Differential Forms II

(Differential forms in differential and algebraic topology)
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Literature

- R. Bott, L.W. Tu: **Differential Forms in Algebraic Topology**, Springer, 1982
- S. Morita: **Geometry of Differential Forms**, American Mathematical Society, 2001
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- Ib H. Madsen, Jorgen Tornehave: **From Calculus to Cohomology**: De Rham Cohomology and Characteristic Classes, Cambridge University Press, 1997.
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Homological algebra, Mayer-Vietoris & Čech-de Rham

<u>Def.:</u> Let R be a ring. A sequence A of R-modules A^i

$$\cdots \longrightarrow A^{i-1} \xrightarrow{d} A^i \xrightarrow{d} A^{i+1} \longrightarrow \cdots$$ and R-module

homomorphisms $d : A^i \to A^{i+1}$ is called a complex if $\boxed{d^2 = 0}$

and it is called an exact sequence if

$$\boxed{\operatorname{Im}[\, d : A^{i-1} \to A^i \,] = \ker[\, d : A^i \to A^{i+1} \,]}.$$

<u>remarks:</u>

- Clearly, we have e.g. a different map $d_i : A^i \to A^{i+1}$ for every i and mainly drop the index 'i' in d_i out of lazyness.
- Our main interest lies in:
  - (i) $R = \mathbb{Z}$, A^i abelian groups and d a group homomorphism,
  - (ii) $R = \mathbb{R}$, A^i vector spaces and d a linear map.
- A collection of abelian groups or vector spaces indexed by an integer is called graded and often viewed as direct sum $\bigoplus_i A_i =: A$
- One (somewhat artificially & unnecessarily) distinguishes between cochain complexes and chain complexes depending on whether d
  increases or decreases dimension / rank / length
  of vec. spaces / groups / modules

The quotient module $\dfrac{\ker[\, d : A^i \to A^{i+1} \,]}{\operatorname{Im}[\, d : A^{i-1} \to A^i \,]}$ is then called the

i'th cohomology group (written as $H^i(A)$ or $H_i(A)$)

Lemma: (i) $0 \longrightarrow M \xrightarrow{f} N$ is exact iff f is injective

(ii) $M \xrightarrow{f} N \longrightarrow 0$ is exact iff f is surjective

proof: (i) Note that $0 \to M$ is a uniquely defined homomorphism that has image 0 in M. This is the kernel of f iff it is injective.

(ii) The kernel of $N \to 0$ is N, which equals the image of f iff f is surjective.

Q: What can be said about the case of an exact sequence $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$?

Def.: An exact sequence of the form $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ is called **short exact sequence**.

examples: ① For any R-module homomorphism $\psi: B \to C$ the sequence

$$0 \longrightarrow \ker \psi \underset{\substack{\uparrow \\ \text{inclusion}}}{\hookrightarrow} B \longrightarrow \operatorname{Im} \psi \longrightarrow 0 \quad \text{is exact.}$$

② For any submodule A of an R-module B the sequence

$$0 \longrightarrow A \underset{\substack{\uparrow \\ \text{inclusion}}}{\hookrightarrow} B \underset{\substack{\uparrow \\ \text{quotient}}}{\longrightarrow} B/A \longrightarrow 0 \quad \text{is exact.}$$

In fact, up to isomorphisms, every short exact sequence is of the forms ①&②.

proof: (of the claimed equivalence) In $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$,

if it is exact, ψ has to be surjective s.t. $C = \operatorname{Im}\psi$. Moreover,

$\ker\psi = \operatorname{Im}\varphi = \varphi(A) \simeq A$ where the last isomorphism is due to

injectivity of φ.

Given a sequence as in (1), we can define $A := \ker\psi \subseteq B$ and argue

that by the '1st isomorphism thm.' $\operatorname{Im}(\psi) \simeq B/\ker\psi \simeq B/A$. $\square$

recall: The length of an R-module M is length of the longest chain of

submodules, i.e. $\operatorname{length}(M) := \sup\{n \in \mathbb{N}_0 \cup \{\infty\} \mid 0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M\}$.

If M is a vector space, then $\operatorname{Length}(M) = \dim(M)$.

The above equivalence shows the following relation between the lengths

of the modules of a short exact sequence $0 \to A \to B \to C \to 0$:

$$\operatorname{length}(B) = \operatorname{length}(A) + \operatorname{length}(C)$$

In general:

Lemma: If $0 \longrightarrow A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} \dots \xrightarrow{d_{n-1}} A^n \longrightarrow 0$ is an exact sequence of

R-modules of finite lengths, then $\boxed{\sum_{i=1}^{n} (-1)^i \operatorname{length}(A^i) = 0}$.

proof: This follows from $\operatorname{length}(A^i) = \operatorname{length}(\ker d_i) + \operatorname{length}(\underbrace{\operatorname{Im} d_i}_{= \ker d_{i+1}})$ $\square$

$\Rightarrow$ Consequently, we can infer one of the lengths from the others!

Lemma: (splitting and gluing exact sequences)

(i) If $A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^3 \xrightarrow{d_4} A^4$ is an exact sequence of R-modules, the two sequences $A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} B \longrightarrow 0$, $0 \longrightarrow B \overset{\text{incl.}}{\hookrightarrow} A^3 \xrightarrow{d_3} A^4$ are also exact if $B := \operatorname{Im} d_2 = \ker d_3$.

(ii) If $A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} B \longrightarrow 0$, $0 \longrightarrow B \overset{\text{incl.}}{\hookrightarrow} A^3 \xrightarrow{d_3} A^4$ are exact where $B \subseteq A^3$ is a submodule, then $A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} A^3 \xrightarrow{d_4} A^4$ is exact.

proof: (i) The 1^{st} sequence is exact at A^2 since $\operatorname{Im} d_1 = \ker d_2$ and exact at B as $B = \operatorname{Im} d_2$. The 2^{nd} sequence is exact at B as the middle map is an inclusion (and thus injective) and exact at A^3 as $B = \ker d_3$.

(ii) Exactness at A^2 follows from $\operatorname{Im} d_1 = \ker d_2$. Moreover, exactness of the 1^{st} sequence at B and of the 2^{nd} at A^3 means that $\operatorname{Im} d_2 = B = \ker d_3$, implying exactness at A^3.

$\square$

Def.: ∘ If A, B are complexes, a cochain map $F: A \to B$ is a collection of homomorphisms $F: A^i \to B^i$ s.t. $F \circ d = d \circ F$, i.e. the diagram

$$\begin{tikzcd} \cdots \arrow[r] & A^i \arrow[r, "d"] \arrow[d, "F"] & A^{i+1} \arrow[r] \arrow[d, "F"] & \cdots \\ \cdots \arrow[r] & B^i \arrow[r, "d"] & B^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

commutes $\forall i$.

∘ A short exact sequence of complexes consists of three complexes A, B, C with cochain maps

$$0 \longrightarrow A \xrightarrow{F} B \xrightarrow{G} C \longrightarrow 0$$

s.t. $0 \longrightarrow A^i \xrightarrow{F} B^i \xrightarrow{G} C^i \longrightarrow 0$ is exact for every i.

remarks: ∘ Note that $F \circ d = d \circ F$ implies that F induces a homomorphism on cohomology $F: H^i(A) \to H^i(B)$, $F[\omega] := [F\omega]$. This is well-defined since $F[\omega + d\eta] =$
$= [F(\omega + d\eta)] = [F(\omega) + dF\eta] = [F(\omega)]$.

Strictly speaking, there are three different types of F in this story, which we could (but do not) denote differently.

∘ In any short exact sequence, F is injective and G surjective.

Lemma: Consider a commutative diagram of homomorphisms of finite-dimensional $\mathbb{F}$-vector spaces of the following type:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\alpha"] & B \arrow[r, "\psi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r, "\psi"] & C \arrow[r] & 0 \end{tikzcd}$$

If the horizontal sequences are exact, then $\boxed{tr[\beta] = tr[\alpha] + tr[\gamma]}$

proof: Let $\{a_i\}_{i=1}^m$ and $\{c_j\}_{j=1}^n$ be bases for A and C, respectively. Surjectivity of ψ allows us to pick $b_j \in B$ s.t. $\psi(b_j) = c_j$. Exactness at B then demands that $\operatorname{Im}\varphi = \ker\psi$ so that $\dim B = \dim\ker\psi + \dim\operatorname{Im}\psi = \dim\operatorname{Im}\varphi + \dim\operatorname{Im}\psi$.

Hence, $b_1, \dots, b_n, \varphi(a_1), \dots, \varphi(a_m)$ is a basis of B.

Commutativity of the diagram leads to

$$\beta(\varphi(a_i)) = \varphi(\alpha(a_i)) \in \operatorname{span}\{\varphi(a_i)\} \quad \text{and}$$

$$\beta(b_j) = \psi^{-1} \circ \gamma \circ \psi(b_j) \quad \text{with} \quad \psi^{-1}: c_j \mapsto b_j .$$

So if we represent β in this basis, the two diagonal blocks are representations of α and γ, resp.. Hence $tr[\beta] = tr[\alpha] + tr[\gamma]$. □

remark: from here one could prove the 'Hopf trace formula' and then proceed to the Lefschetz fixed point thm. We will, however, follow a different route ...

Lemma: (Zigzag Lemma) For any short exact sequence of complexes $0 \longrightarrow A \xrightarrow{F} B \xrightarrow{G} C \longrightarrow 0$ and any corresponding i there is a homomorphism $\delta : H^i(C) \longrightarrow H^{i+1}(A)$ called the connecting homomorphism, s.t. the following sequence is exact: $\dots \xrightarrow{\delta} H^i(A) \xrightarrow{F} H^i(B) \xrightarrow{G} H^i(C) \xrightarrow{\delta} H^{i+1}(A) \xrightarrow{F} \dots$

proof: (idea) The following diagram commutes and has exact rows:

"diagram chasing"

$$\begin{tikzcd}
0 \arrow[r] & A^i \arrow[r, "F"] \arrow[d, "d"] & B^i \arrow[r, "G"] \arrow[d, "d"] & C^i \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r, "F"] \arrow[d, "d"] & B^{i+1} \arrow[r, "G"] \arrow[d, "d"] & C^{i+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{i+2} \arrow[r, "F"] & B^{i+2} \arrow[r, "G"] & C^{i+2} \arrow[r] & 0
\end{tikzcd}$$

Let $c \in C^i$ represent a cohomology class, i.e. $dc = 0$. Surjectivity of G implies $\exists\, b \in B^i : Gb = c$. Then $G d b = d G b = dc = 0$. Hence, $db \in \ker G = \operatorname{Im} F$ s.t. $\exists\, a \in A^{i+1} : Fa = db$. Then again $Fda = dFa = d^2 b = 0$. By injectivity of F this means $da = 0$ s.t. a represents a cohomology class in $H^{i+1}(A)$. δ is then defined as $\delta : H^i(C) \ni [c] \mapsto [a] \in H^{i+1}(A)$, i.e. $\delta[c] = [F^{-1} \circ d \circ G^{-1} c]$

(t.b.p.: well-definedness, linearity, exactness) ...

$\square$

This means that every short exact sequence of cochain complexes $0 \to A \xrightarrow{F} B \xrightarrow{G} C \to 0$ induces a long exact sequence in cohomology. The latter is sometimes written compactly as an exact triangle:

$$\begin{tikzcd} H^*(A) \arrow[rr, "F"] & & H^*(B) \arrow[dl, "G"] \\ & H^*(C) \arrow[ul, "\delta"] & \end{tikzcd}$$

<u>Def.:</u> Let $M = U \cup V$ be a smooth manifold that is the union of two open submanifolds U, V. Given the commutative diagram of inclusions

$$\begin{tikzcd} & U \arrow[dr, "i_1"] & \\ U \cap V \arrow[ur, "j_1"] \arrow[dr, "j_2"'] & & U \cup V \\ & V \arrow[ur, "i_2"'] & \end{tikzcd}$$

the Mayer-Vietoris short exact sequence is defined as

$$\boxed{0 \longrightarrow \Omega(U \cup V) \xrightarrow{i} \Omega(U) \oplus \Omega(V) \xrightarrow{j} \Omega(U \cap V) \longrightarrow 0} \quad (*)$$

where $i(\omega) := (i_1^*(\omega), i_2^*(\omega))$ and $j(\omega_1, \omega_2) := j_1^*(\omega_1) - j_2^*(\omega_2)$.

<u>remark:</u> Here, $\Omega(\dots)$ is understood as de Rham complex, i.e. equipped with the exterior derivative. So $H^*(\dots)$ is de Rham cohomology. A useful convention is that $\Omega^k M = \{0\}$ for all $k \in -\mathbb{N}$.

Thm.: (*) is as the name suggests a short exact sequence of cochain complexes.

It induces a long exact sequence in cohomology (the M.-V. long ex. seq.)

$$\boxed{\dots \xrightarrow{\delta} H^k(U\cup V) \xrightarrow{i} H^k(U)\oplus H^k(V) \xrightarrow{j} H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(U\cup V) \longrightarrow \dots}$$

remark: exactness of (*) is understood as exactness of

$$0 \longrightarrow \Omega^k(U\cup V) \longrightarrow \Omega^k(U)\oplus\Omega^k(V) \longrightarrow \Omega^k(U\cap V) \longrightarrow 0 \quad \forall k.$$

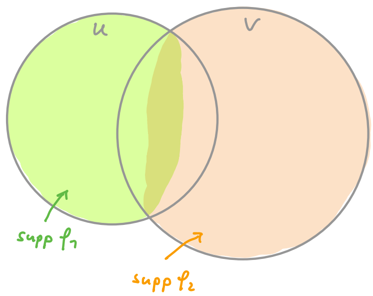
proof: i is injective since every non-zero form on $U\cup V$ has to be non-zero on either U or V. So the sequence is exact at $\Omega(U\cup V)$. Since $j_1^* \circ i_1^* = j_2^* \circ i_2^*$ we have $\operatorname{Im}(i) \subseteq \ker(j)$. Conversely, if $(\omega_1,\omega_2) \in \ker(j)$, then $\omega_1\big|_{U\cap V} = \omega_2\big|_{U\cap V}$ and we can define a k-form $\omega \in \Omega(U\cup V)$ via

$$\omega := \begin{cases} \omega_1 & \text{on } U \\ \omega_2 & \text{on } V \end{cases}$$ so that $(\omega_1,\omega_2) = i(\omega)$ and thus $\operatorname{Im}(i) \supseteq \ker(j)$.

