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Capacity Theorems for Wireless Relay Channels

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Abstract

An achievable rate region for memoryless relay networks is developed based on an existing region for additive white Gaussian noise (AWGN) channels. It is shown that multi–hopping achieves the information–theoretic capacity of wireless relay networks if the relays are in a region near the source terminal, and if phase information is available at the receivers only.

1 Introduction

Network information theory gives the ultimate limits on the performance of communication systems. Although there are still many gaps in our understanding of this theory, the existing results do offer a rich variety of coding strategies that can in principle give large improvements over current schemes. The purpose of this paper is to show that the existing information theory gives the ultimate strategies for certain *relay* networks that might occur in practice.

An information–theoretic model for relay channels was introduced and studied in [1, 2]. Two fundamental coding strategies for a single relay were developed in [3]. These strategies achieve capacity for several classes of channels, as discussed in [3]-[5]. However, none of these classes models a *wireless* environment, and we are primarily interested in attacking the wireless problem.

Our starting point will be the strategy of Theorem 1 in [3] that uses block Markov superposition encoding. This approach was recently generalized to multiple relays by using *multi–hopping* [6]. A simpler coding scheme than in [3] was introduced in [7] by using *regular* block Markov encoding and *backwards* decoding. The regular encoding concept was recently applied to multiple relay channels with additive white Gaussian noise (AWGN) in [8]. However, [8] replaced backwards decoding with a more practical *windowed* decoding. We apply the two concepts of *regular* block Markov encoding and *windowed* decoding to memoryless relay networks in order to generalize the approach of [8] to wider classes of relay networks.

This paper is organized as follows. In Section 2 we define the network model and review a capacity upper bound. In Section 3 we review several multi-hopping strategies for a single relay. We further derive an information-theoretic achievable rate region by modifying the scheme of [8] so as to apply to wider classes of channels. Section 4 gives numerical examples for Gaussian wireless networks, including capacity results when there is phase uncertainty at the transmitters. Section 5 concludes the paper. Note that other recent studies on relaying can be found in [9]–[15].

2 Preliminaries

2.1 Model

The T–terminal relay network has four types of random variables (see [1, 16, 17]): the message W, the channel inputs X_{ti}, $t = 1, 2, \ldots, T-1$, $i = 1, 2, \ldots n$, the channel outputs Y_{ti}, $t = 2, 3, \ldots, T$, $i = 1, 2, \ldots n$, and the message estimate $\hat{W}$. The source terminal (terminal 1) transmits X_{1i} that are a function of W, and the relays use causal coding functions, i.e., relay t's input X_{ti} is a function of its past outputs $Y_t^{i-1} = (Y_{t1}, Y_{t2}, \ldots, Y_{t(i-1)})$. Suppose the relay network is *memoryless* and *time invariant*. This lets one focus attention on the *channel* distribution

$$p(y_2, \ldots, y_T | x_1, \ldots, x_{T-1}) \tag{1}$$

for further analysis. Finally, the destination terminal (terminal T) computes $\hat{W}$ as a function of the channel outputs $Y_T^n = (Y_{T1}, \ldots, Y_{Tn})$.

Suppose that W has B_W bits so that the data rate is $R = B_W/n$ bits per channel use. The *capacity* C of the relay network is the supremum of rates at which reliable communication is possible, i.e., the rates where the destination's message estimate $\hat{W}$ can be made to satisfy $\Pr(\hat{W} \neq W) < \epsilon$ for any positive ϵ. Our goal is to construct coding strategies that communicate reliably at rates close to C.

2.2 Capacity Upper Bound

A capacity upper bound is given by the min–cut bound in [18, p. 445]. Let $\mathcal{T} = \{2, 3, \ldots, T-1\}$ be the set of relay indexes, and let $X_{\mathcal{S}} = \{X_t : t \in \mathcal{S}\}$ for $\mathcal{S} \subseteq \mathcal{T}$. We have the following result.

