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Enhancing Non-Linear Force Density Method through Combinatorial Equilibrium Modelling

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Abstract

The Force Density Method (FDM) is an effective form-finding approach for exploring structural forms in equilibrium based on a given topology and under specified boundary conditions and external forces. Its nonlinear extension, the Non-Linear Force Density Method (NLFDM), was developed to enable structures to satisfy user-defined constraints. However, controlling the NLFDM is often challenging due to the difficulty in defining a suitable initial force density set that does not lead to degenerate results. In contrast, the Combinatorial Equilibrium Modelling (CEM) method offers an interactive and intuitive form-finding process for discrete networks, which offers a relatively high level of stability for designing mixed tension-compression structures. This paper reviews NLFDM and CEM and introduces an adaptive form-finding workflow for constrained discrete networks in static equilibrium that combines the two methods. CEM is employed to generate the initial structure and the force density set to be used as an input for NLFDM, thereby enhancing the controllability of the overall form-finding process; NLFDM ensures the speed and accuracy of the constraint-based optimization process. A case study is used to illustrate the proposed form-finding workflow.

Keywords: Form-finding, Non-Linear Force Density Method, Combinatorial Equilibrium Modelling, initial force density set, constraint-based structural optimization

1. Introduction

The form-finding process allows the generation of new structural forms using either physical or digital models (Boller and D'Acunto [1]). This process entails determining the equilibrium shape of a structure for a given input topology and specified boundary conditions and applied loads. Different ways to solve the equilibrium problem for discrete networks lead to different form-finding methods within the realm of digital form-finding. These methods can be broadly classified into three categories: stiffness matrix methods, geometric stiffness methods, and dynamic equilibrium methods (Veenendaal and Block [2]). To satisfy user-defined constraints (e.g., the length or force magnitude in specific members or reaction forces at the supports), these methods are extended by integrating optimization processes. This paper focuses on two geometric stiffness methods, namely the Force Density Method (FDM) (Schek [3]) with its extension, the Non-Linear Force Density Method (NLFDM) (Schek [3], Malerba et al. [4], Aboul-Nasr and Mourad [5]), and the Combinatorial Equilibrium Modelling (CEM) method (Ohlbrock and D'Acunto [6], Ohlbrock et al. [7]), aiming to develop an adaptive form-finding workflow for constrained discrete networks.

By introducing the notion of force density, representing the force-to-length ratio, the FDM (Schek [3]) addresses the form-finding of discrete networks by solving a set of linear equations. For a given topological diagram $\boldsymbol{T_F}$ (Figure 1 left), which provides the connectivity of different members, vertices are categorized into two groups: fixed vertices V_f and free vertices V_v. Connections between vertices are represented as edges E. Based on $\boldsymbol{T_F}$, given the coordinates vector $\boldsymbol{C_f}$ of the fixed vertices V_f and the external forces matrix $\boldsymbol{P_v}$ applied to the free vertices V_v, the FDM generates the structural form $\boldsymbol{F_F}$ (Figure 1 right) after assigning an initial force density set $\boldsymbol{q}$ to all edges E. The Non-Linear Force Density Method (NLFDM) (Schek [3], Malerba, et al. [4], Aboul-Nasr and Mourad [5]) is used when user-defined constraints, like lengths and force magnitudes of the edges, are taken into consideration. NLFDM deals with multi-constraints form-finding problems with the help of a gradient-based optimization algorithm. The speed and accuracy of this method are guaranteed by directly employing the exact mathematical expression for gradient-based optimization calculations.

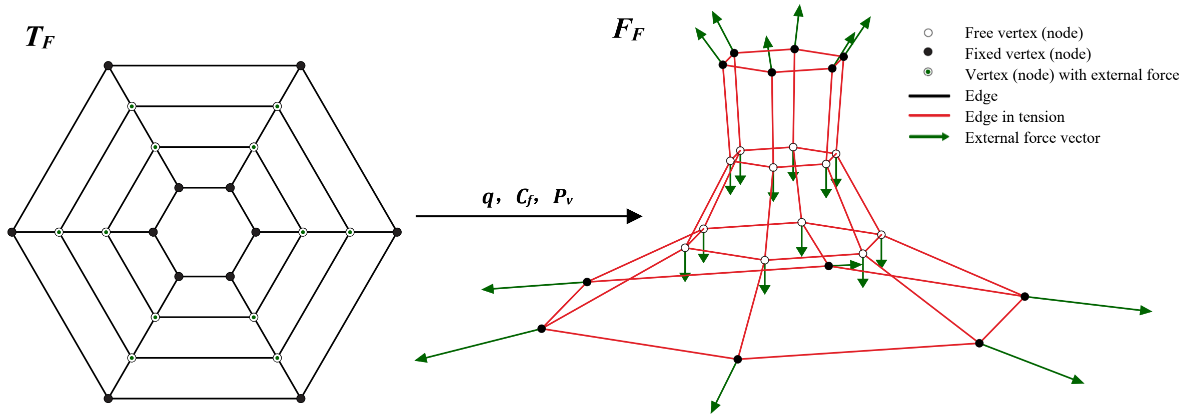


Figure 1: FDM topological diagram of a truncated pyramid-like structure (left); result of the form-finding process based on the FDM (right)

