

NLO QCD renormalization group evolution for nonleptonic $\Delta F = 2$ transitions in the SMEFT

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We present for the first time Next-to-Leading (NLO) QCD renormalization group (RG) evolution matrices for nonleptonic $\Delta F = 2$ transitions in the Standard Model effective field theory (SMEFT). To this end we transform first the known two-loop QCD anomalous dimension matrices (ADMs) of the BSM (Beyond the SM) operators in the so-called Buras Misiak Urban basis into the ones in the common weak effective theory (WET) basis (the so-called Jenkins Manohar Stoffer basis) for which tree-level and one-loop matching to the SMEFT are already known. This subsequently allows us to find the two-loop QCD ADMs for the SMEFT nonleptonic $\Delta F = 2$ operators in the Warsaw basis. Having all these ingredients we investigate the impact of these NLO QCD effects on the QCD RG evolution of SMEFT Wilson coefficients for nonleptonic $\Delta F = 2$ transitions from the new physics scale Λ down to the electroweak scale μ_{ew} . The main benefit of these new contributions is that they allow one to remove renormalization scheme dependences present in the one-loop matchings both between the WET and SMEFT and also between SMEFT and a chosen UV completion. But the Next-to-Leading (NLO) QCD effects, calculated here in the Naive dimensional regularisation minimal subtraction scheme, turn out to be small, in the ballpark of a few percent but larger than one-loop Yukawa top effects when only the $\Delta F = 2$ operators are considered. The more complicated class of nonleptonic $\Delta F = 1$ decays will be presented soon in another publication.

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I. INTRODUCTION

Nonleptonic $\Delta F = 2$ transitions, represented by $K^0 - \bar{K}^0$, $D^0 - \bar{D}^0$, and $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings play very important roles in the tests of the Standard Model (SM) and of the new physics (NP) beyond it [1]. To increase the precision of these tests it is necessary to go beyond the leading order (LO) analyses both in the weak effective theory (WET) and also in the Standard Model effective field theory (SMEFT). To this end it is mandatory to include first in the renormalization group (RG) analyses in these theories the one-loop matching contributions, both between these two theories as well as when passing thresholds at which heavy particles are integrated out. But this is not the whole story, a fact which is forgotten in some recent SMEFT analyses present in the literature. To complete a NLO analysis and remove various renormalization scheme (RS) dependences in the one-loop matching also two-loop anomalous dimensions of all operators in the WET and SMEFT have to be included.

The present status of these efforts in the case of nonleptonic meson $\Delta F = 1$ decays and $\Delta F = 2$ quark mixing processes is as follows:

- (i) The matchings in question are known by now both at tree level [2] and one-loop level [3].¹
- (ii) The one-loop anomalous dimension matrices (ADMs) relevant for the RG in WET [9,10] and SMEFT [11–13] are also known.
- (iii) The two-loop QCD ADMs relevant for RG evolutions for both $\Delta F = 1$ and $\Delta F = 2$ transitions in WET are also known [14–16].

The main goal of the present paper is to extend the QCD RG evolution in the SMEFT for $\Delta F = 2$ transitions beyond the leading order. In fact, at first sight this is straightforward because the $SU(3)_c$ symmetry remains unbroken in the SMEFT, and in the absence of electroweak interactions one could just use the NLO QCD Beyond the SM (BSM) analysis of [14] up to the new physics scale Λ in the so-called Buras Misiak Urban (BMU) operator basis that is useful for NLO QCD calculations. However, in the presence of electroweak interactions, the so-called SMEFT Warsaw basis [17] is more suitable, and it is necessary to perform the QCD renormalization group analysis within the SMEFT in

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¹Previous partial results can be found, for example, in [4–8].

that basis. Therefore we present here for the first time to our knowledge two-loop ADMs for $\Delta F = 2$ four-quark operators of the SMEFT. We will demonstrate that they can be obtained from the two-loop ADMs in the BMU basis [14]. As an intermediate step we will present the ADM relevant for the WET in the so-called Jenkins Manohar Stoffer (JMS) basis [2] which for the matching of the WET to the SMEFT is more useful than the BMU basis.

The main technology presented here can be extended to nonleptonic $\Delta F = 1$ transitions but due to the large number of operators involved [14,15] the corresponding analysis is much more complicated and will be presented in due time in a separate publication. Moreover, while in the case of $\Delta F = 2$ operators the transformation of ADMs from BMU to JMS and SMEFT bases is free from contributions of Fierz-vanishing evanescent operators [16], they contribute in the $\Delta F = 1$ case [18], which complicates the analysis.

Having all these ingredients we investigate numerically the impact of the NLO corrections calculated here on the LO RG evolutions of SMEFT Wilson coefficients for $\Delta F = 2$ transitions from the new physics scale Λ down to the electroweak scale μ_{ew} presented by us in [16]. Including also the effects of top Yukawa couplings at the one-loop level, taken already into account in the latter paper, we find that the main benefit from the present analysis is the removal of QCD renormalization scheme dependences present currently in the one-loop matchings both between the WET and SMEFT and also between SMEFT and a chosen UV completion. But the NLO QCD effects in RG QCD evolution, calculated here in the Naive dimensional regularisation minimal subtraction (NDR- $\overline{\text{MS}}$) scheme [19], turn out to be small, in the ballpark of a few percent but larger than one-loop Yukawa top effects when only the $\Delta F = 2$ operators are considered.

