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Symmetries and Unconventional Order in Quantum Many-Body Systems

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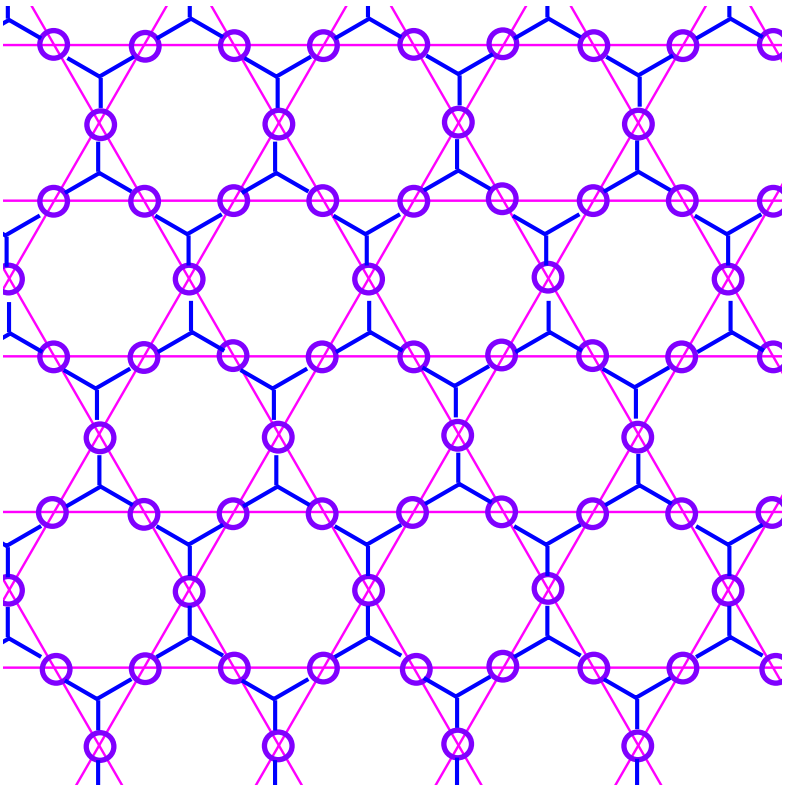

Abstract

This dissertation focuses on methods which aid in the study of exotic phases of matter in quantum many-body systems. For this I look specifically at systems with local disorder that can give rise to the phenomenon of many-body localization as well as two-dimensional systems with certain local symmetries, which I show can potentially lead to the emergence of topologically ordered phases. In this work I use a group-theoretical approach to investigate the topological properties of two-dimensional materials, with a specific focus on the effects of local SU(3) symmetries in the Kagome lattice and enhance this theoretical approach by a numerical analysis that utilizes tensor network constructs. To further investigate the effects of disorder on a quantum many-body system, I derive analytical solutions to time-evolved local observables in many-body localized systems with random perturbation, which I subsequently use to design a tensor network framework by which I combine variational and perturbative solutions for quantum many-body systems to efficiently calculate the time-evolved values of relevant observables.

Zusammenfassung

In dieser Dissertation beschäftige ich mich mit Methoden welche bei der Untersuchung von exotischen Phasen von Quanten-Vielteilchensystemen helfen. Dafür befasse ich mich insbesondere mit Systemen mit lokalen zufälligen Störungen, in welchen das Phänomen der Vielteilchenlokalisierung auftreten kann, sowie mit zweidimensionalen Systemen mit bestimmten lokalen Symmetrien, für welche ich zeige, dass diese zum Auftreten von topologisch geordneten Phasen führen können. In dieser Arbeit benutze ich einen gruppentheoretischen Ansatz, um die topologischen Eigenschaften von zweidimensionalen Materialien zu untersuchen, mit speziellem Fokus auf die Auswirkungen von lokalen SU(3) Symmetrien auf dem Kagome-Gitter. Dieser theoretische Ansatz wurde ergänzt durch eine numerische Untersuchung welche auf Tensornetzwerkmethoden basiert. Um die Effekte von Unordnung auf Quanten-Vielteilchensysteme weiter zu untersuchen, leite ich analytische Lösungen für die Zeitentwicklung lokaler Observablen von Quanten-Vielteilchensystemen gestört durch lokale zufällige Störungen her, welche ich anschließend für die Konzeption eines Tensornetzwerk-Systems benutze, in welchem ich die stör- und variationstheoretische Lösungsansätze für Quanten-Vielteilchensysteme kombiniere um effizient zeitentwickelte Werte einschlägiger Observablen berechnen zu können.

Words of Appreciation

Before getting to the topic at hand, I hope you can allow me to take up some space to express my gratitude to only some of the many people and organizations who have taken me on this hero's journey.

First and foremost, I am truly indebted to my supervisor, Norbert Schuch, for the leadership, patience, and kindness shown to me during my time working on this project. It's been a wild ride, but the end is finally in sight. As fully as I could ever put it, thank you for the whole shebang.

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The people and space of the MPQ Theory Division have been loud and lively, with a million things to do and a thousand more (productive) diversions. For the support of my penchant for finding ever more creative ways to not take the straight line from A to B, my gratitude goes to both my supervisor and the division leader, Ignacio Cirac. I've learned new things and traveled the world in more ways than I could have expected. Thank you for the support that came before I even knew I deserved it.

For always, always having my back and guiding me since I was a confused little pup in a foreign land, I'm sending as much thanks and love as I can muster to Tobias Osborne, my mentor on this project. His group at the Institute for Theoretical Physics at the University of Hannover was my second home and my happy place; I won't name names because everyone I met meant the world and I wish I could have taken you all with me. It was such a unique time — thank you for spending some of it together.

My sanity was kept (mostly) whole by office days with Irene, Asli, David, Caroline, Mohsin, Christoph, Jiří, Anna and sporadic excursions across the street to the European Southern Observatory, where I could think about the other end of the spectrum of physics, hold shows in the planetarium (welcomed by the wonderful Tania, Wolfgang and Alvaro), and remember the joy before the frustration. Thanks also to my former board game and dinner team (Josh, Anne-Laure, Andreas, Matthias) for valiantly choosing distraction, as well as my former ePOD team lead, Lars Lindberg Christensen, for making me promise I wouldn't stay.

I've picked up wisdom and found camaraderie with people I've met along the way, and they were all consequential. The writing groups and the seminars and the committees and the councils, it all mattered. The heartfelt advice (Teresa Fernandez and Andy Williams), the quiet commiseration (Lin Lin Chen), the best of friends who offered me their brain cells when mine weren't available (Zvonimir Domazet), the colleagues who made the industry thing fit right (Isaac De Vlugt and Ludmila Botelho), the theorists by whom I may as well navigate (Tobias Grøsfjeld), and the kindheartedness of Norbert's Quantum Information and Quantum Many-Body Physics group at the University of Vienna. The time I spent at the Centre for Quantum Technologies in Singapore and the time I spent figuring out my entire life after COVID-19 hit. The long-suffering Marius Lewerenz who helped me find the broken pieces and put some of the puzzle back together.

If I tried thanking everyone who touched my soul while I was doing this thing, I'd be writing forever. Suffice it to say you'll always have a friend and I'll always owe you at least one cupcake — even if you don't find yourself in this deficient list of names and places.

The scenic route is often orders of magnitude more interesting, and these have certainly been interesting times.

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Definitions and Abbreviations

Commonly used symbols

$\mathbb{1}$	Identity operator.
$\mathbb{N}$	Set of natural numbers.
$\mathbb{N}_0$	Union of the set of natural numbers and the number zero.
$\mathbb{Z}$	Set of whole numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{C}$	Set of complex numbers.
i	Imaginary unit $i \in \mathbb{C}$, such that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Not to be confused with a counting variable $i \in \mathbb{Z}$.
$\mathcal{H}$	Hilbert space.
H	Hamiltonian; also shown as $\hat{H}$.
1D	One-dimensional.
2D	Two-dimensional.
$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Two-dimensional identity operator; $\sigma_0 = \mathbb{1}_{2\times 2}$.
$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	Pauli X operator; also written as X.
$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	Pauli Y operator; also written as Y.
$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	Pauli Z operator; also written as Z.
δ_{ij}	Kronecker delta function; $\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$
ε_{ijk}	Levi–Civita symbol; $\varepsilon_{ijk} = \begin{cases} +1 & \text{for } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}; \\ -1 & \text{for } (i,j,k) \in \{(3,2,1),(1,3,2),(2,1,3)\}; \\ 0 & \text{for } i=j \text{ or } j=k \text{ or } k=i. \end{cases}$

Mathematical shorthand

$\sum_{j=m}^{n} A_j$	Summation of all terms A_j, where $\mathbb{Z} \ni j \in [m, n]$; $\sum_{j=m}^{n} A_j = A_m + A_{m+1} + A_{m+2} + ... + A_n$.
$\prod_{j=m}^{n} A_j$	Product of all terms A_j, where $\mathbb{Z} \ni j \in [m, n]$; $\prod_{j=m}^{n} A_j = A_m \cdot A_{m+1} \cdot A_{m+2} \cdot ... \cdot A_n$.
$x \oplus y$	Direct sum of x and y.
$\oplus_j = m^n A_j$	Direct summation of all terms A_j, where $\mathbb{Z} \ni j \in [m, n]$.
$x \otimes y$	Tensor product of x and y.
$\otimes_j = m^n A_j$	Tensor product of all terms A_j, where $\mathbb{Z} \ni j \in [m, n]$.
$\mathrm{tr}(A)$	Trace of $n \times n$ matrix A; $\mathrm{tr}(A) = \sum_j = 1^n a_j j$ for $A = (a)_{jk}$.
$\rho_A = \mathrm{tr}_B \rho$	Partial trace of ρ with respect to the system B (for a system consisting out of two subsystems).
$\langle x, y \rangle$	Inner product of x and y.
$n!$	Factorial of $n \in \mathbb{N}$; $n! = \prod_{j=1}^{n} j = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 1$.
$n!!$	Double factorial of $n \in \mathbb{N}$; $n!! = \prod_{j \geq 1}^{n} = n \cdot (n-2) \cdot (n-4) \cdot ....$
$x \in y$	x is an element of y.
$\forall x$	All x.
$\exists x$	There exists an x.
$\forall x \mid B$	B is true for all x.
$A : B$	A is defined by B.
(ξ_1, ξ_2)	This operation acts on some referred to elements, denoted by ξ_1 and ξ_2.
$a \wedge b$	Logical and; both a and b are valid.
$a \vee b$	Logical or; a or b is valid.
A **iff** B	A holds if and only if B does.
$[A, B]$	Commutator; $[A, B] = AB - BA$.
$\{A, B\}$	Anticommutator; $\{A, B\} = AB + BA$.
(AB)	All unique pairs of sites A and B.
$(AB)_n$	All unique pairs of neighboring sites A and B, commonly also shown as $\langle AB \rangle$.
$\langle AB \rangle$	See $(AB)_n$ above.
$(AB)_d$	All unique pairs of non-neighboring sites A and B.

