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Collective Choice from the Probability Simplex with Application to Donor Coordination

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COLLECTIVE CHOICE FROM THE PROBABILITY SIMPLEX WITH APPLICATION TO DONOR COORDINATION

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ABSTRACT

Many real-world scenarios require the distribution of some budget across a set of *public* projects, given the preferences of a set of agents. Depending on whether agents initially own the budget or it is fixed and provided externally, we distinguish between *donor coordination* and *portioning* problems.

In the donor coordination setting, we mainly consider two types of projects (*substitutes* and *complements*) and model the respective preferences via linear and Leontief utilities. We prove that maximizing a weighted product of the agents' utilities leads to efficient and fair outcomes for both cases. Furthermore, we show that these outcomes arise as limits of natural proportional or best response dynamics. For Leontief utilities, the corresponding mechanism is even group-strategyproof and can be characterized based on that property. In addition, we illustrate relations to *public good markets* and gain insights for other utility models.

In portioning, we concentrate on *star-shaped preferences* where each agent has exactly one favorite distribution. For two projects, we show that there is a unique strategyproof and fair mechanism even when arbitrarily restricting the set of admissible preferences. However, for a larger number of projects, we prove that preferences cannot be aggregated in an efficient, strategyproof, and fair way under the predominant model of ℓ_1 *preferences*.

ZUSAMMENFASSUNG

Die zunehmende Verflechtung ganzer Wirtschaften rund um den Globus erfordert mehr denn je Methoden zur Entscheidungsfindung, die nicht nur den Profit aus der Zusammenarbeit maximieren, sondern auch fair gegenüber allen beteiligten Parteien sind. Dabei sollten die oftmals verschiedenen Ziele der einzelnen Parteien nicht außer Acht gelassen werden. Diese Arbeit nutzt Mittel aus der Sozialwahltheorie zur kollektiven Entscheidungsfindung, während individuelle Anreize mithilfe von Spieltheorie modelliert werden.

Im Allgemeinen betrachten wir Probleme, in denen ein Budget unter einer Menge von öffentlichen Projekten verteilt werden soll. Im ersten Teil der Arbeit beschäftigen wir uns mit dem Thema "Spendenkoordination". Hierbei nehmen wir an, dass jeder Agent eine eigene (variable) Spende einbringt. Die Menge aller Alternativen entspricht somit allen möglichen Verteilungen von Geldern auf Projekte und Agenten haben vollständige und transitive Präferenzen über diese, die wir über Nutzenfunktionen repräsentieren. Der zweite Teil behandelt reine Zuteilungsprobleme, in denen ein fixes, extern gegebenes Budget auf die Projekte verteilt werden soll. Beide Teile beginnen mit einer Einführung in das jeweilige Modell und enden mit einer Diskussion der Resultate und offenen Problemen.

In der Spendenkoordination unterscheiden wir grundlegend zwischen zwei verschiedenen Modellen. Zuerst nehmen wir lineare Nutzenfunktionen an und untersuchen die *Nash product rule* (*NASH*), deren zurückgegebene "Nashverteilungen" das Produkt der einzelnen Nutzen der Agenten (gewichtet mit ihren individuellen Beiträgen) maximieren. Dieser Mechanismus ist nicht nur effizient, sondern erfüllt auch starke Fairnesseigenschaften wie den *core* und wir beweisen, dass sich der Nutzen eines Agenten stets mindestens um den Beitrag erhöht, den er zusätzlich spendet, wenn seine kleinste positive Bewertung eines Projekts mindestens 1 ist. Außerdem betrachten wir den Spezialfall, in dem jeder Agent Projekte mit 0 oder 1 bewertet und zeigen, dass ein alternativer Mechanismus, die *conditional utilitarian rule*, überraschenderweise nicht immer den Gesamtnutzen maximiert, wenn wir uns auf eine bestimmte Klasse von Regeln beschränken.

Für den Fall, dass Projekte mit Wohltätigkeitsorganisationen identifiziert werden, erachten wir Leontief Nutzenfunktionen als zutreffender. Wiederum betrachten wir *NASH* und zeigen, dass dieser Mechanismus nicht nur effizient und fair, sondern sogar resistent gegen Manipulationen ganzer Gruppen von Agenten ist und weitere gute Monotonieeigenschaften besitzt. Weiterhin liefert dieser Mechanismus immer das eindeutige Nash-Gleichgewicht zurück und wir zeigen außerdem, dass dieses Gleichgewicht effizient berechenbar ist. Für den Spezialfall von binären Bewertungen aller Agenten für die Projekte erhalten wir alternative Interpretationen von *NASH*. Zusätzlich beweisen wir, dass *NASH* durch Stetigkeit, den core und die bereits erwähnte Manipulationsresistenz charakterisiert wird.

Darüber hinaus geben wir für beide Modelle natürliche Dyamiken an, die jeweils gegen die Menge aller Nashverteilungen konvergieren. Hierbei zeigt sich, dass lineare Nutzenfunktionen einen koordinierten Ansatz erfordern, in dem Agenten

ihren Beitrag proportional zu dem Nutzen verteilen, den sie von einzelnen Projekten bekommen. Für Leontief Nutzenfunktionen hingegen zeigen wir, dass die komplette Verteilung gegen die Nashverteilung konvergiert, wenn Agenten ihre individuellen Beiträge aktualisieren, um ihren individuellen Nutzen zu maximieren.

Anschließend interpretieren wir unsere Resultate im Zusammenhang mit öffentlichen Gütern in Märkten und dem Konzept von *Lindahl equilibrium*. Zudem ziehen wir Schlussfolgerungen für andere Nutzenfunktionen und schaffen Verbindungen zwischen *NASH* und anderen Wohlfahrtsfunktionen.

Im zweiten Teil dieser Arbeit über reine Zuteilungsprobleme führen wir zuerst eine neue Klasse von Nutzenfunktionen (genannt *peak-linear*) auf dem Standardsimplex ein, die eine große Menge an häufig verwendeten Nutzenfunktionen abdeckt und eine Unterklasse der sternförmigen Nutzenfunktionen darstellt.

Während solche Präferenzen auf dem eindimensionalen Standardsimplex auf eine effiziente, faire, und manipulationsresistente Weise aggregiert werden können, zeigen wir für zwei übliche Präferenzmodelle (ℓ_1 and ℓ_∞ Präferenzen), dass dies für höherdimensionale Standardsimplices nicht möglich ist.

PUBLICATIONS

This thesis is based on the following publications and working papers.

DONOR COORDINATION

[1] Funding public projects: A case for the Nash product rule. In *Journal of Mathematical Economics*, 99:102585, 2022 (with F. Brandl, F. Brandt, D. Peters, C. Stricker, and W. Suksompong).

[2] Coordinating charitable donations. 2024. Working paper (with F. Brandt, E. Segal-Halevi, and W. Suksompong).

PORTIONING

[3] Optimal budget aggregation with single-peaked preferences. 2024. Working paper (with F. Brandt, E. Segal-Halevi, and W. Suksompong).

FURTHER RELATED PUBLICATIONS

[4] Settling the score: Portioning with cardinal preferences. 2024. Working paper (with E. Elkind, P. Lederer, W. Suksompong, and N. Teh).

[1] was also published as an extended abstract in the *Proceedings of the 17th International Conference on Web and Internet Economics (WINE)*, 2021 (Brandl et al., 2021a) and was presented online at that conference. This paper was also presented at the 7th Oxford Workshop on Global Priorities Research (June 2021) and a SEA AI Lab seminar in 2022. Preliminary results of this paper were presented at the AAMAS Workshop on Games, Agents, and Incentives (May 2019) (Brandl et al., 2019).

A preliminary version of [2] was published as an extended abstract under the name "Balanced donor coordination" in the *Proceedings of the 24th ACM Conference on Economics and Computation (ACM-EC)*, page 299, 2023 (Brandt et al., 2023) and was presented in London. This paper was also presented at the 3rd Ariel Conference on the Political Economy of Public Policy (September 2022), the joint Microeconomics Seminar of ETH Zurich and the University of Zurich (March 2023), the Bar-Ilan University Computer Science Seminar (April 2023), the Hebrew University of Jerusalem Econ-CS seminar (May 2023), the Bar-Ilan University Game Theory seminar (June 2023), the 9th International Workshop on Computational Social Choice in Beersheba (July 2023), the KIT Conference on Voting Theory and Preference Aggregation (October 2023), the Online Social Choice and Welfare Seminar (January 2024), the Second Vienna-Graz Workshop on (Computational) Social Choice (February 2024), and the 17th Meeting of the Society for Social Choice and Welfare (July 2024).

A preliminary version of [3] was published as an extended abstract under the name "Optimal budget aggregation with single-peaked preferences" in the *Proceedings of the 25th ACM Conference on Economics and Computation (ACM-EC)*, 2024, forthcoming (Brandt et al., 2024) and was presented in New Haven, CT.

A preliminary version of [4] was published in the *Proceedings of the 26th European Conference on Artificial Intelligence (ECAI)*, pages 621–628, 2023 (Elkind et al., 2023) and was presented in Kraków. This paper was also presented at a rump session of the COMSOC Video Seminar (September 2023).

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- my brothers Tim and Pascal, and my parents Sigrid and Markus for their unlimited support.

Feel embraced.

CONTENTS

1 INTRODUCTION

> Our ability to reach unity in diversity will be the beauty and the test of our civilization.
>
> Mahatma Gandhi

These days, economies all around the world are more interdependent than they have ever been. Globalization enables us to join forces to work on global problems such as climate change or protect nature in general. Instead of pursuing individual strategies, we have the opportunity to coordinate approaches and achieve better outcomes for everyone. As an example, consider the *European Green Belt* (see, e.g., Terry et al., 2006), a protected area across Europe from the Barents to the Adriatic and Black Seas.[1] Naturally, each nation has its own interests and incentives to conserve nature in some parts of the country and use other parts economically. A coordinated approach achieved nature conservation in a connected area across Europe.

It is important to make coordinated decisions that do not necessarily maximize gains from cooperation from which only some of the involved parties benefit but aim at solutions that are fair toward all involved factions. Otherwise, disadvantaged parties might terminate the agreement, which is worse for everyone, even advantaged factions, in the long run.

All in all, we are facing problems of collective decision-making originating from *social choice theory* (Arrow, 1951; Arrow et al., 2002, 2011) under the presence of individual incentives which can be modeled via *game theory* (Borel, 1921; von Neumann and Morgenstern, 1944; Osborne and Rubinstein, 1994). Working at the intersection of these two disciplines allows us to understand and balance individual and collective goals.

The problems we want to look at reach from a group of friends that wants to distribute some price money among a set of charities, over cities that want to allocate budgets to public projects, to national tax programs (e.g., *cinque per mille* in Italy) where citizens are allowed to donate part of their personal income tax to charitable organizations. We distinguish between *donor coordination* and *portioning*. In contrast to the latter setting, the budget is not fixed *a priori* in donor coordination as agents might vary the amount of their individual donations.

Once the total amount of contributions is determined, we can consider general applications where the set of alternatives constitutes the probability simplex[2] and agents have complete and transitive preferences over possible outcomes. This is reminiscent of problems from *probabilistic social choice* (e.g., Brandt, 2017) where the set of alternatives is finite and *randomization* leads to the probability simplex.

1 Parts of the belt move along the former Iron Curtain. These areas are particularly well-suited as nature in these former border zones is almost untouched.

2 The total "budget" of 1 does not need to correspond to money. One could also think of timeshares, e.g., how much speaking time each parliament party should receive.

Consequently, agents have ordinal preferences over the finite set of alternatives that need to be extended to preferences over the whole simplex. Apart from preferences represented by linear utilities, the most common extensions violate either completeness (as, e.g., *stochastic dominance* (see, e.g., Blackwell, 1953)) or transitivity (as, e.g., *pairwise comparison* (see, e.g., Blyth, 1972; Aziz et al., 2015)).

A large strand of literature in social choice deals with the Arrovian aggregation of preferences, i.e., deducing collective preferences from a set of individual preferences while satisfying *Pareto optimality* and *independence of irrelevant alternatives*. Pareto optimality requires that alternative a is above alternative b in the collective ranking if all agents prefer a over b. Loosely speaking, the second axiom demands that the comparison of a and b is independent of how agents rank a third alternative c. *Arrow's impossibility theorem* (Arrow, 1951) then states that for three or more alternatives, there exists a dictator, i.e., one particular agent whose preferences always coincide with the chosen collective preferences. Arrow proved this statement for the unrestricted domain of all complete and transitive preferences but later works showed similar impossibilities on restricted domains and for adapted notions of Pareto optimality and independence of irrelevant alternatives (see, e.g., Le Breton and Weymark (2011) and Brandl and Brandt (2020) for an extensive overview of the literature on Arrovian preference aggregation). In particular, Arrow's impossibility still holds for preferences on the probability simplex that are representable by linear utility functions (Le Breton, 1986; Le Breton and Weymark, 2011).

In our settings, we assume the set of alternatives to be fixed and have a divisible resource instead of an "indivisible" choice. Furthermore, an agent's favorite distribution can be located anywhere in the simplex.

Throughout this thesis, agents' preferences are assumed to be complete, transitive, continuous, and convex. Thus, we are able to represent them via continuous and quasi-concave utility functions. The first part of Chapter 2 formally introduces our general setting, and Section 2.1 argues about moving to representations with cardinal utility functions. Moreover, we will often look at certain classes of utility functions in order to capture the central characteristics of specific problems and enable agents to efficiently report their preferences. An overview of all considered classes of utility functions is given in Figure 1.1.

Quantifying Decision Power

Donor coordination problems are particularly well-suited to illustrate our game-theoretic approach taken in Section 2.2. Naturally, each donor is concerned with the allocation of her individual donation as well as the overall distribution of all donations. Her decision power is proportional to the fraction of donations that she contributes. For general decision problems on the probability simplex, we can imagine that each of the n agents is allowed to "control" how a probability of $1/n$ is distributed. This leads to two important concepts for distributions from game theory: *equilibrium* (Definition 2.7) and the *core* (Definition 2.13). Loosely speaking, they guarantee that agents do not have incentives to deviate from the proposed outcome. Thus, these properties admit interpretations as fairness axioms (see Section 2.3.3).

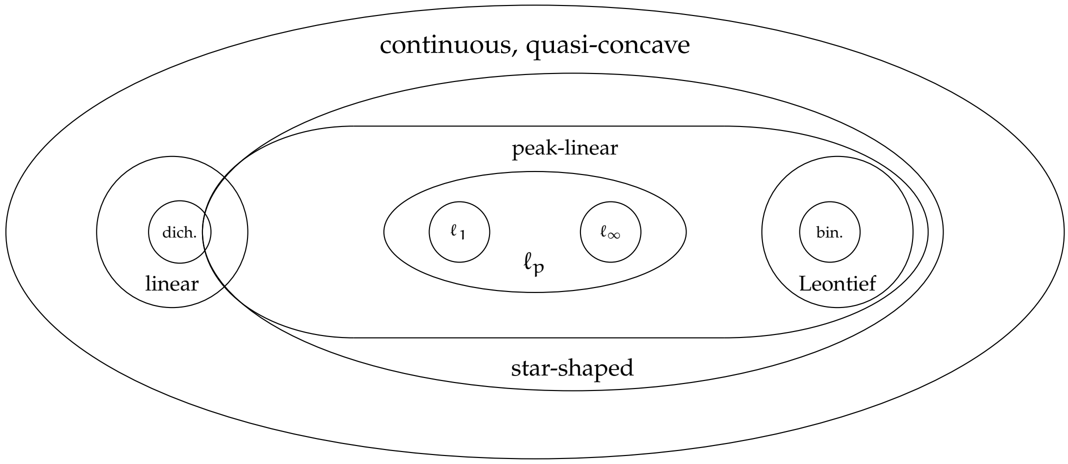


Figure 1.1: Overview of considered classes of utility functions over the probability simplex. Note that linear utilities do not necessarily have a unique peak which is why they are not star-shaped in general. The same holds for the subclass of dichotomous (dich.) utilities. Furthermore, the peaks for binary (bin.) and general Leontief utilities are assumed to be scaled down to distributions.

From General to Specific Preference Models

Results from Debreu (1952) and Scarf (1971) imply the existence of equilibrium and core outcomes for continuous and convex preferences. However, their proofs are non-constructive and do not offer many insights into the structure of such outcomes for specific preferences. Foley (1970) established a useful connection between the core and *Lindahl equilibrium* that was used by Fain et al. (2016) to compute core outcomes (see Chapter 6). In Chapter 7, we relate the unique equilibrium for *Leontief utilities* (Theorem 5.10) to equilibria of other utility models.

However, returning equilibrium distributions is not always necessary to achieve fair outcomes. In fact, for *linear* utilities (Chapter 4), equilibrium distributions might not be Pareto optimal (see Example 2.11). Contrarily, they are unique and define a sensible mechanism for Leontief utilities that maximizes the product of agents' utilities (weighted by their donations for the donor coordination setting). This so-called *Nash product rule* satisfies remarkable efficiency and fairness properties, also for linear utilities. In particular, it returns core outcomes for both utility models (Chapter 6) and is even group-strategyproof for Leontief utilities.

For linear utilities, however, Hylland (1980) showed that efficiency, strategyproofness, and fairness cannot be satisfied simultaneously. The (in-)compatibility of these three properties is another central theme of this thesis, and we show in Chapter 11 that also ℓ_1 or ℓ_∞ preferences lead to impossibilities, unless the probability simplex is one-dimensional (Chapter 10).

Computation and Dynamics

We are not only concerned with the determination of fair and efficient outcomes but also with their computation. For linear utilities, an outcome of the Nash product rule can be approximated to arbitrary precision (the actual outcome might be irrational-valued) via convex programming (see, e.g., Boyd and Vandenberghe, 2004). For Leontief utilities, we prove that the returned outcome is rational-valued

(Lemma 5.19) and can be computed in (pseudo-)polynomial time by leveraging results from Jain (2007).

Furthermore, we propose natural dynamics for both utility models that mirror the respective needs for coordination in Sections 4.2.2 and 5.2.3. For linear utilities, we consider proportional best response dynamics where an agent distributes her individual donation proportional to the utility she receives from each project under the current overall distribution. In this way, a project that is popular among donors usually receives high contributions, and agents donate to it even if it is not their favorite project. These dynamics approximate outcomes of the Nash product rule arbitrarily well in the limit (Theorem 4.21). For Leontief utilities, less coordination is needed, which is illustrated by the fact that best response dynamics converge to the unique equilibrium, i.e., the outcome of the Nash product rule (Theorems 5.24 and 5.29).

Apart from providing an alternative and more insightful method of computing outcomes of the Nash product rule, they are also very important from a practical point of view. First, such dynamics enable a decentralized computation of desirable outcomes where agents update their individual donations according to a simple and comprehensible rule. Second, it is possible to recover the distribution of individual donations from the overall distribution of all donations. Thus, each agent is able to observe the impact of her own donation. Together with its strong fairness guarantees, the Nash product rule becomes a convincing candidate for coordinating donations in practice.

With this thesis, we hope to propose and justify methods to "reach unity in diversity" and contribute to the "beauty" of collective decision-making.

ORGANIZATION OF THIS THESIS

Chapter 2 introduces our theoretical framework, which covers both the donor coordination as well as the portioning setting. These two settings form the two main parts of this thesis. Both parts start with an introduction to the respective setting (Chapters 3 and 9) and conclude with a discussion on the obtained results and directions for future research (Chapters 8 and 12).

Chapters 4 and 5 investigate two different utility models in donor coordination but exhibit a similar structure, at least partially. First, we examine both settings from a game-theoretic perspective. Second, suitable mechanisms are defined and investigated axiomatically. Third, simplified models are explored. Chapter 6 interprets donor coordination in the context of public good markets. Chapter 7 generalizes results to other utility functions and welfare notions.

The remaining chapters on portioning contain a thorough investigation of the problem in the case of two pure alternatives (Chapter 10) as well as impossibility results for specific utility models and higher-dimensional probability simplices (Chapter 11).

UNDERLYING ARTICLES AND ADDITIONAL ACKNOWLEDGMENTS

This thesis is based on the publications and working papers listed on page ix. I would like to thank again all of my co-authors. I further thank Fedor Nazarov for suggesting the proof technique for Theorem 4.10, Martin Bullinger for pointing out a more compact proof for Example 4.15 (Proposition 2 in [1]), Florian Brandl for proposing the redistribution dynamics (Section 5.2.3) together with a proof idea, Igal Milchtaich for pointing out fruitful connections to congestion games used in the proof of Theorem 5.24, and Ido Dagan for suggesting leximin Leontief preferences (Section 7.3). Furthermore, I am grateful to all reviewers of these papers and to all participants of events where these papers were presented for their constructive feedback.

Chapter 3 is in parts based on [1] and [2], Chapter 4 is largely based on [1], Chapter 5 is based on [2] and [3], Chapter 6 contains ideas from [2], and Chapter 7 is based on [2]. Chapter 8 is partially based on [1] and [2]. Chapters 9 to 11 are based on [3] and Chapter 12 is partially based on [3]. Large parts were restructured and significantly extended in order to adapt notation, simplify proofs, and relate results from different papers to one another.

2 FOUNDATIONS

This chapter serves as both an overview of results on continuous and convex preferences and an introduction to the two main settings we want to consider: *donor coordination* and *portioning*. After formally defining continuous and convex preference relations on our set of admissible alternatives, we cautiously move from ordinal to cardinal preferences and discuss the advantages and disadvantages of that approach in Section 2.1. Next, a game-theoretic perspective on social choice is taken with the help of our donor coordination setting. En passant, portioning is interpreted as a special case of it. Section 2.3 defines and classifies axioms for analyzing aggregation mechanisms. Tradeoffs between axioms, one of the main themes of this thesis, are stated for these general preferences in Section 2.4.

Let A be a finite set of $m = |A| \in \mathbb{N}$ *pure alternatives* (e.g., public projects). Given some amount $C \in \mathbb{R}_{>0}$, a *distribution* $\delta = (\delta_x)_{x\in A}$ of C is a vector in $\mathbb{R}^m_{\geqslant 0}$ with $\sum_{x\in A} \delta_x = C$ that describes how C is distributed among pure alternatives. For $C = 1$, distributions correspond to probability distributions (also called lotteries), and the set of alternatives becomes the probability simplex over A. In general, we do not want C to be fixed from the beginning but rather consider a whole interval $[\underline{C}, \bar{C}]$ with $\bar{C} \geqslant \underline{C} \geqslant 0$ and distributions with $\sum_{x\in A} \delta_x \in [\underline{C}, \bar{C}]$. The set of all such distributions, which we denote by $\Delta([\underline{C}, \bar{C}])$ (or $\Delta(C)$ if $\underline{C} = C = \bar{C}$), forms the set of *alternatives* which we will also call *distributions*. For $A' \subseteq A$, define $\delta(A') = \sum_{x\in A'} \delta_x$.

Consider the metric space $(\mathbb{R}^m, d)$ where d is induced by the ℓ_1 norm $\|\cdot\|_1$, i.e., $d(x, y) = \sum_{k=1}^m |x_k - y_k|$ for $x, y \in \mathbb{R}^m$. By definition, $(\Delta([\underline{C}, \bar{C}]), d')$, where d' is the metric induced by d, is also a complete[3] metric space. Furthermore, $(\Delta([\underline{C}, \bar{C}]), d')$ is separable[4] and connected[5].

Let N be a finite set of n agents. We assume that each agent $i \in N$ has a complete[6] and transitive[7] binary relation $\succsim_i$ over $\Delta([\underline{C}, \bar{C}])$, also called a *preference relation*. Furthermore, preferences $\succsim$ are assumed to be continuous in the sense that for every $\delta^* \in \Delta([\underline{C}, \bar{C}])$, the upper and lower contour sets $\{\delta \in \Delta([\underline{C}, \bar{C}]) \mid \delta \succsim \delta^*\}$ and $\{\delta \in \Delta([\underline{C}, \bar{C}]) \mid \delta \precsim \delta^*\}$ are closed. An equivalent and, at least according to Debreu, more intuitive definition of continuity requires that for any sequence $(\delta^k)_{k\in\mathbb{N}}$ in $\Delta([\underline{C}, \bar{C}])$ with $\lim_{k\to\infty} \delta^k = \delta^* \in \Delta([\underline{C}, \bar{C}])$ and any $\delta' \in \Delta([\underline{C}, \bar{C}])$ with $\delta^k \succsim \delta'$ ($\delta^k \precsim \delta'$) for all k, it holds that $\delta^* \succsim \delta'$ ($\delta^* \precsim \delta'$).

A preference relation $\succsim$ is represented by a utility function $u : \Delta([\underline{C}, \bar{C}]) \to \mathbb{R}$ if for any $\delta, \delta' \in \Delta([\underline{C}, \bar{C}])$, it holds that $\delta \succsim \delta'$ if and only if $u(\delta) \geqslant u(\delta')$.

3 A metric space (X, d) is complete if the limit of every Cauchy sequence in X is also an element of X. This property ensures that whenever we talk about a sequence in $(\Delta([\underline{C}, \bar{C}])$ that converges, its limit is also in $(\Delta([\underline{C}, \bar{C}])$.

4 A metric space (X, d) is separable if it contains a countable and dense subset of X. For $(\Delta([\underline{C}, \bar{C}]), d')$, all rational-valued distributions form such a dense subset.

5 A metric space (X, d) is connected if it cannot be partitioned into two disjoint, non-empty, and open sets. As $\Delta([\underline{C}, \bar{C}]) \subset \mathbb{R}^m$ is convex, $(\Delta([\underline{C}, \bar{C}]), d')$ is connected.

6 A binary relation $\succsim$ is complete if, for any $\delta, \delta' \in \Delta([\underline{C}, \bar{C}])$, $\delta \succsim \delta'$ or $\delta \precsim \delta'$.

7 A binary relation $\succsim$ is transitive if, for any $\delta, \delta', \delta'' \in \Delta([\underline{C}, \bar{C}])$, $\delta \succsim \delta'$ and $\delta' \succsim \delta''$ implies $\delta \succsim \delta''$.

The following theorem establishes that, in our case, agents' preferences can indeed be represented by utility functions.

THEOREM 2.1 *(Debreu (1954, Theorem 1))*
Let $\succsim$ be a complete, transitive, and continuous preference relation on a separable and connected metric space (X, d). Then, $\succsim$ can be represented by a real-valued, continuous function on X.

In fact, Debreu proved the above theorem for the more general class of topological spaces.

We further assume preferences to be *convex*, i.e., for all $\delta, \delta', \delta^* \in \Delta([\underline{C}, \bar{C}])$ with $\delta \succsim \delta' \succsim \delta^*$, it holds that $\lambda\delta + (1-\lambda)\delta' \succsim \delta^*$ for all $\lambda \in [0, 1]$.[8] In words, any mixture of the two preferred distributions δ and δ' is still preferred to δ^*. Equivalently, all upper contour sets $\{\delta \in \Delta([\underline{C}, \bar{C}]) \mid \delta \succsim \delta^*\}$ are required to be convex.

This is equivalent to requiring quasi-concavity[9] of any utility function u representing $\succsim$ which can be seen by setting $\delta^* = \delta'$ in the definition of convex preferences.

Convexity, as well as completeness, transitivity, and continuity of preferences, are standard assumptions in economic theory (see, e.g., Mas-Colell et al., 1995), which does not mean that they should not be challenged.[10]

The representation of $\succsim$ via some utility function u is far from unique. In fact, for any continuous and increasing[11] function $g\colon \mathbb{R} \to \mathbb{R}$, $u(\delta) \geqslant u(\delta')$ if and only if $g(u(\delta)) \geqslant g(u(\delta'))$, showing that $g(u)$ represents $\succsim$ equally well. Therefore, in our opinion, when moving from ordinal preferences to specific utility functions, both advantages and disadvantages need to be addressed.

2.1 FROM ORDINAL PREFERENCES TO UTILITY FUNCTIONS — OPPORTUNITIES AND RISKS

From a purely mathematical point of view, utility functions are, in general, easier to handle than ordinal preferences in the form of binary relations: Changes in the utility of different agents can be quantified and used during aggregation as a justification for choosing certain outcomes by comparing these changes, i.e., engaging in *interpersonal comparisons of utility*. So why should one even bother with ordinal preferences?

To answer that question and understand modern points of view, we start by giving a brief historical sketch of the foundations and development of *welfare economics* based on Mandler (1999) and Sen (2017). The interested reader is advised to consult their works for more details.

In its infancy around 1900, welfare economics were influenced by the work of Bentham, who assumed agents' happiness (or utilities) to be cardinally measurable and argued that society should always aim at maximizing the sum of utilities of

8 More generally, a set $S \subseteq \mathbb{R}^n$ is convex if $\lambda x + (1-\lambda)y \in S$ for all $x, y \in S$ and all $\lambda \in [0, 1]$.
9 A (utility) function $u\colon \Delta([\underline{C}, \bar{C}]) \to \mathbb{R}$ is quasi-concave if $u(\lambda\delta + (1-\lambda)\delta') \geqslant \min\{u(\delta), u(\delta')\}$ for all $\delta, \delta' \in \Delta([\underline{C}, \bar{C}])$ and $\lambda \in [0, 1]$.
10 See, e.g., Daskalakis (2025) for a critique of convexity, Aumann (1962) for a critique of completeness, and May (1954); Sen (1969); Blair and Pollak (1982) for a critique of transitivity and continuity.
11 By "increasing", we mean *strictly* increasing.

its members (*greatest happiness principle*). Economic theory at that time did not use cardinal utilities to represent ordinal preferences, as it is often the case nowadays, but rather to quantify (changes in) happiness directly and maximize the sum of utilities as a consequence.

While the goal of maximizing that total welfare itself can be questioned (see Fleurbaey et al. (2008) for a thorough discussion of *utilitarianism*), Robbins (1938) and others argued that there is an inherent problem with interpersonal utility comparisons: If two agents assign the same value to an outcome, this does not automatically imply that both experience the same amount of happiness from it. There is no "common denominator of feeling" (Robbins, 1938, p. 636). This led to the rejection of cardinal utilities as a whole, and welfare economics started to concentrate on ordinal preferences. To quote Arrow (1951, p. 9), "[...] the only meaning the concepts of utility can be said to have is their indications of actual behavior, and, if any course of behavior can be explained by a given utility function, it has been amply demonstrated that such a course of behavior can be equally well explained by any other utility function which is a strictly increasing function of the first. If we cannot have measurable utility, in this sense, we cannot have interpersonal comparability of utilities a fortiori."

Without that possibility, other axioms, most prominently Pareto efficiency, were used to evaluate outcomes, ultimately leading to *Arrow's impossibility theorem* (Arrow, 1951), which shows that preferences cannot be aggregated without violating at least one axiom from a small set of seemingly mild conditions.

However, one crucial and sometimes overlooked assumption is that there are no restrictions on the agents' ordinal preferences (apart from being complete and transitive). An agent that strongly prefers a over b but only slightly prefers b over c cannot report these intensities via $\succsim$. For many applications, especially for a large or even infinite number of alternatives[12], requiring some additional structure of the preferences seems to be sensible, not to say natural.

This is where cardinal utility functions come back into play. For the case when lotteries over pure alternatives are not realizable *ex post* but still need to be compared *ex ante*, von Neumann and Morgenstern showed that agents are *expected utility maximizers*, i.e., compare lotteries via amounts of expected utility when requiring two additional preference axioms. These *von Neumann-Morgenstern utility functions* (sometimes simply referred to as *linear utility functions*) are predominant for such settings in modern economic theory, not least because they are completely specified by their utilities for pure alternatives.

There is always a tradeoff between expressivity and simplicity of agents' preferences. On the one hand, utility models should be powerful enough to represent agents' true preferences reasonably well and explain real-world behavior. On the other hand, agents cannot report valuations for or make comparisons between a large or infinite set of alternatives. Due to the enormous computational power available nowadays, we can aggregate very complex preferences in very complex ways. Paradoxically, using such models might lead to worse results. First, complex utility models are prone to *overfitting*, a well-known phenomenon in machine learning. Loosely speaking, too much importance is assigned to a particular utility function, and another cardinal representation of the same preferences might

12 For example, consider the set of all lotteries over a finite set of alternatives.

result in a completely different outcome. Second, agents should always be able to comprehend their influence on the outcome and its dependence on their reports. This understanding increases acceptance of both the outcome and the aggregation method but becomes more difficult with increasing complexity. In the following chapters, we will further specify and investigate utility models that balance both goals well.

We agree with Mandler that "[...] when properly targeted, nonordinal theorizing can provide plausible psychological foundations for preference theory" (Mandler, 1999, p. 110). Of course, using cardinal utilities comes with the tempting prospect of doing interpersonal utility comparisons. Interestingly, Sen explained that such comparisons can be naturally integrated into the aggregation method by requiring certain *invariance conditions* that induce classes of utility functions for which the aggregation method returns the same outcome. The possibility of making interpersonal utility comparisons is a byproduct rather than a motivation for aggregation methods. He further argued that "[t]he rejection of interpersonal comparisons of utilities in welfare, and in social choice theory [...] was firmly based on interpreting them entirely as comparisons of mental states" (Sen, 2017, p. 22). When comparing, e.g., timeshares or monetary distributions as we do in donor coordination, money or time itself can take the role of the common denominator. Nevertheless, we will state our results for as general utility models as possible and sometimes even become completely independent of the utility function representing $\succsim$ again.

2.2 A GAME-THEORETIC PERSPECTIVE ON SOCIAL CHOICE

Game theory provides a suitable framework for our purposes. Let N be a set of $n = |N| \in \mathbb{N}$ agents and $\mathcal{U}$ be a set of continuous and quasi-concave utility functions $u\colon \Delta([\underline{C}, \bar{C}]) \to \mathbb{R}$. Each agent $i \in N$ has a utility function $u_i \in \mathcal{U}$ representing her preferences over $\Delta([\underline{C}, \bar{C}])$.

A *utility profile* $U = (u_i)_{i \in N}$ specifies a utility function for each agent $i \in N$ where $U(\delta) = (u_i(\delta))_{i \in N}$ denotes the vector of agents' utilities for distribution δ. The set of all utility profiles is denoted by $\mathcal{U}^n$. Similarly, for $N' \subseteq N$, a reduced utility profile $U_{-N'} = (u_i)_{i \in N \setminus N'}$ consists of utility functions for each agent $i \in N \setminus N'$.

First, we focus on donor coordination as that setting nicely illustrates our game-theoretic approach. There, A is interpreted as a set of public projects. Each agent $i \in N$ chooses a contribution C_i from an interval $[0, \bar{C}_i]$, i.e., the amount of money she is willing to contribute to A. Then, $\underline{C} = 0$ and $\bar{C} = \sum_{i \in N} \bar{C}_i$. Thus, $\Delta([0, \bar{C}])$ is formed by all distributions of possible money over projects. Note that we are talking about *public* projects, meaning that agents also benefit from the contributions of others and, in particular, only care about the overall distribution, not the origin of other donations.

A *contribution profile* $C = (C_i)_{i \in N}$ specifies a contribution for each agent. The set of all contribution profiles is denoted by $\mathcal{C} = \prod_{i \in N} [\underline{C}_i, \bar{C}_i]$. Furthermore, for $N' \subseteq N$, $\mathcal{C}^{N'}_{>0}$ denotes the set of contributions where each agent in N' has a *positive* contribution and all other agents contribute zero. We interpret agents with zero contribution as non-participants, i.e., their preferences are not taken into account during aggregation. Consequently, properties of distributions (Section 2.3) only need to hold with respect to all agents with strictly positive contributions, see also Remark 1. For a contribution profile C and $N' \subseteq N$, define $C_{N'} = \sum_{i \in N'} C_i$.

A *profile* $P = (U, C)$ consists of a utility profile and a contribution profile. The set of all profiles is denoted by $\mathcal{P} = \mathcal{U}^n \times \mathcal{C}$.

EXAMPLE 2.2
As an example, consider the following profile with $A = \{a, b, c\}$ and two agents having linear utility functions $u_i(\delta) = \sum_{x \in A} \nu_{i,x} \delta_x$ where valuations and contributions are specified in the following table.

	$\nu_{i,a}$	$\nu_{i,b}$	$\nu_{i,c}$	C_i
Agent 1	1.5	1	0	1
Agent 2	0	1	1.5	1

DEFINITION 2.3
A *mechanism* $f\colon \mathcal{P} \to \Delta([\underline{C}, \bar{C}])$ maps each profile (U, C) to a distribution $f(U, C)$[13] $= \delta \in \Delta(C_N)$.

Throughout this thesis, we will sometimes fix the contribution or the utility profile. With a slight abuse of notation, we then consider accordingly restricted mechanisms and write $f(U) = f(U, \cdot)$ and $f(C) = f(\cdot, C)$, respectively.

For the remainder of this section, let (U, C) be an arbitrary but fixed profile.

By construction, any distribution $\delta \in \Delta(C_N)$ can be decomposed into *individual distributions* $\delta_i = (\delta_{i,x})_{x \in A}$ where $\delta_{i,x}$ can be interpreted as agent i's contribution to project x.

DEFINITION 2.4
A *decomposition* of a distribution $\delta \in \Delta(C_N)$ is a collection of individual distributions $(\delta_i)_{i \in N}$ with $\delta_i = (\delta_{i,x})_{x \in A} \in \Delta(C_i)$ such that $\delta = \sum_{i \in N} \delta_i$.

Moreover, $\delta_x = \sum_{i \in N} \delta_{i,x}$ apparently holds for each $x \in A$. Of course, decompositions are not at all unique in general. Looking at decompositions from the opposite direction, individual distributions form the overall distribution and can be analyzed from a game-theoretic perspective.

DEFINITION 2.5
A *game* is a tuple $(N, S, (u_i)_{i \in N})$ where N is the set of agents, S denotes the set of strategy profiles and each agent i has a utility function $u_i\colon S \to \mathbb{R}$.

Thus, a donor coordination instance can be interpreted as a game where each agent i decides how to distribute C_i over A, i.e., $S_i = \Delta(C_i)$ constitutes her set of strategies and $S = \bigtimes_{i \in N} \Delta(C_i)$. For $N' \subseteq N$, denote by $S_{-N'} = \bigtimes_{i \in N \setminus N'} \Delta(C_i)$ the set of strategy profiles for agents in $N \setminus N'$ consisting of distributions $\delta_{-N'}$. The set of best responses $B_i(\delta_{-i})$ of agent i to δ_{-i} consists of all $\delta_i^* \in \Delta(C_i)$ such that $u_i(\delta_{-i} + \delta_i^*) \geqslant u_i(\delta_{-i} + \delta_i)$ for all $\delta_i \in \Delta(C_i)$.

PROPOSITION 2.6
$B_i(\delta_{-i})$ is convex for all $i \in N$ and $\delta_{-i} \in S_{-i}$.

13 We shorten notation and write f(U,C) instead of f((U,C)).

Proof. For arbitrary $i \in N$ and $\delta_{-i} \in S_{-i}$, assume $\delta_i, \delta_i' \in B_i(\delta_{-i})$. For all $\lambda \in [0,1]$, $u_i(\delta_{-i} + \lambda\delta_i + (1-\lambda)\delta_i') \geqslant \min\{u_i(\delta_{-i} + \delta_i), u_i(\delta_{-i} + \delta_i')\}$ by quasi-concavity of u_i. Thus, $\lambda\delta_i + (1-\lambda)\delta_i' \in B_i(\delta_{-i})$ and the inequality becomes an equality as $\delta_i, \delta_i' \in B_i(\delta_{-i})$. □

Note that best responses are independent of the cardinal representation of the underlying preferences.

An interesting set of distributions are those constellations where no agent has an incentive to deviate from her individual distribution, i.e., distributions δ^* such that $\delta_i^* \in B(\delta_{-i}^*)$ for all $i \in N$.

Definition 2.7
A distribution $\delta^* \in \Delta(C_N)$ is an *equilibrium distribution* if it admits a decomposition $(\delta_i^*)_{i \in N}$ such that $u_i(\delta^*) \geqslant u_i(\delta^* - \delta_i^* + \delta_i)$ for all agents $i \in N$ and $\delta_i \in \Delta(C_i)$.

Such decompositions are equilibria in the original game-theoretic sense (Nash, 1950a; Debreu, 1952).

Example 2.8
In the profile from Example 2.2, $\delta^* = (1,0,1)$ with decomposition $\delta_1^* = (1,0,0)$ and $\delta_2^* = (0,0,1)$ constitutes the unique equilibrium distribution. Note that agents with linear utilities are incentivized to contribute only to their favorite projects. This fact is formalized in Proposition 4.1.

Equilibrium distributions can also be defined for portioning problems. In that setting, $\underline{C} = \bar{C} = 1$, i.e., the set of alternatives consists of all lotteries over A. Although agents do not have contributions in the original sense, imagine that certain contributions or "decision power" are assigned to them. Here, we assume that each agent receives a virtual contribution of $C_i = 1/n$, leading us back to a donor coordination setting with fixed contributions. Thus, portioning mechanisms take only a utility profile as input.

A famous result from Debreu (1952) implies that equilibrium distributions always exist for both of our settings.

Theorem 2.9 *(adapted from Debreu (1952), Arrow and Debreu (1954, Lemma 2.5))*
Every game $(N, \bigtimes_{i \in N} \Delta(C_i), U)$ associated with a profile (U, C) admits an equilibrium distribution.

The existence of an equilibrium distribution is implied by the existence of an equilibrium $(\delta_i^*)_{i \in N}$ which was proven by Debreu (1952) for a larger class of games $(N, S, (u_i)_{i \in N})$. In detail, the set of possible strategies as well as sets of best responses $B_i(s_{-i})$ are assumed to be contractible for every agent i and all $s_{-i} \in S_{-i}$. In our settings, these sets are always convex and thus also contractible (see, e.g., Willard, 1970). In addition, Debreu (1952) allows for dependencies between individual strategies in the sense that an agent's set of admissible strategies might change depending on the chosen strategies of other agents.

Two related questions concern the uniqueness and complexity of computing equilibrium distributions (Rosen, 1965). We will see that answers heavily depend on the structure of specific utility functions and address these topics in the respective chapters. In particular, we postpone the discussion of Rosen (1965) to Section 7.1.

The concept of equilibrium has primarily been applied to *private good markets* where agents are assigned bundles of goods and only benefit from their own, private bundle (see, e.g., Arrow and Debreu, 1954; McKenzie, 1981). In contrast, our settings can be interpreted as a form of *public good markets* where agents can benefit from all "purchased" goods (see Chapter 6). There, equilibria seem less well understood but have become increasingly popular in many areas of social choice in recent years. Examples include the *method of equal shares* (Peters and Skowron, 2020; Peters et al., 2021) in participatory budgeting, the *independent markets mechanism* in portioning (Freeman et al., 2021), or fairness concepts like the *core* in multi-winner voting (Lackner and Skowron, 2023; Haret et al., 2024).

By Theorem 2.9, it is possible to define mechanisms that always return an equilibrium distribution. But why should we do that, or, more generally, how can we assess the quality of a mechanism? In the end, are Nash equilibria more than just fixed points?[14]

We address this criticism by looking at the problem from two different angles. First, we will take an algorithmic approach and consider certain dynamical systems, e.g., *best response dynamics*. There, agents update their strategies (either simultaneously or one after another) by playing best responses with respect to the current overall distribution. The limits of such dynamics are promising candidates for sensible outcomes. Second, we will investigate mechanisms from an axiomatic point of view.

2.3 AXIOMS

This section is dedicated to introducing and discussing desirable properties for mechanisms called *axioms* (see, e.g., Thomson (2023)). Investigating whether a mechanism satisfies certain axioms helps to judge its advantages and disadvantages and identify potential fields of application. Moreover, we are particularly interested in the interplay between axioms and aim at outlining the frontier of what is possible via *characterizations* of mechanisms and *impossibilities* for certain sets of axioms. The axioms for outcomes and mechanisms we want to consider can roughly be classified into three categories: efficiency,[15] robustness to manipulation, and fairness. Tradeoffs between these three kinds of axioms are a recurring theme in social choice (Figure 2.1). A fourth class of axioms concerns the consistency of mechanisms in how changes in the input (should) affect the outcome. As an example, *continuity* requires that small fluctuations in the profile result in only minor changes in the outcome.

Furthermore, we distinguish between *intraprofile* and *interprofile* conditions, a classification proposed by Fishburn (1973). Intraprofile axioms look at one profile at a time and impose restrictions on the set of possible distributions for that instance. Contrarily, interprofile axioms concern relations between outcomes for various profiles. This distinction becomes very convenient when thinking about combinations of axioms of various types. Efficiency and most fairness axioms are

14 This question might be in the spirit of John von Neumann's skepticism towards Nash equilibria (see, e.g., Holt and Roth, 2004).

15 When talking about efficiency, we mean Pareto efficiency.

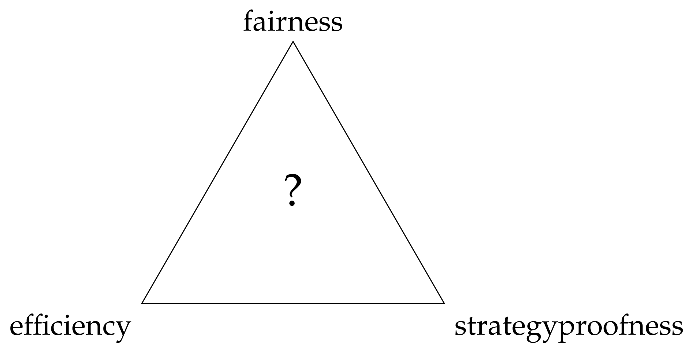


Figure 2.1: The Trinity of Decision-Making: efficiency, strategyproofness, and fairness.

intraprofile conditions, whereas some other fairness notions, strategyproofness, and consistency axioms usually belong to the class of interprofile conditions.

2.3.1 Efficiency

A central goal in collective decision-making is to produce optimal outcomes in the sense that synergies between agents' utilities are optimally taken into account.

DEFINITION 2.10
For a fixed profile (U, C), a distribution $\delta' \in \Delta(C_N)$ *(Pareto) dominates* another distribution $\delta \in \Delta(C_N)$ if $u_i(\delta') \geqslant u_i(\delta)$ for all $i \in N$ and $u_i(\delta') > u_i(\delta)$ for at least one $i \in N$.

A distribution $\delta \in \Delta(C_N)$ is *(Pareto) efficient* if it is not dominated by another distribution $\delta' \in \Delta(C_N)$.

A mechanism f satisfies (Pareto) efficiency if $f(U, C)$ is efficient for each profile $(U, C) \in \mathcal{P}$.

EXAMPLE 2.11
Consider the profile from Example 2.2. The distribution $\delta^* = (1, 0, 1)$ constitutes an equilibrium distribution (with decomposition $\delta^*_{1,a} = 1$ and $\delta^*_{2,c} = 1$) and yields utility 1.5 to both agents. However, it is Pareto dominated by the distribution $(0, 2, 0)$ that improves the utilities of both agents to 2.