Proposition 1 *The T–terminal relay network capacity satisfies*

$$C \leq \max_{p(x_1, x_2, \ldots, x_{T-1})} \min_{\mathcal{S} \subseteq \mathcal{T}} I(X_1 X_{\mathcal{S}}; Y_{\mathcal{S}^C} Y_T \mid X_{\mathcal{S}^C}) \tag{2}$$

where $\mathcal{S}^C$ is the complement of $\mathcal{S}$ in $\mathcal{T}$.

As done here, we adopt the notation of [19, Ch. 2] for mutual information and entropies. Note that the above minimization is over 2^{T-2} bounds. For example, for $T = 3$ we have

$$C \leq \max_{p(x_1, x_2)} \min \left\{ I(X_1; Y_2 Y_3 | X_2),\ I(X_1 X_2; Y_3) \right\}. \tag{3}$$

The minimization quickly becomes difficult to evaluate as T increases. One therefore usually considers only a few of the mutual informations.

Block 1	Block 2	Block 3	Block 4
$\underline{x}_1(1, w_1)$	$\underline{x}_1(w_1, w_2)$	$\underline{x}_1(w_2, w_3)$	$\underline{x}_1(w_3, 1)$
$\underline{x}_2(1)$	$\underline{x}_2(w_1)$	$\underline{x}_2(w_2)$	$\underline{x}_2(w_3)$

Figure 1: Regular block Markov encoding for the single–relay network.

3 Multi–hopping Strategies

3.1 Single Relay

We interpret the strategy of Cover and El Gamal [3, Thm. 1] as a multi–hopping strategy. This strategy achieves any rate up to

$$R_{MH} = \max_{p(x_1,x_2)} \min \left\{ I(X_1; Y_2|X_2),\ I(X_1X_2; Y_3) \right\}. \tag{4}$$

The only difference between (3) and (4) is that Y_3 is included in the first information on the right side of (3).

The rate (4) has in the past been achieved with three different methods. The first is the strategy of [3, Thm. 1] that uses block Markov superposition encoding, random partitioning (binning) and successive decoding. The encoding is done using codebooks of different sizes, and we call this *irregular* block Markov encoding. The second method is a strategy of Willems [7, Ch. 7] that uses block Markov superposition encoding combined with *backwards decoding*. However, the encoding is now done with codebooks of the same size, and we call this *regular* block Markov encoding. The technique is illustrated in Fig. 1, and we proceed to describe its operation.

The message w is divided into B blocks $w_1, w_2, \ldots, w_B$ of 2^{nR} bits each. The transmission is performed in $B+1$ blocks by using codewords $\underline{x}_1(i,j)$ and $\underline{x}_2(i)$ of length n, where i and j range from 1 to 2^{nR}. In the first block, terminal 1 transmits $\underline{x}_1(1, w_1)$ and terminal 2 transmits $\underline{x}_2(1)$. Random coding arguments guarantee that terminal 2 can decode reliably as long as n is large and

$$0 \le R < I(X_1; Y_2|X_2). \tag{5}$$

So suppose terminal 2 correctly obtains w_1. Then in the second block, terminal 1 transmits $\underline{x}_1(w_1, w_2)$ and terminal 2 transmits $\underline{x}_2(w_1)$. Terminal 2 can decode w_2 reliably as long as n is large and (5) is true. One continues in this way until block $B+1$. In this last block, terminal 1 transmits $\underline{x}_1(w_B, 1)$ and terminal 2 transmits $\underline{x}_2(w_B)$.

Consider now the destination (terminal 3), and let $\underline{y}_{3b}$ be its bth block of channel outputs. These blocks are collected until the last block of transmission is completed. Terminal 3 then performs *backwards decoding* by first decoding w_B from $\underline{y}_{3(B+1)}$. Note that $\underline{y}_{3(B+1)}$ depends on $\underline{x}_1(w_B, 1)$ and $\underline{x}_2(w_B)$, which in turn depend only on w_B. One can show (see [7, Ch. 7]) that terminal 3 can decode reliably as long as n is large and

$$0 \le R < I(X_1X_2; Y_3). \tag{6}$$

Terminal 3 next decodes w_{B-1} from $\underline{y}_{3B}$ which depends on $\underline{x}_1(w_{B-1}, w_B)$ and $\underline{x}_2(w_{B-1})$. Since terminal 3 knows w_B, it can again decode reliably as long as (6) is true. One continues in this fashion until all message blocks have been decoded. The overall rate is $R \cdot \frac{B}{B+1}$ bits per use, and by making B large we can get the rate as close to R as desired.