x^*	Complex conjugate of $a + ib = x \in \mathbb{C}$, such that $(a + ib)^* = a - ib$ for $a, b \in \mathbb{R}$.
$\overline{x}$	Complex conjugate of x, also written as x^*.
$\mathbf{a}$	Vector; $\mathbf{a} = (a_1, a_2, a_3, \dots, a_n) = (a_j)$.
A^T	Transpose of matrix $A = (a_{jk})$, such that $A^T = (a_{kj})$.
$A^\dagger$	Adjoint or Hermitian conjugate of matrix A, such that $A^\dagger = (A^T)^*$.
$\lvert\psi\rangle$	Vector; referred to as a ket in quantum physics.
$\langle\psi\rvert$	Dual vector equal to the transposition of $\lvert\psi\rangle$; referred to as a bra in quantum physics.
$\langle\psi \,\vert\, \varphi\rangle$	Scalar obtained by acting with the bra $\langle\psi\rvert$ onto the ket $\lvert\varphi\rangle$; referred to as a braket in quantum physics.

Initialisms

AKLT model	Affleck–Kennedy–Lieb–Tasaki model.
CCSD	Coupled-cluster singles doubles.
DMRG	Density matrix renormalization group.
iMPS	Infinite matrix product state.
irrep	Irreducible representation.
LSMA theorem	Lieb–Schultz–Mattis–Affleck theorem.
MBL	Many-body localization.
MPO	Matrix product operator.
MPS	Matrix product state.
OTOC	Out-of-time-order correlation.
PEPS	Projected entangled pair state.
RVB	Resonating valence bond.
SDE	Stochastic differential equation.
SVD	Singular value decomposition.
TFIM	Transverse-field Ising model.
TN	Tensor network.
TNO	Tensor network operator.

Chapter 1

Introduction and Motivation

Contents

My doctoral work focuses on methods to enable the study of exotic phases in quantum many-body systems. One point of focus is the study of the effects of symmetries on the properties of quantum many-body systems, as well as the possible emergence of topologically ordered phases and their characteristics. The project work began with a group-theoretical approach to the investigation of the topological properties of two-dimensional materials, with a specific focus on the effects of local SU(3) symmetries in the Kagome lattice. This was followed by a numerical analysis that utilized tensor network constructs. Another point of focus in my later work involved analytical derivations of time-evolved observables for a many-body system with random perturbation. My thesis attempts to provide a holistic approach to the study of exotic phases in quantum many-body systems, utilizing an array of different methods from a toolbox of numerical and analytical methods, some of which I newly developed for this purpose.

The chapter you are currently reading, **Chapter 1: Introduction and Motivation**, serves as an introduction in which I introduce the mathematical bones of the problems I address in the work shown in my dissertation, but also present the physical interpretations of various problems and list the contemporary scientific achievements in fields relevant to my topics of research. **Chapter 2: A Perturbative Solution to a Many-Body Localization Hamiltonian** details my analytical work on the perturbation theory of many-body systems with interesting properties, such as the emergence of many-body localization. Some of this work was summarized in [1]. By completing arduous calculations, I discovered the prevalent behavior and values of significant observables in quantum many-body systems — in an exact fashion. **Chapter 3: Creating a Tensor Bridge for Perturbative Many-Body Localization Solutions** shows, in simple terms, the midpoint between the work presented in the previous two chapters. I use the tensor network methodology to create parametrized wave functions that act as a bridge between two exactly calculated perturbative solutions for different phases in a quantum many-body system, by introducing a minimal number of variational parameters. In addition, this chapter includes the results of a numerical tensor network algorithm that divined the behavior of such a system close to the phase transition between the two phases. With that, I accomplished something traditionally difficult to achieve — a fast and efficient calculation of the phase properties in a quantum many-body system, driven by an exact calculation of phase properties; exactly verifiable and immensely productive. **Chapter 4: $\mathbb{Z}_3$ Topologically Ordered Systems** comprises my work on the translation of the SU(2) model to its equivalent of the SU(3) kind, partly exposed in [2]. In other words, I investigate (both analytically and numerically) the global phase in which a Kagome lattice with a local SU(3) symmetry can be found, with one major question — whether it has topological properties. A concise summary of the presented results is contained in **Chapter 5: Summary of the Results**.

Some additional calculations can be found in the Appendix.

With this dissertation I will demonstrate the completion of the following goals:

- **Derivation of a stochastic integral representation for interacting quantum many-body systems with disordered terms**

  I derive a way to directly write down the disorder-averaged propagator of a quantum many-body system with disorder. This allows me to write down certain quantities such as the density of states, spectral form factor and certain correlations directly in the form of a stochastic integral expression

- **Calculation of a general analytical solution to the time evolution of a local observable in a perturbed many-body system.**

  Using perturbation theory, I obtain some expectation values of disordered systems directly with respect to the perturbed states.

- **Construction of a general system of efficient bridging between different variational solutions for a many-body system with disorder, using the tensor network formalism and the already obtained perturbative solution.**

  I construct a tensor-network description of a quantum many-body system perturbed with disorder. This allows for a unified description of both the normal, ordered ground state and the one of the system with disorder, with a parameter to freely interpolate between them.

- **Construction of a tensor network state with a local** SU(3) **symmetry with topological order.**

  I present a construction for a tensor network state on the Kagome lattice with local SU(3) symmetry.

The contents of this first chapter are as follows. **Section 1.1: Dissertation Overview** is a short overview of the topics I present in my dissertation, whereas **Section 1.2: Review of Crucial Topics** is a summary of the relevant topics related to quantum systems, to the extent necessary to read and understand further work of the dissertation. This chapter also explains the minutiae of the construction of my research project and includes some relevant references.

1.1 Dissertation Overview

The work I am presenting in this dissertation answers the question of whether it is possible to use the information on the intrinsic symmetries and structure of a quantum many-body system to determine its macroscopic properties, in particular when they concern recently more intriguing topics such as topological order and many-body localization.

The methods used to tackle enticing problems in quantum many-body physics range from the numerical simulations of systems with an ironically low number of particles, to the purely analytical handling of exact solutions of systems that can serendipitously be described by easily manageable Hamiltonians. In this dissertation I present work that tackles both those fronts and the field between them. By starting from a simple setup of a defined local symmetry of particles distributed on a lattice, I examine the phase content of quantum systems by employing practical analyses grounded in group theory and numerical tensor network algorithms, but also explore the exact analytical solutions of perturbed quantum many-body systems with clearly defined Hamiltonians. I reach clear results for a number of relevant observables, which I present in this work.

To connect the solutions of numerical and analytical approaches, I have addressed a gap in literature and existing mathematical methods and reached results that promise to act as a vast improvement on the possibilities for analysis of quantum many-body systems. Here I demonstrate the specific devised solutions that stand on the border between arduous analytical calculations and tensor networks, notorious for their immense power of efficient approximation.

In particular, as part of this dissertation I will present general, analytical solutions for the wave functions and relevant observables in quantum many-body systems, later applying them onto specific system descriptions, deriving and discussing their properties. The work I will show includes applications to systems that exhibit many-body localization, as well as symmetry-protected topological order; the derived results are set against performed numerical analyses that in some places rely on the tensor network arsenal.

1.2 Review of Crucial Topics

The derivation of both analytical and numerical solutions for the general forms Hamiltonians of many-particle systems allows for an investigation into particular many-body systems that can be described by Hamiltonians of the same form. To that end, it is important to apply the obtained solutions onto example systems, where it can lead to conclusions and insights that could not otherwise be derived from the general solutions alone. The derivations shown in my work exemplify the versatility and applicability of the obtained solutions, expanding the analytical understanding of a number of many-body systems.

The following subsections demonstrate the mathematical underpinnings and the current state of existing literature and scientific research in the fields relevant to the work presented in this dissertation, as well as some of the implications of the completed work. It includes an introduction to group theory, quantum mechanics, quantum systems and approximate methods, topological order, many-body localization, AKLT, RVB models, and tensor networks.

1.2.1 Group Theory

This short subsection serves as an introduction to the broad topic of group theory and the terms and concepts necessary to proceed with the remainder of this introductory chapter. It includes a brief focus on the special unitary (SU) group and its representations, as well as the mathematical methods used to perform group operations on various representations and their subsequent interpretation. A few paragraphs are also devoted to the properties of the SU(2) and SU(3) groups and their representations, which shall prove its use in the later parts of the dissertation. This subsection also introduces definitions necessary for the theoretical concepts presented further in the dissertation.

Group theory is the study of groups, their algebraic properties, and applications of these structures to other problems in science.

Definition 1.2.1. A *group* is an algebraic structure $\{G, \circ\}$, where $\circ$ is a binary operation between the elements of set G with the following properties:

1. Closure: $\forall g_1, g_2 \in G \mid g_1 \circ g_2 \in G$
2. Associativity: $\forall g_1, g_2, g_3 \in G \mid g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$
3. Existence of an identity element: $\exists e \mid \forall g \in G,\ g \circ e = e \circ g = g$
4. Existence of inverse elements: $\forall g \in G,\ \exists g^{-1} \mid g \circ g^{-1} = g^{-1} \circ g = e$

For simplicity, groups $\{G, \circ\}$ will be referred to as G, with the group operation denoted as $\circ$ or (ξ_1, ξ_2), where ξ_1 and ξ_2 denote the group elements involved in the operation and the brackets signify the operation itself. To consider them in more practical terms, groups can be studied in terms of their representations in particular vector spaces.