2.3.2 Strategyproofness

When aggregating preferences (and contributions), agents should not have incentives to manipulate the outcome by misreporting utility functions, as such manipulations might lead to entirely different profiles and worse outcomes with respect to the agents' true preferences.

DEFINITION 2.12
A mechanism f satisfies *group-strategyproofness* if for all $N' \subseteq N$ and all $P = (U, C), P' = (U', C') \in \mathcal{P}$ with $U_{-N'} = U'_{-N'}$ and $C = C'$, either $u_i(f(P)) > u_i(f(P'))$ for at least one $i \in N'$ or $u_i(f(P)) = u_i(f(P'))$ for all $i \in N'$.

A mechanism satisfies *strategyproofness* if the above condition holds for $|N'| = 1$.

In donor coordination, agent i also reports a contribution $C_i \in [\underline{C}_i, \bar{C}_i]$ in addition to her utility function. However, unlike her utility function, there does not seem to be a "true" contribution. Therefore, changes in contribution should be interpreted as admissible choices rather than manipulations and are covered in the subsection on consistency.

2.3.3 Fairness

In contrast to efficiency and strategyproofness, there does not seem to be *the one* fairness notion. Instead, the literature has identified a plethora of axioms of varying kinds. In this subsection, we state fairness properties for general, convex preferences while deferring the definition of other axioms for specific utility models to the respective sections.

The first fairness axiom to be considered is the *core* which originates from cooperative game theory (Scarf, 1967). The following definition is adapted from Scarf (1971) and due to Aumann (1961).

Definition 2.13
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ is in the *core* if there does not exist a subset of agents $N' \subseteq N$ and a distribution $\delta' \in \Delta(C_{N'})$ such that $u_i(\delta' + \tilde{\delta}) > u_i(\delta)$ for all $i \in N'$ and for all $\tilde{\delta} \in \Delta(C_{-N'})$.

In other words, a distribution is in the core if no coalition of agents has an incentive to deviate from the proposed distribution regardless of how the other agents redistribute their contributions. Most works related to and used in our donor coordination setting (e.g., Foley, 1970; Fain et al., 2016) require $u_i(\delta') > u_i(\delta)$ instead. At first sight, this might not seem sensible as agents in N' potentially benefit from the other agents' contributions. However, at least for the considered models, we can always assume the existence of an additional project that all agents value at zero, i.e., they do not gain any utility from the amount of contributions assigned to that project. Nevertheless, in the portioning setting, $u_i(\delta')$ is not well-defined, so we stay with the more general definition, noting that all results in the donor coordination setting hold for both versions.

Maybe surprisingly, it can be shown that the core is always non-empty.

Theorem 2.14 *(adapted from Scarf (1971))*
For every profile (U, C), the core is non-empty.

Core outcomes ensure that each subset of agents is well represented. Inspired by Bogomolnaia et al. (2002) and Aziz et al. (2020), this property can be weakened by restricting N' to coalitions of size 1.

Definition 2.15
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ satisfies *individual fair share* if there does not exist an agent $i \in N$ and a distribution $\delta' \in \Delta(C_i)$ such that $u_i(\delta' + \tilde{\delta}) > u_i(\delta)$ for all $\tilde{\delta} \in \Delta(C_{-i})$. Call $\max_{\delta \in \Delta(C_i)} \min_{\tilde{\delta} \in \Delta(C_{-i})} u_i(\delta + \tilde{\delta})$ agent i's individual fair share, i.e., the amount of utility agent i can achieve by herself independent of other agents.

A mechanism satisfies individual fair share if it returns such a distribution for each profile $(U, C) \in \mathcal{P}$.

Thus, individual fair share provides fairness guarantees for individual agents. Notably, equilibrium distributions satisfy individual fair share as no agent has an incentive to change her individual distribution in a corresponding decomposition.

An even weaker property called *positive share* (Bogomolnaia et al., 2002, 2005)[16] only requires that each agent somehow benefits from the chosen distribution.

Definition 2.16
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ satisfies *positive share* if there does not exist an agent $i \in N$ with $u_i(\delta) = \min_{\delta' \in \Delta(C_N)} u_i(\delta')$ and an individual fair share larger than $\min_{\delta' \in \Delta(C_N)} u_i(\delta')$.
A mechanism satisfies positive share if it returns such a distribution for each profile $(U, C) \in \mathcal{P}$.

The latter condition prevents unjustified violations of positive share by agents with, e.g., constant utility functions.
All in all,

$$\text{core} \implies \text{individual fair share} \implies \text{positive share.}$$

This hierarchy of fairness notions will help us assess the strength of other fairness axioms and compare them.

Example 2.17
The equilibrium distribution $\delta^* = (1, 0, 1)$ from Example 2.2 satisfies individual fair share. However, it is not in the core as the coalition consisting of both agents can deviate to the distribution $(0, 2, 0)$ that increases both utilities.

Note that all previously defined axioms are independent of the actual utility representation and depend solely on the underlying preferences.
Finally, the well-known axiom *anonymity* requires that agents' identities do not matter for the outcome.

Definition 2.18
A mechanism f satisfies *anonymity* if for every profile $P \in \mathcal{P}$ and permutation π of the agents in P, it holds that $f(P) = f(\pi \circ P)$.

2.3.4 Consistency

For utility models that will be investigated in the following chapters, each agent is assumed to report a vector of cardinal valuations $v_i = (v_{i,x})_{x \in A}$ that completely specifies her utility function $u_i \in \mathcal{U}$. Naturally, these vectors admit different interpretations depending on the underlying utility model. Small perturbations of the reports due to agents' uncertainties about their exact valuations should not significantly impact the outcome. This property is called *continuity*. Similarly, in the donor coordination setting, small changes in contribution should not largely affect the outcome.
Monotonicity axioms regarding the agents' preferences and contributions concern the "direction" in which the outcome changes when reports change. For example, it seems plausible that an increase in valuation for a project should not decrease the amount of received contribution.
As these notions vary across the donor coordination and portioning setting, they are defined later in Sections 3.2 and 9.2, respectively.

16 Bogomolnaia et al. (2002, 2005) and Aziz et al. (2020) introduced positive share and individual fair share for dichotomous utilities (Section 4.3). We generalize these notions to arbitrary preferences.

2.4 A GENERAL IMPOSSIBILITY

Returning to the "trinity of decision-making", Zhou (1991) implicitly showed the incompatibility of efficiency, strategyproofness, and fairness for continuous and convex preferences by characterizing the set of all strategyproof mechanisms.

DEFINITION 2.19
A mechanism f is *dictatorial* or a *dictatorship* if there exists an agent $i \in N$ such that $u_i(f(P)) = \max_{\delta \in f(\mathcal{P})} u_i(\delta)$ for all $P \in \mathcal{P}$ where $f(\mathcal{P})$ denotes the image of f.

THEOREM 2.20 *(Zhou (1991))*
For $m \geqslant 3$, any strategyproof mechanism is dictatorial.

As Zhou writes himself, this theorem seems to be the analog of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) from voting.

Barberà and Peleg (1990) showed a similar theorem for continuous preferences. Note that both statements are incomparable. On the one hand, Zhou's result implies the one from Barberà and Peleg (1990) for convex preferences but does not make any statements about continuous but non-convex preferences. On the other hand, Barberà and Peleg (1990)'s proof makes explicit use of non-convex preferences and consequently does not apply to convex preferences. This illustrates that, although an impossibility holds on a large preference domain, restricted domains might allow for more positive results.

Zhou initially shows Theorem 2.20 for the smaller domain of quadratic preferences[17] and stresses that "an important point is that such a preference space should be rich enough so that it is closed under any nonsingular transformation". The classes of utility functions we want to consider in the subsequent chapters do not provide this richness which immediately follows from the fact that usually other, non-dictatorial but strategyproof mechanisms exist, e.g., the *conditional utilitarian rule* for dichotomous preferences (Duddy (2015) and Section 4.3) or the independent markets mechanisms for ℓ_1 preferences (Freeman et al., 2021).

17 Quadratic preferences can be represented by a utility function $u(\delta) = -(\delta - p)^T Q(\delta - p)$ where $Q \in \mathbb{R}^{m \times m}$ is a positive definite matrix and $p \in \Delta(\underline{C}, \bar{C})$.

Part I

DONOR COORDINATION

3 INTRODUCTION TO DONOR COORDINATION

Imagine a group of four friends coming together to discuss how to distribute donations among a set of three projects. Two of them want to donate €200 whereas each of the other half has set aside €100. Furthermore, each of them approves only a subset of the three projects.

Example 3.1
This scenario is summarized in the following profile with a set of three projects $\{a, b, c, d\}$ and the four agents. If agent i approves project x, $v_{i,x} = 1$, otherwise $v_{i,x} = 0$.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	C_i
Agent 1	1	0	0	200
Agent 2	1	1	0	200
Agent 3	1	1	0	100
Agent 4	1	0	1	100

They discuss various possible distributions of their entire budget of €600. The first idea is to allocate €200 to each project. However, one of the first three agents is probably not satisfied as she has to donate part of her contribution to a project she does not approve. Noting that they all approve project a, they could donate their entire budget to project a. In fact, this leads to the unique, efficient distribution when agents' utilities are *linear*. Loosely speaking, each agent only cares how much money is allocated to her approved projects. Such utility functions are sensible when projects can be interpreted as *substitutes*, e.g. when they correspond to three public places for social activities where the four friends can meet. If the projects correspond to charities, that distribution seems less optimal as Agents 2, 3, and 4 are probably quite unhappy as one of their approved charities receives no funding. Moreover, they are willing to move part of their contributions to charities b and c, respectively. This shows that charities should be seen as *complements* rather than substitutes, and we model agents' preferences with the help of Leontief utilities.

This example illustrates the central challenge in donor coordination: finding an efficient distribution of the total budget while respecting the will of individual donors.

On a larger scale, government programs like *cinque per mille* in Italy or *mechanizm 1%* in Poland[18] allow citizens to reallocate part of their personal income tax to nonprofit organizations. In 2023, cinque per mille collected a record of more than €520 million for about 81,000 organizations. Other examples include employee

18 In fact, Poland increased the quota of personal income tax that can be allocated directly to charity from 1% to 1.5% in 2022.

matching programs where companies double their employees' donations to charity. Typically, a donor has to allocate her entire contribution to one project. As a result, large organizations accumulate a great portion of the donations, whereas smaller but still popular projects are often left almost empty-handed.[19]

Enabling citizens to report a set of approved organizations instead and coordinating the distributions of their individual budgets has the potential to increase welfare and incentivize participation in such programs.

3.1 RELATED WORK

DONOR COORDINATION. To the best of our knowledge, Conitzer and Sandholm (2004, 2011) were the first to consider donor coordination from a mechanism design perspective. They introduce a bidding language that allows each agent to condition her contribution on the utility she receives from the distribution of the total budget. Furthermore, they investigate the computational complexity of determining optimal allocations and individual contributions based on the agents' bids and utility functions.

Other works (e.g., Buterin et al., 2019; Wagner and Meir, 2023) achieve stability of the outcome via taxation. In detail, Buterin et al. (2019) proposed a mechanism that determines an allocation where no agent has an incentive to change her individual contribution. However, the sum of contributions usually does not equal the money spent by the allocation, and balance is achieved by introducing taxes on the agents, which are assumed to not influence their utilities. Such a stability notion is motivated by the literature on the *private provision of public goods* (see Chapter 6), where public projects compete with using the money for private purposes. Wagner and Meir (2023) obtained a strategyproof and utilitarian welfare maximizing rule by adapting a Vickrey-Clarke-Groves mechanism (see, e.g., Nisan et al., 2007), again by taxing agents. In contrast to their models, we assume that agents have already set aside some amount of money they would like to donate.

For our donor coordination setting, Brandl et al. (2019) give an overview of sensible distribution rules for *dichotomous preferences* and investigate them axiomatically. Aziz and Ganguly (2021) assume projects to have fixed costs, which is reminiscent of problems from *participatory budgeting* that will be discussed in the context of portioning in Chapter 9.

NASH PRODUCT RULE FOR LINEAR UTILITIES. The idea of maximizing the product of agents' utilities fascinates by its efficiency and fairness guarantees for many problems in social choice theory. Originating from the *Nash bargaining solution* (Nash, 1950b), the Nash product rule is nowadays seen as a compromise between maximizing utilitarian and egalitarian welfare (Moulin, 1988) and is applicable not only to public but also *private* goods.

When private goods are indivisible, i.e., each good can be allocated to only one agent, Caragiannis et al. (2019) showed that the Nash product rule returns an efficient allocation that satisfies *envy-freeness up to one good* (Lipton et al., 2004;

19 See https://pl.wikipedia.org/wiki/Przekazywanie_1%25_podatku_dochodowego_na_rzecz_organizacji_po%C5%BCytku_publicznego_w_Polsce and the references therein for criticism of the Polish program in that regard.

Budish, 2011). In fact, this mechanism is the only welfarist rule that satisfies envy-freeness up to one good (Yuen and Suksompong, 2023). For the case of divisible private goods, the Nash product rule selects the set of all *competitive equilibria from equal incomes* (Eisenberg and Gale, 1959). Hence, it always returns efficient and *envy-free* allocations (Foley, 1967). Connections to such private good markets are explored in Chapter 8 after public good markets have been discussed in Chapter 6.

Returning to public goods, the Nash product rule is closely related to the core. Fain et al. (2016) showed that in the portioning setting, this mechanism returns core outcomes for a large class of utility functions (see Chapter 6 for a detailed discussion). Even for a finite set of alternatives, e.g., in committee voting (Lackner and Skowron, 2023), ideas similar to maximizing Nash welfare are used to approximate the core (see, e.g., Fain et al., 2018; Munagala et al., 2022).

Bogomolnaia et al. (2002, 2005) initiated the study of dichotomous preferences where agents distinguish only between approved and disapproved projects. Since then, this model has received a lot of interest (Duddy, 2015; Brandl et al., 2019; Aziz et al., 2020; Brandl et al., 2021b). Guerdjikova and Nehring (2014) considered that setting from the perspective of *judgment aggregation* (see, e.g., Genest and Zidek, 1986) and characterized the Nash product rule. Dichotomous preferences are covered in Section 4.3.

LEONTIEF UTILITIES. All previously mentioned models have in common that agents receive utility *separately* from each project. However, Example 3.1 illustrates that especially when projects are charities, this assumption should be challenged, and agents normally sympathize with those approved projects that receive low funding. Such preferences are captured by *Leontief utilities* (see, e.g., Varian, 1992; Mas-Colell et al., 1995). These types of utilities are mostly studied in resource allocation problems where they allow for surprisingly positive results regarding efficiency and strategyproofness (e.g., Nicoló, 2004; Ghodsi et al., 2011; Li and Xue, 2013). We defer their discussion to Chapter 8. As far as we know, Leontief utilities have not been considered before in the context of public goods.

In the remainder of this chapter, we repeat our donor coordination setting and give formal definitions for our consistency axioms.

Chapters 4 and 5 cover the two cases when projects are substitutes and complements, respectively. Accordingly, Chapter 4 assumes linear utility functions, whereas Leontief utilities are considered in Chapter 5. Both chapters follow the same structure. First, we investigate equilibrium distributions and efficiency, observing that for linear utilities, coordination among agents is necessary to obtain efficient distributions, while Leontief utilities always allow for efficient equilibrium distributions. Second, it is shown that maximizing the product of agents' utilities gives us strong fairness guarantees for both models. We address questions regarding their axiomatic properties, computability, and natural dynamics converging to such distributions. Third, we restrict valuations to $\{0, 1\}$ and discuss an alternative rule for linear utilities as well as alternative interpretations for Leontief utilities. Finally, we look at possibilities to characterize the Nash product rule for linear utilities and derive a new characterization for the corresponding mechanism with Leontief utilities.

Chapter 6 discusses relations to public good markets, and we show that known results on Lindahl equilibria can be extended to Leontief utilities.

Implications of our results for other utility models, e.g., Cobb-Douglas utilities, are described in Chapter 7. In particular, we prove that equilibrium distributions coincide for Leontief and Cobb-Douglas utilities and establish connections to other welfare notions.

Finally, we conclude the part on donor coordination with a discussion of our results and outline future research directions in Chapter 8.

3.2 PRELIMINARIES

Recall our donor coordination setting. Each agent $i \in N$ has a set of possible contributions $[0, \bar{C}_i]$ and a utility function $u_i \colon \Delta([0, \bar{C}]) \to \mathbb{R}$ representing her preferences over all distributions over the set of m alternatives A which are called *projects* in the following. A profile (U, C) consists of a utility function $u_i \in \mathcal{U}$ and a chosen contribution $C_i \in [0, \bar{C}_i]$ for each agent $i \in N$.

Until now, $u_i \in \mathcal{U}$ has been assumed to be continuous and quasi-concave. In the next section, further restrictions will be imposed on $\mathcal{U}$.

To account for the fact that projects are interpreted as *goods*, we require $u_i(\delta) \geqslant u_i(\delta')$ for all $\delta, \delta' \in \Delta([0, \bar{C}])$ with $\delta_x \geqslant \delta'_x$ for all $x \in A$.[20] Thus, increasing contributions weakly increases agents' utilities.[21] This implies that for every $u_i \in U$, $0_m \in \arg\min_{\delta \in \Delta([0,\bar{C}])} u_i(\delta)$ where 0_m denotes the zero vector (on m projects). In addition, we assume that for each $u_i \in U$ and $C > 0$, there exists $\delta \in \Delta(C)$ such that $u_i(\delta) > u_i(0_m)$.[22] Loosely speaking, each agent is interested in and gains utility from at least one project, a prerequisite for participation, in our opinion.

In the following chapters, we investigate specific utility models. In detail, Chapter 4 assumes linear utility functions, and Chapter 5 considers Leontief utilities. In Chapter 7, we aggregate insights from these two chapters and look at other classes of utility functions. All these models have in common that each agent $i \in N$ is assumed to report a vector $(v_{i,x})_{x \in A}$ where $v_{i,x}$ is her valuation for project x. These valuations *completely* specify her utility function, i.e., there is a one-to-one correspondence between utility functions u_i and valuations v_i. We write $(v^{u}_{i,x})_{x \in A}$ for the vector of valuations corresponding to utility function u_i but omit the superscript whenever that connection is clear from the context.

Consistency Axioms

We now catch up on consistency axioms, starting with continuity with respect to the returned utility vector. We define the distance between two utility functions via the distance between their corresponding valuations. Note that this approach can be justified by the fact that all considered utility functions will be continuous

20 Some authors like Foley refer to this property as *monotonicity*.
21 Note that this property is not well-defined in the context of portioning as increasing contributions to one project automatically decreases contributions to other projects.
22 This assumption allows us to avoid edge cases where some agents are indifferent between all distributions from $\Delta([0, \bar{C}])$.

in their respective valuations. For a fixed contribution profile, we first define continuity with respect to changes in utility functions.

Definition 3.2
For a fixed contribution profile, a mechanism f satisfies *(U-)continuity* if for all $U \in \mathcal{U}^n$,

$$\forall \varepsilon > 0 \; \exists \gamma > 0 \; \forall U' \in \mathcal{U}^n : \sum_{i \in N} \|v_i^u - v_i^{u'}\|_1 < \gamma \implies \|U(f(U)) - U(f(U'))\|_1 < \varepsilon$$

where the ℓ_1 norm of a vector $v = (v_x)_{x \in A}$ is defined as $\|v\|_1 = \sum_{x \in A} |v_x|$. Note that due to the norm equivalence on finite-dimensional vector spaces, choosing the ℓ_1 norm as a distance measure for valuations has no impact on the generality of our results.

Continuity can also be defined for a fixed utility profile and changing contributions.

Definition 3.3
For a fixed utility profile, a mechanism f satisfies *C-continuity* if for all $C \in \mathcal{C}_{>0}^{N'}$

$$\forall \varepsilon > 0 \; \exists \gamma > 0 \; \forall C' \in \mathcal{C}_{>0}^{N'} : \|C - C'\|_1 < \gamma \implies \|U(f(C)) - U(f(C'))\|_1 < \varepsilon$$

for any $N' \subseteq N$.

Remark 1
Agents with zero contribution are interpreted as non-participants and, therefore, ignored by all mechanisms that will be considered in the following. In theory, they could still be used for tiebreaking when multiple (optimal) distributions yield the same utilities to all participating agents. Thus, "jumps" in distribution when agents increase their contribution from zero seem to be natural, not to say unavoidable, and should not be taken into account when talking about C-continuity.

Monotonicity axioms determine the "direction" of changes in the outcome for varying input to some extent. Again, we consider both changes in utility and contribution profiles.

Definition 3.4
For a fixed contribution profile, a mechanism f satisfies *preference-monotonicity* if for every two utility profiles U and U' which are identical except that $v_{i,x}^{u'} > v_{i,x}^{u}$ for one agent i and one project x, we have $f(U')_x \geqslant f(U)_x$.

In other words, increasing valuations for a project cannot reduce the amount of contributions that this project receives.

Definition 3.5
For a fixed utility profile, a mechanism f satisfies *contribution-monotonicity* if for every two contribution profiles C and C' where C' can be obtained from C by increasing the contribution of one agent (possibly from 0), $f(C')_x \geqslant f(C)_x$ for all projects $x \in A$.

In other words, increasing the contributions of the agents cannot decrease the amount of donations that any project receives.

Whenever a mechanism might result in multiple distributions that yield the same returned utility vector, both monotonicity versions are defined in a way such that the above statements have to hold for *some* pair of possibly returned distributions in P and P'.

4 DONOR COORDINATION WITH SUBSTITUTES

In this chapter, utility functions $u_i\colon \Delta([\underline{C},\bar{C}]) \to \mathbb{R}_{\geqslant 0}$ are assumed to be linear, i.e.,

$$u_i(\delta) = \sum_{x\in A} v_{i,x}\delta_x$$

with $v_{i,x} \geqslant 0$ for all $x \in A$ and $i \in N$. Let $A_i = \{x \in A\colon v_{i,x} > 0\}$ the set of agent i's *accepted* or *approved* projects. Furthermore, $A_i^{max} = \{x \in A\colon x \in \arg\max_{y\in A} v_{i,y}\}$ denotes the set of agent i's *favorite* projects. In Section 4.3, we consider the case of *dichotomous preferences* where $v_{i,x} \in \{0,1\}$ for all $i \in N$ and $x \in A$ for which $A_i = A_i^{max}$.

We start with the investigation of equilibrium distributions.

4.1 EQUILIBRIUM DISTRIBUTIONS AND EFFICIENCY

Proposition 4.1
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ is an equilibrium distribution if and only if it admits a decomposition $(\delta_i)_{i\in N}$ such that for all $i \in N$, $\delta_{i,x} = 0$ for all $x \notin A_i^{max}$.

Proof. It is straightforward to see that an agent i is able to beneficially deviate from her individual distribution if and only if $\delta_{i,x} > 0$ for some $x \notin A_i^{max}$. In that case, moving $\delta_{i,x}$ from project x to a project $y \in A_i^{max}$ increases her utility. □

Moreover, best response dynamics immediately result in equilibrium distributions as agents' best responses reduce to moving all their contributions to favorite projects.

A second, maybe less straightforward characterization looks at groups of agents and gives a lower bound on the sum of contributions that need to be assigned to favorite projects.

Proposition 4.2
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ is an equilibrium distribution if and only if for every $N' \subseteq N$, $\sum_{x\in\bigcup_{i\in N'} A_i^{max}} \delta_x \geqslant \sum_{i\in N'} C_i$.

Proof. Brandl et al. (2021b) prove the statement via the max-flow min-cut theorem. We give the proof from [1], which is based on the strong duality theorem.

It is easy to see that the inequalities hold if δ is an equilibrium distribution. For the converse direction, note that δ is an equilibrium distribution if and only if the following linear program P with variables $d = (d_{i,x})_{i\in N, x\in A} \in \mathbb{R}^{n\cdot m}_{\geqslant 0}$ has a solution with value C_N.

$$\begin{array}{lll} \text{primal } (P) & & \\ \max \sum_{i\in N}\sum_{x\in A} d_{i,x} & & \\ \text{s.t. } \sum_{x\in A} d_{i,x} \leqslant C_i & \forall i \in N & \\ \quad\; \sum_{x\notin A_i^{\max}} d_{i,x} \leqslant 0 & \forall i \in N & \\ \quad\; \sum_{i\in N} d_{i,x} \leqslant \delta_x & \forall x \in A & \\ \quad\; d_{i,x} \geqslant 0 & \forall i \in N & \forall x \in A, \end{array}$$

where $d_{i,x}$ represents a possible contribution of agent i to project x.

The dual of the linear program P is

$$\begin{array}{ll} \text{dual } (D) & \\ \min \sum_{i\in N} C_i \tilde{d}_i + \sum_{x\in A} \delta_x \tilde{d}_x & \\ \text{s.t. } \tilde{d}_i + \tilde{d}_x \geqslant 1 & \forall i \in N\, \forall x \in A_i^{\max} \\ \quad\; \tilde{d}_i + \tilde{d}_x \geqslant 1 - \tilde{d}_{n+i} & \forall i \in N\, \forall x \notin A_i^{\max} \\ \quad\; \tilde{d}_i \geqslant 0 & \forall i \in \{1, \dots, 2n\} \\ \quad\; \tilde{d}_x \geqslant 0 & \forall x \in A \end{array}$$

with variables $\tilde{d}_1, \dots, \tilde{d}_{2n}$ and $\tilde{d}_x$ for $x \in A$, and $\tilde{d} \in \mathbb{R}^{2n+m}_{\geqslant 0}$.

Assuming $\sum_{x\in\bigcup_{i\in N'} A_i^{\max}} \delta_x \geqslant \sum_{i\in N'} C_i$ for every $N' \subseteq N$, we claim that there always exists an optimal solution $\tilde{d}^*$ to its dual D such that $\tilde{d}^*_{n+1} = \cdots = \tilde{d}^*_{2n} = 0$. This means that we can reduce D to D' where the second constraint simplifies to $\tilde{d}_i + \tilde{d}_x \geqslant 1$. Looking at the dual of D' called P', we observe that compared to P, the constraint $\sum_{x\notin A_i^{\max}} d_{i,x} \leqslant 0$ for all $i \in N$ is removed. Thus, the optimal value of P' is C_N as $\delta \in \Delta(C_N)$. As all of the stated problems have optimal solutions, the strong duality theorem implies that all four linear programs have the same optimal value C_N, and thus, δ is an equilibrium distribution as then, P has a solution with value C_N.

To prove the claim, let $\tilde{d}$ be an optimal solution to D and A_0 be the set of all $x \in A$ with $\tilde{d}_x = 0$. Thus, for all i with $A_i^{\max} \cap A_0 \neq \emptyset$, we have $\tilde{d}_i \geqslant 1$ and we can set $\tilde{d}_{n+i} = 0$. Denote the set of all such agents by N_0. If $N_0 = N$, we are done.

Otherwise, let $N' = N \setminus N_0$ and $x' = \arg\min_{x\in\bigcup_{i\in N'} A_i^{\max}} \tilde{d}_x$.

Define $\tilde{d}_i' = \tilde{d}_i + \tilde{d}_{x'}$ for all $i \in N'$ and $\tilde{d}_i' = \tilde{d}_i$ for all $i \notin N'$ as well as $\tilde{d}_x' = \tilde{d}_x - \tilde{d}_{x'}$ for all $x \in \bigcup_{i \in N'} A_i^{max}$ and $\tilde{d}_x' = \tilde{d}_x$, otherwise. By construction, $\tilde{d}'$ is still feasible and

$$\begin{aligned}
&\sum_{i \in N} C_i \tilde{d}_i + \sum_{x \in A} \delta_x \tilde{d}_x \\
&= \sum_{i \in N \setminus N'} C_i \tilde{d}_i' + \sum_{i \in N'} C_i(\tilde{d}_i' - \tilde{d}_{x'}) \\
&+ \sum_{x \in \bigcup_{i \in N'} A_i^{max}} \delta_x(\tilde{d}_x' + \tilde{d}_{x'}) + \sum_{x \notin \bigcup_{i \in N'} A_i^{max}} \delta_x \tilde{d}_x' \\
&\geqslant \sum_{i \in N} C_i \tilde{d}_i' - \sum_{i \in N'} C_i \tilde{d}_{x'} + \tilde{d}_{x'} \sum_{i \in N'} C_i + \sum_{x \in A} \delta_x \tilde{d}_x' \\
&= \sum_{i \in N} C_i \tilde{d}_i' + \sum_{x \in A} \delta_x \tilde{d}_x'
\end{aligned}$$

as $\tilde{d}_{x'} \sum_{x \in \bigcup_{i \in N'} A_i^{max}} \delta_x \geqslant \tilde{d}_{x'} \sum_{i \in N'} C_i$ by assumption. Thus, $\tilde{d}'$ is an optimal solution to D, and compared to $\tilde{d}$, the set N_0 is larger. Therefore, iterating this procedure with $\tilde{d} = \tilde{d}'$ until $N_0 = N$, we end up with a solution $\tilde{d}^*$ to D with $\tilde{d}_{n+1}^* = \cdots = \tilde{d}_{2n}^* = 0$. □

For dichotomous preferences, the equivalent property was introduced by Bogomolnaia et al. (2002) and is known as *fair group share* (Bogomolnaia et al., 2002) or *proportional sharing* (Duddy, 2015).

Example 2.11 shows the need for coordination among the agents. The unique equilibrium distribution $(1, 0, 1)$ is Pareto dominated by a coordinated approach in which the agents notice that they both approve project b and assign their contributions to it.

In fact, it has been shown that efficiency of a distribution depends only on its *support*[23] (see, e.g., Aziz et al. (2015) or Greger (2020) for a direct proof in the donor coordination setting).

PROPOSITION 4.3 *(see, e.g., Aziz et al. (2015); Greger (2020))*
For a fixed profile (U, C), if $\delta \in \Delta(C_N)$ is efficient, then each distribution $\delta' \in \Delta(C_N)$ with $\text{supp}(\delta') \subseteq \text{supp}(\delta)$ is also efficient.

A result from Hylland (1980) implies that we cannot hope for an efficient and strategyproof mechanism that additionally meets some fairness criteria.

THEOREM 4.4 *(Hylland (1980))*
For $m \geqslant 3$, any efficient and strategyproof mechanism is dictatorial.

Hylland's original characterization is even stronger as it makes use of a weaker efficiency condition where a distribution δ is dominated by δ' if $u_i(\delta') > u_i(\delta)$ for *all* $i \in N$.

Clearly, dictatorships leave some (non-dictatorial) agents with zero utility, which is why even the weakest fairness criterion considered in Section 2.3.3, namely positive share, has to be violated.

23 The support of a distribution δ is defined as $\text{supp}(\delta) = \{x \in A \colon \delta_x > 0\}$.

Corollary 4.5
For $m \geqslant 3$, there does not exist any mechanism that satisfies efficiency, strategyproofness, and positive share.

Thus, when aiming at efficient and fair mechanisms, we need to sacrifice strategyproofness. We note that agents' valuations are private information, which, in general, complicates the task of finding beneficial manipulations.

4.2 THE NASH PRODUCT RULE

It might not come as a surprise that maximizing the (weighted) product of the agents' utilities is an appealing choice. The *Nash product rule* is defined as

$$\mathit{NASH}(U, C) = \arg\max_{\delta \in \Delta(C_N)} \prod_{i \in N} (u_i(\delta))^{C_i} = \arg\max_{\delta \in \Delta(C_N)} \sum_{i \in N} C_i \log(u_i(\delta))$$

with the convention that $0^0 = 1$ and $0 \log 0 = 0$ in order to ignore agents with zero contribution.[24] The Nash welfare of a distribution δ is given by $\mathit{Nash}(\delta) = \sum_{i \in N} C_i \log(u_i(\delta))$.

In order to show that the Nash product rule is a well-defined mechanism, we prove that it always returns a unique utility vector.

Proposition 4.6
Nash welfare is maximized by a unique utility vector.

Proof. For a fixed profile (U, C), let $\delta, \delta' \in \Delta(C_N)$ be two distributions that maximize Nash welfare. With $\tilde{\delta} = \frac{1}{2}\delta + \frac{1}{2}\delta'$, we observe $u_i(\tilde{\delta}) = \frac{1}{2}u_i(\delta) + \frac{1}{2}u_i(\delta')$ for all $i \in N$ by linearity of the utility functions. Thus,

$$\begin{aligned}
\mathit{Nash}(\tilde{\delta}) &= \sum_{i \in N} C_i \log\left(\frac{1}{2}u_i(\delta) + \frac{1}{2}u_i(\delta')\right) \\
&\geqslant \sum_{i \in N} C_i \left(\frac{1}{2}\log(u_i(\delta)) + \frac{1}{2}\log(u_i(\delta'))\right) = \frac{1}{2}\mathit{Nash}(\delta) + \frac{1}{2}\mathit{Nash}(\delta') \\
&= \mathit{Nash}(\delta)
\end{aligned}$$

where the inequality, which follows from the concavity of the log function, is strict if $u_i(\delta) \neq u_i(\delta')$ for some $i \in N$.

However, $\mathit{Nash}(\tilde{\delta}) = \mathit{Nash}(\delta)$ as δ maximizes Nash welfare by assumption. Therefore, $u_i(\delta) = u_i(\delta')$ has to hold for all $i \in N$. □

Still, there might be multiple distributions that maximize Nash welfare.

Example 4.7
To see that, consider the following profile with $A = \{a, b, c, d\}$ and four agents.

24 We use log to denote the natural logarithm.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	$v_{i,d}$	C_i
Agent 1	1	0	1	0	1
Agent 2	1	0	0	1	1
Agent 3	0	1	1	0	1
Agent 4	0	1	0	1	1

Note that the sum of agents' utilities $\sum_{i \in N} u_i(\delta)$ is constant on $\delta \in \Delta(4)$ and sums up to eight due to symmetric valuations. The weighted inequality of arithmetic and geometric mean (see Section 5.2.2) implies that the Nash welfare is maximized by all distributions δ with $u_i(\delta) = 2$ for all $i \in N$. This is achieved by all convex combinations of $(2, 2, 0, 0)$ and $(0, 0, 2, 2)$. In particular, this example shows that multiple Nash welfare maximizing distributions are not always caused by "clone" projects for which all agents have the same valuations.

All properties of the Nash product rule that will be considered in the following do not depend on the actual distribution but only on the utility vector. Therefore, we leave open how to choose from multiple Nash welfare maximizing distributions but note that our computational methods for computing the Nash product rule sometimes make that decision for us.

4.2.1 Properties

NASH is efficient by construction as a function that is strictly increasing in the agents' utilities (or at least, in the utilities of all agents with positive contribution) is maximized (see, e.g., Moulin, 1988).

We note that the Nash product depends on the cardinal utility representation of agents' preferences. Still, *NASH* is independent of preference intensities in the sense that rescaling agents' valuations, e.g., such that $\min_{x \in A_i} v_{i,x} = 1$ for each agent i does not have an impact on the outcome.[25] This resembles an invariance condition as mentioned by Sen (2017).

We now investigate fairness properties of *NASH*.

Theorem 4.8
For a fixed profile (U, C), all Nash welfare maximizing distributions lie in the core and furthermore admit a decomposition where agents contribute only to their approved projects.

Proof. The proof that all Nash welfare maximizing distributions lie in the core is deferred to Chapter 6 (see Corollary 6.4) in order to interpret that result in a bigger context.

For the second part of the statement, consider the Karush–Kuhn–Tucker (KKT) conditions of the constrained optimization problem of maximizing Nash welfare and write the Lagrangian as

$$\mathcal{L}(\delta, \lambda, \mu_1, \dots, \mu_m) = \sum_{i \in N} C_i \log(u_i(\delta)) + \lambda \left(C_N - \sum_{x \in A} \delta_x \right) + \sum_{x \in A} \mu_x \delta_x,$$

25 When all agents rescale by the same factor, this property coincides with Moulin's *independence of the common utility scale*.

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier for the constraint $\sum_{x \in A} \delta_x = C_N$ and $\mu_x \geqslant 0$ is the multiplier for the constraint $\delta_x \geqslant 0$.

Here, the KKT conditions are not only sufficient but also necessary as Slater's condition holds (see, e.g., Boyd and Vandenberghe, 2004)[26], meaning that any optimal solution δ satisfies the KKT conditions.

By complementary slackness, $\mu_x = 0$ for $\delta_x > 0$. In addition, $\partial\mathcal{L}/\partial\delta_x = 0$, i.e., $\sum_{i \in N} C_i v_{i,x}/u_i(\delta) - \lambda + \mu_x = 0$. This implies $\lambda\delta_x = \sum_{i \in N} C_i \delta_x v_{i,x}/u_i(\delta)$ for all $x \in A$ (in case of $\mu_x > 0$, $\delta_x = 0$). Hence,

$$\lambda \cdot C_N = \sum_{x \in A} \lambda\delta_x = \sum_{x \in A} \sum_{i \in N} C_i \frac{v_{i,x}\delta_x}{u_i(\delta)} = \sum_{i \in N} C_i \frac{u_i(\delta)}{u_i(\delta)} = \sum_{i \in N} C_i = C_N.$$

Thus, $\lambda = 1$, and $\sum_{i \in N} C_i v_{i,x}/u_i(\delta) = 1$ for all $x \in A$ with $\delta_x > 0$.

For each $i \in N$, we can now define an individual distribution $\delta_i \in \Delta(C_i)$ via $\delta_{i,x} = C_i \delta_x v_{i,x}/u_i(\delta)$ for all $x \in A$. By construction, $\text{supp}(\delta_i) \subseteq \{x \in A \colon v_{i,x} > 0\}$ and $\delta_i \in \Delta(C_i)$, since $\sum_{x \in A} \delta_x v_{i,x} = u_i(\delta)$. To see that $\delta = \sum_{i \in N} \delta_i$, note that for $x \in A$ with $\delta_x = 0$ we have $\delta_{i,x} = 0$ for all $i \in N$. Furthermore, for $x \in A$ with $\delta_x > 0$,

$$\sum_{i \in N} \delta_{i,x} = \sum_{i \in N} C_i \delta_x \frac{v_{i,x}}{u_i(\delta)} = \delta_x \sum_{i \in N} C_i \frac{v_{i,x}}{u_i(\delta)} = \delta_x.$$

□

Guerdjikova and Nehring (2014) gave a similar proof for dichotomous preferences and noted that under the stated decomposition, each agent i contributes proportional to $\delta_x v_{i,x}/u_i(\delta)$, i.e., to the utility received from each individual project x. This insight will play an important role for the dynamics presented in Section 4.2.2.

Example 2.11 shows that a stronger notion where agents contribute only to favorite projects is incompatible with efficiency.

So far, we restricted ourselves to fixed contributions.

Definition 4.9

A mechanism f satisfies *contribution incentive-compatibility* if for each $i \in N$ and any two profiles (U, C) and (U, C') with $C_j = C'_j$ for $j \neq i$ and $C'_i \leqslant C_i \leqslant \bar{C}_i$,

$$u_i(f(C_{-i}, C_i)) \geqslant u_i(f(C_{-i}, C'_i)) + (C_i - C'_i) \min_{x \in A_i} v_{i,x}. \tag{1}$$

Contribution incentive-compatibility can be interpreted as a participation axiom and gives explicit lower bounds on an agent's utility gain when increasing her contribution.

Theorem 4.10

NASH satisfies contribution incentive-compatibility.

We need some auxiliary lemmas in order to bind some error terms appearing in the main proof.[27] These lemmas are proven afterward as they use notation from the main proof.

26 This follows directly from the fact that all constraints are affine functions.

27 Brandl et al. (2019) already obtained a proof for dichotomous preferences which can be generalized to linear utilities ([1]).

Proof. First, fix two profiles (U, C) and (U, C') with $C_i = C'_i$ for $i \neq 1$ and $C'_1 \leqslant C_1 \leqslant \bar{C}_1$ and rescale valuations such that for each $i \in N$, $\min_{x \in A_i} v_{i,x} = 1$. This is permitted as *NASH* is invariant under rescaling valuations.

Moreover, we can assume $C'_1 = 0$ w.l.o.g. as Agent 1 with $C'_1 > 0$ can be separated into two agents with the same utility function u_1, one with contribution C'_1 and the other one increasing her contribution from zero to $C_1 - C'_1$. Again, *NASH* is invariant under such separations. Slightly abusing notation, we also write $C_{-1} = C'$ as $C'_1 = 0$ by assumption. We also assume $C_i > 0$ for all $i \in N$ as agents with contribution zero are ignored by *NASH*, and the statement becomes trivial for $C_1 = 0$.

Furthermore, define $g \colon \mathcal{C} \to \Delta(1)$ as $g(C) = NASH(C)/C_N$ for all $C \in \mathcal{C}$. Then, showing (1) for $i = 1$ becomes equivalent to proving

$$u_1(g(C)) \geqslant \frac{1}{C_N}((C_N - C_1)u_1(g(C_{-1})) + C_1) \tag{2}$$

since utility functions are linear.

Second, denote by $\mathcal{P} \subseteq \mathbb{R}^n$ the convex polytope of attainable utility vectors when $\delta \in \Delta(1)$, i.e., $\mathcal{P} = \{U(\delta) \colon \delta \in \Delta(1)\}$, where convexity follows from linearity of the utility functions. Define $\mathcal{F} \colon \mathcal{C} \to \mathcal{P}$ as the function that, given a contribution profile C, returns the Nash welfare maximizing utility vector, i.e., $\mathcal{F}(C) = \max_{U(\delta) \in \mathcal{P}} \sum_{i \in N} C_i \log(u_i(\delta))$ which is unique by Proposition 4.6. Since all but Agent 1's contribution remain fixed, we consider the function $\mathcal{F}_1(C_1) = u_1(g(C_1, C_{-1}))$. By the proof of Theorem 5.15 and Remark 3, $\mathcal{F}_1$ is weakly increasing in C_1.

CASE 1
If $\mathcal{F}_1(C_1) \geqslant 1$, this implies

$$\mathcal{F}_1(C_1) = \frac{1}{C_N}((C_N - C_1)\mathcal{F}_1(C_1) + C_1\mathcal{F}_1(C_1)) \geqslant \frac{1}{C_N}((C_N - C_1)\mathcal{F}_1(0) + C_1), \tag{3}$$

which proves (2).

CASE 2
For $\mathcal{F}_1(C_1) < 1$, we again prove (3) which can be rewritten as $\frac{1-\mathcal{F}_1(C_1)}{1-\mathcal{F}_1(0)} \leqslant \frac{C_N - C_1}{C_N}$. For arbitrary $\tilde{\varepsilon} \in (0, C_1)$, we show $\frac{1-\mathcal{F}_1(C_1)}{1-\mathcal{F}_1(\tilde{\varepsilon})} \leqslant \frac{C_N - C_1 + \tilde{\varepsilon}}{C_N}$ which implies the previous inequality as $\mathcal{F}_1(C_1)$ is weakly increasing in C_1. Taking the log (both sides are positive by assumption) yields

$$\log(1 - \mathcal{F}_1(C_1)) - \log(1 - \mathcal{F}_1(C_1)) \leqslant -(\log(C_N) - \log(C_N - C_1 + \tilde{\varepsilon})).$$

Using $\frac{d}{dz}\log(1 - h(z)) = -\frac{\frac{d}{dz}h(z)}{1-h(z)}$ for smooth h with $1 - h(z) > 0$ for all z as well as $\frac{d}{dz}\log(K + z) = \frac{1}{K+z}$ for all constants K such that $K + z > 0$, it is sufficient to show that

$$-\int_{\tilde{\varepsilon}}^{C_1} \frac{\frac{\partial \mathcal{F}_1(s)}{\partial s}}{1 - \mathcal{F}_1(s)} ds \leqslant -\int_{\tilde{\varepsilon}}^{C_1} \frac{1}{C_N - C_1 + s} ds$$

where we need to derive a lower bound on $\frac{\partial \mathcal{F}_1(s)}{\partial s}$ such that the inequality holds.

Thus, the remainder of this proof is devoted to proving

$$\frac{\partial \mathcal{F}_1(s)}{\partial s} \geqslant \frac{1 - \mathcal{F}_1(s)}{C_N - C_1 + s} \tag{4}$$

for $s \in (\tilde{\varepsilon}, C_1)$, i.e., giving a lower bound on the marginal increase in Agent 1's utility when her contribution increases.

Let $U = \mathcal{F}(C)$. We start by bounding the change in welfare when slightly deviating from the optimal utility vector $U \in \mathcal{P}$ to another utility vector $U + dU \in \mathcal{P}$ and another contribution vector C' with $C'_i = C_i$ for $i \neq 1$ and $C'_1 = C_i + dC_1$. Using the first-order Taylor expansion of the logarithm, we get

$$\begin{aligned}\sum_{i \in N} C'_i \log(U_i + dU_i) = &\sum_{i \in N} C_i \log(U_i) + dC_1 \log(U_1) \\ &+ \sum_{i \in N} C_i \frac{dU_i}{U_i} + dC_1 \frac{dU_1}{U_1} \\ &- \frac{1}{2} \sum_{i \in N} C'_i \left(\frac{dU_i}{\xi_i}\right)^2\end{aligned}$$

for some $\xi_i \in [U_i, U_i + dU_i]$ for each $i \in N$.

Denote

$$\begin{aligned}\phi(dU) &= \sum_{i \in N} C'_i \log(U_i + dU_i) - \sum_{i \in N} C_i \log(U_i) - dC_1 \log(U_1) \\ &= \sum_{i \in N} C_i \frac{dU_i}{U_i} + dC_1 \frac{dU_1}{U_1} - \psi(dU)\end{aligned} \tag{5}$$

where $\psi(dU)$ denotes the second order error term

$$\psi(dU) = \frac{1}{2} \sum_{i \in N} C'_i \left(\frac{dU_i}{\xi_i}\right)^2 = \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{\xi_i}\right)^2 + dC_1 \left(\frac{dU_1}{\xi_1}\right)^2.$$

Let $\mu \in (0, 2)$ be arbitrary, set ε^* such that Lemma 4.14 applies and choose an arbitrary $\varepsilon \in (0, \varepsilon^*)$. For small enough dC_1 and $\|dU\|_1$, the following bounds for $\psi(dU)$ hold:

$$(1 - \varepsilon) \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{U_i}\right)^2 \leqslant \psi(dU) \leqslant (1 + \varepsilon) \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i}{U_i}\right)^2. \tag{6}$$

Next, let $U' = \mathcal{F}(C')$ and $dU' = U' - U$. By definition, dU' maximizes ϕ among all dU such that $U + dU \in \mathcal{P}$. Furthermore, it is possible to find $\varepsilon'' > 0$ such that for all $dU \in \mathbb{R}^n$ with $\|dU\|_1 \leqslant \varepsilon''$, $U + r dU \in \mathcal{P}$ for all $r \in [0, 2]$ if $U + dU \in \mathcal{P}$. Finding such an ε'' is always possible due to the fact that $\mathcal{P}$ is a polytope, i.e., an intersection of a finite number of closed halfspaces in $\mathbb{R}^n$.