Block 1	Block 2	Block 3	Block 4
$\underline{x}_1(1,1,w_1)$	$\underline{x}_1(1,w_1,w_2)$	$\underline{x}_1(w_1,w_2,w_3)$	$\underline{x}_1(w_2,w_3,w_4)$
$\underline{x}_2(1,1)$	$\underline{x}_2(1,w_1)$	$\underline{x}_2(w_1,w_2)$	$\underline{x}_2(w_2,w_3)$
$\underline{x}_3(1)$	$\underline{x}_3(1)$	$\underline{x}_3(w_1)$	$\underline{x}_3(w_2)$

Block 5	Block 6	Block 7	Block 8
$\underline{x}_1(w_3,w_4,w_5)$	$\underline{x}_1(w_4,w_5,w_6)$	$\underline{x}_1(w_5,w_6,1)$	$\underline{x}_1(w_6,1,1)$
$\underline{x}_2(w_3,w_4)$	$\underline{x}_2(w_4,w_5)$	$\underline{x}_2(w_5,w_6)$	$\underline{x}_2(w_6,1)$
$\underline{x}_3(w_3)$	$\underline{x}_3(w_4)$	$\underline{x}_3(w_5)$	$\underline{x}_3(w_6)$

Figure 2: A multi–hopping strategy for the two–relay network.

We remark that the backwards decoding strategy has the disadvantage of incurring a large delay. However, the regular encoding structure is easier to extend to multiple relays than the Cover/El Gamal strategy.

Finally, the third method to achieve (4) is based on a recent strategy of Xie and Kumar [8] that uses regular block Markov encoding, but with the novel idea of *windowed* decoding. Consider again Fig. 1, but suppose terminal 3 decodes w_1 *after block 2* by using a *window* of the two past received blocks $\underline{y}_{31}$ and $\underline{y}_{32}$. One can again show that terminal 3 can decode reliably as long as n is large and

$$0 \leq R < I(X_1 X_2; Y_3). \tag{7}$$

The mutual information (7) has a contribution of $I(X_2;Y_3)$ from $\underline{y}_{32}$, and $I(X_1;Y_3|X_2)$ from $\underline{y}_{31}$. After receiving $\underline{y}_{3b}$, $b \geq 3$, Terminal 3 similarly decodes w_{b-1} by using $\underline{y}_{3(b-1)}$, $\underline{y}_{3b}$, and its past message estimate $\hat{w}_{b-2}^{(3)}$ (which is assumed to be w_{b-2}). The overall rate is again $R \cdot \frac{B}{B+1}$ bits per use, and by making B large we can get the rate as close to R as desired.

Note that the above strategy enjoys the advantages of both the Cover/El Gamal strategy (two block decoding delay) and the Willems strategy (regular block Markov encoding). Furthermore, it is easy to extend windowed decoding to multiple relays, and even to multiple sources.

3.2 Multiple Relays

A natural first approach to multi–hop with several relays is to generalize the Cover–El Gamal strategy. This was done in [6]. However, we here wish to consider the improved strategy of [8], and we generalize this strategy to apply to wider classes of channels. Our generalization is easily modified to apply to certain relay networks with multiple sources (see [20]).

Consider two relays. We divide the message w into B blocks of 2^{nR} bits each. The transmission is performed in $B+2$ blocks by using $\underline{x}_1(i,j,k)$, $\underline{x}_2(i,j)$, and $\underline{x}_3(i)$, where i, j, k range from 1 to 2^{nR}. For example, the encoding for $B = 6$ is depicted in Fig. 2. Terminal 2 can reliably decode w_b after transmission block b if n is large, its past message estimates $\hat{w}_{b-2}^{(2)}, \hat{w}_{b-1}^{(2)}$ were correct, and