In contrast, CEM (Ohlbrock and D'Acunto [6], Ohlbrock et al. [7]) introduces additional information on the topological diagram $\boldsymbol{T_C}$ (Figure 2 left) to guide the form-finding process for discrete networks. In $\boldsymbol{T_C}$, vertices are categorized into three types: origin vertices $\mathbf{V_o}$, normal vertices $\mathbf{V_n}$, and support vertices $\mathbf{V_s}$. All edges **E** are separated into trail edges $\mathbf{E_t}$ and deviation edges $\mathbf{E_d}$, and the user can freely assign the combinatorial state (tension or compression) of their internal forces. The polylines that originate from an origin vertex and terminate at a support vertex are called trails. The segments of trails are trail edges $\mathbf{E_t}$. Deviation edges $\mathbf{E_d}$ link vertices on different trails. Based on $\boldsymbol{T_C}$, under the external force vector $\boldsymbol{P}$ on all vertices, the structural form $\boldsymbol{F_C}$ (Figure 2 right) is constructed sequentially from the position $\boldsymbol{C_o}$ of the origin vertices $\mathbf{V_o}$ by assigning the trail length $\boldsymbol{\lambda}$ and deviation force magnitude $\boldsymbol{\mu}$ for the $\mathbf{E_t}$ and $\mathbf{E_d}$, respectively. As the form-finding process of CEM follows a linear sequence from the $\mathbf{V_o}$ to the $\mathbf{V_s}$, the form-finding process presents a high level of stability, especially for a complex combination of tension-compression internal forces. Two extensions for CEM have been developed to solve structures with user-defined constraints. One extension uses the finite difference approximation method to calculate the gradient and then employs a gradient-based local optimization algorithm for optimization (Ohlbrock et al. [7]). While this method is straightforward to implement, it can suffer from numerical instability because the choice of step size can significantly influence the accuracy of gradients. To solve this problem, the second extension (Pastrana et al. [8]) utilizes the Automatic Differentiation (AD) (Baydin et al. [9]) method for gradient calculations, resulting in a quicker and more precise optimization process.