As *NASH* is C-continuous (Theorem 4.16), we can choose dC_1 small enough such that the function $\Phi\colon [0, 2] \to \mathbb{R}$ with $\Phi(r) = \phi(r dU')$ is well-defined and Lemma 4.14 applies with

$$\alpha = \sum_{i \in N} C_i \frac{dU_i'}{U_i} + dC_1 \frac{dU_1'}{U_1} \quad \text{and} \quad \beta = \frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i'}{U_i} \right)^2.$$

Thus, Lemma 4.14 implies

$$\sum_{i \in N} C_i \frac{dU_i'}{U_i} + dC_1 \frac{dU_1'}{U_1} \geqslant \mu \Phi(1).$$

By Lemma 4.11, $\sum_{i \in N} C_i \frac{dU_i'}{U_i} \leqslant 0$ and thus,

$$dC_1 \frac{dU_1'}{U_1} \geqslant \mu \Phi(1). \tag{7}$$

We established a connection between Agent 1's changes in contribution dC_1 as well as utility dU_1' and the loss in Nash welfare $\phi(dU')$ when going from the optimal utility vector U' (with respect to C') to U.

Next, we derive a lower bound for $\Phi(1)$. To this end, let $\delta = g(C)$. Due to the normalization in the beginning and our assumption that $U_1 < 1$, the amount of contribution assigned to agent 1's approved projects $\delta(A_1)$ has to be less than 1 but also more than 0. As $\delta \in \Delta(1)$, we conclude $\delta_x < 1$ for all $x \in A$. Hence, for small enough $|t| > 0$, the distribution δ^t with

$$\delta_x^t = \begin{cases} (1+t)\delta_x & \text{for all } x \in A_1, \\ \left(1 - \frac{\delta(A_1)}{1-\delta(A_1)} t\right) \delta_x & \text{for all } x \in A \setminus A_1. \end{cases}$$

is well-defined and in $\Delta(1)$ as

$$\begin{aligned} \sum_{x \in A} \delta_x^t &= \sum_{x \in A_1} (1+t)\delta_x + \sum_{x \in A \setminus A_1} \left(1 - \frac{\delta(A_1)}{1-\delta(A_1)} t\right) \delta_x \\ &= (1+t)\delta(A_1) + \left(1 - \frac{\delta(A_1)}{1-\delta(A_1)} t\right)(1 - \delta(A_1)) \\ &= 1. \end{aligned}$$

Let $dU^t = U(\delta^t) - U$. For $|t|$ small enough, we have that $\delta^t \in \Delta(1)$ and $U(\delta^t) = U + dU^t \in \mathcal{P}$. Moreover, $U - dU^t \in \mathcal{P}$ as we can change δ by a small enough amount in the opposite direction to δ^t due to the fact that $\delta_x < 1$ for all $x \in A$, and for $x \in A$ such that $\delta_x = 0$, we have $\delta_x^t = \delta_x$.

Thus, by Lemma 4.11, we obtain $\sum_{i \in N} C_i \frac{dU_i^t}{U_i} = 0$. Hence, for sufficiently small $|t|$, we have

$$\phi(dU^t) \overset{(5)}{=} dC_1 \frac{dU_1^t}{U_1} - \psi(dU^t) \overset{(6)}{\geqslant} dC_1 \frac{dU_1^t}{U_1} - (1+\varepsilon)\frac{1}{2} \sum_{i \in N} C_i \left(\frac{dU_i^t}{U_i}\right)^2. \tag{8}$$

Since $dU_1^t = u_1(\delta^t) - U_1 = (1+t)U_1 - U_1$, we have that $\frac{dU_1^t}{U_1} = t$.

As we normalized valuations in the beginning, we have $U_1 \geqslant \delta(A_1)$ which implies $1 - U_1 \leqslant 1 - \delta(A_1)$. Hence, $-\frac{U_1}{1-U_1} t \leqslant -\frac{\delta(A_1)}{1-\delta(A_1)} t \leqslant \frac{dU_i^t}{U_i} \leqslant t$.

Applying Lemma 4.12 with $\alpha = \frac{U_1}{(1-U_1)} t$, $\beta = t$, and $z_i = \frac{dU_i^t}{U_i}$ to (8) yields

$$\phi(dU^t) \geqslant dC_1 t - (1+\varepsilon)\frac{1}{2}\frac{U_1 C_N}{1-U_1} t^2.$$

Now, let $t := \frac{1-U_1}{U_1 C_N} dC_1$. If dC_1 is small enough, then t is also small enough and, recalling that dU' maximizes ϕ among all $dU \in \mathbb{R}^n$ with $U + dU \in \mathcal{P}$, we get

$$\Phi(1) = \phi(dU') \geqslant \phi(dU^t) \geqslant \frac{1}{2}(1-\varepsilon)\frac{1-U_1}{U_1 C_N}(dC_1)^2.$$

Together with (7), we get

$$dC_1 \frac{dU_1'}{U_1} \geqslant \frac{\mu}{2}(1-\varepsilon)\frac{1-U_1}{U_1 C_N}(dC_1)^2.$$

Rearranging terms (note that $dC_1 > 0$) gives

$$dU_1' \geqslant \frac{\mu}{2}(1-\varepsilon)\frac{1-U_1}{C_N} dC_1.$$

Since $\mu \in (0,2)$ was arbitrary and $\varepsilon > 0$ can be chosen arbitrarily small, it follows that

$$dU_1' \geqslant \frac{1-U_1}{C_N} dC_1.$$

This implies that Agent 1 with increasing contribution (by dC_1) gains utility (dU_1') at least at rate $\frac{1-U_1}{C_N}$. Translating that statement to (4), we can conclude that for arbitrary $s \in (\tilde{\varepsilon}, C_1)$, $\frac{\partial \mathcal{F}_1(s)}{\partial s} \geqslant \frac{1-\mathcal{F}_1(s)}{C_N - C_1 + s}$ indeed holds. □

We complete the proof by showing the auxiliary lemmas.

Lemma 4.11

Let $C \in \mathcal{C}$ with $C_i > 0$ for all $i \in N$. Further, $U = \mathcal{F}(C)$ and $dU \in \mathbb{R}^n$ such that $U + dU \in \mathcal{P}$. Then,

$$\sum_{i \in N} C_i \frac{dU_i}{U_i} \leqslant 0.$$

In case $U - dU \in \mathcal{P}$, equality holds.

Proof. Define the function $\tau\colon [0,1] \to \mathbb{R}$ with $\tau(t) = \sum_{i \in N} C_i \log(U_i + t dU)$. Note that τ is well-defined as $\mathcal{P}$ is convex and $U + dU \in \mathcal{P}$. As $U = \mathcal{F}(C)$, τ attains its maximum at $t = 0$. In addition its right derivative at $t = 0$ exists as $U_i > 0$ for all $i \in N$ and is non-positive since τ attains its maximum at $t = 0$ (as $U = \mathcal{F}(C)$):

$$\frac{\partial \tau}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\left(\sum_{i \in N} C_i \log(U_i + t dU_i)\right)\Big|_{t=0} = \sum_{i \in N} C_i \frac{dU_i}{U_i} \leqslant 0.$$

If $U - dU \in \mathcal{P}$ also holds, then the same argument shows that $-\sum_{i \in N} C_i \frac{dU_i}{U_i} \leqslant 0$ which implies equality. □

Lemma 4.12
Let $C \in \mathcal{C}$, $z \in \mathbb{R}^n$, and $\alpha, \beta > 0$ such that $\sum_{i \in N} C_i z_i = 0$ and $-\alpha \leqslant z_i \leqslant \beta$ for all $i \in N$. Then,

$$\sum_{i \in N} C_i z_i^2 \leqslant \alpha\beta \sum_{i \in N} C_i.$$

Proof. Since $-\alpha \leqslant z_i \leqslant \beta$, we have $\left| z_i - \frac{\beta-\alpha}{2} \right| \leqslant \frac{\beta+\alpha}{2}$. It follows that

$$\begin{aligned} \sum_{i \in N} C_i z_i^2 &= \sum_{i \in N} C_i \left(z_i - \frac{\beta - \alpha}{2} \right)^2 - \left(\frac{\beta - \alpha}{2} \right)^2 \sum_{i \in N} C_i \\ &\leqslant \left(\frac{\beta + \alpha}{2} \right)^2 \sum_{i \in N} C_i - \left(\frac{\beta - \alpha}{2} \right)^2 \sum_{i \in N} C_i \\ &= \alpha\beta \sum_{i \in N} C_i, \end{aligned}$$

where we used $\sum_{i \in N} C_i z_i = 0$ for the first equality. □

The following lemma is required for the proof of Lemma 4.14.

Lemma 4.13
Let $\lambda^* \in (0, \frac{1}{2})$. Then, there are $t \in [1, 2]$ and $\varepsilon^* \in (0, 1)$ such that

$$t - \lambda \frac{1+\varepsilon}{1-\varepsilon} t^2 > 1 - \lambda \quad \text{for all } \lambda \in [0, \lambda^*] \text{ and } \varepsilon \in (0, \varepsilon^*).$$

Proof. For arbitrary $\varepsilon > 0$ and $t \in [1, 2]$,

$$t - \lambda \frac{1+\varepsilon}{1-\varepsilon} t^2 > 1 - \lambda \iff \lambda < \frac{t-1}{\frac{1+\varepsilon}{1-\varepsilon} t^2 - 1} \iff \frac{1+\varepsilon}{1-\varepsilon}(t+1)\lambda < 1.$$

The left side of the last inequality is continuous and increasing in λ, t, and ε. Thus, it is sufficient to find $t \in [1, 2]$ and ε^* such that $\frac{1+\varepsilon^*}{1-\varepsilon^*}(t+1)\lambda^* < 1$. As $\lambda^* < 1/2$, it is possible to find such $t \in (1, 2]$ and $\varepsilon^* \in (0, 1)$ close enough to their lower bounds by continuity. □

Lemma 4.14
For all $\mu \in (0, 2)$ there is $\varepsilon^* \in (0, 1)$ with the following property. For any $\Phi \colon [0, 2] \to \mathbb{R}$ such that $\Phi(1) = \max_{t \in [0,2]} \Phi(t)$ and such that there are $\alpha, \beta \geqslant 0$ and $\varepsilon \in (0, \varepsilon^*)$ with

$$\alpha t - (1+\varepsilon)\beta t^2 \leqslant \Phi(t) \leqslant \alpha t - (1-\varepsilon)\beta t^2$$

for all $t \in [0, 2]$, it holds that $\alpha \geqslant \mu\Phi(1)$.

Proof. If $\mu \leqslant 1$, we have $\mu\Phi(1) \leqslant \Phi(1) \leqslant \alpha$ by assumption and can choose any $\varepsilon^* \in (0, 1)$.

Assume henceforth that $\mu > 1$. Let $\lambda^* := 1 - \frac{1}{\mu} > 0$ and choose $\varepsilon^* > 0$ and $t^* \in [1, 2]$ such that

$$t^* - \lambda \frac{1+\varepsilon}{1-\varepsilon}(t^*)^2 > 1 - \lambda \tag{9}$$

for all $\lambda \in [0, \lambda^*]$ and $\varepsilon \in (0, \varepsilon^*)$, which is possible by Lemma 4.13.

Let Φ, α, β, and ε as in the statement of the lemma.

If $\alpha = 0$ or $\Phi(1) = 0$, we get $\Phi(1) \leqslant 0$ and $\alpha \geqslant \mu\Phi(1)$ holds.

Finally, consider the case $\alpha > 0$ and $\Phi(1) > 0$. Set $\lambda := \frac{\alpha - \Phi(1)}{\alpha} \geqslant 0$ and assume for contradiction that the desired conclusion is not true, i.e., $\alpha < \mu\Phi(1)$. Noting that $\lambda, \lambda^* < 1$ and $\alpha = \frac{\Phi(1)}{1-\lambda}$ as well as $\mu = \frac{1}{1-\lambda^*}$, this is equivalent to $\lambda < \lambda^*$.

The function $\Psi(t) := \alpha t - \Phi(t)$ satisfies $\beta(1-\varepsilon)t^2 \leqslant \Psi(t) \leqslant \beta(1+\varepsilon)t^2$ by assumption. By substituting $t = t^*$ and using $\beta \leqslant \Psi(1)/(1-\varepsilon)$, we have

$$\Psi(t^*) \leqslant \Psi(1)\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2. \tag{10}$$

Consequently,

$$\begin{aligned}
\Phi(t^*) &= \alpha t^* - \Psi(t^*) \\
&\overset{(10)}{\geqslant} \alpha\left(t^* - \frac{\Psi(1)}{\alpha}\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2\right) \\
&= \alpha\left(t^* - \lambda\frac{1+\varepsilon}{1-\varepsilon}(t^*)^2\right) \\
&\overset{(9)}{>} \alpha(1-\lambda) \\
&= \Phi(1).
\end{aligned}$$

This contradicts the assumption that $\Phi(1) = \max_{t \in [0,2]} \Phi(t)$. □

In a public good market, Theorem 4.10 shows that, when using the Nash product rule, agents are incentivized to exclusively contribute to the public goods even in the case when they can also contribute to a private good which is "worth" at most as much as any public good in A_i, see Chapter 6 for a detailed discussion.

A stronger notion that requires max instead of min in Definition 4.9, which we call *strong contribution incentive-compatibility*, is incompatible with efficiency as shown by the following example.

Example 4.15

Assume there are four projects $A = \{a, b, c, d\}$ and three agents.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	$v_{i,d}$	C_i
Agent 1	1.75	0	0	1	1
Agent 2	0	1.75	0	1	1
Agent 3	0	0	1.75	1	1

We show that efficiency and strong contribution incentive-compatibility cannot be satisfied simultaneously.

Assume w.l.o.g $u_3(\delta) = \min_{i\in N} u_i(\delta)$ in the contribution profile $(1,1,1)$, i.e., $\delta \in \Delta(3)$.

Next, consider the contribution profile $(1,1,0)$ where Agents 1 and 2 have already contributed. By efficiency, the chosen distribution δ^{12} has to satisfy $\delta^{12}_a = 0$ or $\delta^{12}_b = 0$. Otherwise, the distribution $\tilde{\delta}^{12} = (\delta^{12}_a - \varepsilon, \delta^{12}_b - \varepsilon, \delta^{12}_d + 2\varepsilon)$ with $\varepsilon = \min\{\delta^{12}_a, \delta^{12}_b\}$ would Pareto dominate δ^{12} as, for $i \in \{1,2\}$, $u_i(\tilde{\delta}^{12}) - u_i(\delta) = -\varepsilon(\nu_{i,a}\delta^{12}_a + \nu_{i,b}\delta^{12}_b) + 2\varepsilon = 0.25\varepsilon$. W.l.o.g., $\delta^{12}_b = 0$. However, $u_2(\delta^{12}) \geqslant 1.75$ by strong contribution incentive-compatibility and thus, $\delta^{12}_d \geqslant 1.75$.

With $u_3(\delta^{12}) = 1.75$, Agent 3's utility has to increase to at least 3.5 when adding her contribution. By assumption, $u_i(\delta) \geqslant 3.5$ for all $i \in N$. This implies, inter alia, $\delta_a > 0$ and $\delta_b > 0$, a contradiction to efficiency by the same argument as above.

Another property between these two notions of contribution incentive-compatibility requires that $u_i(f(C_{-i}, C_i)) \geqslant u_i(f(C_{-i}, C'_i)) + u_i(\delta')$ for a distribution $\delta' \in \Delta(C_i - C'_i)$ with $\delta'_x = \left(C_i - C'_i\right) \cdot (\nu_{i,x}\delta_x)/u_i(\delta)$ for all $x \in A$. Roughly speaking, agent i's gains are proportional to the former distribution δ, which is in the spirit of the dynamics discussed in Section 4.2.2. Computer simulations suggest that *NASH* satisfies this stronger property, but a formal proof is pending.

We now turn our attention to consistency properties of *NASH*.

THEOREM 4.16
NASH is U-continuous and C-continuous.

Proof. Consider an arbitrary sequence of profiles $(U^t \times C^t)_{t\in\mathbb{N}}$ with $U^t \in \mathcal{U}^n$ and $C^t \in \mathcal{C}^{N'}_{>0}$ for all $t \in \mathbb{N}$ with $\lim_{t\to\infty} U^t \times C^t = U \times C \in \mathcal{U}^n \times \mathcal{C}^{N'}_{>0}$. Denote the utility vectors returned by *NASH* as $U(\delta)$ for profile $(U, C) = (u_i(\delta))_{i\in N}$ and $U^t(\delta^t) = (u^t_i(\delta^t))_{i\in N}$ for profile (U^t, C^t) and $t \in \mathbb{N}$. To prove continuity, we need to show that $\lim_{t\to\infty} U^t(\delta^t) = U(\delta)$. The sequence $(U^t(\delta^t))_{t\in\mathbb{N}})$ is bounded as for large enough t, each agent i's utility is lower bounded by $(\min_{x\in A_i} \nu_{i,x} - \varepsilon_{u_i}) \cdot (C_i - \varepsilon_{C_i})$ where $\varepsilon_{u_i}, \varepsilon_C > 0$ can be chosen arbitrarily small as $(U^t)_{t\in\mathbb{N}}$ converges to U and $(C^t)_{t\in\mathbb{N}}$ converges to C by assumption. Theorem 4.8 implies that each agent contributes only to approved projects which yields the lower bound. Similarly, an upper-bound is given by $(\nu_{i,x} + \varepsilon_{u_i}) \cdot (C_i + \varepsilon_{C_i})$ for $x \in A^{max}_i$. As we consider finite numbers of agents, there exist general bounds L_U, H_U, L_C and H_C such that $0 < L_U \leqslant u^t_i(\delta^t) \leqslant H_U$ and $0 < L_C \leqslant C^t_i \leqslant H_C$ for all $i \in N$ and $t \in \mathbb{N}$. By the Bolzano-Weierstrass theorem, there exists a convergent subsequence $(U^{t_k}(\delta^{t_k}))_{k\in\mathbb{N}}$, denote its limit by $U^*(\delta^*)$. Due to the fact that all contributions are positive and each agent has at least one approved project, Nash welfare is continuous on $U^t \times C^t$ and even uniformly continuous on $[L_C, H_C] \times [L_U, H_U]^n$ by the Heine-Cantor theorem. Thus,

$$\lim_{k\to\infty} \sum_{i\in N} C^{t_k}_i \log\left(u^{t_k}_i(\delta^{t_k})\right) = \sum_{i\in N} C_i \log\left(u^*_i(\delta^*)\right)$$

as $\lim_{k\to\infty} C^{t_k} = \lim_{t\to\infty} C^t = C$ and $\lim_{k\to\infty} U^{t_k}(\delta^{t_k}) = U^*(\delta^*)$ by assumption. This also shows that $\delta^* \in \Delta(C_N)$.

Together with the fact that

$$\sum_{i\in N} C^{t_k}_i \log\left(u^{t_k}_i(\delta^{t_k})\right) \geqslant \sum_{i\in N} C^{t_k}_i \log\left(\frac{C^{t_k}_N}{C_N} \cdot u_i(\delta)\right)$$

by optimality of $U^{t_k}(\delta^{t_k})$ for C^{t_k} (we rescaled distribution δ to be in $\Delta(C_N^{t_k})$),

$$\sum_{i \in N} C_i \log(u_i^*(\delta^*)) \geqslant \sum_{i \in N} C_i \log(u_i(\delta)).$$

By Proposition 4.6, $U^*(\delta^*) = U(\delta) = \lim_{t \to \infty} U^t(\delta^t)$ where the last equality follows again from Nash welfare being uniformly continuous on $[L_C, H_C] \times [L_U, H_U]^n$. $\square$

Brandl et al. (2021b) showed that every strategyproof mechanism also satisfies preference-monotonicity for dichotomous preferences. Their proof can be generalized to linear utilities.

Proposition 4.17
Strategyproofness implies preference-monotonicity.

Proof. Let $P = (U, C)$ and $P' = (U', C)$ be two profiles where compared to U, one agent i increased her valuation for one project x in U', i.e., $v'_{i,x} > v_{i,x}$. Let f be any strategyproof mechanism and define $\delta = f(U, C)$ and $\delta' = f(U', C)$.

By strategyproofness for agent i when manipulating from P to P',

$$\sum_{y \in A} v_{i,y}\delta_y = u_i(\delta) \geqslant u_i(\delta') = \sum_{y \in A} v_{i,y}\delta'_y.$$

By strategyproofness for agent i when manipulating from P' to P,

$$v'_{i,x}\delta'_x + \sum_{y \in A \setminus x} v_{i,y}\delta'_y = u'_i(\delta') \geqslant u'_i(\delta) = v'_{i,x}\delta_x + \sum_{y \in A \setminus x} v_{i,y}\delta_y.$$

Adding up both inequalities and identifying identical addends leads to

$$v'_{i,x}\delta'_x + v_{i,x}\delta_x \geqslant v'_{i,x}\delta_x + v_{i,x}\delta'_x.$$

Rearranging terms yields $v'_{i,x}(\delta'_x - \delta_x) \geqslant v_{i,x}(\delta'_x - \delta_x)$ which implies $\delta'_x \geqslant \delta_x$ as $v'_{i,x} > v_{i,x}$. $\square$

Remark 2
For the case of dichotomous preferences, Brandl et al. (2021b) proposed the *sequential utilitarian rule* which satisfies efficiency, monotonicity, and positive share (in fact, each agent contributes only to approved projects), showing that Corollary 4.5 turns into a possibility when weakening strategyproofness to monotonicity. Their proofs can again be generalized to linear utilities, showing that efficiency, monotonicity, and positive share are simultaneously satisfiable.

The following two examples show that *NASH* satisfies neither preference- nor contribution-monotonicity.

Example 4.18 *(Brandl et al. (2019))*
Consider an instance with four projects $A = \{a, b, c, d\}$ and six agents.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	$v_{i,d}$	C_i
Agent 1	1	0	0	0	1
Agent 2	1	1	0	0	1
Agent 3	1	0	1	0	1
Agent 4	0	1	1	1	2
Agent 5	0	1	0	1	2
Agent 6	0	0	1	1	2

NASH returns the distribution $(3,0,0,6)$. If Agent 1 approves project d in addition, Nash welfare is maximized by $(2\gamma,\gamma,\gamma,9-4\gamma)$ with $\gamma = (7-\sqrt{22})/3 > 0.76$. Thus, *NASH* violates preference-monotonicity as project d receives less contribution in total.

Example 4.19
Consider the following example with two projects $A = \{a, b\}$ and three agents.

	$v_{i,a}$	$v_{i,b}$	C_i
Agent 1	1	0	1
Agent 2	0	1	1
Agent 3	1	1	1

NASH returns the distribution $(3/2, 3/2)$. If Agent 1 increases her contribution to 2, Nash welfare is maximized by $(8/3, 4/3)$, where project b receives less contribution in total. Thus, *NASH* violates contribution-monotonicity.

4.2.2 Computation

Due to its formulation as a convex optimization program, *NASH* can be computed to arbitrary precision using convex programming (see, e.g., Bogomolnaia et al., 2005). However, Example 4.18 demonstrates that *NASH* may return distributions with irrational values and, thus, cannot be computed exactly (in the standard binary representation).

We have already seen in the proof of Theorem 4.8 that a *NASH* distribution admits a decomposition where agents' individual distributions are proportional to the utility they receive from each project under the *NASH* distribution. This observation gives rise to a simple, dynamic procedure for approximating *NASH*, similar to proportional response dynamics that converge to equilibrium in Fisher markets for private goods (Zhang, 2011).

For a fixed profile (U, C), consider the mapping $f\colon \Delta(C_N) \to \Delta(C_N)$ defined by

$$(f(\delta))_x = \sum_{i\in N} C_i \frac{v_{i,x}}{u_i(\delta)} \delta_x \qquad \text{for all } \delta \in \Delta(C_N). \tag{11}$$

At any stage of our analysis, we will assume $u_i(\delta) > 0$ for all $i \in N$ such that f is well-defined. The proof of Theorem 4.8 shows that each *NASH* distribution is a fixed point of f. Note that the converse is not true in general.

Example 4.20
Look at the following example with two projects $A = \{a, b\}$ and two agents.

	$v_{i,a}$	$v_{i,b}$	C_i
Agent 1	2	1	1
Agent 2	2	1	1

Obviously, the Nash welfare maximizing distribution is $(2, 0)$, but $(0, 2)$ is another fixed point.

Naturally, we are interested in the stability of *NASH* distributions with respect to f and initial distributions δ^0 that guarantee convergence of the sequence $(\delta^t)_{t\in\mathbb{N}}$ with $\delta^t = f(\delta^{t-1})$.

These questions have been asked before in the literature on optimal portfolios. Cover (1984) showed that the Nash product of δ^t converges to the optimum Nash product if δ^0 has full support, and the sequence $(\delta^t)_{t\in\mathbb{N}}$ converges to a Nash distribution under additional assumptions. There, projects correspond to stocks, and utilities encode their performance.[28] Our setting allows for a more compact argument based on Cover's proof since our model assumes the number of agents to be finite.

Theorem 4.21
Let (U, C) be an arbitrary but fixed profile, and $\delta^0 \in \Delta(C_N)$ be a distribution with full support. Denote by $(\delta^t)_{t\in\mathbb{N}}$ the sequence induced by (11). Then, $(Nash(\delta^t))_{t\in\mathbb{N}}$ converges to the optimum Nash product. If the Nash distribution is unique, $(\delta^t)_{t\in\mathbb{N}}$ converges to *NASH*.

Proof. To see that $u_i(\delta) > 0$ for all $i \in N$ during our analysis, note that δ^0 has full support. Moreover, in all subsequent steps, every agent assigns her contribution only to projects for which she has strictly positive utility. Hence, $u_i(\delta^t) > 0$ for all i and t, and $\delta^t_x = 0$ for a project x implies $v_{i,x} = 0$ for all agents i. We can thus ignore such projects and assume $\delta^t_x > 0$ for all x and t.

W.l.o.g., assume that $C_N = 1$, so that $\delta^t \in \Delta(1)$ for all t.

The proof proceeds in two steps. First, it is shown that the sequence $(Nash(\delta^t))_{t\in\mathbb{N}}$ converges. Second, we prove that every accumulation point of $(\delta^t)_{t\in\mathbb{N}}$ is a Nash distribution.

28 That literature has argued that a portfolio of stocks that maximizes expected log returns produces optimal earnings in the long run (Cover and Thomas, 2006, Chapter 16).

Step 1
For $t \geqslant 1$, we get

$$
\begin{aligned}
Nash(\delta^{t+1}) - Nash(\delta^t) &= \sum_{i \in N} C_i \log\left(\frac{u_i(\delta^{t+1})}{u_i(\delta^t)}\right) \\
&= \sum_{i \in N} C_i \log\left(\sum_{x \in A} \delta_x^{t+1} \frac{v_{i,x}}{u_i(\delta^t)}\right) \\
&\overset{(1)}{=} \sum_{i \in N} C_i \log\left(\sum_{x \in A} \left(\sum_{j \in N} C_j \frac{v_{j,x}}{u_j(\delta^t)}\right) \delta_x^t \frac{v_{i,x}}{u_i(\delta^t)}\right) \\
&\overset{(2)}{\geqslant} \sum_{i \in N} C_i \sum_{x \in A} \delta_x^t \frac{v_{i,x}}{u_i(\delta^t)} \log\left(\sum_{j \in N} C_j \frac{v_{j,x}}{u_j(\delta^t)}\right) \\
&\overset{(3)}{=} \sum_{x \in A} \delta_x^t \sum_{i \in N} C_i \frac{v_{i,x}}{u_i(\delta^t)} \log\left(\sum_{j \in N} C_j \frac{v_{j,x}}{u_j(\delta^t)} \frac{\delta_x^t}{\delta_x^t}\right) \\
&\overset{(4)}{=} \sum_{x \in A} \delta_x^{t+1} \log\left(\frac{\delta_x^{t+1}}{\delta_x^t}\right) \\
&\overset{(5)}{\geqslant} \frac{1}{2\log(2)} \|\delta^{t+1} - \delta^t\|_1^2 \geqslant 0,
\end{aligned}
$$

where (1) and (4) follow from the definition of the dynamic procedure, (2) is an application of Jensen's inequality for concave functions ($\sum_{x \in A} \delta_x^t (v_{i,x}/u_i(\delta^t)) = 1$), (3) changes the summation order, and (5) uses Lemma 11.6.1 of Cover and Thomas (2006), where the left-hand side is the Kullback-Leibler divergence of δ^{t+1} and δ^t.

Hence, $(Nash(\delta^t))_{t \in \mathbb{N}}$ is a weakly increasing sequence. It converges as it is bounded from above by $Nash(\delta^*)$ where δ^* is a Nash distribution.

Step 2
The KKT conditions for this concave optimization problem are sufficient (as argued in the proof of Theorem 4.8), i.e., every $\delta^* \in \Delta(1)$ that satisfies them maximizes Nash welfare. The same proof shows that, for every $x \in A$ with $\mu_x \geqslant 0$, the KKT conditions are given by

$$\sum_{i \in N} C_i \frac{v_{i,x}}{u_i(\delta^*)} + \mu_x = 1 \quad \text{and} \quad [\delta_x^* > 0 \text{ implies } \mu_x = 0].$$

Assume that the dynamic procedure *terminates*, i.e., for some t, $\delta_x^t = \delta_x^{t+1} = \delta_x^t \sum_{i \in N} C_i(v_{i,x}/u_i(\delta^t))$ for all $x \in A$. Recalling that $\delta_x^t > 0$ for all projects $x \in A$ and $t \in \mathbb{N}$, δ^t satisfies the KKT conditions and is a Nash distribution.

In all other cases, let δ' be an accumulation point of $(\delta^t)_{t \in \mathbb{N}}$ and $(\delta^{t_l})_{l \in \mathbb{N}}$ be a subsequence converging to it. We show that δ' is a fixed-point of f. The sequence $(Nash(f(\delta^{t_l})) - Nash(\delta^{t_l}))_{l \in \mathbb{N}}$ converges to 0 by Step 1. Continuity of *Nash* implies $0 = Nash(f(\delta')) - Nash(\delta') \geqslant (2\log(2))^{-1} \|f(\delta') - \delta'\|_1^2$, and so $f(\delta') = \delta'$. Therefore, $\delta'_x = \delta'_x \sum_{i \in N} C_i(v_{i,x}/u_i(\delta'))$, which shows that δ' satisfies the KKT conditions for all x with $\delta'_x > 0$.

Denote by S the set of all accumulation points. S is connected as the step size of

the dynamics converges to 0 by Step 1. As $(Nash(\delta^t))_{t\in\mathbb{N}}$ converges, $Nash(\delta') = Nash(\delta'')$ for any two $\delta', \delta'' \in S$. If there exists a $\delta' \in S$ that has full support, then δ' and consequently, all accumulation points are Nash distributions as $(Nash(\delta^t))_{t\in\mathbb{N}}$ is increasing.
In the remaining cases, every accumulation δ' point is located in a face $T_{\delta'} = \{\delta \in \Delta(1) : \delta'_x = 0 \Rightarrow \delta_x = 0\}$ of $\Delta(1)$ and maximizes the Nash product on this face by the fact that δ' has full support in $T_{\delta'}$. Therefore, $u_i(\delta') = u_i(\delta'')$ for all $i \in N$ and $\delta', \delta'' \in T_{\delta'}$ and even for general $\delta', \delta'' \in S$ by connectivity of S. Assume now that there exist $\delta' \in S$ and $x \in A$ with $\delta'_x = 0$ but $\sum_{i\in N} C_i(v_{i,x}/u_i(\delta')) > 1$. This implies $\lim_{t\to\infty} \sum_{i\in N} C_i(v_{i,x}/u_i(\delta^t)) > 1$ which contradicts $\delta'_x = 0$.

Combining both steps, we conclude that every accumulation point of $(\delta^t)_{t\in\mathbb{N}}$ is a Nash distribution and $(Nash(\delta^t))_{t\in\mathbb{N}}$ converges to the optimum Nash welfare as it is weakly increasing. If the Nash distribution is unique, $(\delta^t)_{t\in\mathbb{N}}$ thus has a unique accumulation point and converges (to the Nash distribution). □

Thus, by simply computing terms of the sequence $(\delta^t)_{t\in\mathbb{N}}$, a Nash distribution can be approximated without resorting to convex programming.

In addition, every distribution δ^t appearing in the sequence (apart from δ_0) admits a decomposition where agents need to contribute only to approved projects, an important property when stopping after a finite number of steps.

Interesting open questions concern the convergence rate and whether the dynamics indeed always converge to a unique distribution, as our simulations suggest.

Regarding the former question, Cover (1984) noted that the approximation error can be bounded via $Nash(\delta^*) - Nash(\delta^t) \leqslant \max_{x\in A} \log\left(\sum_{i\in N} C_i(v_{i,x}/u_i(\delta^t))\right)$.

Regarding the latter question, we were able to show convergence to a Nash distribution for cases where non-uniqueness is induced by two "clone" projects a and b, i.e., the Nash distribution is unique if we were to merge these two projects. Starting the dynamic procedure with the uniform distribution over all projects, we have $\delta^t_a = \delta^t_b$ at each step t, which implies that the dynamic procedure does converge to a Nash distribution that puts equal contributions on a and b.

4.3 DICHOTOMOUS UTILITIES

We conclude this chapter by restricting valuations $v_{i,x}$ to $\{0, 1\}$, i.e., each agent i has a non-empty set of approved projects A_i and $u_i(\delta) = \sum_{x\in A_i} \delta_x$.

Note that Hylland's theorem (Theorem 4.4) does not hold in this smaller domain. On the negative side, efficiency, strategyproofness, and fairness are still incompatible.

THEOREM 4.22 *(Brandl et al. (2021b))*
No mechanism satisfies efficiency, strategyproofness, and positive share[29] when $n \geqslant 6$ and $m \geqslant 4$.

29 For dichotomous preferences, positive share simply demands that each set of approved projects A_i receives positive contribution.

On the positive side, there exist attractive strategyproof mechanisms, above all the *conditional utilitarian rule* (*CUT*) that was introduced by Duddy (2015) and proved to be strategyproof by Aziz et al. (2020). Under *CUT*, each agent spreads her contribution uniformly over all projects in A_i that are approved by most agents where approvals are weighted by contributions.[30] Formally, let $n_x = \sum_{i \in N\colon x \in A_i} C_i$ for $x \in A$ be the total contribution of all agents that approve project x and denote by $A_i^+ = \{x \in A_i \colon n_x \geqslant n_y \text{ for all } y \in A_i\}$. With that,

$$CUT(U, C)_x = \sum_{i \in N\colon x \in A_i^+} \frac{C_i}{|A_i^+|} \qquad \text{for all } x \in A.$$

By construction, *CUT* can be computed efficiently and always returns an equilibrium distribution. For example, $CUT(U, C) = (1/2, 1/2, 3)$ in Example 4.18.

Furthermore, Brandl et al. (2019) showed that *CUT* is contribution incentive-compatible. They also noted that it satisfies a natural weakening of efficiency where only distributions that can be decomposed such that each agent distributed her contribution on approved projects are considered for domination.

Definition 4.23
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ is *efficient among equilibrium distributions* if it is not Pareto dominated by an equilibrium distribution.

A mechanism is efficient among equilibrium distributions if it returns such a distribution for each profile $(U, C) \in \mathcal{P}$.

Theorem 4.24 *(Brandl et al. (2019); Aziz et al. (2020))*
CUT is efficient among equilibrium distributions, strategyproof, and contribution incentive-compatible.

Strong contribution incentive-compatibility coincides with its weaker version for dichotomous preferences, making it an even more desirable axiom. For public good markets, agents are (at least weakly) better off contributing their entire budget, even under the presence of private goods, when contributions are distributed via a contribution incentive-compatible mechanism.

Although equilibrium distributions and the axiom of contribution incentive-compatibility seem closely related for dichotomous preferences, they do not imply one another.

Proposition 4.25
A mechanism that always returns equilibrium distributions may not satisfy contribution incentive-compatibility, and vice versa.

Proof. First, to see that returning equilibrium distributions is not sufficient for contribution incentive-compatibility, consider a rule that always returns an equilibrium distribution with *minimal* weighted utilitarian welfare and thus represents an antipode to *CUT*. Define $A_i^- = \{x \in A_i \colon n_x \leqslant n_y \text{ for all } y \in A_i\}$. Then,

$$ANTICUT(U, C)_x = \sum_{i \in N\colon x \in A_i^-} \frac{C_i}{|A_i^-|} \qquad \text{for all } x \in A.$$

30 Duddy (2015) and Aziz et al. (2020) work in a portioning setting, i.e., all agents have the same "contributions" equal to $1/n$.

This mechanism always returns equilibrium distributions by construction but fails to satisfy contribution incentive-compatibility.

For example, assume we are given $A = \{a, b\}$ and two agents with $A_1 = \{a, b\}$, $A_2 = \{a\}$, and $\bar{C}_1 = \bar{C}_2 = 1$. For $C^{12} = (1, 1)$ and $C^1 = (1, 0)$, $ANTICUT(U, C^1) = (1/2, 1/2)$ and $ANTICUT(U, C^{12}) = (1, 1)$. Since

$$u_2(ANTICUT(U, C^1)) + 1 = 3/2 > 1 = u_2(ANTICUT(U, C^{12})),$$

ANTICUT violates contribution incentive-compatibility.

Second, we construct a mechanism that is contribution incentive-compatible but does not always return equilibrium distributions for the following set of instances with $A = \{a, b, c, d\}$, $N = \{1, 2, 3\}$ and a fixed utility profile.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	$v_{i,d}$	$\bar{C}_i$
Agent 1	1	1	0	0	1
Agent 2	1	0	1	0	1
Agent 3	0	0	0	1	1

We define $f(U, C)$ via

$$\begin{aligned} f(U, C)_a &= \min\{C_1, C_2\}, \\ f(U, C)_b &= C_1 - \min\{C_1, C_2\}, \\ f(U, C)_c &= C_2 - \min\{C_1, C_2\}, \text{ and} \\ f(U, C)_d &= \min\{C_1, C_2\} + C_3. \end{aligned}$$

Then, f does not return an equilibrium distribution for the contribution vector $C^{123} = (1, 1, 1)$ as $f(U, C^{123}) = (1, 0, 0, 2)$. However, one can verify that f is contribution incentive-compatible by distinguishing the cases $C_1 < C_2$ and $C_1 \geqslant C_2$. □

All in all, *CUT* is a valuable alternative to *NASH* in the case of dichotomous preferences, especially when strategyproofness is required.[31]

While *CUT* maximizes weighted utilitarian welfare $\sum_{i \in N} C_i u_i(\cdot)$ among mechanisms that always return equilibrium distributions, the statement does not hold when we require contribution incentive-compatibility instead.

Proposition 4.26
CUT is not weighted utilitarian welfare maximizing among contribution incentive-compatible mechanisms.

Proof. For $A = \{a, b, c, d\}$ and an arbitrary set of agents N with $\bar{C}_i = 1$ for all $i \in N$, we show that the following rule f satisfies contribution incentive-compatibility and the weighted utilitarian welfare of the returned distribution $f(U, C)$ is at least as large as the weighted utilitarian welfare of $CUT(U, C)$ for each profile $(U, C) \in \mathcal{P}$ and larger for some of them. Define $f(U, C) = CUT(U, C)$, unless the two following conditions hold for profile (U, C):

1. $2C_\Delta := n_x = n_y = n_z$ for three projects $x, y, z \in A$ and $n_w > n_x$ for $w \in A \setminus \{x, y, z\}$.

31 By Theorem 4.22, *NASH* violates strategyproofness.

2. For all agents i with $C_i > 0$, $w \in A_i$ implies $A_i = \{w\}$ and $w \notin A_i$ implies $|A_i| = 2$.

For these exceptional profiles, set

$$f(U,C)_w = CUT(U,C)_w + \frac{3}{2}C_\Delta \quad \text{and}$$
$$f(U,C)_x = f(U,C)_y = f(U,C)_z = CUT(U,C)_x - \frac{1}{2}C_\Delta$$

where *CUT* returns the distribution $(n_w, C_\Delta, C_\Delta, C_\Delta)$. We observe that for such profiles, f gives larger weighted utilitarian welfare than *CUT* as $n_w > n_x = n_y = n_z$. It remains to show that f does not violate contribution incentive-compatibility.

For that, assume that some agent i increases her contribution from $C_i \geqslant 0$ to $C_i' = C_i + \varepsilon \leqslant 1$ with $\varepsilon > 0$, moving from profile (U, C) to (U, C').

We distinguish two cases.

Case 1

Assume that (U, C) is an exceptional profile. The case $A_i = A$ is trivial.

If $A_i = \{w\}$, then (U, C') is also exceptional and

$$\begin{aligned} u_i(f(U,C')) &= CUT(U,C')_w + \frac{3}{2}C_\Delta = CUT(U,C)_w + \varepsilon + \frac{3}{2}C_\Delta \\ &= u_i(f(U,C)) + \varepsilon. \end{aligned}$$

For all other A_i's, (U, C') is not exceptional.

If $w \notin A_i$,

$$u_i(f(U,C')) = u_i(CUT(U,C')) \geqslant u_i(CUT(U,C)) + \varepsilon > u_i(f(U,C)) + \varepsilon$$

by contribution incentive-compatibility of *CUT*.

If w.l.o.g., $A_i = \{w, x\}$, then $n_x' > n_y'$ and $n_x' > n_z$. Thus, $CUT(U, C')_x = 2C_\Delta$ and

$$u_i(f(U,C')) = u_i(CUT(U,C')) = n_w + \varepsilon + 2C_\Delta = u_i(f(U,C)) + \varepsilon.$$

Finally, if w.l.o.g., $A_i = \{w, x, y\}$, then $f(U, C')_z = 0$ and $u_i(f(U, C')) = C_N + \varepsilon$.

Case 2

Assume that (U, C) is not exceptional. If (U, C') is not exceptional either, f coincides with *CUT* on these two profiles, and contribution incentive-compatibility cannot be violated. Otherwise, (U, C') is exceptional and $A_i \in \{\{w\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ has to hold. In particular, $n_x' = n_y' = n_z'$.

If $A_i = \{w\}$,

$$u_i(f(U,C')) > u_i(CUT(U,C')) \geqslant u_i(CUT(U,C)) + \varepsilon = u_i(f(U,C)) + \varepsilon$$

by contribution incentive-compatibility of *CUT*.

If w.l.o.g., $A_i = \{x, y\}$, then $n_x = n_y = n_z - \varepsilon$. With $2C'_\Delta = n_z$,

$$u_i(f(U, C')) = CUT(U, C')_x - \frac{1}{2}C'_\Delta + CUT(U, C')_y - \frac{1}{2}C'_\Delta$$
$$= C'_\Delta = \frac{n_x - \varepsilon}{2} + \varepsilon = u_i(f(U, C)) + \varepsilon$$

where the last inequality follows from the fact that only agents with approval sets $\{x, y\}$ contribute to x or y under *CUT* in (U, C). The contributions from that set sum up to $(n_x - \varepsilon)/2$ whereas the contributions of agents with approvals $\{x, z\}$ and $\{y, z\}$ add up to $(n_x + \varepsilon)/2$, respectively, in order to get $n_x = n_y = n_z - \varepsilon$.

All in all, f never violates contribution incentive-compatibility. □

This raises the question of whether there are other sensible mechanisms "between" *CUT* and utilitarian welfare maximization. Axiomatic characterizations of *CUT* and *NASH* using contribution incentive-compatibility are still pending.

For dichotomous preferences, Guerdjikova and Nehring (2014) showed that *NASH*[32] is essentially characterized by always returning a convex set of equilibrium distributions in conjunction with satisfying certain consistency conditions. To be precise, they additionally require that 1) the set of returned distributions does not change when excluding projects that are not approved by anybody, 2) if the sets of returned distributions for two profiles have a non-empty intersection, then exactly the distributions from that intersection are returned for any convex combination of the two profiles with $\lambda \in (0, 1)$, and 3) the mechanism is upper hemicontinuous. To the best of our knowledge, their proof cannot be extended to general linear utilities as it heavily relies on binary valuations.

Bogomolnaia et al. (2002) proved, again for the case of dichotomous preferences, that *NASH* is the only mechanism that satisfies *unanimous fair share*[33] among rules that maximize $\sum_{i \in N} C_i g(u_i(\cdot))$ where $g\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ is continuous, increasing, and strictly concave.

We are not aware of other mechanisms than *NASH* that satisfy efficiency and contribution incentive-compatibility for arbitrary numbers of agents and projects. However, when fixing, e.g., $m = 3$, other such mechanisms can be defined. This shows that at least some other consistency conditions, like continuity or 2) from above, are needed. Furthermore, proving that *NASH* satisfies stronger versions of contribution incentive-compatibility (e.g., the one mentioned in Section 4.2.1) seems desirable and promising for characterizing and better understanding *NASH*.

32 They call the set of all Nash welfare maximizing distributions the *diversity value*.

33 This property is a weakening of the core in the sense that only subsets of agents with the same approval sets are considered in Definition 2.13.

5 DONOR COORDINATION WITH COMPLEMENTS

In this chapter, we consider Leontief utility functions $u_i \colon \Delta([\underline{C}, \bar{C}]) \to \mathbb{R}_{\geqslant 0}$, i.e.,

$$u_i(\delta) = \min_{x \in A_i} \frac{\delta_x}{\nu_{i,x}}$$

with $\nu_{i,x} \geqslant 0$ for all $x \in A$ and $i \in N$ and $A_i = \{x \in A \colon \nu_{i,x} > 0\}$ again denotes the set of *approved* projects. In Section 5.3, we consider the case of *binary weights* where $\nu_{i,x} \in \{0, 1\}$ for all $i \in N$ and $x \in A$.

Before investigating equilibrium distributions, we define the concept of *critical projects* which will turn out to be very useful when thinking about and dealing with Leontief utilities.

Definition 5.1
Given a distribution δ for a fixed profile (U, C), we define the set of agent i's *critical projects* as

$$T_{\delta,i} := \arg\min_{x \in A_i} \frac{\delta_x}{\nu_{i,x}}.$$

Each project $x \in T_{\delta,i}$ is critical for agent i in the sense that the utility of i would decrease if the amount allocated to x were to decrease.

For linear utilities (discussed in Chapter 4), all approved projects are critical in that sense, independent of the actual distribution. This nicely illustrates the fundamental differences between substitutes and complements. Agents receive utility separately from each project for linear utilities, whereas the opposite is true for Leontief utilities.

