with $\mathbb{1}_\Lambda$ as its identity element. Due to the tensor product structure, for any subset $\Lambda_1 \subset \Lambda$, we can embed $\mathcal{A}_{\Lambda_1}$ into $\mathcal{A}_\Lambda$ by identifying each element $X \in \mathcal{A}_{\Lambda_1}$ with $X \otimes \mathbb{1}_{\Lambda \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_1} \otimes \mathcal{A}_{\Lambda \setminus \Lambda_1} = \mathcal{A}_\Lambda$ [84, 87, 89].

By arranging the finite sets in order, the union $\bigcup_{\Lambda \in \mathcal{F}} \mathcal{A}_\Lambda$ is well-defined, and it is called the algebra of *local observables*, denoted by $\mathcal{A}_\Lambda^{loc}$. The closure of this union with respect to the operator norm forms the C^*-algebra of observables for the entire (infinite) spin system on Γ, known as the algebra of *quasi-local observables*, written as $\mathcal{A}$. While we do not delve into the detailed structure of quasi-local algebras in this thesis, readers interested in a deeper understanding are encouraged to consult [7, 84].

In many cases, each site $j \in \Gamma$ is associated with a Hilbert space of the same dimension $d_j \geq 2$, i.e., $\mathcal{H}_{\{j\}} \cong \mathbb{C}^n$, which is the setting we will adopt in this thesis [3]. Additionally, we will focus on one spatial dimensional quantum spin systems, commonly called *quantum spin chains*. A technique to simulate the properties of an infinite system within a finite spin chain is to employ *periodic boundary conditions*. In a chain with N sites, interactions typically occur between neighboring spins. Periodic boundary conditions wrap the chain around such that the last site interacts with the first, forming a ring-like structure. This configuration treats the first and last spins as nearest neighbors, ensuring the system does not have open ends. To connect this setup to physical spin, the dimension n of $\mathbb{C}^n$ is often expressed as $n = 2S + 1$, where S represents a half-integer value. The simplest non-trivial case occurs when $S = 1/2$, giving $n = 2$, which corresponds to a spin-1/2 particle.

A central set of observables in a quantum spin system are the *spin−S matrices* S_x, S_y, and S_z, which are Hermitian matrices representing spin components along three mutually orthogonal directions. These matrices satisfy the canonical commutation relations:

$$[S_u, S_v] = i\varepsilon^{uvk} S_k, \tag{6.2}$$

for $u, v \in \{x, y, z\}$. Here, ε^{ijk} is the Levi-Civita symbol, which equals 1 for cyclic permutations, -1 for anti-cyclic permutations, and 0 otherwise. For example, in qubit systems, which correspond to spin-$\frac{1}{2}$ particles, the observables for the three spin directions are represented by the three Pauli matrices σ_x, σ_y and σ_z, each scaled by a factor of $1/2$.

To describe the dynamics of a quantum spin system, we introduce *interactions*. An interaction is defined as a mapping Φ from finite subsets $Z \in \mathcal{F}$ to the Hermitian elements of $\mathcal{A}$, where each $\Phi(Z) \in \mathcal{A}_Z$. Formally, Φ is a function $\Phi : \mathcal{F} \to \mathcal{A}$ such that for each $Z \in \mathcal{F}$, it holds that $\Phi(Z) \in \mathcal{A}_Z$ and $\Phi(Z)^* = \Phi(Z)$. Each $\Phi(Z)$ represents the interaction energy of the particles within the subset Z [84]. In spin systems, the particles are fixed at the lattice sites, and the total interaction energy within a subset Λ is given by the sum of interaction energies across all subsystems in Λ. This total energy is known as the Hamiltonian $H_\Phi(\Lambda)$, associated with the finite set Λ:

$$H_\Phi(\Lambda) = \sum_{Z \subseteq \Lambda} \Phi(Z), \tag{6.3}$$

where $H_\Phi(\Lambda)$ is Hermitian and belongs to $\mathcal{A}_\Lambda$. Specifying the Hamiltonian, or a family of Hamiltonians, is crucial for defining a quantum spin model. A particularly important class of Hamiltonians is the family of k-local Hamiltonians, which are expressed as sums of terms that act non-trivially on at most k particles while acting as the identity on all other particles. Here, k is a fixed constant.

A specific type of interactions that occur in ν-dimensional lattices $\mathbb{Z}^\nu$ are the *translation-invariant interactions*. These interactions, Φ, satisfy in addition the condition:

$$\Phi(\tau_j(Z)) = \tau_j(\Phi(Z)) \tag{6.4}$$

for all $Z \in \mathcal{F}$ and $j \in \mathbb{Z}^\nu$, where $\tau_j(Z) \coloneqq Z + j$. A quantum spin model is called *nearest neighbor* if $\Phi(Z) \neq 0$ implies that $Z = \{j, j \pm 1\}$, and *finite range* if there exists a positive constant R such that $\Phi(Z) = 0$ whenever $\mathrm{diam}(Z) > R$. The set of interactions in a quantum spin model forms a real vector space when endowed with the standard linear structure. Specifically, for any two interactions Φ_1 and Φ_2, addition is defined as $(\Phi_1 + \Phi_2)(Z) = \Phi_1(Z) + \Phi_2(Z)$, and scalar multiplication is defined as $\alpha\Phi_1(Z) = (\alpha\Phi_1)(Z)$, where α is a real scalar.

The dynamics of any quantum spin system can be formulated in the Heisenberg picture by evolving the associated observables. For a quantum spin system confined to a finite volume $\Lambda \subset \Gamma$, with interaction Φ and Hamiltonian $H_\Phi(\Lambda)$, the time evolution is given by the relation:

$$X \in \mathcal{A}_\Lambda \mapsto \tau_t^\Lambda(X) \in \mathcal{A}_\Lambda, \text{ where } \tau_t^\Lambda(X) = e^{itH_\Phi(\Lambda)} X e^{-itH_\Phi(\Lambda)}. \tag{6.5}$$

For each $t \in \mathbb{R}$, the mapping $t \in \mathbb{R} \mapsto \tau_t^\Lambda$ forms a one-parameter group of (*-)automorphism of the matrix algebra $\mathcal{A}_\Lambda$:

$$\tau_t^\Lambda \circ \tau_s^\Lambda = \tau_{t+s}^\Lambda, \quad \tau_0^\Lambda = \mathbb{1}, \quad \forall t, s \in \mathbb{R}.$$

Under certain conditions, where the interaction Φ decays sufficiently rapidly, the dynamics τ_t^Λ converges strongly as the volume increases $\Lambda \to \mathbb{Z}^\nu$ in a suitable sense. Specifically, for fixed $t \in \mathbb{R}$ and $X \in \mathcal{A}$ it holds that

$$\|\tau_t^{\Lambda_n}(X) - \tau_t(X)\| \to 0 \tag{6.6}$$

as $\Lambda_n \uparrow \mathbb{Z}^\nu$. Here, we use the notion $\Lambda_n \uparrow \mathbb{Z}^\nu$ to describe an increasing and absorbing sequence, meaning that for all n, $\Lambda_n \subseteq \Lambda_{n+1}$, and $\bigcup_{n\geq 1} \Lambda_n = \mathbb{Z}^\nu$. The limiting dynamics $\tau_t(X)$ can be shown to be independent of the choice of the increasing and absorbing sequence. The infinite-volume dynamics $\{\tau_t : t \in \mathbb{R}\}$, similar to the finite-volume scenario, form a strongly-continuous, one-parameter group of automorphisms in $\mathcal{A}$. Additionally, under specific conditions on the decay of the interaction, it can be shown that $\delta(X) = \lim_{\Lambda_n \uparrow \mathbb{Z}^\nu} [H_\Phi(\Lambda_n), X] = \sum_{A \in \mathcal{F}} [\Phi(A), X]$, for $X \in \bigcup_{\Lambda \in \mathcal{F}} \mathcal{A}_\Lambda$, and one writes $\tau_t = e^{it\delta}$. For further details on the sufficient conditions needed to establish the limit in (6.6), depending on the interactions, we refer the reader to [84, 89]. In this thesis, we focus specifically on translation-invariant and two-body interaction spin chains. In this case, the limit in (6.6) is established under the following condition:

$$\sum_{j \in \mathbb{Z}^\nu} \|\Phi(\{0, j\})\| e^{\lambda |j|} \leq \infty, \tag{6.7}$$

for some $\lambda > 0$ (see Chapter 6 of [84]). This condition indicates that the interaction must decrease exponentially with distance.

An example of a translation-invariant, nearest-neighbor model is provided below.

Example 6.1.1 (Quantum Heisenberg model [90])**.** *The quantum Heisenberg model is constructed as follows: For a finite interval $\Lambda = [a, b] \subset \mathbb{Z}$, each site $j \in \Lambda$ is associated with a Hilbert space $\mathcal{H}_{\{j\}} \cong \mathbb{C}^2$. The overall Hilbert space corresponding to the interval Λ is given by*

$$\mathcal{H}_\Lambda = \bigotimes_{j=a}^{b} \mathbb{C}^2 = \mathbb{C}^{2^{b-a+1}},$$

with the associated observable algebra defined as

$$\mathcal{A}_\Lambda = \bigotimes_{j=a}^{b} \mathcal{M}_{2\times 2}(\mathbb{C}) = \mathcal{M}_{2^{b-a+1} \times 2^{b-a+1}}(\mathbb{C}).$$

For each site $j \in \Lambda$ and for any $\gamma = x, y, z$, we associate a self-adjoint observable $\sigma_\gamma^j \in \mathcal{A}_\Lambda$ by defining

$$\sigma_\gamma^j = \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \sigma_\gamma \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}, \tag{6.8}$$

where the identity operator acts on all sites except at the j-th site, where the Pauli matrix σ_γ acts non-trivially. An interaction Φ_H for our model defined on the interval $[a,b]$ is given by

$$\Phi_H(Z) = \begin{cases} -J\vec{\sigma}^j\vec{\sigma}^{j+1} & \text{if } Z = \{j, j+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\vec{\sigma}^j \coloneqq \{\sigma_x^j, \sigma_y^j, \sigma_z^j\}$. This way, the quantum Heisenberg Hamiltonian on Λ is expressed as

$$H_\Phi(\Lambda) = \sum_{Z\subseteq\Lambda} \Phi_H(Z) = -J\sum_{j=a}^{b-1} \vec{\sigma}^j \cdot \vec{\sigma}^{j+1} = -J\sum_{j=a}^{b-1} (\sigma_x^j \otimes \sigma_x^{j+1} + \sigma_y^j \otimes \sigma_y^{j+1} + \sigma_z^j \otimes \sigma_z^{j+1}),$$

where $J \in \mathbb{R}$ is the coupling constant. Depending on the sign of J, the model is referred to as the quantum ferromagnetic *Heisenberg chain for $J > 0$ and the quantum* anti-ferromagnetic *Heisenberg chain for $J < 0$.*

Ground States. An important aspect of studying quantum spin systems is analyzing the spectrum of the Hamiltonian and its eigenstates as the system size increases. The *thermodynamic limit*, where the number of sites approaches infinity, helps to understand the system's behavior at large scales. A common challenge in this area is finding the ground state of a Hamiltonian and determining its corresponding ground-state energy.

Let $\Lambda \subset \Gamma$ be a finite set. Upon fixing an interaction Φ, the Hamiltonian $H_\Phi(\Lambda)$ acts on a finite-dimensional complex Hilbert space $\mathcal{H}_\Lambda$. Since the dimension of $\mathcal{H}_\Lambda$ is finite, the spectrum of the Hamiltonian, denoted as $\text{spec}(H) = \{E_0^\Lambda, E_1^\Lambda, ...\}$, consists of a finite set of real eigenvalues ordered in non-decreasing order. These eigenvalues are referred to as *energy levels*. The smallest eigenvalue E_0^Λ is called the *ground-state energy* of the system, while the *ground state* refers to any state that minimizes the energy, that is

$$\rho_0(H_\Phi(\Lambda)) = \min\{\, \text{tr}[\rho H_\Phi(\Lambda)] \mid \rho \in \mathcal{S}(\mathcal{H}_\Lambda)\,\}. \tag{6.9}$$

The eigenspace $\ker(H_\Phi(\Lambda) - E_0^\Lambda \mathbb{1})$ corresponding to the ground state energy E_0^Λ is known as the *ground state space*. The ground state is unique if, and only if, the smallest eigenvalue is non-degenerate. In the algebraic terms, a state $\omega_\Lambda(X) = \text{tr}[\rho_\Lambda X]$ for all $X \in \mathcal{A}_\Lambda$, is considered a ground state if $\text{ran}(\rho_\Lambda) \subseteq \ker(\mathrm{H}_\Phi(\Lambda) - \mathrm{E}_0^\Lambda \mathbb{1})$. Here, $\rho_\Lambda \in \mathcal{A}_\Lambda$ is a non-negative matrix with trace one. It can be shown that $\omega_\Lambda : \mathcal{A}_\Lambda \to \mathbb{C}$ is a ground state if and only if $\omega_\Lambda(X^*[H_\Phi(\Lambda), X]) \geq 0$, for all $X \in \mathcal{A}_\Lambda$ (for more details see [84]). A similar criterion is used to define the ground state of the infinite volume system: a state $\omega : \mathcal{A} \to \mathbb{C}$ is a ground state of the infinite system dynamics $\tau_t = e^{it\delta}$ if

$$\omega(X^*\delta(X)) \geq 0, \tag{6.10}$$

for all $X \in \bigcup_{\Lambda\in\mathcal{F}} \mathcal{A}_\Lambda$. Finally, the *ground state energy density* is defined by

$$\varepsilon = \lim_{|\Lambda|\to\infty} \frac{E_0^\Lambda}{|\Lambda|}, \tag{6.11}$$

which captures the energy contribution per site in the ground state for large systems.

In this introduction, we have outlined general definitions related to the dynamics of systems in infinite volumes and their corresponding ground states. Our primary focus, however, is on the specific model of translation-invariant infinite quantum spin chains.

6.2 XY model

Having established the mathematical framework for quantum spin systems, we now turn to a concrete example: the XY model. This model is used to explore whether the set of bipartite reduced states of a translation-invariant infinite quantum spin system is semialgebraic, as discussed in Core Article [3].

The quantum XY model is an anisotropic linear spin chain characterized by nearest-neighbor interactions. It is commonly employed in condensed matter physics to study quantum phase transitions. Anisotropy refers to the variation in physical properties depending on the direction of measurement, meaning that anisotropic systems exhibit different behaviors along different axes.

The XY model can be viewed as a special case of the more general Heisenberg model, focusing solely on interactions in the x- and y-directions of the spin space. Unlike the Heisenberg model, which includes interactions in all three spatial directions, the XY model does not have interactions in the z-direction.

In this model, N spin-$\frac{1}{2}$ particles are arranged in a chain, and the Hamiltonian is defined as:

$$H_\gamma = \sum_j [(1+\gamma) S_x^j \otimes S_x^{j+1} + (1-\gamma) S_y^j \otimes S_y^{j+1}], \tag{6.12}$$

with the parameter $\gamma \in (-1, 1)$ characterizing the degree of anisotropy in the xy-plane. The spin operators S_x^j, S_y^j and S_z^j correspond to the Pauli matrices σ_x, σ_y and σ_z each divided by 2. The superscript j denotes the position of the spin in the spin system. As the Hamiltonian only involves the $x-$ and $y-$ components of the spin operators, and because the interaction strengths in these directions differ, this model is referred to as the XY anisotropic model.

This quantum mechanical model is exactly solvable, meaning that its ground state and other properties, such as excitations and free energy, can be fully determined. These derivations are beyond the scope of this thesis, and we refer the interested readers to [81]. Our focus here is on the ground state energy density in the thermodynamic limit, which has also been derived in [81] and shown to be

$$\varepsilon(\gamma) = -\frac{1}{4\pi} \int_{-\pi}^{\pi} [1 - (1-\gamma^2) sin^2 k]^{1/2} dk. \tag{6.13}$$

By setting $1 - \gamma^2 =: z^2$, the above expression simplifies to $\varepsilon(\gamma) = -\frac{1}{\pi} E(z)$, where $E(z)$ is the complete elliptic integral of the second kind.