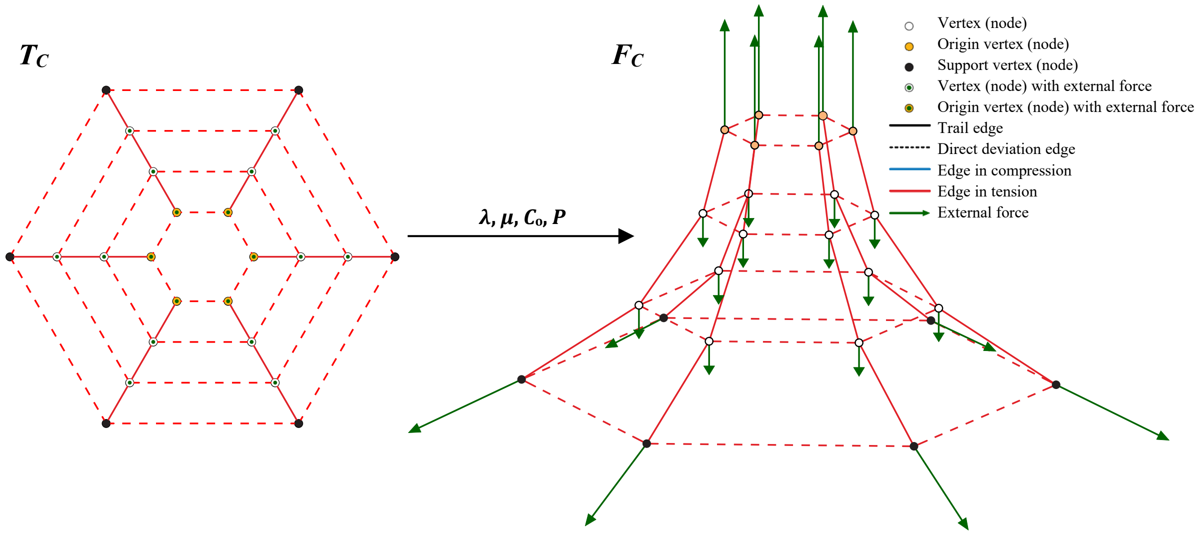


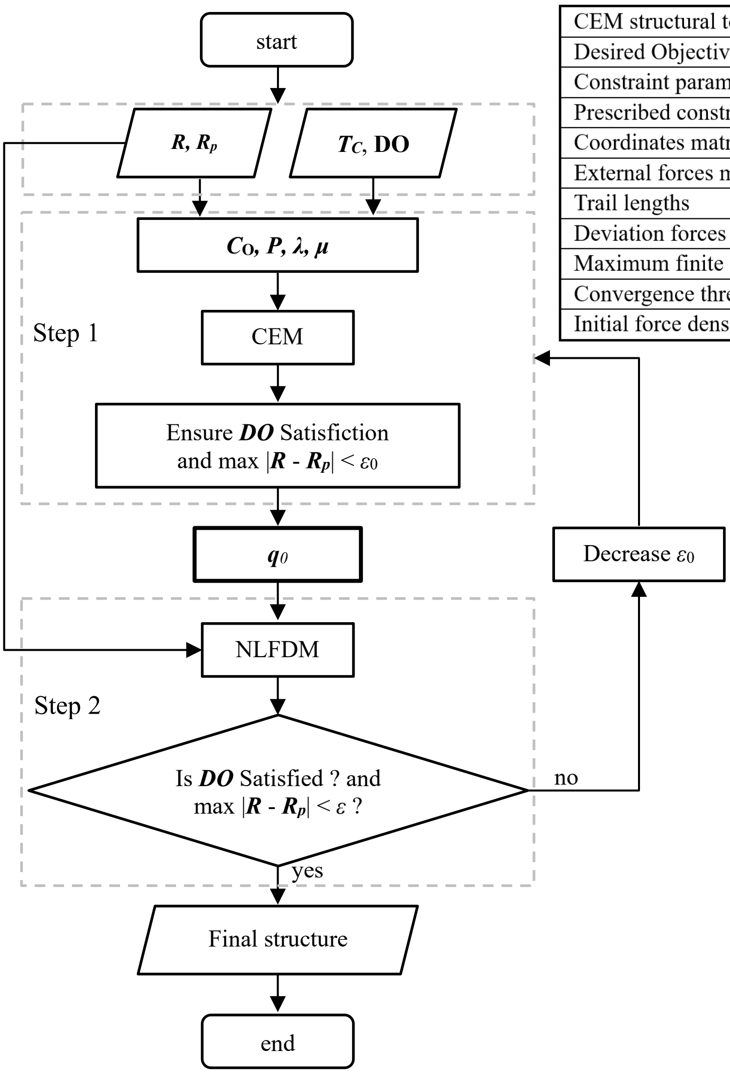
Figure 2: CEM topological diagram of a truncated pyramid-like structure (left); result of the form-finding process based on the CEM (right)

NLFDM and extended CEM framework are effective form-finding methods for constrained discrete networks. However, neither NLFDM nor the extended CEM framework has achieved a fast and accurate optimization process that is also easily controllable by the user. Although the use of symbolic differentiation (SD) (Baydin et al. [8]) has enhanced the speed and accuracy of NLFDM, designers might face challenges in defining an initial set of force densities that meets user expectations for subsequent NLFDM control. This difficulty arises from the abstract concept of force density and the highly non-linear correlation between the generated structural forms and the initial force density set. In the extended CEM framework, while CEM provides an intuitive and interactive form-finding process for defining the initial structure, CEM's sequential and iterative form-finding process does not make form-finding particularly computationally efficient, especially compared to NLFDM. Given the complementary characteristics of NLFDM and CEM, this research develops a controllable, fast, and accurate form-finding workflow for constrained discrete networks by integrating these two methods.

2. Method

2.1. Workflow

The proposed form-finding workflow that combines CEM and NLFDM is illustrated in Figure 3. The design requirements consist of two categories: 1. constraints (e.g., the length of the member, the force of the member, and the reaction force acting on the fixed nodes), which can be satisfied exactly, and 2. desired objectives (e.g., a reasonable shape to avoid degenerate solution) which depend on the designer's needs. Based on the given structural topology $\boldsymbol{T_c}$, desired objectives ***DO***, constraint parameters ***R*****,** and the corresponding prescribed values $\boldsymbol{R_p}$, the form-finding process is as follows: in the first step, the designer manually adjusts the input parameters directly in CEM to generate an initial structure that aligns with ***DO*** and ensure the constraint parameters ***R*** reach the corresponding prescribed values $\boldsymbol{R_p}$ within a user-defined maximum finite difference value ε_0 between ***R*** and $\boldsymbol{R_p}$. Then, the force density set $\boldsymbol{q_0}$ computed from this initial structure serves as the input for NLFDM. In the second step, NLFDM is applied to ensure the structure satisfies the constraints precisely. If the final structure fails to fulfill the ***DO*** (such as substantial changes in overall form) or if NLFDM does not achieve constraint satisfaction within the convergence threshold ε, which is nearly zero, the designer needs to redefine a smaller ε_0 to bring the initial structure closer to the fulfillment of the constraints. Following this adjustment, the above steps are repeated iteratively until a solution is found or convergence is achieved.



CEM structural topological diagram	$\boldsymbol{T_C}$
Desired Objectives	***DO***
Constraint parameters	$\boldsymbol{R}$
Prescribed constraint values	$\boldsymbol{R_p}$
Coordinates matrix of Origin nodes	$\boldsymbol{C_O}$
External forces matrix	$\boldsymbol{P}$
Trail lengths	$\boldsymbol{\lambda}$
Deviation forces magnitude	$\boldsymbol{\mu}$
Maximum finite difference value	ε_0
Convergence threshold	ε
Initial force density set	$\boldsymbol{q_0}$

Figure 3: The proposed form-finding workflow based on the combination of CEM and NLFDM

2.2. Optimization method and constraints

In local optimization problems, the choice of initial values and the method for gradient calculations are crucial factors that significantly influence the optimization process's controllability, speed, and accuracy. In this workflow, CEM visualizes the manual process of searching for an appropriate initial force density set while aligning with the desired design objectives by adjusting its concrete input parameters, thus ensuring great control over the entire process. NLFDM enhances the speed and accuracy of the optimization process by utilizing exact mathematical expressions to compute gradients. As in NLFDM-related research, Newton's method with quadratic convergence speed is adopted in this workflow. At every iteration, the variation of the optimization variables is calculated by minimizing the square of the increment (Schek [3]).