Finally, to show that j is surjective, consider any $\omega \in \Omega^k(U\cap V)$ and let f_1, f_2 be a smooth part. of unity on $U\cup V$ subordinate to $\{U,V\}$.

Define $$\omega_1 := \begin{cases} f_2\,\omega & \text{on } U\cap V \\ 0 & \text{on } U\setminus V \end{cases},$$

$$\omega_2 := \begin{cases} f_1\,\omega & \text{on } V\cap U \\ 0 & \text{on } V\setminus U \end{cases}.$$



Then $$j(\omega_1,-\omega_2) = \omega_1\big|_{U\cap V} + \omega_2\big|_{U\cap V}$$
$$= (f_1+f_2)\,\omega = \omega.$$

So (*) is indeed a short exact sequence of complexes.

Exactness of the M.V. long exact sequence then follows from the Zigzag Lemma.

$\square$

Prop. For $n \geq 1$, $H^k(S^n) \simeq \begin{cases} \mathbb{R} & \text{if } k \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$

proof: We know that $H^0(S^n) \simeq \mathbb{R}$ and $H^1(S^1) \overset{(1)}{\simeq} \mathbb{R}$.

For $n \geq 2$, set $U := S^n \setminus \{(0, \dots, 0, -1)\}$, $V := S^n \setminus \{(0, \dots, 0, 1)\}$.

Then $S^n = U \cup V$, U and V are diffeomorphic to $\mathbb{R}^n$ by stereogr. proj., and $U \cap V$ is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$ and thus to S^{n-1}.

The beginning of the M.-V. long exact sequence is

$$0 \to \underbrace{H^0(S^n)}_{\cong \mathbb{R}} \to \underbrace{H^0(U) \oplus H^0(V)}_{\cong \mathbb{R}^2} \to \underbrace{H^0(U \cap V)}_{\cong \mathbb{R}} \to H^1(S^n) \to \underbrace{H^1(U) \oplus H^1(V)}_{= 0}$$

As the alternating sum of dimensions has to vanish, we conclude $H^1(S^n) \overset{(2)}{=} 0$.

Next consider $n, k \geq 2$ and the part of M.-V. l.e.s.

$$\underbrace{H^{k-1}(U) \oplus H^{k-1}(V)}_{= 0} \to \underbrace{H^{k-1}(U \cap V)}_{\cong H^{k-1}(S^{n-1})} \to H^k(S^n) \to \underbrace{H^k(U) \oplus H^k(V)}_{= 0}$$

So $H^{k-1}(S^{n-1}) \simeq H^k(S^n)$, which proves the claim since it reduces the case $k = n$ to (1) and the case $2 \leq k < n$ to (2). $\square$

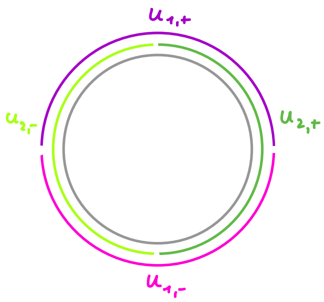
As a second application we show that de Rham cohomology groups are often finite-dimensional:

Def.: An open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of a smooth manifold M is called a **good cover** if for every finite subset $S \subseteq \Lambda$ $\bigcap_{\lambda \in S} U_\lambda$ is either empty or diffeomorphic to $\mathbb{R}^{\dim(M)}$.

remarks: ◦ Equipping M with a Riemannian metric and using 'geodesically convex neighborhoods' one can show that any open cover admits a refinement that is a good cover.

◦ Every compact M admits a finite good cover (i.e. one with $|\Lambda| < \infty$).

example: S^n:



Define the $(2n+2)$ open half spaces

$$\mathbb{R}^{n+1}_{i,\pm} := \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \pm x_i > 0 \}.$$

Then the $(2n+2)$ sets $U_{i,\pm} := S^n \cap \mathbb{R}^{n+1}_{i,\pm}$ are a finite good cover for S^n.

Thm.: If a smooth manifold M admits a finite good cover, then $H^k(M)$ is finite-dimensional for every k.

proof: We use induction on the number n of open sets in a good cover. Suppose the theorem holds for any M with good cover of size $|\Lambda| = n$ (certainly true for $n=1$). Let $\underbrace{U_{n+1}}_{=:V} \cup \underbrace{\bigcup_{\lambda=1}^{n} U_\lambda}_{=:U}$ be a good cover. Then $U \cap V$ admits a finite good cover $U_1 \cap V, \dots, U_n \cap V$. By induction hypothesis, the cohomology groups of U, V and $U \cap V$ are finite-dimensional. Now consider $\dots \to H^{k-1}(U \cap V) \xrightarrow{\delta} H^k(U \cup V) \xrightarrow{i} H^k(U) \oplus H^k(V) \longrightarrow \dots$

Since $\dim \operatorname{Im}(i) \leq \dim H^k(U) \oplus H^k(V) < \infty$ and

$$\dim \ker(i) = \dim \operatorname{Im}(\delta) \leq \dim H^{k-1}(U \cap V) < \infty$$

we have $\dim H^k(U \cup V) = \dim \operatorname{Im}(i) + \dim \ker(i) < \infty$. $\square$

Let $M = M_1 \times M_2$ be a product of smooth manifolds.

How can $H^*_\Omega(M)$ be expressed in terms of $H^*_\Omega(M_1)$ and $H^*_\Omega(M_2)$?

Consider the projections $M_1 \times M_2 \begin{smallmatrix} \xrightarrow{\pi_1} M_1 \\ \xrightarrow{\pi_2} M_2 \end{smallmatrix}$, $\omega \in \Omega^k M_1$ and $\eta \in \Omega^l M_2$.

Then $\pi_1^*(\omega) \wedge \pi_2^*(\eta) \in \Omega^{k+l} M$ is closed if both ω and η are and

it is exact if either ω or η is and the other one is closed (e.g. if $\omega = d\alpha$, then $\pi_1^*(\omega) \wedge \pi_2^*(\eta) = d\pi_1^*(\alpha) \wedge \pi_2^*(\eta) \pm \pi_1^*(\alpha) \wedge \underbrace{d\pi_2^*(\eta)}_{=0 \text{ since } d\eta = 0} = d(\pi_1^*(\alpha) \wedge \pi_2^*(\eta))$)

This shows that $\boxed{(\omega, \eta) \mapsto \pi_1^*(\omega) \wedge \pi_2^*(\eta)}$

after building equivalence classes gives a well-defined bilinear map

$H^k_\Omega(M_1) \times H^l_\Omega(M_2) \longrightarrow H^{k+l}_\Omega(M_1 \times M_2)$ and thus a linear map

$H^k_\Omega(M_1) \otimes H^l_\Omega(M_2) \longrightarrow H^{k+l}_\Omega(M_1 \times M_2)$.

Recall: the tensor product $V \otimes W$ is a vec. space whose basis is $\{v_i \otimes w_j\}$ if $\{v_i\} \subseteq V$, $\{w_j\} \subseteq W$ are bases.

Considering all degrees we obtain a linear map:

$$\boxed{K: \left(\bigoplus_k H^k_\Omega(M_1)\right) \otimes \left(\bigoplus_l H^l_\Omega(M_2)\right) \longrightarrow \bigoplus_m H^m_\Omega(M_1 \times M_2)}$$

Using a Mayer-Vietories argument and the 'Five Lemma' one can prove by induction on the number of elements in a good cover:

Thm.: (Künneth formula) If M_1 and M_2 have finite good covers, then K is an isomorphism. Hence,

$$\boxed{H^m_\Omega(M_1 \times M_2) \cong \bigoplus_{k=0}^m H^k_\Omega(M_1) \otimes H^{m-k}_\Omega(M_2)}$$

and the Betti numbers of M_1, M_2 and $M_1 \times M_2$ are related by:

$$\boxed{\beta_m(M_1 \times M_2) = \sum_{k=0}^m \beta_k(M_1)\beta_{m-k}(M_2)}$$

remark : By recursion this can easily be extended to higher products :

$$\beta_m(M_1 \times \ldots \times M_n) = \sum_{\substack{k \in \{0,\ldots,m\}^n \\ \sum_i k_i = m}} \beta_{k_1}(M_1) \cdot \ldots \cdot \beta_{k_n}(M_n) .$$

example: For the n-torus $T^n := \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$ we can use that $\beta_0(S^1) = \beta_1(S^1) = 1$

to obtain $\beta_m(T^n) = \sum_{\substack{k \in \{0,1\}^n \\ \sum_i k_i = m}} \underbrace{\beta_{k_1}(S^1) \cdot \ldots \cdot \beta_{k_n}(S^1)}_{1} = \binom{n}{m}$.

$\beta_l(S^1) = 0$ for $l \geq 2$

This implies that $\chi(T^n) = \sum_{k=0}^{n} (-1)^k \beta_k(T^n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 1^{n-k}$

$$= (1-1)^n = 0 .$$

Another consequence is that every $[\omega] \in H^k_{dR}(T^n)$ can be represented uniquely by a $\omega_c \in \Omega^k T^n$, $\omega_c := \sum_{1 \leq i_1 < \ldots < i_k \leq n} c_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ with const. coefficients.

Cor.: Let M_1, M_2 be smooth manifolds with finite good covers, then

$\chi(M_1 \times M_2) = \chi(M_1) \cdot \chi(M_2)$.

proof: $\chi(M_1 \times M_2) = \sum_k (-1)^k \beta_k(M_1 \times M_2) = \sum_k \sum_{k_1+k_2=k} (-1)^{k_1+k_2} \beta_{k_1}(M_1) \beta_{k_2}(M_2)$

$$= \sum_{k_1, k_2} (-1)^{k_1+k_2} \beta_{k_1}(M_1) \beta_{k_2}(M_2) = \chi(M_1)\chi(M_2) . \qquad \square$$

Čech cohomology

motivation/spoiler:
- The Mayer-Vietoris argument can be extended to covers by arbitrarily many open sets.
- In case of a 'good cover', the cohomology depends only on the intersection properties of the open sets.

Def.: Let $\mathcal{U} := \{ U_i \}_{i \in I}$ be a cover of a topological space by non-empty open sets. For every $k \in \mathbb{N}_0$ define $\mathfrak{Z}_k := \{ (i_0, \ldots, i_k) \in I^{k+1} \mid U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset \}$.
For every $i \in I^{k+1}$, $r \in \{0, \ldots, k\}$ set $i^{(r)} := (i_0, \ldots, i_{r-1}, i_{r+1}, \ldots i_k) \in I^k$ and
$C^k(\mathcal{U}, \mathbb{R}) := \{ c \in \mathbb{R}^{\mathfrak{Z}_k} \mid \forall \pi \in S_{k+1}\ \forall i \in \mathfrak{Z}_k : c((i_{\pi(0)}, \ldots, i_{\pi(k)})) = \operatorname{sgn}(\pi)\, c(i) \}$.

remarks:
- $i \in \mathfrak{Z}_k$ is called a Čech k-simplex, $c \in C^k(\mathcal{U}, \mathbb{R})$ a Čech k-cochain.
- Finiteness of the cover (i.e., $|I| < \infty$) implies that the vector spaces $C^k(\mathcal{U}, \mathbb{R})$ are finite-dimensional.

examples:
- $C^0(\mathcal{U}, \mathbb{R}) \ni c$, a Čech-0-cochain, assigns a real number to every element $U_i \in \mathcal{U}$.
- $C^1(\mathcal{U}, \mathbb{R}) \ni c$, a Čech-1-cochain, assigns a real number $c((i,j))$ to every ordered non-empty intersection $U_i \cap U_j \neq \emptyset$ s.t. $c((i,j)) = -c((j,i))$.

__Lemma:__ $0 \longrightarrow C^0(\mathcal{U},\mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U},\mathbb{R}) \xrightarrow{\delta} C^2(\mathcal{U},\mathbb{R}) \xrightarrow{\delta} \dots$ becomes a cochain complex, called Čech complex (with real coefficients), when equipped with the 'coboundary operator' $\delta : C^k(\mathcal{U},\mathbb{R}) \longrightarrow C^{k+1}(\mathcal{U},\mathbb{R})$,

$$(\delta c)(i) := \sum_{r=0}^{k+1} (-1)^r \, c(i^{(r)}) .$$

__proof:__ (that $\delta^2 = 0$). For $i \in \mathcal{I}_{k+2}$ and $c \in C^k(\mathcal{U},\mathbb{R})$ we have

$$(\delta \circ \delta c)(i) = \sum_{r=0}^{k+2} (-1)^r (\delta c)(i^{(r)})$$

$$= \sum_{0 \leq s < r \leq k+2} (-1)^{r+s} c(i^{(r,s)}) + \sum_{0 \leq r < s \leq k+2} (-1)^{r+s-1} c(i^{(r,s)}) = 0 .$$

$\square$

__Def.:__ The Čech cohomology groups are defined as

$$\boxed{H^k(\mathcal{U},\mathbb{R}) := \frac{\ker \delta : C^k(\mathcal{U},\mathbb{R}) \to C^{k+1}(\mathcal{U},\mathbb{R})}{\operatorname{Im} \delta : C^{k-1}(\mathcal{U},\mathbb{R}) \to C^k(\mathcal{U},\mathbb{R})}}$$

__remark:__ note the generality: this definition works for any open cover of any top. space.

__example:__ $H^0(\mathcal{U},\mathbb{R}) = \ker \delta : C^0(\mathcal{U},\mathbb{R}) \to C^1(\mathcal{U},\mathbb{R})$ is the space of all $c \in \mathbb{R}^{\mathcal{I}}$ that satisfy $(\delta c)(i,j) = c(i) - c(j) = 0$ whenever $U_i \cap U_j \neq \emptyset$.