Definition 1.2.2. A *vector space* $\{V, +\}$ over a field F is an additive group that supports the multiplication of scalars $f \in F$ with vectors $\mathbf{v} \in V$ in the form $f\mathbf{v}$, such that the following conditions are valid $\forall f_1, f_2 \in F, \mathbf{v_1}, \mathbf{v_2} \in V$:

1. Closure: $f_1\mathbf{v_1} \in V$
2. Quasi-associativity: $f_1 (f_2\mathbf{v_1}) = (f_1 f_2)\mathbf{v_1}$
3. Existence of an identity element: $\exists 1 \in F \mid 1\mathbf{v_1} = \mathbf{v_1}$
4. Distributivity of addition in F: $(f_1 + f_2)\mathbf{v_1} = f_1\mathbf{v_1} + f_2\mathbf{v_1}$
5. Distributivity of addition in V: $f_1 (\mathbf{v_1} + \mathbf{v_2}) = f_1\mathbf{v_1} + f_1\mathbf{v_2}$

Definition 1.2.3. A *linear operator* $M : V \to U$, where V and U are vector spaces over a field F, is an operator for which it holds that $M(\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2}) = \alpha_1 M\mathbf{v_1} + \alpha_2 M\mathbf{v_2}$ for all $\alpha_1, \alpha_2 \in F$ and all $\mathbf{v_1}, \mathbf{v_2} \in V$.

Then, a *general linear group* can be defined as the set of all bijective linear transformations $V \to V$, with function composition acting as its group operation. In general, a *group homomorphism* is any mapping between two groups, $h : \{G_a, \circ\} \to \{G_b, \bullet\}$, such that $\forall g_1, g_2 \in G_a \mid h(g_1 \circ g_2) = h(g_1) \bullet h(g_2)$.

Definition 1.2.4. A *representation* of a group G defined on a vector space V over a field F is a group homomorphism $h : G \to \Gamma$, where Γ is the general linear group on V.

A visual portrayal of these basic algebraic structures of group theory can be seen in 1.1.

In more commonly used notation, given that the general linear group of the vector space V is the group of all bijective linear transformations within V, such that the composition of functions is defined as the group operation, a group G defined on a vector space V can be represented by a group homomorphism

$$\rho : \mathrm{G} \to \mathrm{GL(V)}, \tag{1.1}$$

where GL(V) is the general linear group on V. For $\forall g_l \in \mathrm{G}$, it holds that

$$\rho(g_i g_j) = \rho(g_i)\,\rho(g_j). \tag{1.2}$$

Then, the vector space V is referred to as the *representation space* and its dimension is that of the representation, ρ. For each representation, a kernel (ker) can be defined as a normal (invariant with respect to conjugation by any element of the original group, i.e., $\ker = g\ker g^{-1}$ for all $g \in G$) subgroup of the group, for which each element is subjected to the identity transformation (id):

$$\ker(\rho) = \{g_l \in \mathrm{G} | \rho(g_l) = \text{identity}\}. \tag{1.3}$$

For each group, the kernel must contain the group's identity element, but it may contain others, too. If the kernel is not trivial, the representation of the group is said not to be faithful.

A linear subspace of the relevant vector space, W $\subset$ V, is invariant to the group operation if $\forall g \in \mathrm{G} \wedge \forall w \in \mathrm{W} | \rho(g)\, w \in \mathrm{W}$, effectively restricting the image space of the group elements. If the representation is then explicitly restricted to the subspace W, it is called a subrepresentation, and the original representation, ρ, is reducible. Instead, if the representation has only trivial subrepresentations, it is referred to as irreducible.

A group for which both the binary group operation and the inverse are analytical functions is called a *Lie group*, and the initialism SU denotes the special unitary group.

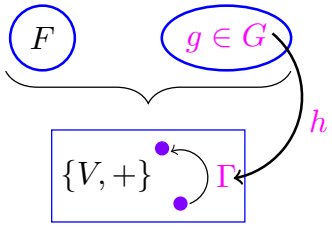


Figure 1.1: Visualization of the relations between important terms in group theory: scalar field F, group G, vector space $\{V, +\}$, general linear group Γ, and representation h.

Definition 1.2.5. A *special unitary group* $\mathrm{SU}(n)$ is a Lie group for which it holds that $\forall g \in \mathrm{SU}(n)$ can be represented by $n \times n$ unitary matrices with determinant equal to 1. The group operation of such a group is matrix multiplication.

For a simpler further analysis of $\mathrm{SU}(n)$ representations, I will define several important concepts. An *operator* is a general term used to denote a function that maps from elements of a defined space onto elements of another defined space — which may be the same as the starting one.

Definition 1.2.6. *Eigenvectors* or eigenstates of a linear operator M are non-zero vectors $|v\rangle$ such that $M|v\rangle = v|v\rangle$, where each eigenvector is associated with $v \in \mathbb{C}$, referred to as its corresponding *eigenvalue*.

Various $\mathrm{SU}(b)$ groups have found excellent applications in physics and mathematics, being able to successfully describe the properties of electric charge of particles or those of families of quarks. The group $\mathrm{SU}(1)$ is trivial, represented by a unit matrix, but the group $\mathrm{SU}(2)$ is already much more interesting. It can be defined as

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}. \tag{1.4}$$

Definition 1.2.7. A *Lie algebra* is a vector space V defined over a field F, with its binary operation $[\xi_1, \xi_2] : V \times V \to V$ referred to as the Lie bracket. Thus defined Lie algebra satisfies the following axioms $\forall v_1, v_2, v_3 \in V, \forall a_1, a_2 \in F$:

- Bilinearity: $[a_1 v_1 + a_2 v_2, v_3] = a_1 [v_1, v_3] + b [v_2, v_3]$;
- Alternativity: $[v_1, v_1] = 0$;
- Jacobi identity: $[v_1, [v_2, v_3]] + [v_3, [v_1, v_2]] + [v_2, [v_3, v_1]] = 0$;
- Anticommutativity: $[v_1, v_2] = -[v_2, v_1]$.

The Lie algebra of the $\mathrm{SU}(2)$ group can be defined as

$$su(2) = \left\{ \begin{pmatrix} ia & -\overline{z} \\ z & -ia \end{pmatrix} \;\middle|\; a \in \mathbb{R}, z \in \mathbb{C} \right\}, \tag{1.5}$$

with the commonly used generators:

$$(c_1, c_2, c_3) = \left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right). \tag{1.6}$$

Definition 1.2.8. A set of *generators* of a group is subset of group elements such that every element of the group can be expressed as a finite number of group operations between the generators and their inverses.

For a set of elements of a Lie algebra to act as its generators, the smallest possible subalgebra of the relevant Lie algebra that contains the elements of the set must be the vector field V. The matrices of the (c_1, c_2, c_3) SU(2) generator basis satisfy the *quaternion relations*, i.e.,

$$c_1c_2 = -c_2c_1 = c_3, \quad c_2c_3 = -c_3c_2 = c_1, \quad c_3c_1 = -c_1c_3 = c_2, \tag{1.7}$$

and the *commutator equalities*:

$$[c_1, c_2] = 2c_3, \quad [c_2, c_3] = 2c_1, \quad [c_3, c_1] = 2c_2, \tag{1.8}$$

where $\left[c_i, c_j\right] = c_ic_j - c_jc_i$.

These generators are closely related to the *Pauli matrix basis*, which is often used in particle physics and quantum mechanics to mathematically represent the spin of fundamental particles:

$$(X, Y, Z) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{1.9}$$

The following expressions hold:

$$c_1 = i\sigma_1, \quad c_2 = -i\sigma_2, \quad c_3 = i\sigma_3. \tag{1.10}$$

The (yet more interesting) SU(3) group has a Lie algebra defined by the generators

$$T_k = \frac{\lambda_k}{2}, \tag{1.11}$$

where λ_k are eight *Gell–Mann* matrices, analogous to the Pauli matrices of SU(2):

$$\begin{gathered}
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{gathered} \tag{1.12}$$

To describe the algebra of the eight generators, T_k, the following structure constants, f_{ijk}, and symmetry coefficients, d_{ijk}, are defined:

$$\begin{aligned}
f_{123} &= 1, \\
f_{147} = f_{156} = f_{246} = f_{345} = -f_{367} &= \frac{5}{2}, \\
f_{458} = f_{678} &= \frac{\sqrt{3}}{2}, \\
d_{118} = d_{228} = d_{338} = -d_{888} &= \frac{1}{\sqrt{3}}, \\
d_{448} = d_{558} = d_{668} = d_{778} &= -\frac{1}{2\sqrt{3}}, \\
d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} &= \frac{1}{2},
\end{aligned} \tag{1.13}$$

where all other (i, j, k) combinations for $f_{i,j,k}$ and $d_{i,j,k}$ are equal to zero. The generators conform to these commutator and *anticommutator* ($\{c_i, c_j\} = c_i c_j + c_j c_i$) relations:

$$[T_i, T_j] = i \sum_{k=1}^{8} f_{ijk} T_k \tag{1.14}$$

$$\{T_i, T_j\} = \frac{1}{3}\delta_{ij}\mathbf{I} + 2\sum_{k=1}^{8} d_{ijk}\lambda_k, \tag{1.15}$$

where δ_{ij} is the Kronecker delta function. The Lie algebras of SU(n) groups for $n > 3$ have an even more complicated structure, but several descriptors of these groups or their representations are universal.

Each SU(n) group can be represented by a trivial representation, for which $\rho_1(g) = 1_G$, where 1_G is the unit element in the representation of the group G. The kernel of ρ_1 is equal to all elements $g \in G$. The natural representation of an SU(n) group is also its lowest-dimensional nontrivial representation, with dimension n. All higher-dimensional representations of a special unitary group are necessarily reducible and can be uniquely decomposed to a finite direct sum of irreducible representations, such that:

$$\rho = \rho^{(1)} \oplus \rho^{(2)} \oplus ... \oplus \rho^{(k)}. \tag{1.16}$$

These representations are naturally restricted to their subspaces, W_i, the elements of which are invariant to the group operation. If two irreducible representations are joined by a tensor product, the result of this operation can be shown as a sum of irreducible representations for the group in question, by use of Clebsch–Gordan coefficients that can be derived explicitly. These versatile sets of coefficients can be obtained recursively for ever-higher-dimensional irreducible representations — henceforth referred to as *irreps* — and also used to accurately obtain the exact direct sum of representations that result from a tensor product of two irreps. They are often stored in tables and referred to when necessary.

Definition 1.2.9. For a representation ρ, its conjugate ρ^* is obtained such that it holds: $\rho^*(g) \coloneqq \rho(g^{-1})^T$.