If requested by the design problem, constraint planes can be introduced in the initial CEM form-finding process to anchor nodes onto designer-defined planes. To maintain these constraints in the final structure, these are introduced into NLFDM in compliance with Newton's method. As shown by Schek [3], the constrained problems are solved using Newton's method to find the roots of Equation 1. As a combination of coordinate constraints, the plane or line constraint is introduced into NLFDM using A_i, B_i, C_i, and D_i as the combination coefficients. The plane constraint function is represented by Equation 2. The gradient of this function with respect to $\boldsymbol{q}$ is obtained using Equation 3 in which $\frac{\partial x_i}{\partial \boldsymbol{q}}$, $\frac{\partial y_i}{\partial \boldsymbol{q}}$, $\frac{\partial z_i}{\partial \boldsymbol{q}}$ are provided by Aboul-Nasr and Mourad [5].

$$\boldsymbol{g} = \boldsymbol{R} - \boldsymbol{R_p} = \boldsymbol{0} \tag{1}$$

$$g_i = A_i x_i + B_i y_i + C_i z_i + D_i = 0 \tag{2}$$

$$\frac{\partial g_i}{\partial \boldsymbol{q}} = A_i \frac{\partial x_i}{\partial \boldsymbol{q}} + B_i \frac{\partial y_i}{\partial \boldsymbol{q}} + C_i \frac{\partial z_i}{\partial \boldsymbol{q}} \tag{3}$$

where g_i, represents the *i*th constraint equation in $\boldsymbol{g}$ corresponding to the *i*th node, with coordinates x_i, y_i, and z_i, respectively, lying in the *i*th plane in which plane coefficients are A_i, B_i, C_i, and D_i, respectively.

3. Implementation: a spiral staircase

Figure 4 shows a topological diagram and boundary conditions of a staircase, which is taken here as a case study to illustrate the proposed form-finding workflow. The project's objective is to design a staircase reaching a height of 3 meters, consisting of 20 steps, each measuring 0.15 meters in height and 1.0 meters in width. The external force applied on each normal vertex is 1kN. Besides, the footprint area of the staircase is limited to the surface Π (Figure 4(b)). Based on this design requirements, a spiral staircase configuration is employed to ensure an elegant and efficient design.

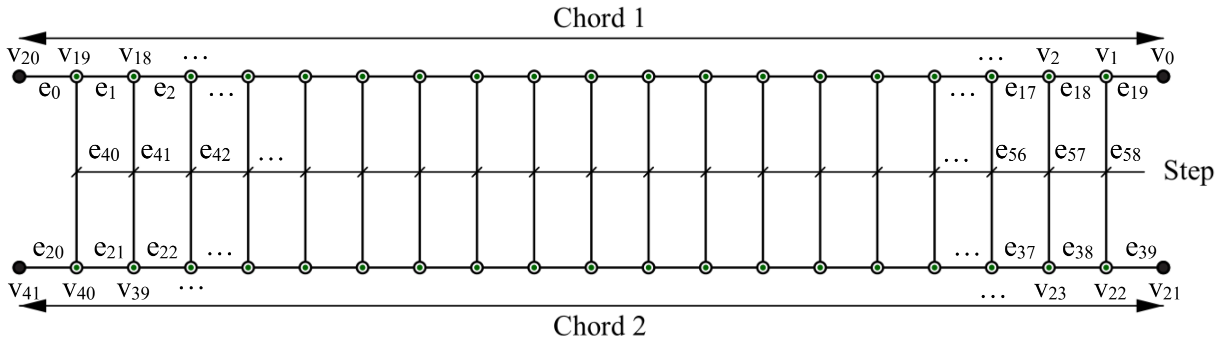
3.1. Form-finding using CEM + NLFDM

Using the proposed form-finding workflow, the form of the staircase is obtained through the following steps: initially, the CEM topology (Figure 5a) of this staircase is established by labeling Chord 1, Chord 2, and the step edges as tension trail edges, compression trail edges, and compression direct deviation edges, respectively. According to the design requirements, the constraints parameters are separated into 3 groups: $\boldsymbol{R_z}$ denotes the *z* coordinates of the nodes of the steps, $\boldsymbol{R_l}$ is the length of the step edges, and $\boldsymbol{R_c}$ is the fixed nodes' position. Since each step needs to be horizontal and 0.15 meters high, the corresponding prescribed values $\boldsymbol{R_{zp}}$ for $\boldsymbol{R_z}$ are obtained. $\boldsymbol{R_l}$ has the prescribed value $\boldsymbol{R_{lp}}$, which is a vector of ones since each step is 1.0m in width. The design coordinates $\boldsymbol{R_{cp}}$ for $\boldsymbol{R_c}$ are shown in Figure 4b. By adjusting the input parameters of the CEM layer by layer, the CEM sequentially builds the overall structure over Π (Figure 5b). The coordinates $\boldsymbol{C_O}$ of origin nodes, the external force $\boldsymbol{P_O}$ applied to original nodes, and the deviation force magnitude $\boldsymbol{\mu}$ are listed in Table 1, Table 2, and Table 3, respectively. Since CEM can get the correct trail length $\boldsymbol{\lambda}$ based on simple geometric calculations when nodes are limited to certain planes (Ohlbrock and D'Acunto [6]), the constraint planes for all steps can be directly applied to the nodes. Subsequently, the derived force density set (Figure 5c) from this structure is utilized as the initial set for NLFDM, to which the plane and length constraints (Schek [3]) for each step are applied. Newton's method is the optimization algorithm with a convergence threshold ε set at 1×10^{-6}. The optimization process concludes in 56 milliseconds on an Intel i9-10900K CPU. Figure 5d displays the result, confirming compliance with design constraints while retaining a form closely aligned with the initial structure.