That is, for every $c \in H^0(\mathcal{U},\mathbb{R})$ there is a locally constant function f s.t. $f|_{U_i} = c(i)$. Hence, for a smooth manifold M, $H^0(\mathcal{U},\mathbb{R}) \cong H^0_{dR}(M)$ if the cover is sufficiently fine (e.g. for a 'good cover').

Lemma: Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an open cover of a smooth manifold M and $\{\varphi_i\}_{i\in I}$ a smooth partition of unity subordinate to $\mathcal{U}$. The map

$$\boxed{C^k(\mathcal{U},\mathbb{R})\to\Omega^k M : c\mapsto \omega_c := \sum_{i\in\mathcal{J}_k} c((i_0,\dots,i_k))\,\varphi_{i_0}\, d\varphi_{i_1}\wedge\dots\wedge d\varphi_{i_k}}$$

is a cochain map. That is, $\boxed{\omega_{\delta c} = d\omega_c}$ so that the map induces a homomorphism on cohomology $\boxed{H^k(\mathcal{U},\mathbb{R})\longrightarrow H^k_{\Omega}(M)}$.

proof:

$$\omega_{\delta c} = \sum_{i\in\mathcal{J}_{k+1}} (\delta c)(i)\,\varphi_{i_0}\, d\varphi_{i_1}\wedge\dots\wedge d\varphi_{i_{k+1}}$$

$$= \sum_{i\in\mathcal{J}_{k+1}}\sum_{r=0}^{k+1} (-1)^r c(i^{(r)})\,\varphi_{i_0}\, d\varphi_{i_1}\wedge\dots\wedge d\varphi_{i_{k+1}}$$

$$= \sum_{(i_0,\dots,i_{k+1})\in I^{k+2}} c((i_1,\dots,i_{k+1}))\,\varphi_{i_0}\, d\varphi_{i_1}\wedge\dots\wedge d\varphi_{i_{k+1}} + \underbrace{\sum_{r=1}^{k+1}(-1)^r \sum_{(i_0,\dots,i_{k+1})\in I^{k+2}} c(i^{(r)})\,\varphi_{i_0}\, d\varphi_{i_1}\wedge\dots\wedge d\varphi_{i_{k+1}}}_{=0}$$

here we define $c(i):=0$ if $i\notin\mathcal{J}_{k+1}$

$\sum_{i\in I} d\varphi_i = 0$

$\sum_{i\in I}\varphi_i = 1$

$$= \sum_{(i_1,\dots,i_{k+1})\in I^{k+1}} c((i_1,\dots,i_{k+1}))\, d\varphi_{i_1}\wedge\dots\wedge d\varphi_{i_{k+1}}$$

$$= d\omega_c$$

□

Thm.: If $\mathcal{U}$ is a good cover of a smooth manifold M, then the map induced on cohomology in the Lemma is an isomorphism. That is,

$$\boxed{H^k(\mathcal{U},\mathbb{R})\cong H^k_{\Omega}(M)}\quad \forall k.$$

Consequences:

- All good covers of M lead to the same Čech cohomology.
- Cohomology only depends on intersection combinatorics of a good cover.
- If M admits a finite good cover, then $H^k_{\Omega}(M)$ are finite-dimensional.

proof idea: One combines the de Rham complex and the Čech complex into a double complex called the **Čech-de Rham complex**.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(M) & \xrightarrow{d} & \Omega^1(M) & \xrightarrow{d} & \Omega^2(M) & \xrightarrow{d} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C^0(\mathcal{U},\mathbb{R}) & \longrightarrow & & \longrightarrow & & \longrightarrow & & \longrightarrow & \cdots \\
\downarrow \delta & & \downarrow & & \downarrow & & \downarrow & & \\
C^1(\mathcal{U},\mathbb{R}) & \longrightarrow & & \longrightarrow & & \longrightarrow & & \longrightarrow & \cdots \\
\downarrow \delta & & \downarrow & & \downarrow & & \vdots & & \\
C^2(\mathcal{U},\mathbb{R}) & \longrightarrow & & \longrightarrow & & \cdots & C^k(\mathcal{U},\Omega^p) & & \\
\downarrow \delta & & \downarrow & & \downarrow & & & & \\
\cdots & & \cdots & & \cdots & & \cdots & &
\end{array}$$

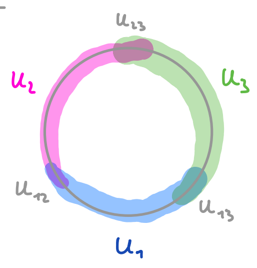
$C^k(\mathcal{U},\Omega^p)$ space of all p-forms defined on sets $U_{i_0} \cap \ldots \cap U_{i_k}$

This is constructed s.t.

- the first row is the de Rham complex
- the first column is the Čech complex
- all other rows and columns are exact sequences

Then a Mayer-Vietoris type diagram chasing argument can be carried out that shows that $H^k(\mathcal{U},\mathbb{R}) \cong H^k_\Omega(M)$. □

example: For S^1:



$C^0(\mathcal{U},\mathbb{R}) \simeq \{(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3\}$ where $\alpha_i \in \mathbb{R}$ is assigned to U_i

$C^1(\mathcal{U},\mathbb{R}) \simeq \{(\gamma_{12},\gamma_{23},\gamma_{13}) \in \mathbb{R}^3\}$ where γ_{ij} - " - to $U_i \cap U_j$

$\delta_0 : C^0 \longrightarrow C^1$, $\delta_0 : \alpha \mapsto \gamma$ s.t. $\gamma_{ij} = (\delta_0\alpha)_{ij} = \alpha_i - \alpha_j$

Then $\ker(\delta_0 : C^0 \to C^1) = \{(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R} \mid \alpha_1 = \alpha_2 = \alpha_3\} \simeq \mathbb{R}$

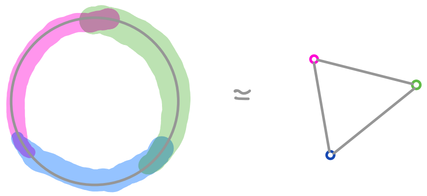
s.t. **$H^0(\mathcal{U},\mathbb{R}) \simeq \mathbb{R}$**. As $\ker(\delta_1 : C^1 \to C^2 = \{0\}) \simeq C^1 \simeq \mathbb{R}^3$

and $\operatorname{Im}\delta_0 \simeq \operatorname{Im}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \simeq \mathbb{R}^2$, we have **$H^1(\mathcal{U},\mathbb{R}) \simeq \mathbb{R}$**

open problem: the minimal nr. of elements of a good cover of a manifold (its so-called covering type) is only known for the simplest examples. It is unknown for the Klein bottle (7 or 8) and for the two-holed torus (surface of genus 2, where it between 6 and 10).

remarks:
- As $C^k(\mathcal{U}, \mathbb{R})$ are finite-dimensional (for a finite cover $\mathcal{U}$) the computation of Čech cohomology is linear algebra from the start — in contrast to de Rham cohomology, where $\Omega^k M$ is infinite dimensional.

- Note that the Čech k-simplex is indeed (combinatorially) a simplex. In the case of S^1:



An abstract simplicial complex is a family of sets that is closed under taking subsets.

The simplicial complex corresponding to an open cover of a top. space is called the nerve (or nerve complex) of the cover. Leray's nerve theorem states that the nerve of a good cover of a top. space X is a simplicial complex whose geometric realization is homotopy equivalent to X (and thus has the same (co)homology).

Poincaré duality revisited

In order to formulate a more general version of the Poincaré duality thm. (that does not require the manifold to be compact) we need a variant of de Rham cohomology that considers only compactly supported diff. forms.

Def.: For a smooth manifold M, we define:

$$\Omega_c^k M := \{ \omega \in \Omega^k M \mid \operatorname{supp}(\omega) = \overline{\{p \in M \mid \omega_p \neq 0\}} \text{ is compact} \}$$

$$H_c^k(M) := \frac{\ker\left(d : \Omega_c^k M \longrightarrow \Omega_c^{k+1} M\right)}{\operatorname{Im}\left(d : \Omega_c^{k-1} M \longrightarrow \Omega_c^k M\right)}$$ the compactly supported de Rham cohomology.

remarks:
- For compact M, clearly $H_c^k(M) = H_{\Omega}^k(M)$.
- $\Omega_c^k M$ is a vector space s.t. $d : \Omega_c^k M \longrightarrow \Omega_c^{k+1} M$ so the def. makes sense.

  However, there is an issue with 'functoriality':

  If $f : M \to N$ is smooth and $\omega \in \Omega_c^k N$ then $\operatorname{supp}(f^*\omega) \subseteq f^{-1}(\operatorname{supp}(\omega))$ may not be compact. So one has to restrict the class of maps:

Def.: A map $f : M \to N$ is called proper if preimages of compact sets under f are compact.

Cor.: (1) If $f : M \to N$ is a proper smooth map, then the pullback under f is a cochain map $f^* : \Omega_c^k N \to \Omega_c^k M$ and thus induces a homomorphism $f^* : H_c^k(N) \longrightarrow H_c^k(M)$.

(2) $H_c^k(M)$ is invariant under proper homotopies. In particular, if M and N are homeomorphic, then $H_c^k(M) \cong H_c^k(N)$.

The proofs follow the ones of $\Omega^k M$, $H^k M$ exactly. The last point is due to the fact that homeomorphisms are proper maps.

Some differences between $H^k_c(M)$ and $H^k_\Omega(M)$:

(i) $k=0$: $H^0_c(M)$ consists of all $f \in C^\infty(M)$ for which $df=0$ and $\operatorname{supp}(f)$ is compact. This means that on any non-compact component of M, f has to be zero. So

$$\dim(H^0_c(M)) = \# \text{ of } \underline{\text{compact}} \text{ connected components}$$

(ii) $H^k_c(M)$ is not a homotopy invariant since for instance by (i) we get $H^0_c(\{0\}) \simeq \mathbb{R}$ but $H^0_c(\mathbb{R}^n) \simeq \{0\}$ for any $n \in \mathbb{N}$.

(iii) Mayer-Vietoris: the pullback-by-the-inclusion idea that considers restrictions does no longer work i.g.. However, it can be replaced by a push-forward-by-the-inclusion idea since every compactly supported k-form can be extended by zero. In this way one obtains a M.V. exact sequence that goes in the 'opposite direction' within the k-th level:

$$\dots \longrightarrow H^k_c(U\cap V) \longrightarrow H^k_c(U) \oplus H^k_c(V) \longrightarrow H^k_c(U\cup V) \longrightarrow H^{k+1}_c(U\cap V) \longrightarrow \dots$$

This again enables a proof of the Künneth formula.

(iv) $H^1_c(\mathbb{R}) \simeq \mathbb{R}$ (compared to $H^1_\Omega(\mathbb{R}) \simeq \{0\}$). To see this consider the integration map $\int_\mathbb{R} : \Omega^1_c(\mathbb{R}) \to \mathbb{R}$, $\omega \mapsto \int_\mathbb{R} \omega$. This is linear & surjective. Moreover, if ω is exact, i.e. there is a compactly supported $f \in C^\infty(\mathbb{R})$ s.t. $df = \omega$, then by the fundam. thm. of calc. $\int_\mathbb{R} \omega = 0$. So $\int_\mathbb{R}$ induces a surjective homomorphism $\int_\mathbb{R} : H^1_c(\mathbb{R}) \to \mathbb{R}$.

This is also injective: if $\omega \in \Omega^1_c(\mathbb{R})$ is s.t. $d\omega = 0$ and $\int_\mathbb{R} \omega = \int_\mathbb{R} f(t)\,dt = 0$, then $g(t) := \int_{-\infty}^t f(\tau)\,d\tau$ is in $\Omega^0_c(\mathbb{R})$ and s.t. $dg = \omega$. So $[\omega] = [0]$ in $H^1_c(\mathbb{R})$. Consequently, $\int_\mathbb{R} : H^1_c(\mathbb{R}) \to \mathbb{R}$ is an isomorphism.

Generalizing this idea leads to the following:

Def.: Let M be a smooth oriented n-dim. manifold (without boundary), and $k \in \{0, \dots, n\}$. We define the Poincaré pairing

$$\boxed{H^k_\Omega(M) \times H^{n-k}_c(M) \longrightarrow \mathbb{R}, \quad ([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta}$$

and the related Poincaré duality operator

$$\boxed{P^k_M : H^k_\Omega(M) \longrightarrow \left(H^{n-k}_c(M)\right)^*: \ [\omega] \mapsto \left([\eta] \mapsto \int_M \omega \wedge \eta \right)}$$

example: If M is connected, then P^0_M maps $1 \in \mathbb{R} \simeq H^0_\Omega(M)$ to $\left(\eta \mapsto \int_M \eta\right) \in \left(H^n_c(M)\right)^*$.

Thm.: (Poincaré duality) Let M be a smooth oriented n-dim. manifold (without boundary), and $k \in \{0, \dots, n\}$. Then the Poincaré duality operator is a vector space isomorphism. Consequently, $\boxed{H^k_\Omega(M) \simeq \left(H^k_c(M)\right)^*}$.

remark: This can be proven via a Mayer-Vietoris argument. If M has a finite good cover, then this can be done by induction on the number of elements in a good cover. In fact, under this additional assumption, we get:

Cor.: Let M be a smooth oriented n-dim. manifold (without boundary) with finite good cover, and $k \in \{0, \dots, n\}$. Then the Poincaré pairing is a nondegenerate bilinear map s.t. $\boxed{\dim H^k_\Omega(M) = \dim H^{n-k}_c(M)}$.

remark: This uses that (i) finite good cover implies that $\dim(H^k_\Omega(M)), \dim(H^{n-k}_c(M)) < \infty$ and (ii) for any finite dim. vector space V, we have $V^* \simeq V$.

examples: $M = \mathbb{R}^n$: $\quad H^k_\Omega(M) \simeq \begin{cases} \mathbb{R}, & k=0 \\ 0, & k \neq 0 \end{cases}$. So $H^k_c(M) \simeq \begin{cases} \mathbb{R}, & k=n \\ 0, & k \neq n \end{cases}$

$M = S^n$: $\quad H^k_\Omega(M) \simeq H^k_c(M) \simeq \begin{cases} \mathbb{R}, & k \in \{0, n\} \\ 0 \end{cases}$

M connected, oriented n-dim.: $\quad H^0_\Omega(M) \simeq \mathbb{R} \simeq H^n_c(M)$

and $\quad H^n_\Omega(M) = \left\{ \begin{matrix} \mathbb{R}, & M \text{ compact} \\ 0, & M \text{ non-compact} \end{matrix} \right\} \simeq H^0_c(M)$

remark: Orientability is crucial for Poincaré duality. E.g. for the Möbius strip $M = [0,1] \times (0,1)/\sim$ we have ($\rightarrow$ exercise)

$$H^0_{\Omega}(M) \simeq \mathbb{R} \qquad\qquad H^2_c(M) \simeq 0$$

$$H^1_{\Omega}(M) \simeq \mathbb{R} \qquad \text{but} \qquad H^1_c(M) \simeq 0$$

$$H^2_{\Omega}(M) \simeq 0 \qquad\qquad H^0_c(M) \simeq 0$$

More generally, one can show that on any non-orientable manifold closed top forms are always exact. That is, if M is any non-orientable n-dim. smooth manifold, then $H^n_{\Omega}(M) \simeq 0 \simeq H^n_c(M)$.