$$0 \leq R < I(X_1; Y_2 | X_2 X_3). \tag{8}$$

Terminal 3 decodes w_{b-1} by using $\underline{y}_{3(b-1)}$, $\underline{y}_{3b}$, and $\hat{w}_{b-2}^{(3)}$, $\hat{w}_{b-3}^{(3)}$. This can be done reliably if n is large, if $\hat{w}_{b-2}^{(3)} = w_{b-2}$, $\hat{w}_{b-3}^{(3)} = w_{b-3}$, and if

$$0 \leq R < I(X_1 X_2; Y_3 | X_3). \tag{9}$$

The mutual information (9) has a contribution of $I(X_2; Y_3|X_3)$ from $\underline{y}_{3b}$ and $\hat{w}_{b-2}^{(3)}$, and a contribution of $I(X_1; Y_3|X_2 X_3)$ from $\underline{y}_{3(b-1)}$, $\hat{w}_{b-2}^{(3)}$, and $\hat{w}_{b-3}^{(3)}$. Assuming correct decoding, Terminal 3 knows w_{b-1} after transmission block b, and can encode the messages as shown in Fig. 2.

Finally, terminal 4 decodes w_{b-2} by using $\underline{y}_{4(b-2)}$, $\underline{y}_{4(b-1)}$, $\underline{y}_{4b}$ and $\hat{w}_{b-4}^{(4)}$, $\hat{w}_{b-3}^{(4)}$. This can be done reliably if n is large, the past message estimates were correct, and

$$0 \leq R < I(X_1 X_2 X_3; Y_4). \tag{10}$$

This mutual information has a contribution of $I(X_3; Y_4)$ from $\underline{y}_{4b}$, $I(X_2; Y_4|X_3)$ from $\underline{y}_{4(b-1)}$ and $\hat{w}_{b-3}^{(4)}$, and $I(X_1; Y_4|X_2 X_3)$ from $\underline{y}_{4(b-2)}$, $\hat{w}_{b-3}^{(4)}$ and $\hat{w}_{b-4}^{(4)}$. The overall rate is $R \cdot \frac{B}{B+2}$, so by making B large we can get the rate as close to R as desired.

We remark that one could alternatively use backwards decoding, but then one must transmit using nested blocks to allow the intermediate relays (terminal 3 in Fig. 2) to decode before the destination. The result is an excessive decoding delay. The Xie/Kumar windowed decoding is a much more elegant and practical technique.

It is clear that this strategy generalizes to T–terminal relay networks, and we prove the following theorem. Let $\pi(\cdot)$ be a permutation on $\mathcal{T} = \{2, 3, \ldots, T-1\}$. We further define $\pi(1) = 1$, $\pi(T) = T$, and $\pi(i:j) = \{\pi(i), \pi(i+1), \ldots, \pi(j)\}$.

Theorem 1 *The T–terminal relay network capacity is at least*

$$R_{MH} = \max_{\pi(\cdot)} \min_{1 \leq m \leq T-1} I(X_{\pi(1:m)}; Y_{\pi(m+1)} \mid X_{\pi(m+1:T-1)}) \tag{11}$$

where one can choose any distribution on $(X_1, X_{\mathcal{T}})$.

Theorem 1 is basically due to Xie and Kumar [8]. Although [8] considers only AWGN channels, and hence expresses the rate region in a different form than (11), it was clear that windowed decoding applies to wider classes of relay networks. Note that we have expressed Theorem 1 using only permutations rather than the level sets of [6, 8]. This is because one need consider only permutations to maximize the *rate*. However, we remark that to minimize the *delay* for a given rate, one will need to consider level sets again. This occurs, e.g., if one relay is at the same location as another.

As an application of Theorem 1, for $T = 4$ there are two permutations on $\mathcal{T} = \{2, 3\}$, and the rate (11) is

$$R_{MH} = \max \left\{ \begin{array}{l} \min\{I(X_1; Y_2|X_2 X_3), I(X_1 X_2; Y_3|X_3), I(X_1 X_2 X_3; Y_4)\}, \\ \min\{I(X_1; Y_3|X_2 X_3), I(X_1 X_3; Y_2|X_2), I(X_1 X_2 X_3; Y_4)\} \end{array} \right\}. \tag{12}$$