6.3 Matrix Product States and (C^*-)Finitely Correlated States

Matrix Product States (MPS) provide a compact and powerful representation of quantum states, particularly effective in describing quantum many-body systems, especially quantum spin chains. In a quantum system of K sites arranged on a chain, a general quantum state $|\psi\rangle$ in a computational basis $|i\rangle$ can be written as:

$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_K} c_{i_1,i_2,\ldots,i_K} |i_1 i_2 \ldots i_K\rangle. \tag{6.14}$$

Given that the number of states in the Hilbert space of a $K-$particle system scales exponentially, the number of parameters in the coefficient $c_{i_1,i_2,\ldots,i_K}$ also scales exponentially. MPS addresses this challenge by approximating the multiple-dimensional array $c_{i_1,i_2,\ldots,i_K}$ with a product of matrices, reducing the parameter count and simplifying computation. Thus, the expression in (6.14) can be rewritten as:

$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_K} \underbrace{\sum_{a_1,a_2,\ldots,a_K} A_{i_1}^{a_K,a_1} A_{i_2}^{a_1,a_2} \ldots A_{i_K}^{a_{K-1},a_K}}_{c_{i_1,i_2,\ldots,i_K}} |i_1 i_2 \ldots i_K\rangle, \tag{6.15}$$

where $A_i^{a_{l-1},a_l} \equiv A_i^{[l]}$ represents the $D_{l-1} \times D_l$ matrix associated with site $l \in \{1,\ldots,K\}$. Here, $i-$indices are the *physical indices*, while a_l-indices are *virtual indices* of dimension D. Physical indices relate to the physical system itself, while virtual indices are introduced in the MPS construction and are traced out at the end. The boundary condition affects the indexing: with periodic boundaries, a_K appears as both the first and last index in (6.15), since the first and last indices are contracted.

To obtain the MPS form of $c_{i_1,i_2,\ldots,i_K}$, we can proceed as follows. Each of the K spins arranged on a chain is assigned two *virtual* spins of dimension D. Initially, each neighboring pair of virtual spins from adjacent sites is in an un-normalized maximally entangled state $|E\rangle = \sum_{\gamma=1}^{D} |\gamma\,\gamma\rangle$, known as the *entangled bond*. So, the maximally entangled state occupies a Hilbert space of dimension D, corresponding to the virtual bond dimension. Next, apply the map

$$\mathcal{F} = \sum_{i=1}^{n} \sum_{\gamma,\nu=1}^{D} A_{i,\gamma,\nu} |i\rangle\langle\gamma,\nu|$$

to each site of the system, projecting neighboring unentangled sites together. Here, the γ and ν indices correspond to the virtual systems. Let A_i represent the $D\times D$ matrix with elements $A_{i,\gamma,\nu}$: the coefficients of the final state in a product basis are given by $\mathrm{tr}[A_{i_1} A_{i_2} \ldots A_{i_K}]$. Generally, the entangled state dimension and the map $\mathcal{F}$ may vary across sites, so we denote the matrix for site $l \in \{1,\ldots,K\}$ as $A_i^{[l]}$ with dimension $D_l \times D_{l+1}$ [82].

Definition 6.3.1 (Matrix Product States (MPS))**.** *A pure quantum state $|\psi\rangle \in \mathbb{C}^{\otimes n^K}$ modeling a system of K sites, each with an n-dimensional Hilbert space, is called a* Matrix Product State (MPS) *if it can be expressed as*

$$|\psi\rangle = \sum_{i_1,i_2,\ldots,i_K} \mathrm{tr}[A^{[1]}_{i_1} A^{[2]}_{i_2} \ldots A^{[K]}_{i_K}] |i_1 i_2 \ldots i_K\rangle, \tag{6.16}$$

The state in (6.16) is generally not normalized, and its MPS representation is not unique. Every translation-invariant (TI) state can be represented by a translation-invariant MPS [82]. It was shown in [91] that every state can be represented as an MPS if the bond dimensions D_l are sufficiently large. Typically, however, a state is referred to as an MPS if it has a compact representation with a small bond dimension $D = \max_l D_l$, that does not grow with K, commonly known as a $D-$dimensional MPS. In [92] it is studied how MPS can approximate the local properties of translation-invariant states in systems as $K \to \infty$. Given a translation-invariant state ρ with $K = \infty$, let $\tau := \mathrm{tr}_{\neg(1,2)}[\rho]$ be the bipartite reduced density operator of ρ, obtained by tracing out all particles except two neighboring ones, say 1 and 2. The goal is to show the existence of a $D-$dimensional MPS with a corresponding reduced density operator τ_D that effectively approximates τ, providing a meaningful bound from the above on the difference $\|\tau - \tau_D\|$. This bound is shown to decrease as the bond dimension D increases, indicating an improved approximation with larger D (see Part B, Lemma 3 in [92]).

To introduce *finitely correlated states*, we start by letting $\mathcal{M}$ be a finite-dimensional C^*-algebra of observables associated with each site i in a translation-invariant, infinite quantum spin chain. We may think of $\mathcal{M}$ as the algebra of $d_{\mathcal{M}} \times d_{\mathcal{M}}$ matrices.

Definition 6.3.2 (Finitely Correlated States (FCS) [4,83])**.** *A linear realization of a translation invariant state ω is a quadtruple $(W, e, \mathbb{E}, \rho)$, where W denotes a finite dimensional vector space, $\mathbb{E}$ a linear map $\mathbb{E} : \mathcal{M} \ni M \to \mathbb{E}_M \in Hom(W)$, e is an element of W, and ρ is a linear functional in W^*, satisfying:*

$$\omega(M_1 \otimes M_2 \ldots \otimes M_n) = \rho \circ \mathbb{E}_{M_1} \circ \ldots \circ \mathbb{E}_{M_n}(e), \tag{6.17}$$

$$\rho \circ \mathbb{E}_{\mathbb{1}} = \rho, \quad \mathbb{E}_{\mathbb{1}}(e) = e \tag{6.18}$$

A translation-invariant state ω that admits such a realization is called a finitely correlated state.

The space W is often referred to as a *memory* space. For details on the existence of realizations $(W, e, \mathbb{E}, \rho)$ of ω see [4,83].

A particular class of finitely correlated states is formed by those generated through iterative applications of a quantum operation on a quantum memory system. These states admit a so-called *C^*-realization* and are known as *C^*-finitely correlated states.*

Definition 6.3.3 (C^*-Finitely Correlated States (C^*-FCS) [4,83])**.** *A C^*-realization of a translation invariant state ω is a realization $(\mathcal{A}, \mathbb{1}_{\mathcal{A}}, \mathcal{E}, \rho_0)$ of ω, where $\mathcal{A}$ is a C^*-algebra, $\mathbb{1}_{\mathcal{A}}$ denotes the identity in $\mathcal{A}$, $\mathcal{E} : \mathcal{M} \otimes \mathcal{A} \to \mathcal{A}$ is a completely positive and unital map, and ρ_0 is a positive functional on $\mathcal{A}$, with $\rho_0(\mathbb{1}) = 1$. A state ω is referred to as a C^*-finitely correlated state if it admits a C^*-realization.*

The existence of finitely correlated states that are not C^*-finitely correlated was demonstrated by Fanizza, Lumbreras, and Winter in [45] using transcendental techniques. We provide more details on their proof technique in Section 4.5.

In [4] Fannes, Nachtaergaele, and Werner use C^*-finitely correlated states as trial states for the ground state variational problem of a given interaction. By employing a parameter-free algebraic ansatz, they prove that the energy of a C^*-finitely correlated ground state is necessarily an algebraic number. They apply this result to the antiferromagnetic spin-1/2 Heisenberg chain, for whom by that time, the exact ground state energy density had been computed and shown to be a transcendental number.
For any two-body Hamiltonian, the corresponding ground-state energy density in the thermodynamic limit is obtained by minimizing over the set of all reduced bipartite density operators. When applied to the antiferromagnetic spin-1/2 Heisenberg chain, their approach shows that, while the matrix elements of the interaction are algebraic, the ground-state energy is transcendental. This implies that the ground state of the antiferromagnetic spin-1/2 Heisenberg chain cannot be a C^*-finitely correlated state. This argument applies to any parameter-free algebraic ansatz but breaks down when transcendental parameters are introduced. Building on this idea, we study the geometry of the set of reduced bipartite density operators arising from a translation-invariant infinite quantum spin chain in Core Article [3], incorporating transcendental functions and sets into the analysis.

Chapter 7

Summary of Core Publications

7.1 Transcendental properties of entropy-constrained sets

Vjosa Blakaj and Michael M. Wolf

In the study of information-theoretic quantities with asymptotic operational characterizations, a natural question arises: can these quantities also be described through a single-shot characterization? Such a characterization might involve optimization over an ancillary system. When the underlying expressions are algebraic and the ancillary system is finite, the resulting sets are semialgebraic. This work provides a criterion to distinguish semialgebraic sets from non-semialgebraic ones. The criterion builds on two key insights: (i) analytic continuations of algebraic functions have a finite number of branches, and (ii) for sets defined by level sets of differentiable functions, the normal space can be derived from the gradient of the defining function. These ideas underpin Theorem 1. This criterion is applied to von Neumann entropy, $S(\rho) \coloneqq -\operatorname{tr}[\rho \log \rho]$, and its level sets in Section 5. For dimensions $d \geq 3$, we demonstrate that these level sets are transcendental everywhere. This result precludes any semialgebraic single-shot characterization of von Neumann entropy, such as those involving catalytic state transformations with finite ancillary systems. Section 6 extends this analysis by considering an additional constraint, like the distance of the state to a specific target, its energy, or the expected value of any observable. Theorem 4 proves that such constrained sets remain transcendental in dimensions $d \geq 3$. We apply this result to the level set of relative entropy, $S(\rho\|\sigma) \coloneqq \operatorname{tr}[\rho \ln \rho] - \operatorname{tr}[\rho \ln \sigma]$, with fixed (but arbitrary) σ, and show that it is transcendental in these dimensions. Section 4 provides an interlude, where we show that slightly weaker results can also be achieved using transcendental number theory. Here, we establish a dichotomy in entropy values using Baker's result and show that they tend to be either rational or transcendental, thus avoiding the intermediate range of irrational algebraic values.

This work originated from discussions between my supervisor, Michael M. Wolf, and me. I contributed significantly to developing the ideas and conducting the scientific research, with a particular focus on Sections 3, 4, and 5. Additionally, I was responsible for writing most of the manuscript.

7.2 Transcendental properties of entropy-constrained sets II

Vjosa Blakaj and Chokri Manai

This second Core Article extends the investigation into the (im)possibility of single-letter formulas by focusing on the transcendental nature of the level sets of relative entropy, mutual information, and α-Rényi entropies. We complete the analysis of relative entropy ($S(\rho\|\sigma) = \mathrm{tr}[\rho(\log\rho - \log\sigma)]$) level sets, showing that they are transcendental in all dimensions $d \geq 3$, regardless of whether σ is fixed, ρ is fixed, or both states are allowed to vary. In Section 4, using the transcendental nature of von Neumann entropy-constrained sets we prove the transcendence of the mutual information, $I(\rho_{AB}) \coloneqq S(\rho_{AB}\|\rho_A \otimes \rho_B)$, level sets. These sets are transcendental whenever the smallest dimension among the subsystems' Hilbert spaces is at least 3, except at the extremal values of mutual information, where they remain semialgebraic.

Section 6 focuses on quantum α-Rényi entropies, $S_\alpha(\rho) \coloneqq \frac{1}{1-\alpha}\log\mathrm{tr}[\rho^\alpha]$, revealing a dependence on the parameter $\alpha \in (0,1)\cup(1,\infty)$. If $\alpha \in \mathbb{Q}$, the level sets are semialgebraic, but for $\alpha \notin \mathbb{Q}$, they are transcendental everywhere for $d \geq 3$.

In order to classify entropy-constrained sets as semialgebraic or transcendental, our study employs tools from algebraic geometry and complex analysis, including the Tarski-Seidenberg theorem and properties of algebraic functions. The proofs rely on the detection of irrational poles and the non-algebraic behavior of certain functions. These findings highlight the mathematical limits of algebraic frameworks in bounded resource theories and reinforce the distinction between single-shot and asymptotic settings in quantum information theory.

We apply these results to the study of catalytic transformations within resource theories. Specifically, to catalytic transformations involving diagonalized quantum states (characterized by relative entropy) and catalytic LOCC transformations (described through entanglement entropy). The transcendental nature of these level sets shows the absence of universal bounds on the dimensions of catalysts required for such transformations. Moreover, it restricts the feasibility of semialgebraic single-shot characterizations in settings with bounded ancillary systems.

The idea for this work evolved from my first article with my supervisor, Michael M. Wolf, where we sought to extend the results established in Core Article [1] to cover additional entropic quantities. I am the main author of this article and contributed significantly to proving the results. My focus included the motivating example in the Introduction, Theorem 1 (parts 1 and 3 and partially part 2) and its application, the analysis of the extremal cases of mutual information, parts of Theorem 2, and Section 5, except for part 3 of Theorem 3. I was also responsible for writing the article.

7.3 On the set of reduced states of translation invariant, infinite quantum systems

Vjosa Blakaj and Michael M. Wolf

A further application of our transcendental toolbox is the study of the mathematical structure of the set of reduced two-body density matrices in translation-invariant, infinite quantum spin chains. Determining the ground state energy per site for such a chain with nearest-neighbor interactions is a fundamental problem in quantum physics. This task essentially involves minimizing the expectation value of the interaction over the set of two-spin reduced density matrices associated with a translation-invariant state of the chain. However, even for two-dimensional local Hilbert spaces, this set cannot be precisely characterized.

In the main result of this paper, Theorem 2, we show that this set is not semialgebraic. We prove this by contradiction and checking its implications. If the set were semialgebraic and the Hamiltonians were parameterized by an algebraic family, then the mapping from the parameter interval to the ground state energy density would also be semialgebraic. This conclusion follows from the Tarski-Seidenberg theorem, which states that semialgebraic properties are preserved under quantifier elimination. Consequently, the problem of determining the ground state energy could theoretically be reduced to solving a system of polynomial equations and inequalities. However, we provide a counterexample: the anisotropic XY model. In this case, the ground state energy density in the thermodynamic limit is linked to the complete elliptic integral of the second kind, which is shown to be a transcendental function over $\mathbb{R}$.

The non-algebraic nature of this set implies that any algebraic approximation would require an infinite sequence of steps, as no finite-level approximation can capture the set exactly. This finding highlights the inherent limitations of algebraic techniques for exploring ground state properties in many-body quantum systems. Furthermore, we examine whether incorporating elementary transcendental functions could yield a finite and precise characterization of the set. We provide evidence against this through connections to undecidability results and Schanuel's conjecture in transcendental number theory.

The idea for this work originated while my supervisor and I were working on Core Article [1] of this thesis. We drew inspiration from the work of Fannes, Nachtergaele, and Werner [4], who showed that the ground state of the antiferromagnetic spin-1/2 Heisenberg model cannot be a C^*-finitely correlated state: its energy is transcendental, whereas an algebraic ansatz can yield only algebraic energies. This argument breaks down when transcendental parameters are introduced. Here, we extend this approach by considering transcendental sets and functions, rather than merely numbers, and investigate the set of reduced two-body states in translation-invariant, infinite quantum spin chains.

I am the main author of this article and contributed extensively to both the analysis of the results and the writing of the manuscript. My role involved interpreting the findings, ensuring the coherence of the mathematical arguments, and presenting the results in a clear and structured manner.

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Appendix A

Core Articles

A.1 Transcendental properties of entropy-constrained sets

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Transcendental Properties of Entropy-Constrained Sets

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Abstract. For information-theoretic quantities with an asymptotic operational characterization, the question arises whether an alternative single-shot characterization exists, possibly including an optimization over an ancilla system. If the expressions are algebraic and the ancilla is finite, this leads to semialgebraic level sets. In this work, we provide a criterion for disproving that a set is semialgebraic based on an analytic continuation of the Gauss map. Applied to the von Neumann entropy, this shows that its level sets are nowhere semialgebraic in dimension $d \geq 3$, ruling out algebraic single-shot characterizations with finite ancilla (e.g., via catalytic transformations). We show similar results for related quantities, including the relative entropy, and discuss under which conditions entropy values are transcendental, algebraic, or rational.