Our paper is organized as follows. In Sec. II we derive the general relation between QCD RG evolutions in two different operator bases for the case of $\Delta F = 2$ Wilson coefficients. Subsequently we also give the corresponding relations for one-loop and two-loop ADMs. Using these formulas we summarize in Sec. II B the one-loop and two-loop ADMs for $\Delta F = 2$ transitions in the BMU, JMS, and SMEFT bases as well as the matching matrices between these bases. Subsequently in Sec. IV we analyze numerically the size of the NLO QCD corrections calculated here by presenting the RG evolution matrices in the WET and SMEFT at both LO and NLO in QCD. We summarize in Sec. V.

II. BASIC FORMULAS FOR NLO QCD RG EVOLUTION

A. Evolution matrix

The RG evolution matrix for nonleptonic transitions in the BMU basis for SM and BSM operators is known

including one-loop and two-loop QCD contributions. It is given as follows:

$$\hat{U}_{\text{BMU}}(\mu_{\text{had}}, \mu_{\text{ew}}) = \left[\hat{1} + \hat{J}_{\text{BMU}} \frac{\alpha_s(\mu_{\text{had}})}{4\pi} \right] \hat{U}_{\text{BMU}}^{(0)}(\mu_{\text{had}}, \mu_{\text{ew}}) \times \left[\hat{1} - \hat{J}_{\text{BMU}} \frac{\alpha_s(\mu_{\text{ew}})}{4\pi} \right], \quad (1)$$

where $\hat{U}_{\text{BMU}}^{(0)}$ is the RS-independent LO evolution matrix. On the other hand, \hat{J}_{BMU} stems from the RS-dependent two-loop ADMs, which makes them sensitive to the renormalization scheme considered. This scheme dependence is canceled by one of the matching at μ_{ew} and by the one of the hadronic matrix elements at μ_{had} . Explicit general expressions for $\hat{U}_i^{(0)}$ and \hat{J}_i in terms of the coefficients of the one-loop and two-loop perturbative expansions for the ADM $\hat{\gamma}$ and the QCD β -function can be found including their derivations in Chap. 5 of [1]. They will be listed in Sec. III D.

It should be stressed that all two-loop ADMs and the corresponding values of \hat{J}_i are given in our paper in the NDR- $\overline{\text{MS}}$ scheme as defined in [19] with evanescent operators entering two-loop calculations defined by the so-called Greek method. The details in the context of WET and SMEFT are discussed in Appendix E of [16].

Our goal is to obtain an analogous expression in the SMEFT, that is,

$$\hat{U}_{\text{SMEFT}}(\mu_{\text{ew}}, \Lambda) = \left[\hat{1} + \hat{J}_{\text{SMEFT}} \frac{\alpha_s(\mu_{\text{ew}})}{4\pi} \right] \hat{U}_{\text{SMEFT}}^{(0)}(\mu_{\text{ew}}, \Lambda) \times \left[\hat{1} - \hat{J}_{\text{SMEFT}} \frac{\alpha_s(\Lambda)}{4\pi} \right]. \quad (2)$$

To this end we notice that this evolution depends only on \hat{J}_{SMEFT} and $\hat{U}_{\text{SMEFT}}^{(0)}$ without any explicit dependence on one- and two-loop anomalous dimensions of the SMEFT operators. This gives us a hint that it should be possible to obtain \hat{J}_{SMEFT} and $\hat{U}_{\text{SMEFT}}^{(0)}$ from the known \hat{J}_{BMU} and $\hat{U}_{\text{BMU}}^{(0)}$ without knowing explicitly one-loop and two-loop ADMs in the SMEFT. It turns out that this is indeed possible but as an intermediate step we should first find \hat{J}_{JMS} and $\hat{U}_{\text{JMS}}^{(0)}$ from \hat{J}_{BMU} and $\hat{U}_{\text{BMU}}^{(0)}$. We will perform this intermediate step in what follows.

To reach this goal we first present the general formula that allows one to obtain \hat{J}_B and $\hat{U}_B^{(0)}$ in a given basis B from \hat{J}_A and $\hat{U}_A^{(0)}$ in the basis A for which these two objects, as in the BMU basis, are already known.

Defining the transformation matrix \hat{R} between these two operator bases through

$$\vec{Q}_B = \hat{R} \vec{Q}_A, \quad \vec{C}_A = \hat{R}^T \vec{C}_B, \quad \hat{R} = \hat{R}_0 + \frac{\alpha_s(\mu)}{4\pi} \hat{R}_1, \quad (3)$$

one finds then that in the case of $\Delta F = 2$ operators, the absence of Fierz-vanishing operator contributions in the NDR- $\overline{\text{MS}}$ scheme implies $\hat{R}_1 = \hat{O}$ [16]. A straightforward calculation results then in the relations we were looking for:

$$\boxed{\hat{J}_B = (\hat{R}_0^{-1})^T \hat{J}_A \hat{R}_0^T, \quad \hat{U}_B^{(0)}(\mu_1, \mu_2) = (\hat{R}_0^{-1})^T \hat{U}_A^{(0)}(\mu_1, \mu_2) \hat{R}_0^T.} \quad (4)$$

Let us illustrate these formulas on the two cases of interest.