Naturally, every agent has at least one critical project. The following equivalences hold for each agent i and project x such that either $\delta_x > 0$ or $\nu_{i,x} > 0$:

$$\begin{aligned} x \in T_{\delta,i} \quad &\Leftrightarrow \quad \delta_x = \nu_{i,x} \cdot u_i(\delta) \qquad \text{and} \\ x \notin T_{\delta,i} \quad &\Leftrightarrow \quad \delta_x > \nu_{i,x} \cdot u_i(\delta). \end{aligned} \tag{12}$$

For every group of agents $N' \subseteq N$, we denote by $T_{\delta,N'}$ the set of projects critical to at least one member of N'.

5.1 EQUILIBRIUM DISTRIBUTIONS AND EFFICIENCY

Critical projects allow for a nice characterization of equilibrium distributions.

Proposition 5.2
For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ is an equilibrium distribution if and only if it admits a decomposition $(\delta_i)_{i \in N}$ such that $\delta_{i,x} = 0$ for every project $x \notin T_{\delta,i}$, i.e., each agent contributes only to her critical projects.

Proof. To show that each equilibrium distribution admits such a decomposition, suppose that, in every decomposition of δ, some agent i contributes to a project $y \notin T_{\delta,i}$. Fix a decomposition $(\delta_i)_{i\in N}$ of δ. Since $\delta_y > 0$, there exists an $\varepsilon > 0$ such that $\delta_y - \varepsilon > \nu_{i,y} \cdot u_i(\delta)$ by (12). Thus, agent i can reduce δ_{iy} by ε and distribute it equally among all projects in $T_{\delta,i}$. This strictly increases agent i's utility. Therefore, δ is not an equilibrium distribution.

For the converse direction, suppose δ admits a decomposition in which each agent i only contributes to projects in $T_{\delta,i}$. In every other individual distribution of agent i, she contributes less money to at least one such project $y \in T_{\delta,i}$. Since $\delta_y > 0$, by (12), the original distribution to project y was $\delta_y = \nu_{i,y} \cdot u_i(\delta)$, implying that less than $\nu_{i,y} \cdot u_i(\delta)$ is allocated to y under the new distribution. Therefore, agent i' utility is smaller than $u_i(\delta)$, and the deviation is not beneficial. □

EXAMPLE 5.3

Consider the following example with three projects $A = \{a, b, c\}$ and two agents.

	$\nu_{i,a}$	$\nu_{i,b}$	$\nu_{i,c}$	C_i
Agent 1	1	1	0	1
Agent 2	0	1	1	1

Then, $\delta^* = (2/3, 2/3, 2/3)$ with $\delta_1^* = (2/3, 1/3, 0)$ and $\delta_2^* = (0, 1/3, 2/3)$ is the unique equilibrium distribution. Projects a and b are critical for Agent 1, whereas projects b and c are critical for Agent 2.

COROLLARY 5.4

In an equilibrium distribution, every project that receives a positive amount is critical for at least one agent.

Proposition 5.2 implies that an equilibrium distribution satisfies an even stronger equilibrium property.

COROLLARY 5.5

In every equilibrium distribution (and associated decomposition), no *group of agents* can deviate without making at least one of its members worse off. Consequently, every equilibrium distribution lies in the core.

This is because *any* deviation decreases the contribution to a critical project of at least one group member. This equilibrium notion is slightly stronger than Aumann's *strong equilibrium* and implies membership of equilibrium distributions in the core.

Moreover, the set of efficient distributions can be characterized as follows.

PROPOSITION 5.6

For a fixed profile (U, C), a distribution $\delta \in \Delta(C_N)$ is efficient if and only if every project $x \in \text{supp}(\delta)$ is critical for some agent.

Proof. To show that efficiency of δ implies that every project in $\text{supp}(\delta)$ is critical for some agent, assume that some project $x \in \text{supp}(\delta)$ is not critical for any agent. Since $\delta_x > 0$, there exists an $\varepsilon > 0$ such that $\delta_x - \varepsilon \geqslant \nu_{i,x} \cdot u_i(\delta)$ for all agents $i \in N$ by (12). We claim that the distribution δ' with $\delta'_x = \delta_x - (1 - 1/m)\varepsilon$ and

$\delta'_y = \delta_y + (1/m)\varepsilon$ for all $y \in A \setminus x$ Pareto dominates δ. We even prove $u_i(\delta') > u_i(\delta)$ for every agent $i \in N$. Indeed,

$$u_i(\delta') = \min\left(\frac{\delta'_x}{\nu_{i,x}}, \min_{y \in A_i \setminus x} \frac{\delta'_y}{\nu_{i,y}}\right)$$

where $\delta'_x/\nu_{i,x} := \infty$ for $\nu_{i,x} = 0$ and only the second term needs to be considered. Both terms are larger than $u_i(\delta)$ for every $i \in N$. The first term $\delta'_x/\nu_{i,x}$ is larger than $u_i(\delta)$ by our choice of ε. For all other projects $y \in A_i$, $\delta'_y/\nu_{i,y} > \delta_y/\nu_{i,y} \geqslant u_i(\delta)$ by construction. Hence, δ is not efficient.

For the converse direction, suppose that every project $x \in \text{supp}(\delta)$ is critical for some agent. Let δ' be any distribution different than δ. Since the sum of both distributions is the same (C_N), there exists a project $y \in \text{supp}(\delta)$ with $\delta'_y < \delta_y$. Let $i_y \in N$ be an agent for whom y is critical in δ. Then the utility of i_y is strictly smaller in δ' as

$$\begin{aligned} u_{i_y}(\delta') &\leqslant \frac{\delta'_y}{\nu_{i_y,y}} && \text{(by definition of Leontief utilities)} \\ &< \frac{\delta_y}{\nu_{i_y,y}} && (\delta'_y < \delta_y \text{ and } \nu_{i_y,y} > 0 \text{ by definition of } y \text{ and } i_y) \\ &= u_{i_y}(\delta) && \text{(by (12), since } y \text{ is critical for } i_y \text{ in } \delta) \end{aligned}$$

so δ' does not dominate δ. Hence, δ is efficient. □

Despite this characterization, the set of efficient distributions fails to be convex, just as in the case of linear utilities (see Bogomolnaia et al., 2005). In Example 5.3, $\delta = (1,1,0)$ and $\delta' = (0,1,1)$ are both efficient distributions, but not $0.5\delta + 0.5\delta' = (1/2, 1, 1/2)$ as project b is not critical for any agent (Proposition 5.6).

Moreover, Proposition 5.6 and Corollary 5.4 imply efficiency of all equilibrium distributions.

Corollary 5.7
Every equilibrium distribution is efficient.

Corollary 5.7 starkly contrasts linear utilities, where we observed that coordination is indispensable for achieving efficiency. Our preceding results for Leontief utilities already suggest the concept of equilibrium to be (at least in some sense) stronger here. Therefore, looking at mechanisms that return equilibrium distributions (which are guaranteed to exist by Theorem 2.9) seems fruitful.

Furthermore, equilibrium distributions can also be characterized among efficient distributions via some fairness property for groups in the spirit of Proposition 4.2.

Lemma 5.8
For a fixed profile (U, C), if δ is no equilibrium distribution but efficient, then N can be partitioned into two disjoint groups of agents, N_+ and $N_- = N \setminus N_+$, such that

$$\delta(T_{\delta,N_-}) < C_{N_-} \quad \text{and} \tag{13}$$

$$\delta(T_{\delta,N_+} \setminus T_{\delta,N_-}) > C_{N_+}. \tag{14}$$

Proof. Let $(\delta_i)_{i \in N}$ be any decomposition of δ. Construct a directed graph G in which the nodes correspond to agents, and there is an arc $i \to j$ if and only if $\delta_{i,T_{\delta,j}} > 0$, that is, agent i contributes to a critical project of agent j. We call the arc $i \to j$ *strong* if $\delta_i(T_{\delta,j} \setminus T_{\delta,i}) > 0$, that is, agent i contributes to a project that is critical for j but not for i. Otherwise, we call the arc $i \to j$ *weak*. As δ is no equilibrium distribution, there is an agent, say Agent 1, who contributes to a project $x \notin T_{\delta,1}$ by Proposition 5.2. As δ is efficient, x is critical to some other agent, say Agent 2, by Proposition 5.6, so G contains a strong arc $1 \to 2$.

If the strong arc is part of a directed cycle, then we can move a sufficiently small amount ε along the cycle without changing δ. In detail, suppose w.l.o.g. that the cycle is $1 \to 2 \to \cdots \to k \to 1$, where the involved projects are $x_1 \in T_{\delta,1}$, $x_2 \in T_{\delta,2} \setminus T_{\delta,1}$, $x_3 \in T_{\delta,3}$, $x_4 \in T_{\delta,4}$, $\ldots$, $x_k \in T_{\delta,k}$. We assume that x_2 is in $T_{\delta,2} \setminus T_{\delta,1}$ since the arc $1 \to 2$ is strong. In particular, x_2 must be different than x_1. The other arcs may be strong or weak, and some of the x_i may coincide. For every $i \in \{1, \ldots, k-1\}$, move some small amount $\varepsilon > 0$ from $\delta_{i,x_{i+1}}$ to δ_{i,x_i} and move the same ε from δ_{k,x_1} to δ_{k,x_k}. Note that the decomposition changes, but the overall δ remains the same. Increase ε until one arc of the cycle disappears or the strong arc becomes weak. Repeat this cycle-removal procedure until all strong arcs are not part of any directed cycle. This process is guaranteed to terminate since in each cycle removal, either the respective strong arc becomes weak, or the cycle it is part of is removed. Furthermore, no new (strong) arcs are created as agents do not contribute to additional projects, and the overall distribution δ and the set of critical projects do not change.

Let G be the graph of the resulting decomposition. As the total distribution is still δ, which is efficient but no equilibrium distribution, G still has at least one strong arc, say from agent $j \to k$. Let N_+ be the set of agents accessible from k via a directed path (where $k \in N_+$), and let $N_- := N \setminus N_+$. As $j \to k$ is not part of any directed cycle, $j \in N_-$.

Due to the strong arc $j \to k$, agents of N_- "waste" some of their own contributions on critical projects of N_+, that are not critical for themselves. Moreover, the critical projects of N_- do not receive any donations from agents of N_+, since they are not accessible from N_+. This proves (13).

In contrast, the agents in N_+ spend all their contributions on their own critical projects that are not critical for agents outside N_+. In addition, they receive some donations from agents of N_-. This proves (14). □

Lemma 5.8 can be used for an alternative proof of Lemma 5.11 below (see [2]). It is also central to the characterization obtained in Section 5.4.

Next, we show that every efficient utility vector is generated by at most one distribution.

Proposition 5.9
For a fixed profile (U, C), let $\delta \in \Delta(C_N)$ and $\delta' \in \Delta(C_N)$ be efficient distributions inducing the same utility vector, i.e., $u_i(\delta) = u_i(\delta')$ for all $i \in N$. Then, $\delta = \delta'$.

Proof. For each $x \in \text{supp}(\delta)$, there is an agent for whom x is critical by Proposition 5.6. Denote one such agent by i_x. Then,

$$\begin{aligned}
\delta_x &= \nu_{i_x,x} \cdot u_{i_x}(\delta) && \text{(by (12) as } x \text{ is critical for } i_x\text{)} \\
&= \nu_{i_x,x} \cdot u_{i_x}(\delta') && \text{(by assumption)} \\
&\leqslant \delta'_x && \text{(by definition of Leontief utilities).}
\end{aligned}$$

The same inequality $\delta_x \leqslant \delta'_x$ trivially holds also for all $x \notin \text{supp}(\delta)$ by efficiency of δ and δ'. Since both distributions sum up to C_N, this implies $\delta = \delta'$. □

Consequently, an efficient mechanism can also be interpreted as a mapping from profiles to utility vectors.

In the remainder of this section, we prove the following theorem.

THEOREM 5.10
Every profile admits a unique equilibrium distribution. This distribution maximizes Nash welfare.

We start by giving an alternative proof for equilibrium existence that is tailored to our setting and establishes the connection to Nash welfare.

LEMMA 5.11
For a fixed profile, every distribution that maximizes Nash welfare constitutes an equilibrium distribution.

Proof. For a fixed profile (U, C), Nash welfare optimization can be presented as the following maximization problem, with variables $(\tilde{u}_i)_{i \in N}$ for the utilities and $(d_x)_{x \in A}$ for the distribution:

$$\begin{aligned}
\max \sum_{i \in N} & C_i \log \tilde{u}_i \\
\text{s.t. } & \tilde{u}_i \leqslant d_x / \nu_{i,x} && \forall i \in N, x \in A_i \\
& \sum_{x \in A} d_x = C_N \\
& \tilde{u}_i \geqslant 0 && \forall i \in N \\
& d_x \geqslant 0 && \forall x \in A.
\end{aligned}$$

As we can always give each agent a positive utility, and a zero utility leads to $-\infty$ for the objective, we can assume that $\tilde{u}_i$ is always positive, and therefore, the constraints $\tilde{u}_i \geqslant 0$ can be ignored.

Moreover, $d_x > 0$ for every x positively valued by at least one agent, whereas other projects can be excluded from consideration. Therefore, we can also neglect the constraints $d_x \geqslant 0$.

Now, the Lagrangian $\mathcal{L}(\tilde{u}, d, \lambda, \ldots, \mu_{i,x}, \ldots)$ is equal to

$$\sum_{i \in N} C_i \log \tilde{u}_i + \sum_{i \in N, x \in A_i} \mu_{i,x}(\tilde{u}_i - d_x / \nu_{i,x}) + \lambda \left(\sum_{x \in A} d_x - C_N \right).$$

The first derivative conditions (besides the constraints) are:

$$C_i/\tilde{u}_i + \sum_{x \in A_i} \mu_{i,x} = 0 \qquad \forall i \in N \tag{15}$$

$$-\sum_{i \in N_x} \mu_{i,x}/\nu_{i,x} + \lambda = 0 \qquad \forall x \in A \tag{16}$$

where $N_x := \{i \in N : x \in A_i\}$. The complementary slackness conditions imply that

$$\mu_{i,x} \cdot (\tilde{u}_i - d_x/\nu_{i,x}) = 0 \qquad \forall i \in N, x \in A_i. \tag{17}$$

That is, whenever $\tilde{u}_i < d_x/\nu_{i,x}$, which means that x is not critical for agent i, we must have $\mu_{i,x} = 0$.

We claim that $\delta_{i,x} := -\mu_{i,x} \cdot \tilde{u}_i$ for $x \in A_i$ and $\delta_{i,x} := 0$ for $x \notin A_i$ defines a decomposition satisfying the requirements of Proposition 5.2.

Equation (15) implies that, for all $i \in N$, $\sum_{x \in A_i} \delta_{i,x} = -\tilde{u}_i \sum_{x \in A_i} \mu_{i,x} = C_i$.

Equation (16) implies that, for all $x \in A$, $\sum_{i \in N_x} \delta_{i,x} = -\sum_{i \in N_x} \mu_{i,x} \cdot \tilde{u}_i = -\sum_{i \in N_x} \mu_{i,x} \cdot \delta_x/\nu_{i,x} = -\lambda \cdot \delta_x$. Summing over all x on both sides gives $\sum_{i \in N} C_i = -\lambda \cdot \sum_{x \in A} d_x = -\lambda \cdot C_N$, so $\lambda = -1$. Therefore, $\sum_{i \in N} \delta_{i,x} = -\lambda \cdot \delta_x = \delta_x$.

Equation (17) implies that $\delta_{i,x} = 0$ if project $x \in A_i$ is not critical for agent i.

All in all, δ is an equilibrium distribution by Proposition 5.2. □

Lemma 5.12
For each profile, every equilibrium distribution maximizes Nash welfare.

Proof. Let δ^* be an equilibrium distribution for a fixed profile (U, C). For any distribution δ, we derive an upper bound for $Nash(\delta)$ in terms of δ^*. We show that this upper bound is maximized when $\delta = \delta^*$ and is equal to $Nash(\delta)$ for $\delta = \delta^*$. Thus, $Nash(\delta) \leqslant Nash(\delta^*)$ so δ^* maximizes Nash welfare.

Formally, let $(\delta_i^*)_{i \in N}$ be any decomposition of δ^* satisfying Proposition 5.2, and let $N_{\delta^*,x} := \{i \in N : x \in T_{\delta^*,i}\}$ be the set of agents for whom x is critical in δ^*.

For every distribution δ with $Nash(\delta) > -\infty$, we have

$$\begin{aligned}
Nash(\delta) &= \sum_{i \in N} C_i \log(u_i(\delta)) \\
&\overset{(1)}{=} \sum_{i \in N} \left(\sum_{x \in T_{\delta^*,i}} \delta^*_{i,x} \right) \log(u_i(\delta)) \\
&\overset{(2)}{\leqslant} \sum_{i \in N} \sum_{x \in T_{\delta^*,i}} \delta^*_{i,x} \log\left(\frac{\delta_x}{\nu_{i,x}}\right) \\
&\overset{(3)}{=} \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta^*_{i,x} \log\left(\frac{\delta_x}{\nu_{i,x}}\right)\right) \\
&= \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta^*_{i,x} \right) \log(\delta_x) - \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta^*_{i,x} \log(\nu_{i,x}) \right) \\
&= \sum_{x \in A} \delta^*_x \log(\delta_x) - \sum_{x \in A} \left(\sum_{i \in N_{\delta^*,x}} \delta^*_{i,x} \log(\nu_{i,x}) \right)
\end{aligned}$$

where (1) follows from the fact that δ^* is an equilibrium distribution, (2) holds as $u_i(\delta) \leqslant \delta_x/\nu_{i,x}$ for all $x \in T_{\delta^*,i}$, and we changed summation order in (3).

We claim that, for every fixed δ^*, the last expression is maximized for $\delta = \delta^*$.

The second term is independent of δ.

As for the first term $\sum_{x\in A} \delta^*_x \log(\delta_x)$, consider the optimization problem of maximizing $\sum_{x\in A} \delta^*_x \log(\delta_x)$ subject to $\sum_{x\in A} \delta_x = \sum_{x\in A} \delta^*_x$. Note that δ^* is a constant in this problem. Its Lagrangian is

$$\sum_{x\in A} \delta^*_x \log(\delta_x) + \lambda \cdot \left(\sum_{x\in A} \delta^*_x - \sum_{x\in A} \delta_x \right).$$

Setting the derivative to 0 with respect to δ_x gives $\delta^*_x/\delta_x = \lambda$ for all $x \in A$. As $\sum_{x\in A} \delta_x = \sum_{x\in A} \delta^*_x$, we must have $\lambda = 1$, so $\delta = \delta^*$. Thus,

$$\mathit{Nash}(\delta) \leqslant \sum_{x\in A} \delta^*_x \log(\delta^*_x) - \sum_{x\in A} \left(\sum_{i\in N_{\delta^*,x}} \delta^*_{i,x} \log(\nu_{i,x}) \right).$$

For $\mathit{Nash}(\delta^*)$, the same derivation holds, but the inequality becomes an equality as δ^* is an equilibrium distribution, implying that $\delta^*_{i,x} > 0$ only if $u_i(\delta^*) = \delta^*_x/\nu_{i,x}$. Therefore, $\mathit{Nash}(\delta) \leqslant \mathit{Nash}(\delta^*)$, so δ^* maximizes Nash welfare. □

Since the logarithm function is strictly concave, Proposition 7.3 in Section 7.2 implies that there is a unique distribution that maximizes Nash welfare.

Hence, Lemmas 5.11 and 5.12 entail uniqueness of the equilibrium distribution which additionally maximizes Nash welfare, as claimed in Theorem 5.10.

5.2 THE NASH PRODUCT RULE

As a consequence of Theorem 5.10, the Nash product rule (*NASH*)[34] is a well-defined mechanism and always returns the equilibrium distribution for Leontief utilities.

5.2.1 Properties

We have already observed that *NASH* is efficient (Corollary 5.7) and always returns a core outcome (Corollary 5.5). Fascinatingly, *NASH* is also strategyproof. We prove that *NASH* is even group-strategyproof with the help of the following lemma.

LEMMA 5.13
Let δ and δ' be two distributions, and $i \in N$ an agent.

(a) If $u_i(\delta') \geqslant u_i(\delta)$, then every project in $T_{\delta,i}$ receives at least as much funding in δ', i.e., $\delta'_y \geqslant \delta_y$ for all $y \in T_{\delta,i}$.

(b) Similarly, if $u_i(\delta') > u_i(\delta)$, then $\delta'_y > \delta_y$ for all $y \in T_{\delta,i}$.

34 In [2], we call this mechanism the *equilibrium distribution rule* (*EDR*).

Proof. For (a), note that for every project $y \in T_{\delta,i}$, we have

$$\begin{aligned}
\delta_y &= \nu_{i,y} \cdot u_i(\delta) && \text{(by (12), as } y \text{ is critical for agent } i \text{ in } \delta) \\
&\leqslant \nu_{i,y} \cdot u_i(\delta') && \text{(by assumption)} \\
&= \nu_{i,y} \cdot \min_{x \in A_i} \frac{\delta'_x}{\nu_{i,x}} && \text{(by definition of Leontief utilities)} \\
&\leqslant \nu_{i,y} \cdot \frac{\delta'_y}{\nu_{i,y}} && \text{(since } y \in T_{\delta,i} \subseteq A_i) \\
&= \delta'_y.
\end{aligned}$$

For (b), the first inequality becomes strict. □

THEOREM 5.14
NASH satisfies group-strategyproofness.

Proof. Suppose by contradiction that some group of agents has a successful manipulation, and let $G \subseteq N$ be an inclusion-maximal such group. For an arbitrary profile $P = (U, C)$, denote by $P' = (U', C)$ the profile after a successful manipulation by G, and by δ and δ' the respective equilibrium distributions. As the manipulation succeeds, $u_j(\delta') \geqslant u_j(\delta)$ for all $j \in G$ and $u_i(\delta') > u_i(\delta)$ for at least one $i \in G$. By Lemma 5.13, $\delta'_x \geqslant \delta_x$ for every project x that belongs to $T_{\delta,j}$ for some $j \in G$, and $\delta'_x > \delta_x$ for every project x in $T_{\delta,i}$. This implies

$$\delta'\left(\bigcup_{j\in G} T_{\delta,j}\right) > \delta\left(\bigcup_{j\in G} T_{\delta,j}\right). \tag{18}$$

We write both equilibrium distributions as decompositions $\delta = \sum_{i\in N} \delta_i$ and $\delta' = \sum_{i\in N} \delta'_i$ satisfying Proposition 5.2. Since $C'_G \leqslant C_G$, inequality (18) above must hold for the individual distribution of at least one agent $k \in N \setminus G$, that is,

$$\delta'_k\left(\bigcup_{j\in G} T_{\delta,j}\right) > \delta_k\left(\bigcup_{j\in G} T_{\delta,j}\right).$$

Consequently, at least one project $y \in \bigcup_{j\in G} T_{\delta,j}$ has $\delta'_{k,y} > \delta_{k,y}$. By Proposition 5.2, project y must be critical for agent k in δ'. Therefore,

$$\begin{aligned}
\nu_{k,y} \cdot u_k(\delta') &= \delta'_y && \text{(by (12), as } y \text{ is critical for agent } k \text{ in } \delta') \\
&\geqslant \delta_y && \text{(as } y \in T_{\delta,j} \text{ for some } j \in G) \\
&\geqslant \nu_{k,y} \cdot u_k(\delta),
\end{aligned}$$

so agent k's utility is not decreased by the group's manipulation. Consequently, k could be added to G which contradicts the maximality of G.

We conclude that no group of agents has a successful manipulation and thus, *NASH* is group-strategyproof. □

Moreover, the above proof shows that if the total contribution C_G decreases, then the utility of at least one agent in G has to *strictly* decrease under *NASH* as

$\sum_{i\in G} \delta'_i \left(\bigcup_{j\in G} T_{\delta,j}\right) < \sum_{i\in G} \delta_i \left(\bigcup_{j\in G} T_{\delta,j}\right)$ and the above argument applies. In particular, an agent receives *strictly* more utility when she increases her contribution. The interpretation of *NASH* as the Nash product rule even allows us to give a lower bound on the utility gain when contributions increase.

THEOREM 5.15
Under *NASH*, if the contribution of some agent increases by $Z > 0$, then her utility increases by a factor of at least $(C_N + Z)/C_N$.

Proof. Let $P' = (U, C')$ be the profile where, compared to $P = (U, C)$, one agent i increased her contribution by $Z > 0$. Thus, $C'_N = C_N + Z$. Let $\delta^P \in \Delta(C_N)$ and $\delta^{P'} \in \Delta(C'_N)$ be the respective equilibrium distributions.

We claim that $u_i(\delta^{P'})/u_i(\delta^P) \geqslant C'_N/C_N$. For that, define $\delta = (C'_N/C_N) \cdot \delta^P$ and $\delta' = (C'_N/C_N) \cdot \delta^{P'}$ such that $\delta \in \Delta(C'_N)$ and $\delta' \in \Delta(C_N)$. Denote by $\mathit{Nash}_P^{\exp}(\delta)$ the weighted product of agents' utilities in profile P and distribution δ (the exponent of Nash welfare as previously defined). Then,

$$\begin{aligned}
1 &\leqslant \frac{\mathit{Nash}_{P'}^{\exp}(\delta^{P'})}{\mathit{Nash}_{P'}^{\exp}(\delta)} && \text{(by maximality of } \delta^{P'} \text{ in } \Delta(C'_N))\\
&= \frac{\mathit{Nash}_{P}^{\exp}(\delta^{P'})}{\mathit{Nash}_{P}^{\exp}(\delta)} \cdot \frac{u_i(\delta^{P'})^Z}{u_i(\delta)^Z} && \text{(as } C'_j = C_j + Z)\\
&= \left(\frac{C'_N}{C_N}\right)^{C_N} \cdot \frac{\mathit{Nash}_{P}^{\exp}(\delta')}{\mathit{Nash}_{P}^{\exp}(\delta)} \cdot \frac{u_i(\delta^{P'})^Z}{u_i(\delta)^Z} && \text{(as } \delta^{P'} = \frac{C'_N}{C_N} \cdot \delta')\\
&= \frac{\mathit{Nash}_{P}^{\exp}(\delta')}{\mathit{Nash}_{P}^{\exp}(\delta^P)} \cdot \frac{u_i(\delta^{P'})^Z}{u_i(\delta)^Z} && \text{(as } \delta = \frac{C'_N}{C_N} \cdot \delta^P)\\
&\leqslant \frac{u_i(\delta^{P'})^Z}{u_i(\delta)^Z} && \text{(by maximality of } \delta^{P} \text{ in } \Delta(C_N))
\end{aligned}$$

Thus, $u_i(\delta^{P'}) \geqslant u_i(\delta) = \frac{C'_N}{C_N} \cdot u_i(\delta^P)$ and the claim is proved.

If $u_i(\delta^P) > 0$, then the auxiliary claim implies that $u_i(\delta^{P'}) > u_i(\delta^P)$. Otherwise, $u_i(\delta^P) = 0$ implies $C_i = 0$, and $C'_i = Z > 0$ implies $u_i(\delta^{P'}) > 0$ by the equilibrium property, so again $u_i(\delta^{P'}) > u_i(\delta^P)$. □

REMARK 3
The above proof is independent of Leontief utilities and only requires utility functions to be homogeneous of degree 1.[35] In particular, Theorem 5.15 also holds for linear utilities. In contrast to contribution incentive-compatibility, we have a multiplicative bound instead of an explicit utility gain. Interestingly, the stronger version of contribution incentive-compatibility mentioned in Section 4.2.1 is also of a multiplicative form which seems to be more in the spirit of Nash welfare.

Next, we show that *NASH* also satisfies a variety of consistency properties, cementing its state as the favorable mechanism in the case of Leontief utilities.

THEOREM 5.16
NASH satisfies U-continuity and C-continuity.

35 A function $u\colon \mathbb{R}^m \to \mathbb{R}^m$ is homogeneous of degree 1 if $u(\lambda x) = \lambda u(x)$ for all $\lambda \geqslant 0$.

Proof. We use a similar argument as in the proof of Theorem 4.16. Suppose we are given a sequence of profiles $(U^t, C^t)_{t\in\mathbb{N}}$ converging to (U, C), i.e., $\lim_{t\to\infty} v^t_{i,x} = v_{i,x}$ for every agent $i \in N$ and project $x \in A$ as well as $\lim_{t\to\infty} C^t = C$.

Let $\delta = NASH(U, C)$ and $\delta^t = NASH(U^t, C^t)$ for all $t \in \mathbb{N}$ and denote by $U(\delta) = (u_i(\delta))_{i\in N}$ and $U^t(\delta^t) = (u^t_i(\delta^t))_{i\in N}$ the respective utility vectors returned by *NASH*. Analogously to the proof of Theorem 4.16, upper and lower bounds for $u^t_i(\delta^t)$ and C^t_i can be derived, and even bounds for δ^t_x can be given. Note that in contrast to linear utilities, a unique distribution is returned. By boundedness, it suffices to show that every convergent subsequence of $(\delta^t)_{t\in\mathbb{N}}$ converges to δ. Such a subsequence $(\delta^{t_k})_{k\in\mathbb{N}}$ has to exist by the Bolzano-Weierstrass theorem. Denote its limit by δ^* which is in $\Delta(C_N)$ as $\lim_{k\to\infty} C^{t_k} = C$. Our goal is to show that $\delta^* = \delta$. Denote by $Nash_P(\delta)$ the Nash welfare of distribution δ in profile P.

CASE 1
Assume $\delta_x > 0$ for all projects $x \in A$.

We have $Nash_{(U^{t_k},C^{t_k})}((C^{t_k}_N/C_N)\cdot\delta) \leqslant Nash_{(U^{t_k},C^{t_k})}(\delta^{t_k})$ for every $k \in \mathbb{N}$ by definition of *NASH* on (U^t, C^t) and take limits on both sides as $k \to \infty$.

For the left-hand side, the utility $u^{t_k}_i(\delta)$ is a minimum of ratios $\delta_x/v^{t_k}_{i,x}$ for every $i \in N$, where all numerators are at least ε_δ, for some $\varepsilon_\delta > 0$. As for each agent i, there exists at least one project x with $v_{i,x} > 0$, also $v^{t_k}_{i,x} > 0$ for k large enough as $\lim_{k\to\infty} v^{t_k}_{i,x} = v_{i,x}$ by assumption. For the same reasons, $v^{t_k}_{i,x} < \varepsilon_u$ for all $i \in N$, $x \notin A_i$ and k large enough for arbitrarily small $\varepsilon_u > 0$. Thus, k can be chosen large enough such that projects with $v^{t_k}_{i,x} < \varepsilon_u$ do not affect the minimum and can be ignored. Therefore, the minimum is determined only by ratios with $v^{t_k}_{i,x} \geqslant \varepsilon_u$. In this domain, the ratios are continuous, and their minimum is continuous, too. Hence, $\lim_{k\to\infty} u^{t_k}_i(\delta) = u_i(\delta)$, and the limit of the product at the left-hand side equals $Nash_{(U,C)}(\delta)$.

For the right-hand side, we claim that $\delta^{t_k}_x$ is bounded away from zero for large enough k. Indeed, for each project $x \in A$, there has to exist at least one agent i_x with $v_{i_x,x} > 0$ as $\delta_x > 0$ for all projects $x \in A$. Choosing k large enough ensures that project x receives at least some contribution $\delta^{t_k}_x > \varepsilon_\delta$, otherwise, agent i_x would have an incentive to transfer some of her contribution to x which is not possible as $\delta^{t_k} = NASH(U^{t_k}, C^{t_k})$. This also implies $\delta^*_x > 0$ for all $x \in A$. Now, we can apply the same argument as above to show that $\lim_{k\to\infty} u^{t_k}_i(\delta^{t_k}) = u_i(\delta^*)$, and the limit of the product at the right-hand side is $Nash_{(U,C)}(\delta^*)$.

Thus, we have $Nash_{(U,C)}(\delta) \leqslant Nash_{(U,C)}(\delta^*)$. By definition of *NASH* and uniqueness of the equilibrium distribution, we get $\delta^* = \delta$.

CASE 2
Suppose $\delta_x = 0$ for some project $x \in A$.

Define $A^0 = \{x \in A \colon \delta_x = 0\}$ and $A^+ = A \setminus A^0$. As δ maximizes Nash welfare for (U, C), we have $v_{i,x} = 0$ for every $i \in N$ and $x \in A^0$. Otherwise, an agent i with $v_{i,x} > 0$ would receive zero utility under the equilibrium distribution returned by *NASH*, a contradiction. Consequently, $\lim_{k\to\infty} v^{t_k}_{i,x} = 0$ for every $i \in N$ and $x \in A^0$ and the amount allocated to projects in A^0 by δ^{t_k} also tends to zero for $k \to \infty$. Hence, $\delta_x = 0$ implies $\delta^*_x = 0$.

In the following, we measure utilities and Nash welfare only with respect to projects in A^+. We know that $\delta^{t_k}(A^+)$ converges to 1 for $k \to \infty$. By definition of

NASH on (U^{t_k}, C^{t_k}), we have $Nash_{(U^{t_k},C^{t_k})}((\delta^{t_k}(A^+)/C_N)\cdot\delta) \leqslant Nash_{(U^{t_k},C^{t_k})}(\delta^{t_k})$. As in Case 1, we take the limit of both sides, the left-hand side equals $Nash_{(U,C)}(\delta)$, and the right-hand side is $Nash_{(U,C)}(\delta^*)$. These two quantities remain the same even if we take the alternatives in A^0 back into account for Nash welfare as all agents assign zero to alternatives from A^0 in (U, C). Hence, as in Case 1, we get $Nash_{(U,C)}(\delta) \leqslant Nash_{(U,C)}(\delta^*)$ and $\delta^* = \delta$ in the end.

All in all, $\delta^* = \delta$ for both cases, proving that $\lim_{t\to\infty} \delta^t = \delta$. □

An important property from the perspective of project managers is *preference-monotonicity*, which requires that for every agent i and project $x \in A$, δ_x weakly increases when $v_{i,x}$ increases (see Definition 3.4).

For linear utilities, strategyproofness implies preference-monotonicity (see Proposition 4.17). This does not hold for Leontief utilities, even when valuations are binary. Nevertheless, *NASH* is preference-monotonic.

Theorem 5.17
NASH satisfies preference-monotonicity.

Proof. Let P and P' be two profiles where one agent i increases her valuation for one project x from P to P', i.e., $v'_{i,x} > v_{i,x}$ and $v'_{i,y} = v_{i,y}$ for all $y \in A \setminus x$. Let δ and δ' be the respective equilibrium distributions. We need to show that $\delta'_x \geqslant \delta_x$.

Let u_i and u'_i be agent i's Leontief utility functions in the two profiles.

By definition of Leontief utilities, $u'_i(\delta) = \min\{u_i(\delta), \delta_x/v'_{i,x}\}$. We consider two cases, depending on which of the two expressions within the minimum is larger.

Case 1
Assume $u_i(\delta) < \delta_x/v'_{i,x}$. Then, $u'_i(\delta) = u_i(\delta)$, and all projects in $T_{\delta,i}$ remain critical for i in the new profile. Therefore, by Proposition 5.2, δ is still an equilibrium distribution for P'. By uniqueness of the equilibrium distribution, $\delta'_x = \delta_x$.

Case 2
Suppose $u_i(\delta) \geqslant \delta_x/v'_{i,x}$. By definition of Leontief utilities,

$$\frac{\delta'_x}{v'_{i,x}} \geqslant u'_i(\delta').$$

By strategyproofness (Theorem 5.14),

$$u'_i(\delta') \geqslant u'_i(\delta).$$

By definition of Leontief utilities,

$$u'_i(\delta) = \min\left\{u_i(\delta), \frac{\delta_x}{v'_{i,x}}\right\} = \frac{\delta_x}{v'_{i,x}},$$

as $u_i(\delta) \geqslant \delta_x/v'_{i,x}$ by assumption.

Combining these three inequalities yields $\delta'_x \geqslant \delta_x$, as desired.

In addition, increasing the valuation $v_{i,x}$ of one agent i for one project x cannot increase Nash welfare of the equilibrium distribution. Otherwise, the equilibrium distribution of the new profile would yield higher Nash welfare for the original profile than the original equilibrium distribution. Nash welfare might remain constant in case x is not among agent i's critical projects.

Similarly, the utility of agent i under the equilibrium distribution cannot increase. If agent i's utility with the new valuation is larger under the new equilibrium, her utility under her original valuations is also larger for the new equilibrium distribution and thus, reporting exactly these new valuations constitutes a beneficial manipulation. This would contradict strategyproofness of *NASH* (Theorem 5.14). However, agent i's utility might remain constant in case x is not among her critical projects.

For some applications, it seems desirable to ensure that increased contributions do not result in the redistribution of funds that have already been allocated. For example, if agents arrive or increase their contributions over time, ideally, the mechanism only needs to take care of the additional contributions. Hence, such a mechanism can be implemented as an incremental process in which projects are able to use allocated donations immediately. This is referred to as contribution-monotonicity (see Definition 3.5).

THEOREM 5.18
NASH satisfies contribution-monotonicity.

Proof. We show the equivalent statement that for any two profiles $P = (U, C)$ and $P' = (U, C')$ with $C'_i \geqslant C_i$ for all $i \in N$, $\delta'_x \geqslant \delta_x$ holds for all projects $x \in A$ where δ and δ' are the equilibrium distributions corresponding to profiles P and P', respectively. Fix decompositions of δ and δ' into individual distributions satisfying Proposition 5.2.

Let A^-, $A^=$, and A^+ be the sets of all projects $x \in A$ with $\delta'_x < \delta_x$, $\delta'_x = \delta_x$, and $\delta'_x > \delta_x$, respectively. Assume for contradiction that A^- is non-empty. Thus, $\sum_{i \in N} \delta'_i(A^-) < \sum_{i \in N} \delta_i(A^-)$, so there has to be an agent $i \in N$ with $\delta'_i(A^-) < \delta_i(A^-)$, and a project $y \in A^-$ with $\delta'_{i,y} < \delta_{i,y}$. But $\delta'_i(A) = C'_i \geqslant C_i = \delta_i(A)$, so $\delta'_i(A^= \cup A^+) > \delta_i(A^= \cup A^+)$, so there has to be a project $z \in A^= \cup A^+$ with $\delta'_{i,z} > \delta_{i,z} \geqslant 0$. By Proposition 5.2, projects z and y are critical for i under δ' and δ, respectively. This implies that $v_{i,z} > 0$ and $v_{i,y} > 0$. Therefore,

$$\frac{\delta'_z}{v_{i,z}} \leqslant \frac{\delta'_y}{v_{i,y}} < \frac{\delta_y}{v_{i,y}} \leqslant \frac{\delta_z}{v_{i,z}},$$

where the first and last inequalities follow from the definition of critical projects. This implies $\delta'_z < \delta_z$, a contradiction to $z \in A^= \cup A^+$. □

REMARK 4
Theorem 5.18 yields an alternative proof for the uniqueness of equilibrium distributions, which does not rely on the equivalence with Nash welfare optimality. If δ and δ' are equilibrium distributions for the same profile, then both $\delta_x \geqslant \delta'_x$ and $\delta'_x \geqslant \delta_x$ have to hold for every project $x \in A$, which implies $\delta = \delta'$.

5.2.2 Computation

The equilibrium distribution can be computed by solving a convex program as it maximizes Nash welfare similar to the case of linear utilities. However, an *exact* computation of the equilibrium distribution cannot be excluded a priori as equilibrium distributions are rational-valued (Lemma 5.19). In fact, we prove that *NASH* can be computed exactly in pseudo-polynomial time in Theorem 5.20.

Note that the equilibrium distribution does not change when individual valuations are rescaled. Similarly, rescaling contributions preserves the share each project receives. Thus, for the sake of simplicity, we assume throughout this section that all valuations and contributions are natural numbers. We denote $v_{max} := \max_{i \in N, x \in A} v_{i,x}$. We prove that the equilibrium distribution can be computed in time $\mathrm{poly}(n, m, \log_2(v_{max}), C_N)$. If all contributions are equal, w.l.o.g. $C_i = 1$ for all $i \in N$, then the run-time is polynomial in the binary encoding length of the input.

As a first step, we prove that the equilibrium distribution δ^* and its utility profile (which we denote by u^*) are rational with a bounded binary encoding length.

Lemma 5.19
If agents' valuations $v_{i,x}$ and contributions C_i are natural numbers, then the equilibrium distribution δ^* and its utility profile u^* are rational-valued.

Moreover, the binary encoding length of (δ^*, u^*) is bounded by a polynomial function of the binary encoding length of $v_{i,x}$ and C_i.

Proof. For each $i \in N$, let T_i be a non-empty set of projects. Consider the following linear program (LP), with variables $\tilde{u}_i$ (for $i \in N$), d_x (for $x \in A$), and $d_{i,x}$ (for $i \in N$ and $x \in A$):

$$
\begin{aligned}
d_x &= \tilde{u}_i \cdot v_{i,x} && \text{for all } i \in N, x \in T_i \\
d_x &\geqslant \tilde{u}_i \cdot v_{i,x} && \text{for all } i \in N, x \notin T_i \\
\sum_{x \in T_i} d_{i,x} &= C_i && \text{for all } i \in N \\
d_{i,x} &= 0 && \text{for all } i \in N, x \notin T_i \\
\sum_{i \in N} d_{i,x} &= d_x && \text{for all } x \in A \\
d_{i,x} &\geqslant 0 && \text{for all } i \in N, x \in A.
\end{aligned}
$$

Every potential solution to this LP indicates a distribution $\delta_x = d_x$ for all x, with a decomposition $\delta_{i,x} = d_{i,x}$ for all $i \in N$ and $x \in A$, such that each agent i contributes only to projects in T_i, and all projects in T_i are critical for i. By Proposition 5.2, such a distribution has to coincide with the equilibrium distribution.

The equilibrium distribution δ^* is a solution to the above LP whenever $T_i = T_{\delta^*,i}$ for all $i \in N$. By assumption, the coefficients of this LP are all rational. Therefore, by well-known properties of linear programming, the LP has a rational solution, with binary encoding length bounded by a polynomial function of the representation length of its coefficients. We can even give an explicit bound on the representation length of δ^*. Given the equilibrium distribution δ^*, we construct an undirected graph where vertices correspond to projects, and there is an edge between $x, y \in A$ if and only if there exists an agent $i \in N$ with $x, y \in T_{\delta^*,i}$. Each component of that graph can be considered separately, as the sum of contributions to that component equals the sum of contributions of agents for whom parts of the component are critical projects.

Thus, let $A' \subseteq A$ be a subset of m' projects forming a component and $N' \subseteq N$ the subset of agents contributing to projects in A'. Given some $x_1 \in A'$, there needs to be at least one other project x_2 that can be reached in one step, i.e., there exists an agent $i_1 \in N'$ such that $\delta^*_{x_1}/v_{i_1,x_1} = \delta^*_{x_2}/v_{i_1,x_2}$. Hence, $\delta^*_{x_2} = (v_{i_1,x_2}/v_{i_1,x_1}) \cdot \delta^*_{x_1}$. Next, there exists another project $x_3 \in A'$ that can be reached in one step from either x_1 or x_2. In general, given $k < m'$ connected projects from A', there needs to be another one that can be reached in one step from one of the k projects.

This gives a system of linear equations where each $\delta^*_{x_j}$ can be written in terms of $\delta^*_{x_1}$. In detail, $\delta^*_{x_2}$ requires only the two valuations v_{i_1,x_1} and v_{i_1,x_2}, $\delta^*_{x_3}$ requires at most four valuations and so on. Considering this "worst case" in terms of representation length together with $\sum_{x \in A} \delta^*_x = \sum_{i \in N'} C_i$, $\delta^*_{x_1}$ can be written as the fraction of $\sum_{i \in N'} C_i$ and $1 + v_{i_1,x_2}/v_{i_1,x_1} + (v_{i_1,x_2}/v_{i_1,x_1}) \cdot (v_{i_2,x_3}/v_{i_2,x_2}) + \ldots$ resulting in a binary encoding length of at most $\log_2(C_N \cdot {v_{max}}^{m-1})$ for the nominator and $\log_2(m \cdot {v_{max}}^{m-1}))$ for the denominator where we upper-bounded m' by m, $\sum_{i \in N'} C_i$ by C_N, and $v_{i,x}$ by v_{max}.

As x_1 was chosen arbitrarily and Leontief utilities coincide with (weighted) distributions to certain projects, (δ^*, u^*) can be represented by $n + m$ times the derived length for $\delta^*_{x_1}$ plus an additional n times the length of v_{max} for the valuations on critical projects. □

Lemma 5.19 cannot be applied directly for computing the equilibrium distribution in polynomial time as the proof requires us to know $T_{\delta^*,i}$. We could loop over all possible $T_{\delta^*,i}$, but this would require exponential time due to the exponential number of possible sets.

The remainder of this section is dedicated to proving the following theorem.

Theorem 5.20
For rational valuations and contributions, *NASH* can be computed exactly in pseudo-polynomial[36] time. If, in addition, all contributions are equal, *NASH* can be computed exactly in polynomial time.

To prove (pseudo) polynomial-time computability, we use Theorem 13 of Jain (2007).

Lemma 5.21 *(Jain (2007))*
Let S be a convex set given by a strong separation oracle, and $\phi > 0$ an integer.

There is an oracle-polynomial time and ϕ-linear time algorithm which either

1. concludes that there is no point in S with binary encoding length at most ϕ, or

2. produces a point in S with binary encoding length at most $P(n) \cdot \phi$, where $P(n)$ is a polynomial.

36 Pseudo-polynomial time computability requires the running time of an algorithm to be polynomial only in the input *size*, but not necessarily in the *binary encoding length* of the input which is needed for polynomial-time computability

In order to apply Lemma 5.21, we need to find a strong separation oracle for the equilibrium distribution.

For every positive rational number z_0, we define a convex set $S(z_0) \subseteq \mathbb{R}^{n+m}$, where the variables are $\tilde{u}_i$ for $i \in N$ and d_x for $x \in A$:

$$\begin{aligned} \prod_{i=1}^{n} \tilde{u}_i{}^{C_i} &\geqslant z_0 && \\ d_x &\geqslant \tilde{u}_i \cdot v_{i,x} && \text{for all } i \in N, x \in A \\ \sum_{x \in A} d_x &= C_N && \\ \tilde{u}_i &\geqslant 0 && \text{for all } i \in N \\ d_x &\geqslant 0 && \text{for all } x \in A. \end{aligned}$$

The set $S(z_0)$ contains all pairs (δ, u) such that δ is a feasible distribution, u is its utility profile, and the Nash product is at least z_0. A *strong separation oracle for* $S(z_0)$ is a function that accepts as input a rational vector $y' = (\tilde{u}'_1, \dots, \tilde{u}'_n, d'_1, \dots, d'_m)$. It should return either an assertion that $y' \in S(z_0)$, or a hyperplane that separates y' from $S(z_0)$ (i.e., a rational vector c such that $c \cdot y' < c \cdot y$ for all $y \in S(z_0)$).