Contents

1. Introduction

Semialgebraic sets are ubiquitous in quantum information theory. This is to some extent due to the fact that a large part of the theory is formulated in finite-dimensional vector spaces and that many often used constraints (like positivity of operators) are semialgebraic. Another reason for the prolificness of semialgebraic sets is the diverse closure properties of such sets—especially the Tarski-Seidenberg theorem (1.4 and 2.2 in [1]), which shows that arbitrary quantifiers ($\exists$, $\forall$) over sets that are themselves semialgebraic can be used without leaving the semialgebraic world. This world can be left, however, when using non-algebraic functions or limits, for instance those related to unbounded dimensions or to an asymptotic number of copies in information theoretic contexts. To put it boldly, quantification over an integer variable is the nemesis of the semialgebraic world.

The present work is devoted to deciding whether or not certain sets are semialgebraic. We will first derive a general criterion that is especially suited for sets that are implicitly defined as preimages of functions and then apply it to the level sets of the entropy function and related quantities.

We want to begin, however, with a motivating example in which the question 'semialgebraic or not?' arises rather naturally. Consider the following case of catalytic state transformations, which was studied in [2,3]:

For an initial state ρ' on $\mathbb{C}^d$, define $\mathcal{S}_n$ as the set of all states ρ on $\mathbb{C}^d$ with the property that for any $\epsilon > 0$ there is a state σ on $\mathbb{C}^n$ and a unitary U on $\mathbb{C}^d \otimes \mathbb{C}^n$ such that the reduced states of $\rho_{12} := U(\rho' \otimes \sigma)U^*$ satisfy $\rho_2 = \sigma$ and $\|\rho_1 - \rho\|_1 \leq \epsilon$. In other words, $\mathcal{S}_n$ is the set of states that can be reached (approximately) from ρ' with the help of an n-dimensional 'catalyst'. As a second set, consider the set $\mathcal{S}$ of all states ρ on $\mathbb{C}^d$ whose entropy is larger or equal to the one of ρ'. It follows from the subadditivity of the entropy [2,3] that

$$\mathcal{S}_n \subseteq \mathcal{S}, \tag{1}$$

and the question arises whether equality holds in Eq.(1) for some n depending on d. There are various ways of addressing this question and coming to the conclusion that this is not the case despite the fact that $\mathcal{S}_\infty = \mathcal{S}$ [3]. The arguably simplest one would be to exclude equality by arguing that $\mathcal{S}_n$ is semialgebraic (courtesy of Tarski-Seidenberg), while $\mathcal{S}$ is not. After all, the definition of $\mathcal{S}$ involves the logarithm, which is the paradigm of a non-algebraic function. While, as we see later, this way of reasoning is essentially correct, a more careful argument is required. Consider for instance $d = 2$. In this case, $\mathcal{S}$ has an alternative semialgebraic characterization as the set of states for which $\mathrm{tr}\left[\rho^2\right] \leq c$ for a suitable constant c. Hence, for $d = 2$ entropy-constrained sets *are* semialgebraic and the transcendental nature of the logarithm only becomes relevant in the interrelation (and not in the intra-relation) of level sets. As we will show below, this changes when $d > 2$ where $\mathcal{S}$ becomes indeed

transcendental. This does not only rule out equality in Eq.(1) but any kind of semialgebraic characterization of entropy constrained sets.

The use of transcendentality as a simple argument for ruling out specific representations has a long but difficult to trace history. Here we want to mention at least [4], where it was shown that the ground state of the antiferromagnetic spin 1/2 Heisenberg chain cannot be a finitely correlated state, since its energy is transcendental.

An interesting problem that is at least superficially related to the one we consider is the question whether *almost-entropic regions*, which are specified by linear information inequalities for the Shannon entropy, are semialgebraic [5].

Outline of the paper. After laying out the notation and collecting some preliminaries, Theorem 1 will provide a general result that discriminates semialgebraic sets from non-semialgebraic ones. It is guided by two simple observations: (i) analytic continuations of algebraic functions have at most a finite number of branches, and (ii) if a set is given as the level set of some differentiable function, then the normal space of this set is particularly easy to access via the gradient of the defining function. The resulting theorem is then applied to the von Neumann entropy in Sect. 5 and to related sets (involving other constraints or the relative entropy) in Sect. 6. The interlude in Sect. 4 shows that a related albeit slightly weaker result can also be obtained using transcendental number theory. The main observation in Sect. 4 is a dichotomy of entropy values: these tend to be either transcendental or rational—avoiding the irrational algebraic regime in between.

2. Preliminaries

We call a function $f : \mathbb{R}^n \to \mathbb{R}^k$ *algebraic* over a subfield $\mathbb{F} \subseteq \mathbb{R}$ if for each of its k component functions f_i there is a polynomial $p_i \in \mathbb{F}[y, x_1, \ldots, x_n]$ such that $y = f_i(x) \Leftrightarrow p_i(y, x_1, \ldots, x_n) = 0$. We will usually apply this concept only locally, i.e., to a neighborhood of a point on the graph of f.

By $H_d \subseteq \mathbb{C}^{d\times d}$ we denote the space of Hermitian $d \times d$ matrices and by $P_d \subseteq H_d$ the set of positive definite matrices with non-degenerate spectrum. The subset of trace-one matrices will be denoted by $D_d \subseteq P_d$. That is, D_d is the set of density matrices that are non-degenerate and have full rank. We can identify H_d with $\mathbb{R}^{d^2}$ by collecting all relevant real and imaginary parts of matrix entries into one vector. The resulting map is then a vector-space isomorphism and we write $\nu : \mathbb{R}^{d^2} \to H_d$ for its inverse. P_d is an open subset of H_d with respect to the usual topology that is induced by any norm on H_d. Accordingly, $\nu^{-1}(P_d)$ becomes an open subset of $\mathbb{R}^{d^2}$ and thus a smooth d^2-dimensional submanifold. A function $f : H_d \to \mathbb{R}^k$ will be called algebraic, if $f \circ \nu$ is algebraic.

A subset of $\mathbb{R}^n$ is called semialgebraic if it can be defined by a finite number of polynomial equations and inequalities. Unless otherwise stated, all

the involved polynomials are over $\mathbb{R}$. A map between semialgebraic sets is called semialgebraic if its graph is a semialgebraic set.

3. Properties of Semialgebraic Sets

In this section, we will have a closer look at general properties of semialgebraic sets—aiming at criteria that allow us to show that certain subsets of an Euclidean space are not semialgebraic.

Every semialgebraic set $S \subseteq \mathbb{R}^n$ is the disjoint union

$$\dot{\bigcup}_{i=0}^{p} M_i = S \tag{2}$$

of finitely many *Nash submanifolds* of $\mathbb{R}^n$, i.e., submanifolds that are at once smooth and semialgebraic (see 2.3 in [6]). We can choose this decomposition (a.k.a. *stratification*) such that each M_i is connected, but we can also choose it such that i is the manifold-dimension of M_i. A *Nash map* between Nash submanifolds of Euclidean spaces is a map that is both smooth and semialgebraic. For more details on semialgebraic sets, Nash manifolds, and Nash maps we refer to [1,7].

Denoting by $N_x M_i$ the normal space of M_i at $x \in M_i$, we define the *normal bundle* of S as

$$NS := \bigcup_{i=0}^{p} \{(x,y) | x \in M_i,\ y \in N_x M_i\} \subseteq \mathbb{R}^n \times \mathbb{R}^n.$$

In the following, our main focus will be on the so-called *Gauss map*, which maps a point on a manifold to its normal space. One reason behind this is that if the manifold is implicitly defined as the preimage of some function (such as the level sets of the entropy), the Gauss map is often easier to handle than the manifold itself. Moreover, it has the following simple but crucial property:

Lemma 1. *If $S \subseteq \mathbb{R}^n$ is semialgebraic, then for any corresponding Nash submanifold $M_i \subseteq S$ the map from $x \in M_i$ to the orthogonal projector $P(x)$ onto $N_x M_i$ is a Nash map. In particular, the normal bundle NS is semialgebraic.*

Proof. Consider the squared distance function $\eta : \mathbb{R}^n \ni x \mapsto \frac{1}{2} \inf\{\|x-y\|^2 | y \in M_i\}$. Since M_i is a smooth submanifold of $\mathbb{R}^n$, according to Theorem 3.1 in [8], the Hessian $\nabla^2 \eta(x)$ represents the orthogonal projection $P(x)$ onto the normal space $N_x M_i$ and it depends smoothly on $x \in M_i$. Moreover, as M_i is in addition semialgebraic, Proposition 2.2.8 in [1] implies that η is semialgebraic. Since derivatives of semialgebraic functions remain semialgebraic (Sect. 4 in [9] or Proposition 2.9.1 in [1]), $x \mapsto \nabla^2 \eta(x) = P(x)$ is semialgebraic and smooth, hence a Nash map. The normal bundle NS is a union of p graphs of such maps and therefore a semialgebraic set. □

Theorem 1. *Let M be a Nash submanifold of $\mathbb{R}^n$, and $P(x) \in \mathbb{R}^{n \times n}$ the orthogonal projector onto the normal space of M at $x \in M$. For any pair of Nash maps $g : I \subseteq \mathbb{R} \to M$, I an open interval, and $h : \mathbb{R}^{n \times n} \to \mathbb{R}$ define $f := h \circ P \circ g : I \to \mathbb{R}$. Then*

(1) *f is analytic and algebraic over* $\mathbb{R}$,
(2) *the global analytic function obtained from f by analytic continuation has a compact Riemann surface and, in particular, a finite number of branches.*

Proof. Lemma 1 implies that f is a composition of Nash maps and thus itself a Nash map.[1] Proposition 8.1.8 in [1] states that a function from a semialgebraic set into $\mathbb{R}$, like f, is a Nash map if and only if it is analytic and algebraic over $\mathbb{R}$. This proves (1).

In order to arrive at (2) we use that any real analytic function f on an interval I can be extended uniquely to a complex analytic function in an open neighborhood of $I \times \{i0\} \subseteq \mathbb{C}$ and further to a global analytic function by analytic continuation. Regarding the polynomial $p \in \mathbb{R}[y, x]$ that governs the algebraic relation $p(f(x), x) = 0$ as an element in $\mathbb{C}[y, x]$, the complex analytic extension of f is still algebraic: the same polynomial relation interpreted over $\mathbb{C}$ locally defines a unique function ([10], Chap. 8) that has to coincide with the (extension of) f by the identity principle. Algebraic analytic functions are known to have at most a finite number of branches (Theorem 4, p.306 in [10]) and a compact Riemann surface (I.§2 in [11]). □

In order to show transcendentality of entropy-constrained sets, the idea is to use the last part of (2) in Theorem 1 and to seek a contradiction to the fact that the logarithm-function has an infinite number of branches.

4. Transcendental Entropy Values

Before we discuss transcendental sets and functions, we will make a brief excursion to the related topic of transcendental numbers and analyze under which conditions the entropy of a single state is transcendental, algebraic or rational. $\overline{\mathbb{Q}}$ will denote the field of algebraic numbers. The main tool from transcendental number theory on which our results are based on is:

Lemma 2. (Baker's theorem [12]) *Let $\lambda_1, \ldots, \lambda_n \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ be such that $\ln \lambda_1, \ldots, \ln \lambda_n$ are linearly independent over $\mathbb{Q}$. Then $1, \ln \lambda_1, \ldots, \ln \lambda_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

Equipped with this instrument we obtain the following dichotomies:

Theorem 2. *Let $\rho \in \mathbb{C}^{d \times d}$ be a density matrix and $S(\rho) := -\mathrm{tr}\,[\rho \log_b \rho]$ where $b > 1$ is the base of the logarithm.*

(1) *If $b = e$ (i.e., $\log_b = \ln$) and the eigenvalues of ρ are algebraic, then $S(\rho)$ is either zero or transcendental.*
(2) *If b is algebraic and the eigenvalues of ρ are rational, then $S(\rho)$ is either rational or transcendental.*

[1] Since the composition of smooth maps is smooth, and the composition of semialgebraic maps is semialgebraic (Proposition 2.2.6, [1]), it follows that the composition of Nash maps is a Nash map.

Remark: Note that the eigenvalues are in particular algebraic if the matrix elements of ρ are, since then $\det(\lambda\mathbb{1}-\rho)=0$ becomes a polynomial in $\overline{\mathbb{Q}}[\lambda]$.

Proof. Let $\lambda_1,\ldots,\lambda_d$ be the eigenvalues of ρ.

(1): A consequence of Lemma 2 is that, under the assumptions of the lemma, any $\overline{\mathbb{Q}}$-linear combination of $\ln\lambda_1,\ldots,\ln\lambda_n$ is either zero or transcendental. This can be seen by induction over n. For $n=1$ it follows immediately from Lemma 2. The induction step from n to $n+1$ can be shown by contradiction: suppose some $\overline{\mathbb{Q}}$-linear combination

$$\beta := \sum_{i=1}^{n+1} \alpha_i \ln\lambda_i, \quad \alpha_i \in \overline{\mathbb{Q}} \tag{3}$$

is a non-zero algebraic number β. Then by Lemma 2 one of the $\ln\lambda_i$ is a $\mathbb{Q}$-linear combination of the other n and can thus be replaced by them in Eq.(3), which would contradict the induction hypothesis.

(2): If we define $a := \prod_i \lambda_i^{-\lambda_i}$ where the product runs over all nonzero eigenvalues, then by assumption $a \in \overline{\mathbb{Q}}$ and $S(\rho) = (\ln a)/(\ln b)$. Hence, if $S(\rho)\in\overline{\mathbb{Q}}$, then $\ln a$ and $\ln b$ would be $\overline{\mathbb{Q}}$-linearly dependent. By Lemma 2 this would imply that they are $\mathbb{Q}$-linearly dependent, which in turn implies that their fraction $S(\rho)$ must be rational. □

A simple consequence of Theorem 2 is that when using the natural logarithm, level sets of the entropy cannot be semialgebraic over $\overline{\mathbb{Q}}$ unless the entropy is zero. Next, we show that something similar is true over $\mathbb{R}$.

5. Entropy-Surfaces are Nowhere Semialgebraic

In this section, we present the application of Theorem 1 to von Neumann entropy level sets. The sets of states that have extremal entropy (0 or $\ln d$) are evidently (semi)algebraic in any dimension d. Moreover, as we have discussed in introduction, all level sets of the entropy are semialgebraic for $d=2$. The following shows that this is no longer the case for $d>2$.

Theorem 3. *For any $d\geq 3$ and $c\in(0,\ln d)$ the set of $d\times d$ density operators whose von Neumann entropy is equal to c is nowhere semialgebraic. That is, if $\mathcal{S} := \{\rho\in H_d | \rho\geq 0, \mathrm{tr}[\rho]=1, -\mathrm{tr}[\rho\ln\rho]=c\}$, then for any open set $V\subset H_d$ the set $\mathcal{S}\cap V$ is not semialgebraic unless it is empty.*

Remarks: [1.] Theorem and proof use the natural logarithm. However, the same statement holds true as well for any other base, since changing the base of the logarithm is equivalent to changing the value of c.

[2.] Here and in the following section, we only consider the case of equality "$=c$". Since boundaries of semialgebraic sets are semialgebraic, the same result holds true for the inequalities "$<c$", "$\leq c$", "$>c$" and "$\geq c$".

Proof. Since we consider neighborhoods of states of constant entropy, we can restrict ourselves to sufficiently small subsets and thereby assume w.l.o.g. that $(\mathcal{S}\cap V) \subset D_d$. In this way, we can regard $\mathcal{S}\cap V$ as a smooth (d^2-2)-dimensional submanifold of D_d, since any $c \in (0, \ln d)$ is a regular value of the entropy function.

Next, we employ the map $\Phi : D_d \to \mathbb{R}^{d-1} \times \mathbb{R}^{d^2-d}$ from Lemma 4 in "Appendix A" that maps any $\rho \in D_d$ onto a vector whose first $d-1$ components are distinct eigenvalues of ρ. As proven in Lemma 4, we can choose Φ such that it becomes an algebraic diffeomorphism onto its range when restricted to a sufficiently small neighborhood. Consequently, if $(\mathcal{S}\cap V)$ is semialgebraic, then its image $M := \Phi(\mathcal{S} \cap V)$ would be a Nash submanifold of $\mathbb{R}^n$, $n := d^2 - 1$, to which Theorem 1 applied (cf. Fig. 1). We now assume that this is the case and seek for a contradiction.