Example 1

Here

$$A = \text{BMU}, \quad B = \text{JMS}, \quad \hat{R}_0^T \equiv \hat{M}_0, \quad (5)$$

and we find

$$\boxed{\hat{J}_{\text{JMS}} = \hat{M}_0^{-1} \hat{J}_{\text{BMU}} \hat{M}_0, \quad \hat{U}_{\text{JMS}}^{(0)}(\mu_1, \mu_2) = \hat{M}_0^{-1} \hat{U}_{\text{BMU}}^{(0)}(\mu_1, \mu_2) \hat{M}_0.} \quad (6)$$

Example 2

Here

$$A = \text{JMU}, \quad B = \text{SMEFT}, \quad \hat{R}_0^T \equiv \hat{K}_0, \quad (7)$$

and we find

$$\boxed{\hat{J}_{\text{SMEFT}} = \hat{K}_0^{-1} \hat{J}_{\text{JMS}} \hat{K}_0, \quad \hat{U}_{\text{SMEFT}}^{(0)} = \hat{K}_0^{-1} \hat{U}_{\text{JMS}}^{(0)} \hat{K}_0.} \quad (8)$$

where we suppressed the scales in the last relation. The reason is that the SMEFT and WET are valid at different energy scales. Therefore, to use the second relation in the equation above in the basic formula (2) one should properly adjust the scales in $\hat{U}_{\text{JMS}}^{(0)}$. To test it one can use the one-loop ADMs of [11–13].

In summary, starting with the known RG NLO evolution in the BSM basis one cannot only find the corresponding NLO evolution in WET in the JMS basis but also the NLO evolution in the SMEFT.

This formulation is general but in the case at hand, while $\hat{R}_0 \neq \hat{1}$, the transformation from the JMS to the SMEFT basis simplifies drastically. With $\hat{K}_0 = \hat{1}$ one has

$$\boxed{\hat{J}_{\text{SMEFT}} = \hat{J}_{\text{JMS}}, \quad \hat{U}_{\text{SMEFT}}^{(0)} = \hat{U}_{\text{JMS}}^{(0)}.} \quad (9)$$

This demonstrates the advantage of the JMS basis over the BMU basis when the matching to the SMEFT is considered.

B. One-loop and two-loop ADMs in SMEFT

Despite the possibility of finding the NLO RG evolution matrix in the basis B from the one in the basis A without knowing anomalous dimensions in basis B, it is useful to find these anomalous dimensions if one wants to include subsequently the effects of Yukawa couplings.

The ADM has the following perturbative expansion:

$$\hat{\gamma}(\alpha_s) = \frac{\alpha_s}{4\pi} \hat{\gamma}^{(0)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{\gamma}^{(1)} + \mathcal{O}(\alpha_s^3). \quad (10)$$

We find then

$$\hat{\gamma}_{\text{JMS}}^{(0)} = \hat{R}_0 \hat{\gamma}_{\text{BMU}}^{(0)} \hat{R}_0^{-1}, \quad \hat{\gamma}_{\text{JMS}}^{(1)} = \hat{R}_0 \hat{\gamma}_{\text{BMU}}^{(1)} \hat{R}_0^{-1} \quad (11)$$

and

$$\hat{\gamma}_{\text{SMEFT}}^{(0)} = \hat{\gamma}_{\text{JMS}}^{(0)}, \quad \hat{\gamma}_{\text{SMEFT}}^{(1)} = \hat{\gamma}_{\text{JMS}}^{(1)}. \quad (12)$$

Starting then with the ADMs in the BMU basis one can find the ADMs in the JMS basis and subsequently ADMs in the SMEFT. However, there is one subtlety to be taken care of: The anomalous dimension matrix $\hat{\gamma}$ used by us and in the most literature including [1] and in particular in the BMU basis in [14] governs the RG evolution of the matrix elements of the operators involved, while the evolution of the corresponding Wilson coefficients is governed by the transposed matrix $\hat{\gamma}^T$. But the authors that introduced the JMS and SMEFT bases decided to denote by $\hat{\gamma}$ the one which usually would be called $\hat{\gamma}^T$.

In what follows we will within QCD use the standard notation so that the RG evolution of operator matrix elements will be governed by $\hat{\gamma}$ and the evolution of Wilson coefficients by γ^T . Therefore when listing ADMs and \hat{J} in the JMS and SMEFT bases we will use the standard notation used also in the BMU basis in [14]. The resulting QCD RG evolution matrices at LO and NLO obviously do not depend on this choice. We will also see at the end of the next section that the LO Yukawa effects in the RG evolution can also be incorporated in the presence of NLO QCD corrections in a straightforward manner.

III. ONE-LOOP AND TWO-LOOP ADMs IN BMU, JMS, and SMEFT BASES

A. BMU

We begin with the BMU basis [14] for which the complete ADMs at NLO in QCD have been calculated in [14].² The BMU basis consists in full generality of $(5+3) = 8$ physical operators belonging to the five distinct sectors (VLL, SLL, LR, VRR, SRR). However, SLL and SRR operators, violating hypercharge conservation are not allowed within the SMEFT at dimension-six level, and we will not consider them in what follows. Adopting the w_{Cxf} convention [21], the remaining four operators are $(ij = sd, db, sb, cu)$,³

$$\begin{aligned} Q_{VLL}^{ij} &= [\bar{d}_i \gamma_\mu P_L d_j] [\bar{d}_i \gamma^\mu P_L d_j], \\ Q_{VRR}^{ij} &= [\bar{d}_i \gamma_\mu P_R d_j] [\bar{d}_i \gamma^\mu P_R d_j], \\ Q_{LR,1}^{ij} &= [\bar{d}_i \gamma_\mu P_L d_j] [\bar{d}_i \gamma^\mu P_R d_j], \\ Q_{LR,2}^{ij} &= [\bar{d}_i P_L d_j] [\bar{d}_i P_R d_j], \end{aligned} \quad (13)$$

which are built exclusively out of color-singlet currents $[\bar{d}_i^\alpha \dots d_j^\alpha] [\bar{d}_i^\beta \dots d_j^\beta]$, where α, β denote color indices. This feature is very useful for calculations in Dual QCD [22,23], because their matrix elements in the large- N_c limit can be obtained directly without using Fierz identities.