LEMMA 5.22
For every rational $z_0 > 0$, there is a polynomial-time strong separation oracle for the convex set $S(z_0)$.

Proof. Given a rational vector $y' = (\tilde{u}'_1, \dots, \tilde{u}'_n, d'_1, \dots, d'_m)$, we first check whether the point satisfies the linear constraints $d'_x \geqslant \tilde{u}'_i \cdot v_{i,x}$, $\sum_{x \in A} d'_x = C_N$, $\tilde{u}'_i \geqslant 0$, and $d'_x \geqslant 0$. If one of these constraints is violated, the constraint itself yields a separating hyperplane. As the number of these constraints is polynomial in n and m, all of them can be checked in polynomial time.

It remains to handle the case that all linear constraints are satisfied, whereas the nonlinear constraint is violated, i.e.,

$$\prod_{i=1}^{n} (\tilde{u}'_i)^{C_i} < z_0. \tag{19}$$

Recall that the C_i's are assumed to be natural numbers. Therefore, the above condition can be checked exactly using arithmetic operations on rational numbers. The binary encoding length of the product is polynomial in the binary encoding length of $\tilde{u}'_i$ and in C_i. Due to the latter fact, it is, in fact, pseudo-polynomial in the input size. However, if all contributions are equal, they can be ignored and the representation stays polynomial in the input size.

To construct a separating hyperplane, we use an idea similar to Jain (2007). Define the vector c to have the coefficient $(1/C_N) \cdot (C_i/\tilde{u}'_i)$ for each variable $\tilde{u}'_i$, and the coefficient 0 for each variable d_x. Note that the encoding length of c is polynomial in the input size. For every vector $y = (u_1, \dots, u_n, d_1, \dots, d_m)$,

$$c \cdot y = \frac{1}{C_N} \cdot \sum_{i=1}^{n} C_i \frac{u_i}{u'_i}.$$

Substituting $y = y'$ yields $c \cdot y' = 1$. We now prove that $c \cdot y > 1$ for every $y \in S(z_0)$.

Indeed, $c \cdot y$ is a weighted arithmetic mean of the n positive numbers $(\tilde{u}_i/\tilde{u}'_i)$, with weights C_i. By the weighted AM-GM inequality[37], this sum is at least as large as their weighted geometric mean, i.e.,

$$\frac{1}{C_N} \cdot \sum_{i=1}^{n} C_i \frac{\tilde{u}_i}{\tilde{u}'_i} \geqslant \left(\prod_{i=1}^{n} \left(\frac{\tilde{u}_i}{\tilde{u}'_i} \right)^{C_i} \right)^{1/C_N} = \frac{(\prod_{i=1}^{n} (\tilde{u}_i)^{C_i})^{1/C_N}}{(\prod_{i=1}^{n} (\tilde{u}'_i)^{C_i})^{1/C_N}}.$$

As $y = (\tilde{u}_1, \ldots, \tilde{u}_n, d_1, \ldots, d_m)$ is in $S(z_0)$, it satisfies $(\prod_{i=1}^{n} (\tilde{u}_i)^{C_i}) \geqslant z_0$. Lower bounding the right-hand side above gives

$$\frac{1}{C_N} \cdot \sum_{i=1}^{n} C_i \frac{\tilde{u}_i}{\tilde{u}'_i} \geqslant \frac{{z_0}^{1/C_N}}{(\prod_{i=1}^{n} (\tilde{u}'_i)^{C_i})^{1/C_N}} = \left(\frac{z_0}{\prod_{i=1}^{n} (\tilde{u}'_i)^{C_i}} \right)^{1/C_N},$$

which is larger than 1 by (19). Hence, $c \cdot y > 1$, so c is indeed a separating hyperplane. □

Lemma 5.22 allows us to apply Lemma 5.21 to $S(z_0)$. Thus, the equilibrium distribution can be computed by applying binary search to z_0 in the following way.

1. Initialize L as the Nash product of some arbitrary distribution (e.g., the uniform distribution).

2. Initialize H as some upper bound on the maximum Nash product, e.g. the Nash product resulting from the (unrealistic) distribution in which each agent i divides C_N optimally (in proportion to v_i).

3. Let ϕ be an upper bound on the binary encoding length of (δ^*, u^*), derived in Lemma 5.19.

4. Set $z_0 := (L + H)/2$. Note that both L and H can be encoded in length polynomial in the input size, so the same applies to z_0.

5. Apply Lemma 5.21 to $S(z_0)$, using Lemma 5.22 for the strong separation oracle.

   CASE 1
   Lemma 5.21 yields outcome 1 ("no point in $S(z_0)$ with binary encoding length at most ϕ"). We know that $S(z_0)$ does not contain an equilibrium distribution. This means that the Nash product of the equilibrium distribution is lower than z_0. We set $H := z_0$ and return to step 4.

   CASE 2
   Lemma 5.21 yields outcome 2 ("a point in $S(z_0)$ with binary encoding length at most $P(n) \cdot \phi$"). In particular, we have a distribution δ with Nash product at least z_0.

37 The weighted inequality of arithmetic and geometric mean states that for any number $n \in \mathbb{N}$ of nonnegative, real numbers $x_1, \ldots, x_n$ and weights $w_1, \ldots, w_n$ with $W = \sum_{i \in N} w_i$, it holds that $(\sum_{i=1}^{n} w_i x_i)/W \geqslant \sqrt[W]{\prod_{i=1}^{n} x_i^{w_i}}$ with equality if and only if all x_i's with $w_i > 0$ are equal.

We check whether δ is an equilibrium distribution (this can be done in polynomial time). If it is, we return δ and finish. Otherwise, we set $L := z_0$ and return to step 4.

As the binary encoding length of (δ^*, u^*) is at most ϕ, the binary encoding length of the maximum Nash product is at most $\sum_i C_i \log_2(u_i^*) \leqslant C_N \cdot \phi$. Therefore, after at most $C_N \cdot \phi$ steps, the binary search is guaranteed to terminate with an equilibrium distribution.

5.2.3 Spending Dynamics

So far, we have taken the viewpoint and assumed the presence of a central authority that collects agents' valuations and contributions and then either distributes the latter among the projects or gives recommendations to each agent on how to distribute her individual contribution.

In this section, we show that equilibrium distributions are also attained as the limits of natural multi-round processes without a central authority, where agents spend their contribution one after another in a myopically optimal way. Agents need not reveal their preferences explicitly, but they have to be able to observe the donations made in previous rounds.

To this end, we consider infinite processes in which agents repeatedly play best responses against the strategies of other agents from previous rounds. We first analyze redistribution dynamics where contributions remain fixed, and agents are allowed to redistribute their contribution whenever it is their turn. As we will see in Theorem 5.24, the distribution converges to the equilibrium distribution under a very mild condition on the sequence of agents. We then consider continuous spending dynamics in which there is a constant flow of contributions from each agent, e.g. when each agent i has set aside a monthly budget C_i. We focus on the case of round-robin sequences and show that the relative overall distribution (or, equivalently, the average distribution over all rounds) converges to the equilibrium distribution when agents can observe only the distribution given by the last $n-1$ rounds (Theorem 5.29).[38]

First of all, these convergence results show stability of the equilibrium distribution in the sense that best response dynamics return to it from arbitrary initial distributions. Furthermore, they allow us to make statements in more flexible settings where the set of participating agents, as well as their preferences and contributions, can change over time. The finite number of donations contributed up to a certain point will always be outweighed by the infinite number of donations that follow. Hence, even with occasional changes to the profile, the relative overall distribution keeps converging towards an equilibrium distribution of the current profile.

Redistribution dynamics

First, consider a dynamics in which contributions remain fixed and agents repeatedly redistribute them after observing the current overall distribution.

38 The formal statement is stronger as not only the relative overall distribution but also the distribution given by the last n rounds, converges to the equilibrium distribution.

Formally, denote by δ^* the equilibrium distribution and by δ^t the distribution at round t (along with its associated decomposition), e.g., δ^0 equals the null vector as no agent $i \in N$ has yet distributed her contribution C_i.

In each round t, allow one agent i_t to (re-)distribute her entire contribution in such a way that her utility is maximized for the new distribution δ^{t+1}, i.e.,

$$\delta_{i_t}^{best} := \arg \max_{\delta_{i_t} \in \Delta(C_{i_t})} u_{i_t}\left(\delta_{i_t} + \sum_{j \neq i_t} \delta_j^t\right)$$
$$\delta^{t+1} := \delta_{i_t}^{best} + \sum_{j \neq i_t} \delta_j^t.$$

LEMMA 5.23
For every round t and agent i_t, there is a unique best response $\delta_{i_t}^{best}$.

Proof. As a best response corresponds to a solution of a maximization problem over the closed and bounded set of possible distributions $\delta_{i_t} + \sum_{j \neq i_t} \delta_j^t$ with the continuous objective function u_{i_t}, existence is guaranteed.

To show uniqueness, observe that for the distribution in round $t+1$ (which for simplified notation we denote by $\delta := \delta^{t+1}$), we have $\delta_{i_t}(T_{\delta,i_t}) = C_{i_t}$, that is, agent i_t distributes all of her contribution on her critical projects in δ. In any other response δ'_{i_t}, agent i_t must contribute less to at least one project of T_{δ,i_t}. Therefore, her utility must be lower than $u_{i_t}(\delta)$, so δ'_{i_t} cannot be a best response. □

Before turning to the main result on the convergence of the dynamics, we apply it to Example 5.3.

Suppose agents take turns updating their individual distribution, starting with Agent 1. She initially distributes uniformly over her approved projects, i.e., $\delta^1 = (1/2, 1/2, 0)$. Agent 2 then distributes according to his best response $(0, 1/4, 3/4)$ resulting in $\delta^2 = (1/2, 3/4, 3/4)$. Next, Agent 1 updates her individual distribution such that $\delta^3 = (5/8, 5/8, 3/4)$, and so on. In the limit, they arrive at the equilibrium distribution $\delta^* = (2/3, 2/3, 2/3)$.

THEOREM 5.24
Given a profile P, let $\mathcal{S} = (i_0, i_1, i_2, \dots)$ be an infinite sequence of agents updating their individual distributions via best responses such that there is a bound $K \in \mathbb{N}$ on the maximal number of rounds an agent has to wait until she is allowed to redistribute. Then, the redistribution dynamics converge to the equilibrium distribution, i.e., $\lim_{t \to \infty} \delta^t = \delta^*$.

The proof proceeds in two steps: First, we will show that the amount an arbitrary agent wants to redistribute converges to 0. Second, we will conclude that this can only be the case if the dynamics converge to the equilibrium distribution.

For the first step, we define a real-valued function Φ on the set of individual distributions, such that whenever some agent deviates to a best response, Φ strictly increases.[39]

39 The definition of Φ is inspired by the definition of *ordinal potential functions*, which were originally introduced to prove the existence of pure strategy Nash equilibria in congestion games (Rosenthal, 1973; Monderer and Shapley, 1996) and have since then been widely used to prove convergence to equilibrium (Milchtaich, 1996, 2000, 2004). However, an ordinal potential function increases when-

Define

$$\Phi(\delta_1, \ldots, \delta_n) := \sum_{i \in N} \sum_{x \in A_i} \delta_{i,x} \log\left(\frac{\nu_{i,x}}{\delta_x}\right). \tag{20}$$

Note that Φ is well-defined, as $\delta_x = 0$ implies $\delta_{i,x} = 0$ for all $i \in N$, and $x \in A_i$ implies $\nu_{i,x} > 0$.

LEMMA 5.25

For any best-response sequence $\mathcal{S}$, it holds that $\Phi(\delta^{t+1}) > \Phi(\delta^t)$ for all t.

Proof. First, observe that an agent's best response going from δ^t to δ^{t+1} can be described by the following continuous process: as long as the agent spends a positive amount on a non-critical project, transfer money from such a project to all critical projects equally until either (i) at least one more charity becomes critical or (ii) the agent no longer spends a positive amount on a non-critical project. This process can be interpreted as a sequence of transfers, where each transfer of amount $\varepsilon > 0$ goes from a charity x with higher weighted distribution to a project y $\left(\delta_x/\nu_{i,x} > \delta_y/\nu_{i,y}\right)$ such that after the transfer, the weighted distribution of the former project remains at least as high as that of the latter: $(\delta_x - \varepsilon)/\nu_{i,x} \geqslant (\delta_y + \varepsilon)/\nu_{i,y}$.

For each t, since the difference between δ^t and δ^{t+1} is caused by transfers, and each amount ε transferred from one project to another one causes a change of ε in distribution for both projects, it suffices to prove that each transfer increases Φ, i.e., $\Phi(\delta^\varepsilon) - \Phi(\delta) > 0$ for arbitrary $\delta \in \Delta(C_N)$ and $\varepsilon > 0$ where δ and δ^ε denote the distributions before and after the transfer.

To see this, note that

$$\begin{aligned}
\Phi(\delta^\varepsilon) - \Phi(\delta) &= (\delta_{i,x} - \varepsilon) \log\left(\frac{\nu_{i,x}}{\delta_x - \varepsilon}\right) + (\delta_{i,y} + \varepsilon) \log\left(\frac{\nu_{i,y}}{\delta_y + \varepsilon}\right) \\
&\quad + \sum_{j \in N \setminus i \colon \delta_{j,x} > 0} \delta_{j,x} \log\left(\frac{\nu_{j,x}}{\delta_x - \varepsilon}\right) \\
&\quad + \sum_{j \in N \setminus i \colon \delta_{j,y} > 0} \delta_{j,y} \log\left(\frac{\nu_{j,y}}{\delta_y + \varepsilon}\right) \\
&\quad - \delta_{i,x} \log\left(\frac{\nu_{i,x}}{\delta_x}\right) - \delta_{i,y} \log\left(\frac{\nu_{i,y}}{\delta_y}\right) \\
&\quad - \sum_{j \in N \setminus i \colon \delta_{j,x} > 0} \delta_{j,x} \log\left(\frac{\nu_{j,x}}{\delta_x}\right) - \sum_{j \in N \setminus i \colon \delta_{j,y} > 0} \delta_{j,y} \log\left(\frac{\nu_{j,y}}{\delta_y}\right) \\
&= \sum_{j \in N \colon \delta_{j,x} > 0} \delta_{j,x} \log\left(\frac{\delta_x}{\delta_x - \varepsilon}\right) + \sum_{j \in N \colon \delta_{j,y} > 0} \delta_{j,y} \log\left(\frac{\delta_y}{\delta_y + \varepsilon}\right) \\
&\quad + \varepsilon \left(\log\left(\frac{\nu_{i,y}}{\delta_y + \varepsilon}\right) - \log\left(\frac{\nu_{i,x}}{\delta_x - \varepsilon}\right)\right) \\
&= \delta_x \log\left(\frac{\delta_x}{\delta_x - \varepsilon}\right) + \delta_y \log\left(\frac{\delta_y}{\delta_y + \varepsilon}\right)
\end{aligned}$$

---

ever a player plays a "better response", whereas our Φ increases only when a player plays a best response. In fact, our dynamics cannot have an ordinal potential function, as there might exist cycles of better responses.

$$
\begin{aligned}
&+\varepsilon\left(\log\left(\frac{\nu_{i,y}}{\delta_y+\varepsilon}\right)-\log\left(\frac{\nu_{i,x}}{\delta_x-\varepsilon}\right)\right)\\
&>0
\end{aligned}
$$

as the last term is nonnegative by $(\delta_x - \varepsilon)/\nu_{i,x} \geqslant (\delta_y + \varepsilon)/\nu_{i,y}$ and the first two terms sum up to something strictly positive which can be seen by using the bound $\log(1+x) > x/(1+x)$ for $x > -1$ and $x \neq 0$:

$$
\begin{aligned}
&\delta_x \log\left(\frac{\delta_x}{\delta_x-\varepsilon}\right)+\delta_y \log\left(\frac{\delta_y}{\delta_y+\varepsilon}\right)\\
>&\delta_x \cdot \frac{\frac{\varepsilon}{\delta_x-\varepsilon}}{1+\frac{\varepsilon}{\delta_x-\varepsilon}}+\delta_y \cdot \frac{\frac{-\varepsilon}{\delta_x+\varepsilon}}{1+\frac{-\varepsilon}{\delta_x+\varepsilon}}\\
=&\delta_x \cdot \frac{\varepsilon}{\delta_x}+\delta_y \cdot \frac{-\varepsilon}{\delta_y}\\
=&0. \qquad \square
\end{aligned}
$$

Φ is bounded on $\Delta(C_N)$ as

$$
\begin{aligned}
\Phi(\delta_1,\dots,\delta_n) &= \sum_{i\in N}\sum_{x\in A_i} \delta_{i,x}\log\left(\frac{\nu_{i,x}}{\delta_x}\right)\\
&= \sum_{i\in N}\sum_{x\in A_i} \delta_{i,x}\log(\nu_{i,x}) - \sum_{x\in A}\delta_x\log(\delta_x)\\
&\leqslant \sum_{i\in N}\sum_{x\in A_i} \delta_{i,x}\log(\nu_{i,x}) + \frac{m}{e} < \infty
\end{aligned}
$$

where we used $x\log(x) \geqslant -1/e$ for the inequality.

Therefore, the sequence $(\Phi(\delta^t))_{t\in\mathbb{N}}$ has to converge to some limit. We denote this limit by ϕ^*.

Next, we show that the amount an arbitrary agent wants to redistribute converges to 0. By assumption, there exists a round $T \leqslant K$ by which all agents have already appeared at least once in $\mathcal{S}$. It is sufficient to prove the theorem for the subsequence starting at T. Therefore, from now on, we assume without loss of generality that at round $t = 0$, all agents have already appeared at least once in $\mathcal{S}$, and thus, have contributed the entire amount C_i.

Denote the amount of shifted contributions in round t by c_t:

$$
c_t := \frac{1}{2}\|\delta^t - \delta^{t+1}\|_1.
$$

When moving from δ^t to δ^{t+1} in round t, agent i_t redistributes c_t from a set of projects $A^-_{i_t}$ to another set $A^+_{i_t}$ with $A^+_{i_t} \cap A^-_{i_t} = \emptyset$. Since the agent is only allowed to redistribute her individual distribution, $c_t \leqslant \delta^t_{i_t}(A^-_{i_t})$. Furthermore, since she redistributes according to her best response, she gives money only to projects that are critical to her in the new distribution, so $\delta^{t+1}_{i_t,x} = 0$ for all $x \in A$ with $\delta^{t+1}_x/\nu_{i_t,x} > u_{i_t}(\delta^{t+1})$ and $u_{i_t}(\delta^{t+1}) = \delta^{t+1}_{x^+}/\nu_{i_t,x^+}$ for every $x^+ \in A^+_{i_t}$. An illustrative example is given in Figure 5.1. In particular, $\delta^{t+1}_{x^-}/\nu_{i_t,x^-} \geqslant u_{i_t}(\delta^{t+1}) = \delta^{t+1}_{x^+}/\nu_{i_t,x^+}$ for all $x^- \in A^-_{i_t}$ and $x^+ \in A^+_{i_t}$.

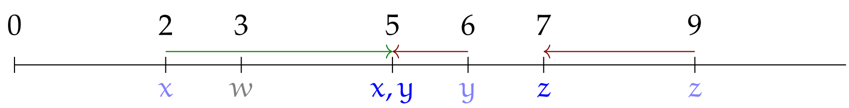


Figure 5.1: An instance with four projects (named w, x, y, z), $\delta^t = (3,2,6,9)$, and an agent i_t with $\delta^t_{i_t} = (0,2,2,2)$ and valuations $v_{i_t} = (0,1,1,1)$. Then, $\delta^{best}_{i_t} = (0,5,1,0)$, $\delta^{t+1} = (3,5,5,7)$, $c_t = 3$, $A^-_{i_t} = \{y,z\}$, and $A^+_{i_t} = \{x\}$.

Define $d_i(\delta)$ as the amount of contribution that would be shifted by an agent i if the current distribution (along with its associated decomposition) were δ and it was her turn to respond. Note that we define $d_i(\delta)$ for all agents, not only the one who actually plays her best response. In particular, $d_{i_t}(\delta^t) = c_t$ for all t. Note also that δ is the equilibrium distribution if and only if $d_i(\delta) = 0$ for all $i \in N$.

Lemma 5.26
For any sequence $\mathcal{S}$, round $t \geqslant 0$, and agent $j \in N$,

$$d_j(\delta^t) \leqslant d_{i_t}(\delta^t) + d_j(\delta^{t+1}).$$

Proof. If $d_j(\delta^t) \leqslant d_{i_t}(\delta^t)$, the statement holds trivially. Hence, assume that $d_j(\delta^t) > d_{i_t}(\delta^t)$. In particular, $j \neq i_t$.

Let $\tilde{\delta}^{t+1}_j$ and $\tilde{\delta}^{t+1}$ be the (hypothetical) individual distribution of agent j and the overall distribution had she been able to implement her best response at round t.

Denote the sets of projects that would be affected by agent j's best response at δ^t by $A^-_j := \{x^- \in A_j : \tilde{\delta}^{t+1}_{j,x^-} < \delta^t_{j,x^-}\}$ and $A^+_j := \{x^+ \in A_j : \tilde{\delta}^{t+1}_{j,x^+} > \delta^t_{j,x^+}\}$. Then,

$$\frac{\tilde{\delta}^{t+1}_{x^-}}{v_{j,x^-}} \geqslant \frac{\tilde{\delta}^{t+1}_{x^+}}{v_{j,x^+}} \text{ for all } x^- \in A^-_j \text{ and } x^+ \in A^+_j, \text{ and} \tag{21}$$

$$\tilde{\delta}^{t+1}_{j,x} = 0 \text{ for all } x \in A \text{ with } u_j(\tilde{\delta}^{t+1}) < \frac{\tilde{\delta}^{t+1}_x}{v_{j,x}} \tag{22}$$

hold by definition of best responses.

Now, a lower bound for $d_j(\delta^{t+1})$ is given by the amount shifted from projects in A^-_j under j's best response in round $t+1$. Again, denote by $\tilde{\delta}^{t+2}_j$ and $\tilde{\delta}^{t+2}$ agent j's best response in round $t+1$ and the corresponding overall distribution. Note that both (21) and (22) hold also with $t+1$ replaced by $t+2$.

Consider first the special case in which agent i_t did not change her contribution to projects in $A^-_j \cup A^+_j$, that is, $\delta^t_x = \delta^{t+1}_x$ for all $x \in A^-_j \cup A^+_j$. If $d_j(\delta^{t+1}) < d_j(\delta^t)$, then a smaller amount is transferred from projects in A^-_j to projects in A^+_j in agent j's best response at δ^{t+1} than in her best response at δ^t, so by (21), there exist projects $x^- \in A^-_j$ and $x^+ \in A^+_j$ such that $\tilde{\delta}^{t+2}_{x^-} > \tilde{\delta}^{t+2}_{x^+} \geqslant v_{j,x^+} \cdot u_i(\tilde{\delta}^{t+2})$ and thus, $\tilde{\delta}^{t+2}_{j,x^-} > 0$. This contradicts (22) with $t+2$ instead of $t+1$. Thus, $d_j(\delta^{t+1}) \geqslant d_j(\delta^t)$ and the claim follows.

Consider now the general case, in which agent i_t may have changed her contribution to some charities in $A^-_j \cup A^+_j$. We claim that the total transfer of i_t and then j (i.e., $d_{i_t}(\delta^t) + d_j(\delta^{t+1})$) cannot be less than the transfer if j were to act alone (i.e., $d_j(\delta^t)$). The reason is similar to the previous paragraph. If this total transfer is less than $d_j(\delta^t)$, then there exist projects $x^- \in A_j^-$ and $x^+ \in A_j^+$ such that $\tilde{\delta}_{x^-}^{t+2} > \tilde{\delta}_{x^+}^{t+2} \geqslant \nu_{j,x^+} \cdot u_j(\tilde{\delta}^{t+2})$ and $\tilde{\delta}_{j,x^-}^{t+2} > 0$, which is a contradiction. Hence, $d_{i_t}(\delta^t) + d_j(\delta^{t+1}) \geqslant d_j(\delta^t)$, as desired. □

Intuitively, Lemma 5.26 can be interpreted as a "triangle inequality". The left-hand side denotes the direct distance from δ^t towards agent j's optimal redistribution. The right-hand side denotes the distance along an indirect path that first goes to δ^{t+1} and then proceeds from there towards j's optimal redistribution.

For any agent $j \in N$ and round t, denote the next round j will get the chance to redistribute her contribution by $t' \leqslant t + K - 1$. Then,

$$\begin{aligned}
\sum_{\ell=t}^{t'} c_\ell &= \sum_{\ell=t}^{t'} d_{i_\ell}(\delta^\ell) \\
&\geqslant \sum_{\ell=t}^{t'} \left(d_j(\delta^\ell) - d_j(\delta^{\ell+1})\right) && \text{(by Lemma 5.26)} \\
&= d_j(\delta^t) - d_j(\delta^{t'+1}) \\
&= d_j(\delta^t) && \text{(as } d_j(\delta^{t'+1}) = 0 \text{ after j's best response).}
\end{aligned}$$

Corollary 5.27
For all rounds t, $\sum_{\ell=t}^{t+K-1} c_\ell \geqslant \max_{i \in N} d_i(\delta^t)$.

Combining Lemma 5.25 with Corollary 5.27 yields the following lemma.

Lemma 5.28
For any sequence $\mathcal{S}$ and agent $j \in N$, $\lim_{t\to\infty} d_j(\delta^t) = 0$.

Proof. We prove the equivalent statement that $\lim_{t\to\infty} \max_{i\in N} d_i(\delta^t) = 0$. Assume for contradiction that there exists $\gamma > 0$ such that for all $T > 0$ there exists $T' \geqslant T$ with $\max_{i\in N} d_i(\delta^{T'}) \geqslant \gamma$. Recall that ϕ^* is the limit of the increasing function $\Phi(\delta^t)$ as $t \to \infty$. Choose some T such that $\phi^* - \Phi(\delta^T) < \gamma^2/(4C_N K^2(m-1)^2)$ and $T' \geqslant T$ with $\max_{i\in N} d_i(\delta^{T'}) \geqslant \gamma$. By Corollary 5.27, $\sum_{\ell=T'}^{T'+K-1} c_\ell \geqslant \max_{i\in N} d_i(\delta^{T'}) \geqslant \gamma$. Thus, there exists some $t \in \{T', \ldots, T'+K-1\}$ with $c_t \geqslant \gamma/K$. Consequently, agent i_t transfers at least $\varepsilon = \gamma/(K(m-1))$ from some project x to some other project y in round t.

The upper bound on $\log(1+x)$ from Lemma 5.25 can be refined to $\log(1+x) > x/(1+x) + x^2/(2+x)^2$ for $x > -1$ and $x \neq 0$, so we get

$$
\begin{aligned}
\Phi(\delta^{t+1}) - \Phi(\delta^{T'}) &\geqslant \Phi(\delta^{t+1}) - \Phi(\delta^{t}) \\
&> \delta_x^t \frac{\left(\frac{\varepsilon}{\delta_x^t - \varepsilon}\right)^2}{\left(2 + \frac{\varepsilon}{\delta_x^t - \varepsilon}\right)^2} + \delta_y^t \frac{\left(\frac{-\varepsilon}{\delta_y^t + \varepsilon}\right)^2}{\left(2 + \frac{-\varepsilon}{\delta_y^t + \varepsilon}\right)^2} \\
&= \delta_x^t \frac{\varepsilon^2}{(2\delta_x^t - \varepsilon)^2} + \delta_y^t \frac{\varepsilon^2}{(2\delta_y^t + \varepsilon)^2} \\
&> \delta_x^t \frac{\varepsilon^2}{(2\delta_x^t - \varepsilon)^2} \\
&> \frac{\varepsilon^2}{4\delta_x^t} > \frac{\varepsilon^2}{4C_N} > \phi^* - \Phi(\delta^T) \\
&> \phi^* - \Phi(\delta^{T'}).
\end{aligned}
$$

This implies $\Phi(\delta^{t+1}) > \phi^*$. But this is impossible, since $\Phi(\delta^t)$ is increasing in t and converges to ϕ^*.

Thus, $\lim_{t\to\infty} d_i(\delta^t) = 0$ for every agent i. □

We can now complete the proof of Theorem 5.24.

Proof of Theorem 5.24. For any $\mathcal{S}$, since $(\delta^t)_{t\in\mathbb{N}}$ is an infinite sequence in the closed set of distributions in the (bounded) simplex $\Delta(C_N)$, the Bolzano-Weierstrass theorem states that it has a convergent subsequence $(\delta^{t_k})_{k\in\mathbb{N}}$ with limit $\delta \in \Delta(C_N)$. Furthermore, by Lemma 5.28, $\lim_{k\to\infty} d_i(\delta^{t_k}) = 0$ for *any* convergent subsequence, implying $d_i(\lim_{t\to\infty} \delta^{t_k}) = 0$ for every agent $i \in N$, and so $\lim_{t\to\infty} \delta^{t_k} = \delta^*$ for every convergent subsequence $(\delta^{t_k})_{k\in\mathbb{N}}$. Thus, $\lim_{t\to\infty} \delta^t = \delta^*$. □

Remark 5
For binary Leontief utilities, Φ simplifies to $\Phi(\delta_1, \dots, \delta_n) = -\sum_{x\in A} \delta_x \log(\delta_x)$, i.e., lexicographic improvements of δ increase Φ. Thus, $\widehat{\Phi}(\delta) := -\sum |\delta_x - \delta_y|$, where the sum is taken over all (unordered) pairs of distinct projects $x, y \in A$, can be chosen. Moreover, $\widehat{\Phi}$ has the advantage that it increases linearly (not only quadratically) in the redistributed amounts. It can then be shown that Theorem 5.24 holds for *any* sequence $\mathcal{S}$ in which each agent appears infinitely often.

Round-robin spending dynamics

We now move on to a model in which there is a constant flow of donations, and each agent repeatedly donates her contribution C_i when it is her turn. To this end, fix some order of the agents (say, $1, 2, \dots, n$) and denote by $\delta_{i,x}^t$ the total amount of contributions of agent i to project x until round t. At each round $t \geqslant 0$, agent

$i_t = 1 + (t \mod n)$ donates C_{i_t} in such a way that her utility is maximized with respect to the previous donation of each other agent, i.e.,

$$\begin{aligned}
\delta_t^{best} &:= \arg \max_{\delta_{i_t} \in \Delta(C_{i_t})} u_{i_t}\left(\delta_{i_t} + \sum_{t-n<s<t} \delta_s^{best}\right) \\
\delta^{t+1} &:= \delta^t + \delta_t^{best} \\
\delta_{i_t}^{t+1} &:= \delta_{i_t}^t + \delta_t^{best},
\end{aligned}$$

where the distribution of the contribution of agent i_t in round t is denoted by δ_t^{best}.[40]

Lemma 5.23 still applies, showing that each agent's best response is unique. To compare δ^t with the equilibrium distribution δ^* (where each agent only contributed once), we scale δ_i^t by the number of donations of agent i until round t, which equals $\lfloor (t+n-i)/n \rfloor$.

To illustrate the process, consider again Example 5.3 where agents take turns and repeatedly contribute $C_i = 1$. The first two steps of the dynamics coincide with the redistribution dynamics, i.e., $\delta^1 = (1/2, 1/2, 0)$ and $\delta^2 = (1/2, 3/4, 3/4)$. Then, Agent 1 contributes $C_1 = 1$ again, leading to $\delta^3 = (9/8, 9/8, 3/4)$.

Theorem 5.29
Given a profile P, the continuous round-robin spending dynamics converge to the equilibrium distribution, i.e.,

$$\lim_{t\to\infty} \sum_{i \in N} \frac{1}{\lfloor (t+n-i)/n \rfloor} \delta_i^t = \delta^*.$$

Proof. For every t, note that δ_t^{best} is the same distribution as the best response of agent i_t under the redistribution dynamics with round-robin sequence $\mathcal{S}$. Thus, Theorem 5.24 implies that the sum of the last n individual distributions (one per agent) converges to the equilibrium distribution, i.e., $\lim_{t\to\infty} \sum_{k=t-n+1}^{t} \delta_k^{best} = \delta^*$. Consequently, for t being a multiple of n, the sum

$$\sum_{i \in N} \frac{1}{\lfloor (t+n-i)/n \rfloor} \delta_i^t = \sum_{i \in N} \frac{n}{t} \delta_i^t = \frac{n}{t} \sum_{\ell=1}^{t/n} \sum_{k=(\ell-1)n}^{\ell n - 1} \delta_k^{best}$$

converges to δ^* as $t \to \infty$. As for arbitrarily large t not being a multiple of n, donations from rounds $\lfloor n/t \rfloor, \dots, t-1$ only have an arbitrarily small impact on $\sum_{i \in N} \frac{1}{\lfloor (t+n-i)/n \rfloor} \delta_i^t$ for large t, convergence to δ^* holds for the whole sequence. □

5.3 BINARY LEONTIEF UTILITIES

Again, we consider the special case of binary valuations, i.e., $v_{i,x} \in \{0, 1\}$ for all agents $i \in N$ and projects $x \in A$ separately as they allow for further insights into the structure of the equilibrium distribution, leading to new interpretations and

40 We here assume that the "observation window" of each agent is given by the last $n-1$ rounds. Computer simulations suggest that convergence also holds for larger observation windows.

additional properties of *NASH*. Thus, agent i's utility from a distribution δ is given by

$$u_i(\delta) = \min_{x \in A_i} \delta_x.$$

For each project $x \in A$, we denote by $N_x \subseteq N$ the set of agents who approve x. For a subset of agents $N' \subseteq N$, we denote $A_{N'} := \bigcup_{i \in N'} A_i$ as the set of projects approved by at least one member of N'. Note that, for every project $x \in A$ and every agent $i \in N_x$, $\delta_x \geqslant u_i(\delta)$.

Recall from Section 4.3 that for linear utilities with binary weights, a distribution is in equilibrium if and only if each agent contributes only to approved projects. Obviously, this equivalence no longer holds for binary Leontief utilities. Nevertheless, the concept of agents contributing only to approved projects is still important.

Definition 5.30
For a fixed profile (U, C) with binary valuations, a distribution δ is *implementable*[41] if it admits a decomposition $(\delta_i)_{i \in N}$ such that $\delta_{i,x} = 0$ for every agent $i \in N$ and project $x \notin A_i$.

Implementability can be used to establish two appealing alternative interpretations of *NASH* for binary weights.

5.3.1 Egalitarianism for Projects

We aim at a rule that distributes contributions to the projects as equally as possible while still respecting the preferences of the donors. One mechanism that comes to mind selects a distribution that, among all implementable distributions, maximizes the smallest amount allocated to a project. Subject to this, it maximizes the second-smallest allocation to a project, and so on.

Definition 5.31
Given two vectors v, w of the same size, we say that v *is leximin-higher than* w (denoted $v \succ_{lex} w$) if the smallest value in v is larger than the smallest value in w, or the smallest values are equal, and the second-smallest value in v is larger than the second-smallest value in w, and so on. $v \succeq_{lex} w$ means that either $v \succ_{lex} w$ or the multiset of values in v is the same as the one in w.

The leximin order (w.r.t. projects) on the closed and convex set of implementable distributions is connected, every two vectors are comparable, and there exists a unique maximal element (otherwise, any convex combination of two different maximal elements would yield another distribution that is leximin-higher). It turns out that this maximal element coincides with the equilibrium distribution.

Theorem 5.32
With binary weights, *NASH* chooses the leximin-maximal (w.r.t. projects) among implementable distributions.

41 Brandl et al. (2021b) called such distributions *decomposable*. We stick with the "original" name *implementability* (Brandl et al., 2019) as we think it better reflects the role of this axiom, at least for Leontief utilities.

Proof. Let δ be the leximin-maximal distribution among implementable distributions. Suppose for contradiction that δ is not the equilibrium distribution. By Proposition 5.2, there is an agent $i \in N$ who contributes to a non-critical project $x \in A_i$, that is, $\delta_{i,x} > 0$ and $\delta_x > u_i(\delta)$. Let $y \in A_i$ be a critical project of agent i under δ, that is, $\delta_y = u_i(\delta)$.

If agent i now moves $0.5(\delta_x - \delta_y)$ from x to y, the resulting distribution is still decomposable, as both x and y are in A_i. It is leximin-higher than δ as $\delta_x > \delta_y$, which contradicts the leximin-maximality of δ. □

Remarkably, this new interpretation of *NASH* ignores the agents' utility structure and does not directly take into account individual contributions. Instead, they enter indirectly through the implementability constraints.

We will later show that the equilibrium distribution also maximizes other welfare objectives among all implementable distributions (Theorem 7.6).

Theorem 5.32 implies that *NASH* can be computed by solving the following program, with variables $d_{i,x}$ for all $i \in N, x \in A$.

$$\begin{aligned}
&\operatorname{lex\,max\,min}\{d_x\}_{x\in A} && \\
&\quad\text{s.t. } d_x = \sum_{i\in N} d_{i,x} && \text{for all } x \in A \\
&\quad\sum_{x\in A_i} d_{i,x} = C_i && \text{for all } i \in N \\
&\quad d_{i,x} = 0 && \text{for all } i \in N, x \notin A_i \\
&\quad d_{i,x} \geqslant 0 && \text{for all } i \in N, x \in A_i,
\end{aligned}$$

where "lex max min" refers to finding a solution vector that is maximal in the leximin order subject to the constraints, and the second and third constraints represent implementability.

It is well-known that such leximin optimization with k objectives and linear constraints can be solved by a sequence of k linear programs (see, e.g., Ehrgott, 2005, Sect. 5.3).

Corollary 5.33
With binary weights, the equilibrium distribution can be computed by solving at most m linear programs.

5.3.2 Egalitarianism for Agents

Switching the roles of agents and projects, one could also consider rules that are egalitarian from the point of view of the agents and consider the leximin order w.r.t. agents' utilities.

Theorem 5.34
With binary weights, *NASH* chooses the leximin-maximal (w.r.t. utilities) among implementable distributions.

Proof. By uniqueness of the equilibrium distribution (Theorem 5.10), it is sufficient to show that every leximin-maximal among implementable distributions δ coincides with the equilibrium distribution. Suppose for contradiction that δ is no equilibrium distribution. Then, some agent $i \in N$ has to contribute to a non-critical project $x \in A_i$, that is, $\delta_{i,x} > 0$ and $\delta_x > u_i(\delta)$.

Set $\varepsilon := \min\{\delta_{i,x}, \ \delta_x - u_i(\delta)\}$. By assumption, $\varepsilon > 0$. Construct a new distribution δ' from δ by changing only δ_i. Remove ε from project x and add $\varepsilon/|A_i|$ to every project in A_i (including x). Agent i's utility increases by $D/|A_i|$ as

$$u_i(\delta') = \min\{\delta'_x, \min_{y \in A_i \setminus x} \delta'_y\} = u_i(\delta) + \frac{\varepsilon}{|A_i|}$$

where $\delta'_x \geqslant u_i(\delta)$ follows from the definition of ε. Moreover, any decrease in the utility of some agent j has to be caused by the decrease in the distribution to x, so x must be a critical project for agent j in δ', i.e., $u_j(\delta') = \delta'_x \geqslant u_i(\delta') > u_i(\delta)$.

Thus, moving from δ to δ' strictly increases the number of agents with utility larger than $u_i(\delta)$, and the utility of each agent with utility at most $u_i(\delta)$ in δ does not decrease. Therefore, the utility vector induced by δ' is lexicographically preferred to the one induced by δ. Since δ' is decomposable, this contradicts the leximin-maximality of δ. □

Theorem 5.34 implies that the equilibrium distribution can be computed by solving the following program, with variables $\tilde{u}_i$ for all $i \in N$ and $d_{i,x}$ for all $i \in N, x \in A_i$.

$$\begin{aligned}
\operatorname{lex\,max} \min\{\tilde{u}_i\}_{i \in N} & \\
\text{s.t. } \tilde{u}_i \leqslant d_{i,x} & \qquad \text{for all } i \in N, x \in A_i \\
\sum_{x \in A_i} d_{i,x} = C_i & \qquad \text{for all } i \in N \\
d_{i,x} = 0 & \qquad \text{for all } i \in N, x \notin A_i \\
d_{i,x} \geqslant 0, \tilde{u}_i \geqslant 0 & \qquad \text{for all } i \in N, x \in A_i.
\end{aligned}$$

By the same argument as for leximin maximization w.r.t. projects, this program can be solved using at most n linear programs.

All in all, we have obtained two additional algorithms for computing the equilibrium distribution in the case of binary valuations which consist of m and n linear programs, respectively.

These insights do not generalize to Leontief utilities with arbitrary valuations.

EXAMPLE 5.35
Consider the following example with three projects $A = \{a, b, c\}$ and two agents.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	C_i
Agent 1	1	2	0	6
Agent 2	0	1	1	6

The leximin-maximal distribution w.r.t. projects (subject to implementability) is $\delta = (4, 4, 4)$ with decomposition $\delta_1 = (4, 2, 0)$ and $\delta_2 = (0, 2, 4)$, but Agent 1 contributes to non-critical project a.

The leximin-maximal distribution w.r.t. agents (subject to implementability) is $\delta' = (3, 6, 3)$ with decomposition $\delta'_1 = (3, 3, 0)$ and $\delta'_2 = (0, 3, 3)$, but Agent 2 contributes to non-critical project b.

It can be verified that the equilibrium distribution is $\delta^* = 0.2 \cdot (12, 24, 24)$ with decomposition $\delta_1^* = 0.2 \cdot (12, 18, 0)$ and $\delta_2^* = 0.2 \cdot (0, 6, 24)$.

5.4 CHARACTERIZATION OF *NASH*

In this section, we show that *NASH* admits an appealing characterization via strategyproofness and fairness.

THEOREM 5.36
NASH is the only U-continuous mechanism that satisfies group-strategyproofness and always returns core distributions.

REMARK 6
Group-strategyproofness, in combination with the core, ensures that the mechanism is also efficient. To see this, note that for fixed contribution profile C, each distribution $\delta \in \Delta(C_N)$ is attainable. For the profile in which $v_i = \delta$ for all $i \in N$, such a mechanism has to return δ. Otherwise, the set of all agents could deviate to δ which violates the core property. Thus, in case an outcome is not efficient, the set of all agents has the power to switch to the Pareto dominating distribution, which contradicts group-strategyproofness. In fact, it can be shown that every core distribution is already efficient (see [3]).

Theorem 5.16, Theorem 5.14, and Corollary 5.5 show that *NASH* satisfies all properties in the theorem statement.

We observe that none of the axioms includes changes in contributions. Our proof does not make use of constraints induced by changes in the contribution profile and we fix the contribution profile to C in the following. In addition, we consider only manipulations that do not change the *sum* of an agent i's valuations $\sum_{x \in A} v_{i,x}$. First, this ensures that the proof also works for the portioning setting. Second, the profile U together with all other profiles where each agent i's valuations sum up to $\sum_{x \in A} v^u_{i,x}$ forms a compact set in $\mathbb{R}^n_{\geq 0}$.[42] By the Heine-Cantor theorem, f is *uniformly continuous* on this set of profiles. Thus, we are allowed to apply uniform continuity instead of U-continuity in our proof.

Let f be a mechanism satisfying the properties from Theorem 5.36. The proof is divided into three lemmas and has the following structure. Starting at an arbitrary utility profile U, we first show in Lemma 5.37 that moving to a "key" profile U^* cannot change the outcome, i.e., $f(U^*) = f(U)$. Next, Lemma 5.38 states that $f(U^*) = NASH(U^*)$. Finally, Lemma 5.39 proves $NASH(U^*) = NASH(U)$, which completes the proof as we then have $f(U) = NASH(U)$.

Define $\delta = f(U)$ and denote by U^* the profile with valuations

$$v^*_{i,x} = \begin{cases} v_{i,x}/v_i(T_{\delta,i}) & \text{for } x \in T_{\delta,i} \\ 0 & \text{for } x \notin T_{\delta,i}, \end{cases}$$

where $v_i(T_{\delta,i}) := \sum_{x \in T_{\delta,i}} v_{i,x}$. That is, in U^*, each agent changed her valuations so that they are nonzero only on her critical projects (under δ).

42 In case each agent's valuations sum up to 1, we are in the portioning setting.

For example, suppose we are given a set of four projects $A = \{a, b, c, d\}$ and a profile with equilibrium distribution $\delta = (0.1, 0.1, 0.6, 0.2)$ where one agent i has valuations $v_i = (0.1, 0.2, 0.3, 0.4)$. Then, $T_{\delta,i} = \{b, d\}$, $v_i(T_{\delta,i}) = 0.6$, and $v_i^* = (0, 1/3, 0, 2/3)$.

As δ lies in the core, without loss of generality, we can assume that $\delta_x = 0$ if and only if $v_{i,x} = 0$ for all $i \in N$. To see this, note that if $\delta_x = 0$ for some $x \in A$, then $u_i(\delta) = 0$ for all agents with $v_{i,x} > 0$, meaning that such agents could gain utility from "leaving" and distributing their contributions on their own. Moreover, $v_{i,x} = 0$ for all $i \in N$ implies that $\delta_x = 0$, otherwise the contribution on x could be distributed uniformly over all other projects instead, which increases the utilities of all agents.

Lemma 5.37

If f satisfies the properties from Theorem 5.36, then $f(U^*) = f(U) = \delta$ and the sets of critical projects do not change, that is, $T_{\delta,i} = T^*_{\delta,i}$ for every $i \in N$.

Proof. We change the valuations of each agent in turn. For a fixed agent i, we change v_i towards v_i^* gradually, to some $\tilde{v}_i := \lambda v_i^* + (1-\lambda)v_i$, for some $\lambda \in (0, 1]$ to be computed later. Then, we proceed along this line until we reach v_i^*. In particular, for an agent i with $T_{\delta,i} = A$, it holds that $v_i = v_i^*$. In the above example, $\lambda = 0.3$ gives $\tilde{v}_i = (0.07, 0.24, 0.21, 0.48)$. If $v_i = v_i^*$, it is clear that the distribution does not change, so assume that $v_i \neq v_i^*$. The change from v_i to $\tilde{v}_i$ has a simple structure:

- $\tilde{v}_{i,x} > v_{i,x}$ for all $x \in T_{\delta,i}$, and the ratio $\tilde{v}_{i,x}/v_{i,x} = \lambda/v_i(T_{\delta,i}) + (1-\lambda) =: \lambda^+$ is a constant independent of x (in the example, $\lambda^+ = 1.2$), and

- $\tilde{v}_{i,x} < v_{i,x}$ for all $x \notin T_{\delta,i}$ with $v_{i,x} > 0$, and the ratio $\tilde{v}_{i,x}/v_{i,x} = (1-\lambda) =: \lambda^-$ is again independent of x (in the example, $\lambda^- = 0.7$).