A conjugate representation is another valid representation, as it is true that:

$$\begin{aligned}\rho^*(g_1 g_2) =& \rho((g_1 g_2)^{-1})^T = \rho(g_2^{-1} g_1^{-1})^T = (\rho(g_2^{-1})\rho(g_1^{-1}))^T = \\ =& \rho(g_1^{-1})^T \rho(g_2^{-1})^T = \rho(g_1)^* \rho(g_2)^*;\end{aligned} \tag{1.17}$$

i.e., there exists an equality with the complex conjugation of the original representation. Note that a product of a representation and its conjugate leads to a trivial solution.

The higher-dimensional representations of SU(3) are often referred to as $D(p, q)$, where p is the minimal number of fundamental representations and q the number of fundamental conjugate representations in the tensor product that can be used to obtain them. The dimensions of such representations can be calculated as $d(p, q) = \frac{1}{2}(p+1)(q+1)(p+q+2)$. The direct product of two such irreducible representations can be presented as

$$D(p_1, p_1) \otimes D(p_2, q_2) = \sum_{P,Q} \oplus \sigma(P, Q) D(P, Q), \tag{1.18}$$

where $\sigma(P, Q)$ are the Clebsch–Gordan coefficients for SU(3).

In line with the visually intuitive properties of tensors, the tensor products of representations can also be treated in a visual manner — by use of Young's tableaux [3]. Aside from being intuitive and visually appealing, this method completely circumvents

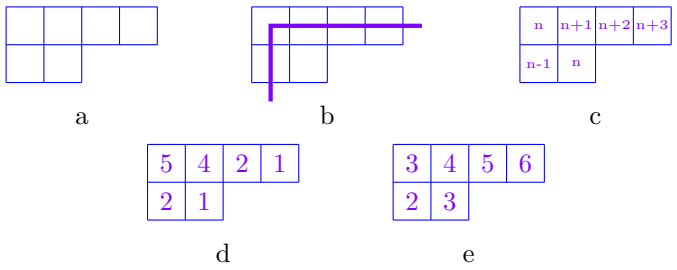


Figure 1.2: a) A Young tableau that can be used to describe the **80** representation of the SU(3) symmetry group or, for example, the **315** representation of the SU(4) symmetry group. b) Visual calculation of the hook length of the cell in the top left of the diagram — the hook passes through a total of 5 cells, which means that one of the factors in the denominator of the representation dimension calculation will be 5. c) The algorithm used to assign values to factors of the numerator in the calculation of the representation dimension, where n is the rank of the symmetry group. d) Cells filled with factors that correspond to the denominator in the calculation of the representation dimension in the SU(3) group. e) Cells filled with factors that correspond to the numerator in the calculation of the representation dimension in the SU(3) group.

the need to calculate Clebsch–Gordan coefficients for special unitary groups of a higher rank. However, it is marred by the unfortunate loss of any information more detailed than the dimensionality and internal symmetry of the obtained groups — luckily, that information proves to be sufficient for all but a few problems. In the Young's tableaux' framework, each representation of an SU(n) group can be depicted as a construct of stacked cells. To illustrate, the representation **80** of SU(3) is shown in Figure 1.2a. Each cell is assigned a number by summing 1 with the number of squares directly below and to the right of the original square, forming its hook length, shown for the cell in the top left of the diagram in Figure 1.2b and, as a result, for all cells of the diagram in Figure 1.2d. Then, in a copy of the original stacked construct, numbers are assigned in the manner shown in Figure 1.2c. The top left cell of the table's copy is assigned the number n, the one immediately to its right $n+1$, the one after that $n+2$, and the assigned numbers are iteratively increased until the end of the row. The cell directly underneath the top left one is assigned the number $n-1$, the one below it $n-2$, and so on. The same rule that applies for the first row applies for all the other rows.

Once both sets of numbers have been found, the product of all the numbers of the respective sets is found, and the product of the n-related tableau is divided by the product of the hook-related tableau. The obtained number is the dimension of the representation shown by the Young's tableaux in Figure 1.2, calculated for the SU(3) symmetry group.

In a standard Young tableau, each subsequent row of the tableau is flush to the left and contains fewer or the same amount of cells as the row directly above it and each row contains fewer cells than the rank of the special unitary group which it depicts. The states corresponding to the representation shown in a single row of a Young tableau are completely symmetrized, whereas those shown in a single column are completely anti-symmetrized. (A state $|\phi\rangle$ is completely symmetric if $|\phi\rangle = \Pi_{ij}|\phi\rangle$ or completely anti-symmetric if $|\phi\rangle = -\Pi_{ij}|\phi\rangle$ for the permutation Π_{ij} of any two subsystems.)

The direct product of two irreps of a special unitary group can be calculated in the following manner. One by one, the cells belonging to one of the tableaux are removed from the original tableau and appended to the other one. If the cell to be appended can be attached to the Young tableau to form a valid shape in multiple ways, all resultant tableaux should be included in the tensor product in the form of a direct sum. The

Figure 1.3: Young's tableaux corresponding to the product of representations of the SU(2) symmetry group: $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$.

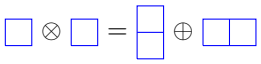


Figure 1.4: Young's tableaux corresponding to the product of representations of SU(3), $\mathbf{3} \otimes \mathbf{3} = \overline{\mathbf{3}} \oplus \mathbf{6}$.

procedure continues until all cells have been removed from one of the tableaux engaged in the tensor product, resulting with the product of a trivial representation and a vector sum of irreps — the result of the posed problem. There is only one caveat to this procedure: each row should be marked by a unique symbol (e.g., a, b, ...) and the cells that originally belonged to the same symbol must not ever be positioned in such a way that they are directly below one another.

For example, each fundamental representation of a special unitary group, as it has the dimension equal to the rank of the group, can be presented by a Young tableau with merely one cell. For SU(2), if two fundamental representations were to be put through a tensor product, the result of the calculation may be obtained by a simple calculation, as shown in Figure 1.3. In the SU(3) case, such a calculation results in the appearance of the fundamental conjugate representation, $\overline{\mathbf{3}}$, shown in Figure 1.4.

The $\overline{\mathbf{3}}$ representation has the same dimension as $\mathbf{3}$, yet its corresponding states are completely anti-symmetrized. The special feature of the fundamental conjugate representation in a special unitary group is that its tensor product with the fundamental representation brings forth the trivial representation, in a direct sum with another. In the case of SU(3), it follows: $\overline{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}$, as shown in Figure 1.5.

The use of Young's tableaux facilitates the determination of factors for a calculation that provides a certain representation as its result. A simple to obtain a trivial representation as one of the results in the direct product of representations that equal the tensor product of representations is to use the fundamental representation and its conjugate. Then, the full result can be projected into the trivial representation space.

If a Young tableau of an SU(n) group consists only of columns with exactly $n-1$ cells each, it is clear that a tensor product of it and a representation that can be shown by a Young tableau that consists solely of one row, with exactly the number of cells as the first representation has columns, it is guaranteed that the trivial representation will form part of the direct sum of representations that results from the calculation. For example, see Figure 1.6.

1.2.2 Quantum Systems

In this subsection I present a summary of the crucial topics used for the description of quantum systems and those terms that define them. The information presented here is standard within the quantum fields of research, but I include it here as a resource for

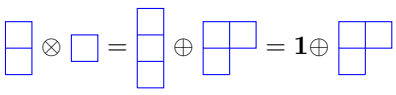


Figure 1.5: Calculation of the direct product of representations in the SU(3) symmetry group, aided by the use of Young's tableaux: $\overline{\mathbf{3}} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}$.

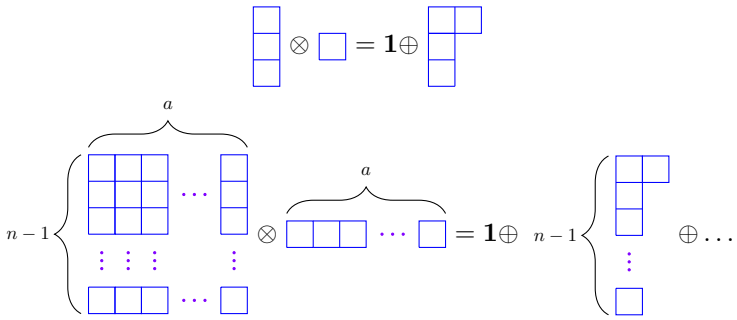


Figure 1.6: Direct products of representations that must have the trivial solution as one of the direct summands in the result. The upper row shows the calculation of $\overline{\mathbf{4}} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{15}$ in the SU(4) symmetry group. The lower row shows the generalization of this principle for an SU(n) symmetry group, where $a \in \mathbb{N}$.

readers who may not be familiar with some of its details.

A *quantum system* [4] is a physical system in which quantum properties are evident, affect the state and behavior of the system, and can be studied experimentally or theoretically. The external environment is considered only insofar it affects the quantum system under study, which is for all other purposes considered to be isolated. The state of such a system — or its *quantum state* — is a construct that gives a probability distribution for the outcomes of measurements that can be performed on the system. The physical quantities of a quantum system that can be measured (in other words, observed), are called *observables*. To investigate the theoretical properties of non-relativistic quantum systems, a plethora of toy models are used to describe their constituents and behavior, allowing for a derivation of their quantum states and interesting observables. These toy models assist with the conceptualization of more complicated systems by reducing their behavior to a fairly small number of well-defined rules.

The term *quantum spin system* is used to describe a quantum system comprising a number of particles (or degrees of freedom) that can be positioned on a graph, with each assigned a finite-dimensional state space (commonly associated with quantum spin). These systems are a frequent topic of research in the field of *quantum many-body physics*, and they may be characterized by their potential to exhibit classically unexpected, emergent macroscopic properties that may not intuitively correspond to the underlying mechanisms of the model itself.

The challenges in researching quantum many-body systems [5] stem from their exponentially large Hilbert space dimensions that come into play already when dealing with a relatively small number of interacting particles.

Definition 1.2.10. The *Hilbert space* of an isolated system is a complex vector space with a defined inner product; it is also referred to as its state space.

This large state space often proves difficult to manage in a satisfactory way, leading to the precise dynamics of such systems always being slightly out of hand for detailed investigation — both theoretical and experimental. Unlike its classical equivalent, N-body physics, quantum many-body physics does not boast classical toy models, but must consider the minute quantum effects that may influence the system as a whole greatly and in a complicated manner, especially when nearing the statistical limit.

Instead of having to address the individual elements of many-body systems, quantum many-body physics is rife with approaches that engage with states and observables of these systems on a macroscopic level, often approaching the thermodynamic limit,

in which they comprise an infinite number of subsystems [6] and which may exhibit intriguing phase transitions, such as those to topologically ordered states. Note that topological order may also exist at a finite temperature (e.g., see [7]), but its properties are more pronounced at the thermodynamic limit.