The one-loop and two-loop anomalous dimensions are given as follows:

$$\begin{aligned} \hat{\gamma}_{\text{BMU}}^{(0)} &= \begin{pmatrix} 6 - \frac{6}{N_c} & 0 & 0 & 0 \\ 0 & 6 - \frac{6}{N_c} & 0 & 0 \\ 0 & 0 & \frac{6}{N_c} & 12 \\ 0 & 0 & 0 & -6N_c + \frac{6}{N_c} \end{pmatrix}, \\ \hat{\gamma}_{\text{BMU}}^{(1)} &= \begin{pmatrix} \hat{\gamma}_{VLL}^{(1)} & 0 & 0 & 0 \\ 0 & \hat{\gamma}_{VRR}^{(1)} & 0 & 0 \\ 0 & 0 & (\hat{\gamma}_{LR}^{(1)})_{11} & (\hat{\gamma}_{LR}^{(1)})_{12} \\ 0 & 0 & (\hat{\gamma}_{LR}^{(1)})_{21} & (\hat{\gamma}_{LR}^{(1)})_{22} \end{pmatrix}. \end{aligned} \quad (14)$$

Here

$$\begin{aligned} \hat{\gamma}_{VLL}^{(1)} &= \hat{\gamma}_{VRR}^{(1)} \\ &= -\frac{19}{6}N_c - \frac{22}{3} + \frac{39}{N_c} - \frac{57}{2N_c^2} + \frac{2}{3}N_f - \frac{2}{3N_c}N_f, \end{aligned} \quad (15)$$

and

²In the so-called SUSY basis this calculation has been performed in [20].

³We use the ordering of flavors as in [16] but different papers use different conventions, which has to be taken into account.

$$(\hat{\gamma}_{LR}^{(1)})_{11} = \frac{137}{6} + \frac{15}{2N_c^2} - \frac{22}{3N_c}N_f, \quad (16)$$

$$(\hat{\gamma}_{LR}^{(1)})_{12} = \frac{200}{3}N_c - \frac{6}{N_c} - \frac{44}{3}N_f, \quad (17)$$

$$(\hat{\gamma}_{LR}^{(1)})_{21} = \frac{71}{4}N_c + \frac{9}{N_c} - 2N_f, \quad (18)$$

$$(\hat{\gamma}_{LR}^{(1)})_{22} = -\frac{203}{6}N_c^2 + \frac{479}{6} + \frac{15}{2N_c^2} + \frac{10}{3}N_cN_f - \frac{22}{3N_c}N_f. \quad (19)$$

Here N_c is the number of colors with $N_c = 3$ in QCD. N_f is the number of quark flavors, $N_f = 3, 4, 5$ in the WET, and $N_f = 6$ in the SMEFT. The numerical solutions for evolution matrices for $ij = sd, db, sb$ are given in [24].

B. JMS

The JMS basis has been introduced to facilitate the classification of the complete WET operator basis [2] for the purpose of matching from SMEFT onto WET. The relevant $\Delta F = 2$ operators are

$$\begin{aligned} [Q_{dd}^{VLL}]_{ijij} &= Q_{VLL}^{ij}, & [Q_{dd}^{VRR}]_{ijij} &= Q_{VRR}^{ij}, \\ [Q_{dd}^{V1,LR}]_{ijij} &= Q_{LR,1}^{ij}, \\ [Q_{dd}^{V8,LR}]_{ijij} &= [\bar{d}_i \gamma_\mu P_L T^A d_j] [\bar{d}_i \gamma^\mu P_R T^A d_j] \\ &= -\frac{1}{2N_c}Q_{LR,1}^{ij} - Q_{LR,2}^{ij}, \end{aligned} \quad (20)$$

where T^A are $SU(3)_c$ color generators of the fundamental representation.

Using the relations above one finds first

$$\hat{R}_0 = \hat{R}_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2N_c} & -1 \end{pmatrix}. \quad (21)$$

Using subsequently (11) and the results in the BMU basis we find one-loop and two-loop ADMs in the JMS basis

$$\hat{\gamma}_{\text{JMS}}^{(0)} = \begin{pmatrix} 6 - \frac{6}{N_c} & 0 & 0 & 0 \\ 0 & 6 - \frac{6}{N_c} & 0 & 0 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & -\frac{6}{N_c} C_F & -6N_c + \frac{12}{N_c} \end{pmatrix},$$

$$\hat{\gamma}_{\text{JMS}}^{(1)} = \begin{pmatrix} (\hat{\gamma}_{\text{JMS}}^{(1)})_{11} & 0 & 0 & 0 \\ 0 & (\hat{\gamma}_{\text{JMS}}^{(1)})_{22} & 0 & 0 \\ 0 & 0 & (\hat{\gamma}_{\text{JMS}}^{(1)})_{33} & (\hat{\gamma}_{\text{JMS}}^{(1)})_{34} \\ 0 & 0 & (\hat{\gamma}_{\text{JMS}}^{(1)})_{43} & (\hat{\gamma}_{\text{JMS}}^{(1)})_{44} \end{pmatrix}, \quad (22)$$

where we have still used the BMU notation for $\hat{\gamma}$ so that in contrast to [9]

$$\mu \frac{d}{d\mu} \vec{C}_{\text{JMS}} = \hat{\gamma}_{\text{JMS}}^T \vec{C}_{\text{JMS}}, \quad (23)$$