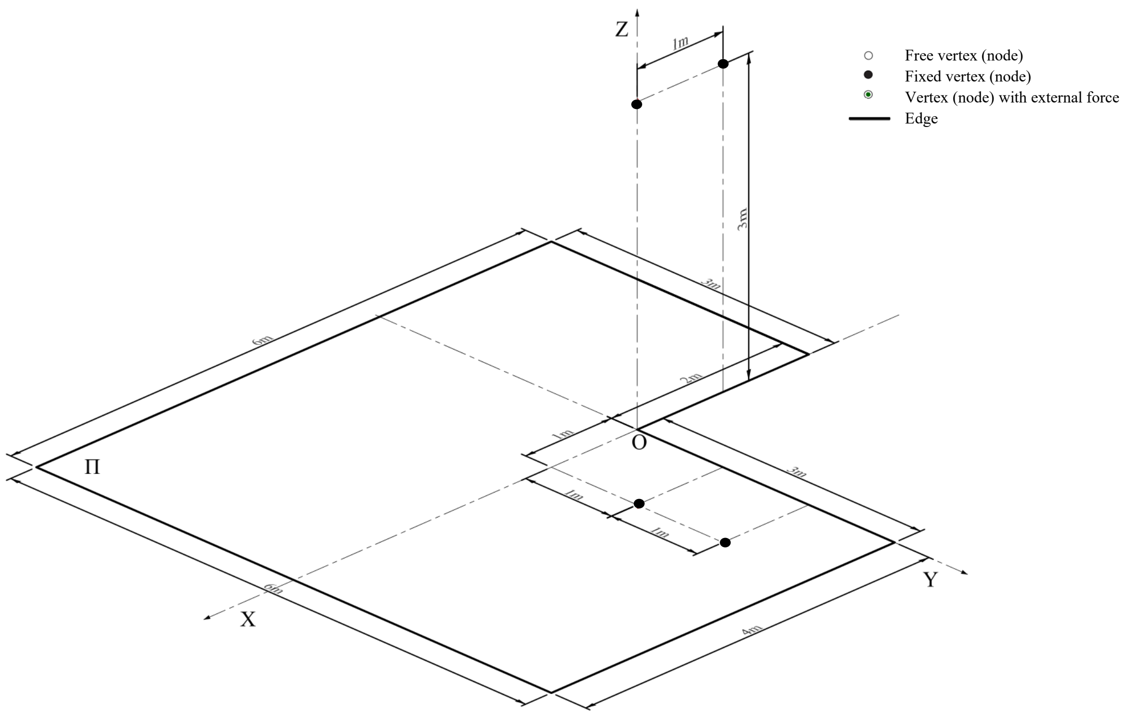
3.2. Form-finding using NLFDM alone

A second experiment is carried out to compare the proposed workflow with a conventional one that employs NLFDM only. Within the NLFDM form-finding process, adjusting the force density value in each edge individually to achieve the desired form poses a considerable challenge. Therefore, as shown in Figure 4 (a), all edges are categorized into three groups (Chord 1, Chord 2, Step) according to their respective static role in the staircase. The same force density value is uniformly applied to all edges in each group for form-finding purposes. Table 4 lists four initial force density sets for NLFDM related to four case studies. The convergence threshold ε is set to 1×10^{-6}. The plane constraint and length constraint are applied in this process. The corresponding initial and final structures are depicted in Figure 6.

Setting the force density in Chord 1 to 30 kN/m (Case 1) results in structures that do not fulfill the design objectives (Figures 6a and 6b). Increasing the force density in Chord 1 from 30 kN/m to 50 kN/m (Case 2) brings about a complete change in the resulting form, demonstrating significant non-linearity of the method. However, when the force density in Chord 1 is further increased from 50 kN/m to 70 kN/m (Case 3), the initial and final structures shift to the opposite side (Figures 6e and 6f). Therefore, identifying suitable initial force density values in FDM for the three groups of edges to meet designers' requirements poses a challenge. Only the initial force density set of Case 4 can yield a reasonable final result. In summary, utilizing CEM to search for the initial force density value is quick and efficient, unlike trial-and-error adjustments to different force density sets as in NLFDM alone. Furthermore, while grouping edges can greatly simplify and speed up searching for a reasonable form compared to individual adjustment of each edge's force density, designers might miss numerous opportunities to explore novel design solutions.