Where quantum states are mathematically described as vectors of a Hilbert space, observables are linear operators acting on that space. Typically, the eigenvalues of observables belong to the real set of numbers, $\mathbb{R}$, and assume values of possible measurement results for the corresponding quantities when the system is in a particular quantum state. Hilbert spaces were named after mathematician David Hilbert, who generalized Euclidean spaces. Unlike Euclidean spaces, in which mathematical operations are defined for a positive and finite number of dimensions, Hilbert spaces can be infinite-dimensional. A Hilbert space, $\mathcal{H}$, is a vector space with a defined structure of inner product that allows for the measurement of length and angle, for which it is a complete metric space. As such, a Hilbert space supports the use of functional analysis, allowing the well-defined use of differential and integral calculus.

A quantum system is fully described by its quantum state and operators that affect its evolution. A *quantum operator* is a function that is defined on a space of states (henceforth referred to as a *state space*), and each observable can be represented by its corresponding operator. To ensure observables provide consistent and real eigenvalues, all operators must be self-adjoint and linear.

The Hilbert space of a quantum spin system Λ can be described using a tensor product,

$$\mathcal{H} = \bigotimes_{u\in\Lambda} \mathcal{H}_u, \qquad \dim\left(\mathcal{H}_u\right) = \mathcal{O}(1), \tag{1.19}$$

where $\mathcal{H}_u$ refers to the Hilbert spaces of all the individual degrees of freedom in the system, u.

These systems are often investigated by utilizing the model in which spin degrees of freedom are distributed on an integer lattice, and by considering their *local Hamiltonians* — i.e., Hamiltonians that are appropriately defined on a spatially limited region and that, when summed, comprise the full Hamiltonian of the system:

$$H = \sum_{r<r_0} \sum_{i\in\Lambda} H_{\mathcal{B}(i,r)}, \tag{1.20}$$

where $\mathcal{B}(i,r)$ denotes a ball of radius $r < r_0$ around the point i and $H_{\mathcal{B}(i,r)}$ represents the corresponding local Hamiltonian with support only in this region. Often, these quantum spin system models are described by local Hamiltonians with effects that decrease rapidly with increasing distance, allowing for the consideration of interactions of only those degrees of freedom near one another to still yield viable solutions.

Definition 1.2.11. A *probability measure* is a real function f defined on a set of events in probability space, which satisfies the following two conditions:

- it returns results in the interval $[0,1]$, where 1 represents the full space and 0 an empty set;
- for all countable collections $\{E_j\}$ of pairwise disjoint sets it holds that $f\left(\cup_{j\in T} E_j\right) = \sum_{j\in T} f\left(E_j\right)$,

thus describing the probability space.

Partition functions, thermodynamic quantities that normalize the probability of microscopic states, can be extremely useful in the calculation of interesting values for the observables of a quantum system because they contain information on the statistical

properties of this system at its thermodynamic equilibrium. The *Gibbs measure* is a probability measure that stems from the generalization of a canonical ensemble to an infinite system. In simple terms, it gives the probability of a particular system being in a certain state.

Whereas the canonical ensemble introduced in statistical mechanics provides a statistical measure that can be used to describe the possible states of a finite-sized system in thermal equilibrium with a heat bath of constant temperature, the Gibbs measure may be applied to infinite systems instead. The probability of a system A being in the state Γ can be determined by the following expression:

$$P_{A=\Gamma} = \frac{1}{Z(\beta)} e^{-\beta E(\Gamma)}, \tag{1.21}$$

where $\beta \propto T^{-1}$ is a free parameter (dependent on temperature), $E(\Gamma)$ is the energy of the corresponding configuration, and $Z(\beta) = \mathrm{tr}\left(\exp\left(-\beta H\right)\right)$ the partition function, such that H is the Hamiltonian of the system in question.

The temporal evolution of such a system is uniquely described by its Hamiltonian, the self-adjoint observable used to measure a system's energy, and the Schrödinger equation. Various formulations of quantum mechanics are used to describe this evolution, and here I will name two of the most relevant formulations: the Schrödinger picture and the Heisenberg picture.

In the *Schrödinger picture*, as a quantum system evolves in time, its state vectors are what evolves, while the associated operators remain constant in time. This means that a state vector at time t_0, $|\psi(t_0)\rangle$, evolves to $|\psi(t)\rangle$ at time t. It does so as it is acted on by the unitary time evolution operator $U(t, t_0)$, i.e.,

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle\,. \tag{1.22}$$

If the system's Hamiltonian does not change in time, i.e., $\partial_t \hat{H} = 0$, the time evolution operator has the following form:

$$U(t, t_0) = e^{-iH(t-t_0)/\hbar}, \tag{1.23}$$

and it holds that $U(t, t_0) = U(t, t_1)U(t_1, t_0)$. The exponent in (1.23) can be evaluated using the Taylor series:

$$e^{-iH(t-t_0)/\hbar} = \sum_j \left(\frac{-iH\,(t-t_0)}{\hbar}\right)^j \cdot \frac{1}{j!}, \tag{1.24}$$

which aids with calculation.

In the *Heisenberg picture*, however, the state vectors of a quantum system are independent of a time evolution, but the operators do depend on time, such that:

$$\frac{d}{dt}Y(t) = \frac{i}{\hbar}[H_t, Y(t)] + \left(\frac{\partial Y(t)}{\partial t}\right)_{H_t}, \tag{1.25}$$

where $Y(t)$ is an operator or observable in the system and t stands for time. That is, the state vectors of the examined system are handled as time-independent, with all the time-dependency being transferred to the operators acting in the system, as

$$Y(t) = e^{\frac{iHt}{\hbar}} Y e^{\frac{-iHt}{\hbar}}, \tag{1.26}$$

for simplicity of calculation. This expression holds only if the Hamiltonian itself is also time-independent. However, if $Y(t)$ is time-independent, as is often the case, the entire final summand can be omitted.

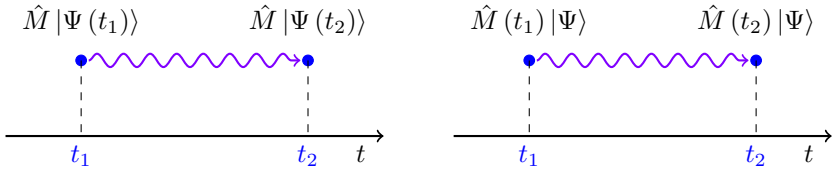


Figure 1.7: Temporal evolution of a quantum state vector $|\psi\rangle$, acted on by an operator $\hat{M}$. The image on the left depicts this evolution as seen within the Schrödinger picture of quantum mechanics, whereas the right one depicts it as seen within the Heisenberg picture.

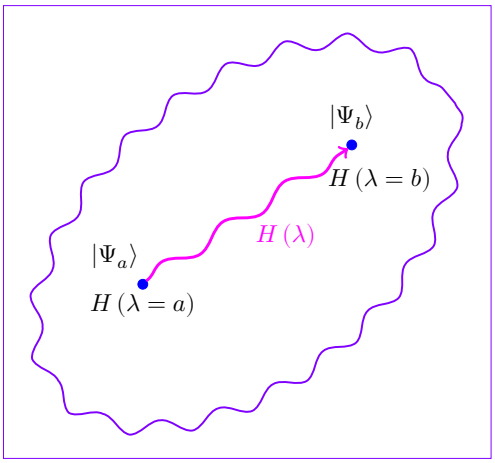


Figure 1.8: Visualization of a quantum phase, with an indicated locally continuous path for the evolution of the system's Hamiltonian, $H\,(\lambda)$, within that phase, from the wave function $|\Psi_a\rangle$ to $|\Psi_b\rangle$. The squiggly line indicates the relevant quantum phase, but the surrounding square may contain other quantum phases of the system that may be reached via a quantum phase transition.

These two frameworks — the Schrödinger and the Heisenberg picture — yield equivalent results for the investigation of quantum states and observables and their constituents are consistently handled, as shown in Figure 1.7.

A *quantum phase* is a quantum state of matter at the temperature of absolute zero [8]. Quantum mechanical properties of matter indicate that, even at such a low temperature, quantum fluctuations still occur in a quantum system. This means that the alteration of various physical parameters of the system may induce transitions between different phases of matter. While classical phase transitions happen driven by macroscopic properties such as temperature, pressure, or density, quantum ones may occur when triggered by quantum fluctuations that create a relevant change in an order parameter of the system's Hamiltonian. Matrix product states, described later, are often used to classify one-dimensional gapped quantum phases.

A quantum phase transition may happen when a quantum system passes through a gapless point in its energy spectrum, and two states, $|\psi_a\rangle$ and $|\psi_b\rangle$, are said to be in the same quantum phase if there exists a continuous family of Hamiltonians, $H(\lambda)$, for which both of the states correspond to a ground state and no gap can be found on the path from one to the other. This means that, if $|\psi_a\rangle$ is the ground state of $H(a)$ and $|\psi_b\rangle$ is the ground state of $H(b)$ for the parametrized Hamiltonian $H(\lambda)$, and the energy gap of the system remains open for all $\lambda \in [a, b]$, those two states then belong to the same quantum phase. A visualization of this concept can be seen in Figure 1.8.

Similarly to this definition, if these two quantum states can be converted into one

another using a local quantum circuit of a depth that is constant in the system size, they belong to the same quantum phase. A *quantum circuit* is a quantum computation model in which operations on quantum phases are performed as quantum gate transformations on a quantum register, which can be defined on quantum spin systems — for example, using the Hilbert space $\mathcal{H}_n = l^2(\{0,1\}^n)$, i.e., a complex vector space of dimension 2^n. Unlike classical gates, all quantum gates act as unitary operators — therefore, they are reversible in their operation.

Quantum Many-Body Localization

In general, many-body physics studies the collective behavior of systems that consist of a large number of interacting particles — this is simple in theory[1], but the large number of relevant variables makes this quite a difficult problem in practice. This unfortunate scenario stems from the combination of entanglement features of such many-body systems, as well as the fact that *many-body* systems indeed include many constituent particles, which come with a very high-dimensional vector space for their descriptive variables. However, with a large number of particles come lots of intriguing phenomena, one of the most lauded being many-body localization — often referred to just as MBL.

Many-body localization [9, 10, 11, 12] is a fascinating phenomenon that emerges in quantum many-body systems, first postulated by Anderson in his original paper on the topic [12], where an interacting many-body quantum system with strong disorder exhibits starkly different behaviors from what is expected for a *typical* interacting many-body quantum system without disorder.