as emphasized above. Here

$$\begin{aligned} (\hat{\gamma}_{\text{JMS}}^{(1)})_{11} &= (\hat{\gamma}_{\text{JMS}}^{(1)})_{22} \\ &= \frac{(N_c - 1)(171 - 19N_c^2 - N_c(63 - 4N_f))}{6N_c^2}, \end{aligned} \quad (24)$$

$$(\hat{\gamma}_{\text{JMS}}^{(1)})_{33} = \frac{21}{2} \left(-1 + \frac{1}{N_c^2} \right), \quad (25)$$

$$(\hat{\gamma}_{\text{JMS}}^{(1)})_{34} = \frac{6}{N_c} - \frac{200N_c}{3} + \frac{44N_f}{3}, \quad (26)$$

$$(\hat{\gamma}_{\text{JMS}}^{(1)})_{43} = \frac{(N_c^2 - 1)(9 - 208N_c^2 + 22N_c N_f)}{6N_c^3}, \quad (27)$$

$$(\hat{\gamma}_{\text{JMS}}^{(1)})_{44} = \frac{27 + 679N_c^2 - 203N_c^4 - 88N_c N_f + 20N_c^3 N_f}{6N_c^2}, \quad (28)$$

and $C_F = (N_c^2 - 1)/(2N_c)$. We have checked that the obtained $\hat{\gamma}_{\text{JMS}}^{(0)}$ agrees after transposition with the results in [9].

C. SMEFT

1. Operators

In the SMEFT there are five operators that can contribute to $\Delta F = 2$ processes at the dimension-six level

$$\begin{aligned} \mathcal{O}_{qq}^{(1)} &= (\bar{q}_p \gamma_\mu q_r)(\bar{q}_s \gamma^\mu q_t), & \mathcal{O}_{qq}^{(3)} &= (\bar{q}_p \gamma_\mu \tau^I q_r)(\bar{q}_s \gamma^\mu \tau^I q_t), \\ \mathcal{O}_{dd} &= (\bar{d}_p \gamma_\mu d_r)(\bar{d}_s \gamma^\mu d_t), & \mathcal{O}_{qd}^{(1)} &= (\bar{q}_p \gamma_\mu q_r)(\bar{d}_s \gamma^\mu d_t), \\ \mathcal{O}_{qd}^{(8)} &= (\bar{q}_p \gamma_\mu T^A q_r)(\bar{d}_s \gamma^\mu T^A d_t). \end{aligned} \quad (29)$$

The relevant SMEFT Wilson coefficients are

$$B = \{C_{qq}^{(1)} + C_{qq}^{(3)}, C_{aa}, C_{qa}^{(1)}, C_{qa}^{(8)}\}, \quad (30)$$

in the down ($a = d$) and up ($a = u$) sectors, respectively [4]. At tree level for $B_{s,d} - \bar{B}_{s,d}$ and $K^0 - \bar{K}^0$ mixing one finds the following matching conditions at μ_{ew} in the *down-basis*:

$$\begin{aligned} [C_{dd}^{V,LL}]_{ijij} &= -[C_{qq}^{(1)}]_{ijij} - [C_{qq}^{(3)}]_{ijij}, & [C_{dd}^{V,RR}]_{ijij} &= -[C_{dd}]_{ijij}, \\ [C_{dd}^{V1,LR}]_{ijij} &= -[C_{qd}^{(1)}]_{ijij}, & [C_{dd}^{V8,LR}]_{ijij} &= -[C_{qd}^{(8)}]_{ijij}, \end{aligned} \quad (31)$$

and for $D^0 - \bar{D}^0$ mixing in the *up-basis*:

$$\begin{aligned} [C_{uu}^{V,LL}]_{ijij} &= -[\hat{C}_{qq}^{(1)}]_{ijij} - [\hat{C}_{qq}^{(3)}]_{ijij}, & [C_{uu}^{V,RR}]_{ijij} &= -[\hat{C}_{uu}]_{ijij}, \\ [C_{uu}^{V1,LR}]_{ijij} &= -[\hat{C}_{qu}^{(1)}]_{ijij}, & [C_{uu}^{V8,RR}]_{ijij} &= -[\hat{C}_{qu}^{(8)}]_{ijij}, \end{aligned} \quad (32)$$

where we have neglected contributions from modified Z-coupling operators. Note that we use the Hamiltonian for WET to define Wilson coefficients contrary to [2,3], who use the Lagrangian, and in consequence minus signs are present in the matching conditions.

2. QCD anomalous dimensions

The ADMs in this case are, as given in (12), the same as for the JMS basis. Inspecting the RG evolution for $[\hat{C}_{qq}^{(1)}]_{ijij}$ and $[\hat{C}_{qq}^{(3)}]_{ijij}$ one finds that the sum for dd or uu indices from these two Wilson coefficients evolves without being affected by other operators and only this sum matches on the VLL operator in the JMS basis. While this can be verified explicitly by using the RG equations (35) and (36) in [16], the inclusion of flavor diagonal gluon exchanges at the two-loop level cannot change this property.

3. Top Yukawa anomalous dimension

In this subsection we report the LO ADM resulting from top Yukawa interactions. It reads for a given sector with flavor indices $ij = sd, db, sb, cu$ in the down-basis [4]:

$$\hat{\gamma}_{y_i}^{(0)} = \frac{y_i^2}{16\pi^2} \begin{pmatrix} r_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} r_{ij} & 0 \\ 0 & 0 & 0 & \frac{1}{2} r_{ij} \end{pmatrix}, \quad (33)$$

with the combination $r_{ij} = |V_{ii}|^2 + |V_{ij}|^2$. In the up-basis the ADM is zero and the corresponding mixing effects only come into play when considering back-rotation [25] at the electroweak scale.