Let ρ be any state in $\mathcal{S} \cap V$. The manifold M is the preimage of c under the map $F : \mathbb{R}^n \to \mathbb{R}$,

$$F(x) := -\left(1 - \sum_{i=1}^{d-1} x_i\right) \ln \left(1 - \sum_{i=1}^{d-1} x_i\right) - \sum_{i=1}^{d-1} x_i \ln x_i,$$

in a neighborhood of $\Phi(\rho)$.

Standard results in differential geometry tell us that the normal space of M at $x \in M$ is the one-dimensional space spanned by the gradient $\nabla F(x)$. The components of the gradient are $\nabla F(x)_i = \ln\left(1 - \sum_{j=1}^{d-1} x_j\right) - \ln x_i$ for $i < d$ and zero otherwise. In order to apply Theorem 1 we define a Nash map $h : \mathbb{R}^{n\times n} \to \mathbb{R}$, $h(X) := \sqrt{X_{11}/X_{22}}$. Composing this with the projector $P(x)$

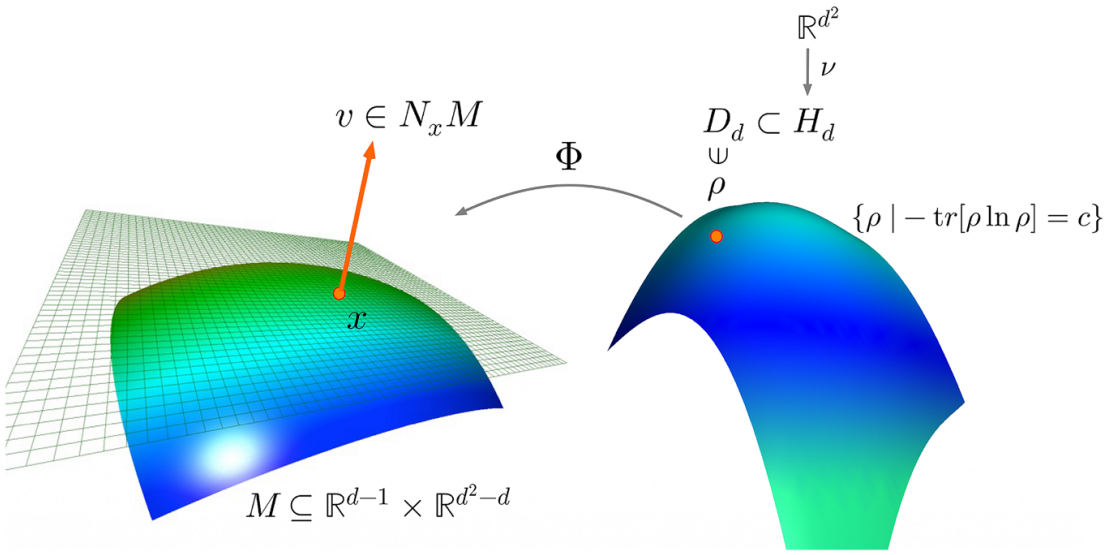

FIGURE 1. Visualization for Lemmas 3, 4 and the proof of Theorem 3. M is the considered manifold that is obtained via the diagonalizing algebraic diffeomorphism Φ acting locally on the set of states of constant entropy

onto the normal space we obtain (under the assumption that $d \geq 3$):

$$h \circ P(x) = \left| \frac{\nabla F(x)_1}{\nabla F(x)_2} \right|. \tag{4}$$

We can assume that the components of x, i.e., the eigenvalues of $\Phi^{-1}(x)$, are in ascending order for all x in the considered neighborhood. In this way, the absolute value in Eq.(4) can be neglected, as the quotient is positive.

Now consider a path on M parameterized on an open interval I by $g : \mathbb{R} \supset I \to M$ that goes through $\xi := \Phi(\rho)$ so that $g(\lambda_1) := (\lambda_1, \lambda_2, \xi_3, \ldots, \xi_n)$ where $\lambda_2 = \lambda_2(\lambda_1)$ is implicitly defined by demanding $g(I) \subset M$. Here, the implicit function theorem guarantees the existence of $\lambda_2(\lambda_1)$ as solution to $F(\lambda_1, \lambda_2, \xi_3, \ldots, \xi_n) = c$. If M is a Nash manifold, then $\lambda_1 \mapsto \lambda_2(\lambda_1)$ is algebraic and g is a Nash map on a neighborhood of ξ_1 at which $g(\xi_1) = \Phi(\rho)$.

According to Theorem 1 the analytic continuation of the function

$$f(\lambda_1) := h \circ P \circ g(\lambda_1) = \frac{\ln\left(\frac{\lambda_1}{w-\lambda_1-\lambda_2(\lambda_1)}\right)}{\ln\left(\frac{\lambda_2(\lambda_1)}{w-\lambda_1-\lambda_2(\lambda_1)}\right)}, \quad \text{where} \quad w := 1 - \sum_{j=3}^{d-1} \xi_j \tag{5}$$

must lead to a global analytic function with only finitely many branches—if the hypothesis of M being semialgebraic is valid. In the remaining part of the proof, we will show that this is not that case and that f gives rise to an infinite number of branches.

Let $z, y : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ denote the functions $z(\lambda_1) := \frac{\lambda_1}{w-\lambda_1-\lambda_2(\lambda_1)}$ and $y(\lambda_1) := \frac{\lambda_2(\lambda_1)}{w-\lambda_1-\lambda_2(\lambda_1)}$, respectively. Under the assumption that $\lambda_1 \mapsto \lambda_2(\lambda_1)$ is an algebraic function, and using the closure properties of the set of algebraic functions (Sect. 4 in [9], Theorem 6.4 in [13]), z and y are algebraic functions as well. As such, they can be regarded as global analytic functions defined on the entire complex plane up to a finite number of points (Theorem 3.1, [14]) and with at most finitely many branches.

Consider the analytic continuation of f along a closed path γ in the complex plane that starts at ξ_1, bypasses all of the finitely many singularities and is such that the image $z(\gamma)$ goes around the origin once and returns to the initial function value. Since algebraic functions have algebraic inverses, such a path always exists. As y has a finite number of branches, there is a $k \in \mathbb{N}$ such that also y returns to the same function value if we run through γ k-times in a row. After continuously tracing the path γ nk times for any $n \in \mathbb{Z}$, the branches of the complex logarithm give rise to a change of the value of the function f from the initial $f(\xi_1)$ to

$$\frac{2\pi i k n + \ln z(\xi_1)}{2\pi i m(n) + \ln y(\xi_1)}, \tag{6}$$

where $m : \mathbb{Z} \to \mathbb{Z}$ is some function that takes into account how many times the image of the path under y has enclosed the origin. As shown in Lemma 5 in "Appendix B", if we run over all $n \in \mathbb{Z}$, then Eq.(6) represents an infinite number of function values as long as $f(\xi_1)$ is irrational. In case $f(\xi_1)$ happens to be rational, we apply the same argument albeit with the starting point ξ_1

slightly shifted to a point $\tilde{\xi}_1 \in I$ for which $f(\tilde{\xi}_1) \notin \mathbb{Q}$. That such a $\tilde{\xi}_1$ exists in any neighborhood of ξ_1 is implied by the fact that f is continuous and non-constant, since $f'(\xi_1) \neq 0$, which is proven in Lemma 6 in "Appendix B". □

From the proof of the above theorem, we can see that the same outcome holds true for the classical case where the Shannon entropy is used instead of the von Neumann entropy.

6. Additional Constraints and Relative Entropy

In this section, we want to extend the result by first allowing for an additional constraint and then showing a similar result for the relative entropy. Additional constraints could concern the distance of the state to a specific target, its energy or the expectation value of any observable. Mathematically, we will describe those using a differentiable function $h : P_d \to \mathbb{R}$ for which we denote by $\nabla h(\rho) \in H_d$ the gradient, i.e., the operator that is related to the Fréchet derivative $h'(\rho)$ via $h'(\rho) : H_d \ni X \mapsto \mathrm{tr}\,[X \nabla h(\rho)]$.

Theorem 4. *For $d \geq 3$, $c_1 \in (0, \ln d)$, $c_2 \in \mathbb{R}$ and $h \in C^1(P_d, \mathbb{R})$ let ρ be an element of*

$$\mathcal{S} := \{\rho \in P_d \mid \mathrm{tr}\,[\rho] = 1, -\mathrm{tr}\,[\rho \ln \rho] = c_1, h(\rho) = c_2\},$$

with $[\rho, \nabla h(\rho)] \neq 0$. Then $\mathcal{S}$ is not semialgebraic in any neighborhood of ρ.

Remark: The condition $[\rho, \nabla h(\rho)] \neq 0$ is stronger than necessary but conveniently serves the purpose of the proof. It can be interpreted as a first-order-way of imposing that $h(\rho)$ does not solely depend on the spectrum of ρ.

Proof. Without loss of generality, we can consider a sufficiently small neighborhood in which $[\rho, \nabla h(\rho)] \neq 0$ holds for each of its elements. We begin with convincing ourselves that within such a neighborhood $\mathcal{S}$ is a regular C^1-submanifold of P_d. To this end, we regard $\mathcal{S}$ as the preimage of $c := (1, c_1, c_2)$ under $f : P_d \to \mathbb{R}^3$, $f(\rho) := (\mathrm{tr}\,[\rho], S(\rho), h(\rho))$. Then c is a regular value, and thus $\mathcal{S}$ a regular C^1-submanifold, if the three involved gradients $\mathbb{1}, \nabla h(\rho)$ and $\nabla S(\rho) = -\mathbb{1} - \ln \rho$ (Lemma VI.9 in [15]) are linearly independent. This is, however, guaranteed by $[\rho, \nabla h(\rho)] \neq 0$.

Next, we adopt the viewpoint of $\mathcal{S}$ as an intersection of two manifolds, namely the level set of the von Neumann entropy studied in Theorem 3 and the manifold $h^{-1}(\{c_2\})$. More precisely, we will consider a submersion of those manifolds into a space that is relevant for the argument in the proof of Theorem 3. The considered submersion is given by $\psi : P_d \to \mathbb{R}^d$, $\psi_i(\rho) := \Psi_i(\rho)$, where Ψ is the diagonalizing algebraic diffeomorphism from Lemma 3. We claim for the tangent space of one of the submersed manifolds that

$$T_{\psi(\rho)}\Big[\psi \circ h^{-1}\big(\{c_2\}\big)\Big] = \psi'(\rho)\, T_\rho\Big[h^{-1}\big(\{c_2\}\big)\Big] = \mathbb{R}^d. \tag{7}$$

Before we prove Eq.(7), let us see why Eq.(7) completes the proof of the theorem: the tangent space $T_{\psi(\rho)}\big[\psi(\mathcal{S})\big]$ of the intersected manifold is equal to

the intersection of the tangent spaces of the individual manifolds.[2] However, by Eq.(7), the tangent space associated to the additional constraint is the entire space $\mathbb{R}^d$ so that this 'intersection' becomes void and $T_{\psi(\rho)}\big[\psi(\mathcal{S})\big]$ is, in fact, the tangent space of the manifold already studied in Theorem 3. Consequently, the same argument applies.

Now let us show Eq.(7). The first equality just uses the Fréchet derivative as pushforward between tangent spaces. For the second equality we exploit that $\psi_i'(\rho) : X \mapsto \langle\varphi_i|X|\varphi_i\rangle$, where $\{|\varphi_i\rangle\}$ is the eigenbasis of ρ (see proof of Lemma 3). Moreover, $T_\rho\big[h^{-1}\big(\{c_2\}\big)\big]$ is the orthogonal complement of $\nabla h(\rho)$. Seeking for a contradiction, suppose the last equation in Eq.(7) does not hold. Then, there would exist a non-zero $b \in \mathbb{R}^d$ such that $\sum_i b_i\langle\varphi_i|X|\varphi_i\rangle = 0$ holds for any $X \perp \nabla h(\rho)$. In other words, $B := \sum_i b_i|\varphi_i\rangle\langle\varphi_i|$ would have to be proportional to $\nabla h(\rho)$. This is impossible since $[\rho, B] = 0$, whereas $[\rho, \nabla h(\rho)] \neq 0$. □

As an application of Theorem 4 let us show transcendentality of the level sets of the relative entropy. The *relative entropy* of two density operators ρ, σ on $\mathbb{C}^d$ is defined as $S(\rho\|\sigma) := \operatorname{tr}[\rho \ln \rho] - \operatorname{tr}[\rho \ln \sigma]$ whenever $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$ and $S(\rho\|\sigma) := \infty$ otherwise.

Corollary 1. *For any $c > 0$, $d \geq 3$, any positive definite density matrix $\sigma \in H_d$ and any open subset $U \subseteq H_d$ the set*

$$\mathcal{R} := \big\{\rho \in D_d \mid S(\rho\|\sigma) = c\big\} \cap U \tag{8}$$

is not semialgebraic in H_d unless it is empty.

Proof. We use that $S(\rho\|\sigma) = -S(\rho) - \operatorname{tr}[\rho \ln \sigma]$ and that $\rho \in D_d$ excludes the extremal values 0 and $\ln d$ of the von Neumann entropy. If $\sigma = \mathbb{1}/d$, then the set $\mathcal{R}$ reduces to a level set of the von Neumann entropy so that Theorem 3 applies. If $\sigma \neq \mathbb{1}/d$, we claim that there is a $\tilde{\rho} \in \mathcal{R}$ that does not commute with σ and thus $[\tilde{\rho}, \ln \sigma] \neq 0$.

In order to show this, note that $\mathcal{R}$ is a manifold of dimension $d^2 - 2$. The manifold of all density operators commuting with $\sigma \neq \mathbb{1}/d$, which we denote by C_σ, however, can be shown to have dimension at most $(d-1)^2$: since the commutant is a proper von Neumann subalgebra and since the largest such subalgebra is isomorphic to the matrix algebra $\mathcal{M}_{(d-1)} \oplus \mathcal{M}_1$ (cf. Theorem 5.6 in [16]), the dimension of the submanifold of density matrices that are contained in this subalgebra is at most $(d-1)^2$. This implies that for all $d \geq 3$ we have $\dim(C_\sigma) < \dim(\mathcal{R})$ so that $\mathcal{R}$ must contain elements outside the commutant of σ. Let $\tilde{\rho}$ be one such element.

We restrict the focus to a neighborhood of $\tilde{\rho}$ in which no element commutes with σ. If $\mathcal{R}$ were semialgebraic in that neighborhood, then $\mathcal{R}$ intersected with the affine space of all ρ for which $\operatorname{tr}[(\rho - \tilde{\rho}) \ln \sigma] = 0$ would be semialgebraic as well. However, this intersection is covered by Theorem 4 with $h(\rho) := \operatorname{tr}[\rho \ln \sigma]$ so that $\nabla h(\tilde{\rho}) = \ln \sigma$ and therefore $[\tilde{\rho}, \nabla h(\tilde{\rho})] \neq 0$. □

[2] Strictly speaking, this requires that the manifolds intersect transversally, but as one of the tangent spaces is the entire space, the intersection is trivially transversal.

7. Outlook

We expect variants of the presented arguments to also apply to many other entropic quantities that appear in classical and quantum information theory. Since the obtained results then rule out algebraic one-shot characterizations with finite ancilla, it may be interesting to investigate known characterizations of this type, such as the ones in [17–20], from this angle.

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Appendix A: Diagonalizing Algebraic Diffeomorphisms

In this appendix, we show that the diagonalization employed in the proof of Theorem 3 can be done by means of an algebraic diffeomorphism.

Lemma 3. *For any $\rho \in P_d$ there is an open neighborhood $U \subseteq P_d$ and a map $\Psi : U \to \mathbb{R}^{d^2}$ such that (i) Ψ is algebraic over $\mathbb{Q}$, (ii) Ψ is a diffeomorphism onto its range, and (iii) $\{\Psi_i(X)\}_{i=1}^{d} = \operatorname{spec}(X)$ for any $X \in U$.*

Proof. As demanded by (iii), we define $\Psi_i(X)$ to be the i'th eigenvalue of X for $i = 1, \ldots, d$. This leads to algebraic functions over $\mathbb{Q}$ due to the polynomial relation $\det\big(\Psi_i(X)\mathbb{1} - X\big) = 0$.