(a) Topological diagram of the staircase: edges are grouped into three categories: Chord 1, Chord 2, and Step



(b) Boundary conditions of the staircase

Figure 4: Configuration of the staircase

Table 1: Positions ***Co*** of origin nodes [m]

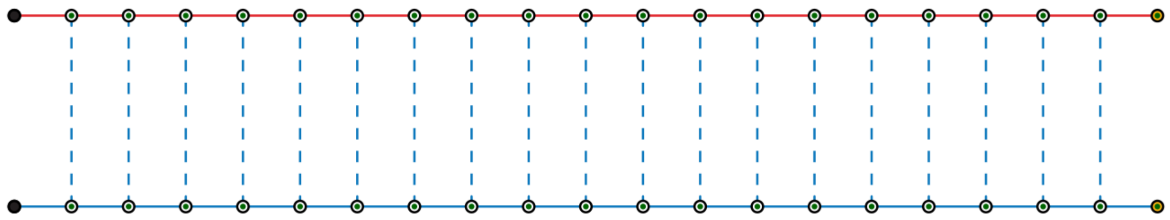
	C_x	C_y	C_y
v_0	-1.00	0.00	3.00
v_{21}	0.00	0.00	3.00

Table 2: External forces ***Po*** applied to origin nodes [kN]

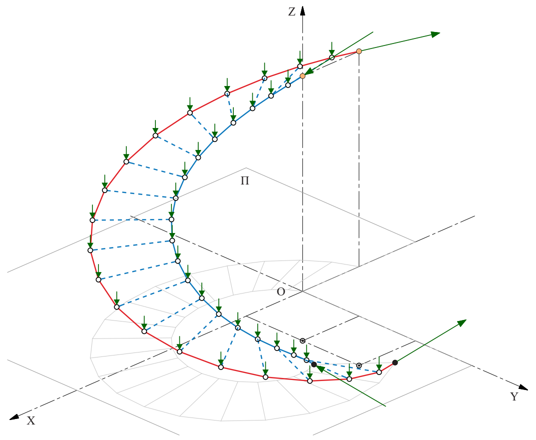
	P_x	P_y	P_z
v_0	142.4	63.6	65.1
v_{21}	-114.5	-67.4	-104.6

Table 3: Deviation forces $\boldsymbol{\mu}$ [-kN]

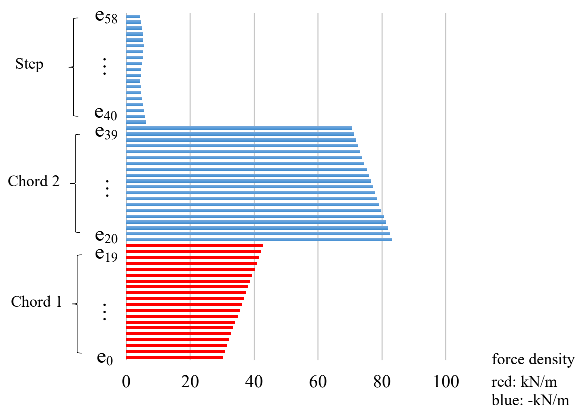
e_{58}	e_{57}	e_{56}	e_{55}	e_{54}	e_{53}	e_{52}	e_{51}	e_{50}	e_{49}	e_{48}	e_{47}	e_{46}	e_{45}	e_{44}	e_{43}	e_{42}	e_{41}	e_{40}
39.1	40.2	41.2	42.3	43.4	44.5	45.5	46.6	47.7	48.8	49.8	50.9	52.0	53.0	54.1	55.2	56.3	57.3	58.4