Indeed, this anomalous dimension matrix can be extracted from formulas (30)–(33) in [16]. It is the coefficient of

$$L = \frac{1}{(4\pi)^2} \ln\left(\frac{\mu_{\text{ew}}}{\Lambda}\right), \quad (34)$$

in the solution of RG equations retaining only the first leading logarithm and neglecting the μ -dependence of y_t . Defining

$$[\mathcal{C}_{qq}^{(1+3)}]_{ijij} \equiv [\mathcal{C}_{qq}^{(1)}]_{ijij} + [\mathcal{C}_{qq}^{(3)}]_{ijij}, \quad (35)$$

these equations read in the down-basis

$$[\mathcal{C}_{qq}^{(1+3)}]_{ijij}(\mu_{\text{ew}}) = [\mathcal{C}_{qq}^{(1+3)}]_{ijij} + y_t^2 \left[\lambda_t^{ik} [\mathcal{C}_{qq}^{(1+3)}]_{kjiij} + \lambda_t^{kj} [\mathcal{C}_{qq}^{(1+3)}]_{ikij} \right] L, \quad (36)$$

$$[\mathcal{C}_{qd}^{(1)}]_{ijij}(\mu_{\text{ew}}) = [\mathcal{C}_{qd}^{(1)}]_{ijij} + y_t^2 \left[\frac{\lambda_t^{ik}}{2} [\mathcal{C}_{qd}^{(1)}]_{kjiij} + \frac{\lambda_t^{kj}}{2} [\mathcal{C}_{qd}^{(1)}]_{ikij} \right] L, \quad (37)$$

$$[\mathcal{C}_{qd}^{(8)}]_{ijij}(\mu_{\text{ew}}) = [\mathcal{C}_{qd}^{(8)}]_{ijij} + y_t^2 \left[\frac{\lambda_t^{ik}}{2} [\mathcal{C}_{qd}^{(8)}]_{kjiij} + \frac{\lambda_t^{kj}}{2} [\mathcal{C}_{qd}^{(8)}]_{ikij} \right] L, \quad (38)$$

where a summation over k is implied and where we only considered the $\Delta F = 2$ operators in (29). We have suppressed the argument of the NP scale Λ in the Wilson coefficients and SM parameters on the right-hand side (RHS) to simplify the notation. The μ dependence of y_t will be included in the next subsection.

D. Explicit expression for the evolution matrix

1. Pure QCD

We can now find the NLO QCD evolution matrix in any of the bases considered by us using the general formulas that we recall here in the case of SMEFT for completeness. Simplifying the notation by dropping the subscript SMEFT we have

$$\hat{U}(\mu_{\text{ew}}, \Lambda) = \left[1 + \hat{J} \frac{\alpha_s(\mu_{\text{ew}})}{4\pi} \right] \hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda) \left[1 - \hat{J} \frac{\alpha_s(\Lambda)}{4\pi} \right]. \quad (39)$$

Here $\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda)$ denotes the usual LO RG evolution matrix that is explicitly given as follows:

$$\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda) = \hat{V} \left(\left[\frac{\alpha_s(\Lambda)}{\alpha_s(\mu_{\text{ew}})} \right]^{\frac{\vec{\gamma}^{(0)}}{2\beta_0}} \right)_D \hat{V}^{-1}, \quad (40)$$

where \hat{V} diagonalizes $\hat{\gamma}^{(0)T}$,

$$\hat{\gamma}_D^{(0)} = \hat{V}^{-1} \hat{\gamma}^{(0)T} \hat{V}, \quad (41)$$

and $\vec{\gamma}^{(0)}$ is the vector containing the diagonal elements of the diagonal matrix $\hat{\gamma}_D^{(0)}$.

The NLO matrix \hat{J} is given by

$$\hat{J} = \hat{V} \hat{H} \hat{V}^{-1} \quad (42)$$

with

$$(H)_{ij} = \delta_{ij} (\gamma^{(0)})_i \frac{\beta_1}{2\beta_0^2} - \frac{G_{ij}}{2\beta_0 + (\gamma^{(0)})_i - (\gamma^{(0)})_j}, \quad (43)$$

where

$$\hat{G} = \hat{V}^{-1} \hat{\gamma}^{(1)T} \hat{V}, \quad (44)$$

with the two-loop matrix $\hat{\gamma}^{(1)}$ found using (11) and $\beta_1 = \frac{34}{3} N_c^2 - \frac{10}{3} N_c N_f - 2 C_F N_f$ [26].

Setting $N_c = 3$ and $N_f = 6$ we find

$$\hat{J}_{\text{SMEFT}}^{(6)} = \hat{J}_{\text{JMS}}^{(6)} = \begin{pmatrix} 1.37 & 0 & 0 & 0 \\ 0 & 1.37 & 0 & 0 \\ 0 & 0 & 1.47 & 2.75 \\ 0 & 0 & 16.60 & 6.92 \end{pmatrix},$$

$$\hat{J}_{\text{BMU}}^{(6)} = \begin{pmatrix} 1.37 & 0 & 0 & 0 \\ 0 & 1.37 & 0 & 0 \\ 0 & 0 & -1.30 & -1.38 \\ 0 & 0 & -16.60 & 9.69 \end{pmatrix}. \quad (45)$$

The corresponding matrices for $N_f = 4$ and $N_f = 5$ can be found in Appendix.