From eigenvalue perturbation theory of Hermitian matrices (an adapted version of Theorem 2.3 in [21] where the left and right eigenvectors coincide)

we know that $\Psi_i(\rho + tA) = \Psi_i(\rho) + \mathrm{tr}\,[A|\varphi_i\rangle\langle\varphi_i|]\,t + \mathcal{O}(t^2)$ holds in the limit $t \to 0$, where $|\varphi_i\rangle$ is the normalized eigenvector of ρ corresponding to $\Psi_i(\rho)$. This implies that the derivative $\Psi_i'(\rho) : H_d \to \mathbb{R}$ acts as $A \mapsto \mathrm{tr}\,[A|\varphi_i\rangle\langle\varphi_i|]$ and since the φ_i's form an orthonormal basis, the d derivatives $\Psi_1'(\rho), \ldots, \Psi_d'(\rho)$ are linearly independent.

To construct the remaining component functions Ψ_i with $i > d$, consider the map $\tilde{\Psi} : \mathbb{R}^{d^2} \to \mathbb{R}^d \times \mathbb{R}^{d^2}$,

$$\tilde{\Psi}(x) := \Big(\Psi_1\big(\nu(x)\big), \ldots, \Psi_d\big(\nu(x)\big), x\Big),$$

where ν is the parametrization of Hermitian matrices as defined in Sect. 2. The Jacobi matrix $\tilde{J}$ that represents the derivative of $\tilde{\Psi}$ at $\nu^{-1}(\rho)$ has the form $\tilde{J} = \binom{C}{\mathbb{1}}$ and is of dimension $(d + d^2) \times d^2$, where C is a $d \times d^2$ matrix that represents the derivatives of the d eigenvalues. As we have seen above, those are linearly independent, so that C has rank d. Hence, we can find d rows within the $\mathbb{1}$-block of $\tilde{J}$ that can be erased so that the resulting square matrix is non-singular. Denote this square matrix by J and the map that selects the rows by $s : \mathbb{R}^d \times \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ so that $J = s\tilde{J}$. The derivative of the map $\Psi := s \circ \tilde{\Psi} \circ \nu^{-1}$ at ρ then has full rank so that by the inverse function theorem there exists an open neighborhood $U \ni \rho$ such that $\Psi : U \to \Psi(U)$ is a diffeomorphism. □

Based on this lemma we can now show an analogous result that incorporates the constraint $\mathrm{tr}\,[\rho] = 1$.

Lemma 4. *For any $\rho \in D_d$ there is an open neighborhood $V \subseteq D_d$ and a map $\Phi : V \longrightarrow \mathbb{R}^{d-1} \times \mathbb{R}^{d^2-d}$, such that Φ is (i) algebraic over $\mathbb{Q}$, (ii) a diffeomorphism onto its range and (iii) $\{\Phi(X)\}_{i=1}^{d-1}$ are distinct eigenvalues of X for any $X \in V$.*

Proof. To construct such a map we compose the algebraic diffeomorphism Ψ from Lemma 3 with the following maps: $\iota : D_d \longrightarrow P_d$ denotes the inclusion map of the set D_d into the larger set P_d and $\pi : \mathbb{R}^d \times \mathbb{R}^{d^2-d} \longrightarrow \mathbb{R}^{d-1} \times \mathbb{R}^{d^2-d}$ the projection map that discards the d-th component of the input. By Φ we denote the composition $\Phi := \pi \circ \Psi \circ \iota : V \longrightarrow \mathbb{R}^{d-1} \times \mathbb{R}^{d^2-d}$. Being the composition of three smooth and algebraic maps, Φ is smooth and algebraic. It is also bijective onto its range with inverse $\Phi^{-1} = \Psi^{-1} \circ \widehat{\pi}$, where

$$\widehat{\pi} : (\lambda_1, ..., \lambda_{d-1}, x) \mapsto \Big(\lambda_1, ..., \lambda_{d-1}, 1 - \sum_{j=1}^{d-1} \lambda_j, x\Big) \quad \text{with } x \in \mathbb{R}^{d^2-d}.$$

As a composition of smooth maps Φ^{-1} is smooth. Since a bijective smooth map with smooth inverse is a diffeomorphism, this concludes the proof. □

Appendix B: Technical Lemmas

Lemma 5. *Let $a, b \in \mathbb{R} \setminus \{0\}$ be such that $\frac{a}{b} \notin \mathbb{Q}$, $k \in \mathbb{N}$ and $m : \mathbb{Z} \longrightarrow \mathbb{Z}$. Then*

$$\left| \left\{ \frac{a + 2\pi ikn}{b + 2\pi im(n)} \right\}_{n \in \mathbb{Z}} \right| = \infty.$$

Proof. Choose $n, \hat{n} \in \mathbb{Z}$ such that $n \neq \hat{n}$. Let $m := m(n)$ and $\hat{m} := m(\hat{n})$, and suppose that these give rise to the same value, i.e.,

$$\frac{a + 2\pi ikn}{b + 2\pi im} = \frac{a + 2\pi ik\hat{n}}{b + 2\pi i\hat{m}}. \tag{9}$$

By bringing together the imaginary parts we obtain from Eq.(9)

$$a(\hat{m} - m) + bk(n - \hat{n}) = 0.$$

Since $n - \hat{n} \neq 0$ we see that this is only possible if $\frac{a}{b} \in \mathbb{Q}$. In other words, if $\frac{a}{b} \notin \mathbb{Q}$, then different $n \in \mathbb{Z}$ lead to different values. □

For the following lemma and its proof we use the notation of the proof of Theorem 3.

Lemma 6. *The function $f : \mathbb{R} \supset I \to \mathbb{R}$ defined in Eq.(5) satisfies $f'(\xi_1) \neq 0$.*

Proof. With $C(\lambda_1, \lambda_2) := F(\lambda_1, \lambda_2, \xi_3, \ldots, \xi_n)$ the function $\lambda_2(\lambda_1)$ is implicitly defined as solution to $C\big(\lambda_1, \lambda_2(\lambda_1)\big) = c$. The implicit function theorem provides us with the derivative $\lambda_2'(\lambda_1) = -f(\lambda_1)$. Hence, $f'(\xi_1) = -\lambda_2''(\xi_1)$. We once again invoke implicit differentiation in the form

$$0 \stackrel{!}{=} \frac{d^2}{(d\lambda_1)^2} C\big(\lambda_1, \lambda_2(\lambda_1)\big) = v^T H v + \partial_2 C\big(\lambda_1, \lambda_2(\lambda_1)\big)\lambda_2''(\lambda_1), \tag{10}$$

with $v := \big(1 \;\; \lambda_2'(\lambda_1)\big)^T$ and the Hessian $H_{ij} := \partial_i \partial_j C\big(\lambda_1, \lambda_2(\lambda_1)\big)$. The latter can be computed to $H_{ij} = -\delta_{ij}\lambda_i^{-1} - (w - \lambda_1 - \lambda_2)^{-1}$ and is thus negative definite, which reflects the strict concavity of the entropy function. Therefore $v^T H v < 0$ so that Eq.(10) implies $\lambda_2''(\lambda_1) \neq 0$ for any λ_1 in a neighborhood of ξ_1. □

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A.2 Transcendental properties of entropy-constrained sets II

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Transcendental properties of entropy-constrained sets II

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ABSTRACT

In this work, we address the question of the impossibility of certain single-letter formulas by exploiting the semi-algebraic nature of various entropy-constrained sets. The focus lies on studying the properties of the level sets of relative entropy, mutual information, and Rényi entropies. We analyze the transcendental structure of the set of states in which one of the aforementioned entropy quantities is fixed. Our results rule out (semi)algebraic single-shot characterizations of these entropy measures with bounded ancilla for both the classical and quantum cases.

I. INTRODUCTION

Algebraic geometry has been a useful tool in the study of various problems in quantum information theory.[1–4] The distinction between the semialgebraic and the transcendental world has proven useful in separating the single-shot and asymptotic settings.[3] Finite resource theories are often described by polynomial (in)equalities since large parts of the underlying theory are described by relatively small, simple mathematical objects defined on finite-dimensional vector spaces. Infinite resources, on the other hand, fall outside this regime and could be associated with the transcendental world, and the difficulty of studying such regimes arises from quantification over infinite dimensional underlying structures.[2]

Our main motivation for this work lies in the question of the impossibility of single-letter formulas, especially for asymptotically defined quantities. We study this by exploiting the transcendental properties of certain entropy-constrained sets, specifically, the level sets of relative entropy, mutual information and Rényi entropies. The results we present here are based on several characterizations of algebraic functions and on the fact that von Neumann's entropy-constrained sets are nowhere semialgebraic. The latter was proved in Ref. 3 exploiting the fact that the analytic continuation of algebraic functions has at most a finite number of branches.

There have been many attempts to determine whether the entropy quantities that usually characterize the asymptotic regime can be given operational meaning in the single-shot setting.[5–9] One of the many interesting problems that could be investigated from a (semi)algebraic point of view is that of entanglement catalysis. These catalytic state transformations have been studied in Refs. 9 and 10 and are defined as follows:

For a given bipartite entangled state shared between two parties, say, Alice and Bob, $|\psi\rangle\langle\psi|^{AB}$ on $\mathbb{C}^{m_A}\otimes\mathbb{C}^{m_B}$, define $\mathcal{C}_{m,d}$ as the set of all entangled states $|\phi\rangle\langle\phi|^{AB}$ on $\mathbb{C}^{m_A}\otimes\mathbb{C}^{m_B}$ such that for any $\varepsilon>0$ there is a state $\tau^{A'B'}$ on $\mathbb{C}^{d_{A'}}\otimes\mathbb{C}^{d_{B'}}$ and a LOCC protocol Λ satisfying

$$\mathrm{tr}_{AB}[\Lambda(|\psi\rangle\langle\psi|^{AB}\otimes\tau^{A'B'})]=\tau^{A'B'}, \tag{1}$$

$$\|\,|\phi\rangle\langle\phi|^{AB}-\mathrm{tr}_{A'B'}[\Lambda(|\psi\rangle\langle\psi|^{AB}\otimes\tau^{A'B'})]\|_1\leq\varepsilon, \tag{2}$$

$$\|\Lambda(|\psi\rangle\langle\psi|^{AB}\otimes\tau^{A'B'})-|\phi\rangle\langle\phi|^{AB}\otimes\tau^{A'B'}\|_1\leq\varepsilon. \tag{3}$$

TABLE I. Overview of the semialgebraic behavior of entropy level sets.

Semialgebraic nature of entropy measures		
Function	Space dimension	Level set
von Neumann entropy	$d = 2$	(Semi)algebraic everywhere
von Neumann entropy	$d \geq 3$	Transcendental everywhere
Relative entropy $S(\rho\|\sigma)$ with σ fixed	$d \geq 3$	Transcendental everywhere
Relative entropy $S(\rho\|\sigma)$ with ρ fixed	$d \geq 3$	Transcendental everywhere
Relative entropy $S(\rho\|\sigma)$	$d \geq 3$	Transcendental everywhere
Mutual information $I(\rho_{AB}) := S(\rho_{AB}\|\rho_A \otimes \rho_B)$	$\min(d_A, d_B) \geq 3$	Transcendental
Rényi entropy in the limit $\alpha \to 0$	$d \geq 2$	(Semi)algebraic everywhere
Rényi entropy in the limit $\alpha \to \infty$	$d \geq 2$	(Semi)algebraic everywhere
Rényi entropy with $\alpha \in \mathbb{Q}^+\backslash\{1\}$	$d \geq 2$	(Semi)algebraic everywhere
Rényi entropy with $\alpha \in \mathbb{R}^+\backslash\mathbb{Q}$	$d = 2$	(Semi)algebraic everywhere
Rényi entropy with $\alpha \in \mathbb{R}^+\backslash\mathbb{Q}$	$d \geq 3$	Transcendental everywhere

For simplicity, we include Eq. (2), although it is a consequence of Eq. (3) because the trace norm satisfies the data processing inequality. Such an LOCC multicopy protocol Λ is described by the following three steps (as explained in the Proof of Theorem 1 in Ref. 9): as a first step, Alice performs rank 1 projective measurement on her auxiliary system and depending on the outcome, the other parties apply a certain LOCC protocol Γ, or not. Then Alice continues with an application of a unitary on the auxiliary system, and as a last step, all parties perform a SWAP unitary. By the results in Ref. 11 the LOCC protocol Γ is in fact equivalent to a strategy involving *only* a single (generalized) measurement by Alice, followed by a one-way communication of the result to Bob. In other words, $\mathcal{C}_{m,d}$ is the set of states that can be reached (approximately) from $|\psi\rangle\langle\psi|^{AB}$ using a bounded catalyst $\tau^{A'B'}$, and all the aforementioned transformations on which the set definition is based may be embedded into a bounded vector space and can be written via means of polynomial (in)equalities.

As a second set, consider the set $\mathcal{C}_m$ of all pure states on $\mathbb{C}^{m_A} \otimes \mathbb{C}^{m_B}$ whose entanglement entropy is smaller than or equal to that of the initial state $|\psi\rangle\langle\psi|^{AB}$. If no bound is assumed on the dimension of the catalyst, then it is known that $\mathcal{C}_{m,\infty} = \mathcal{C}_m$.[9,10] The question now arises whether there is equality between these two sets for some d as a function of m. One way to settle this question and rule out equality is to state that due to the Tarski-Seidenberg theorem (arbitrary quantifiers $(\exists, \forall)$ over sets that are semialgebraic do not change the nature of semialgebraic sets, 1.4 and 2.2 in Ref. 12), the set $\mathcal{C}_{m,d}$ is semialgebraic, while the set $\mathcal{C}_m$ is not, as shown in Ref. 3. This observation shows that there is no universal bound on the dimension of the catalyst for the catalytic LOCC transformations presented in Ref. 9. In the case of uncorrelated entanglement catalysis, originally introduced by Jonathan and Plenio,[13] Daftuar proved the non-existence of a universal bound on the catalyst dimension by relying on a different proof method (see Ref. 14). However, our results not only rule out the equality between the sets $\mathcal{C}_m$ and $\mathcal{C}_{m,d}$ but also show that no semialgebraic characterization of such entropic quantities is possible as long as a bounded ancilla is considered.

Outline of the paper. After establishing the necessary notations for the paper in Sec. II, and specifying the proof method in the following subsection, we give the complete proof of the surfaces of the relative entropy in Sec. III. This is divided into three parts, with all variables considered accordingly. Following the same train of thought, we analyze the nature of α-Rényi entropy-constrained sets in Sec. V and show that these sets are nowhere semialgebraic when parameterized by an irrational number. On the other hand, when this parameter is a rational number, the α-Rényi entropy level sets are everywhere semialgebraic. The argument in Sec. IV provides a global result for the level sets of mutual information. We give an overview of the (semi)algebraic nature of the level sets of entropy measures studied here and in Ref. 3, without considering their extremal values, in Table I.

Proof method. To establish transcendental properties of entropy-constrained sets, we mainly invoke ideas and techniques from algebraic geometry and complex analysis. For instance, the well-known Tarski-Seidenberg theorem is used several times to derive the semialgebraic nature of various sets. The result of Ref. 3 on the transcendence of von Neumann's entropy-constrained sets is of particular importance for the study of the level set of mutual information. However, the proofs on α-Rényi entropies and relative entropy require substantially new ideas. In these cases, on the one hand, we take advantage of the nature of algebraic singularities, which precludes the occurrence of poles of irrational degrees. On the other hand, a careful and detailed analysis invoking several ideas from complex analysis allows us to establish irrational poles of the local level functions of the constrained sets.

II. NOTATION

A set $S \subseteq \mathbb{R}^n$ is called *semialgebraic* if it is defined by a finite number of polynomial equations and inequalities; otherwise, the set is called *transcendental*. Unless differently stated, all polynomials involved are over $\mathbb{R}$ (making use of the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$). A function $h : \mathbb{R}^m \to \mathbb{R}^n$ will be called *algebraic* over a subfield $\mathbb{F} \subseteq \mathbb{R}$ if for each of its n component functions h_i there exist a polynomial $p_i \in \mathbb{F}[y, x_1, \ldots, x_m]$ such that $y = h_i(x) \Leftrightarrow p_i(y, x_1, \ldots, x_m) = 0$. $H_d \subseteq \mathbb{C}^{d\times d}$ denotes the space of Hermitian $d \times d$ matrices and $P_d \subseteq H_d$ is

the set of positive definite matrices. By $D_d \subseteq P_d$ we denote the subset of trace one matrices, i.e., D_d is the set of density matrices that are non-degenerate and have full rank, and by $\mathcal{U}_d$ we denote the set of $d \times d$ unitary matrices.