4 Wireless Examples

We illustrate the rates achieved by Theorem 1 with the following wireless channel. Suppose that at time i terminal t receives the symbol

$$Y_{ti} = Z_{ti} + \sum_{s \neq t} \frac{A_{sti}}{\sqrt{d_{st}^{\alpha}}} X_{si} \tag{13}$$

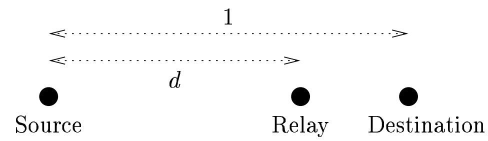

Figure 3: A single relay on a line.

where d_{st} is the distance between terminals s and t, α is an attenuation exponent, A_{sti} is a complex fading random variable, and Z_{ti} is independent and identically distributed (i.i.d.) complex Gaussian noise with zero mean, unit variance, and i.i.d. real and imaginary parts. The X_{si} are complex random variables satisfying the power constraints $\mathrm{E}[|X_{si}|^2] \le P_s$ for all s and i. We will use mainly the free–space attenuation exponent $\alpha = 2$. We further use logarithms to the base 2 so that our rate units are bits per use.

We discuss two kinds of fading:

- No fading: $A_{sti} = 1$ for all s, t, and i.
- Phase fading: $A_{sti} = e^{j\theta_{sti}}$ where θ_{sti} is uniformly distributed over $[0, 2\pi)$. The θ_{sti} are jointly independent of each other and all other random variables. We write $\underline{\theta}_{ti}$ for the vector of phases at terminal t.

For the latter case, we assume that terminal t knows only its *own* fading coefficients. That is, terminal t knows A_{sti} for all s and i, but it does not know $A_{st'i}$ for $t' \neq t$. Note that we are here considering the *fast* (or *ergodic*) fading capacity. One could also develop similar results for *slow* (or *quasi–static*) fading.

4.1 No Fading and One Relay

Suppose we have a single relay with no fading and $\alpha = 2$. Let ρ be the correlation coefficient of X_1 and X_2, and suppose it is a real number. For Gaussian input distributions, we compute

$$R_{MH} = \max_{0 \le \rho \le 1} \min \left\{ \log\left(1 + \frac{P_1}{d_{12}^2}(1 - |\rho|^2)\right), \log\left(1 + \frac{P_1}{d_{13}^2} + \frac{P_2}{d_{23}^2} + \frac{2\rho\sqrt{P_1 P_2}}{d_{13} d_{23}}\right) \right\}. \quad (14)$$

As an example, suppose the source, relay and destination are aligned as in Fig. 3, where $d_{12} = d$, $d_{23} = 1 - d$, and $d_{13} = 1$. Fig. 4 plots various bounds for $P_1 = P_2 = 10$. The curve labeled MH gives the multi–hopping rates. Also shown are the rates when the relay is turned off, and when the strategy of [3, Theorem 6] is used (labeled DP for *destination pooling*). However, only half the power is being consumed when the relay is off as compared to the other cases. Finally, the figure plots rates for the strategy where the relay transmits $X_{2i} = c \cdot Y_{2(i-1)}$, where c is a scaling factor chosen so that $E[|X_{2i}|^2] \le P_2$. This strategy is called "amplify–and–forward" in [13, p. 80], and it turns the relay channel into a unit–memory intersymbol interference (ISI) channel. The curve labeled "amplify–and–forward" shows the capacity of this channel.

4.2 Phase Fading and One Relay

Consider next the case $A_{st} = e^{j\theta_{st}}$ where θ_{st} is known only to terminal t for all s. The result is that R_{MH} in (4) becomes

$$\max_{p(x_1,x_2)} \min \left\{ I(X_1; Y_2 | X_2\, \theta_{12}),\ I(X_1 X_2; Y_3 |\, \theta_{13}\, \theta_{23}) \right\} \quad (15)$$