2. Including top Yukawa effects

Until now we succeeded to find the NLO QCD RG evolution in the SMEFT but also the evolution due to the top Yukawa has to be taken into account. But the two-loop ADM for $\Delta F = 2$ operators including top Yukawa couplings is not known at present. Therefore, we can only combine the known LO evolution due to Yukawa couplings with the NLO QCD evolution just found. The full evolution is then given by

$$[\hat{U}(\mu_{\text{ew}}, \Lambda)]_{\text{QCD}+y_t} = \left[\hat{1} + \hat{J} \frac{\alpha_s(\mu_{\text{ew}})}{4\pi} \right] [\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda)]_{\text{QCD}+y_t} \times \left[\hat{1} - \hat{J} \frac{\alpha_s(\Lambda)}{4\pi} \right], \quad (46)$$

where the label QCD + y_t indicates that besides QCD also Yukawa contributions have been taken into account. Note that \hat{J} contains only QCD contributions. As NLO corrections due to top Yukawa effects are unknown, it is legitimate to proceed in this manner.

There are two routes to find the LO evolution matrix in this formula. If one is interested only in the numerical result, one can simply replace the LO QCD evolution matrix in (39) by the one present in the usual computer codes like WILSON [27] or DsixTools [28,29] for LO RG evolution in the SMEFT.

However, as demonstrated in [30] a very accurate analytic formula for the RG evolution including scale dependence of y_t can be found. To this end we note that

$$[\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda)]_{\text{QCD}+y_t} = [\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda)]_{\text{QCD}} [\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda)]_{y_t}, \quad (47)$$

with the QCD evolution matrix given in (40). To find the second matrix one can simply follow the *case C* in Appendix E of [1] that is a particular case of the general formulas in [30]. We find

$$[\hat{U}^{(0)}(\mu_{\text{ew}}, \Lambda)]_{y_t} = \exp \left[\frac{1}{8\pi} \frac{\hat{\gamma}_t}{\gamma_m^{(0)} - \beta_0} \left([\alpha_s(\Lambda)]^{\frac{\gamma_m^{(0)}}{\beta_0} - 1} - [\alpha_s(\mu_{\text{ew}})]^{\frac{\gamma_m^{(0)}}{\beta_0} - 1} \right) \right], \quad (48)$$

where the diagonal matrix $\hat{\gamma}_t$,

$$\hat{\gamma}_t = \hat{b} y_t^2(\mu_0) [\alpha_s(\mu_0)]^{\frac{\gamma_m^{(0)}}{\beta_0}}, \quad \gamma_m^{(0)} = 8, \quad (49)$$

with the diagonal matrix $\hat{b} = \text{diag}(1, 0, 1/2, 1/2)$ deduced from (33) and an arbitrary scale μ_0 , which we choose to be 160 GeV in our numerical analysis.

IV. NUMERICAL ANALYSIS

To illustrate the importance of NLO QCD corrections within the SMEFT with respect to the LO ones and top quark Yukawa effects as well as LO and NLO QCD effects within the WET, we derive in the following numerical expressions for the evolution matrices in the WET and SMEFT. The results in this section have been obtained using the analytic expressions derived in the previous section.

A. WET

Setting $\mu_{\text{had}} = 1.3$ GeV and $\mu_{\text{ew}} = 160$ GeV and using the threshold scale $\mu_5 = 4.2$ GeV between $N_f = 5$ and $N_f = 4$ we find

$$[\hat{U}_{\text{JMS}}^{(0)}]_{\text{QCD}} = \begin{pmatrix} 0.76 & 0 & 0 & 0 \\ 0 & 0.76 & 0 & 0 \\ 0 & 0 & 1.10 & 0.31 \\ 0 & 0 & 1.38 & 2.71 \end{pmatrix},$$

$$[\hat{U}_{\text{JMS}}]_{\text{QCD}} = \begin{pmatrix} 0.76 & 0 & 0 & 0 \\ 0 & 0.76 & 0 & 0 \\ 0 & 0 & 1.24 & 0.57 \\ 0 & 0 & 2.02 & 3.59 \end{pmatrix}. \quad (50)$$

Here $\hat{U}_{\text{JMS}}^{(0)}$ and \hat{U}_{JMS} are LO and NLO evolution matrices in the JMS basis, respectively. We observe that in the LR sector the NLO effects are large, and it is mandatory to include them in any phenomenological analysis.

B. SMEFT

Here we study the evolution between $\mu_{\text{ew}} = 160$ GeV and $\Lambda = 10$ TeV for various cases in the $N_f = 6$ flavor theory.

1. Pure QCD evolution

$$[\hat{U}_{\text{SMEFT}}^{(0)}]_{\text{QCD}} = \begin{pmatrix} 0.89 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 \\ 0 & 0 & 1.02 & 0.10 \\ 0 & 0 & 0.43 & 1.52 \end{pmatrix},$$

$$[\hat{U}_{\text{SMEFT}}]_{\text{QCD}} = \begin{pmatrix} 0.89 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 \\ 0 & 0 & 1.02 & 0.12 \\ 0 & 0 & 0.46 & 1.57 \end{pmatrix}. \quad (51)$$

As expected, due to a much slower running of α_s and its smaller value than in WET, QCD effects are significantly smaller, and this applies in particular to NLO QCD effects.

2. Pure Yukawa evolution

When only the Yukawa running in the SMEFT at one loop is considered, the resulting evolution matrix reads

$$[\hat{U}_{\text{SMEFT}}^{(0)}]_{y_t} = \begin{pmatrix} 0.98 & 0 & 0 & 0 \\ 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0.99 & 0 \\ 0 & 0 & 0 & 0.99 \end{pmatrix}, \quad (52)$$

where we have used $y_t(\mu = 160 \text{ GeV}) = 0.94$.