III. RELATIVE ENTROPY CONSTRAINED SETS

The *relative entropy* between two density operators ρ, σ is defined as $S(\rho\|\sigma) := \mathrm{tr}[\rho \ln \rho] - \mathrm{tr}[\rho \ln \sigma]$ whenever $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ (which is always the case for $\rho, \sigma \in D_d$) and is $+\infty$ otherwise.

Theorem 1. *For any $c > 0$, $d \geq 3$ the set of $d \times d$ positive definite density operators whose relative entropy is equal to c is nowhere semialgebraic. More precisely, the following sets*

$$\forall \sigma \in D_d \qquad \mathcal{R}_1 := \{\rho \in D_d \mid S(\rho\|\sigma) = c\}, \tag{4}$$

$$\forall \rho \in D_d \qquad \mathcal{R}_2 := \{\sigma \in D_d \mid S(\rho\|\sigma) = c\}, \tag{5}$$

$$\textit{and} \qquad \mathcal{R}_3 := \{(\rho, \sigma) \in D_d \times D_d \mid S(\rho\|\sigma) = c\} \tag{6}$$

are nowhere semialgebraic. That is, for any open subset $U \subseteq H_d$ ($V \subseteq H_d \times H_d$) the sets $\mathcal{R}_i \cap U$, for $i = 1, 2$ ($\mathcal{R}_3 \cap V$), are not semialgebraic unless they are empty.

Proof. We distinguish between the following cases:

(1) That for any positive definite density matrix $\sigma \in D_d$ and any open subset $U \subseteq H_d$, the set $\mathcal{R}_1 := \{\rho \in D_d \mid S(\rho\|\sigma) = c\} \cap U$ is not semialgebraic in H_d unless it is empty, was established in Ref. 3.

(2) We now examine the case where the roles of ρ and σ are reversed. For any positive definite density matrix $\rho \in D_d$, assume that the set $\mathcal{R}_2 := \{\sigma \in D_d \mid S(\rho\|\sigma) = c\}$ is semialgebraic everywhere. We proceed by contradiction.

The above set can be rewritten as $\mathcal{R}_2 := \{\sigma \in D_d \mid \mathrm{tr}[\rho \ln \sigma] = \tilde{c}\}$, where $\tilde{c} := -c - S(\rho)$. Let V denote an open subset in H_d. Any $\sigma \in \mathcal{R}_2 \cap V$ can be written as $\sigma = U\,\mathrm{diag}(\sigma)U^*$ for some unitary $U \in \mathcal{U}_d$. Here, $\mathrm{diag}(\sigma)$ denotes the diagonal matrix consisting of the eigenvalues of σ. By Lemma 3 in Ref. 3 there exists a local algebraic diffeomorphism $\Phi : \sigma \mapsto (D := \mathrm{diag}(\sigma), U)$ which maps each $\sigma \in \mathcal{R}_2 \cap V$ to a vector whose d first components are the eigenvalues of the density matrix σ. After specifying a unitary U, the set

$$\mathcal{M} := \left\{\lambda \in \mathbb{R}^{d^2}_{>0} \mid \sum_{i=1}^{d} \lambda_i = 1, \sum_{i=1}^{d} a_i \log \lambda_i = \tilde{c}\right\} \tag{7}$$

where $a_i := (U^* \rho U)_{ii}$, is semialgebraic according to Lemma 1 in the Appendix B. Note that $\mathcal{M}$ is a smooth submanifold of $\mathbb{R}^{d^2}$. As we will show later, we can always choose $a_1, a_2, a_d \neq 0$ such that $\frac{a_1 + a_d}{a_2} \notin \mathbb{Q}$. For any $\lambda \in \mathbb{R}^{d^2-1}_{>0}$ we define

$$f(\lambda) := a_1 \log \lambda_1 + a_2 \log \lambda_2 + a_d \log\left(1 - \sum_{i=1}^{d-1} \lambda_i\right) - \tilde{c} + \sum_{i=3}^{d-1} a_i \log \lambda_i. \tag{8}$$

The manifold $\mathcal{M}$ is characterized by the positive roots of the above equation. After fixing $\lambda_3, \ldots, \lambda_{d-1}$ to $x_3, \ldots, x_{d-1}$ by a further application of Lemma 1, the set defined by $f(\lambda_1, \lambda_2, x_3, \ldots, x_{d-2}) = 0$ defines an algebraic curve. In this situation, the roots of Eq. (8) give rise to the local algebraic curve

$$\lambda_1^{a_1} \lambda_2^{a_2} (c - \lambda_1 - \lambda_2)^{a_d} = \beta \tag{9}$$

for some $\beta, c \in \mathbb{R}$. The implicit function theorem guarantees the existence of the function $\lambda_2 = g(\lambda_1)$ as a solution of (9), for a function g.

Moreover, by a standard argument based on analytic continuation, the whole Riemann surface characterized by (9) still forms an algebraic curve. We observe that there exists at least one branch of the Riemann surface such that $|\lambda_1| \to \infty$ and $\lambda_2 \sim \lambda_1^{-\frac{a_1+a_d}{a_2}}$ with one particular branch of the complex function $z \mapsto z^{-\frac{a_1+a_d}{a_2}}$. Here and in the following, we write $f \sim g$ to denote the asymptotic equality of two functions f, g at a point $a \in \mathbb{C} \cup \{\infty\}$, i.e., $\lim_{z \to a} f(z)/g(z) = w$ for some $w \in \mathbb{C}$.

Coming back to the proof, we consequently find an unbounded open set $U \subset \mathbb{C}^2$ and an algebraic function g such that a pair of complex numbers $(\lambda_1, \lambda_2) \in U$ is a solution to (9) if and only if we have functional dependence $\lambda_2 = g(\lambda_1)$ and, moreover, the asymptotic equality $g(\lambda_1) \sim \lambda_1^{-\frac{a_1+a_d}{a_2}}$ as $\lambda_1 \to \infty$ holds true in U. Considering the inversion $w(x) := \frac{1}{g(1/x)}$, we obtain an algebraic function $w(\lambda_1) \sim \lambda_1^{\frac{a_1+a_d}{a_2}}$ as $\lambda_1 \to 0$. This leads to a nonalgebraic singularity at 0, since we have chosen $\frac{(a_1+a_d)}{a_2} \notin \mathbb{Q}$. This is not possible due to Theorem 4 in the Appendix A.

It remains to show that in any neighborhood of a unitary U_0, there exists another unitary U which satisfies $a_1, a_2, a_d \neq 0$ such that $\frac{a_1+a_d}{a_2} \notin \mathbb{Q}$ with the notation from above. Note that the submanifold of unitaries characterized by $a_i = 0$ is a set of measure zero with respect to the Haar measure. Similarly, for $d \geq 3$ and a fixed $\lambda \in \mathbb{Q}$ the set of unitaries with $a_1 + a_d = \lambda a_2$ is again a set of measure zero. This is still true for the union over all rational $\lambda \in \mathbb{Q}$. In particular, the set of all unitaries with $a_1, a_2, a_d \neq 0$ such that $\frac{a_1+a_d}{a_2} \notin \mathbb{Q}$ forms a dense subset, completing the proof.

(3) For the last case, we study the set $\mathcal{R}_3 := \{(\rho, \sigma) \in D_d \times D_d \mid S(\rho\|\sigma) = c\} \cap V$, for any open subset $V \subseteq H_d \times H_d$. Suppose the set $\mathcal{R}_3$ is semialgebraic. By Lemma 1 in the Appendix B, for any $\sigma_0 \in D_d$, the set $\{\rho \in D_d \mid S(\rho\|\sigma_0) = c\}$ would also be semialgebraic. This contradicts (1).

□

Remark 1.

(1) We have used the natural logarithm for the formulation of the theorem and its proof but note that the same result holds for any other base of the logarithm since the change in the base of the logarithm corresponds to a change in the value of c.
(2) In the theorem above, only the case of equality "=c" is considered, but the same result holds for the inequalities "<c", "≤c," ">c" and "≥c" since the boundaries of semialgebraic sets are semialgebraic.
(3) The same result holds if the classical relative entropy is used instead of the quantum relative entropy.

Applications: An immediate application of the above theorem would be to examine the results presented in Ref. 8 from the "semialgebraic" perspective. There, relative entropy was given an operational meaning beyond its conventional interpretation in the asymptotic framework. It has been shown that the catalytic transformation between pairs of quantum states can be characterized using only the relative entropy. Specifically, given two pairs of commutative quantum states (ρ, σ) and (ρ', σ'), the pair (ρ, σ) is transformed to (ρ', σ') by using a catalyst consisting of a pair of distributions (ξ, η) in conjunction with (ρ, σ). The target pair is then generated via a classical channel $\mathcal{N}$ acting on $(\rho \otimes \xi, \sigma \otimes \eta)$ such that the first and second marginals of $\mathcal{N}(\rho \otimes \xi)$ are ρ' and ξ, respectively, and $\mathcal{N}(\sigma \otimes \eta) = \sigma' \otimes \eta$, where η is the uniform distribution on the support of ξ.

Moreover, $\mathcal{N}(\rho \otimes \xi)$ is required to be close in relative entropy to the product of its marginals, $\rho' \otimes \xi$. Whether this is also true for pairs of general quantum states and quantum catalysts remains an open question. However, we would like to elaborate on a scenario in which the distinction between the semialgebraic and the transcendental world provides insight into such transformations.

Note that the same operational meaning for the relative entropy as above applies if the condition of proximity between $\mathcal{N}(\rho \otimes \xi)$ and $\rho' \otimes \xi$ is now replaced by e.g., the trace distance between $\mathcal{N}(\rho \otimes \xi)$ and $\rho' \otimes \xi$:

$$T(\mathcal{N}(\rho \otimes \xi), \rho' \otimes \xi) := \frac{1}{2}\|\mathcal{N}(\rho \otimes \xi) - \rho' \otimes \xi\|_1. \tag{10}$$

Indeed, assuming that $\forall \gamma > 0$, $\exists$ a channel $\mathcal{N}$ (and an appropriately chosen catalyst, depending on γ) such that $D(\mathcal{N}(\rho \otimes \xi)\|\rho' \otimes \xi) < \gamma$, Pinsker's inequality[15] yields

$$T(\mathcal{N}(\rho \otimes \xi), \rho' \otimes \xi) \leq \sqrt{\gamma}. \tag{11}$$

Conversely, for any $\beta \in (0, 1)$ assume that $T(\mathcal{N}(\rho \otimes \xi), \rho' \otimes \xi) \leq \beta$. Using the continuity bound given by Winter[16] and Eq. (16) in Lemma 5.8 from Ref. 17 we have:

$$|D(\mathcal{N}(\rho \otimes \xi)\|\rho' \otimes \xi)| \leq \beta \log |A| + \sqrt{2\beta} =: \gamma, \tag{12}$$

where A denotes the Hilbert space on which ρ and ρ' live.

Now, for an initial pair of commuting states (ρ, σ) on $\mathbb{C}^d \times \mathbb{C}^d$, define $\mathcal{R}_n$ as the set of all pairs of commuting states (ρ', σ') on $\mathbb{C}^d \times \mathbb{C}^d$ with the property that for any $\varepsilon \in (0, 1)$ and $\gamma \in (0, 1)$ there is a pair of probability distributions (ξ, η) on $\mathbb{C}^n \times \mathbb{C}^n$ and a channel $\mathcal{N}$ on $\mathbb{C}^d \otimes \mathbb{C}^n$ such that the reduced states of $\mathcal{N}(\rho \otimes \xi)$ satisfy $\mathcal{N}(\rho \otimes \xi)_2 = \xi$ and $\frac{1}{2}\|\rho' - \mathcal{N}(\rho \otimes \xi)_1\|_1 \leq \varepsilon$. Furthermore, $\mathcal{N}(\sigma \otimes \eta) = \sigma' \otimes \eta$, where η is the uniform distribution on the support of ξ and $T(\mathcal{N}(\rho \otimes \xi), \mathcal{N}(\rho \otimes \xi)_1 \otimes \xi) < \gamma$. In other words, $\mathcal{R}_n$ is the set of pairs of states that can be reached (approximately) from (ρ, σ) with the help of an n-dimensional "catalyst." As a second set, we consider the set $\mathcal{R}$ of all pairs of states (ρ', σ') on $\mathbb{C}^d \times \mathbb{C}^d$ whose relative entropy is less than or equal to that of (ρ, σ). The question would be whether these two sets are the same for some n as a function of d. One way to answer this question is to state that the set $\mathcal{R}_n$ is semialgebraic according to the Tarski-Seidenberg theorem, while $\mathcal{R}$ is not, as shown in Theorem 1. This distinction excludes the equality between the sets $\mathcal{R}_n$ and $\mathcal{R}$, and any kind of semialgebraic characterization of the relative entropy as long as bounded ancillary systems are considered.

IV. MUTUAL INFORMATION CONSTRAINED SETS

To lighten the notation, in this section, we denote by $\mathcal{H}_A$ and $\mathcal{H}_B$ two complex Hilbert spaces of finite dimension d_A and d_B for systems A and B, respectively. By $\mathcal{S}(\mathcal{H}_\cdot) \equiv \mathcal{S}_\cdot$ we denote the set of states associated with the Hilbert space $\mathcal{H}_\cdot$. The *quantum mutual information* of a

bipartite state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$, quantifying the correlations between subsystems A and B, is defined as

$$I(\rho_{AB}) = I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = S(\rho_{AB} \| \rho_A \otimes \rho_B), \tag{13}$$

where $S(\rho_A)$ and $S(\rho_B)$ denote the von Neumann entropy of the marginals $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ and $\rho_B \in \mathcal{S}(\mathcal{H}_B)$ of ρ_{AB}, respectively.

Recall that $0 \le I(A:B) \le 2\log[\min\{d_A, d_B\}]$, where the lower bound follows from the positivity of the relative entropy and the upper bound is due to the strong subadditivity of the relative entropy. The set of states that have extremal mutual information (0 or $2\log[\min\{d_A, d_B\}]$) are (semi)algebraic in each dimension d.

Due to the strict positivity of the relative entropy, the case $c = 0$ coincides with $\rho_{AB} = \rho_A \otimes \rho_B$. Consequently, this set is characterized by linear constraints and is thus (semi)algebraic. For the other extreme value, we distinguish the cases: $(i) d_A = d_B$ and $(ii) d_A \neq d_B$.

For the first case, note that the level set consists of pure states ρ_{AB} whose marginal ρ_A satisfies $S(\rho_A) = \log d_A$. Both these conditions are (semi)algebraic. For the second case, assume without loss of generality that $d_A < d_B$. Note that we can write $I(A:B) = S(A) - S(A|B)_\rho = S(A) + S(A|E)_\psi$, where $|\psi\rangle_{ABE} \equiv |\psi\rangle$ is a purification of the state ρ_{AB} to some environment E (following the Proof of Theorem 11.5.1[18]). We take advantage of the fact that (see Exercise 11.8.9. in Ref. 18)

$$S(A|E) = -S\left(\mathrm{tr}_B[|\psi\rangle\langle\psi|] \Big\| \left(\frac{\mathbb{1}_A}{d_A} \otimes \mathrm{tr}_{AB}[|\psi\rangle\langle\psi|]\right)\right) + \log\, d_A \tag{14}$$

is maximal ($\log d_A$) if and only if $\mathrm{tr}_B[|\psi\rangle\langle\psi|] = \frac{\mathbb{1}_A}{d_A} \otimes \mathrm{tr}_{AB}[|\psi\rangle\langle\psi|]$. Then, the level set of mutual information for $c = 2\log d_A$ has the following form:

$$\mathcal{I} = \{\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \mid \exists\, |\psi\rangle_{ABE} \text{ s. t. } \mathrm{tr}_E[|\psi\rangle\langle\psi|] = \rho_{AB}, \\ \mathrm{tr}_B[|\psi\rangle\langle\psi|] = \frac{\mathbb{1}_A}{d_A} \otimes \mathrm{tr}_{AB}[|\psi\rangle\langle\psi|]\Big\}.$$

An application of the Tarski-Seidenberg theorem to the above set shows that it is semialgebraic. For other values of c and other dimensions, the answer is given in the following theorem.