Cor. / Def.: Let M be oriented smooth n-dim. and $\iota : S \hookrightarrow M$ a oriented k-dim. submanifold that is top. closed in M. Then there is a unique $[\omega] \in H^{n-k}_{\Omega}(M)$, called the Poincaré dual of S in M, s.t. $\forall\, [\eta] \in H^k_c(M)$:

$$\int_S \eta := \boxed{\int_S \iota^*\eta = \int_M \eta \wedge \omega.}$$

proof: As $S \subseteq M$ is closed $\operatorname{supp}(\eta|_S)$ is closed not only in S but also in M. Since $\operatorname{supp} \eta|_S \subseteq \operatorname{supp}(\eta) \cap S$ is a closed subset of a compact set, $\iota^*\eta$ also has compact support on S, so $\int_S \iota^*\eta$ is well defined.

By Stokes' thm. it induces a linear functional $H^k_c(M) \longrightarrow \mathbb{R}$, i.e. an element of $(H^k_c(M))^*$. Using the inverse of the Poincaré duality operator $H^{n-k}_{\Omega}(M) \longrightarrow (H^k_c(M))^*$ gives a unique cohomology class $[\omega] \in H^{n-k}_{\Omega}(M)$ s.t.

$$\int_S \iota^* \eta = \int_M \eta \wedge \omega. \qquad \square$$

<u>examples:</u> ① If M is compact and oriented, we can take $S = M$. So the Poincaré dual of M in M is $[1] \in H^0_{\Omega}(M)$.

② Let M be oriented, and T be a orientable, top. closed submanifold of M with boundary $\partial T =: S$. Then the Poincaré dual of S in M is 0: using Stokes' thm. we get $\forall\, [\eta] \in H^{n-k}_c(M)$:

$$\int_M \eta \wedge \omega_S = \int_S \eta = \int_{\partial T} \eta = \int_T d\eta = 0$$

ω_S Poincaré dual of S in M

The Poincaré dual behaves nicely under diffeomorphisms:

<u>Prop.:</u> Let M be oriented smooth n-dim., $f: M \to M$ an orientation-preserving diffeomorphism, and $\omega_S \in H^{n-k}_{\Omega}(M)$ the Poincaré dual of $S \subseteq M$. Then $\boxed{\omega_S = f^* \omega_{f(S)}}$.

remark: If $f: M \to M$ is orientation-reversing, then $\omega_S = -f^*\omega_{f(S)}$.

proof: The characterizing property of the Poincaré dual gives

$\forall\, \eta \in H_c^k(M)$:

$$\int_M \eta \wedge \omega_{f(S)} = \int_{f(S)} \eta = \int_S f^*\eta = \int_M f^*\eta \wedge \omega_S$$

At the same time $\displaystyle\int_{M = f(M)} \eta \wedge \omega_{f(S)} = \int_M f^*\eta \wedge f^*\omega_{f(S)}$

By uniqueness of the Poincaré dual (as cohomology class), $\omega_S = f^*\omega_{f(S)}$.

$\square$

Cor.: Let M be oriented smooth n-dim., $f: M \to M$ an orientation-preserving diffeomorphism that is homotopic to the identity, and $S \subseteq M$ a top. closed oriented submanifold. Then S and $f(S)$ have the same Poincaré dual in M.

proof: By the previous prop., we know that $\omega_S = f^*\omega_{f(S)}$.

However, since $f \simeq id$, we have $f^* = id^*: H_{\Omega}^{n-k}(M) \to H_{\Omega}^{n-k}(M)$.

So $\omega_S = \omega_{f(S)}$.

$\square$

Def.: Let M be an oriented, n-dim. smooth manifold with finite good cover, and $S \subseteq M$ a k-dim. compact, oriented submanifold. The compact Poincaré dual of S in M is the unique $[\omega] \in H_c^{n-k}(M)$ for which $\forall\, [\eta] \in H_{\Omega}^{k}(M)$:

$$\boxed{\int_S \eta = \int_M \eta \wedge \omega .}$$

remarks:
- Compactness of S is assumed so that $\int_S \eta$ is well-defined for all η.
- Existence and uniqueness follow from the fact that the 'Poincaré pairing' on the r.h.s. is non-degenerate.
- If M is compact, then 'Poincaré dual' = 'comp. Poincaré dual'.

Thm.: (Localization principle)

Let M be an oriented, n-dim. smooth manifold with finite good cover, and $S \subseteq M$ a k-dim. compact, oriented submanifold. For every open neighborhood U of S there is a representative $\omega \in \Omega_c^{n-k}(M)$ of the compact Poincaré dual of S in M s.t. $\operatorname{supp}(\omega) \subseteq U$.

remark: the same holds for the Poincaré dual of any top. closed submanifold, but this requires a different proof strategy.

<u>proof:</u> As a compact submanifold of U, S has a compact Poincaré dual $[\tilde{\omega}] \in H_c^{n-k}(U)$ in U. As $\tilde{\omega}$ has compact support, we can extend it to $\omega \in \Omega_c^{n-k}(M)$ s.t. $\omega|_U = \tilde{\omega}$, $\omega|_{M\setminus U} = 0$ and $\iota_2^* \omega = \tilde{\omega}$.

Then $\forall\, [\eta] \in H^k_\Omega(M)$:

$$\int_S \iota^* \eta = \int_S \iota_1^* (\underbrace{\iota_2^* \eta}_{\in \Omega^k U}) = \int_U \iota_2^* \eta \wedge \tilde{\omega}$$

$$= \int_U \iota_2^* (\eta \wedge \omega) = \int_M \eta \wedge \omega .$$

$\iota_1 : S \hookrightarrow U$
$\iota_2 : U \hookrightarrow M$
$\iota := \iota_2 \circ \iota_1 : S \hookrightarrow M$

Hence, $[\omega] \in H_c^{n-k}(M)$ is the compact Poincaré dual of S in M. □

<u>Examples</u> (aiming at fixed point theory):

1. Let M be a compact, oriented smooth n-dim. manifold and $\Delta := \{ (x,x) \mid x \in M \} \subset M \times M$ the 'diagonal submanifold' of $M \times M$. What is the Poincaré dual of Δ in $M \times M$?

We denote it by $[\rho]$ and note that $[\rho] \in H^n_\Omega(M \times M)$ as $\dim(M) - \dim(\Delta) = 2n - n = n$.

Let $\pi_i : M \times M \to M$ be the canonical projections onto the i'th factor with $i \in \{1,2\}$. If $\left\{ [\omega_i^j] \mid i \in \{1, \dots, \beta_j := \dim(H^j_\Omega(M))\} \right\}$ is a basis of $H^j_\Omega(M)$, Poincaré pairing gives a dual basis $\left\{ [\nu_i^{n-j}] \mid i \in \{1, \dots, \beta_{n-j} = \beta_j\} \right\}$ s.t. $\int_M \omega_i^j \wedge \nu_k^{n-j} = \delta_{ik}$.

From the Künneth formula and its derivation we know that

$(\pi_1^* \omega_i^j) \wedge (\pi_2^* \nu_k^{n-j})$ represents a basis of $H^n_\Omega(M \times M)$.

So $[\varphi] = \sum_{i,j,k} c_{i,j,k} \, [(\pi_1^* \omega_i^j) \wedge (\pi_2^* \nu_k^{n-j})]$ for some $c_{ijk} \in \mathbb{R}$.

By definition of the Poincaré dual we have

$$\underbrace{\int_{M\times M} \eta \wedge \varphi}_{=:\,LHS} = \underbrace{\int_\Delta \eta}_{=:\,RHS} \quad \text{in particular for } \eta := (\pi_1^* \nu_s^{n-r} \wedge \pi_2^* \omega_t^r)$$

<u>RHS:</u> define $\iota : M \to M\times M$, $\iota(x) := (x,x)$. Then $\pi_1 \circ \iota = \pi_2 \circ \iota = id$

so that
$$\int_\Delta \eta = \int_M \underbrace{\iota^* (\pi_1^* \nu_s^{n-r} \wedge \pi_2^* \omega_t^r)}_{= (\pi_1 \circ \iota)^* \nu_s^{n-r} \wedge (\pi_2 \circ \iota)^* (\omega_t^r)}$$

$$= \int_M \nu_s^{n-r} \wedge \omega_t^r = (-1)^{r\cdot(n-r)} \int_M \omega_t^r \wedge \nu_s^{n-r}$$

$$= (-1)^{r\cdot(n-r)} \delta_{t,s}$$

<u>LHS:</u> inserting η and φ gives:

$$\int_{M\times M} \eta \wedge \varphi = \sum_{ijk} c_{ijk} \int_{M\times M} (\pi_1^* \nu_s^{n-r} \wedge \pi_2^* \omega_t^r) \wedge (\pi_1^* \omega_i^j) \wedge (\pi_2^* \nu_k^{n-j})$$

$$= \ldots = c_{srt} (-1)^{n\cdot r}$$

So $c_{srt} = (-1)^{n\cdot r + r\cdot(n-r)} \delta_{t,s} = (-1)^r \delta_{t,s}$ and thus

$$\boxed{[\varphi] = \sum_{j,k} (-1)^j \, [(\pi_1^* \omega_k^j) \wedge (\pi_2^* \nu_k^{n-j})]}\ .$$

(2) Let $f: M \to M$ be a smooth function on a compact, oriented n-dim. M and $\Gamma_f := \{ (x, f(x)) \mid x \in M \} \subseteq M \times M$ its graph.

Following (1) we compute its Poincaré dual $[\varphi_f] \in H^n_{\Omega}(M \times M)$.

Again let ω_i^j represent a basis of $H^j_{\Omega}(M)$ s.t. $\int_M \omega_i^j \wedge \nu_k^{n-j} = \delta_{ik}$

and $(\pi_1^* \omega_i^j) \wedge (\pi_2^* \nu_k^{n-j})$ represent a basis of $H^n_{\Omega}(M \times M)$.

We expand $f^*: H^j_{\Omega}(M) \to H^j_{\Omega}(M)$ as $f^*([\omega_i^j]) = \sum_k F^j_{ki} \omega_k^j$ s.t.

$F^j_{ki} = \int_M f^*(\omega_i^j) \wedge \nu_k^{n-j}$ and

$[\varphi_f] = \sum_{i,j,k} c_{i,j,k} \left[(\pi_1^* \omega_i^j) \wedge (\pi_2^* \nu_k^{n-j}) \right]$

By Poincaré duality:

$$\underbrace{\int_{M \times M} \eta \wedge \varphi_f}_{=:\, LHS} = \underbrace{\int_{\Gamma_f} \eta}_{=:\, RHS} \quad \text{in particular for } \eta := (\pi_1^* \nu_s^{n-r} \wedge \pi_2^* \omega_t^r)$$

As before, $LHS = c_{srt} (-1)^{n \cdot r}$. For RHS we use the orient. pres. diffeomorphism $\gamma: M \to \Gamma_f$, $x \mapsto (x, f(x))$ for which $\pi_1 \circ \gamma = id$, $\pi_2 \circ \gamma = f$.

Then
$$\int_{\Gamma_f = \gamma(M)} \pi_1^* \nu_s^{n-r} \wedge \pi_2^* \omega_t^r = \int_M \gamma^* \pi_1^* \nu_s^{n-r} \wedge \gamma^* \pi_2^* \omega_t^r$$
$$= \int_M \nu_s^{n-r} \wedge f^*(\omega_t^r) = \sum_k F^r_{kt} \int_M \nu_s^{n-r} \wedge \omega_k^r$$
$$= (-1)^{r(n-r)} F^r_{st}.$$

So
$$\boxed{\begin{aligned} [\varphi_f] &= \sum_{i,j,k} (-1)^j F^j_{ik} \left[(\pi_1^* \omega_i^j) \wedge (\pi_2^* \nu_k^{n-j}) \right] \\ &= \sum_{j,k} (-1)^j \, \pi_1^* f^*(\omega_k^j) \wedge \pi_2^* \nu_k^{n-j} \end{aligned}}$$

Def.: Let $f: M \to M$ be a smooth map on a smooth n-dim. manifold M with finite-dim. $H^*_\Omega(M)$ (e.g. with M admitting a finite good cover).