Now, consider the ratios $\delta_x/v_{i,x}$ versus the ratios $\delta_x/\tilde{v}_{i,x}$ for $x \in A_i$. For each $x \in T_{\delta,i}$, we have $\delta_x/v_{i,x} > \delta_x/\tilde{v}_{i,x}$ since $\tilde{v}_{i,x} > v_{i,x}$, whereas for each $x \notin T_{\delta,i}$, we have $\delta_x/v_{i,x} < \delta_x/\tilde{v}_{i,x}$ as $\tilde{v}_{i,x} < v_{i,x}$. Furthermore, for all $x \in T_{\delta,i}$, the ratios $\delta_x/\tilde{v}_{i,x}$ remain equal (as $\tilde{v}_{i,x}/v_{i,x}$ is constant) and smallest when moving from v to $\tilde{v}$. This implies that $\tilde{T}_{\delta,i} = T_{\delta,i}$.

Moreover, the entire ordering of projects by the ratio $\delta_x/v_{i,x}$ is identical to the ordering of projects by the ratio $\delta_x/\tilde{v}_{i,x}$ as the smallest ratio is divided by $\lambda^+ > 1$ and the other ratios are divided by $\lambda^- < 1$. In other words, suppose we partition the projects into subsets according to the ratios $\delta_x/v_{i,x}$, and denote the subset with the smallest ratio by $T_{\delta,i,1} \equiv T_{\delta,i}$, the subset with the second smallest ratio by $T_{\delta,i,2}$, and so on. Then, $T_{\delta,i,r} = \tilde{T}_{\delta,i,r}$ for all $r \geqslant 1$.

COMPUTING λ. We pick λ sufficiently small such that no new project becomes critical for i. Specifically, set

$$\varepsilon := \frac{\min_{x \in T_{\delta,i}, y \notin T_{\delta,i}} (\delta_y v_{i,x} - \delta_x v_{i,y})}{\sum_{x \in A} v_{i,x}} \leqslant \min_{x \in T_{\delta,i}, y \notin T_{\delta,i}} \frac{\delta_y v_{i,x} - \delta_x v_{i,y}}{v_{i,x} + v_{i,y}}.$$

Note that $\varepsilon > 0$, as $\delta_y/v_{i,y} > \delta_x/v_{i,x}$ by definition of critical projects.

By uniform continuity of f, there exists $\gamma > 0$ such that $\|f(U) - f(U')\|_1 < 2\varepsilon$ for all U' with $\|U - U'\|_1 \leqslant \gamma$.[43] Set

$$\lambda := \min\left(1, \frac{\gamma}{\|\nu_i - \nu_i^*\|_1}\right),$$

and define $\tilde{U}$ as a profile identical to U except that i changes her valuations from ν_i to $\tilde{\nu}_i := \lambda\nu_i^* + (1-\lambda)\nu_i$. Note that $\|U - \tilde{U}\|_1 = \lambda\|\nu_i - \nu_i^*\|_1 \leqslant \gamma$, so $\|\delta - \tilde{\delta}\|_1 < 2\varepsilon$, where $\delta = f(U)$ and $\tilde{\delta} = f(\tilde{U})$.

The choice of ε ensures that $T_{\tilde{\delta},i} \subseteq T_{\delta,i}$, as for arbitrary $x \in T_{\delta,i}$ and $y \notin T_{\delta,i}$ it holds that $\tilde{\delta}_x < \delta_x + \varepsilon$ and $\tilde{\delta}_y > \delta_y - \varepsilon$, so

$$\begin{aligned}\frac{\tilde{\delta}_y}{\nu_{i,y}} &> \frac{\delta_y - \varepsilon}{\nu_{i,y}} \geqslant \frac{\delta_y}{\nu_{i,y}} - \frac{\delta_y\nu_{i,x} - \delta_x\nu_{i,y}}{\nu_{i,y}(\nu_{i,x}+\nu_{i,y})} = \frac{\delta_y\nu_{i,y} + \delta_x\nu_{i,y}}{\nu_{i,y}(\nu_{i,x}+\nu_{i,y})} = \frac{\delta_y + \delta_x}{\nu_{i,x}+\nu_{i,y}} \\ &= \frac{\delta_x\nu_{i,x} + \delta_y\nu_{i,x}}{\nu_{i,x}(\nu_{i,x}+\nu_{i,y})} = \frac{\delta_x}{\nu_{i,x}} + \frac{\delta_y\nu_{i,x} - \delta_x\nu_{i,y}}{\nu_{i,x}(\nu_{i,x}+\nu_{i,y})} \geqslant \frac{\delta_x + \varepsilon}{\nu_{i,x}} > \frac{\tilde{\delta}_x}{\nu_{i,x}}.\end{aligned}$$

So every y which is not critical for agent i under δ cannot be critical for her under $\tilde{\delta}$. Therefore, $T_{\tilde{\delta},i} \subseteq T_{\delta,i} = \tilde{T}_{\delta,i}$.

PROVING THAT THE OUTCOME DOES NOT CHANGE. Consider a manipulation of agent i who manipulates between reporting ν_i and $\tilde{\nu}_i$. Strategyproofness for agent i implies both $u_i(\delta) \geqslant u_i(\tilde{\delta})$ and $\tilde{u}_i(\tilde{\delta}) \geqslant \tilde{u}_i(\delta)$.

The latter condition implies that, for every alternative $x \in T_{\delta,i}$,

$$\begin{aligned}\frac{\delta_x}{\tilde{\nu}_{i,x}} &= \tilde{u}_i(\delta) && \text{(as } x \in T_{\delta,i} = \tilde{T}_{\delta,i}) \\ &\leqslant \tilde{u}_i(\tilde{\delta}) && \text{(by strategyproofness)} \\ &\leqslant \frac{\tilde{\delta}_x}{\tilde{\nu}_{i,x}} && \text{(by definition of Leontief utilities).}\end{aligned}$$

So $\delta_x \leqslant \tilde{\delta}_x$ for each alternative $x \in T_{\delta,i}$. Together with $T_{\tilde{\delta},i} \subseteq T_{\delta,i}$, this implies $u_i(\delta) \leqslant u_i(\tilde{\delta})$. Therefore, $u_i(\delta) = u_i(\tilde{\delta})$. Furthermore, if $\tilde{u}_i(\tilde{\delta}) > \tilde{u}_i(\delta)$, then $\tilde{\delta}_x > \delta_x$ for all $x \in \tilde{T}_{\delta,i} \supseteq T_{\tilde{\delta},i}$, which means that $u_i(\delta) < u_i(\tilde{\delta})$, contradicting $u_i(\delta) = u_i(\tilde{\delta})$. Thus, $\tilde{u}_i(\tilde{\delta}) = \tilde{u}_i(\delta)$.

Moreover, if the utility of some other agent i' increases, group-strategyproofness is violated for the pair $\{i, i'\}$, as this pair could profitably manipulate from δ to $\tilde{\delta}$. Similarly, if the utility of some other agent i' decreases, group-strategyproofness is again violated for the pair $\{i, i'\}$, as this pair could profitably manipulate from $\tilde{\delta}$ to δ. Thus, $u_j(\delta) = u_j(\tilde{\delta})$ for all $j \in N$. Since δ is efficient with respect to U (see Remark 6), so is $\tilde{\delta}$. By Proposition 5.9, $\delta = \tilde{\delta}$.

Applying this argument repeatedly, we get a sequence of profiles (U^k) with $U^0 = P$ where ν_i^k lies on the line $\lambda\nu_i^* + (1-\lambda)\nu_i$ for every k. It remains to show that (ν^k) reaches ν_i^* after a finite number of steps. For that, consider the expression in the definition of ε:

$$\min_{x \in T_{\delta,i},\, y \notin T_{\delta,i}} (\delta_y\nu_{i,x} - \delta_x\nu_{i,y}).$$

43 We measure the distance between two profiles as in Definition 3.2.

As v_i approaches v_i^*, $v_{i,x}$ increases and $v_{i,y}$ decreases while δ and $T_{\delta,i}$ stay the same, so overall the expression increases. Thus, we can take the ε (and the corresponding γ) from the first step for every step. Furthermore, $\|U^k - U^{k+1}\|_1 = \gamma$ (unless $\lambda = 1$, but then we have reached v_i^*) implying that we reach v_i^* after at most $\lceil \|v_i - v_i^*\|_1/\gamma \rceil$ steps as we move on a line of length $\|U^k - U^{k'}\|_1 = \sum_{\ell=k}^{k'-1} \|U^\ell - U^{\ell+1}\|_1$ for $k' \geqslant k$.

After the first agent has reached her desired peak v_i^*, we turn to the next agent and repeat the procedure. In that way, we eventually arrive at U^*.

To see that $T^*_{\delta,i} = T_{\delta,i}$ for all $i \in N$, note that for every non-critical project $x \notin T_{\delta,i}$ we have $v^*_{i,x} = 0$, so $x \notin T^*_{\delta,i}$. Furthermore, for any critical project $x \in T_{\delta,i}$ and any other $y \in T_{\delta,i}$,

$$\frac{\delta_x}{v^*_{i,x}} = \frac{\delta_x \cdot v_i(T_{\delta,i})}{v_{i,x}} = \frac{\delta_y \cdot v_i(T_{\delta,i})}{v_{i,y}} = \frac{\delta_y}{v^*_{i,y}},$$

so $x, y \in T^*_{\delta,i}$. Therefore, $T_{\delta,i} = T^*_{\delta,i}$. □

Lemma 5.38

Let U^* be a utility profile and δ be a distribution in which every agent values every non-critical project at 0, i.e., $x \notin T^*_{\delta,i}$ implies $v^*_{i,x} = 0$ for any agent i. If δ is in the core, then $\delta = \mathit{NASH}(U^*)$.

Proof. Let U^* be an arbitrary profile and let $\delta \neq \mathit{NASH}(U^*)$ be a distribution such that $x \notin T^*_{\delta,i}$ implies $v^*_{i,x} = 0$. In particular, δ does not maximize Nash welfare. By Lemma 5.8, there exists a group $N^- \subseteq N$ of agents such that the total amount given to projects critical for some agent from N^- is less than C_{N^-}, that is,

$$\delta(T^*_{\delta,N_-}) < C_{N^-}, \tag{23}$$

where $T^*_{\delta,N_-} := \bigcup_{i \in N_-} T^*_{\delta,i}$. We will now show that the core property is violated for N^-. This is clear if $\delta(T^*_{\delta,N_-}) = 0$, so assume that $\delta(T^*_{\delta,N_-}) > 0$.

Define a new distribution in which only alternatives in T^*_{δ,N_-} are funded via

$$\delta' := \begin{cases} (C_{N^-}/\delta(T^*_{\delta,N_-})) \cdot \delta_x & \text{for } x \in T^*_{\delta,N_-} \\ 0 & \text{for } x \notin T^*_{\delta,N_-}. \end{cases}$$

For every $i \in N_-$, as $v^*_{i,x} = 0$ for $x \notin T^*_{\delta,N_-} \supseteq T^*_{\delta,i}$, the utility $u_i^*(\delta')$ equals $(C_{N^-}/\delta(T^*_{\delta,N_-})) \cdot u_i^*(\delta)$, which is larger than $u_i^*(\delta)$ by (23). By construction, $\delta' \in \Delta(C_{N^-})$ proving that the group of agents in N_- can move to a better outcome δ' with their contributions, showing that δ is not in the core. □

Lemma 5.39

Let U^* and U be profiles where $T^*_{\delta,i} = T_{\delta,i}$ for $\delta = \mathit{NASH}(U^*)$ and all $i \in N$. Then, $\mathit{NASH}(U) = \delta$.

Proof. As δ maximizes Nash welfare in U^*, there exists a decomposition $(\delta_i)_{i\in N}$ such that $d_{i,x} = 0$ for every $i \in N$ and $x \notin T^*_{\delta,i}$ by Proposition 5.2. Due to

$T^*_{\delta,i} = T_{\delta,i}$, the same decomposition proves that δ also maximizes Nash welfare in U by Proposition 5.2, and $NASH(U) = \delta$. □

Proof of Theorem 5.36. Let U be an arbitrary profile, and U^* a modified profile defined as in Lemma 5.37. Then,

$$f(U) \overset{\text{Lemma 5.37}}{=} f(U^*) \overset{\text{Lemma 5.38}}{=} NASH(U^*) \overset{\text{Lemma 5.39}}{=} NASH(U),$$

where Lemma 5.39 uses the fact that the sets of critical alternatives under δ did not change when moving from U to U^*. □

Concerning independence of the axioms, it is straightforward to see that the core property is required for Theorem 5.36 since the conjunction of the remaining axioms is satisfied by e.g., a mechanism that returns the uniform distribution over all projects for all profiles. The necessity of group-strategyproofness can be shown by slightly perturbing the outcome of *NASH* when $m = 2$. We conjecture that U-continuity is also required for the characterization.

6 RELATIONS TO PUBLIC GOOD MARKETS AND LINDAHL EQUILIBRIUM

Essentially, our projects are public goods that are funded by the agents. This bears a strong resemblance to *public good markets*, which (mainly) concern the problem of financing public goods under the presence of *private goods* (see, e.g., Moore (2007) for an overview). Loosely speaking, agents have the possibility to spend their contribution on some private good from which only they can benefit. There, the central question is how to ensure the funding of public goods and avoid the common phenomenon of "free-riding" where each individual relies on other agents to finance the public goods while saving her own contribution for private purposes such that in the end, no public goods are funded at all. This is reminiscent of a prisoner's dilemma where private and public goods correspond to "defect" and "cooperate". Classical works (e.g., Samuelson, 1954; Bergstrom et al., 1986) often include production costs, restrict themselves to one public good, and investigate its funding with respect to agents' utility functions and varying contributions. In light of the prisoner's dilemma, it is not surprising that free-riding is a serious problem when having rational agents. This has also been observed in behavioral economics (see, e.g., Lang et al., 2018) where such settings are known as *public goods games* and investigated experimentally. As a consequence, further incentives like rewards for cooperation are included in experimental setups (see, e.g., Ostrom et al., 1992; Andreoni et al., 2003).

For our donor coordination setting, we assume that each agent i has set aside a certain budget $\bar{C}_i$, either voluntarily or as part of, e.g., their personal income tax, and only decides how much she wants to contribute. In particular, saving contributions or allocating them to private projects is impossible. Therefore, we are concerned with how contributions are allocated *among* public projects instead of how much contributions are allocated *to* them.

Nevertheless, as already mentioned, contribution incentive-compatibility (Section 4.2.1) admits a very interesting interpretation when both public and private goods are available, and agents have dichotomous utilities. In that case, this axiom states that it is always (at least weakly) better to contribute to public projects, i.e., reporting $C_i = \bar{C}_i$ to the mechanism, than to spend some of the money privately (either on one of the public or a private project). One might argue that for linear utilities, a private project can be interpreted as a public project that is valued at zero by all other agents. Efficiency would then ensure that this project does not receive any funding at all.[44] Note that this observation does not generalize to other utility functions, e.g., Leontief utilities, as contribution-incentive compatibility implicitly assumes that agents' utilities are quasi-linear with respect to the money set aside by an agent (see, e.g., Moore, 2007). On top of that, contribution incentive-compatibility as defined in Definition 4.9 seems less interesting

44 Given that an agent shares at least one approved project with another agent, an efficient mechanism always assigns zero contribution to projects no other agent approves.

for utility functions other than linear ones, at least in the donor coordination setting, as we would compare apples and oranges and demand quasi-linearity while knowing that the utility structure is entirely different. It is not sensible to require quantitatively similar utility gains for linear and Leontief utilities.

So far, we have not talked about a central feature of markets: *prices*. Foley (1970) took up on an idea by Lindahl (1919), who proposed that agents should pay for a public good proportional to the derived benefits. In detail, if each agent faces an individual vector of prices for public goods[45] and agrees on the overall distribution δ given these prices, then δ constitutes a *Lindahl equilibrium* in case the price vectors satisfy some additional properties. We adapt the definition from Fain et al. (2016) for public good markets without private goods to our setting.

Definition 6.1
For a given profile (U, C) and individual price vectors $p_i = (p_{i,x})_{x \in A} \in \mathbb{R}^m_{\geqslant 0}$ for each $i \in N$, a distribution $\delta \in \Delta(C_N)$ is a *Lindahl equilibrium* if

(i) For every agent $i \in N$, $\delta \in \max_{z \in \mathbb{R}^m_{\geqslant 0}} u_i(z)$ subject to $\sum_{x \in A} p_{i,x} z_x = C_i$, and

(ii) For every project $x \in \text{supp}(\delta)$, $\sum_{i \in N} p_{i,x} = 1$ and for every project $x \notin \text{supp}(\delta)$, $\sum_{i \in N} p_{i,x} \leqslant 1$.

Remark 7
Given that agents' utilities are weakly increasing in the allocated contributions (and strictly increasing for some), agents are incentivized to use their whole contribution in (i), turning the inequality from Fain et al. (2016) into an equality.

Furthermore, for (ii), Fain et al. (2016) require δ to maximize a hypothetical profit of the form $\sum_{x \in A} (\sum_{i \in N} p_{i,x}) \cdot z_x - \|z\|_1$ where $z \in \mathbb{R}^m_{\geqslant 0}$. This implies that $\sum_{i \in N} p_{i,x} := K$ is constant for $x \in \text{supp}(\delta)$ and $\sum_{i \in N} p_{i,x} \leqslant K$ for $x \notin \text{supp}(\delta)$. Summing up all equalities from (i) yields $K \cdot \|\delta\|_1 = C_N$ and thus, $K = 1$.

Remark 8
One might wonder why agents are allowed to choose $z \in \mathbb{R}^m_{\geqslant 0}$ instead of $z \in \Delta(C_N)$ in (i). For that, consider two projects and two agents with linear utilities and $v_1 = (1, 0)$, $v_2 = (0, 1)$, and $C_i = 1$. Then, the equilibrium distribution is $(1, 1)$ but with prices $p_1 = (3/4, 5/12)$, $p_2 = (1/4, 7/12)$, and $\delta \in \max_{z \in \Delta(2)} u_i(z)$ subject to $\sum_{x \in A} p_{i,x} z_x = 1$, both agents choose $\delta = (1/2, 3/2)$. In particular, Agent 1 is forced to allocate only $1/2$ to her approved project as she needs to afford an allocation of total size 2.

The first part of Definition 6.1 shows that given her individual budget C_i and facing her individual prices p_i, each agent i agrees on δ.

The second part simply ensures that prices for projects in $\text{supp}(\delta)$ sum up to 1, whereas the ones for projects outside of the support cannot exceed those.

Such price vectors seem purely theoretical as they are hard to justify in practice. Nonetheless, research still investigates the concept of Lindahl equilibrium (Gul and Pesendorfer, 2022; Munagala et al., 2022) as all Lindahl equilibria lie in the core (see, e.g., Foley, 1970). We give a quick proof that is adapted to our donor coordination setting.

45 Foley notes the reversed roles compared to private good markets where the price vector is the same for all agents but they receive individual (private) goods.

Proposition 6.2 *(Foley (1970))*
Every Lindahl equilibrium lies in the core.

Proof. Assume for contradiction that a Lindahl equilibrium $\delta \in \Delta(C_N)$ is not part of the core. Then, there exists a subset of agents $N' \subseteq N$ and a distribution $\delta' \in \Delta(C_{N'})$ such that $u_i(\delta') > u_i(\delta)$ for all $i \in S$.

By (i),

$$\sum_{x \in A} p_{i,x}\delta'_x > \sum_{x \in A} p_{i,x}\delta_x = C_i$$

for all $i \in N'$. Summing up these inequalities over all $i \in N'$ yields

$$\sum_{x \in A} \left(\sum_{i \in N'} p_{i,x} \right) \cdot \delta'_x > C_{N'}.$$

By (ii), $\sum_{i \in N'} p_{i,x} \leqslant 1$. Combining both inequalities implies $\sum_{x \in A} \delta'_x > C_{N'}$, which contradicts $\delta' \in \Delta(C_{N'})$. □

Furthermore, Fain et al. (2016) showed an interesting connection to maximizing Nash welfare.

Theorem 6.3 *(Fain et al. (2016))*
For a fixed profile (U, C), assume agents' utilities are given by utility functions $u_i \colon \Delta(C_N) \to \mathbb{R}^m_{\geqslant 0}$ that are homogeneous of degree 1 and of the form $u_i(\delta) = \sum_{x \in A} v_{i,x} g_x(\delta_x)$ where $g_x \colon [0, C_N] \to \mathbb{R}^m_{\geqslant 0}$ are differentiable, weakly increasing, and concave.

Then, every distribution that maximizes Nash welfare constitutes a Lindahl equilibrium.

In particular, Theorem 6.3 covers linear utilities.

Corollary 6.4 *(Fain et al. (2016))*
For linear utilities, every Nash welfare maximizing distribution constitutes a Lindahl equilibrium and lies in the core.

We observe that $p_{i,x} = C_i \cdot v_{i,x}/u_i(\mathit{NASH}(U, C))$ for every $i \in N$ and $x \in A$ forms a valid system of prices in case of linear utilities. In particular, agent i's price for project x is proportional to her valuation $v_{i,x}$ whereas (ii) corresponds to the KKT conditions for maximizing Nash welfare from the proof of Theorem 4.8. This induces an alternative proof to the one given by Fain et al. (2016).

Theorem 6.3 can be extended to Leontief utilities.

Proposition 6.5
For Leontief utilities, δ maximizes Nash welfare (i.e., constitutes an equilibrium distribution) if and only if it is a Lindahl equilibrium distribution.

Proof. We first show that the equilibrium distribution is also a Lindahl equilibrium distribution. Let δ be the equilibrium distribution with equilibrium decomposition $(\delta_i)_{i \in N}$. Without loss of generality, we can assume that every project is valuable to at least one agent, and therefore $\delta_x > 0$ for all $x \in A$. The prices are set as $p_{i,x} = \delta_{i,x}/\delta_x$ for all $i \in N$ and $x \in A$. Then, $\sum_{x \in A} p_{i,x}\delta_x = \sum_{x \in A} \delta_{i,x} = C_i$. By

Proposition 5.2, $p_{i,x} = 0$ for every project x not critical for agent i. Therefore, any other $z \in \mathbb{R}^m_{\geqslant 0}$ with $\sum_{x \in A} p_{i,x} z_x = C_i$ must allocate a smaller amount to some project critical for i, and hence yield a smaller utility for i. So (i) holds.

For (ii), simply note that for every $x \in \operatorname{supp}(\delta)$, $\sum_{i \in N} \delta_{i,x}/\delta_x = 1$ by construction. Hence, δ is a Lindahl equilibrium.

We now show that every Lindahl equilibrium is an equilibrium distribution. Let z be a Lindahl equilibrium with prices $p_1, \ldots, p_n \in \mathbb{R}^m_{\geqslant 0}$. Again, we can assume without loss of generality that every project is valuable to at least one agent. Hence, $z_x > 0$ for all $x \in A$ as otherwise, at least one agent would receive zero utility, and (i) would be violated for that agent.

Furthermore, $p_{i,x} = 0$ for $x \notin T_{z,i}$. Otherwise, as $z_x > 0$, agent i spends a positive amount on x and could use it on her critical projects instead to improve her utility.

We claim that for any group of agents $N' \subset N$, $C_{N'} \leqslant \delta(T_{z,N'})$. To see this, note that

$$
\begin{aligned}
C_{N'} &= \sum_{i \in N'} C_i = \sum_{i \in N'} \sum_{x \in A} p_{i,x} z_x && \left(\text{as } C_i = \sum_{x \in A} p_{i,x} z_x\right) \\
&= \sum_{i \in N'} \sum_{x \in T_{z,N'}} p_{i,x} z_x && (\text{as } p_{i,x} = 0 \text{ for } x \notin T_{z,i}) \\
&= \sum_{x \in T_{z,N'}} \sum_{i \in N'} p_{i,x} z_x \\
&\leqslant \sum_{x \in T_{z,N'}} z_x && \left(\text{as } \sum_{i \in N'} p_{i,x} \leqslant 1 \text{ for all } x \in A\right) \\
&= z(T_{z,N'}) .
\end{aligned}
$$

By (13) in Lemma 5.8, z is the equilibrium distribution. □

As the equilibrium distribution is unique, Proposition 6.5 also implies uniqueness of the Lindahl equilibrium.

Corollary 6.6
For Leontief utilities, there exists a unique Lindahl equilibrium.

These two equilibrium notions do not coincide for general utility functions, as we will see in the next chapter.

7 IMPLICATIONS FOR OTHER UTILITY FUNCTIONS

Our results on Leontief utilities from Chapter 5 allow for further insights regarding equilibrium distributions for various utility functions and welfare-maximizing distributions for various welfare notions.

7.1 COBB-DOUGLAS UTILITIES

Given valuations $(\nu_{i,x})_{x \in A}$ of agent i, her Cobb-Douglas utility function is defined as

$$u_i(\delta) = \prod_{x \in A} \delta_x^{\nu_{i,x}}.$$

By taking the logarithm, we observe that these continuous and convex preferences are also represented by $u_i(\delta) = \sum_{x \in A} \nu_{i,x} \log(\delta_x)$. Both Cobb-Douglas and Leontief utility functions belong to the class of utility functions with *constant elasticity of substitution* (see, e.g., Arrow et al., 1961; Varian, 1992) which again form a subclass of homogeneous utility functions.

Cobb-Douglas functions were originally used to model the dependence of production on labor and capital (Cobb and Douglas, 1928). Thus, labor and capital correspond to two projects.[46] The respective weights determine the proportional change in production when the input changes and the "optimal ratio" between both factors.

Cobb-Douglas utilities bear a strong resemblance to Nash welfare. However, note that utility functions represent agents' preferences over distributions and are motivated by the "nature" of the considered projects. In contrast, social welfare functions like Nash welfare represent "preferences" of the social aggregator over the set of admissible utility vectors. They are motivated by axioms, e.g., fairness properties like anonymity.

Surprisingly, the equilibrium distribution is unaffected if the agents' Leontief utility functions are replaced with Cobb-Douglas utility functions with the same valuations.

Proposition 7.1
Given values $(\nu_{i,x})_{i \in N, x \in A}$, a distribution is in equilibrium for Leontief utility functions if and only if it is in equilibrium for Cobb-Douglas utility functions.

Proof. We show that Proposition 5.2 (with the same definition of critical projects) also holds for Cobb-Douglas utilities with $u_i(\delta) = \sum_{x \in A} \nu_{i,x} \cdot \log(\delta_x)$.

46 In the standard representation of the Cobb-Douglas production function, another (constant) factor indicates the ratio of production to the weighted product of labor and capital.

First, we prove that every equilibrium distribution with respect to Cobb-Douglas utilities is also an equilibrium distribution with respect to Leontief utilities via contraposition.

Suppose that, in every decomposition of δ, some agent i with Leontief utilities contributes to a project $y \notin T_{\delta,i}$. Fix a decomposition $(\delta_i)_{i \in N}$ of δ. By assumption, $v_{i,x}/\delta_x > v_{i,y}/\delta_y$ for any $x \in T_{\delta,i}$. Agent i can move a sufficiently small amount ε from $\delta_{i,y}$ to a project $x \in T_{\delta,i}$, resulting in a new individual distribution δ_i^ε which increases her utility as

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{u_i(\delta - \delta_i + \delta_i^\varepsilon) - u_i(\delta)}{\varepsilon} \\ =& \lim_{\varepsilon \to 0} v_{i,x} \cdot \frac{\log(\delta_x + \varepsilon) - \log(\delta_x)}{\varepsilon} + v_{i,y} \cdot \frac{\log(\delta_y - \varepsilon) - \log(\delta_y)}{\varepsilon} \\ =& \frac{v_{i,x}}{\delta_x} - \frac{v_{i,y}}{\delta_y} > 0. \end{aligned}$$

Therefore, δ is not an equilibrium distribution.

On the one hand, Theorem 2.9 states that there has to exist at least one equilibrium distribution for Cobb-Douglas utilities. On the other hand, Theorem 5.10 and the above argument imply that there can be at most one equilibrium distribution. All in all, there is a unique equilibrium distribution for Cobb-Douglas utilities which has to coincide with the equilibrium distribution for Leontief utilities. □

Remark 9

Via a similar argument, Proposition 7.1 also holds for utility functions of the form $u_i(\delta) = \sum_{x \in A} v_{i,x} g(\delta_x)$ for $v_{i,x} \in \{0, 1\}$ and $g\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ being an increasing and strictly concave function which is, for the sake of simplicity, differentiable on $\mathbb{R}_{>0}$. Essentially, agents are incentivized to move contributions to approved projects with the lowest total contribution, as in the case of binary Leontief utilities.

In the spirit of Proposition 7.1, the convergence results for Leontief utilities (Section 5.2.3) also apply to Cobb-Douglas as well as utility functions from Remark 9 as not only equilibrium distributions but also best responses coincide. In addition, Proposition 7.1 and Theorem 5.20 show that the equilibrium distribution for Cobb-Douglas utilities can be computed efficiently.

The simplicity of these generalizations shows the power of the game-theoretic approach based on the independence of the specific utility functions chosen to represent agents' preferences.

However, the equilibrium distribution can violate efficiency for Cobb-Douglas utilities.

Example 7.2

Assume there are three projects $A = \{a, b, c\}$ and two agents having Cobb-Douglas utilities.

	$v_{i,a}$	$v_{i,b}$	$v_{i,c}$	C_i
Agent 1	1	1	0	6
Agent 2	0	1	1	6

The equilibrium distribution $\delta^* = (4, 4, 4)$ results in utilities $u_1(\delta^*) = u_2(\delta^*) = 4 \cdot 4 = 16$. However, the distribution $\delta = (3, 6, 3)$ provides more Cobb-Douglas utility to both agents as $u_1(\delta) = u_2(\delta) = 3 \cdot 6 = 18$.

In the above example, δ also constitutes the Lindahl equilibrium, which follows from the fact that maximizing Nash welfare for Cobb-Douglas utilities is equivalent to choosing the distribution that is proportional to the sum of valuations for individual projects (Fain et al., 2016), weighted by the individual contributions. Thus, Proposition 6.5 does not hold for Cobb-Douglas utilities.

Given Proposition 7.1, it is possible to leverage results from Rosen (1965) to obtain an alternative proof for the uniqueness of equilibrium distributions in the case of Leontief utilities. In detail, Rosen gives a non-constructive proof for the existence and uniqueness of equilibrium distributions in case utility functions are smooth (particularly continuously differentiable) and strictly concave, as in the case of Cobb-Douglas utilities.

7.2 WELFARE FUNCTIONS MAXIMIZED BY *NASH*

In this section, agents are assumed to have binary Leontief utilities.

Based on the observation that *NASH* coincides with both the Nash product rule (Theorem 5.10) and the mechanism for binary Leontief preferences that returns the leximin-maximal distribution with respect to agents' utilities and subject to implementability (Theorem 5.34), a natural question to ask is which other welfare notions are maximized by *NASH* subject to implementability when agents have Leontief preferences.

For this, we take a closer look at g-*welfare-maximizing* functions (subject to implementability).

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. The g-*welfare* of a distribution δ is defined as the following weighted sum:

$$g\text{-welfare}(\delta) = \sum_{i \in N} C_i \cdot g(u_i(\delta))$$

where the u_i's are assumed to be binary Leontief utility functions.

Quantifying welfare enables us to compare and rank all possible utility vectors, which induces a social welfare ordering over all efficient distributions $\delta \in \Delta(C_N)$ by Proposition 5.9.

Inversely, every continuous social welfare ordering without any "welfare dependencies"[47] between the agents' utilities can be represented by a g-welfare function, see Moulin (1988) for a detailed discussion. Weighting agents by their contributions, we arrive at the very expressive class of g-welfare functions.

A distribution is called g-*welfare-maximizing* if it maximizes the g-welfare, i.e., it always chooses a maximal element of the corresponding social welfare ordering. For concave g,[48] g-welfare is concave and a g-welfare-maximizing distribution can

47 This property is known as *separability (see, e.g., Moulin, 1988).*

48 Equivalently, the induced social welfare ordering satisfies the *Pigou-Dalton principle* (see, e.g., Moulin, 1988).

be found by solving the corresponding constrained convex program. Moreover, strict concavity of g implies that g-welfare is maximized by a unique distribution.

Proposition 7.3
For every strictly concave, strictly increasing function g, there is a unique g-welfare-maximizing distribution.

Proof. Assume for contradiction that there exist two different g-welfare-maximizing distributions δ and δ'. Since both distributions are efficient, by Lemma 5.9 they induce two different utility vectors $(u_i(\delta))_{i\in N}$ and $(u_i(\delta'))_{i\in N}$.

Then, for any $\lambda \in (0, 1)$,

$$\begin{aligned} g\text{-welfare}\left(\lambda\delta + (1-\lambda)\delta'\right) &= \sum_{i\in N} C_i \cdot g\left(u_i\left(\lambda\delta + (1-\lambda)\delta'\right)\right) \\ &\geqslant \sum_{i\in N} C_i \cdot g\left(\lambda \cdot u_i(\delta) + (1-\lambda)\cdot u_i(\delta')\right) \\ &> \lambda \sum_{i\in N} C_i \cdot g(u_i(\delta)) + (1-\lambda) \sum_{i\in N} C_i \cdot g(u_i(\delta')) \\ &\geqslant \min(g\text{-welfare}(\delta'), g\text{-welfare}(\delta)) \end{aligned}$$

where the first inequality follows from the convexity of Leontief preferences (see Proposition 9.9) and the second inequality holds due to strict concavity of g.

This contradicts the assumption that δ and δ' are g-welfare-maximizing. □

Clearly, every g-welfare-maximizing distribution is efficient.

Moreover, uniqueness and efficiency are retained even when maximizing among implementable distributions.

Proposition 7.4
Let g be any strictly increasing function, and let δ be a distribution that maximizes g-welfare among implementable distributions. Then, δ is unique and efficient.

Proof. Uniqueness is proved analogously to Proposition 7.3, noting that the mixture of two implementable distributions is again implementable.

Suppose for contradiction that δ is not efficient. By Lemma 5.6, there exists a project $x \in \text{supp}(\delta)$ which is not critical for any agent. Then, one agent who contributes to x is able to shift a small amount from x uniformly to the set of her approved projects such that x is still not critical for any agent. The resulting distribution is still implementable, and Pareto dominates δ as all agents are better off. This contradicts the maximality of δ in g-welfare. □

Note that uniqueness holds only within the set of implementable distributions. There might exist non-implementable distributions with the same g-welfare, as shown by the following example.

Example 7.5
Assume there are two projects $A = \{a, b\}$ and two agents.

	$v_{i,a}$	$v_{i,b}$	C_i
Agent 1	1	0	2
Agent 2	0	1	1

Let $g(z) = -1/z$. Then, $\delta = (2, 1)$ maximizes g-welfare among implementable distributions as δ is the unique implementable distribution. However, it yields the same g-welfare $(-2 \cdot 1/2 - 1 \cdot 1/1 = -2)$ as the non-implementable distribution $\delta' = (3/2, 3/2)$ with g-welfare$(\delta') = -2 \cdot 2/3 - 1 \cdot 2/3 = -2$.

The Nash product rule is often considered a compromise between maximizing utilitarian welfare ($\sum_{i \in N} C_i \cdot u_i$) and egalitarian welfare (maximizing the utility of the agent with the smallest utility). To illustrate that, consider the family of g-welfare functions $\sum_{i \in N} C_i \cdot \operatorname{sgn}(p) \cdot u^p$ for $p \neq 0$ where the limit $p \to 0$ corresponds to $\sum_{i \in N} C_i \cdot \log(u_i)$ and $p \to -\infty$ approaches egalitarian welfare.

Theorem 5.34 already hints at a deeper connection between such welfare functions when considering Leontief utilities.

THEOREM 7.6
Let $g : \mathbb{R}_{\geqslant 0} \to \mathbb{R} \cup \{-\infty\}$ be a function that satisfies the following conditions:

(i) g is strictly increasing on $\mathbb{R}_{\geqslant 0}$ and differentiable on $\mathbb{R}_{>0}$, and

(ii) $xg'(x)$ is non-increasing on $\mathbb{R}_{>0}$.

Then, the equilibrium distribution maximizes g-welfare among all implementable distributions.

Property (i) ensures that social welfare increases when an individual's utility increases, and small changes in individual utilities only cause small changes in the total social welfare. Property (ii) implies that increasing utilities are discounted "at least logarithmically" when being translated to welfare.

The proof requires some additional definitions and lemmas and has the following structure.

First, we show that it is sufficient to prove the statement for *reduced* profiles (Definition 7.11 and Lemma 7.12), which are profiles in which each agent approves only projects that receive the same amount under the equilibrium distribution. Second, we prove that, in any reduced profile, the equilibrium distribution δ^* maximizes g-welfare, not only in the set of implementable distributions but even in a larger set of *weakly implementable* distributions (Definition 7.9). To do this, we show that, for any weakly implementable distribution $\delta \neq \delta^*$, there exists a modification δ', which is weakly implementable but has a higher g-welfare than δ.

Define $[p] := \{1, 2, \ldots, p\}$ for each positive integer p.

DEFINITION 7.7
Given any distribution δ, define $\mathcal{P}(\delta)$ as a partition of the projects into subsets allocated the same amount. That is, $\mathcal{P}(\delta) := (X_1, \ldots, X_p)$ for some integer $p \geqslant 1$, where $\bigcup_{k=1}^{p} X_k = A$, and for each $k \in [p]$, all projects in X_k receive the same amount, $\delta_x = w_k$ for all $x \in X_k$, and the amounts are ordered such that $0 \leqslant w_1 < \cdots < w_k$.

Note that $w_1 = 0$ if and only if there exist projects that receive no funding.

LEMMA 7.8
Let δ^* be the equilibrium distribution, and $(X_1^*, \dots, X_p^*) = \mathcal{P}(\delta^*)$ be the corresponding partition of the projects. For each $k \geqslant 1$, let N_k^* be the set of agents who approve one or more projects from X_k^* but do not approve any project from $\bigcup_{\ell<k} X_\ell^*$. Then, in the equilibrium distribution, agents from N_k^* contribute only to projects from X_k^*, i.e.,

$$\delta^*(X_k^*) = C_{N_k^*} \text{ and } w_k^* = \frac{C_{N_k^*}}{|X_k^*|} = \frac{\delta^*(X_k^*)}{|X_k^*|}.$$

Proof. The utility of all agents in N_k^* is w_k^*, so the set of their critical projects is contained in X_k^*. In the equilibrium distribution, they contribute only to projects in X_k^* by Proposition 5.2.

All projects in X_k^* receive the same amount, so this amount has to be $C_{N_k^*}/|X_k^*|$. □

Note that all projects that are not approved by any agent (or approved only by agents who contribute 0) are in X_1^* and $w_1^* = C_{N_1^*} = 0$.

DEFINITION 7.9
A distribution δ is called *weakly implementable* if it has a decomposition in which each agent i only contributes to projects x with $\delta_x^* \geqslant u_i(\delta^*)$, where δ^* denotes the equilibrium distribution.

With binary weights, $x \in A_i$ implies $\delta_x^* \geqslant u_i(\delta^*)$, so every implementable distribution is also weakly implementable. Therefore, it is sufficient to prove that δ^* maximizes g-welfare among all weakly implementable distributions.

The set of weakly implementable distributions is still convex and admits the following characterization.

LEMMA 7.10
A distribution δ is weakly implementable if and only if, for every $\ell \in [p]$,

$$\delta\left(\bigcup_{k=\ell}^{p} X_k^*\right) \geqslant \delta^*\left(\bigcup_{k=\ell}^{p} X_k^*\right). \tag{24}$$

Proof. A distribution δ is weakly implementable if and only if there exists a decomposition of δ where for every $\ell \in [p]$, agents of N_ℓ^* only contribute to projects from $\bigcup_{k=\ell}^{p} X_k^*$. This holds if and only if $\delta\left(\bigcup_{k=\ell}^{p} X_k^*\right) \geqslant \sum_{k=\ell}^{p} C_{N_\ell^*}$ for every $\ell \in [p]$. By Lemma 7.8, this is equivalent to the condition $\delta\left(\bigcup_{k=\ell}^{p} X_k^*\right) \geqslant \delta^*\left(\bigcup_{k=\ell}^{p} X_k^*\right)$ for every $\ell \in [p]$. □

To simplify the proof of Theorem 7.6, we introduce the following class of profiles.

DEFINITION 7.11
A profile is called *reduced* if, in its equilibrium distribution δ^*, for every agent i, there exists a $k \in [p]$ such that $A_i \subseteq X_k^*$, that is, all projects approved by an agent belong to the same class in the partition induced by δ^*.

Note that in a reduced profile, all projects approved by agent i receive the same amount $u_i(\delta^*)$ in the equilibrium distribution, and therefore, are all critical for agent i, i.e., $T_{\delta^*,i} = A_i$ for all $i \in N$.

LEMMA 7.12
If Theorem 7.6 holds for reduced profiles, then it holds for all profiles.

Proof. Let P be any profile, and δ^* its equilibrium distribution. Let P' be its reduced profile where, compared to P, every agent i has removed her approval from every project x with $\delta^*_x > u_i(\delta^*)$. Then, δ^* is the equilibrium distribution for P' (by the same decomposition). By assumption, Theorem 7.6 is true for P', so δ^* maximizes g-welfare among all distributions that are weakly implementable with respect to P'. Since the equilibrium distribution is the same in P and P', the set of weakly implementable distributions is the same, too.

The profile P differs from P' by having additional approvals, which could only decrease the maximal possible g-welfare. But δ^* yields the same welfare in P and P'. Therefore, δ^* necessarily maximizes g-welfare among all distributions that are weakly implementable with respect to P, too. □

Proof of Theorem 7.6. Based on Lemma 7.12, we assume without loss of generality that we are given a reduced profile. Let $X^*_1, \dots, X^*_p$, and $N^*_1, \dots, N^*_p$ be the partition of projects and agents induced by the equilibrium distribution δ^*, and $w^*_1 < \dots < w^*_p$ the corresponding allocations. By Lemma 7.8, each project in X^*_k receives $w^*_k = \delta^*(X^*_k)/|X^*_k|$, and every agent $i \in N^*_k$ has utility w^*_k. Since the profile is reduced, $T_{\delta^*,i} = A_i \subseteq X^*_k$ for all $i \in N^*_k$.

Let δ be any weakly implementable distribution different than δ^*. We prove that δ does not maximize g-welfare among weakly implementable distributions by deriving a modification δ' of δ, which is weakly implementable but has a higher g-welfare than δ.

Since $\delta \neq \delta^*$ and both distributions sum up to C_N, there must be projects $x^-, x^+ \in A$ with $\delta_{x^-} < \delta^*_{x^-}$ and $\delta_{x^+} > \delta^*_{x^+}$, respectively. Consequently, one of the following two cases has to apply:

CASE 1
If $\delta(X^*_k) = \delta^*(X^*_k)$ for all $k \in [p]$, let $X^*_r = X^*_s$ $(r = s)$ be a class that contains a project x^- with $\delta_{x^-} < \delta^*_{x^-}$ (and thus, also a project x^+ with $\delta_{x^+} > \delta^*_{x^+}$).

CASE 2
Otherwise, let r be the largest index in $[p]$ for which $\delta(X^*_r) \neq \delta^*(X^*_r)$. Weak implementability of δ and Lemma 7.10 imply that $\delta(X^*_r) > \delta^*(X^*_r)$. As $\delta(X^*_k) = \delta^*(X^*_k)$ for all $k > r$, there must be an $s \leqslant r$ such that there exists a project x^- in X^*_s with $\delta_{x^-} < \delta^*_{x^-}$. Choose $s \leqslant r$ to be the largest index with this property.

In both cases, we define $X^- \subseteq X^*_s$ as the set of all projects x in X^*_s with $\delta_x < \delta^*_x$, and $X^+ \subseteq X^*_r$ as the set of all projects x in X^*_r with $\delta_x > \delta^*_x$. Both sets must be non-empty by construction. The case $r > s$ is depicted in Figure 7.1.

Starting from δ, transfer a sufficiently small amount ε uniformly from X^+ to X^- and call the resulting distribution δ'. We choose ε small enough such that it does not change the order relations between projects inside and outside X^+ and X^-, i.e., for all $x^- \in X^-$ and $x^+ \in X^+$, $\delta'_{x^+} > \delta'_x$ for all $x \in A$ with $\delta_{x^+} > \delta_x$,

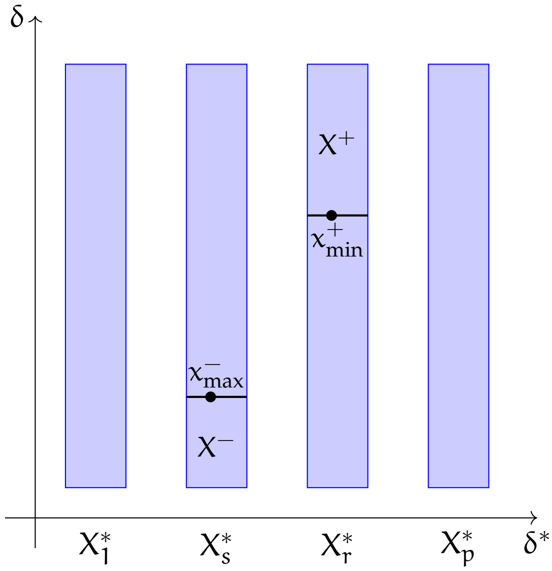


Figure 7.1: Project sets in the proof of Theorem 7.6, for the case $r > s$. The horizontal position of a project denotes its allocation in δ^*, and the vertical position denotes its allocation in δ.

and analogously, $\delta'_{x^-} < \delta'_x$ for all $x \in A$ with $\delta_{x^-} < \delta_x$. In particular, since $\delta_{x^+} > \delta^*_{x^+} \geqslant \delta^*_{x^-} > \delta_{x^-}$, we have $\delta'_{x^+} > \delta'_{x^-}$.

We claim that δ' is weakly implementable. By Lemma 7.10, it suffices to show that (24) holds for δ', that is, $\delta'\left(\bigcup_{k=\ell}^{p} X^*_k\right) \geqslant \delta^*\left(\bigcup_{k=\ell}^{p} X^*_k\right)$ for every $\ell \in [p]$. Note that $\delta'\left(\cup_{k=\ell}^{p} X^*_k\right) = \delta\left(\cup_{k=\ell}^{p} X^*_k\right)$ for all $\ell \leqslant s$ and all $\ell \geqslant r+1$, so for these indices, (24) for δ' follows from weak implementability of δ. Therefore, it remains to prove (24) for $\ell \in \{s+1, \dots, r\}$. This set is non-empty only when $s < r$, which happens only in the second case above.