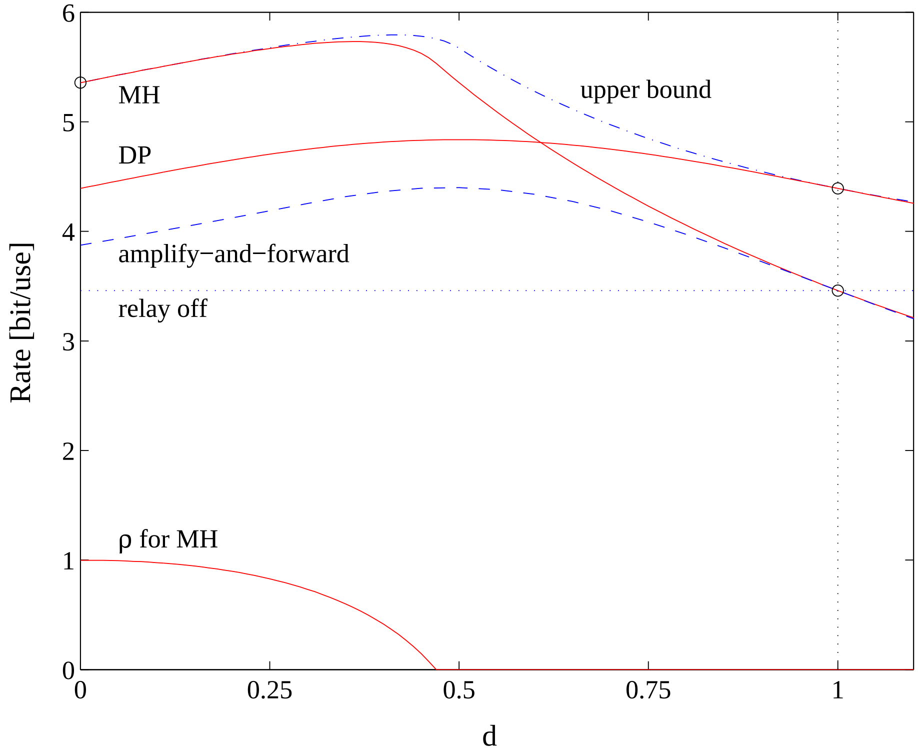

Figure 4: Rates for the single–relay network with $P_1 = P_2 = 10$.

where the θ_{st} appear in the conditioning. We have

$$I(X_1; Y_2|X_2\,\theta_{12}) = \int_0^{2\pi} \frac{d\phi}{2\pi} I(X_1; Y_2|X_2, \theta_{12} = \phi). \tag{16}$$

One can similarly express $I(X_1X_2; Y_3|\theta_{13}\,\theta_{23})$. Further, for a fixed covariance matrix of X_1 and X_2, a conditional maximum entropy theorem (see [21]) tells us that

$$H(Y_2 \mid X_2, \theta_{12} = \phi_{12})$$
$$H(Y_3 \mid \theta_{13} = \phi_{13}, \theta_{23} = \phi_{23})$$

are maximized by making $p(x_1, x_2)$ Gaussian for any choice of the phases. The maximization (15) is thus

$$\max_{\rho}\ \min \left\{ \log\left(1 + \frac{P_1}{d_{12}^2}(1 - |\rho|^2)\right), \right.$$
$$\left. \int_0^{2\pi} \frac{d\phi_{13} d\phi_{23}}{(2\pi)^2} \log\left(1 + \frac{P_1}{d_{13}^2} + \frac{P_2}{d_{23}^2} + \frac{2\,\Re(\rho\, e^{j(\phi_{13}-\phi_{23})})\sqrt{P_1 P_2}}{d_{13}\, d_{23}}\right) \right\}. \tag{17}$$

where $\Re(x)$ is the real part of x, and where we have used $\alpha = 2$. The integral in (17) does not depend on the phase of ρ, so we can replace $\Re(\rho\, e^{j(\phi_{13}-\phi_{23})})$ with $|\rho| \cos\phi_{13}$. The integral evaluates to (see [22, p. 59])

$$\log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right)$$

where $a = 1 + P_1/d_{13}^2 + P_2/d_{23}^2$ and $b = 2|\rho|\sqrt{P_1 P_2}/(d_{13}d_{23})$. The maximizing ρ in (17) is therefore zero, and we have

$$R_{MH} = \min\left\{\log\left(1 + \frac{P_1}{d_{12}^2}\right), \log\left(1 + \frac{P_1}{d_{13}^2} + \frac{P_2}{d_{23}^2}\right)\right\}. \tag{18}$$

The optimality of $\rho = 0$ can alternatively be established as follows. The second mutual information in (17), denoted $I(\rho)$, does not depend on the phase of ρ, so we have $I(\rho) = [I(\rho) + I(-\rho)]/2$. But $\log(x)$ is concave in x and $\Re(x)$ is linear in x, so Jensen's inequality gives

$$[I(\rho) + I(-\rho)]/2 \leq I(0).$$

This shows that $\rho = 0$ is best for both informations in (17).