The **Lefschetz number** of f is defined as:

$$\boxed{L(f) := \sum_{j=0}^{n} (-1)^j \, \mathrm{tr}\left[f^* : H^j_\Omega(M) \to H^j_\Omega(M) \right]}$$

remark: From the definition we obtain two important properties:

1. If $f \simeq g$ are homotopic, then $\boxed{L(f) = L(g)}$
2. If ϕ is a diffeomorphism, then $\boxed{L(\phi \circ f \circ \phi^{-1}) = L(f)}$

Thm.: If $f: M \to M$ is smooth on a compact, oriented manifold M, $\Delta := \{ (x,x) \mid x \in M \} \subseteq M \times M$ and $[\varphi_f] \in H^n_\Omega(M \times M)$ is the Poincaré dual of the graph Γ_f in $M \times M$, then

$$\boxed{\int_\Delta \varphi_f = L(f)} .$$

proof:

$$\int_\Delta \varphi_f := \int_{\Delta = \alpha(M)} \iota^* \varphi_f = \int_M \alpha^* \iota^* \varphi_f = \sum_{i,j,k} (-1)^j F^j_{ik} \underbrace{\int_M w^j_i \wedge \nu^{n-j}_k}_{= \delta_{ik}}$$

$\iota: \Delta \to M \times M$ inclusion

$\alpha: M \to \Delta$, $x \mapsto (x,x)$

s.t. $\pi_i \circ \iota \circ \alpha = \mathrm{id}_M$

$$= \sum_j (-1)^j \underbrace{\mathrm{tr}[F^j]}_{= \mathrm{tr}\left[f^* : H^j_\Omega(M) \to H^j_\Omega(M) \right]}$$

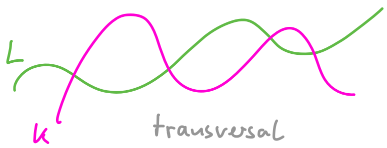
$\square$

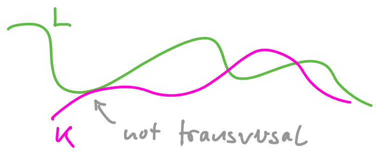
Excursion into Intersection theory

<u>Def.:</u> Let K, L be submanifolds of a smooth manifold M.

K and L are transversal in M and we write $K \pitchfork L$ if

$$\boxed{T_pK + T_pL = T_pM} \quad \forall p \in K \cap L.$$





<u>Prop.:</u> If K, L are transversal submanifolds of M, then $K \cap L$ is a submanifold of M with

$$\boxed{\operatorname{codim}(K \cap L) = \operatorname{codim}(K) + \operatorname{codim}(L)}$$

In particular, if $\dim(K) + \dim(L) = \dim(M)$ and M is compact, then $K \cap L$ is a finite set (as a discrete subset of a compact set is finite) and $T_pK \oplus T_pL = T_pM$ $\forall p \in K \cap L$.

Transversality is 'generic' and can be achieved by 'small perturbations'. This is the content of many transversality theorems. E.g.:

<u>Prop.:</u> Let K, L be smooth submanifolds of $\mathbb{R}^n$. Then $K \pitchfork (L + x)$ for a.e. $x \in \mathbb{R}^n$.

Thm.: Let K, L be compact, oriented, transversal submanifolds of an oriented smooth manifold M. The Poincaré dual $[\omega_{K\cap L}] \in H^*_\Omega(M)$ of $K \cap L$ in M can be expressed by the Poincaré duals of K and L as $\boxed{\omega_{K\cap L} = \omega_K \wedge \omega_L}$.

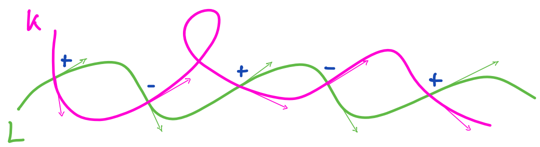
remarks:
- Defining an orientation of $K\cap L$ from K, L and M requires an ordering of K and L. In this way, $\omega_{K\cap L} = \omega_{L\cap K} \cdot (-1)^{\operatorname{codim}(K)\cdot \operatorname{codim}(L)}$.
- Since $\operatorname{degree}(\omega_{K\cap L}) = \operatorname{codim}(K\cap L) = \operatorname{codim}(K) + \operatorname{codim}(L) = \deg(\omega_K) + \deg(\omega_L)$, the wedge product is the natural guess for the Poincaré dual of $K\cap L$ in M. We skip the proof that it really does the job.

Def.: Let K, L be two oriented compact submanifolds of an oriented manifold M s.t. $\dim(K) + \dim(L) = \dim(M)$ and $K \pitchfork L$.

For any $p \in K\cap L$, let $A := (a_1, \dots, a_n)$ and $B := (b_1, \dots, b_n)$ be positively oriented bases of T_pK and T_pL, respectively.

With $\varepsilon(p) := \begin{cases} +1 \\ -1 \end{cases}$ if (A,B) is $\begin{Bmatrix} \text{positively} \\ \text{negatively} \end{Bmatrix}$ oriented in T_pM

define the intersection number $\boxed{I(K,L) := \sum_{p\in K\cap L} \varepsilon(p)}$.



$\varepsilon(p)$ is the orientation of $K\cap P$ at p.

Corr. Let K, L be compact, oriented, transversal submanifolds of an oriented, compact smooth manifold M with Poincaré duals ω_K and ω_L and $\dim(K) + \dim(L) = \dim(M)$. Then

$$\boxed{I(K,L) = \int_M \omega_K \wedge \omega_L} \, .$$

proof: As $\omega_K \wedge \omega_L$ is the Poincaré dual of $K \cap L$ and $[1] \in H^0_c(M)$, we can write

$$\int_M \omega_K \wedge \omega_L = \int_{K \cap L} 1 \underset{\substack{\uparrow \\ K\cap L \text{ finite}}}{=} \sum_{p \in K \cap L} \varepsilon(p)$$

where $\varepsilon(p) \in \pm 1$ is the orientation assigned to p. $\square$

Now consider the case where $K := \Delta$ and $L := \Gamma_f$ for a smooth map $f: M \to M$. Then $\Delta \cap \Gamma_f$ corresponds to the set of fixed points of f.

Def.:
- A fixed point $p \in M$ of a smooth map $f: M \to M$ is called non-degenerate if $d_p f : T_pM \to T_pM$ does not have 1 as an eigenvalue, i.e. $\det(d_p f - \mathbb{1}) \neq 0$.
- f is a Lefschetz map if all its fixed points are non-degenerate

Prop.: Let $f: M \to M$ be a smooth map on a compact, oriented M.

1) f has only non-deg. fixed points iff $\Delta \pitchfork \Gamma_f$.

2) If $\Delta \pitchfork \Gamma_f$, then $\boxed{I(\Delta, \Gamma_f) = \sum_{p = f(p)} \underbrace{\operatorname{sgn} \det(d_p f - \mathbb{1})}_{= \varepsilon(p)}}$

proof: Let $p = f(p)$, and $e_1, \dots, e_n$ a positively oriented basis of T_pM

determining positively oriented bases

$$(e_1, e_1), \dots, (e_n, e_n) \text{ of } T_{(p,p)} \Delta,$$

$$(e_1, d_pf\, e_1), \dots, (e_n, d_pf\, e_n) \text{ of } T_{(p,p)} \Gamma_f \text{ and}$$

$$(e_1, 0), \dots, (e_n, 0), (0, e_1), \dots, (0, e_n) \text{ of } T_{(p,p)} M \times M.$$

The map from the latter to the former two

$T_{(p,p)} M \times M \longrightarrow T_{(p,p)} \Delta \oplus T_{(p,p)} \Gamma_f$ is represented by a matrix $\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & d_pf \end{pmatrix}$

$\Delta \pitchfork \Gamma_f$ iff this is an isomorphism which in turn is equivalent

to $0 \neq \det \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & d_pf \end{pmatrix} \underset{\uparrow}{=} \det \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ 0 & d_pf - \mathbb{1} \end{pmatrix} = \det(d_pf - \mathbb{1})$. Its sign decides

subtract upper rows from lower ones

wether the orientation of $T_{(p,p)} M \times M$ matches the one of $T_{(p,p)} \Delta \oplus T_{(p,p)} \Gamma_f$.

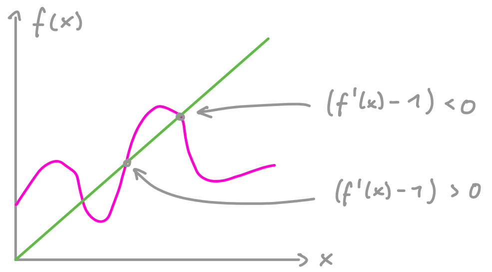
$\square$

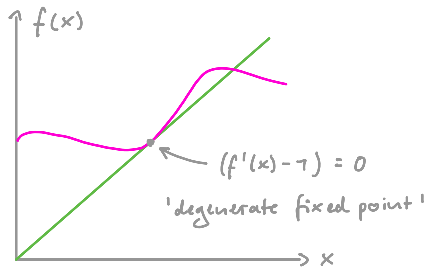
remark: note that the choice of orientation on T_pM does not matter

since a change would result in cancelling signs.

Lefschetz fixed point theorem

recall: $x \in M$ is a fixed point of $f: M \to M$ if $x = f(x)$, which is equivalent to $(x,x) \in \Gamma_f \cap \Delta$.





Thm.: (Lefschetz fixed point thm.)

Let $f: M \to M$ be smooth on a compact, orientable manifold M.

Then f has a fixed point if $L(f) \neq 0$.

proof: Suppose there is no fixed point, i.e. $\Gamma_f \cap \Delta = \emptyset$. Then $U := M \times M \setminus \Delta$ is open and contains Γ_f. According to the localization principle, there is a representative $\varphi_f \in \Omega^n(M \times M)$ of the (compact) Poincaré dual $[\varphi_f] \in H^n_c(M \times M)$ of the graph Γ_f in $M \times M$ s.t. $\operatorname{supp}(\varphi_f) \subseteq U$.

In other words, $\varphi_f \big|_\Delta = 0$. Then $L(f) = \int_\Delta \varphi_f = 0$. $\square$

This theorem can be extended in several directions:

- One can exploit that $L(f)$ is invariant under homotopies of f and e.g. deform f s.t. all its fixed points become non-degenerate (in which case f is a **Lefschetz map**)

- For Lefschetz f we can use that

$$\boxed{L(f) = \int_\Delta \varphi_f = \int_{M\times M} \varphi_f \wedge \varphi_\Delta = I(\Gamma_f, \Delta) = \sum_{p=f(p)} \operatorname{sgn}\big(\det(\mathbb{1} - d_p f)\big)}$$

  So $|L(f)|$ is a lower bound on the number of fixed points.

  (If $f: \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic, then $\det(d_p f - \mathbb{1}) > 0$ for every fixed point s.t. $L(f)$ becomes the nr. of fixed points. In this way, one can e.g. get 'Bezout's thm.' as a corollary)

- Cases with boundary can be reduced to cases without by

  (i) using a homotopy to ensure that ∂M contains no fixed point

  (ii) gluing together two copies of M along the boundary s.t. $M \dot\cup M/\sim$ is without boundary.

- Nonorientable cases can be reduced to orientable ones by

  (i) embedding $M \hookrightarrow \mathbb{R}^N$ and 'thickening' M in the normal direction. The resulting M' is then orientable (since $\mathbb{R}^N$ is) and with the projection $\pi: M' \to M$, $f \circ \pi$ and f have the same fixed pts.

As a result one obtains:

Thm.: (Lefschetz - Hopf fixed point thm.)

Let $f: M \to M$ be smooth on a compact manifold M with boundary.

(i) f is smoothly homotopic to a Lefschetz map (which has the same Lefschetz number).

(ii) If f is Lefschetz, then $\boxed{L(f) = \sum_{p = f(p)} \operatorname{sgn}(\det(\mathbb{1} - d_p f))}$

remarks:
- We emphasize again that $L(f) = L(g)$ if $f \simeq g$ homotopic.
- If $f \simeq id$, then $\boxed{L(f) = \chi(M)}$ as $tr[id^*: H^i_\Omega(M) \to H^i_\Omega(M)] = \beta_i(M)$.

  So $\chi(M)$ can be interpreted as self-intersection number:

  $\chi(M) = 0$ iff M can be displaced from itself by a map homotopic to the identity.

Example: Let $M := U(n) := \{ V \in \mathbb{C}^{n \times n} \mid V^* V = \mathbb{1} \}$. Then $\chi(M) = 0$.

proof: consider $V := e^{iH} \in U(n) \setminus \{\mathbb{1}\}$ for some $H = H^* \in \mathbb{C}^{n \times n}$, and $f: U(n) \to U(n)$, $U \mapsto UV$. Then f has no fixed point since $f(U) = U \Leftrightarrow UV = U \Leftrightarrow V = \mathbb{1}$.

Moreover, $f \simeq id$ via $U \mapsto U \cdot \exp[i t H]$, $t \in [0,1]$.

So $0 = L(f) = L(id) = \chi(M)$. $\square$

Clearly, this applies to every compact connected Lie group.

Cor.: Let M be a compact smooth manifold with boundary and $\chi(M) \neq 0$. Then every smooth map $f: M \to M$ that is homotopic to the identity has a fixed point.

proof: $f \simeq id$ implies that $L(f) = L(id)$. The result follows from $L(id) = \chi(M) \neq 0$. $\square$

recall: E.g. $\chi(S^{2n}) = 2 \quad \forall n \in \mathbb{N}$.

Lemma: If $f: M \to M$ is smooth on a connected, compact manifold with boundary, then $\boxed{tr[f^*: H^0_\Omega(M) \to H^0_\Omega(M)] = 1}$.

proof: If $\omega \in \Omega^0 M$ is s.t. $[\omega] \in H^0_\Omega(M)$, then $\exists c \in \mathbb{R}\ \forall p \in M: \omega(p) = c$.

Since $(f^*\omega)(p) = \omega(f(p)) = c$, we have $f^*: [\omega] \mapsto [\omega]$, so

$f^* = id: H^0_\Omega(M) \to H^0_\Omega(M)$. $\square$

Prop.: Let M be a smooth, connected, compact manifold with boundary that satisfies $H^k_\Omega(M) \simeq 0 \quad \forall k > 0$. Then every smooth $f: M \to M$ has a fixed point.

proof: $\Lambda(f) = tr[f^*: H^0_\Omega(M) \to H^0_\Omega(M)] = 1$. $\square$

Cor.: (Brouwer fixed point thm.) Let M be a contractible compact smooth manifold with boundary. Then every continuous map $f: M \to M$ has a fixed point.

proof: Suppose there was no fixed point. Using compactness we can approximate f by a smooth map $\tilde{f}: M \to M$ that also has no fixed point. However, $\Lambda(\tilde{f}) = 1$ since $\beta_k(M) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$. $\lightning$ $\square$

For the real projective space $\mathbb{R}P^n := S^n/\sim$ where $x \sim (-x)$ one can show that $H^*_\Omega(\mathbb{R}P^n) \simeq \begin{cases} H^*_\Omega(\mathbb{R}^n), & n \text{ even} \\ H^*_\Omega(S^n), & n \text{ odd} \end{cases}$. This implies:

Cor.: For n even, every continuous map $f: \mathbb{R}P^n \longrightarrow \mathbb{R}P^n$ has a fixed point.

remark: $\mathbb{R}P^n$ is not contractible for any $n \in \mathbb{N}$. So Brouwer's fixed point theorem does not apply.