Theorem 2. *Let* $\min(d_A, d_B) \ge 3$ *and* $c \in (0, 2\min\{\log d_A, \log d_B\})$. *Then the set of density matrices with constrained mutual information*

$$\mathcal{I} := \{\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \mid I(\rho_{AB}) := S(\rho_{AB} \| \rho_A \otimes \rho_B) = c\} \tag{15}$$

where $\rho_A := \mathrm{tr}_B[\rho_{AB}]$ *and* $\rho_B := \mathrm{tr}_A[\rho_{AB}]$ *is not semialgebraic unless it is empty.*

Proof. Without loss of generality, let $3 \le d_A \le d_B$ and $0 < c < 2\log d_A$ be a fixed real number. We proceed by contradiction and assume that the level set of the quantum mutual information is a nonempty semialgebraic set for our choice of c.

If the level set

$$\mathcal{I} := I^{-1}(c) := \{\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \mid I(\rho_{AB}) = c\}$$

was a semialgebraic set, then so would be (Sec. 2, Ref. 12) the set

$$\mathcal{C} := \{(\rho_{AB}, \rho_A, \rho_B) \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \mathcal{S}(\mathcal{H}_A) \times \mathcal{S}(\mathcal{H}_B) \mid I(\rho_{AB}) = c\} \\ \cap \{(\rho_{AB}, \rho_A, \rho_B) \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \mathcal{S}(\mathcal{H}_A) \times \mathcal{S}(\mathcal{H}_B) \mid \mathrm{tr}_B[\rho_{AB}] = \rho_A\} \\ \cap \{(\rho_{AB}, \rho_A, \rho_B) \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \mathcal{S}(\mathcal{H}_A) \times \mathcal{S}(\mathcal{H}_B) \mid \mathrm{tr}_A[\rho_{AB}] = \rho_B\}.$$

If we further intersect $\mathcal{C}$ with the semialgebraic set

$$\mathcal{P} := \left\{\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \mid \mathrm{tr}\left[\rho_{AB}^2\right] = 1\right\},$$

we get another semialgebraic set. Now, for any $(\rho_{AB}, \rho_A, \rho_B) \in \mathcal{M} := \mathcal{C} \cap \mathcal{P}$ we have $I(\rho_{AB}) = 2S(\rho_A)$, and therefore $\mathcal{M}$ takes the form

$$\mathcal{M} = \Big\{(\rho_{AB}, \rho_A, \rho_B) \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \mathcal{S}(\mathcal{H}_A) \times \mathcal{S}(\mathcal{H}_B) \Big| \\ \mathrm{tr}_B[\rho_{AB}] = \rho_A, \mathrm{tr}_A[\rho_{AB}] = \rho_B, \mathrm{tr}\left[\rho_{AB}^2\right] = 1, S(\rho_A) = \frac{c}{2}\Big\}.$$

As a corollary of the second form of the Tarski-Seidenberg theorem (2.1.2, Ref. 19) the image of $\mathcal{M}$ by the projection on the space of the second coordinate, which is given by $\{\rho_A \in \mathcal{S}(\mathcal{H}_A) \mid S(\rho_A) = \tilde{c}\}$ due to the well-known purification lemma, where $\tilde{c} := \frac{c}{2}$, is a semialgebraic set. This is not possible because von Neumann's entropy-constrained sets for dimensions $d \ge 3$ are everywhere transcendental.[3] □

V. RÉNYI ENTROPY CONSTRAINED SETS

For $\rho \in D_d$ the quantum Rényi entropy of order $\alpha \in (0,1) \cup (1,\infty)$ is defined as

$$S_\alpha(\rho) := \frac{1}{1-\alpha} \log_b \operatorname{tr}\left[\rho^\alpha\right]. \tag{16}$$

As generally known, in the limit $\alpha \to 1$ the Rényi entropy reduces to the von Neumann entropy, the level set of which was fully analyzed in Ref. 3. We will extend these results to the α-Rényi entropy-constrained sets for any $\alpha \geq 0$. We start with the two remaining limit cases.

(1) *Max entropy (Hartley entropy)*: Let $\mathcal{S}(\mathcal{H})$ denote the set of density operators for the Hilbert space $\mathcal{H}$. Then, the max entropy for $\rho \in \mathcal{S}(\mathcal{H})$ is defined as follows

$$S_0(\rho) := \lim_{\alpha \to 0} S_\alpha(\rho) = \log_b \operatorname{rank}(\rho). \tag{17}$$

For any $a \in \mathbb{R}$ and $d \geq 2$ the level set $\mathcal{S}_0 := \{\rho \in \mathcal{S}(\mathcal{H}) \mid S_0(\rho) = a\} = \{\rho \in \mathcal{S}(\mathcal{H}) \mid \operatorname{rank}(\rho) = c\}$, where $c := b^a \in \mathbb{R}$, is semialgebraic everywhere, i.e., for any open set $H \subseteq H_d$ the set $\mathcal{S}_0 \cap H$ is semialgebraic because the rank of the density matrix ρ is the size of the largest non-vanishing minor.[2]

(2) *Min entropy*:

$$S_\infty(\rho) := \lim_{\alpha \to \infty} S_\alpha(\rho) = \log_b \|\rho\|, \tag{18}$$

where $\|\cdot\|$ denotes the operator norm. For any $a \in \mathbb{R}$ and $d \geq 2$, the set $\mathcal{S} \cap W := \{\rho \in D_d \mid \|\rho\| = b^a\} \cap W$ is again semialgebraic for any open set $W \subset H_d$, as the set of Hermitian matrices with bounded norm.[2]

Let us further remark that for any $\alpha \in (0,1) \cup (1,\infty)$ the set of states with extremal Rényi entropy 0 or $\log d$ are (semi)algebraic in each dimension d. Indeed, the states with vanishing Rényi entropy are exactly the pure states, and $S_\alpha(\rho) = \log d$ corresponds to the set of the maximally mixed states, both of which exhibit an algebraic characterization. Similarly, for $d = 2$ all level sets are semialgebraic since any constraint of the form $S_\alpha(\rho) = c$ is equivalent to a constraint on the eigenvalues $\{\lambda_1, \lambda_2\} = \{\gamma, 1-\gamma\}$ of ρ, which in turn can be formulated as roots of a quadratic polynomial. The following theorem characterizes the remaining nontrivial situations.

Theorem 3. *For any $d \geq 3$ and $c \in (0, \log d)$, the set of $d \times d$ density operators with the Rényi entropy being fixed to c is transcendental everywhere if the order $\alpha \notin \mathbb{Q}$, and otherwise it is semialgebraic. That is, if*

$$\mathcal{S}_\alpha := \{\rho \in D_d \mid S_\alpha(\rho) = c\}, \tag{19}$$

then for any open subset $W \subset H_d$ the set $\mathcal{S}_\alpha \cap W$ is not semialgebraic for $\alpha \in (\mathbb{R} \backslash \mathbb{Q}) \cap [(0,1) \cup (1,\infty)]$ unless it is empty.

Proof. We distinguish the following cases:

(1) For $d \geq 3$ and $\alpha \in \mathbb{N} \cap [(0,1) \cup (1,\infty)]$ we write $S_\alpha(\rho) = c$ as $\operatorname{tr}[\rho^\alpha] = v$ for $v := b^{c(1-\alpha)} \in \mathbb{R}$, which is a polynomial over the field of real numbers. As a consequence, the level sets of the α – *natural* Rényi entropy $\mathcal{S}_\alpha$ are everywhere (semi)algebraic.

(2) For $d \geq 3$, $\alpha \in \mathbb{Q} \cap [(0,1) \cup (1,\infty)]$ and $a, b \in \mathbb{N}$, the set $\mathcal{S}_\alpha = \left\{\rho \in D_d \mid \operatorname{tr}\left[\rho^{a/b}\right] = v\right\}$ can be rewritten as

$$\mathcal{S}_\alpha := \{\, \rho \in D_d \mid \exists X \geq 0, X^b = \rho, \operatorname{tr}\left[X^a\right] = v\} \tag{20}$$

An application of the Tarski-Seidenberg theorem to the above set yields the claim.

(3) For $d \geq 3$ and $\alpha \in (\mathbb{R} \backslash \mathbb{Q}) \cap [(0,1) \cup (1,\infty)]$ the set (19) reduces to $\mathcal{S}_\alpha := \{\rho \in D_d \mid \operatorname{tr}[\rho^\alpha] = v\}$, $v := b^{c(1-\alpha)}$, which we assume to be everywhere semialgebraic. The proof then follows the same idea as that of relative entropy. Everything boils down to the analysis of the implicit equation

$$\lambda_1^\alpha + \lambda_2^\alpha + (\gamma - \lambda_1 - \lambda_2)^\alpha = \beta \tag{21}$$

which by assumption defines an algebraic curve, where $\gamma := 1 - \sum_{i=3}^{d-1} \lambda_i \in \mathbb{R}$, and $\beta := v - \sum_{i=3}^{d-1} \lambda_i \in \mathbb{R}$. Note that we can assume $\beta \neq 2(\gamma/2)^\alpha$ in the case $c \notin \{0, \log d\}$.

We claim that there is a complex solution of Eq. (21) with $\lambda_1 + \lambda_2 = \gamma$, which we denote by $(x, y := \gamma - x)$. This follows from the following observations. Considering the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) := z^\alpha + (\gamma - z)^\alpha$ – or to be more precise its branches – the open mapping theorem yields that its range is an open subset of the complex plane $\mathbb{C}$. On the other hand, the irrationality of α and the resulting infinite branches imply the density of its range. Indeed, let us write $z = |z|e^{i\phi}$ and $z' = \gamma - |z| = |z'|e^{i\phi'}$ and we observe that if ϕ and ϕ' are irrational and rationally independent, all branches of f together lead to an image which is dense in the annulus $\{w \mid ||z|^\alpha - |z'|^\alpha| \leq |w| \leq |z|^\alpha + |z'|^\alpha\}$. Making use of the transcendence of the trigonometric functions, such complex numbers z are themselves dense in $\mathbb{C}$. Combining these considerations, one concludes the existence of a solution (x, y) as above.

The idea is now that if we fix a branch of the algebraic curve (21) which contains the solution (x, y), then we find a non-algebraic behavior close to (x, y) which amounts to the desired contradiction. Let us make this intuition precise. First, the tuple (x, y) is regular, as the partial derivatives do not vanish. Thus, in a sufficiently small neighborhood of (x, y) the solutions of (21) may be represented in the form $y' = g(x')$, where g is by assumption an algebraic function. We now expand the function g around x using its characterizing identity (21). We proceed iteratively until we "detect" the non-algebraic singularity. Let us demonstrate the first step. Note that $x \neq y$ and we assume that both x, y do not vanish. A first-order Taylor expansion yields

$$\begin{aligned} &\alpha x^{\alpha-1}(x - x') + \alpha y^{\alpha-1}(y - y') + (x + y - x' - y')^{\alpha} \\ &= o(x - x') + o(y - y'). \end{aligned} \tag{22}$$

If $\alpha < 1$, one sees that the left-hand side of (22) can only cancel up to the first order if $(y - y') \sim (x - x')^{\alpha}$, which shows that g cannot be an algebraic function. If $\alpha > 1$, one obtains $(y - y') = -\frac{x^{\alpha-1}}{y^{\alpha-1}}(x - x') + o(x - x')$, which we plug into (21) and continue with a second order Taylor expansion. After finitely many steps, we arrive at an expansion of the form

$$(y' - y) \sim \sum_k c_k (x' - x)^k + \delta (x' - x)^{\alpha} + o((x - x')^{\alpha})$$

for some complex constants c_k and $\delta \neq 0$. It easily follows then that the $(k + 1)$th derivative of g has a non-algebraic singularity at x. Theorem 4 from the Appendix A completes the proof. □

Remark 2. From the above proof, it is clear that the same result holds if classical Rényi entropy is used instead of quantum Rényi entropy.

VI. OUTLOOK

Variants (or in some cases direct consequences) of the arguments presented here and in Ref. 3 can be extended to divergence measures that appear in classical and quantum information theory.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Vjosa Blakaj: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal). **Chokri Manai**: Formal analysis (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: ALGEBRAIC FUNCTIONS

In this section, we summarize some characterizations of algebraic functions that we use to show the transcendental nature of the entropy-constrained sets studied in this paper. One of the most well-known characterizations of algebraic functions is that they have a compact Riemann surface.[20]

Another characterization of the algebraic functions is given in terms of *algebraic singularities.*[21]

Definition 1 (Ref. 21.) A singular point z_0 of $f(z)$ is called algebraic if in a neighborhood of z_0 the function can be represented by a Puiseaux series

$$f(z) = \sum_{l=M}^{\infty} a_l (z - z_0)^{l/n}$$

where n $(n > 0)$ and M are integers, and $a_M \neq 0$. When $M < 0$, this point is called a critical pole.

To show the transcendence of α-Rényi entropy-constrained sets and relative entropy-constrained sets, we use the following theorem and look for the contradiction to the fact that non-algebraic functions have more than algebraic singularities.

Theorem 4 (Theorem 3.1 Ref. 21.) *A global analytic function $f(z)$, $z \in \mathbb{C} \cup \{\infty\}$ is algebraic if and only if all its singular points are isolated and it has finitely many algebraic singular elements.*

APPENDIX B: SEMIALGEBRAIC SETS

Lemma 1. Let $A \subseteq \mathbb{R}^{m+n}$ be a semialgebraic set. Then, for every $x_0 \in \mathbb{R}^m$ the set $\{y \in \mathbb{R}^n \mid (x_0, y) \in A\}$ is semialgebraic.

Proof. Follows directly from the definition of semialgebraic sets. □

Remark 3. The same result holds if we fix any other component from the entries of the defining set.

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A.3 On the set of reduced states of translation invariant, infinite quantum systems

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On the set of reduced states of translation invariant, infinite quantum systems

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Abstract

The set of two-body reduced states of translation invariant, infinite quantum spin chains can be approximated from inside and outside using matrix product states and marginals of finite systems, respectively. These lead to hierarchies of algebraic approximations that become tight only in the limit of infinitely many auxiliary variables. We show that this is necessarily so for any algebraic ansatz by proving that the set of reduced states is not semialgebraic. We also provide evidence that additional elementary transcendental functions cannot lead to a finitary description.

Keywords Quantum spin chains · Ground states · Transcendental · Quantum marginals

Mathematics Subject Classification 81Q35 · 81P16

Contents

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1 Introduction

Quantum correlations become non-trivially constrained when considering parts of a larger system that obeys symmetry rules. This fact is exhibited most prominently by the *monogamy of entanglement*. More generally, it is manifestly inherent in all *quantum marginal problems* that concern the consistency of reduced density matrices with global, often symmetry-based, constraints. A central motivation for studying these problems is the fact that ground-state energies in quantum chemistry [10] and condensed matter physics [23] depend solely on the reduced density matrices under global symmetry constraints. Also, phase transitions and symmetry breaking are reflected in the geometry of the set of reduced density matrices [25].

In this paper, we consider the set of reduced two-body density matrices that arise in translationally invariant, infinite quantum spin chains. In these systems, the ground state energy density of a nearest-neighbor interaction Hamiltonian could in principle be obtained by optimizing over all admissible reduced states. However, even in the case of two-dimensional local Hilbert spaces, this set is not known exactly. There are inner approximations based on mean-field and matrix-product techniques [23] and outer approximations based on relaxations of constraints [5, 14, 17]. When unraveling these approximations, which are usually expressed directly as algorithms for bounding ground state energy densities, they all correspond to *semialgebraic* sets of reduced density matrices, that is, sets that can be described by a finite number of algebraic (in-)equalities. Our aim is to prove that the set of interest, i.e., the set of admissible two-body reduced density matrices, is itself not semialgebraic. In the last section, we show that the situation is most likely even worse (or more interesting) by providing evidence that additional elementary transcendental functions are still not sufficient to allow an explicit description by finite means.