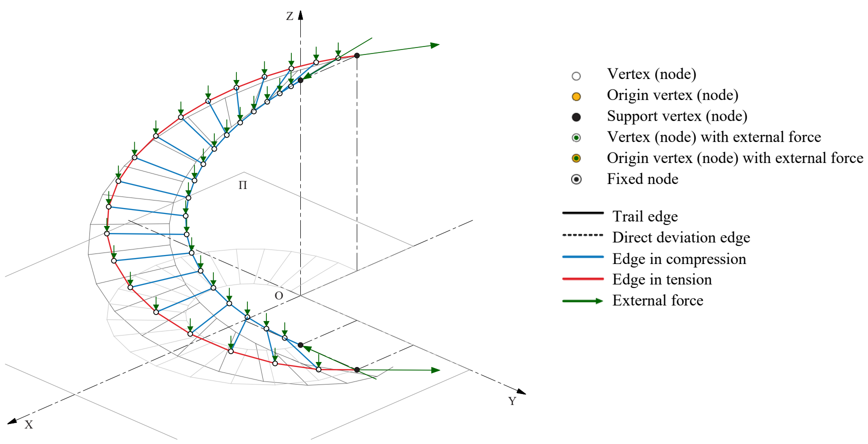
(a) CEM topology of the staircase



(b) Initial structural form generated by CEM



(c) Force density distribution in the initial structure

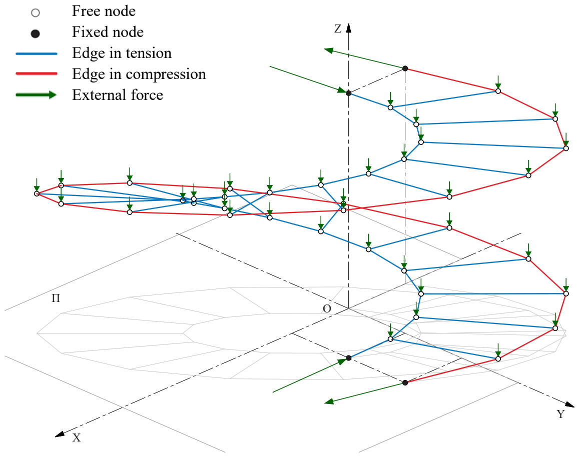


(d) Final structural form optimized by NLFDM

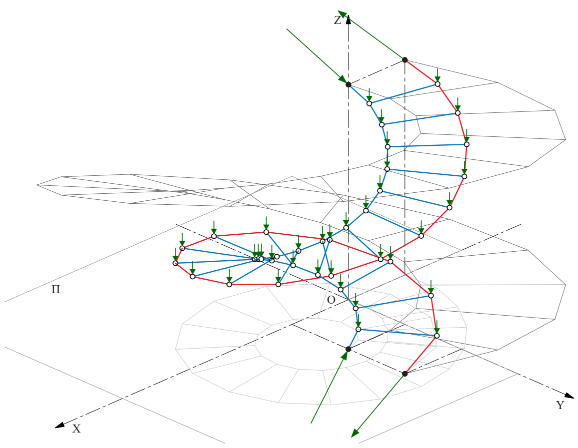
Figure 5: Form-finding based on the proposed workflow

Table 4: Four initial force density sets for NLFDM [kN/m]

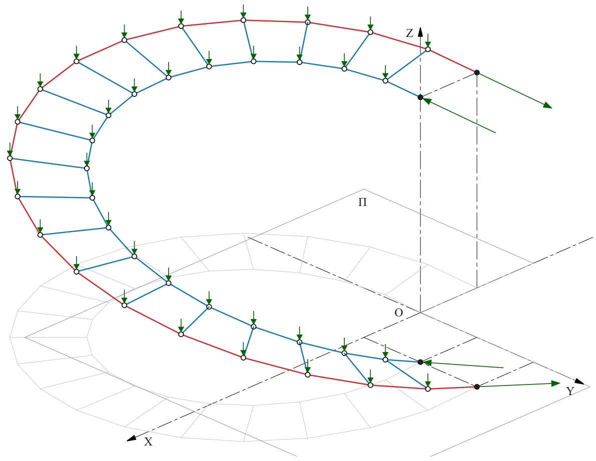
	Chord 1	Chord 2	Step
Case 1	30	-70	-10
Case 2	50	-70	-10
Case 3	70	-70	-10
Case 4	40	-70	-5



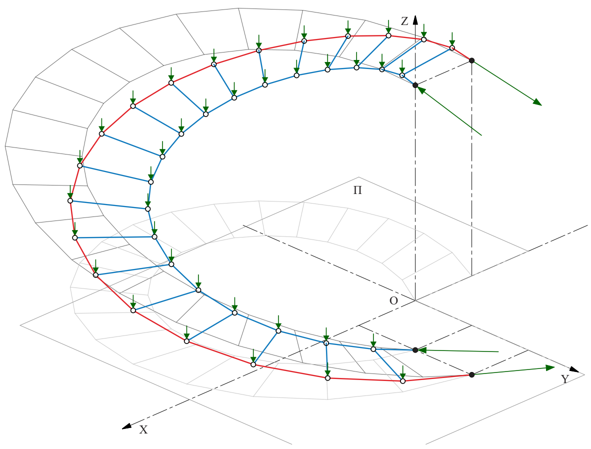
(a) Case 1: initial structural form



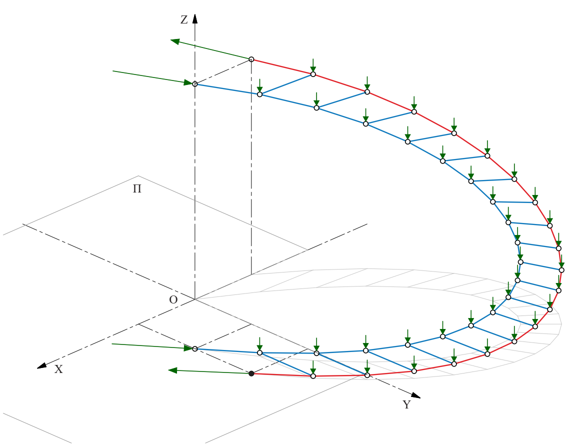
(b) Case 1: final structural form



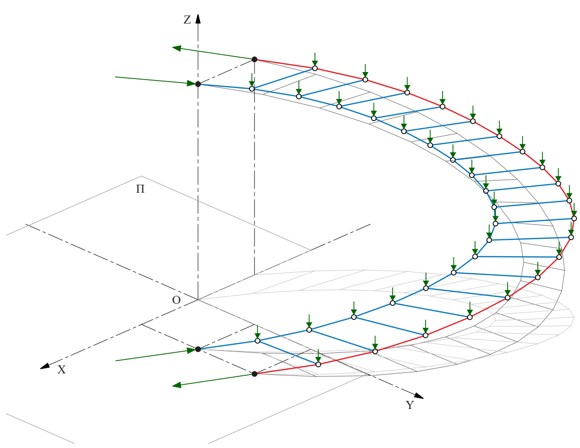
(c) Case 2: initial structural form of Case 2