Degree theory

Thm./Def.: Let M, N be smooth oriented manifolds of the same dimension n and with finite good covers. If N is connected and $f: M \to N$ a smooth proper map, there is a unique $\deg(f) \in \mathbb{R}$, called the **degree** of f, s.t.

$\forall \omega \in \Omega_c^n N$:
$$\boxed{\int_M f^*\omega = \deg(f) \int_N \omega \, .}$$

remarks:
- Note that any continuous map $f: M \to N$ is proper if M is compact.
- $\deg(f)$ is also known as Brouwer degree / topological degree / mapping degree

proof: Since f is proper, the pullback induces a map $f^*: H_c^n(N) \longrightarrow H_c^n(M)$.

Poincaré duality together with connectedness of N implies that

$H_c^n(N) \cong \mathbb{R}$ and $H_c^n(M) \cong \mathbb{R}^m$, where $m := \#$ connected comp. of M.

Specifically, we get that

$$H_c^n(N) \ni [\omega] \longmapsto \int_N \omega \in \mathbb{R} \, ,$$

$$H_c^n(M) \ni [\eta] \longmapsto \left(\int_{M_i} \eta \right)_{i=1}^m \in \mathbb{R}^m$$

(with $M = \bigcup_{i=1}^m M_i$, M_i connected) are vect. space isomorphisms. So we can define

$\deg(f)$ via the com. diagram:

$$\begin{tikzcd}
H_c^n(N) \arrow[r, "f^*"] \arrow[dd, "\int_N"', "\text{isom.}"] & H_c^n(M) \arrow[d, "\text{isom.}"', "\int_{M_i}"] \\
 & \mathbb{R}^m \arrow[d, "\text{hom.}"', "v \mapsto \sum_{i=1}^m v_i"] \\
\mathbb{R} \arrow[r, "\text{mult. by } \deg(f)", "(\text{isom. if } m=1)"'] & \mathbb{R}
\end{tikzcd}$$

Since any $\omega \in \Omega^n_c(N)$ and also any $f^*\omega \in \Omega^n_c(M)$ is a closed form (as $n = \dim(N) = \dim(M)$) they are representatives of cohomology classes and $\deg(f)\int_N \omega = \sum_i \int_{M_i} f^*\omega = \int_M f^*\omega$.

Uniqueness follows by considering any ω with $\int_N \omega \neq 0$. □

example: If $f: M \to N$ is a diffeomorphism that preserves or reverses orientation, then $\deg(f) = 1$ or $\deg(f) = -1$, resp. since

$$\int_M f^*\omega = \pm\int_{f(M)} \omega = \underbrace{\pm}_{\deg(f)}\int_N \omega .$$

Thm.: Let M, N, K be oriented smooth n-dim. manifolds with finite good covers, and N and K connected. If $M \overset{f}{\underset{g}{\rightrightarrows}} N \xrightarrow{h} K$ are proper smooth maps, then:

(i) Homotopy invariance: $\boxed{f \simeq g \Rightarrow \deg(f) = \deg(g)}$ proper homotopy!

(ii) Multiplicativity: $\boxed{\deg(h \circ g) = \deg(h) \cdot \deg(g)}$

(iii) If $y \in N$ is any regular value of f, then:

$$\boxed{\deg(f) = \sum_{p \in f^{-1}(\{y\})} \operatorname{sgn}(\det(d_p f))}$$

recall: y is a regular value if $\forall p \in f^{-1}(\{y\}) : \det(d_p f) \neq 0$. By Sard's thm. the set of regular values is open and dense.

In particular, $\boxed{\deg(f) \in \mathbb{Z}}$

remark: note that if the manifolds are compact, then 'proper' and 'finite good cover' are guaranteed by compactness.

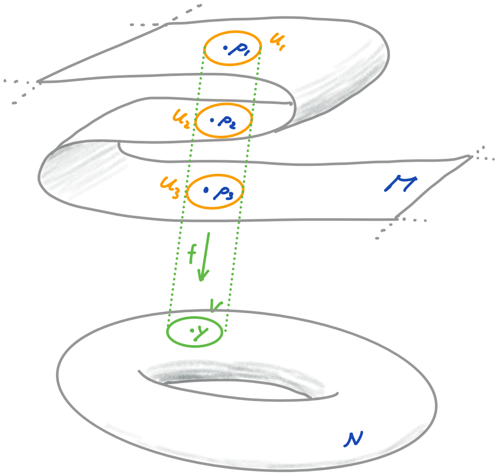
proof: (i) If there is a proper homotopy between f & g, then $f^* = g^* : H^n_c(N) \to H^n_c(M)$. Since the degree only depends on this induced map, we have $\deg(f) = \deg(g)$.

(ii) For any $[\omega] \in H^n_c(M)$ by def. & uniqueness of the degree:

$$\int_K (h \circ g)^* \omega = \int_K g^*(h^* \omega) = \deg(g) \int_N h^* \omega = \underbrace{\deg(g) \deg(h)}_{= \deg(h \circ g)} \int_M \omega .$$

(iii) By the 'regular value thm.' $f^{-1}(\{y\})$ is a smooth submanifold of dimension $\dim(M) - \dim(N) = 0$. So it is a discrete set, which is finite due to the fact that f is proper. So $f^{-1}(\{y\}) = \{p_1, \dots, p_k\} \subseteq M$. Since $\det(d_{p_i} f) \neq 0$ there are open neighborhoods $U_i \ni p_i$ s.t. $f|_{U_i}$ is a diffeomorphism onto a neighborhood of y. W.l.o.g. we assume the U_i's disjoint and s.t. $f(U_i) = V \; \forall i$ and $f^{-1}(V) = \bigcup_i U_i$. Pick any $\omega \in \Omega^n_c(V)$ with

$$1 = \int_V \omega = \int_N \omega$$



Then $f^*\omega = \sum_{i=1}^{k} (f^*\omega)\big|_{U_i}$

where each $(f^*\omega)\big|_{U_i}$ has

compact support in U_i.

So $\deg(f) = \deg(f) \int_N \omega$

$$= \int_M f^*\omega$$

$$= \sum_{i=1}^{k} \int_{U_i} (f^*\omega)\big|_{U_i}$$

$f\big|_{U_i}: U_i \to V$ is diffeom.

$$= \sum_{i=1}^{k} \operatorname{sgn}\left(\det\left(d_{p_i} f\right)\right). \qquad \square$$

remarks: ◦ note that the proof of (iii) still works if $f^{-1}(\{y\}) = \emptyset$: in this case, we can choose V s.t. $f^{-1}(V) = \emptyset$, which implies $f^*\omega = 0$ for $\omega \in \Omega^n_c(V)$.

◦ $\deg(f)$ can be interpreted as the nr. of windings of M around N by f.

example: Regard S^1 as $\{ z \in \mathbb{C} \mid |z| = 1 \}$, $f: S^1 \to S^1$, $z \mapsto z^n$ for some $n \in \mathbb{Z}$.

If $\omega := \frac{1}{2\pi} d\theta$ is the standard volume form on S^1, then $f^*\omega = n \cdot \omega$.

So $\deg(f) = n$.

Prop.: Let M, N be smooth n-dim. oriented manifolds with finite good covers and N connected. If a smooth proper map $f: M \to N$ is not surjective, then $\deg(f) = 0$.

proof: Suppose $y \in N \setminus f(M)$. Then y is a regular value with $f^{-1}(\{y\}) = \emptyset$. So $\deg(f) = 0$. $\square$

From here we can obtain a generalization of the fundamental theorem of algebra:

Thm.: Let $f: M \to N$ be a proper map between oriented non-compact, n-dim. manifolds with finite good cover, where N is connected. If f is orientation preserving (and thus non-singular) outside a compact set C, then f is surjective.

proof: Since f is proper, $f^{-1}(\underbrace{f(C)}_{\text{compact due to cont.}})$ is compact. Hence, there is a point $x \in M \setminus f^{-1}(f(C))$, which then satisfies $y := f(x) \notin f(C)$.

Then y is a regular value (since all critical points lie in C) and

$$\deg(f) = \sum_{p \in f^{-1}(\{y\})} \operatorname{sgn}\det(d_p f) \underset{\substack{\uparrow \\ p \notin C \text{ so } \det(d_p f) > 0}}{=} |f^{-1}(\{y\})| > 0$$

$f^{-1}(\{y\}) \neq \emptyset$

According to the previous Cor., f must be surjective. $\square$

remarks: ∘ If $M \setminus \{\text{critical points}\}$ is connected, we can replace 'orientation preserving' by 'non-singular', since $M \setminus \{\text{critical points}\} \ni p \mapsto \operatorname{sgn}\det(d_p f)$ is then constant $+1$ or -1.

However, in particular if $\dim(M)=1$, this may not be connected.

E.g. for $M=N=\mathbb{R}$, $f(x)=x^2$, $\{\text{critical points}\}=\{0\}$ and despite this being compact, f is not surjective.

Lemma: Let $F: \mathbb{C}^n \to \mathbb{C}^n$ be represented by $f: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ when representing $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ in terms of real and imaginary part. If F is holomorphic, then $\forall p \in \mathbb{R}^{2n}$: $\det(d_p f) \geq 0$.

proof: Holomorphic means that the derivative at each point is given by a complex linear map. At any given point let this be represented by a complex Jacobian matrix $Z = X + iY$ with $X, Y \in \mathbb{R}^{n \times n}$. The Jacobian of f is then $J = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} = U^* \begin{pmatrix} \bar{Z} & 0 \\ 0 & Z \end{pmatrix} U$ where $U := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{pmatrix}$ is a unitary. Hence, $\det(J) = |\det(Z)|^2 \geq 0$.

□

So for holomorphic maps, we can replace 'orientation preserving' by 'non-singular'. The fundamental thm. of algebra then becomes a special case of the above thm. due to the following:

Lemma: Every non-constant polynomial $f: \mathbb{C} \to \mathbb{C}$ is a proper map.

proof: Since $|f(z)| \to \infty$ as $|z| \to \infty$, the preimage of bounded sets has to be bound. Due to continuity, -"- of closed -"- to be closed. Since for $\mathbb{C}$ we have compact = closed + bounded, f is a proper map. $\square$

remark: note that this also implies that the set of critical points $C := \{ z \in \mathbb{C} \mid f'(z) = 0 \}$ of any non-constant polynomial is compact since f' is a ain a polynomial (and $C = \emptyset$ if f' is const.).

In some cases $\deg(f)$ has a close relation to the Lefschetz number:

Prop.: Let M be a connected compact oriented n-dim. manifold and $f: M \to M$ a smooth map. Then

$$\boxed{tr[f^*: H^n_{\Omega}(M) \to H^n_{\Omega}(M)] = \deg(f)}$$

If $M = S^n$ or if n is odd and $M = \mathbb{R}P^n$, then:

$$\boxed{\Lambda(f) = 1 + (-1)^n \deg(f)}$$

proof: Due to compactness of M, $H^n_c(M) = H^n_{\Omega}(M)$ and by Poincaré-duality $\dim(H^n_{\Omega}(M)) = 1$. By definition of $\deg(f)$ we have for any $\omega \in H^n_{\Omega}(M)$: $f^*\omega = \deg(f)\,\omega$. So $tr[f^*: H^n_{\Omega}(M) \to H^n_{\Omega}(M)] = \deg(f)$.

For $M \in \{S^n, \mathbb{R}P^{2k+1}\}$ we have $H^m_{\Omega}(M) = \{0\}$ for all $m \notin \{0, n\}$.

Moreover, $tr[f^*: H^0_{\Omega}(M) \to H^0_{\Omega}(M)] = 1$ due to connectedness. $\square$

The degree can also serve as an obstruction to extending a map:

Prop.: Let $F: N \to M$ be smooth between compact, connected, oriented manifolds, where $\dim(M) = n = \dim(N) - 1$ and N has a boundary ∂N.

Then $f := F|_{\partial N}$ has $\deg(f) = 0$.

proof: Consider $\omega \in \Omega^n M$ with $\int_M \omega = 1$. Then

$$\deg(f) = \int_{\partial N} f^* \omega \underset{\text{Stokes}}{=} \int_N d F^* \omega = \int_N F^* d\omega = 0. \qquad \square$$

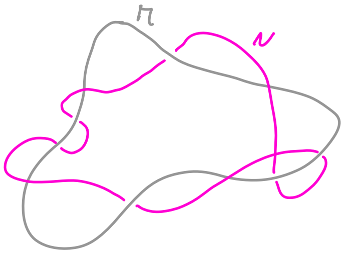
The degree of maps into S^n is particularly important. Partly due to:

Thm.: (Hopf degree theorem) Let M be a compact, connected, oriented n-dim. manifold and $f, g: M \to S^n$ two smooth maps.

$$\boxed{\underset{\text{homotopic}}{f \simeq g} \iff \deg(f) = \deg(g)}$$

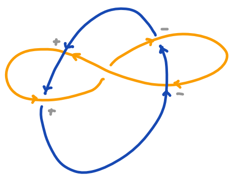
Def.: Let $M, N \subseteq \mathbb{R}^{m+n+1}$ be two disjoint, closed, oriented submanifolds of dimensions $\dim(M) = m$ and $\dim(N) = n$. Their linking number is defined as

$$\boxed{\begin{aligned} &l(M,N) := \deg\left(F: M \times N \to S^{n+m}\right), \\ &\text{where} \quad F(x,y) := \frac{x-y}{\|x-y\|}. \end{aligned}}$$



If $m, n \geq 1$, then using homotopy invariance of the degree, one can for instance show that if M is contractible to a point without intersecting N, then $l(M, N) = 0$.