3. QCD+Yukawa evolution

In this subsection we consider the combination of QCD and Yukawa running effects at LO and NLO. The corresponding evolution matrices read

$$[\hat{U}_{\text{SMEFT}}^{(0)}]_{\text{QCD}+y_t} = \begin{pmatrix} 0.87 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 \\ 0 & 0 & 1.01 & 0.10 \\ 0 & 0 & 0.43 & 1.51 \end{pmatrix},$$

$$[\hat{U}_{\text{SMEFT}}]_{\text{QCD}+y_t} = \begin{pmatrix} 0.88 & 0 & 0 & 0 \\ 0 & 0.89 & 0 & 0 \\ 0 & 0 & 1.01 & 0.11 \\ 0 & 0 & 0.44 & 1.54 \end{pmatrix}, \quad (53)$$

where for both matrices we have used Eq. (46), setting \hat{J} to zero in the LO case, and keeping \hat{J} in the computation of \hat{U}_{SMEFT} .

We observe that when only $\Delta F = 2$ operators are considered, the impact of top-Yukawa effects is very small. It is significantly larger when $\Delta F = 1$ operators are also included in the analysis. A detailed discussion of these effects has been presented in [16].

V. CONCLUSIONS

The main results of our paper are as follows:

- (i) General formulas for the relation of QCD RG evolution matrices at LO and NLO for $\Delta F = 2$ Wilson coefficients between two different operator bases, given in Eq. (4).
- (ii) The relation of QCD one-loop and two-loop anomalous dimension matrices between BMU and JMS bases reported in (11) and JMS and SMEFT bases in Eq. (12).
- (iii) The two-loop QCD ADMs for $\Delta F = 2$ operators in the JMS WET basis and in the SMEFT Warsaw basis. They are given in Eqs. (22)–(28) and (12).
- (iv) Master formulas for QCD RG evolution matrices for the Wilson coefficients of $\Delta F = 2$ operators in the SMEFT Warsaw basis at the NLO were derived. They are given in Eq. (39).

- (v) Generalization of these formulas to include top Yukawa effects at the one-loop level.

These findings allow for a general and scheme-independent QCD analysis of nonleptonic $\Delta F = 2$ processes in the SMEFT and WET at NLO. In a given UV completion in which the Wilson coefficients have been calculated at a NP scale Λ , our master formulas allow us to calculate them at the μ_{ew} scale. The inclusion of NLO QCD corrections in the RG evolution in the WET from the hadronic scale μ_{had} to the electroweak scale μ_{ew} allows a correct matching of Wilson coefficients to the matrix elements calculated by lattice QCD (LQCD) or other nonperturbative methods sensitive to renormalization scheme dependences. The use of the JMS basis, on the other hand, allows one to generalize this formula to the SMEFT, because in this basis the tree-level matching of SMEFT onto WET [2] and the one-loop matching [3,4] are known.

The main messages from the numerical analysis in Sec. IV are as follows:

- (i) The NLO QCD corrections to the $\Delta F = 2$ RG evolution matrices within WET are substantial.
- (ii) The NLO QCD corrections to the $\Delta F = 2$ RG evolution matrices within SMEFT are small, in the ballpark of a few percent.
- (iii) Even smaller are top-Yukawa effects if only $\Delta F = 2$ operators are included in the analysis.

The small NLO QCD corrections to the $\Delta F = 2$ matrices within the SMEFT could be considered at first sight disappointing. However, it should be realized that they have been calculated in the NDR scheme but could be larger in a different RS. This also does not preclude significant one-loop matching contributions, which in principle could be larger than the QCD effects found here. However, to combine these one-loop matching conditions with the NLO effects calculated here they have to be calculated in the NDR scheme. Only then can RS independent results be obtained.

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APPENDIX: $\hat{J}_{\text{JMS}}^{(N_f)}$ AND $\hat{J}_{\text{BMU}}^{(N_f)}$

For convenience we report in this appendix the \hat{J} matrices for $N_f = 5, 4$ flavors in the JMS and BMU bases, obtained from Eq. (42),

$$\hat{J}_{\text{JMS}}^{(5)} = \begin{pmatrix} 1.63 & 0 & 0 & 0 \\ 0 & 1.63 & 0 & 0 \\ 0 & 0 & 1.67 & 2.44 \\ 0 & 0 & 17.04 & 5.12 \end{pmatrix}, \quad \hat{J}_{\text{JMS}}^{(4)} = \begin{pmatrix} 1.79 & 0 & 0 & 0 \\ 0 & 1.79 & 0 & 0 \\ 0 & 0 & 2.43 & 2.10 \\ 0 & 0 & 21.15 & 3.18 \end{pmatrix}, \quad (\text{A1})$$

$$\hat{J}_{\text{BMU}}^{(5)} = \begin{pmatrix} 1.63 & 0 & 0 & 0 \\ 0 & 1.63 & 0 & 0 \\ 0 & 0 & -1.17 & -1.39 \\ 0 & 0 & -17.04 & 7.96 \end{pmatrix}, \quad \hat{J}_{\text{BMU}}^{(4)} = \begin{pmatrix} 1.79 & 0 & 0 & 0 \\ 0 & 1.79 & 0 & 0 \\ 0 & 0 & -1.10 & -1.39 \\ 0 & 0 & -21.15 & 6.71 \end{pmatrix}. \quad (\text{A2})$$

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