Our choices of r and s ensure that $\delta\left(\cup_{k=r}^{p} X^*_k\right) > \delta^*\left(\cup_{k=r}^{p} X^*_k\right)$ and $\delta\left(\cup_{k=1}^{s} X^*_k\right) < \delta^*\left(\cup_{k=1}^{s} X^*_k\right)$. For ε sufficiently small, the same inequalities hold between δ' and δ^*. Moreover, for $s < \ell \leqslant r$,

$$\begin{aligned}
\delta'\left(\cup_{k=\ell}^{p} X^*_k\right) &= \delta'\left(\cup_{k=r}^{p} X^*_k\right) + \delta'\left(\cup_{k=\ell}^{r-1} X^*_k\right) \\
&> \delta^*\left(\cup_{k=r}^{p} X^*_k\right) + \delta\left(\cup_{k=\ell}^{r-1} X^*_k\right) \\
&\geqslant \delta^*\left(\cup_{k=\ell}^{p} X^*_k\right)
\end{aligned}$$

where the first inequality holds since $\delta'\left(\cup_{k=r}^{p} X^*_k\right) > \delta^*\left(\cup_{k=r}^{p} X^*_k\right)$ and $\delta'(X^*_k) = \delta(X^*_k)$ for all $k \notin \{r, s\}$, and the second inequality holds as, for each $k \in \{s+1, \dots, r-1\}$, all projects x in X^*_k satisfy $\delta_x \geqslant \delta^*_x$ by definition of s. Therefore, by Lemma 7.10, δ' is still weakly implementable.

We now analyze the effect of this redistribution on the agents' utilities. For that, we prove an auxiliary claim on critical projects of agents under δ. Define $x^+_{\min} \in \arg\min_{x^+ \in X^+} \delta_{x^+}$ as a project from X^+ with minimal allocation in δ and $x^-_{\max} \in \arg\max_{x^- \in X^-} \delta_{x^-}$ as a project from X^- with maximal contribution in δ (see Figure 7.1).

CLAIM. For every agent $i \in N$, either $T_{\delta,i} \cap X^- = \emptyset$ or $T_{\delta,i} \subseteq X^-$. Similarly, either $T_{\delta,i} \cap X^+ = \emptyset$ or $T_{\delta,i} \subseteq X^+$.

PROOF OF CLAIM. We prove the claim for X^-. The proof for X^+ is analogous. By definition of critical projects, $T_{\delta,i} \subseteq A_i$. Since the profile is reduced, A_i is contained in a single partition class. If this partition class is not the one that contains X^-, namely X_s^*, then $T_{\delta,i} \cap X^- = \emptyset$. Otherwise, $T_{\delta,i} \subseteq X_s^*$. Now, if $u_i(\delta) > \delta_{x^-_{\max}}$, then $\delta_x > \delta_{x^-_{\max}}$ for every $x \in T_{\delta,i}$, so $T_{\delta,i} \cap X^- = \emptyset$. If $u_i(\delta) \leqslant \delta_{x^-_{\max}}$, then $\delta_x \leqslant \delta_{x^-_{\max}}$ for every $x \in T_{\delta,i}$, so $T_{\delta,i} \subseteq X^-$.

BACK TO PROOF OF THEOREM. Denote by "losers" the agents who lose utility from the redistribution. The claim implies that all losers have $T_{\delta,i} \subseteq X^+$. Consequently, each of them loses $\varepsilon/|X^+|$. Moreover, all losers have $A_i \subseteq X^+$ due to the fact that $A_i \subseteq X_r^*$ (since the profile is reduced), and $\delta_{x_A} \geqslant \delta_{x_T} \geqslant \delta_{x^+_{\min}}$ for all $x_A \in A_i$ and $x_T \in T_{\delta,i}$. Therefore, in the equilibrium distribution, all losers give all their contributions to projects in X^+. This implies that the contributions of all losers sum up to at most $\delta^*(X^+) = w_r^* \cdot |X^+|$.

Then, for every loser i,

$$g(u_i(\delta)) - g(u_i(\delta')) \leqslant g\left(\delta_{x^+_{\min}}\right) - g\left(\delta_{x^+_{\min}} - \frac{\varepsilon}{|X^+|}\right) \tag{25}$$

by concavity of g, which follows from the assumption that $xg'(x)$ is non-increasing.

Denote by "gainers" the agents who gain utility from the redistribution. The claim implies that every agent with $T_{\delta,i} \cap X^- \neq \emptyset$ is a gainer as each of them gains $\varepsilon/|X^-|$. Moreover, every agent with $A_i \cap X^- \neq \emptyset$ is a gainer due to the fact that $A_i \cap X^- \neq \emptyset$ implies $\delta_{x_A} \leqslant \delta_{x^-_{\max}}$ for at least one project $x_A \in A_i$, and $\delta_{x_T} \leqslant \delta_{x_A}$ for all projects $x_T \in T_{\delta,i}$. Therefore, in the equilibrium distribution, every agent who contributes a positive amount to at least one project in X^- must be a gainer. Hence, the contributions of all gainers have to sum up to at least $\delta^*(X^-) = w_s^* \cdot |X^-|$.

Then, for every gainer i,

$$g(u_i(\delta')) - g(u_i(\delta)) \geqslant g\left(\delta_{x^-_{\max}} + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta_{x^-_{\max}}\right) \tag{26}$$

by concavity of g.

Therefore, by (25) and (26), the increase in g-welfare from δ to δ' is at least

$$\begin{aligned} & w_s^* \cdot |X^-| \cdot \left[g\left(\delta_{x^-_{\max}} + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta_{x^-_{\max}}\right)\right] \\ & - w_r^* \cdot |X^+| \cdot \left[g\left(\delta x^+_{\min}\right) - g\left(\delta_{x^+_{\min}} - \frac{\varepsilon}{|X^+|}\right)\right]. \end{aligned} \tag{27}$$

Since g is strictly concave,

$$g\left(\delta_{x^-_{\max}} + \frac{\varepsilon}{|X^-|}\right) - g\left(\delta_{x^-_{\max}}\right) > \frac{\varepsilon}{|X^-|} \cdot g'\left(\delta_{x^-_{\max}} + \frac{\varepsilon}{|X^-|}\right) \qquad \text{and}$$

$$g\left(\delta_{x^+_{\min}}\right) - g\left(\delta_{x^+_{\min}} - \frac{\varepsilon}{|X^+|}\right) < \frac{\varepsilon}{|X^+|} \cdot g'\left(\delta_{x^+_{\min}} - \frac{\varepsilon}{|X^+|}\right).$$

Plugging this into (27), we get that the increase in g-welfare is larger than

$$w_s^* \cdot |X^-| \cdot \frac{\varepsilon}{|X^-|} \cdot g'\left(\delta_{x_{\max}^-} + \frac{\varepsilon}{|X^-|}\right) - w_r^* \cdot |X^+| \cdot \frac{\varepsilon}{|X^+|} \cdot g'\left(\delta_{x_{\min}^+} - \frac{\varepsilon}{|X^+|}\right).$$

By our choice of ε, we have $w_r^* = \delta^*_{x_{\min}^+} < \delta_{x_{\min}^+}$, so $w_r^* < \delta_{x_{\min}^+} - \varepsilon/|X^+|$ for sufficiently small ε. Similarly, $w_s^* = \delta^*_{x_{\max}^-} > \delta_{x_{\max}^-} + \varepsilon/|X^-|$ for sufficiently small ε.

Therefore, the increase in g-welfare is larger than

$$\begin{aligned} &\varepsilon \cdot \left(\delta_{x_{\max}^-} + \frac{\varepsilon}{|X^-|}\right) \cdot g'\left(\delta_{x_{\max}^-} + \frac{\varepsilon}{|X^-|}\right) \\ -&\varepsilon \cdot \left(\delta_{x_{\min}^+} - \frac{\varepsilon}{|X^+|}\right) \cdot g'\left(\delta_{x_{\min}^+} - \frac{\varepsilon}{|X^+|}\right). \end{aligned} \tag{28}$$

By our choice of ε, $\delta_{x_{\max}^-} + \varepsilon/|X^-| < \delta_{x_{\min}^+} - \varepsilon/|X^+|$. By the assumption on g, $xg'(x)$ is non-increasing in x. Therefore, the expression in (28) is at least 0, so the increase in g-welfare from δ to δ' is larger than 0. This means that δ does not maximize g-welfare.

Since δ was any weakly implementable distribution different than δ^*, we conclude that δ^* maximizes g-welfare subject to weak implementability in any reduced profile. By Lemma 7.12, the same is true in any profile. □

Theorem 7.6 holds for all functions of the form $\sum_{i \in N} C_i \cdot \mathrm{sgn}(p) \cdot u^p$ with $p < 0$. However, it does not generalize for $p > 0$. In particular, the statement holds for a smaller set of g's than related results by Bogomolnaia and Moulin (2004) would suggest (see also Chapter 8).

Proposition 7.13
For each $p > 0$, maximizing the g-welfare with respect to $g(u) = u^p$ subject to decomposability does not always return the equilibrium distribution.

Proof. For a fixed $p > 0$, consider a profile consisting of two agents with binary Leontief utilities and approval sets $A_1 = \{a\}$ and $A_2 = \{a, b\}$, and respective contributions $C_1 = \max\left((2^{p-1} \cdot p)^{-1/p}, 2\right)$ and $C_2 = 1$. Since $C_1 \geqslant 2$, the equilibrium distribution is $(C_1, 1)$. We claim that the implementable distribution $(C_1 + 1, 0)$ yields higher g-welfare, that is,

$$\begin{aligned} &C_1 \cdot g(C_1 + 1) + 1 \cdot g(0) > C_1 \cdot g(C_1) + 1 \cdot g(1) \\ \iff &C_1 \cdot (g(C_1 + 1) - g(C_1)) > 1. \end{aligned}$$

For every $p \geqslant 1$, g is convex, so

$$\begin{aligned} &g(C_1 + 1) - g(C_1) \geqslant g'(C_1) \cdot 1 = p \cdot C_1^{p-1} \\ \implies &C_1 \cdot (g(C_1 + 1) - g(C_1)) \geqslant p \cdot C_1^p \geqslant p \cdot 2^p \geqslant 2 > 1. \end{aligned}$$

For every $0 < p < 1$, g is strictly concave, so

$$\begin{aligned} g(C_1 + 1) - g(C_1) > g'(C_1 + 1) \cdot 1 &= p \cdot (C_1 + 1)^{p-1} \\ &> p \cdot (2C_1)^{p-1} \end{aligned}$$

since $p - 1 < 0$ and $C_1 > 1$. Thus,

$$C_1 \cdot (g(C_1 + 1) - g(C_1)) > 2^{p-1} \cdot p \cdot C_1^p$$
$$\geqslant 2^{p-1} \cdot p \cdot \left(\left(\frac{1}{2^{p-1} p} \right)^{\frac{1}{p}} \right)^p = 1.$$

In both cases, the equilibrium distribution does not maximize g-welfare. □

Theorem 7.6 stresses the fact that *NASH* can be motivated not only from a game-theoretic and axiomatic point of view but also from a welfarist perspective.

7.3 LEXIMIN LEONTIEF UTILITIES

Leontief utilities take only the minimum of $\delta_x / v_{i,x}$ for $x \in A$ into account. By breaking ties between distributions lexicographically, indifference classes can be resolved, leading to a refinement of Leontief preferences. More precisely, each agent i ranks all distributions according to the leximin relation (Definition 5.31) among the vectors $(\delta_x / v_{i,x})_{x \in A}$.

Remarkably, all of our results for Leontief preferences carry over to these refined leximin Leontief preferences by carefully adapting our proofs (see [2] and [3]). For instance, a manipulation of an agent could potentially change the outcome in a way such that the agent is indifferent between both distributions under Leontief preferences but strictly prefers the latter outcome under leximin Leontief preferences. This example shows that strategyproofness for Leontief preferences does not automatically imply strategyproofness for refined preferences.

However, leximin Leontief preferences are discontinuous[49] which is why they do not fall within our framework of continuous and convex preferences. Still, they can be seen as a constant reminder that even basic assumptions on agents' preferences, like continuity, could and should be challenged.

49 E.g., consider an instance with three projects and an agent with leximin Leontief preferences and valuations $(1/2, 1/3, 1/6)$. This agent prefers $\delta^\varepsilon = (1/4 + \varepsilon, 7/12 - \varepsilon, 1/6)$ to $(1/4, 1/2, 1/4)$ for every $\varepsilon \in (0, 5/12)$ but prefers $(1/4, 1/2, 1/4)$ to $\lim_{\varepsilon \to 0} \delta^\varepsilon = (1/4, 7/12, 1/6)$.

8 DISCUSSION

This chapter summarizes our results on donor coordination, draws interesting connections to other works, which seem to be only loosely related at first sight, and points towards some promising directions for future research.

LINEAR AND LEONTIEF UTILITIES

We start with a comparison of our results for linear and Leontief utilities. As already argued in Example 3.1, we take the view that both utility models represent reasonable preferences, depending on the type of considered projects. We observed that efficiency, strategyproofness, and fairness in the form of positive share are incompatible for linear and even dichotomous utilities. Furthermore, equilibrium distributions might lead to inefficient outcomes, showing the strong need for coordination. Leontief utilities yield a unique equilibrium distribution with remarkable properties, and the Nash product rule is characterized by continuity, group-strategyproofness, and the core (Theorem 5.36). It is unclear whether group-strategyproofness can be weakened in that characterization.

Open Problem 1
Is *NASH* already characterized by continuity, strategyproofness, and the core for Leontief utilities?

For linear utilities, a characterization of *NASH* based on contribution-incentive compatibility seems worthwhile.

Open Problem 2
Characterize *NASH* with the help of contribution-incentive compatibility for linear utilities.

Strengthening that axiom might simplify the previous problem.

Open Problem 3
Does *NASH* satisfy a stronger version of contribution-incentive compatibility (like the one from Section 4.2.1) for linear utilities?

Table 8.1 gives an overview of axioms satisfied by the Nash product rule for both utility models. For completeness, we also included *CUT* as it constitutes a valid alternative rule in case of dichotomous preferences. In the following, we explain special entries of that table as well as results that were not proven in the preceding chapters. As shown by Theorem 4.8 and Example 2.11, *NASH* returns equilibrium distributions for dichotomous preferences but not for general linear utilities. *CUT* chooses only outcomes that are efficient among equilibrium distributions, which is why that rule violates the core property. Contribution-incentive compatibility does not seem to be a sensible axiom for Leontief utilities as mentioned

	NASH (linear)	*NASH* (Leontief)	*CUT* (dich.)
Equilibrium	(dich.)	✓	✓
Efficiency	✓	✓	(eq.)
Strategyproofness	✗	✓	✓
Core	✓	✓	✗
Contribution-incentive comp.	✓	n.a.	✓
Efficient computability	(✓)	✓	✓
U-continuity	✓	✓	n.a.
C-continuity	✓	✓	✗
Preference-monotonicity	✗	✓	✓
Contribution-monotonicity	✗	✓	✗

Table 8.1: Axiomatic properties of *NASH* for linear and Leontief utilities, and *CUT* for dichotomous utilities.

in Chapter 6.[50] *NASH* with linear utilities might return irrational distributions (see Example 4.18), but convex programming allows us to at least approximate Nash welfare maximizing distributions arbitrarily well. Furthermore, note that U-continuity is defined for valuations from a continuous set and therefore does not make sense for binary valuations.[51] Finally, *CUT* violates C-continuity as well as contribution-monotonicity since arbitrarily small changes in contribution can change the sets A_i^+ resulting in an entirely different distribution but satisfies preference-monotonicity by Proposition 4.17.

DYNAMICS

Next, we turn to the dynamics converging to Nash welfare maximizing distributions for both utility models (Sections 4.2.2 and 5.2.3). For linear utilities, we have already established a connection to portfolio theory. Interestingly, these dynamics also admit an interpretation as a *replicator equation* (Taylor and Jonker, 1978) used in *evolutionary game theory* (see, e.g., Maynard Smith, 1982; Hofbauer and Sigmund, 1998, for an overview). In detail, rewriting (11) for $t \geqslant 0$ and $x \in A$ yields

$$\begin{aligned}\delta_x^{t+1} - \delta_x^t &= \delta_x^t \left(\sum_{i \in N} C_i \frac{v_{i,x}}{u_i(\delta)} - 1 \right) \\ &= \delta_x^t \left(\sum_{i \in N} C_i \frac{v_{i,x}}{u_i(\delta)} - \sum_{y \in A} \delta_y \left(\sum_{i \in N} C_i \frac{v_{i,y}}{u_i(\delta)} \right) \right)\end{aligned}$$

for all $\delta^t \in \Delta(C_N)$ such that $u_i(\delta^t) > 0$ for all $i \in N$ with $C_i > 0$.

50 Consider one agent that values two projects at 1. With Leontief utilities, she has to split her contribution C_1 equally among those projects and only gains $C_1/2$. Even adapted versions of that axiom immediately clash with efficiency.

51 On a technical level, *CUT* satisfies U-continuity for dichotomous utilities as Definition 3.2 is satisfied for any $\gamma \in (0, 1)$.

Interpreted as population dynamics, the pure alternatives $x \in A$ correspond to different types of a population. The fitness of a type x for a specific population δ is given by $\sum_{i \in N} C_i(v_{i,x}/u_i(\delta))$. Since the average fitness is always equal to 1, the proportion of types with fitness larger than 1 increases, whereas the proportion of types with fitness less than 1 decreases in the population. Under an equilibrium distribution δ^*, all types with $\delta^*_x > 0$ have the same fitness (namely 1), which is also displayed by the KKT-conditions of maximizing Nash welfare (see the proof of Theorem 4.8) where $\sum_{i \in N} C_i(v_{i,x}/u_i(\delta^*)) = 1$ and all $x \notin \text{supp}(\delta)$ do not have a fitness higher than 1. We are excited about possibilities to apply tools from evolutionary game theory to our dynamics.

Open Problem 4
Investigate the Nash dynamics for linear utilities regarding the existence of a limit and convergence rates.

Rosen (1965) has already considered a continuous and gradient-based version of the best response dynamics we investigated for Leontief utilities and related utility functions (see Sections 5.2.3 and 7.1) where agents update their strategies *simultaneously*. He showed that these systems converge to the unique equilibrium distribution under quite general assumptions, one being that utility functions are strictly concave. In contrast to Rosen (1965), our dynamics are discrete, and agents update their individual distributions one after another. For our best response dynamics, we proved convergence to the equilibrium distribution (Theorem 5.24) under a weak assumption on the sequences of agents updating their individual distributions. Observing that this additional assumption is neither required for binary Leontief utilities nor for functions from Remark 9, we assume it can be dropped.

Open Problem 5
Does Theorem 5.24 hold for *any* sequence where each agent appears infinitely often?

PRIVATE GOOD MARKETS

As already mentioned in Chapter 3, Nash welfare maximization is related to finding (market) equilibria not only for public good markets (Chapter 6) but also various private good markets (see, e.g., Jain and Vazirani, 2010). In these *pure exchange economies*,[52] a finite number of resources needs to be divided among a set of agents with individual budgets. The goal is to price the private goods in a way such that there exists an allocation of all resources to the agents, and each agent can buy her assigned bundle but cannot afford any other strictly preferred bundle. Then, prices and the corresponding allocation form a *competitive* or *Walrasian equilibrium*, and existence is guaranteed for the case of divisible goods (see, e.g., Varian, 1992) in these so-called *Fisher markets*.

Eisenberg (1961) proved that competitive equilibria for Fisher markets can be found by maximizing the Nash product when utility functions are continuous,

52 The set of goods is fixed and aspects regarding their production are not considered.

concave, homogeneous, and non-constant[53] (see also Codenotti and Varadarajan, 2007).

This is reminiscent of the connection to Lindahl equilibrium established by Fain et al. (2016) (see Theorem 6.3). However, there are some fundamental differences between equilibrium distributions and the Nash product rule for public good markets. First, maximizing Nash welfare does not always result in an equilibrium distribution (see, e.g., Example 2.11). Second, equilibrium distributions admit different structures even for Leontief utilities. Lemma 5.19 shows that the equilibrium distribution is always rational-valued in our setting, whereas this does not hold for Fisher markets (Codenotti and Varadarajan, 2004). For linear utilities, there always exists a rational-valued Nash welfare maximizing distribution for Fisher markets but not necessarily for our public good markets (see Example 4.18). Finally, Ghodsi et al. (2011) showed that the Nash product rule violates strategyproofness for Leontief utilities in Fisher markets.

Nevertheless, some ideas from private good markets are also very useful for public good markets, like the concept of competitive equilibrium leading to the notion of Lindahl equilibrium or Jain's approach to computing an equilibrium for Fisher markets with linear utilities that can be adapted to prove efficient computability of *NASH* for Leontief utilities (Theorem 5.20).

BINARY LEONTIEF UTILITIES

For the case of binary Leontief utilities, Theorem 7.6 indicates that restricting the set of possible distributions, e.g., to implementable distributions in order to ensure some fairness, suffices to recover the equilibrium distribution as the welfare-maximizing outcome for various notions.

Assuming that agents exclusively contribute to approved projects, this can be interpreted as a (many-to-many) matching problem on a bipartite graph where agents (and their contributions) need to be assigned to projects with unlimited capacity. Theorem 5.32 shows that the equilibrium distribution maximizes egalitarian welfare of the projects, assuming they have dichotomous utilities (Section 4.3). Thus, the equilibrium distribution coincides with a solution for such matching problems proposed by Bogomolnaia and Moulin (2004), who also showed that their solution constitutes a competitive equilibrium from equal incomes from the project managers' point of view.

Property (ii) from Theorem 7.6 requires the function g to be "at least as concave" as the log function. Bogomolnaia et al. (2002) and Aziz et al. (2020) noted that for dichotomous utilities, g-welfare-maximizing satisfies individual fair share for exactly these g's. Proposition 7.13 shows that at least when restricting ourselves to functions of the form $g(u) = \text{sgn}(p) \cdot u^p$ with $p \in \mathbb{R}$, Property (ii) can be replaced by the axiom of positive share.

Open Problem 6
Which fairness properties are satisfied by *NASH* or other mechanisms independent of the underlying utility model?

53 These functions include linear, Leontief, and Cobb-Douglas utilities.

All in all, Theorem 7.6 illustrates various tools for achieving fairness. The probably most straightforward approach reduces the set of admissible outcomes via some fairness axioms like implementability. Furthermore, the mechanism itself is usually (and sometimes implicitly) constructed based on some fairness axioms (e.g., Property (ii)). Finally, restricting expressible preferences (e.g., by assuming binary valuations in specific utility models) helps to concentrate on the main features of preferences for certain problems. Combining these ideas often results in mechanisms with very desirable properties like *NASH*.

Part II

PORTIONING

9 INTRODUCTION TO PORTIONING

Of course, funding public projects is not restricted to donor coordination. On the contrary, many real-world scenarios concern the allocation of some fixed and externally provided budget, e.g., when a city wants to let its citizens decide on how to distribute money among a set of city projects. This form of direct democracy has received a lot of attention under the name *participatory budgeting* (Cabannes, 2004) in recent years (see, e.g., Aziz and Shah, 2021, for an overview).

In practice, participatory budgeting starts with proposing and preselecting feasible projects (Rey et al., 2021). Like most of the social choice literature, we focus on the second stage, where the budget needs to be distributed based on the agents' preferences over a fixed set of projects.

Aziz and Shah (2021) define a taxonomy of participatory budgeting problems based on the possible degrees of completion of projects. The perhaps most researched model assumes projects to have costs and thus be either fully implemented or not funded at all. Normally, each agent reports her approved projects, and the most common mechanism, sometimes called *greedy*, first orders projects by the number of approvals. Given that order, it funds the project with the highest number of approvals, then the one with the second-highest number, and so on, while skipping projects with higher costs than the remaining budget. When thinking about city projects, this mechanism disadvantages suburban projects as they naturally accumulate fewer approvals than projects in areas with a higher population density. Peters and Skowron (2020) and Peters et al. (2021) recently introduced the *method of equal shares*, which provides better fairness for smaller projects and agents that approve those.

In portioning, projects can receive and benefit from any amount of money, which is more realistic, e.g., when projects correspond to whole areas of public interest like education, safety, and public transport. Therefore, the set of possible distributions coincides with the probability simplex.[54]

Example 9.1
Assume there are three city projects $A = \{a, b, c\}$ and three citizens where $p_{i,x}$ denotes the value of project x at agent i's favorite distribution.

	$p_{i,a}$	$p_{i,b}$	$p_{i,c}$
Agent 1	1/3	1/3	1/3
Agent 2	1/2	1/3	1/6
Agent 3	0	1/2	1/2

For linear utilities, no efficient distribution assigns money to project c as it is dominated by project b. However, the favorite distributions of all agents indicate

54 W.l.o.g, we normalize the total budget to 1.

that they want c to receive at least some funding. Thus, linear utilities do not seem sensible when agents' favorite distributions are not degenerate.

The predominant model assumes ℓ_1 preferences (Definition 9.5) where an agent prefers distribution δ over δ' if the ℓ_1 distance from δ' to her favorite distribution is at least as large as the one from δ. For example, Agent 3 strictly prefers $(1/4, 1/2, 1/4)$ to $(0, 1, 0)$ but is indifferent between $(1/2, 1/4, 1/4)$ and $(1/2, 1/2, 0)$.

Leontief utilities can also be investigated by scaling the sum of valuations to 1.

The challenges remain the same as in donor coordination: We want to define a framework where agents can efficiently report their preferences and find mechanisms that aggregate them in an efficient, fair, and maybe even strategyproof way.

9.1 RELATED WORK

STRATEGYPROOFNESS FOR CARTESIAN PRODUCT DOMAINS. Inspired by Moulin (1980), many works (Border and Jordan, 1983; Barberà et al., 1993, 1997) investigated strategyproofness for problems where the set of alternatives is formed by some Cartesian product, e.g., $[0, 1]^m$ or $\mathbb{R}^m$. Their results state that all strategyproof rules decompose into median rules for each coordinate, entailing incompatibilities when considered in conjunction with efficiency and anonymity (Peters et al., 1992).[55] These problems are fundamentally different from portioning, at least with respect to strategyproofness, due to the normalization constraint, which prohibits a separate treatment of each coordinate.

PEAK-BASED PREFERENCES. In order to enable agents to report their preferences over an infinite set of alternatives, certain assumptions about the structure of preferences need to be made. Standard models include linear utilities (see Chapter 4) or extending rankings over the set of pure alternatives to preferences over all lotteries (Brandt, 2017).[56] These models have in common that every agent has a pure alternative as her favorite outcome, which seems rather restrictive (Intriligator, 1973). We follow Intriligator's "refreshing change from typical approaches" (Fishburn, 1975, p. 297) and also consider models where agents' favorite alternatives are not necessarily pure. In fact, most of the above-mentioned literature on Cartesian product domains as well as Leontief utilities (see Chapter 5 and Section 9.3) assume agents to have unique peaks anywhere in that domain. Dutta et al. (2002) showed that for $m \geqslant 3$ and single-peaked, continuous, and strictly convex preferences, dictatorships are the only surjective and strategyproof mechanisms.

Nehring and Puppe (2022, 2023) explore a "frugal aggregation" model where agents' peaks are known, but their preferences are not completely specified but only restricted by general assumptions like convexity. They investigate so-called *ex ante* Condorcet winners, where pairwise comparisons between two alternatives x and y are based on the number of agents that prefer x to y and vice versa, which is

55 A notable exception is the case $m = 2$ and n odd, for which Peters et al. (1992) showed that such a coordinatewise median rule is characterized by these three properties.

56 We need to differentiate between settings with a finite set of alternatives where randomization is introduced to increase fairness and models where the set of possible outcomes is indeed the probability simplex.

an interval rather than an explicit number since multiple preference profiles are admissible for each set of peaks. If more than half of the agents have the same peaks, the corresponding alternative constitutes the unique Condorcet winner, showing that this approach does not give any fairness guarantees to minorities.[57] Interestingly, they show that for their model of separably convex preferences, the set of local *ex ante* Condorcet winners coincides with the set of alternatives that minimize the sum of ℓ_1 distances to the agents' peaks.

ℓ_1 PREFERENCES. To the best of our knowledge, Lindner et al. (2008) were the first to consider ℓ_1 preferences in portioning.[58] They proved that minimizing the sum of ℓ_1 distances leads to an efficient and strategyproof mechanism (see also Lindner, 2011; Goel et al., 2019).

Freeman et al. (2021) realized that the idea of Moulin's generalized median rules (see Theorem 10.1) can be generalized to strategyproof *moving phantom mechanisms* for $m > 2$. Compared to Moulin's characterization, the $n + 1$ phantoms are not fixed but increase continuously and independently from each other from 0 to 1 over time. By the intermediate value theorem, there exists a point in time where the m medians[59] sum up to 1, i.e., form a valid outcome. Freeman et al. (2021) identified the unique efficient mechanism within this class as the rule that minimizes the sum of ℓ_1 distances. Furthermore, they defined the *independent markets* mechanism, which is inefficient but satisfies *proportionality* (see Section 9.4).

However, not all strategyproof mechanisms are moving phantoms, as proven by [3] and de Berg et al. (2024). Caragiannis et al. (2022) generalize proportionality to arbitrary profiles via the ℓ_1 distance between the returned outcome and the mean of the agents' peaks, whereas Freeman and Schmidt-Kraepelin (2024) use the ℓ_∞ distance. An overview of various axioms and mechanisms for ℓ_1 preferences is given in [4].

BELIEF AGGREGATION. Interpreting agents' peaks as probabilistic beliefs over m states of the world leads to the research area of *belief aggregation* or *opinion pooling* (see, e.g., Dietrich and List, 2017). These states correspond to interconnected events, which are reflected in the agents' preferences. Thus, different utility functions are usually considered, although there are some overlaps with portioning, e.g., when all states can be ordered on a line.

In the remainder of this chapter, we formally introduce the portioning setting and define several (classes of) utility models as well as another fairness axiom.

In Chapter 10, we consider arbitrary subsets of single-peaked preferences for $m = 2$ and show, inter alia, that preferences can be aggregated in an efficient, strategyproof, and fair way. Chapter 11 proves that such a mechanism cannot exist for $m \geqslant 3$, $n \geqslant 3$ and ℓ_1 or ℓ_∞ preferences. Finally, Chapter 12 concludes the part on portioning by discussing results and outlining promising directions for future research.

57 Nehring and Puppe (2022) stress that their focus lies on aggregating preferences via voting theory.
58 Barberà et al. (1993, 1997) already investigated a form of ℓ_1 preferences for Cartesian product domains.
59 For each alternative x, take the median of the $n + 1$ phantom voters and the coordinatewise peaks $p_{i,x}$ for each agent i.

9.2 PRELIMINARIES

Recall our portioning setting. Each agent $i \in N$ has a continuous and quasi-concave utility function $u_i \colon \Delta \to \mathbb{R}$ representing her preferences over all distributions over the set of m pure alternatives A which we denote by Δ.[60] A profile $P = U$ consists of a utility function $u_i \in \mathcal{U}$ for each agent where $\mathcal{U}$ denotes the set of all admissible utility functions.

In contrast to donor coordination, where agents need to compare distributions of different "sizes", their preferences are restricted to the probability simplex, i.e., $\underline{C}_N = \bar{C}_N = 1$. In particular, we work with a fixed set of agents since we cannot simply set the contribution of an agent to zero to signal non-participation. Instead of reporting valuations for each alternative, we expect each agent i to have and report their favorite distribution, their unique "peak" $p_i = (p_{i,x})_{x \in A} \in \Delta$ with $p_i = \arg\max_{\delta \in \Delta} u_i(\delta)$. Similar to the case of having valuations as reports, we investigate specific utility models where peaks and utility functions admit a one-to-one correspondence. A notable exception is Chapter 10. The peak corresponding to utility function u_i is denoted by p_i^u, but the superscript is usually omitted. Generally, we assume that, when fixing a utility model, a profile consists of a set of reported peaks $p_i \in \Delta$, i.e., $P = (p_i)_{i \in N}$.

Consistency Axioms

The definitions for (U-)continuity and preference-monotonicity can be adapted from Section 3.2 whereas axioms based on contributions are not applicable in portioning.

Definition 9.2
A mechanism f satisfies *continuity* if for all $U \in \mathcal{U}^n$,

$$\forall \varepsilon > 0\ \exists \gamma > 0\ \forall U' \in \mathcal{U}^n : \sum_{i \in N} \|p_i^u - p_i^{u'}\|_1 < \gamma \implies \|U(f(U)) - U(f(U'))\|_1 < \varepsilon.$$

Definition 9.3
A mechanism f satisfies *preference-monotonicity* if for every two utility profiles U and U' which are identical except that $p_{i,x}^{u'} > p_{i,x}^{u}$ for one agent i and one alternative x and $p_{i,y}^{u'} \leqslant p_{i,y}^{u}$ for all $y \in A \setminus x$, we have $f(U')_x \geqslant f(U)_x$.

We require $p_{i,y}^{u'} \leqslant p_{i,y}^{u}$ for all $y \in A \setminus x$ to account for the fact that $p_i \in \Delta$. If $p_{i,y}^{u'} > p_{i,y}^{u}$ for some $y \neq x$ holds in addition, it does not seem meaningful to rule out that $f(U')_y > f(U)_y$ but $f(U')_x < f(U)_x$.

9.3 STAR-SHAPED AND PEAK-LINEAR UTILITY FUNCTIONS

To achieve the previously mentioned one-to-one correspondence between peaks and utility functions, several specific utility models with certain properties are

60 We simplify notation and write Δ instead of $\Delta(1)$ as in the donor coordination setting.

investigated in the following. They all share that an agent's utility increases when "walking" toward her peak. Formally, for any distribution $\delta \neq p_i^u$ and all $\lambda \in (0, 1)$,

$$u_i(p_i^u) > u_i(\lambda p_i^u + (1-\lambda)\delta) > u_i(\delta). \tag{29}$$

Such utility functions are sometimes referred to as *star-shaped preferences* (e.g., Border and Jordan, 1983) and constitute a generalization of single-peaked preferences (Black, 1948) for $m > 2$. To be precise, the considered utility functions even belong to a subclass of star-shaped preferences, named *peak-linear* utilities ([3]).

DEFINITION 9.4
A utility function is peak-linear if for any distribution $\delta \in \Delta$ and $\lambda \in [0, 1]$,

$$u_i(\lambda p_i^u + (1-\lambda)\delta) = \lambda u_i(p_i^u) + (1-\lambda)u_i(\delta). \tag{30}$$

Loosely speaking, an agent's utility for a distribution depends linearly on the fraction of her peak "included" in that distribution as well as the remainder δ.

Inserting (30) in (29) directly shows that every peak-linear utility function is also star-shaped.

From Definition 9.4, it is straightforward to see that a peak-linear utility function u_i is completely determined by its value at the peak $u_i(p_i)$ and values for boundary distributions δ with $\text{supp}(\delta) \not\subseteq \text{supp}(p_i)$.

Of course, assuming continuity and quasi-concavity imposes restrictions on the values for boundary distributions. In particular, the utility function restricted to the boundary must still be continuous and quasi-concave.

ℓ_p *preferences*

A natural idea for specifying utilities with a unique favorite distribution is to measure the distance to the peak using some metric $d : \Delta \times \Delta \to \mathbb{R}$. Then, agent i's utility for a distribution δ can be defined as $u_i(\delta) = -d(p_i^u, \delta)$.

Especially ℓ_p norms, given by $\|\delta\|_p := \left(\sum_{x \in A} |\delta_x|^p\right)^{1/p}$ for $1 \leqslant p \leqslant \infty$, belong to the most studied norm-based utility functions.

DEFINITION 9.5
For $1 \leqslant p \leqslant \infty$, an agent i has ℓ_p *preferences* if $u_i(\delta) = -\|p_i^u - \delta\|_p$. For $p = \infty$, set $u_i(\delta) = -\max_{x \in A} |p_{i,x} - \delta_x|$.

We will also sometimes refer to these preferences as ℓ_p disutilities.

In particular, the special case of $p = 1$, i.e., $u_i(\delta) = -\sum_{x \in A} |p_{i,x} - \delta_x|$ has received a lot of attention (Lindner et al., 2008; Goel et al., 2019; Freeman et al., 2021). Such ℓ_1 preferences admit a nice interpretation in portioning, which can be seen when considering an alternative utility representation.[61]

PROPOSITION 9.6 *(Goel et al. (2019))*
Agent i's ℓ_1 preferences are also represented by $u_i(\delta) = \sum_{x \in A} \min\{p_{i,x}, \delta_x\}$.

Therefore, $p_{i,x}$ can be interpreted as a cap for agent i's utility concerning alternative x. Her utility obtained from δ_x increases linearly until $\delta_x = p_{i,x}$ and stays constant for larger δ_x.

61 Goel et al. (2019) call these two utility models "ℓ_1 costs" and "overlap utilities", respectively.

Proposition 9.7
For $1 \leqslant p \leqslant \infty$, ℓ_p preferences are continuous, convex, and can be represented by a peak-linear utility function.

Proof. Assume agent i has ℓ_p preferences. Consider utility functions $u_i(\delta) = \left(\sum_{x \in A} |p_{i,x} - \delta_x|^p\right)^{1/p}$ for $p \in [1, \infty)$ and $u_i(\delta) = -\max_{x \in A} |p_{i,x} - \delta_x|$ for $p = \infty$. Continuity follows from the fact that each utility function is a composition of continuous functions. Furthermore, quasi-concavity is implied by the fact that norms are convex by definition, resulting in the considered utility functions being concave. We still need to show peak-linearity.

We first look at the special case of ℓ_∞ preferences. Note that for any distribution $\delta \in \Delta$ and $\lambda \in [0,1]$,

$$\begin{aligned}
u_i(\lambda p_i + (1-\lambda)\delta) &= -\max_{x \in A} |p_{i,x} - (\lambda p_{i,x} + (1-\lambda)\delta_x)| \\
&= -\max_{x \in A} |(1-\lambda)(p_{i,x} - \delta_x)| \\
&= 0 + (1-\lambda) u_i(\delta) \\
&= u_i(p_i) + (1-\lambda) u_i(\delta).
\end{aligned}$$

We now consider arbitrary $p \in [1, \infty)$. Then, for any distribution $\delta \in \Delta$ and $\lambda \in [0,1]$,

$$\begin{aligned}
u_i(\lambda p_i + (1-\lambda)\delta) &= -\left(\sum_{x \in A} |p_{i,x} - (\lambda p_{i,x} + (1-\lambda)\delta_x)|^p\right)^{1/p} \\
&= -\left(\sum_{x \in A} |(1-\lambda)(p_{i,x} - \delta_x)|^p\right)^{1/p} \\
&= 0 + (1-\lambda) u_i(\delta) \\
&= u_i(p_i) + (1-\lambda) u_i(\delta).
\end{aligned}$$

□

Therefore, ℓ_p preferences are also star-shaped for $1 \leqslant p \leqslant \infty$. This does not hold for arbitrary preferences induced by some metric, e.g., consider the *trivial metric*, $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ otherwise.

Leontief preferences

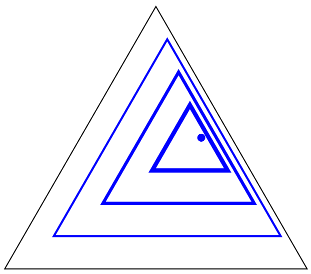
Chapter 5 introduced Leontief utility functions for the donor coordination setting. Normalizing valuations to 1 directly leads to a valid utility function in portioning.

Definition 9.8
An agent i has *Leontief preferences* if $u_i(\delta) = \min_{x \in A :\, p^u_{i,x} > 0} \delta_x / p^u_{i,x}$.

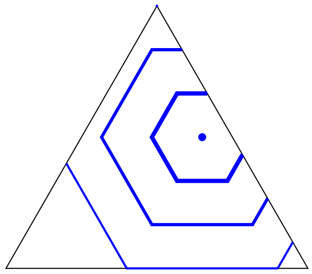
In the following, we omit the condition $p^u_{i,x} > 0$ and set $\delta_x / 0 = \infty$ for all $\delta_x \in [0,1]$.

It is straightforward to see that agent i's utility is maximized solely by $\delta = p_i$. Moreover, Leontief utilities are peak-linear.

Proposition 9.9
Leontief preferences are continuous, convex, and can be represented by a peak-linear utility function.



(a) Leontief utilities



(b) ℓ_1 disutilities

Figure 9.1: Illustration of indifference classes for Leontief utilities and ℓ_1 disutilities for three alternatives when the ideal distribution is $(0.1, 0.4, 0.5)$ (represented by the blue point). The black triangle is the simplex of distributions among three alternatives. Its vertices represent the degenerate distributions $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. For each type of utilities, the peak forms an indifference class by itself, and three other indifference classes are displayed with different line widths.

Proof. Continuity follows from the fact that a minimum over continuous functions is again continuous.[62]

For any $\delta, \delta' \in \Delta$ and $\lambda \in [0,1]$,

$$\begin{aligned} u_i(\lambda\delta + (1-\lambda)\delta') &= \min_{x \in A} \frac{\lambda\delta_x + (1-\lambda)\delta'_x}{p_{i,x}} \\ &\geqslant \min_{x \in A} \frac{\lambda\delta_x}{p_{i,x}} + \min_{x \in A} \frac{(1-\lambda)\delta'_x}{p_{i,x}} \\ &= \lambda u_i(\delta) + (1-\lambda)u_i(\delta'), \end{aligned}$$

showing that Leontief utility functions are even concave.

To show peak-linearity, note that for any $\delta \in \Delta$ and $\lambda \in [0,1]$,

$$\begin{aligned} u_i(\lambda p_i + (1-\lambda)\delta) &= \min_{x \in A} \frac{\lambda p_{i,x} + (1-\lambda)\delta_x}{p_{i,x}} \\ &= \lambda + (1-\lambda) \min_{x \in A} \frac{\delta_x}{p_{i,x}} \\ &= \lambda u_i(p_i) + (1-\lambda)u_i(\delta). \end{aligned}$$

□

In particular, Leontief utilities are zero on boundary distributions and, thus, attach more importance to a proportional representation of all alternatives than ℓ_1 preferences. Their indifference curves are illustrated in Figure 9.1.

All results from Chapter 4 that do not involve flexible contributions directly carry over to portioning, showing that maximizing Nash welfare (*NASH*) is the best approach for achieving efficient, strategyproof, and fair outcomes. In particular, the mechanism that always returns the Nash welfare maximizing distribution is characterized by continuity, group-strategyproofness, and the core (see Section 5.4).

Such positive results do not hold for ℓ_1 and ℓ_∞ preferences as shown in Chapter 11, at least for $m \geqslant 3$.

62 In particular, discontinuity points at alternatives x with $p_{i,x} = 0$ do not matter for the minimum.

9.4 PROPORTIONALITY

In the context of ℓ_1 preferences, Freeman et al. (2021) introduced another fairness axiom called *proportionality* which determines the outcome for profiles where all agents have their peaks at pure alternatives.

DEFINITION 9.10
A profile $P \in \mathcal{P}$ is called *single-minded* if $p_{i,x} \in \{0,1\}$ for all $i \in N$ and $x \in A$.

Proportionality demands the outcome to be proportional to the agents' peaks in single-minded profiles.

DEFINITION 9.11
A mechanism f satisfies *proportionality* if for all single-minded profiles $P \in \mathcal{P}$, it holds that $f(P)_x = \sum_{i \in N} p_{i,x}/n$.

We proceed by placing proportionality into our hierarchy of fairness axioms (see Section 2.3.3). On the one hand, proportionality is an intraprofile condition that applies only to single-minded profiles, so it cannot be stronger than positive share. On the other hand, this axiom requires that (unique) core outcomes are returned for single-minded profiles, at least for certain utility models.

PROPOSITION 9.12
With ℓ_1 preferences, ℓ_∞ preferences, or Leontief utilities, proportionality requires to return the unique core outcome for single-minded profiles.

Proof. Assume that a mechanism f is not proportional for some single-minded profile $P \in \mathcal{P}$. Set $\delta = f(P)$, and let $N' \subseteq N$ be a maximal subset of agents where all agents in N' put probability 1 on the same alternative x^* and proportionality is violated, i.e. $\delta_x < r$ for $r = |N'|/n$.

Note that for all utility models from the statement, the utilities of all agents in N' only depend on and are increasing in the amount of probability assigned to alternative x^*. Specifically, $u_i(\delta) = -2(1-\delta_{x^*})$ for ℓ_1 preferences, $u_i(\delta) = -(1-\delta_{x^*})$ for ℓ_∞ preferences, and $u_i(\delta) = \delta_{x^*}$ for Leontief preferences. Setting $\delta' \in \Delta(r)$ with $\delta'_{x^*} = r$ and $\delta'_x = 0$ for all $x \in A \setminus x^*$, we have $u_i(\delta' + \tilde{\delta}) > u_i(\delta)$ for all $\tilde{\delta} \in \Delta(1-r)$ and all three utility models as $\delta'_{x^*} + \tilde{\delta}_{x^*} > \delta_{x^*}$. Thus, agents from N' have a beneficial deviation, and δ is not in the core. However, Theorem 2.14 implies the existence of a core outcome, which then coincides with the distribution required by proportionality. □

COROLLARY 9.13
With ℓ_1 preferences, ℓ_∞ preferences, or Leontief utilities, the core implies proportionality.

Note that the proof of Proposition 9.12 does not work for other ℓ_p preferences, e.g., ℓ_2 disutilities.

EXAMPLE 9.14
Consider the following example with three projects $A = \{a, b, c\}$ and three agents.

	$p_{i,a}$	$p_{i,b}$	$p_{i,c}$
Agent 1	1	0	0
Agent 2	1	0	0
Agent 3	0	1	0

A rule that returns $\delta = (0.64, 0.18, 0.18)$ violates proportionality but δ is in the core as for both $i \in \{1,2\}$, $u_i(\delta) > -\sqrt{1/5}$, but e.g., for $\tilde{\delta} = (0,1,0)$ and $\delta' = (1,0,0)$, we have $(2/3) \cdot \delta' + (1/3) \cdot \tilde{\delta} = (2/3, 1/3, 0)$ with $u_i((2/3,1/3,0)) = -\sqrt{2/9} < -\sqrt{1/5}$.

In fact, proportionality does not seem to be a very natural notion for such preferences, as the proportionality guarantee for single-minded agents, who put 1 on alternative x^*, concerns only the probability put on alternative x^*, whereas agents with ℓ_2 preferences care also about the distribution on other alternatives.

Obviously, individual fair share does not imply proportionality as for single-minded profiles, this axiom requires only $\delta_x \geqslant 1/n$ for each alternative, which is the peak of at least one agent.

All in all, proportionality is weaker than the core but incomparable to individual fair share and positive share.

Proportionality appears as our fairness axiom when proving impossibilities for ℓ_1 and ℓ_∞ preferences in Chapter 11 for $m \geqslant 3$.