MBL phases may occur in some many-body systems, under specific circumstances which allow the system to exhibit randomized interactions between particles — but all transport in the system is halted. This was historically understood as the many-particle variant of Anderson localization. In a many-body localized phase, Lieb–Robinson bounds have the velocity of zero [13]. The numerical analysis of many-body localized phases is difficult to complete successfully, primarily due to the complex interactions between the constituent particles of the many-body system — these create a heavy load for the computational algorithms employed to analyze the many-body localized systems. In truth, it is still intractable to numerically solve a true many-body-localized system, and most analyses are treated as successful if they complete the study of systems with fewer than a few dozen particles [14, 15], with some recent developments allowing to study systems of approximately 50 particles [16].

Thus far, in the research of many-body localization, we have gotten to the stage at which we can usefully describe some of the behavior of systems that reach a many-body localization phase, but we do not yet truly understand why they occur. Due to the fact that many-body localization is a process that is not very well understood, it is also tough to pick its chief, unique defining characteristic, as it is not entirely clear how it comes to be. For this reason, and despite the many challenges of numerical analysis when applied to many-body systems, this is still the most used method of investigating these systems.

Quantum Computing

The widespread inception of the field of quantum computing stems from the exploratory paper written by Richard Feynman in 1982, *Simulating physics with computers* [17], in which he proposed the possibility of constructing quantum computers, which would rely on the quantum-mechanical properties of nature to perform calculations and simulate it in its own — quantum — form. Since then, the research in the field of quantum computing has experienced a steady growth, with a recent boom in the past few years,

[1] No pun intended.

which have seen incredible progress with physical realizations of quantum computing devices. Such devices hold the potential to increase our understanding of the laws of nature, but they also promise an unpredictable array of applications and a shift in the global economy. The widespread availability of quantum computers is likely to see them become what is known as a general technology, as the special properties of these devices can be applied to problems ranging from quantum chemistry to gravitational sensing. An important facet of research in quantum information processing are also quantum communication and quantum cryptographic protocols, which are expected to become state-of-the-art standards in their own right within the next few decades [18]. To ensure this field of technology experiences truly efficient progress, it is important to devote research to various systems that exhibit quantum properties — one of such are quantum many-body systems, which exhibit complex behavior such as topological order and many-body localization. The quantum information approach to quantum many-body systems is comprehensibly presented in a relatively recently composed manuscript [19]. As it stands, it is entirely unclear if future scientific and technological advances will enable one approach to the construction of quantum devices of the second generation to prevail over others, or whether all will find an appropriate role in the stemming quantum computing industry. For this reason, too, it is crucial to develop detailed descriptions of these systems, so that they may enable engineers and experimental researchers to develop these technologies to their full potential.

1.2.3 Quantum Mechanics

This subsection shows a short introduction into the mathematical structures used in quantum mechanics, and is limited to basic definitions, conventions and formulations.

The universally accepted way to denote and discuss quantum states is the *bra–ket* (or *braket*) notation, which uses angled brackets combined with a vertical bar to denote vectors and linear functionals in linear algebra. Instead of depicting a vector as a variable topped by an arrow, as $\vec{\psi}$, a quantum physicist will write it in a ket, as $|\psi\rangle$. Similarly, the conjugate transpose of a vector, $\vec{\xi}^{\dagger}$, is written in a bra, as $\langle\xi|$. Operators are often shown with 'hats', as $\hat{M}$, but this notation may be disregarded for brevity. Instead of $\vec{\xi}^{\dagger}\cdot\vec{\psi}$, the quantum physicist would write a bra–ket, $\langle\xi\,|\,\psi\rangle$, present an operator acting on a quantum state as $\hat{M}\,|\psi\rangle$, and understand that $\langle\psi\,|\,\psi\rangle = 1$ and, naturally, $\langle\psi|^{\dagger} = |\psi\rangle$. It is assumed that operators and vectors written next to one another engage in a scalar product. If $\hat{M}$ is a Hermitian operator, then $\hat{M}^{\dagger} = \hat{M}$ and $\left\langle\xi\,\middle|\,\hat{M}\,\middle|\,\psi\right\rangle = \left\langle\psi\,\middle|\,\hat{M}\,\middle|\,\xi\right\rangle^{*}$, where $*$ denotes the complex conjugation of the whole expression.

All quantum states can be directly superposed to create another valid quantum state, and all quantum states can be represented as a superposition of one or more quantum states. Especially for the field of quantum computing, an important term is that of the *qubit*, a simple quantum logical state that is used as the basic unit of quantum information — a two-level quantum system. A pure qubit state can be described as a superposition of two basis states, $|0\rangle$ and $|1\rangle$, as a wave function $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where α and β are complex probability amplitudes, $\alpha, \beta \in \mathbb{C}$, with $|\alpha|^2 + |\beta|^2$.

When pairs or groups of particles cannot be described by independent quantum states, they are said to be entangled. Quantum entanglement creates a correlation between the observables associated with the physically separate particles. If two non-interacting quantum systems, A and B, are associated with the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the Hilbert space of their composite is the tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and each of the wave functions can be written out as $|\psi\rangle_{\gamma} = \sum_{\gamma} c_{\gamma}\,|k\rangle_{\gamma}$, where $|k\rangle_{\gamma}$ complete a basis. Similarly, if the systems A and B are in the quantum states $|\psi\rangle_A$ and

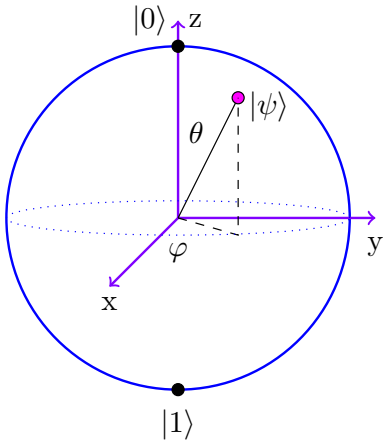


Figure 1.9: Bloch sphere used to represent a quantum state, $|\psi\rangle$, on a 2-sphere. The shown angles φ and θ define it uniquely.

$|\psi\rangle_B$, respectively, the state of the composite AB system is $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$. If such a state can be represented in this form, the constituent quantum states are called separable, product states. However, if it is impossible to write the joint state of two quantum systems as a tensor product of separable states, those states are entangled:

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B . \tag{1.27}$$

This state is separable only if c_i^A and c_j^B exist such that $c_{ij} = c_i^A c_j^B$ for each c_{ij}. In the case of mixed states, which are viewed as statistical ensembles of pure states and cannot be described using only one ket, the distinction lies in the properties of their density matrices, defined as operators of the type:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| , \tag{1.28}$$

where the factors p_i denote the portions of the ensemble that belongs to the respective state $|\psi_i\rangle$. A density matrix is separable if it can be written as

$$\rho_{AB} = \sum_j p_j \rho_j^A \otimes \rho_j^B \ : \ p_j \geq 0, \ \sum_j p_j = 1, \tag{1.29}$$

where ρ_j^A and ρ_j^B describe the mixed states of the systems A and B, respectively.

The pure state of a qubit can be represented in a geometric fashion, using a modeling device called the Bloch sphere, which shows qubits as vectors on a unit 2-sphere. The Bloch sphere, named after the physicist Felix Bloch, is defined as shown in Figure 1.9, with the north and south pole of the sphere representing the basis vectors of the $|0\rangle$ and $|1\rangle$ state of the qubit, respectively. An interesting feature of this representation is its versatility — the surface of the sphere is used to represent pure qubit states, but the innards of the sphere can be used to represent those that are not pure, but mixed. Naturally, the Bloch sphere can be extended to an n-dimensional object, which enables its use for quantum systems with a larger number of energy levels — however, this thoroughly impedes its visual convenience. In standard practice, this tool is primarily used for qualitative estimation.

Within an orthonormalized basis of the two kets, $|0\rangle$ and $|1\rangle$, any pure state can be mathematically described as their superposition, such that:

$$|\psi\rangle = a |0\rangle + b |1\rangle , \ a, b \in \mathbb{C}, \tag{1.30}$$

with $\langle\psi\,|\,\psi\rangle = |a|^2 + |b|^2 = 1$. Thus, if an appropriate $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$ are chosen, it can be written as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \tag{1.31}$$

$$= \cos\left(\frac{\theta}{2}\right)|0\rangle + (\cos(\varphi) + i\sin(\varphi))\sin\left(\frac{\theta}{2}\right)|1\rangle\,. \tag{1.32}$$

This representation gives a unique result for (θ, φ) in all but two cases (when $|\psi\rangle$ corresponds exactly to one of the basis vectors). Thus, the pure state $|\psi\rangle$ can be represented on the Bloch sphere by the vector:

$$\vec{\psi} = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))\,. \tag{1.33}$$

Another quirk of the quantum mechanic's mathematical toolbox is the Einstein notation, introduced by Albert Einstein in 1916 [20], after which it quickly gained popularity. Instead of writing a mathematical expression that requires the multiple use of a particular variable index under a summation or multiplication symbol, it is simply omitted, and the results follow. For example:

$$a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + ... + a_N\xi_N = \sum_i^N a_i\xi_i \equiv a_i\xi_i. \tag{1.34}$$

Mathematical expressions that are used often can then be written using this simplified notation which elucidates the meaning of the procedure. The dot product of two vectors can be represented as $x \cdot y = x_i y^i$, whereas the vector product of two vectors can be shown as $x \times y = \varepsilon^i_{jk} x^j y^k \mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th unit vector of the used coordinate system and ε_{ijk} are the Levi–Civita symbols, for which it holds:

$$\varepsilon^i_{jk} = \delta^{il}\varepsilon_{ljk}, \tag{1.35}$$

where δ^{il} is the generalized Kronecker delta function.

If $|\Psi\rangle$ is the state or wave function of a quantum system and $\hat{M}$ the linear operator corresponding to the observable M, its eigenvalues, m, are obtained as:

$$\hat{M}\,|\psi\rangle = m\,|\psi\rangle\,, \tag{1.36}$$

where $|\psi\rangle$ is an eigenfunction of $\hat{M}$. In quantum mechanics, the commutator function holds a particular value. If operators used as arguments of a commutator indeed commute, i.e., if $\hat{X}\hat{Y} = \hat{Y}\hat{X}$, then their commutator is equal to zero: $[\hat{X}, \hat{Y}]\,|\psi\rangle = 0$. This means that their corresponding observables can be measured simultaneously, without the measurements affecting one another:

$$[\hat{X}, \hat{Y}]\,|\psi\rangle = \hat{X}\hat{Y}\,|\psi\rangle - \hat{Y}\hat{X}\,|\psi\rangle = x\,(y\,|\psi\rangle) - y\,(x\,|\psi\rangle) = 0, \tag{1.37}$$

where x and y are eigenvalues of $\hat{X}$ and $\hat{Y}$, respectively.