Our work is inspired by an argument of Fannes, Nachtergaele, and Werner. In [12] they show that the ground state of the antiferromagnetic spin 1/2 Heisenberg chain cannot be a C^*-finitely correlated state: Its energy is transcendental, while the algebraic ansatz can only lead to algebraic energies. This argument holds for any parameter-free algebraic ansatz but fails as soon as transcendental parameters are allowed. For this reason, we invoke transcendental sets and functions rather than numbers.

2 (Semi-)algebraic preliminaries

We begin with reviewing some mathematical concepts and introducing terms and notations.

A function $f : I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ is called *algebraic* over a subfield $\mathbb{F} \subseteq \mathbb{R}$ if there is a polynomial $p \in \mathbb{F}[y, x]$ and an interval $J \subseteq \mathbb{R}$ s.t. $y = f(x) \Leftrightarrow p(y, x) = 0$ holds for all $(x, y) \in I \times J$. A function that is not piecewise algebraic over $\mathbb{F}$ is called *transcendental* over $\mathbb{F}$. We emphasize the field $\mathbb{F}$ since these notions are often understood with $\mathbb{F} = \mathbb{Q}$ in mind, but we will need $\mathbb{F} = \mathbb{R}$.

A set $A \subseteq \mathbb{R}^n$ is called *semialgebraic* over a subring $R \subseteq \mathbb{R}$ if it can be defined by a finite number of polynomial equalities and inequalities with coefficients in R. A

powerful tool for showing that a set is semialgebraic is the Tarski-Seidenberg theorem, which enables the use of quantifiers without leaving the semialgebraic world:

Theorem 1 (Tarski-Seidenberg quantifier elimination (cf.[6])) *Let $R \subseteq \mathbb{R}$ be a subring and $\{p_j(v,x)\}_{j=1}^{l}$ a finite set of polynomial equalities and inequalities with coefficients in R and variables $(v,x) \in R^k \times R^n$. If $\phi(v,x)$ is a Boolean combination (using $\wedge, \vee, \neg$) of the p_j's and*

$$\Psi(x) := \big(Q_1 v_1 \dots Q_k v_k : \phi(v,x)\big), \quad Q_i \in \{\exists, \forall\}, \tag{1}$$

then there exists a quantifier-free formula $\psi(x)$ consisting of a Boolean combination of finitely many polynomial (in-)equalities with coefficients in R, s.t.

$$\forall x : \ \big(\psi(x) \Leftrightarrow \Psi(x)\big). \tag{2}$$

Moreover, there exists an effective algorithm that constructs ψ from Ψ.

In our case, $\mathbb{R}^n$ will be a real vector space of complex Hermitian matrices, represented in some Hermitian operator basis. The choice of the basis does not matter as long as $R = \mathbb{R}$ is considered.

A function is called *semialgebraic* over R if its graph is a semialgebraic set. If $f : \mathbb{R} \supseteq I \to \mathbb{R}$ is semialgebraic over $\mathbb{R}$, then it is piecewise algebraic over $\mathbb{R}$ (cf. Sec.2.5.2 in [6]).

3 Reduced density matrices

Consider a multipartite quantum system of spins aligned on a one-dimensional chain such that each spin is assigned a Hilbert space $\mathbb{C}^d$. Our interest lies in the reduced density matrices of neighboring spins that are admissible under the constraint of global translation symmetry.

Definition 1 Let Red_d denote the subset of density operators on $\mathbb{C}^d \otimes \mathbb{C}^d$ s.t. for every $N \in \mathbb{N}$ there is a density operator $\rho^{(N+1)}$ on $(\mathbb{C}^d)^{\otimes(N+1)}$ with the property that all its successive bipartite reduced states are equal to ρ, i.e.,

$$\mathrm{tr}_{\neg(i,i+1)}\big[\rho^{(N+1)}\big] = \rho \qquad \forall i \in \{1, \dots, N\}. \tag{3}$$

More informally, $\rho \in \mathrm{Red}_d$ if and only if ρ can be extended in a translational invariant way to an infinite quantum spin chain.

Proposition 1 Red_d *is convex and compact.*

Proof Convexity follows readily from the convexity of the set of density matrices and the linearity of the partial trace. Moreover, boundedness is inherited from the set of all density operators. In order to see closedness, we write $\rho_{i,i+1}^{(N+1)} := \mathrm{tr}_{\neg(i,i+1)}\big[\rho^{(N+1)}\big]$ and define

$$f\left(\rho^{(N+1)}\right) := \sum_{i,j=1}^{N} \left\| \rho_{i,i+1}^{(N+1)} - \rho_{j,j+1}^{(N+1)} \right\| \tag{4}$$

on the set of $(N+1)$-partite density operators. Since f is continuous, the set $S_{N+1} := f^{-1}(\{0\})$ is closed and thus compact. Hence, $R_{N+1} := \mathrm{tr}_{\neg(1,2)}[S_{N+1}]$ is compact as the image of a compact set under a continuous map. Finally, since R_{N+1} is the set of bipartite density operators that can be extended to an $(N+1)$-partite state with equal successive bipartite marginals, we see that $\mathrm{Red}_d = \bigcap_{N\in\mathbb{N}} R_{N+1}$ is compact as it is an intersection of compact sets. □

Given a two-body Hamiltonian described by a Hermitian operator h acting on $\mathbb{C}^d \otimes \mathbb{C}^d$, we define $h_{i,i+1} := \mathbb{1}^{\otimes(i-1)} \otimes h \otimes \mathbb{1}^{\otimes(N-i-1)}$ acting on $(\mathbb{C}^d)^{\otimes N}$. The corresponding *ground state energy density* in the *thermodynamic limit* $(N \to \infty)$ is then

$$\epsilon := \lim_{N\to\infty} \frac{1}{N} \inf_{\rho^{(N)}} \sum_{i=1}^{N-1} \mathrm{tr}\left[\rho^{(N)} h_{i,i+1}\right], \tag{5}$$

where the infimum is taken over all N-partite density operators $\rho^{(N)}$.

The following is well known [23], but we provide proof for a more self-consistent and coherent presentation.

Proposition 2 *For every two-body Hamiltonian h the corresponding ground state energy density in the thermodynamic limit is given by*

$$\epsilon = \min_{\rho\in\mathrm{Red}_d} \mathrm{tr}\left[\rho h\right]. \tag{6}$$

Proof First note that compactness of Red_d (Proposition1) guarantees the existence of a minimizer. The r.h.s. of Eq. (6) can be seen to be an upper bound on ϵ by restricting the infimum in Eq. (5) to density operators with equal bipartite marginals. This leads to

$$\epsilon \leq \lim_{N\to\infty} \frac{N-1}{N} \min_{\rho\in\mathrm{Red}_d} \mathrm{tr}\left[\rho h\right] = \min_{\rho\in\mathrm{Red}_d} \mathrm{tr}\left[\rho h\right].$$

In order to obtain an inequality in the other direction, we add an additional interaction between the N'th and the first particle so that the resulting overall Hamiltonian becomes cyclic on a ring of size N. Since the change in energy is at most $\|h\|$, we can bound

$$\epsilon \geq \lim_{N\to\infty} \frac{1}{N} \left[-\|h\| + \min_{\rho^{(N)}} \sum_{i=1}^{N} \mathrm{tr}\left[\rho^{(N)} h_{i,i+1}\right] \right] \tag{7}$$

$$= \lim_{N\to\infty} \frac{1}{N} \left[-\|h\| + N \min_{\rho\in\mathrm{Red}_d} \mathrm{tr}\left[\rho h\right] \right] = \min_{\rho\in\mathrm{Red}_d} \mathrm{tr}\left[\rho h\right]. \tag{8}$$

Here, we have identified the $N+1$'st with the first site in Eq. (7) and used for Eq. (8) that the energy of a translational invariant Hamiltonian with periodic boundary condition is minimized by a translational invariant state, that is, in particular, by a state with equal bipartite marginals. □

Theorem 2 *For $d \geq 2$, the set Red_d is not semialgebraic over $\mathbb{R}$.*

Proof We will show this by proof of contradiction, starting from the assumption that Red_d is semialgebraic. On an interval $I \subseteq \mathbb{R}$ consider a mapping $I \ni \gamma \mapsto h(\gamma)$ into the set of Hermitian matrices on $\mathbb{C}^d \otimes \mathbb{C}^d$ with the property that each matrix entry of $h(\gamma)$ is a polynomial function of γ. We may, for instance, consider a simple affine relation of the form $h(\gamma) = h_0 + \gamma h_1$.

Regarding $h(\gamma)$ as a parameter-dependent interaction Hamiltonian, the graph of the corresponding ground state energy density $\epsilon(\gamma)$ on the interval I (a semialgebraic set) can be expressed as

$$\begin{aligned} \big\{\big(\epsilon(\gamma), \gamma\big) \;\big|\; \gamma \in I\big\} \;=\; \big\{(E, \gamma) \mid E \in \mathbb{R},\ \gamma \in I, & \qquad (9a)\\ \exists \rho \in \mathrm{Red}_d :\ \mathrm{tr}\big[\rho h(\gamma)\big] = E, & \qquad (9b)\\ \forall \sigma \in \mathrm{Red}_d :\ \mathrm{tr}\big[\sigma h(\gamma)\big] \geq E\ \big\}. & \qquad (9c) \end{aligned}$$

Under the assumptions that h is polynomial and Red_d semialgebraic over $\mathbb{R}$, the Tarski-Seidenberg theorem guarantees that the set defined in Eq. (9) is semialgebraic since all equations are polynomial and all quantifiers run over semialgebraic sets. The fact that the quantifiers in Eq. (9) are not at the beginning is not an obstacle since every first-order formula can be brought to *prenex normal form*, which is used in Theorem 1.

Consequently, $I \ni \gamma \mapsto \epsilon(\gamma)$ is a semialgebraic function and therefore piecewise algebraic over $\mathbb{R}$. Hence, the proof is completed by any example whose ground state energy density is transcendental over $\mathbb{R}$ (despite the fact that $\gamma \mapsto h(\gamma)$ is polynomial). The next section will provide such an example for the case $d = 2$. This also covers the case of higher dimensions by simply embedding smaller Hilbert spaces into larger ones. □

4 Transcendental ground state energy density

As a special example for the completion of the proof of Theorem 2, we consider the anisotropic XY-model, which is specified by the two-body Hamiltonian

$$h(\gamma) := (1-\gamma)\sigma_x \otimes \sigma_x + (1+\gamma)\sigma_y \otimes \sigma_y, \quad \gamma \in (-1, 1). \tag{10}$$

Theorem 3 *In the thermodynamic limit, the ground state energy density $\epsilon(\gamma)$ of the anisotropic XY-model is an analytic function that is transcendental over $\mathbb{R}$.*

Proof The ground state energy density in the thermodynamic limit was derived in [18] and shown to be

$$\epsilon(\gamma) = -\frac{1}{4\pi}\int_0^{\pi/2} \Big[1 - \underbrace{(1-\gamma^2)}_{=:z^2} \sin^2(k)\Big]^{1/2} dk =: -\frac{1}{4\pi} E(z), \tag{11}$$

where $E(z)$ is the *complete elliptic integral of the second kind*. Suppose that ϵ is algebraic over $\mathbb{R}$ in some neighborhood. Since the set of algebraic functions is closed under composition, the same would be true for E. The function E is known to be an analytic solution to the *hypergeometric differential equation*

$$z(1-z)E'' + \big(c - (a+b+1)z\big)E' = abE, \tag{12}$$

for $(a, b, c) = (-1/2, 1/2, 1)$ (see 17.3.10 and 15.5.1 in [1]). Due to the analyticity of E, we can extend the view to the complex plane such that a supposed polynomial relation over $\mathbb{R}$ would imply one over $\mathbb{C}$.

Differential equations of the kind of Eq. (12) have an algebraic solution if and only if they have a finite monodromy group (cf. Thm. 5.11 in [13]). For Eq. (12), these cases were completely characterized by Schwarz in [22], who determined a complete list of corresponding triples (a, b, c) (summarized, e.g., in Tab.2 of [4]), which does not contain $(a, b, c) = (-1/2, 1/2, 1)$. Consequently, E and therefore also ϵ are transcendental functions over $\mathbb{R}$. □

5 Extensions and implications

The fact that Red_d is not semialgebraic over $\mathbb{R}$ implies that no algebraic ansatz (with or without transcendental parameters) can describe the set exactly without invoking a limit of infinitely many variables. The best-known example is the set of matrix product states with increasing bond dimension. This raises the question of whether supplementing algebraic methods with transcendental functions could enable a finitary, explicit description of Red_d. In [11], for instance, entropy constraints, which are known to be transcendental [3, 8], are used in addition.

Evidence against such finitary descriptions we can see from two sides: First, functions related to elliptic integrals [as in Eq. (11)] can often be shown to be non-elementary [15, 16] in the sense that they cannot be expressed in terms of algebraic and elementary transcendental functions—neither explicitly nor implicitly. At the moment, however, we do not know exactly what the implications are for Red_d. We will therefore pursue a second line of thought more closely.

We want to allow the additional use of the exponential function exp (and thereby implicitly also of ln). To this end, denote by $\mathbb{R}_{\exp} := (\mathbb{R}, +, \cdot, \exp, <, 0, 1)$ the ordered field of real numbers with exponentiation, and call a subset $A \subseteq \mathbb{R}^n$ *definable* in the first-order language of $\mathbb{R}_{\exp}$ if $a \in A \Leftrightarrow \Phi(a)$ for some first-order formula Φ. That is, there is a $k \in \mathbb{N}$ and a Boolean combination ϕ of finitely many (in-)equalities of polynomial[1] and exponential functions in $n + k$ real variables, s.t.

$$\Phi(a) = Q_1 b_1 \cdots Q_k b_k \; \phi(a_1, \ldots, a_n, b_1, \ldots, b_k), \tag{13}$$

where each $Q_i \in \{\exists, \forall\}$ is a quantifier. Unlike for $\mathbb{R}$ there is no quantifier elimination for $\mathbb{R}_{\exp}$ [24]. However, in [20] Macintyre and Wilkie proved that the first-order theory of $\mathbb{R}_{\exp}$ is decidable conditioned on the validity of the following:

Conjecture 1 (Schanuel's conjecture (cf. Chap.21 in [19])) *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then $\{z_1, \ldots, z_n, \exp z_1, \ldots, \exp z_n\}$ contains at least n algebraically independent numbers.*[2]

[1] with algebraic coefficients.

[2] A set of numbers is called *algebraically independent* if there is no nonzero multivariate polynomial with coefficients in $\mathbb{Q}$ that has these numbers as roots.

Schanuel's conjecture can be considered *the* central conjecture of transcendental number theory. The best-known proven special case is the one where all z_i's are algebraic numbers. Then this becomes the content of the Lindemann-Weierstrass theorem.

Returning to the set of reduced density matrices, we obtain the following:

Theorem 4 *Assuming Schanuel's conjecture, there exists a* $d \in \mathbb{N}$ *such that the set* Red_d *is not definable in the first-order language of* $\mathbb{R}_{\exp}$.

Proof By the results of [2, 9] there is a $d \in \mathbb{N}$ and a family of interaction Hamiltonians h on $\mathbb{C}^d \otimes \mathbb{C}^d$ with matrix entries in $\mathbb{Q}[\sqrt{2}]$ (and thus expressible in $\mathbb{R}_{\exp}$) such that the ground state energy density problem

$$\forall \rho \in \mathrm{Red}_d : \ \mathrm{tr}\,[h\rho] > 0 \tag{14}$$

is undecidable. Assuming that Red_d was definable in the first-order language of $\mathbb{R}_{\exp}$, then Eq. (14) would be expressible as a sentence in the first-order language of $\mathbb{R}_{\exp}$. So according to the result of Macintyre and Wilkie [20], Eq. (14) would be decidable if we assume Schanuel's conjecture. □

As shown in [7], Schanuel's conjecture in Theorem 4 can be replaced by the so-called *transfer conjecture*, which also implies decidability of the first-order theory of $\mathbb{R}_{\exp}$. Moreover, there are closely related results that do not rely on any unproven conjecture. For instance, it was proven in [21] that sentences in the first-order theory of $\mathbb{R}_{\mathrm{trans}}$, for a transcendental function trans $\in \{\exp, \ln, \arctan, \ldots\}$, are decidable if only the outermost quantified variable occurs in the transcendental function.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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