Similar arguments show that $\rho = 0$ is also best for the capacity upper bound (3). This leads to the following theorem.

Theorem 2 *Multi–hopping achieves capacity with phase fading if the relay is in a region near the source terminal. More precisely, if*

$$P_1/d_{13}^\alpha + P_2/d_{23}^\alpha \leq P_1/d_{12}^\alpha \tag{19}$$

then the capacity is

$$C = \log\left(1 + P_1/d_{13}^\alpha + P_2/d_{23}^\alpha\right). \tag{20}$$

The condition (19) is always satisfied for a range of d_{12} near zero. For example, for the geometry of Fig. 3 with $\alpha = 2$ and $P_1 = P_2$, the bound (19) is $-0.883 \leq d \leq 0.469$.

4.3 Phase Fading and Many Relays

Consider next T terminals with phase fading. Evaluating (2) and (11), we find that it is best to make the X_t, $t = 1, 2, \ldots, T-1$, Gaussian and independent. We further have the following generalization of Theorem 2.

Theorem 3 *Multi–hopping achieves capacity with phase fading if*

$$\sum_{t=1}^{T-1} \frac{P_t}{d_{tT}^\alpha} \leq \max_{\pi(\cdot)} \min_{1 \leq m \leq T-2} \sum_{t \in \pi(1:m)} \frac{P_t}{d_{t\pi(m+1)}^\alpha} \tag{21}$$

and the resulting capacity is

$$C = \log\left(1 + \sum_{t=1}^{T-1} \frac{P_t}{d_{tT}^\alpha}\right). \tag{22}$$

Note that the minimization in (21) does not include $m = T-1$.

The condition (21) is satisfied if all the relays are near the source. For example, consider a two-dimensional geometry and suppose all the relays are in a circle of radius d around the source. Then if the destination is a distance 1 from the source, we have $d_{tT} \geq 1-d$. Suppose further that $P_t = P$ for all t. The bound (21) tells us that multi–hopping achieves capacity if

$$d \leq \frac{1}{(T-1)^{1/\alpha}+1}. \tag{23}$$

The relays must therefore be in a circle of radius about $T^{-1/2}$ about the source for large T and $\alpha = 2$.

As another geometric example, consider a linear network as in Fig. 3 but with $T-2$ relays placed regularly to the right of the source at $d_{1t} = (t-1)d$, $2 \leq t \leq T-1$, where $0 \leq d < 1/(T-1)$ (see also [8, Sec. 2]). Suppose again that $P_t = P$ for all t. The bound (21) ensures that multi–hopping achieves capacity if d satisfies

$$\sum_{t=1}^{T-1} \frac{1}{[1-(t-1)d]^\alpha} \leq \frac{1}{d^\alpha}. \tag{24}$$

For $\alpha \geq 2$ and large T, one can show that d can be made close to $1/(T-1)$. The result is $C \approx \alpha \log(T)$ bits per use, i.e., capacity grows logarithmically in the number of terminals (or relays). Other related logarithmic scaling laws were obtained in [8] and [14].

5 Concluding Remarks

We reviewed and developed several multi–hopping strategies for relay networks. There are clearly many directions for further work. First and foremost, the fundamental problem of the capacity of the single–relay channel has been open for decades. It is, however, encouraging that capacity is known for some practical wireless cases. Second, one can extend the above capacity results to other kinds of fading such as Rayleigh fading [20]. Third, one can determine the outage capacity for slow (or quasi–static) fading. Finally, the strategies can be extended to networks with multiple sources, for example the multi-access relay channels of [11], or the broadcast relay channels of [20].

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