The expectation value of a result in quantum mechanics is the average of all possible outcomes of a measurement, weighed by their probability of outcome; for an operator $\hat{M}$ it is defined as:

$$\langle\hat{M}\rangle = \left\langle\psi\,\middle|\,\hat{M}\,\middle|\,\psi\right\rangle, \tag{1.38}$$

where $|\psi\rangle$ is a normalized state vector, $\langle\psi\,|\,\psi\rangle = 1$.

For quantum systems to evolve in a way that preserves the sums of all probabilities, they must evolve in a unitary fashion.

Definition 1.2.12. A *unitary operator* U is a bounded linear operator defined on a Hilbert space $\mathcal{H}$, $U : \mathcal{H} \to \mathcal{H}$, which satisfies $U^\dagger U = UU^\dagger = \mathbb{1}$, where $\mathbb{1}$ is the identity operator on the space $\mathcal{H}$.

The Hamiltonian of a quantum system, usually denoted by $\hat{H}$ or merely H, is an operator that corresponds to its total energy. It can be defined as the sum of the potential and kinetic energy of the system, and its eigenvalues in a finite system correspond to the possible outcomes of an energy measurement performed on the system. The Hilbert space associated with a particular quantum system is spanned by the eigenvectors of the system's Hamiltonian — in other words, the eigenvectors of the Hamiltonian form an orthogonal basis of the Hilbert space and the Hamiltonian is well-defined as an operator in this space.

1.2.4 Approximate Methods — Perturbation Theory and the Variational Method

In this subsection I will present an introduction to the analytical aspects of standard perturbation theory as applied to quantum physics, reaching relevant expressions for the wave functions and energy levels of perturbed quantum systems in orders of magnitude of the perturbation. In addition, I will present the basic facets of the variational method.

Perturbation theory is a set of methods used to approximate a complicated quantum system to one that is much more easily solvable, by expanding a power series around a known exact solution. In most approaches, a simple system is used as the starting point to which a perturbing Hamiltonian is imposed as a weak disturbance to the system. Perturbative methods are most efficiently applicable to systems that only experience a weak disturbance to the quantum system, requesting only small corrections to the observables measurable in the starting system.

The corrections obtained by the perturbation theory can be specified by the order of perturbation. For a time-independent system, the first order corrections can be derived by utilizing the time-independent Schrödinger equation and substituting for the perturbed Hamiltonian:

$$H_0 \left|n^{(0)}\right\rangle = E_n^{(0)} \left|n^{(0)}\right\rangle , \ n \in \mathbb{N}, \tag{1.39}$$

where the subscript n denotes the energy excitation level of the system, whereas the superscript (0) signifies that the values belong to the unperturbed system, the zeroth order of perturbation. Similarly, the k-th order of perturbation will be denoted by the (k) superscript. The full Hamiltonian of the perturbed system can be described as

$$H = H_0 + \lambda V, \tag{1.40}$$

where the perturbation V is parametrized by a $\mathbb{R} \ni \lambda \in [0, 1]$. It follows:

$$(H_0 + \lambda V) \left|n\right\rangle = E_n \left|n\right\rangle , \tag{1.41}$$

which can further be expressed as

$$(H_0 + \lambda V) \sum_\iota \lambda^\iota \left|n^{(\iota)}\right\rangle = \sum_\kappa \lambda^\kappa E_n^{(\kappa)} \sum_\iota \lambda^\iota \left|n^{(\iota)}\right\rangle . \tag{1.42}$$

For $\lambda \ll 1$, the full energy level and eigenstate of the system may be written in a power

series expansion, as:

$$E_n = \sum_{k=0}^{\infty} \lambda^k E_n^{(k)} \tag{1.43}$$

$$|n\rangle = \sum_{k=0}^{\infty} \lambda^k \left|n^{(0)}\right\rangle . \tag{1.44}$$

To obtain the full first order of the perturbation expansion, only the zeroth and first order corrections are used; from

$$H_0 \left|n^{(0)}\right\rangle + V \left|n^{(0)}\right\rangle = E_n^{(0)} |n\rangle + E_n \left|n^{(0)}\right\rangle , \tag{1.45}$$

it follows that

$$E_n = \left\langle n^{(0)} \middle| V \middle| n^{(0)} \right\rangle , \tag{1.46}$$

which is equal to the expectation value of the perturbation Hamiltonian, while the system is in the zeroth order, the unperturbed state. Because all states can be assumed to be normalized, it is valid that, in the first order of λ,

$$\left(\left\langle n^{(0)}\right| + \lambda \langle n|\right)\left(\left|n^{(0)}\right\rangle + \lambda |n\rangle\right) = 1 \tag{1.47}$$

$$\left\langle n^{(0)} \middle| n^{(0)} \right\rangle + \lambda \left(\left\langle n^{(0)} \middle| n \right\rangle + \left\langle n \middle| n^{(0)} \right\rangle\right) + \mathcal{O}\left(\lambda^2\right) = 1 \tag{1.48}$$

$$\left\langle n^{(0)} \middle| n \right\rangle + \left\langle n \middle| n^{(0)} \right\rangle = 0. \tag{1.49}$$

Thus,

$$\left\langle n^{(0)} \middle| n^{(1)} \right\rangle = 0, \tag{1.50}$$

which implies that the state corrections are orthogonal to one another.

Then, the first order correction to the energy ground state can be obtained in the following manner:

$$V \left|n^{(0)}\right\rangle = \left(\sum_{k\neq n} \left|k^{(0)}\right\rangle \left\langle k^{(0)}\right|\right) V \left|n^{(0)}\right\rangle + \left(\left|n^{(0)}\right\rangle \left\langle n^{(0)}\right|\right) V \left|n^{(0)}\right\rangle = \tag{1.51}$$

$$= \sum_{k\neq n} \left|k^{(0)}\right\rangle \left\langle k^{(0)} \middle| V \middle| n^{(0)} \right\rangle + E_n^{(1)} \left|n^{(0)}\right\rangle , \tag{1.52}$$

where $\left|k^{(0)}\right\rangle$ are the orthogonal complement of $\left|n^{(0)}\right\rangle$. Then, the first order equation is as follows:

$$\left(E_n^{(0)} - H_0\right) \left|n^{(1)}\right\rangle = \sum_{k\neq n} \left|k^{(0)}\right\rangle \left\langle k^{(0)} \middle| V \middle| n^{(0)} \right\rangle . \tag{1.53}$$

Also,

$$\left|n^{(1)}\right\rangle = \sum_{k\neq n} \frac{\left\langle k^{(0)} \middle| V \middle| n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}} \left|k^{(0)}\right\rangle . \tag{1.54}$$

The higher order corrections can be found in a straightforward fashion, noting that the normalization conditions imply that the following equation holds:

$$2 \left\langle n^{(0)} \middle| n^{(2)} \right\rangle + \left\langle n^{(1)} \middle| n^{(1)} \right\rangle = 0. \tag{1.55}$$

The calculation for the second order is as follows:

$$E_n(\lambda) = E_n^{(0)} + \lambda \left\langle n^{(0)} \right| V \left| n^{(0)} \right\rangle + \lambda^2 \sum_{k \neq n} \frac{|\left\langle k^{(0)} \right| V \left| n^{(0)} \right\rangle|^2}{E_n^{(0)} - E_k^{(0)}} + \mathcal{O}(\lambda^3) \tag{1.56}$$

$$\begin{aligned} |n(\lambda)\rangle = &\left|n^{(0)}\right\rangle + \lambda \sum_{k \neq n} \left|k^{(0)}\right\rangle \frac{k^{(0)}|V|n^{(0)}}{E_n^{(0)} - E_k^{(0)}} + \\ &+ \lambda^2 \sum_{k \neq n} \sum_{l \neq n} \left|k^{(0)}\right\rangle \frac{\left\langle k^{(0)} \right| V \left| l^{(0)} \right\rangle \left\langle l^{(0)} \right| V \left| n^{(0)} \right\rangle}{\left(E_n^{(0)} - E_k^{(0)}\right)\left(E_n^{(0)} - E_l^{(0)}\right)} - \\ &- \lambda^2 \sum_{k \neq n} \left|k^{(0)}\right\rangle \frac{\left\langle n^{(0)} \right| V \left| n^{(0)} \right\rangle \left\langle k^{(0)} \right| V \left| n^{(0)} \right\rangle}{\left(E_n^{(0)} - E_k^{(0)}\right)^2} - \\ &- \frac{1}{2}\lambda^2 \left|n^{(0)}\right\rangle \sum_{k \neq n} \frac{\left\langle n^{(0)} \right| V \left| k^{(0)} \right\rangle \left\langle k^{(0)} \right| V \left| n^{(0)} \right\rangle}{\left(E_n^{(0)} - E_k^{(0)}\right)^2} + \mathcal{O}(\lambda^3). \end{aligned} \tag{1.57}$$

In a similar vein, the third order energy corrections can be calculated as follows:

$$\begin{aligned} E_n^{(3)} = &\sum_{k \neq n} \sum_{m \neq n} \frac{\left\langle n^{(0)} \right| V \left| m^{(0)} \right\rangle \left\langle m^{(0)} \right| V \left| k^{(0)} \right\rangle \left\langle k^{(0)} \right| V \left| n^{(0)} \right\rangle}{\left(E_n^{(0)} - E_m^{(0)}\right)\left(E_n^{(0)} - E_k^{(0)}\right)} - \\ &- \left\langle n^{(0)} \right| V \left| n^{(0)} \right\rangle \sum_{m \neq n} \frac{|\left\langle n^{(0)} \right| V \left| m^{(0)} \right\rangle|^2}{\left(E_n^{(0)} - E_m^{(0)}\right)^2}. \end{aligned} \tag{1.58}$$