10 THE CASE OF TWO ALTERNATIVES

Many impossibility results that include strategyproofness hold only for three or more alternatives (e.g., Gibbard, 1973; Satterthwaite, 1975; Zhou, 1991).

For two pure alternatives $A = \{a, b\}$, the set of outcomes becomes one-dimensional and can be identified with the unit interval $[0, 1]$, where the endpoints 0 and 1 correspond to the pure alternatives a and b, respectively. Thus, agent i's peak p_i can be identified with a scalar in $[0, 1]$ representing her favorite distribution $[1 - p_i : a, p_i : b]$. Compared to three or more alternatives, manipulations are restricted to a line, imposing fewer restrictions on a strategyproof mechanism. Furthermore, the intermediate value theorem for continuous functions implies convexity of the image of f for two alternatives, meaning that the set of all possible images coincides with the set of all closed intervals in $[0, 1]$.

In fact, aggregating preferences over the unit interval, sometimes referred to as the (one-dimensional) *facility location problem* (e.g., Aziz et al., 2022), is a whole research area in itself (see also Moulin, 1980; Border and Jordan, 1983; Ching, 1997; Berga and Serizawa, 2000; Massó and de Barreda, 2011; Freeman et al., 2021; Jennings et al., 2024). We discuss these works and compare them to our results after the respective theorems.

In this chapter, we deviate from the assumption that $\mathcal{U}$ has to contain *exactly* one utility function per peak in $[0, 1]$ and demand that $\mathcal{U}$ contains *at least* one utility function per peak.[63] This generalization is possible because the mechanisms we characterize satisfy strategyproofness and further desirable properties without relying on any knowledge of the agents' utility functions except their peaks. Furthermore, we do not consider specific utility models (e.g., ℓ_1 preferences) but the set of all star-shaped utility functions.

10.1 CHARACTERIZING GENERALIZED MEDIAN RULES

For two pure alternatives, the class of star-shaped utilities coincides with the well-studied class of *single-peaked utilities*, in the following denoted by $\mathcal{U}^{SP}$. Under the assumption that $\mathcal{U} = \mathcal{U}^{SP}$, Moulin (1980) characterized the set of all strategyproof mechanisms as generalized median (med) rules.[64] This characterization assumes that the rules have to handle *all* functions in $\mathcal{U}^{SP}$. As a consequence, it no longer holds when restricting $\mathcal{U}$ to a strict subset of $\mathcal{U}^{SP}$. In principle, allowing rules to handle only a subset of $\mathcal{U}^{SP}$ may enable a greater selection of strategyproof rules, similar to, e.g., Zhou's characterization of strategyproof mechanisms that does not hold when restricting the domain to ℓ_1 preferences (Freeman et al., 2021).

63 The "at least one" requirement is needed to allow agents to misreport their peak to any other peak in $[0, 1]$. Note that we still require our mechanism to only depend on the peaks, a property which is called *tops-onlyness*.

64 Ehlers et al. (2002) obtain similar results when outcomes are *probability distributions* over some interval in $\mathbb{R}$ or the whole real line.

In the following, we prove that imposing continuity on the mechanism leads us back to Moulin's characterization, even when considering an *arbitrary* subdomain $\mathcal{U} \subseteq \mathcal{U}^{SP}$.

THEOREM 10.1
For $m = 2$ and any $\mathcal{U} \subseteq \mathcal{U}^{SP}$, a continuous mechanism f satisfies anonymity and strategyproofness if and only if there exist $\alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n$ in $[0,1]$ such that

$$f(P) = \text{med}(p_1, \ldots, p_n, \alpha_0, \ldots, \alpha_n).$$

In particular, continuity is required as shown by the following mechanism f for one agent:

$$f(p) := \begin{cases} 0 & p < 0.5; \\ 1 & p \geqslant 0.5. \end{cases}$$

This discontinuous mechanism satisfies anonymity and strategyproofness but cannot be represented as a generalized median rule.

Before proving Theorem 10.1, we show an interesting result on the interplay between continuity and strategyproofness.

LEMMA 10.2
For $m = 2$ and any $\mathcal{U} \subseteq \mathcal{U}^{SP}$, if two continuous and strategyproof mechanisms yield the same distribution for all single-minded profiles, then they yield the same distribution for all profiles.

Proof. Let f and g be two continuous and strategyproof mechanisms. Let P be a profile for which $f(P) \neq g(P)$. We prove that there is a single-minded profile P' for which $f(P') \neq g(P')$.

At a high level, the proof works as follows. Step by step, each agent with peak on the left side of $f(P)$ moves her peak closer and closer to 0, and each agent with peak on the right side moves to 1. Continuity and strategyproofness imply that $f(P)$ cannot change in the process. Finally, for all agents with peaks at $f(P)$, move their peaks to the alternative that is not "separated" from the peak by $g(P)$. In the process, $f(P)$ can only move further away from $g(P)$.

Assume that $f(P) < g(P)$, the case $f(P) > g(P)$ can be handled analogously and denote $\delta = f(P)$.

Partition the set of agents into four groups: $N = N^{01} \cup N^- \cup N^= \cup N^+$, where $N^{01} = \{i \in N : p_i \in \{0,1\}\}$, $N^- = \{i \in N \setminus N^{01} : p_i < \delta\}$, $N^= = \{i \in N \setminus N^{01} : p_i = \delta\}$, and $N^+ = \{i \in N \setminus N^{01} : p_i > \delta\}$. Our overall goal is to "move" all agents to N^{01} while keeping the chosen distribution different from $g(P)$.

Take any agent $i \in N^-$, and consider the function $F : [0,1] \to [0,1]$ defined by $F(p) = f(p, P_{-i})$. Since f is continuous, so is F. Note that $F(p_i) = f(P) = \delta > p_i$. We now prove $F(p) = \delta$ also for all $p < p_i$ by contradiction.

CASE 1
If $\delta < F(p)$, then $p < p_i < F(p_i) < F(p)$, so agent i with peak at p could beneficially manipulate from p to p_i, contradicting strategyproofness.

CASE 2
If $F(p) < \delta$, then by the intermediate value theorem, there exists $p' \in [p, p_i)$ with $F(p') = \max\{p_i, F(p)\}$, as F is continuous and $p_i < F(p_i)$. Thus, agent i with peak at p_i can beneficially change her reported peak to p' since $p_i \leqslant F(p') < F(p_i) = \delta$.

In particular, $F(0) = \delta = f(P)$. Denote the profile where agent i changed her peak to 0 by $P^{\{i\}}$. Then, $f(P^{\{i\}}) = F(0) = f(P)$. The same argument applies to all other agents from N^-, so $f(P^{N^-}) = f(P)$, where P^{N^-} denotes the profile resulting from P after all agents in N^- moved their peak to 0. Also, $g(P^{N^-}) = g(P)$, as all agents from N^- were also on the left side of $g(P)$ due to $f(P) < g(P)$, so moving them further left cannot change the distribution returned by g.

For agents $i \in N^+$ and the same function F, one can show analogously that $F(p) = \delta$ for all $p \geqslant p_i$, and the outcome remains δ when i moves her peak to 1. Therefore, $f(P^{N^- \cup N^+}) = f(P)$, where $P^{N^- \cup N^+}$ denotes the profile resulting from P after all agents in N^- moved their peak to 0 and all agents in N^+ moved their peak to 1. Also, $g(P^{N^- \cup N^+}) \geqslant g(P^{N^-})$ as moving peaks to the right cannot move the outcome to the left by similar arguments as for the two cases from above. Therefore, $f(P^{N^- \cup N^+}) < g(P^{N^- \cup N^+})$ still holds.

We now consider an agent $i \in N^=$ with $p_i = \delta = f(P) < g(P) \leqslant g(P^{N^- \cup N^+})$.

Again, define $F'(p) := f(p, P_{-i}^{N^- \cup N^+})$, which is continuous as f is continuous. By strategyproofness, $F'(p) \leqslant \delta$ for $p \leqslant p_i$, since $F'(p) > \delta$ would imply $p < p_i = F'(p_i) < F'(p)$, so agent i with peak at p could beneficially manipulate from p to p_i. Thus, with $P^{N^- \cup N^+ \cup \{i\}}$ denoting the profile where agent i moved her peak to 0, $f(P^{N^- \cup N^+ \cup \{i\}}) \leqslant f(P^{N^- \cup N^+}) < g(P^{N^- \cup N^+}) = g(P^{N^- \cup N^+ \cup \{i\}})$. If $f(P^{N^- \cup N^+ \cup \{i\}}) = f(P^{N^- \cup N^+})$, repeat the procedure with the next agent from $N^=$. If $f(P^{N^- \cup N^+ \cup \{i\}}) < f(P^{N^- \cup N^+})$, all remaining agents from $N^=$ now have their peak on the right side of $f(P^{N^- \cup N^+ \cup \{i\}})$ and can move their peak to 1 without changing the chosen distribution. Again, the outcome from g can only move to the right or stay fixed.

In the end, all agents' peaks are at 0 or 1 but $f(P') < g(P')$, where $P' = P^{N^- \cup N^+ \cup N^=}$ denotes the profile after all agents in $N^- \cup N^+ \cup N^=$ have moved their peaks to a pure alternative. Thus, f and g return different distributions for the single-minded profile P'. □

Remark 10

Moving peaks to the extremes 0 and 1 is reminiscent of a property called *uncompromisingness* by Border and Jordan (1983), which states that the outcome cannot change when agents from N^- and N^+ move their peaks to the respective extremes. They show that uncompromisingness implies continuity but in general concentrate on strategyproof and unanimous[65] (which imply uncompromising) mechanisms where peaks and outcomes are elements in $\mathbb{R}^m$ but not necessarily in Δ. Contrarily, we assume continuity and obtain uncompromisingness.

We are now ready to prove Theorem 10.1.

Proof of Theorem 10.1. The "if" direction is obvious. For the other direction and any $k \in \{0, \dots, n\}$, let P_k be some single-minded profile in which some k agents have their peak at 1 and the other $n - k$ agents have their peak at 0. Set $\alpha_k = f(P_k)$ and note that due to anonymity, α_k does not depend on the agents' identities.

As f is strategyproof, $\alpha_k \leqslant \alpha_{k+1}$ for all $k \in \{0, \dots, n-1\}$. Otherwise, some agent with peak at 0 in profile P_k could gain from reporting her peak at 1.

65 A mechanism $f\colon (\mathbb{R}^m)^n \to \mathbb{R}^m$ is unanimous if $f(P) = p$ for all $P = (p_i)_{i \in N}$ with $p_i = p = p_j$ for all $i, j \in N$.

Define $g(P) = \text{med}(p_1, \ldots, p_n, \alpha_0, \ldots, \alpha_n)$. Then, for any $k \in \{0, \ldots, n\}$, $g(P_k) = \alpha_k$ since n arguments of the median ($n-k$ peaks and k α's) are at most α_k and n arguments are at least α_k by construction. This means that f and g coincide on all single-minded profiles. By Lemma 10.2, $f = g$. □

In addition to Moulin (1980) and Border and Jordan (1983), various related characterizations have been shown. Ching (1997) proved, *inter alia*, that tops-onlyness can be replaced with continuity of the mechanism in Moulin's characterization. Berga and Serizawa (2000) characterize generalized median rules as the only strategyproof and surjective mechanism on a subset of single-peaked preferences that they call *minimally-rich*. Massó and de Barreda (2011) show that when restricting the domain to symmetric[66] single-peaked preferences, the class of strategyproof and anonymous mechanisms additionally includes certain disturbed generalized median rules with discontinuity points.

In contrast to all of these existing results, Theorem 10.1 applies not only to specific subdomains but to *any* subdomain of single-peaked preferences, whether symmetric or not.

10.2 CHARACTERIZING THE UNIFORM PHANTOM RULE

Freeman et al. (2021) showed that the only distribution of $n+1$ phantoms that ensures proportionality in addition to all axioms from Theorem 10.1 is to distribute the peaks uniformly, i.e., $\alpha_k = k/n$ for $k \in \{0, \ldots, n\}$, which defines the *uniform phantom rule* (see also Caragiannis et al., 2016). Aziz et al. (2022) strengthened this result by pointing out that continuity, strategyproofness, and proportionality suffice for characterizing the uniform phantom mechanism for symmetric single-peaked preferences.[67]

Recently, Jennings et al. (2024) showed that the uniform phantom mechanism is the unique mechanism that satisfies strategyproof and proportionality among rules that handle all preferences in $\mathcal{U}^{SP}$. Again, we present a characterization that holds for every subset of $\mathcal{U}^{SP}$.

Theorem 10.3
For $m = 2$ and any $\mathcal{U} \subseteq \mathcal{U}^{SP}$, the only continuous mechanism that satisfies strategyproofness and proportionality is the uniform phantom mechanism.

Proof. Note that proportionality specifies the outcome on all single-minded profiles. Thus, by Lemma 10.2, there exists a unique continuous, strategyproof, and proportional mechanism. Freeman et al. (2021) showed that the uniform phantom mechanism satisfies all of these axioms. □

There exists an interesting connection of *NASH* to the uniform phantom mechanism for ℓ_1 preferences, which follows immediately from the mechanisms' axiomatic properties. For $m = 2$, Leontief utilities, as well as ℓ_1 preferences, are

66 Agent i's preferences are symmetric if they can be represented by a utility function u_i with $u_i(\delta) = u_i(\delta')$ if and only if $|p_i - \delta| = |p_i - \delta'|$ for $\delta, \delta' \in [0, 1]$.

67 Note that proportionality already contains some form of anonymity. In single-minded profiles, proportionality requires picking a specific distribution that is independent of the agents' identities.

subsets of $\mathcal{U}^{SP}$, each of which contains one utility function per peak. Both mechanisms are continuous, strategyproof, and proportional. Therefore, they need to be equivalent on their respective utility models by Theorem 10.3.

Corollary 10.4
With two alternatives, *NASH* for Leontief utilities is equivalent to the uniform phantom mechanism for ℓ_1 preferences.

For $m = 2$, an outcome δ is efficient for a profile P if and only if $\min_{i \in N} p_i \leqslant \delta \leqslant \max_{i \in N} p_i$ (see, e.g., [4]). Obviously, the uniform phantom mechanism satisfies that property. Thus, in the case of two alternatives, it is possible to aggregate utilities in an efficient, strategyproof, and fair way, even without knowledge of the specific underlying utility model.

11 IMPOSSIBILITIES FOR THREE AND MORE ALTERNATIVES

Unfortunately, the positive results obtained in Chapter 10 do not carry over to larger numbers of pure alternatives. This chapter is dedicated to proving incompatibilities of efficiency, strategyproofness, and the rather weak fairness condition of proportionality when agents have ℓ_1 or ℓ_∞ preferences.

11.1 ℓ_1 PREFERENCES

Under ℓ_1 preferences, Freeman et al. (2021) observed that the utilitarian welfare maximizing mechanism is the only efficient mechanism in their class of moving phantom mechanisms. However, maximizing utilitarian welfare violates weak fairness axioms such as proportionality leading to the conjecture that there is an "inherent tradeoff between Pareto optimality and proportionality" (Freeman et al., 2021, p. 1) under the presence of strategyproofness. We formalize their intuition by proving an impossibility with tight bounds for n and m.

THEOREM 11.1
With ℓ_1 preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geqslant 3$ and $n \geqslant 3$.

For the proof, we consider disutilities d_i (the ℓ_1 distance to an agent's peak p_i) instead of utilities, i.e., $d_i(\delta) = \|p_i - \delta\|_1$. It is very helpful to keep the following observations in mind.

OBSERVATION 1
With ℓ_1 preferences, any agent i with $p_{i,x} = 1$ for some $x \in A$ receives disutility $d_i(\delta) = 2 - 2\delta_x$ for any $\delta \in \Delta$, regardless of the distribution on alternatives other than x. Loosely speaking, agent i is indifferent to how probability is distributed among alternatives other than x.

OBSERVATION 2
With ℓ_1 preferences, $d_i(\delta) \geqslant 2 \cdot |p_{i,x} - \delta_x|$ for all $i \in N$ and $x \in A$.

Proof of Theorem 11.1. We start with the case $m = 3$ and $n = 3$. For $m = 3$, we set $A = \{a, b, c\}$. For simplicity, we number profiles by a superscript (k). We denote the disutility function of agent i in profile k by $d_i^{(k)}$, and the returned distribution in profile k by $\delta^{(k)}$. Sometimes, $\delta^{(k)}$ cannot be determined completely. In these cases, we give lower or upper bounds on the entries of $\delta^{(k)}$.

Consider first the following two profiles. The outcome in Profile 2 has to be $(1/3, 1/3, 1/3)$ by proportionality. For Profile 1, the last row of the table gives lower bounds on $\delta_a^{(1)}, \delta_b^{(1)}$ and an upper bound on $\delta_c^{(1)}$, which we justify below.

Profile 1

	a	b	c
	1/2	1/2	0
	0	1	0
	0	0	1
$\delta^{(1)}$	$\geqslant$1/6	$\geqslant$1/2	$\leqslant$1/3

Profile 2

	a	b	c
	1	0	0
	0	1	0
	0	0	1
$\delta^{(2)}$	1/3	1/3	1/3

As Agent 1 can manipulate between Profile 1 and Profile 2, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies

$$d_1^{(1)}(\delta^{(1)}) \leqslant d_1^{(1)}(\delta^{(2)}) = 2/3 \quad \text{and} \tag{31}$$

$$d_1^{(2)}(\delta^{(1)}) \geqslant d_1^{(2)}(\delta^{(2)}) = 4/3. \tag{32}$$

By (31) and Observation 2, $\delta_a^{(1)} \geqslant 1/6$ (implying $\delta_b^{(1)} \leqslant 5/6$), $\delta_b^{(1)} \geqslant 1/6$, and $\delta_c^{(1)} \leqslant 1/3$. By (32) and Observation 1, $\delta_a^{(1)} \leqslant 1/3$, implying $\delta_b^{(1)} + \delta_c^{(1)} \geqslant 2/3$, and thus $\delta_b^{(1)} \geqslant 1/3$.

The left figure below illustrates both inequalities. The blue area corresponds to the set of distributions $\delta^{(1)}$ with $d_1^{(1)}(\delta^{(1)}) \leqslant 2/3$ whereas the red area consists of all distributions $\delta^{(1)}$ satisfying (32). Strategyproofness requires $\delta^{(1)}$ to be inside the intersection of the two areas, i.e., the purple region. Hence,

$$1/6 \leqslant \delta_a^{(1)} \leqslant 1/3, \qquad 1/3 \leqslant \delta_b^{(1)} \leqslant 5/6, \qquad \text{and} \qquad 0 \leqslant \delta_c^{(1)} \leqslant 1/3.$$

By efficiency, we can even show that $\delta_b^{(1)} \geqslant 1/2$. Otherwise, as $\delta_a^{(1)} > 0$, some small amount could be moved from a to b. Agent 3 is indifferent due to Observation 1 and Agent 2 strictly gains. Furthermore, this does not change Agent 1's disutility as $\delta_b^{(1)} < 1/2$.

Next, we consider the following two profiles.

Profile 3

	a	b	c
	1/4	3/4	0
	0	1	0
	0	0	1
$\delta^{(3)}$	0	2/3	1/3

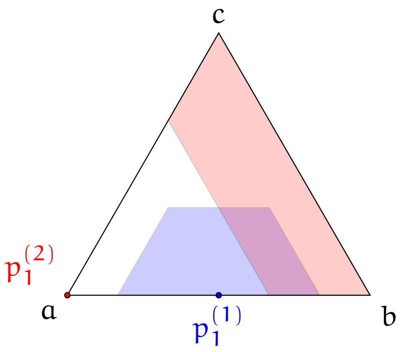
Profile 4

	a	b	c
	0	1	0
	0	1	0
	0	0	1
$\delta^{(4)}$	0	2/3	1/3

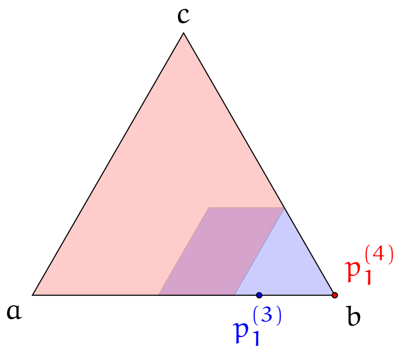
The outcome in Profile 4 follows from proportionality. We now prove that the outcome in Profile 3 must be identical. As Agent 1 can manipulate between Profile 3 and Profile 4, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(3)}(\delta^{(3)}) \leqslant d_1^{(3)}(\delta^{(4)}) = 2/3 \quad \text{and} \tag{33}$$

$$d_1^{(4)}(\delta^{(3)}) \geqslant d_1^{(4)}(\delta^{(4)}) = 2/3. \tag{34}$$



(a) Inequalities (31) and (32)



(b) Inequalities (33) and (34)

By (33), $\delta_c^{(3)} \leqslant 1/3$, implying $\delta_a^{(3)} + \delta_b^{(3)} \geqslant 2/3$. By (34), $\delta_b^{(3)} \leqslant 2/3$. Graphically, strategyproofness for Agent 1 implies that $\delta^{(3)}$ has to be in the purple region in the right figure above.

However, by efficiency, if $\delta_a^{(3)} > 0$ then $\delta_b^{(3)} \geqslant 3/4$. Otherwise, some small amount can be moved from a to b. Agent 3 is indifferent due to Observation 1 and Agent 2 strictly gains. Furthermore, this does not change Agent 1's disutility as $\delta_b^{(3)} < 3/4$. Therefore, $\delta_a^{(3)} = 0$ must hold, and the only outcome compatible with strategyproofness is $\delta^{(3)} = (0, 2/3, 1/3)$.

Now that we know $\delta^{(3)}$, we consider a manipulation of Agent 1 from Profile 3 to Profile 1. Strategyproofness implies

$$d_1^{(3)}(\delta^{(1)}) \geqslant d_1^{(3)}(\delta^{(3)}) = 2/3.$$

But the bounds we already have for $\delta^{(1)}$ imply that $d_1^{(3)}(\delta^{(1)}) \leqslant 2/3$ as $\delta_a^{(1)} \geqslant 1/6$ and $\delta_b^{(1)} \geqslant 1/2$. Therefore, $d_1^{(3)}(\delta^{(1)}) = 2/3$ together with $\delta_a^{(1)} = 1/6$ and $\delta_b^{(1)} = 1/2$. Hence, $\delta^{(1)} = (1/6, 1/2, 1/3)$.

Finally, we consider Profiles 5 and 6.

Profile 5

	a	b	c
	1/2	1/2	0
	0	1	0
	0	1/2	1/2
$\delta^{(5)}$			

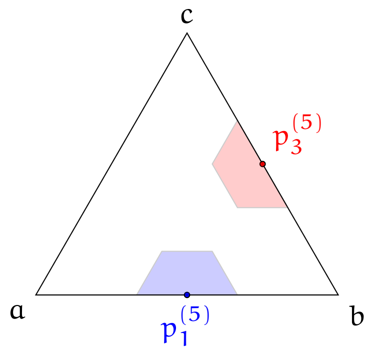
Profile 6

	a	b	c
	1	0	0
	0	1	0
	0	1/2	1/2
$\delta^{(6)}$	1/3	1/2	1/6

The distribution $\delta^{(6)}$ is determined by arguments analogous to those for $\delta^{(1)}$, reasoning about Agent 3 instead of Agent 1.

We now consider a manipulation of Agent 1 from Profile 5 to Profile 6. It follows from strategyproofness that $d_1^{(5)}(\delta^{(5)}) \leqslant d_1^{(5)}(\delta^{(6)}) = 1/3$, which implies that $\delta_c^{(5)} \leqslant 1/6$. Similarly, we consider a manipulation of Agent 3 from Profile 5 to Profile 1. It follows from strategyproofness that $d_3^{(5)}(\delta^{(5)}) \leqslant d_3^{(5)}(\delta^{(1)}) = 1/3$, which implies that $\delta_c^{(5)} \geqslant 1/2 - 1/6 = 1/3$, a contradiction.

Graphically, both inequalities are shown in the figure on the right. The blue area on the right contains the points satisfying the first inequality, and the red area on the left contains the points satisfying the second inequality. It is evident that the two inequalities cannot be satisfied simultaneously.



This example can be extended to arbitrary numbers of alternatives and agents in the following way.

To increase the number of alternatives, simply add alternatives x^+ with $p_{i,x^+} = 0$ for all agents $i \in N$. These new alternatives do not affect the argument, as efficiency ensures that none of them ever receives a positive amount.

Adding agents is more involved, as our proof relies on explicit distributions induced by proportionality and thus depends on the number of agents. However, we note that, throughout the proof, Agent 2 always has the same peak, which puts all mass on alternative b. Therefore, when adding agents i^+ with $p_{i^+,b} = 1$, we can run through the exact same proof but with adapted distributions.

Note that in contrast to the case $n = 3$, we now also need to denote the number of agents with certain peaks in a profile.

Consider first the following two profiles.

	Profile 1′				Profile 2′		
# agents	a	b	c	# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0	1	1	0	0
$n-2$	0	1	0	$n-2$	0	1	0
1	0	0	1	1	0	0	1
$\delta^{(1)}$	$\geqslant 1/2n$	$\geqslant (2n-3)/2n$	$\leqslant 1/n$	$\delta^{(2)}$	$1/n$	$(n-2)/n$	$1/n$

The outcome in Profile 2′ has to be $(1/n, (n-2)/n, 1/n)$ by proportionality. We now justify the bounds on the outcome in Profile 1′. As Agent 1 can manipulate between Profile 1′ and Profile 2′, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(1)}(\delta^{(1)}) \leqslant d_1^{(1)}(\delta^{(2)}) = 2/n, \tag{35}$$

$$d_1^{(2)}(\delta^{(1)}) \geqslant d_1^{(2)}(\delta^{(2)}) = (2n-2)/n. \tag{36}$$

By (35), $\delta_a^{(1)} \geqslant 1/2n$ (implying $\delta_b^{(1)} \leqslant (2n-1)/2n$), $\delta_b^{(1)} \geqslant (2n-5)/2n$, and $\delta_c^{(1)} \leqslant 1/n$. By (36), $\delta_a^{(1)} \leqslant 1/n$, implying $\delta_b^{(1)} + \delta_c^{(1)} \geqslant (n-1)/n$, and thus $\delta_b^{(1)} \geqslant (n-2)/n$.

By efficiency, we can even show that $\delta_b^{(1)} \geqslant (2n-3)/2n$. Otherwise, as $\delta_a^{(1)} > 0$, some small amount could be moved from a to b. Agent n is indifferent due to Observation 1 and Agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not change agent 1's disutility as $\delta_b^{(1)} < (2n-3)/2n$.

Next, we consider Profiles 3 and 4. The outcome in Profile 4′ follows from proportionality. We now prove that the outcome in Profile 3′ must be the same.

Profile 3′

# agents	a	b	c
1	$1/(n+1)$	$n/(n+1)$	0
$n-2$	0	1	0
1	0	0	1
$\delta^{(3)}$	0	$(n-1)/n$	$1/n$

Profile 4′

# agents	a	b	c
1	0	1	0
$n-2$	0	1	0
1	0	0	1
$\delta^{(4)}$	0	$(n-1)/n$	$1/n$

As Agent 1 can manipulate between Profile 3′ and Profile 4′, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(3)}(\delta^{(3)}) \leqslant d_1^{(3)}(\delta^{(4)}) = 2/n, \quad (37)$$
$$d_1^{(4)}(\delta^{(3)}) \geqslant d_1^{(4)}(\delta^{(4)}) = 2/n. \quad (38)$$

By (37), $\delta_c^{(3)} \leqslant 1/n$, implying $\delta_a^{(3)} + \delta_b^{(3)} \geqslant (n-1)/n$. By (38), $\delta_b^{(3)} \leqslant (n-1)/n$.

However, by efficiency, if $\delta_a^{(3)} > 0$ then $\delta_b^{(3)} \geqslant n/(n+1)$. Otherwise, some small amount can be moved from a to b. Agent n is indifferent due to Observation 1 and Agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not change agent 1's disutility as $\delta_b^{(3)} < n/(n+1)$. Therefore, $\delta_a^{(3)} = 0$ must hold, and the only outcome compatible with strategyproofness is $\delta^{(3)} = (0, (n-1)/n, 1/n)$.

Now that we know $\delta^{(3)}$, we consider a manipulation of Agent 1 from Profile 3′ to Profile 1′. Strategyproofness implies

$$d_1^{(3)}(\delta^{(1)}) \geqslant d_1^{(3)}(\delta^{(3)}) = 2/n.$$

But the bounds we already have for $\delta^{(1)}$ imply that $d_1^{(3)}(\delta^{(1)}) \leqslant 2/n$ as $\delta_a^{(1)} \geqslant 1/2n$ and $\delta_b^{(1)} \geqslant (2n-3)/2n$. Therefore, $d_1^{(3)}(\delta^{(1)}) = 2/n$ together with $\delta_a^{(1)} = 1/2n$ and $\delta_b^{(1)} = (2n-3)/2n$. Hence, $\delta^{(1)} = (1/2n, (2n-3)/2n, 1/n)$.

Finally, we consider the following two profiles.

Profile 5′

# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\delta^{(5)}$			

Profile 6′

# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\delta^{(6)}$	$1/n$	$(2n-3)/2n$	$1/2n$

The distribution $\delta^{(6)}$ is determined by arguments analogous to those for $\delta^{(1)}$, reasoning about Agent n instead of Agent 1.

We now consider a manipulation of Agent 1 from Profile 5′ to Profile 6′. It follows from strategyproofness that

$$d_1^{(5)}(\delta^{(5)}) \leqslant d_1^{(5)}(\delta^{(6)}) = 1/n,$$

which implies that $\delta_c^{(5)} \leqslant 1/2n$. Similarly, we consider a manipulation of Agent n from Profile 5 to Profile 1. It follows from strategyproofness that

$$d_3^{(5)}(\delta^{(5)}) \leqslant d_3^{(5)}(\delta^{(1)}) = 1/n,$$

which implies that $\delta_c^{(5)} \geqslant 3/2n - 1/2n = 1/n$, a contradiction. □

Remark 11

The bounds $m \geqslant 3$ and $n \geqslant 3$ in Theorem 11.1 are tight. Indeed, there exists a moving phantom mechanism that satisfies strategyproofness, proportionality, and range-respect (Freeman et al., 2021), and it is known that range-respect and efficiency coincide when $m = 2$ or $n = 2$ (see [4]).

The three axioms required for the impossibility are independent. Indeed, all axioms except proportionality are satisfied by the mechanism that maximizes utilitarian welfare (Lindner et al., 2008), all axioms except efficiency are satisfied by the independent markets mechanism (Freeman et al., 2021), and all axioms except strategyproofness are satisfied by mechanisms that are dictatorial on all profiles that are not determined by proportionality.

Freeman et al. (2021) posed the question of whether every anonymous, neutral,[68] continuous, and strategyproof mechanism can be represented as a moving phantom mechanism. While such a characterization could simplify the previous proof, it does not hold in general (see, e.g., de Berg et al., 2024).

11.2 ℓ_∞ PREFERENCES

ℓ_1 preferences take a special role among ℓ_p disutilities in terms of efficiency: indifference curves partially move along distributions with a constant sum on "approved" ($p_{i,x} > 0$) alternatives. As an example, consider $A = \{a, b, c, d\}$ and agent i with peak $p_i = (1/2, 1/2, 0, 0)$. With ℓ_1 preferences, she is indifferent between all distributions δ with $\delta_a + \delta_b = 1/2$. Having a second agent j with $p_j = (0, 0, 0, 1)$, every efficient distribution δ with $q_a + q_b = 1/2$ has to allocate probability $1/2$ on alternative d. By contrast, when considering, e.g., ℓ_2 preferences, it also matters for agent i *how* $1/2$ is distributed on c and d. As a result, more distributions become efficient, which weakens the role of efficiency for a potential impossibility when $p > 1$.

We proceed by proving an impossibility for the utility model at the other end of the spectrum, namely ℓ_∞ preferences. These preferences behave similarly to ℓ_1 disutilities (Observation 1), which is helpful when arguing about efficiency.

Observation 3

With ℓ_∞ preferences, any agent i with $p_{i,x} = 1$ for some $x \in A$ receives disutility $d_i(\delta) = 1 - \delta_x$ for any $\delta \in \Delta$, regardless of the distribution on alternatives other than x. Loosely speaking, agent i is indifferent to how probability is distributed among alternatives other than x.

68 A mechanism f satisfies *neutrality* if for every profile $P \in \mathcal{P}$ and every permutation π of the alternatives resulting in profile P', it holds that $f(P') = \pi \circ f(P)$.

THEOREM 11.2
With ℓ_∞ preferences, no mechanism satisfies efficiency, strategyproofness, and proportionality when $m \geqslant 3$ and $n \geqslant 3$.

Proof. The proof uses the same profiles as the one for Theorem 11.1, but needs more involved arguments when reasoning about efficiency and extending the argument from $m = 3$ to a larger number of alternatives.[69]

First, fix $m = 3$ and $n \geqslant 3$. We use the same notation as in the proof of Theorem 11.1. We start by considering Profiles 1 and 2.

Profile 1

# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	0	1
$\delta^{(1)}$	$\geqslant 1/2n$	$\geqslant (2n-3)/2n$	$\leqslant 1/n$

Profile 2

# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	0	1
$\delta^{(2)}$	$1/n$	$(n-2)/n$	$1/n$

The outcome in Profile 2 has to be $(1/n, (n-2)/n, 1/n)$ by proportionality. We now justify the bounds on the outcome in Profile 1. As Agent 1 can manipulate between Profile 1 and Profile 2, strategyproofness requires that Agent 1 does not gain from either manipulation. This implies that

$$d_1^{(1)}(\delta^{(1)}) \leqslant d_1^{(1)}(\delta^{(2)}) = 1/n, \tag{39}$$

$$d_1^{(2)}(\delta^{(1)}) \geqslant d_1^{(2)}(\delta^{(2)}) = (n-1)/n. \tag{40}$$

By (39), $\delta_a^{(1)} \geqslant 1/2n$ (implying $\delta_b^{(1)} \leqslant (2n-1)/2n$), $\delta_b^{(1)} \geqslant (2n-5)/2n$, and $\delta_c^{(1)} \leqslant 1/n$. By (40), $\delta_a^{(1)} \leqslant 1/n$, implying $\delta_b^{(1)} + \delta_c^{(1)} \geqslant (n-1)/n$, and thus $\delta_b^{(1)} \geqslant (n-2)/n$.

By efficiency, we can even show that $\delta_b^{(1)} \geqslant (2n-3)/2n$. Otherwise, $\delta_b^{(1)} < (2n-3)/2n$ and $\delta_c^{(1)} + \delta_a^{(1)} > 3/2n$, and some small amount can be moved from a to b. Agent n is indifferent due to Observation 3 and Agents $2, \ldots, n-1$ strictly gain. Furthermore, this does not increase Agent 1's disutility as $d_1^{(1)}(\delta^{(1)}) \geqslant \delta_c^{(1)} > 3/2n - \delta_a^{(1)}$ and $\delta_b^{(1)} < (2n-3)/2n$. Hence, $\delta_b^{(1)} \geqslant (2n-3)/2n$.

Profile 3

# agents	a	b	c
1	$1/(n+1)$	$n/(n+1)$	0
$n-2$	0	1	0
1	0	0	1
$\delta^{(3)}$			

Profile 4

# agents	a	b	c
1	0	1	0
$n-2$	0	1	0
1	0	0	1
$\delta^{(4)}$	0	$(n-1)/n$	$1/n$

69 For ℓ_∞ preferences, an efficient distribution might allocate a positive amount to an alternative x with $p_{i,x} = 0$ for all $i \in N$. For example, let $m = 4$ and $n = 2$ with peaks $(1/2, 1/4, 1/4, 0)$ and $(1/4, 1/2, 1/4, 0)$, respectively. Then, $(3/8, 3/8, 1/8, 1/8)$ is efficient, as the maximal distance is $1/8$ for both agents and if Agent 1 is better off in distribution δ, this implies $\delta_1 > 3/8$ which decreases Agent 2's utility.

Assume for contradiction that $\delta_c^{(1)} \leqslant 3/4n$.

Consider a manipulation of Agent 1 from Profile 3 to Profile 1. Note that $d_1^{(3)}(\delta^{(1)}) \leqslant 3/4n$ with the bounds established for $\delta^{(1)}$. By strategyproofness for Agent 1, $\delta_c^{(3)} \leqslant 3/4n$.

By efficiency, $\delta_b^{(3)} \geqslant n/(n+1)$. Otherwise, $\delta_b^{(3)} < n/(n+1)$ and $\delta_a^{(3)} > 1/(n+1) - \delta_c^{(3)}$, and some small amount can be moved from a to b. Agent n is indifferent due to Observation 3 and Agents $2, \dots, n-1$ strictly gain. Furthermore, this does not increase Agent 1's disutility as $d_1^{(3)}(\delta^{(3)}) \geqslant \delta_c^{(3)} > 1/(n+1) - \delta_a^{(3)}$ and $\delta_b^{(3)} < n/(n+1)$. Hence, $\delta_b^{(3)} \geqslant n/(n+1)$. However, as $n/(n+1) > (n-1)/n$, this contradicts strategyproofness for Agent 1 manipulating from Profile 4 to Profile 3, where $\delta^{(4)} = (0, (n-1)/n, 1/n)$ follows from proportionality.

	Profile 5		
# agents	a	b	c
1	1	0	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\delta^{(5)}$	$\leqslant 1/n$	$\geqslant (2n-3)/2n$	$\geqslant 1/2n$

	Profile 6		
# agents	a	b	c
1	$3/2n$	$(2n-3)/2n$	0
$n-2$	0	1	0
1	0	$(2n-3)/2n$	$3/2n$
$\delta^{(6)}$			

Therefore, $\delta_c^{(1)} > 3/4n$ has to hold. By analogous arguments with reversed roles of Agents 1 and n, the same bounds for $\delta^{(5)}$ as well as $\delta_a^{(5)} > 3/4n$ hold.

Since $\delta_b^{(1)} \geqslant (2n-3)/2n$, we must have $\delta_a^{(1)} < 3/4n$. Consider a manipulation of Agent n from Profile 6 to Profile 1. Note that $d_n^{(6)}(\delta^{(1)}) < 3/4n$, as $\delta_b^{(1)} \geqslant (2n-3)/2n$ and $\delta_c^{(1)} > 3/4n$. By strategyproofness for Agent n, we have $d_n^{(6)}(\delta^{(6)}) \leqslant d_n^{(6)}(\delta^{(1)}) < 3/4n$, and thus $\delta_a^{(6)} < 3/4n$. Finally, consider a manipulation of Agent 1 from Profile 6 to Profile 5. Analogously, $d_1^{(6)}(\delta^{(6)}) \leqslant d_1^{(6)}(\delta^{(5)}) < 3/4n$, which implies $\delta_a^{(6)} > 3/2n - 3/4n = 3/4n$, a contradiction.

We finish the proof by showing that this argument remains valid under the addition of alternatives x^+ with $p_{i,x^+} = 0$ for all agents $i \in N$. We prove that no efficient mechanism puts positive probability on new alternatives x^+ with $p_{i,x^+} = 0$ for all agents $i \in N$ in any of the six profiles used in the proof.

Proportionality directly implies that adding such an alternative x^+ to Profiles 2 and 4 does not change the distribution.

Next, consider Profile 1. If $|(2n-3)/2n - \delta_b^{(1)}| < d_1^{(1)}(\delta^{(1)})$, or $|(2n-3)/2n - \delta_b^{(1)}| = d_1^{(1)}(\delta^{(1)})$ and $\delta_b^{(1)} < (2n-3)/2n$, moving some amount of probability from x^+ to b cannot increase Agent 1's disutility, does not change Agent n's disutility by Observation 3, and decreases the disutilities of all other agents. Therefore, such a redistribution would correspond to a Pareto improvement. Note that $|(2n-3)/2n - \delta_b^{(1)}| = d_1^{(1)}(\delta^{(1)})$ and $\delta_b^{(1)} = (2n-3)/2n$ cannot hold simultaneously, as $\delta_{x^+}^{(1)} > 0$. Hence, the only remaining case we need to consider is $|(2n-3)/2n - \delta_b^{(1)}| = d_1^{(1)}(\delta^{(1)})$ and $\delta_b^{(1)} > (2n-3)/2n$. This implies $3/2n - \delta_a^{(1)} \leqslant d_1^{(1)}(\delta^{(1)})$ and thus, $\delta_a^{(1)} + \delta_b^{(1)} \geqslant 3/2n - d_1^{(1)}(\delta^{(1)}) + (2n-3)/2n + d_1^{(1)}(\delta^{(1)}) = 1$, contradicting $\delta_{x^+}^{(1)} > 0$. The argument for Profile 5 works analogously.

For Profile 3, moving some amount of probability from x^+ to b can only potentially increase the disutility of one agent, namely Agent 1 if $\delta_b^{(3)} \geqslant n/(n+1)$ and $d_1^{(3)}(\delta^{(3)}) = \delta_b^{(3)} - n/(n+1)$. But then, $1/(n+1) - \delta_a^{(3)} \leqslant d_1^{(3)}(\delta^{(3)})$ and again, $\delta_a^{(3)} + \delta_b^{(3)} = 1$, contradicting $\delta_{x^+}^{(3)} > 0$.

Finally, for Profile 6, moving some amount of probability from x^+ to b can only potentially increase the disutilities of two agents, namely Agents 1 and n, if $\delta_b^{(6)} \geqslant (2n-3)/2n$ and $d_k^{(6)}(\delta^{(6)}) = \delta_b^{(6)} - (2n-3)/2n$ holds for at least one $k \in \{1, n\}$, without loss of generality for $k = 1$. But then, $3/2n - \delta_a^{(6)} \leqslant d_1^{(6)}(\delta^{(6)})$ and again, $\delta_a^{(6)} + \delta_b^{(6)} = 1$, contradicting $\delta_{x^+}^{(6)} > 0$. □

Remark 12

Theorem 11.2 requires $m \geqslant 3$, since for $m = 2$, all metrics are equivalent (and thus induce the same preferences, see Chapter 10), and there are mechanisms that satisfy all requirements (see Remark 11). Moreover, $n \geqslant 3$ is required because, for $m = 3$, the ℓ_∞ and ℓ_1 metrics are equivalent since the ℓ_1 distance is always twice the ℓ_∞ distance. Therefore, for $m = 3$ and $n = 2$, the same mechanisms satisfy all the requirements.

Similar to the impossibility for ℓ_1 preferences, we expect all axioms to be independent. However, to the best of our knowledge, this does not follow from existing results, as ℓ_∞ preferences have been studied significantly less than ℓ_1.

12 DISCUSSION

In this chapter, we summarize our results on portioning and outline some exciting ideas for future work including some concrete open problems.

Theorem 11.1 and Theorem 11.2 together with Corollary 9.13 imply that efficient and strategyproof mechanisms cannot always return core outcomes for ℓ_1 as well as ℓ_∞ preferences.

Corollary 12.1
With ℓ_1 or ℓ_∞ preferences, no mechanism satisfies efficiency, strategyproofness, and always returns core outcomes when $m \geqslant 3$ and $n \geqslant 3$.

On the one hand, this raises the question of whether Corollary 12.1 can be generalized and similar impossibility results hold for arbitrary ℓ_p preferences with $p \geqslant 1$.

Open Problem 7
For which ℓ_p preferences is the combination of efficiency and strategyproofness compatible with the core?

Theorem 5.36 shows that Corollary 12.1 does not generalize to all star-shaped or peak-linear utility models as we obtained a possibility for Leontief utilities. Understanding (im-)possibilities on a structural level might enable us to make statements for whole classes of preferences and utility models similar to the frugal aggregation approach by Nehring and Puppe.

On the other hand, sticking with ℓ_1 (or ℓ_∞) preferences, one might ask whether efficiency and strategyproofness are compatible when weakening fairness requirements.

Open Problem 8
With ℓ_1 or ℓ_∞ preferences, are efficiency and strategyproofness compatible with individual fair share?

An axiomatic analysis of known mechanisms for ℓ_1 preferences is given in [4]. To the best of our knowledge, minimizing the sum of ℓ_1 distances to the agents' peaks is the only known efficient and strategyproof mechanism. Unfortunately, this rule does not even satisfy positive share.

Example 12.2
Consider the following example with two projects $A = \{a, b\}$ and three agents.

	$p_{i,a}$	$p_{i,b}$
Agent 1	1	0
Agent 2	1	0
Agent 3	0	1

The sum of ℓ_1 distances is minimized for $(1, 0)$, which gives minimal utility to Agent 3.

Furthermore, we are not aware of any mechanism that always returns core outcomes, although such a mechanism has to exist by Theorem 2.14.

Open Problem 9
Give an explicit mechanism that always returns core outcomes for ℓ_1 preferences.

Investigating equilibrium distributions (where each agent has contribution $1/n$) could lead to new insights as well as new rules. In particular, returning equilibrium distributions guarantees individual fair share (see Section 2.3.3). Interestingly, Freeman et al. (2021) showed that the outcome of their independent markets mechanisms is a Nash equilibrium of a voting game where each agent is allowed to choose a number between 0 and 1 separately for each alternative, and the final outcome is proportional to the received scores. Note that in contrast to the Nash equilibria we want to consider, each agent's scores do not need to sum up to $1/n$, but the sum can be anywhere between 0 and m.

The Nash product rule is not directly applicable to ℓ_1 distances as disutilities are measured. However, using the alternative utility representation from Proposition 9.6 circumvents these problems.

Open Problem 10
Investigate the Nash product rule for ℓ_1 preferences.

Similarly, (best response) dynamics can be explored for ℓ_1 preferences. Due to the separability of ℓ_1 preferences,[70] we think that the behavior of such dynamics is closer to the ones for linear utilities than to the ones for Leontief utilities.

Preference-monotonicity for ℓ_1 preferences has already been studied by Freeman et al. (2021) and [4] under the name of (score-)monotonicity. Freeman et al. (2021) proved that all moving phantom mechanism, in particular the independent markets mechanism and the rule which minimizes the sum of ℓ_1 distances, satisfy preference-monotonicity. In [4], we show that all common "coordinate-wise" mechanisms that aggregate scores (based on the agents' peaks) separately for each alternative satisfy preference-monotonicity. Moreover, the *average rule* which returns the vector of average scores $\left(1/m \sum_{i \in N} p_{i,x}\right)_{x \in A}$ is the only coordinate-wise mechanism that satisfies continuity, anonymity, and score-unanimity[71] if $m \geqslant 4$ (see [4]). This indicates that although agents' preferences might be separable, we should not restrict ourselves to mechanisms with similar structures a priori but always have an eye for the bigger picture.

70 Agents receive utility separately from each alternative.
71 A mechanism f satisfies *score-unanimity* if $f(P)_x = \gamma$ for each profile P and alternative x with $p_{i,x} = \gamma$ for all $i \in N$.

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