However, e.g. the Whitehead link has $l(M, N) = 0$ although it is not 'isotopic to the unlink'.



The winding number is a special case of the linking number, where N is a single point.

Vector fields & flows

Recall: A smooth **vector field** X on a smooth manifold M can equivalently be characterized as

$$\boxed{\begin{array}{l}\text{a smooth map} \\ X : M \longrightarrow TM \\ \quad p \longmapsto X_p\end{array}} \quad \text{or} \quad \boxed{\begin{array}{l}\text{a linear derivation} \\ X : C^\infty(M) \longrightarrow C^\infty(M) \\ \quad f \longmapsto (p \mapsto X_p f)\end{array}}$$

(caution: the same symbol is used for both)

The space $\mathfrak{X}(M)$ of all smooth vector fields on M is a **Lie algebra**.

That is a vector space with a bilinear, alternating map $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$, called **Lie bracket**, that satisfies the **Jacobi identity** $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

In this case, $[X,Y] = XY - YX$.

Def.: **(Pullback of a vector field)** Let $f : M \to N$ be a local diffeomorphism and $Y \in \mathfrak{X}(N)$. The **pullback** of Y by f is defined as the vector field $f^*(Y) \in \mathfrak{X}(M)$ that maps

$$\boxed{M \ni p \longmapsto (d_p f)^{-1} Y_{f(p)} \in T_p M}$$

remark: for a general smooth map, $f^*(Y)$ cannot be defined consistently.

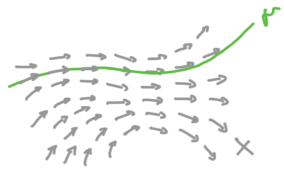
Def.: A curve $\gamma : (a,b) \to M$ is called an integral curve of a vector field $X \in \mathfrak{X}(M)$ if $\forall t \in (a,b)$:

$$\boxed{\dot\gamma(t) = X_{\gamma(t)}}$$

or equivalently, for any $f \in C^\infty(M)$:

$$\boxed{(f \circ \gamma)'(t) = X_{\gamma(t)} f}$$

X is called complete if each of its integral curves can be defined $\forall t \in \mathbb{R}$.



If the image of γ is in a chart (U,x),

and $X_p = \sum_i v_i(p) \frac{\partial}{\partial x_i}\Big|_p$, $x_i(t) := x_i \circ \gamma(t)$ then

$\frac{d}{dt} x_i(t) = v_i(\gamma(t)) = (v_i \circ x^{-1})(x_1(t), \dots, x_n(t))$ is a system of ODEs for $x_i(t)$.

Given an initial value, this will have a unique (maximal) solution.

Note that a reparametrization of an integral curve is i.g. not an integral curve anymore. However, for any $p \in M$ we can choose an integral curve, denoted by $\gamma_p : I_p \to M$, s.t. $\gamma_p(0) = p$.

This leads to a map $\phi(\underbrace{t,p}_{\in\, \mathbb{R}\times M}) := \underbrace{\gamma_p(t)}_{\in\, M}$, s.t.

for $\phi_t(p) := \phi(t,p)$ we have:

$$\phi_0 = id \quad \text{and} \quad \phi_t \circ \phi_s = \phi_{t+s} \quad (\text{for suitable } t,s)$$

This motivates the following:

Def.: Let M be a smooth manifold, U an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$. A smooth map $\phi: U \to M$ with $\phi_t := \phi(t, \cdot): M \to M$ is called a *flow* on M if

(i) $\phi_0 = id$

(ii) $\phi_t \circ \phi_s = \phi_{t+s}$ whenever defined.

The *infinitesimal generator* of a flow ϕ is the vector field

$$X: C^\infty(M) \to C^\infty(M), \quad f \mapsto \left(M \ni p \mapsto \frac{\partial}{\partial t}\Big|_{t=0} f \circ \phi(t,p) \right)$$

A flow is called *global* if $U = \mathbb{R} \times M$.

remarks:

- X is a linear derivation as a result of its definition via a derivative. If $\gamma_p(t) := \phi(t,p)$, then $\gamma_p(0) = p$ and $X_p = \dot{\gamma}_p(0)$.
- The term *local flow* is sometimes used to emphasize that a flow is not necessarily global. A *maximal flow* is one for which U cannot be extended further.

Results on existence & uniqueness of ODE solutions lead to:

Thm.: $\boxed{\text{For every smooth vector field } X \text{ on a smooth manifold } M \text{ there is a unique maximal flow whose inf. generator is } X.}$

In particular: $\boxed{\text{complete vector field} \longleftrightarrow \text{global flow}}$

Deciding whether this is the case may not be easy, but there are useful/insightful sufficient conditions:

Prop.: Let X be a smooth vector field on M.

(i) $\boxed{\operatorname{supp}(X) := \overline{\{p \in M \mid X_p \neq 0\}} \text{ compact} \Rightarrow X \text{ is complete}}$

(ii) If $\gamma: I \to M$ is an integral curve with max. domain I, then

$$\boxed{\overline{\gamma(I)} \text{ compact} \Rightarrow I = \mathbb{R}}$$

remark: In particular, if M is compact, then every $X \in \mathfrak{X}(M)$ is a complete vector field.

Def.: On a smooth manifold M we define the diffeomorphism group

$$\operatorname{Diff}(M) := \{ F: M \to M \mid F \text{ is } C^\infty\text{-diffeomorphism} \}$$

Cor.: For any flow ϕ on M the map $\mathbb{R} \ni t \longmapsto \phi_t \in \operatorname{Diff}(M)$ is a group homomorphism (from $(\mathbb{R},+)$ into $\operatorname{Diff}(M)$ with composition)

So if M is compact, every smooth vector field generates a commutative one-parameter subgroup of transformations.

Thm.: (Poincaré-Hopf I) On a compact, connected smooth manifold M there exists a nowhere-vanishing vector field $X \in \mathfrak{X}(M)$ iff $\chi(M) = 0$.

proof: (of the 'only if' part, which does not require connectedness.) Suppose X is nowhere vanishing and ϕ is the corresponding flow. Then all ϕ_t are homotopic (with homotopy ϕ). Due to compactness and the fact that $X_p \neq 0$ $\forall p$ there is an $\varepsilon > 0$ s.t. ϕ_ε has no fixed point. So

$$0 = L(\phi_\varepsilon) \underset{\substack{\uparrow \\ \phi_\varepsilon \simeq \phi_0 = id}}{=} L(id) = \chi(M). \qquad \square$$

remark: non compact manifolds always admit nowhere vanishing vector fields.

Cor.: (Hairy ball thm.) On an even dimensional sphere S^{2n} there is no nowhere vanishing vector field.

remark: ... and therefore no Lorentzian metric.

proof: $\chi(S^{2n}) = 2$. $\square$

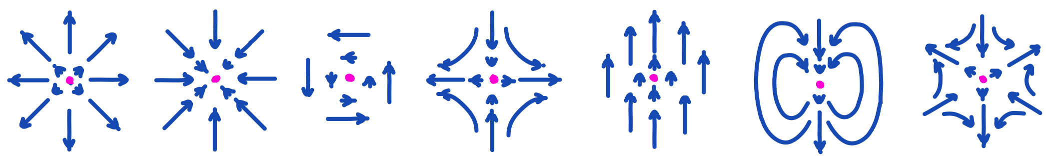
An approach for proving the 'if part' in the Poincaré-Hopf thm. is better understood when considering a more quantitative version.

Def.: Let M be a smooth manifold, $X \in \mathfrak{X}(M)$, and $p \in M$ an isolated zero of X. Let $f : B := \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \} \to M$ extend to a local diffeomorphism s.t. $f(0) = p$ is the only zero of X in $f(B)$.

Define the index $\quad \mathrm{index}(X, p) := \deg(\varphi) \quad$ where $\varphi : S^{n-1} \to S^{n-1}$

$$\varphi(y) := \frac{f^*(X)_y}{\| f^*(X)_y \|}$$

Zeros of a vector field and the corresponding index:



Thm.: (Poincaré-Hopf II) Let M be a compact smooth manifold, and $X \in \mathfrak{X}(M)$ with only a finite set of zeros $Z := \{ p \in M \mid X_p = 0 \}$. Then

$$\boxed{\chi(M) = \sum_{p \in Z} \mathrm{index}(X, p)} \; .$$

remarks:
- This still holds for manifolds with boundary if X is outward-pointing at the boundary.
- An alternative/equivalent way of also defining the index and proving the theorem as corollary of Lefschetz-Hopf is:

$$\chi(M) = L(\mathrm{id}) = L(\phi_\varepsilon) \underset{\text{assuming non-degeneracy}}{=} \sum_{p = \phi_\varepsilon(p)} \underbrace{\mathrm{sgn}(\det(\mathbb{1} - d_p \phi_\varepsilon))}_{\equiv \, \mathrm{index}(X, p)}$$

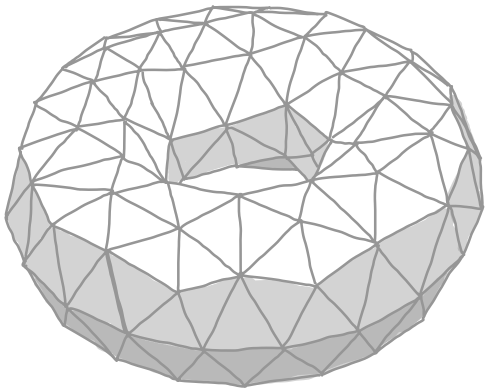
Def.: A triangulation of a topological space M is a homeomorphism between the geometric realization of a simplicial complex and M.

remark: For smooth manifolds, triangulations always exist and can be chosen s.t. the restriction to individual simplexes is smooth.

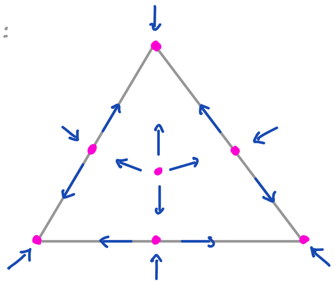
One can construct a vector field X s.t.

(i) every simplex σ is assigned to a zero with $\operatorname{index}(X, p) = (-1)^{\dim(\sigma)}$

(ii) there are no other zeros.



For instance:



The Poincaré-Hopf theorem then gives:

Thm.: For any smooth n-dim. manifold M:

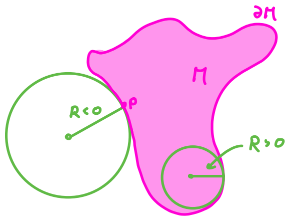
$$\boxed{\chi(M) = \sum_{i=0}^{n} (-1)^i k_i}$$ where k_i is the nr. of i-dim. simplices in a triangulation of M.

remarks:
- for $n=2$ this gives the famous $\chi = V - E + F$
- needless to say, but the k_i's depend on the choice of triangulation while $\chi(M)$ doesn't.

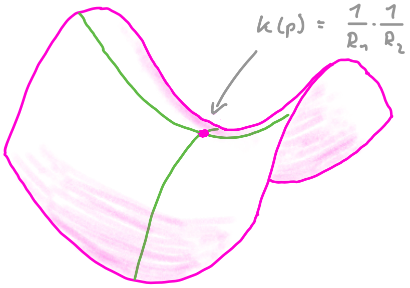
Def.: Let $M \subseteq \mathbb{R}^{n+1}$ be a compact $n+1$-dim. smooth manifold with boundary ∂M. The Gauss map $\nu: \partial M \to S^n$ is s.t. $\nu(p)$ is the unique outward pointing unit vector that is orthogonal to the tangent plane of ∂M at p. The Gauss curvature of ∂M at p is $\boxed{k(p) := \det(d_p\nu)}$.

remarks:

- Since we can identify $T_{\nu(p)} S^n \cong \nu(p)^\perp \cong T_p \partial M$, we can regard $d_p\nu : T_p\partial M \to T_p \partial M$ s.t. $\det(d_p\nu)$ makes sense.

- 

  For $n=1$ the curvature at p is $k(p) = \frac{1}{R}$ where R is the radius of a ball tangent to the curve at p. In n dimensions, there are n 'principal curvatures', which are the eigenvalues of $d_p\nu$.



- The standard volume form $vol_{\partial M} \in \Omega^n \partial M$ can be expressed in terms of the Gauss map as $(vol_{\partial M})_p (v_1, \dots, v_n) = \det(\nu(p), v_1, \dots, v_n)$ where we view $v_1, \dots, v_n, \nu(p) \in \mathbb{R}^{n+1}$ geometrically.

With $(\mathrm{vol}_{S^n})_x(f_1,\dots,f_n) = \det(x, f_1,\dots,f_n) \quad \forall x \in S^n$ we obtain

$$\begin{aligned}(\nu^* \mathrm{vol}_{S^n})_p(v_1,\dots,v_n) &= (\mathrm{vol}_{S^n})_{\nu(p)}(d_p\nu\, v_1, \dots, d_p\nu\, v_n) \\ &= \det(\nu(p), d_p\nu\, v_1, \dots, d_p\nu\, v_n) \\ &= \det(d_p\nu)\, \det(\nu(p), v_1,\dots,v_n) \\ &= \kappa(p)\, (\mathrm{vol}_{\partial M})_p(v_1,\dots,v_n).\end{aligned}$$

So $$\boxed{\kappa\, \mathrm{vol}_{\partial M} = \nu^*(\mathrm{vol}_{S^n})}.$$

<u>Thm.:</u> (Gauss-Bonnet) If ∂M is an even-dim. boundary of an $n+1$-dim compact smooth submanifold $M \subseteq \mathbb{R}^{n+1}$, then

$$\boxed{\int_{\partial M} \kappa\, \mathrm{vol}_{\partial M} = \tfrac{1}{2} \mathrm{Vol}(S^n)\, \chi(\partial M)}$$

where $\mathrm{Vol}(S^n) := \int_{S^n} \mathrm{vol}_{S^n} \underset{n=2m}{\overset{\uparrow}{=}} \frac{2^{n+1} m!}{n!} \pi^m = \begin{cases} 4\pi, & n=2 \\ \frac{8}{3}\pi^2, & n=4 \end{cases}.$

<u>note:</u> While the l.h.s. is geometrical, the r.h.s. is purely topological.