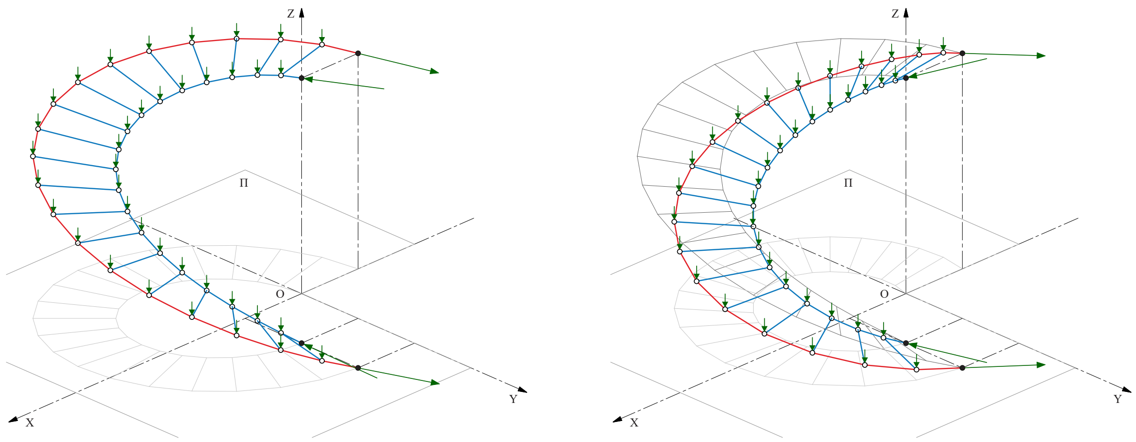
(d) Case 2: final structural form



(e) Case 3: initial structural form



(f) Case 3: final structural form



(g) Case 4: initial structural form

(h) Case 4: final structural form

Figure 6: Four initial structural forms and corresponding NLFDM form-finding results

3.3. Form-finding using CEM alone

For a comparative analysis between NLFDM and the extended CEM framework in which AD is used for gradient calculations, the same initial structure (Figure 4 a) is used as the starting point of the optimization process. In addition to those constraints applied in NLFDM, node position constraints need to be added to the extended CEM framework to ensure CEM's support nodes reach fixed positions. A convergence threshold of 1×10^{-6} is set, with SLSQP (Kraft [10]) serving as the optimization algorithm. The result, comparable to the one in Figure 4, is shown in Figure 7. Convergence is achieved within 17.5 seconds, with a threshold of 1×10^{-2} rather than 1×10^{-6}. In contrast to the proposed form-finding workflow, this process is considerably slower and less accurate, indicating NLFDM's efficiency as an optimization method for CEM.

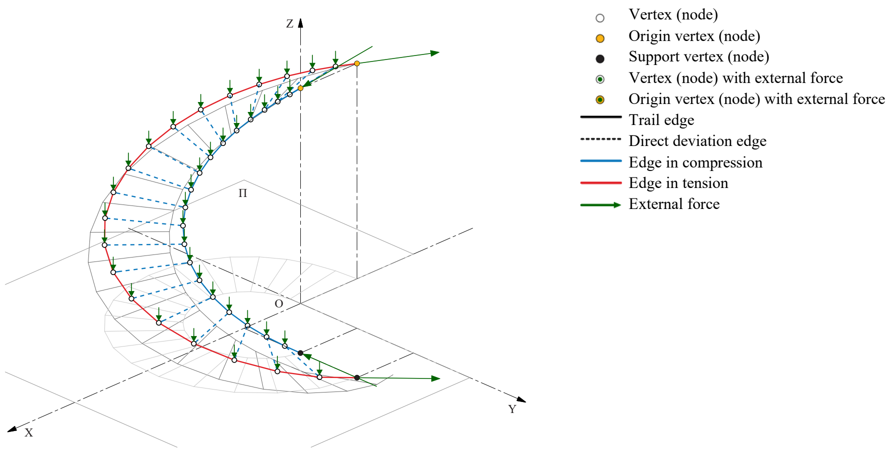


Figure 7: Result of the form-finding based on the extended CEM framework

4. Conclusion

This paper presented an adaptive form-finding workflow for constrained discrete networks in the conceptual design phase based on CEM and NLFDM. In this workflow, CEM is used to generate a user-expected initial structure. Then, NLFDM ensures precise satisfaction of the constraints by utilizing the force density set computed from this initial structure as the initial values. Compared to the NLFDM form-finding process alone, this workflow offers greater controllability as CEM serves as the initial structure generator to align with the design requirements. At the same time, NLFDM tends to be faster and more accurate than the extended CEM alone. As a result, this adaptive form-finding workflow facilitates a controllable, rapid, and accurate form-finding process for constrained discrete networks.

By introducing new optimization constraints, targets, or algorithms, this workflow can be enhanced in terms of integrity, automation, and controllability. In this context, implementing new constraints to guarantee the smoothness of the result would be one possible improvement.

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