<u>proof:</u> $$\int_{\partial M} \kappa\, \mathrm{vol}_{\partial M} = \int_{\partial M} \nu^*(\mathrm{vol}_{S^n}) = \deg(\nu) \underbrace{\int_{S^n} \mathrm{vol}_{S^n}}_{\mathrm{Vol}(S^n)}$$

Since, by Sard's thm., reg. values are open and dense, there is a pair $\{y, -y\} \subseteq S^n$ of regular values of ν. Then

$$\deg(\nu) = \sum_{p \in \nu^{-1}(\{y\})} \mathrm{sgn}(\det(d_p\nu)) \underset{Z := \nu^{-1}(\{y,-y\})}{\overset{\uparrow}{=}} \frac{1}{2} \sum_{p \in Z} \mathrm{sgn}(\det(d_p\nu))$$

Now construct a vector field X_p on ∂M by projecting y onto $T_p\partial M$. Since $X_p = 0 \Leftrightarrow y \perp T_p\partial M \Leftrightarrow p \in Z$, Poincaré-Hopf leads to

$$\chi(\partial M) = \sum_p \operatorname{index}(X, p).$$

A closer look reveals that $\operatorname{index}(X,p) = \begin{cases} \operatorname{sgn}(\det(d_p\nu)), & \text{if } \nu(p) = y \\ (-1)^n \quad \text{---}, & \text{if } \nu(p) = -y \end{cases}$

So if n is even, then $\deg(\nu) = \frac{1}{2}\chi(\partial M)$. $\square$

For odd-dim. compact hypersurfaces, we have $\chi(\partial M) = 0$ and the statement is not true. However, a slightly different strategy leads to:

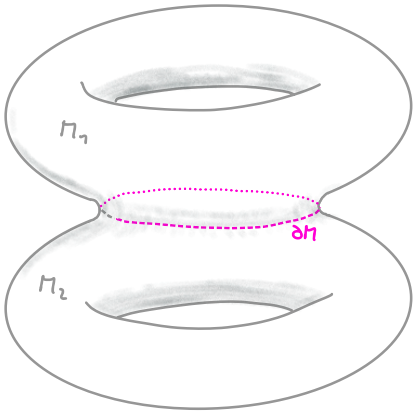
Thm.: (Gauss-Bonnet II) If ∂M is the boundary of an $n+1$-dim compact smooth submanifold $M \subseteq \mathbb{R}^{n+1}$, then

$$\boxed{\int_{\partial M} K \, \mathrm{vol}_{\partial M} = \mathrm{Vol}(S^n)\, \chi(M)}.$$

Instead of proving this (which can again be done by exploiting Poincaré-Hopf to show that $\deg(\nu) = \chi(M)$), we show how the two theorems imply each other if n is even:

Lemma: Let M be a compact orientable manifold with boundary ∂M. If M has odd dimension, then $\boxed{2\chi(M) = \chi(\partial M)}$.

proof: (sketch) We take two copies M_1 and M_2 of M and glue them together at the boundary. The resulting manifold M_{12} is then an odd-dimensional orientable compact manifold with an open cover



$U_1 \cup U_2 = M_{12}$ s.t. $U_i \simeq M$ and $U_1 \cap U_2 \simeq \partial M$. Hence,

$$\underset{\substack{\uparrow \\ \text{Poincaré duality}}}{0 =} \chi(M_{12}) = \underbrace{\chi(U_1) + \chi(U_2)}_{2\chi(M)} - \underbrace{\chi(U_1 \cap U_2)}_{\chi(\partial M)}$$

$$= 2\chi(M) - \chi(\partial M)$$

$\square$

Fiber bundles – a quick walk-through

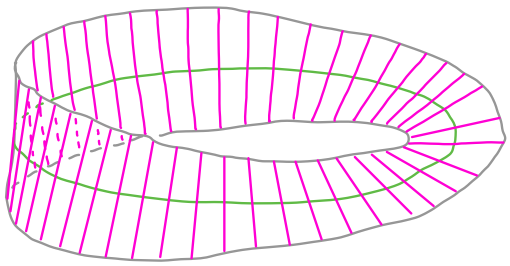
Loosely speaking, a fiber bundle is a topological space E that looks locally like a product $B \times F$.

<u>Def.:</u> Let E, B, F be topological spaces and $\pi: E \to B$ a continuous surjection. (E, B, π, F) is a fiber bundle with typical fiber F if for every $p \in B$ there is an open neighborhood $U \subseteq B$ and a homeomorphism $\varphi: \pi^{-1}(U) \to U \times F$ s.t. the following commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

- A smooth fiber bundle is one for wich E, B, F are smooth manifolds and all involved maps are smooth.

- B : base space
- E : total space
- π : (bundle) projection
- $\pi^{-1}(p) =: F_p$: fiber over p
- $\{(U_\alpha, \varphi_\alpha)\}$: local trivialization

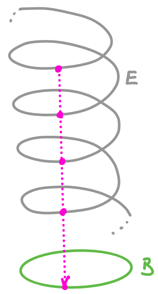


Möbius strip:
$B = S^1$
$F = [0,1]$

- A section of a fiber bundle is a cont. map $\sigma: B \to E$ s.t. $\pi \circ \sigma = id_B$

<u>examples:</u>

- A product space $E = B \times F$ is a trivial fiber bundle.
- The tangent bundle TM of a smooth manifold is an instance of a vector bundle, i.e. a fiber bundle, where F is a vector space.
- The Klein bottle with $B = S^1 = F$ is an instance of a sphere bundle, i.e. a fiber bundle, where F is a sphere.
- A covering space is a fiber bundle for which π is a local homeomorphism, and consequently F a discrete space.



E.g.: ◦ $E = S^n$ is a two-fold covering of $B = \mathbb{R}P^n$ with $\pi : S^n \longrightarrow \mathbb{R}P^n$, $x \longmapsto [x]$ where $x \sim -x$.

◦ $E = SU(2)$ is a two-fold covering of $B = SO(3)$.

$\Big\}$ here $F = \mathbb{Z}_2$

- Smooth sections on TM are exactly the vector fields.

<u>Def.:</u> Two (smooth) fiber bundles $\pi : E \to B$, $\pi' : E' \to B$ with typical fiber F are isomorphic if there is a $\left.\begin{matrix}\text{(diff)}\\ \text{hom}\end{matrix}\right\}$eomorphism $\phi : E \to E'$ s.t. $\pi' \circ \phi = \pi$.

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

A bundle is trivializable (or just trivial) if it is isomorphic to the trivial bundle $E' = B \times F$.

<u>Prop.:</u> Every (smooth) fiber bundle with contractible B is trivializable.

Def.:
- A (smooth) vector bundle is a (smooth) fiber bundle where the typical fiber and each $F_x := \pi^{-1}(\{x\})$ is a vector space and where the homeomorphisms φ can be chosen s.t. $\varphi^{-1}(x, \cdot) : F \to F_x$ is a vector space isomorphism.
- The rank of a vector bundle is the dimension of F.
- Two (smooth) vector bundles over the same B are isomorphic if there exists a (smooth) continuous map $\phi : E \to E'$ s.t. $\pi = \pi' \circ \phi$ and ϕ maps each F_x as vector space isomorphically onto F_x'.
- A vector bundle (E', B', π', F') is a subbundle of a vector bundle (E, B, π, F) if $E' \subseteq E$ and each fiber F_x' is a vector subspace of F_x.

remarks:
- Although not evident from the above characterization, a (smooth) vector bundle isomorphism $\phi : E \to E'$ is s.t. ϕ^{-1} is again a (smooth) v.b. isomorphism.
- Analogous to Whitney's embedding thm.: every smooth vector bundle over a smooth manifold B is a subbundle of a trivial vector bundle. To make this more precise, we introduce the following:

Def.: The Whitney sum of two vector bundles (E_i, B, π_i, F_i), $i \in \{1,2\}$ is the vector bundle $(E_1 \oplus E_2, B, \pi, F_1 \oplus F_2)$ with $E_1 \oplus E_2 := \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}$ and $\pi : (e_1, e_2) \mapsto \pi_1(e_1)$.

Note that we can regard e.g. E_1 as a subbundle of $E_1 \oplus E_2$ via $\iota : E_1 \to E_1 \oplus E_2$, $\iota : (p, v) \longmapsto ((p,v), (p,0))$.

Thm.: For every smooth vector bundle (E, B, π, F) there is a smooth vector bundle $(\tilde{E}, B, \tilde{\pi}, \tilde{F})$ s.t. their Whitney sum $E \oplus \tilde{E}$ is trivial.

An important example of such a pair of 'Whitney sum inverse' vector bundles is the tangent bundle & normal bundle.

Def.: Let $M \subseteq W$ be an embedded smooth submanifold. The normal bundle NM of M in W is defined as the vector bundle $NM \xrightarrow{\pi} M$ where $NM := \bigcup_{p \in M} \{ N_pM := T_pW / T_pM \}$ and $\pi : N_pM \mapsto p$.

remark: If $\Delta \subseteq M \times M$ is the diagonal submanifold, then $N\Delta$ and $T\Delta$ are isomorphic vector bundles.

Thm.: If M is a smooth manifold embedded in some $\mathbb{R}^N$, then $TM \oplus NM$ is trivial (with typical fiber $\mathbb{R}^N$).

A general criterion for a vector bundle to be trivial is the following:

Prop.: A rank k vector bundle (E, B, π, F) is trivializable iff there exist k continuous sections $s_i : B \to E$ s.t. for all $p \in B$: $s_1(p), \dots, s_k(p)$ are linearly independent.

proof: If E is isomorphic to $B \times \mathbb{R}^k$, then we can set $s_i(p) := (p, e_i)$ for any basis $e_1, \dots, e_k$ of $\mathbb{R}^k$.

Conversely, we define $\phi : E \to B \times \mathbb{R}^k$ s.t. for any $(p, v) \simeq x \in E$ with $v =: \sum_{i=1}^{k} v_i \, s_i(p)$ we set $\phi(x) := (p, (v_1, \dots, v_k))$. $\square$

remark: So a tangent bundle TM is trivializable (in which case the manifold M is called parallelizable) iff there are $\dim(M) =: k$ vector fields $X^{(1)}, \dots, X^{(k)} \in \mathfrak{X}(M)$ s.t. $\forall p \in M : \operatorname{span}\{X_p^{(i)}\}_{i=1}^{k} = T_pM$.

Note that a parallelizable manifold is automatically orientable.

Cor.: Let G be a Lie group (i.e. a group that is also a smooth manifold with smooth group operations). G is parallelizable.

proof: For any $g \in G$ define $L_g : G \to G$, $h \mapsto g \cdot h$ (which is smooth also in g) and let $v_1, \dots, v_n$ be a basis of T_eG (with 'e' the identity of G).

Then for any $p \in G$, $X_p^{(i)} := d_e L_p v_i$ forms a basis of T_pG. $\square$

Thm.: Let $E \xrightarrow{\pi} M$ be a smooth vector bundle of rank r over an n-dim. smooth manifold M.

(i) $\boxed{H^k_{\Omega}(E) \simeq H^k_{\Omega}(M)}$ $\forall k$

(ii) If E, M are oriented and have finite good cover, then

$\boxed{H^k_c(E) \simeq H^{k-r}_c(M)}$ $\forall k$ (Thom duality)

proof: (i) By considering the zero section $s_0 : M \to E$, $x \mapsto (x, 0)$,

we see that E is homotopy equivalent to M since

$\pi \circ s_0 = id_M$ and $s_0 \circ \pi \simeq id_E$ via the homotopy

$H : \mathbb{R} \times E \to E$, $(t, (p,v)) \mapsto (p, tv)$.

(ii) Using Poincaré duality twice together with (i) we get:

$$H^k_c(E) \simeq H^{n+r-k}_{\Omega}(E) \simeq H^{n+r-k}_{\Omega}(M) \simeq H^{k-r}_c(M). \qquad \square$$

Def.: If M is a compact, connected, oriented smooth manifold and E an oriented smooth vector bundle over M of rank r, we define the Thom class $\tau(E) \in H^r_c(E)$ as the compact Poincaré dual of M in E (embedded via the zero section s_0) and the Euler class $e(E) \in H^r_{\Omega}(M)$ as $s_0^*(\tau(E))$.

In the definition of the Euler class we could have used any smooth section:

Lemma: $e(E) = s^*(\tau(E))$ for any smooth section $s: M \to E$

proof: Since s is homotopic to s_0 via $H: \mathbb{R} \times M \longrightarrow E$,

$$(t,p) \mapsto t\, s(p) + (1-t)\, s_0(p). \qquad \square$$

Thm.: Let M be an oriented, compact, connected smooth manifold and $E \xrightarrow{\pi} M$ an oriented smooth vector bundle. If E admits a nowhere vanishing smooth section, then $e(E) = 0$.

proof: Let $s: M \to E$ be such a smooth section, and let $\tau \in \Omega_c^r(E)$ be such that $[\tau] \in H_c^r(E)$ is the Thom class.

Due to the compactness of M and the support of τ, we can choose a $c \in \mathbb{R}$ s.t. the range of $\tilde{s} := c \cdot s$ has empty intersection with $\mathrm{supp}(\tau)$. Thus $e(E) = \tilde{s}^*([\tau]) = [\tilde{s}^* \tau] = 0$. $\square$

remark: If $E = TM$ is the tangent bundle, then $e(TM) = \chi(M) \cdot \mu$ where $\mu \in \Omega^n(M)$ is any volume form of M with $\int_M \mu = 1$.

Hence, $\boxed{\int_M e(TM) = \chi(M)}$ s.t. the thm. generalizes the result that $\chi(M) = 0$ if there exists a n.w.v. vector field.

This is the Gauss-Bonnet-Chern thm.

The Euler class is an example of a 'characteristic class'.

Informally, a characteristic class is a mapping $(E \xrightarrow{\pi} B) \longrightarrow H^*(B)$ that associates to every bundle a cohomology class of its base space in a way that is invariant under bundle isomorphisms.