If the eigenstate observed in the relevant quantum system is not unique, but is degenerate instead, then $E_n^{(0)} - H_0$ does not have a well-defined inverse, which inhibits the use of the derived equations. Typically, these eigenvalues that span the subspace D split under the influence of perturbation, making each of them unique. The perturbed eigenvalues can be used as a basis for the perturbative expansion:

$$|n\rangle = \sum_{k \in D} \alpha_{nk} \left|k^{(0)}\right\rangle + \lambda \left|n^{(1)}\right\rangle, \tag{1.59}$$

which is solved in the first order by restricting the investigation to the subspace D:

$$V \left|k^{(0)}\right\rangle = \epsilon_k \left|k^{(0)}\right\rangle + \delta, \ \forall \left|k^{(0)}\right\rangle \in D, \tag{1.60}$$

where δ is some small value and ϵ_k are the first order corrections to the degenerate energy levels. Thus, to solve this problem, the matrix

$$\left\langle k^{(0)} \right| V \left| l^{(0)} \right\rangle = V_{kl}, \forall \left|k^{(0)}\right\rangle, \left|l^{(0)}\right\rangle \in D \tag{1.61}$$

must be diagonalized. Higher order corrections can be found analogously, from

$$\left(E_n^{(0)} - H_0\right)\left|n^{(1)}\right\rangle = \sum_{k \notin D} \left(\left\langle k^{(0)} \right| V \left| n^{(0)} \right\rangle\right) \left|k^{(0)}\right\rangle. \tag{1.62}$$

Then,

$$\left|n^{(1)}\right\rangle = \sum_{k \notin D} \frac{\left\langle k^{(0)} \right| V \left| n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}} \left|k^{(0)}\right\rangle, \tag{1.63}$$

and its effect on the degenerate states is of the order of $\mathcal{O}(\lambda)$.

In conjunction with the perturbative approach, an approach often used in quantum mechanics to find suitable approximations for the relevant (usually ground-state) wave functions of complicated quantum systems is the variational method. With the variational method, one would choose a trial wave function that depends on one or a multitude of variational parameters, after which the values of these parameters are modified and searched for such that the expectation value of the corresponding energy of the quantum system becomes minimized. Thereby obtained energy of the system corresponds to the upper bound of the true relevant energy level of the system for the corresponding ground state.

If the Hamiltonian of the system has a discrete spectrum, it holds that

$$\sum_{\lambda_1,\lambda_2 \in \text{spectrum}(H)} \left\langle \psi_{\lambda_1} \middle| \psi_{\lambda_2} \right\rangle = \delta_{\lambda_1,\lambda_2}, \tag{1.64}$$

where λ_i are the eigenvalues of the system and ψ_{λ_i} their corresponding eigenfunctions, with

$$\hat{H} \left| \psi_\lambda \right\rangle = \lambda \left| \psi_\lambda \right\rangle. \tag{1.65}$$

When the lowest energy of the system exists and can be denoted by E_0, the expectation value of the Hamiltonian can be referred to as follows:

$$\langle \psi | H | \psi \rangle = \sum_{\lambda_1,\lambda_2 \in \text{spectrum}(H)} \left\langle \psi \middle| \psi_{\lambda_1} \right\rangle \left\langle \psi_{\lambda_1} \middle| H \middle| \psi_{\lambda_2} \right\rangle \left\langle \psi_{\lambda_2} \middle| \psi \right\rangle = \tag{1.66}$$

$$= \sum_{\lambda \in \text{spectrum}(H)} \lambda | \langle \psi_\lambda | \psi \rangle |^2 \geq \tag{1.67}$$

$$\geq \sum_{\lambda \in \text{spectrum}(H)} E_0 | \langle \psi_\lambda | \psi \rangle |^2 = E_0 \tag{1.68}$$

To normalize this ansatz wave function, it must hold that:

$$\langle \psi(\alpha_i) | \psi(\alpha_i) \rangle = 1, \tag{1.69}$$

and the following function must be minimized:

$$\epsilon(\alpha_i) = \langle \psi(\alpha_i) | H \psi(\alpha_i) \rangle. \tag{1.70}$$

In general, it can then be defined that:

$$\epsilon(\psi) = \frac{\left\langle \psi \middle| \hat{H} \middle| \psi \right\rangle}{\langle \psi | \psi \rangle}. \tag{1.71}$$

According to the described variational principle, ϵ is greater or equal to that of the ground state, E_0 — equal only when ψ is exactly equal to the ground state wave function of the quantum system under investigation.

1.2.5 Topological Order

In terms of quantum phases, topologically ordered phases represent specific *long-range quantum entanglement patterns* in quantum systems — this subsection provides a concise description thereof.

The detailed classical description of phase transitions of matter was presented by Lev Landau's symmetry-breaking theory [21] in the first half of the twentieth century. Within this theory — and the later-developed extension Landau worked on with Vitaly

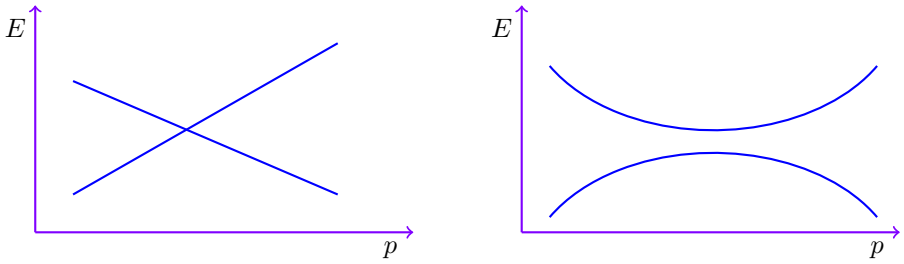


Figure 1.10: A representation of the eigenvalues of the ground state and first excited state of a Hamiltonian $H(p) = H_0 + pH_1$ which depends on a dimensionless coupling constant p, and where H_0 and H_1 commute and are independent of p. The figure on the left shows a level crossing, and the figure on the right shows an avoided level crossing.

Lazarevich Ginzburg (named aptly the Ginzburg–Landau theory) — classical phases of matter can be classified using their physical properties and local order parameters. Phase transitions are then recognized by the change of the physical symmetries in the system. Commonly, the parameters that can be used to describe (and thus predict) these phase transitions depend on temperature, and a transition may occur when the system is in the minimum of its free energy.

However, the later observation of phases that cannot be described by the use of such a parameter allowed for a new perspective. Notably, this concerns the fractional quantum Hall effect [22, 23], in which phases that correspond to certain system filling factors and are associated with different physical properties cannot be described accurately using the classical, Ginzburg–Landau theory of symmetry breaking, which would assign the same classical phase to two distinct phases of matter. The observation of such phases, which are known to emerge at temperatures nearing absolute zero, eventually led to their description through the topological quantum field theory (which was also utilized to describe chiral spin states) and to their dubbing as *topologically ordered phases* [24, 25]. Note that topological order is a phenomenon that is still not sufficiently well understood at temperatures (much) above absolute zero, but its spectral gap above the degenerate energy of the ground state and its inherent robustness to perturbation may prove to be a stable conduit for the development of quantum memory devices or algorithms that incorporate quantum error correction [26]. The non-local observables which remain robust to perturbation within a topologically ordered system are named *topological invariants*.

In general, the term quantum phases is awarded to phases of matter found at the temperature of absolute zero, which correspond to the (degenerate) ground states of topologically ordered systems [8]. For a gapped system (i.e., a system with a gap between the energy of the ground state and that of the first excited state) described by a parametrized Hamiltonian $H(p)$, where p is a dimensionless parameter that couples to a conserved quality, it may hold that $H(p) = H_0 + pH_1$ and $[H_0, H_1] = 0$, which would make it possible to diagonalize H_0 and H_1 simultaneously. Then, the eigenvalues of $H(p)$ would depend on p, but its eigenfunctions would not. Within this system, there can exist a critical value of the parameter, p_c, for which the ground state energy becomes non-analytic, allowing for an excited state to assume the role of the ground state.

The point of non-analyticity in a finite sized lattice presents a level crossing, whereas at the infinite lattice limit such a point may represent either a level crossing or an avoided level crossing (see Figure 1.10, inspired by [8]). Any non-analytic point in the ground state energy of an infinite lattice system is identified as a *quantum phase transition* [8]. In other words, for a quantum phase transition to occur, the spectral gap above the ground state of a quantum spin system must close. Such phase transitions

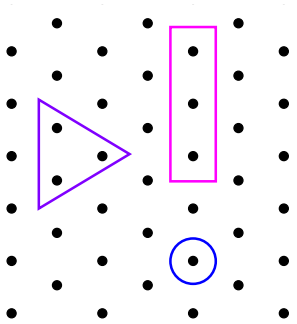


Figure 1.11: Example of a lattice with uniform distance between nearest neighbors, with three examples of regions of different sizes. The purple triangle-shaped lattice belongs to $\mathcal{S}(2)$, the pink rectangle-shaped one to $\mathcal{S}(3)$, and the blue circle-shaped one to $\mathcal{S}(1)$.

may happen naturally at the temperature of absolute zero, where quantum effects that warrant such transitions become predominant. When a system goes through a quantum phase transition, the nature of the correlations in its ground state usually changes.

Topologically ordered phases in quantum spin systems are a phenomenon most commonly described by comparison with the classical Ginzburg–Landau theory of ordered phases in materials. Whereas the parameters of order that govern the symmetry states of materials that can be accurately described by the Ginzburg–Landau theory have been studied in detail and are fairly well understood (such as its polarization, magnetization, crystal lattice deformation, or the wave function of electron pairs in superconductors [27]), the theoretical approach to topologically ordered phases is still not sufficiently conclusive. The concept of topological order is tied to the existence of global variables which are robust to local perturbations of the system's Hamiltonian. This ordering is not yet well understood at temperatures above absolute zero, and it is currently approached from a number of different angles. In this subsection, the description of topological order found in the recent works of Sergey Bravyi, Matthew Hastings, and Spyridon Michalakis [26, 28] will be presented in conjunction with the general notions and implications of topological order.

We observe a system of quantum particles distributed on the sites of an N-dimensional lattice Λ of linear dimension L, with periodic boundary conditions and its Hilbert space represented as stated in (1.19). If $\mathcal{S}(r)$ is defined as the set of all blocks $A \subseteq \Lambda$ with linear size r, where $r > 0$ (see Figure 1.11), it is valid that $\mathcal{S}(L) = \Lambda$, and $\mathcal{S}(r) = \emptyset$ for $r > L$.

Considering a coarse-grained lattice, such that the unperturbed Hamiltonian of the system, H_0, involves only interactions between particles inside of blocks $A \in \mathcal{S}(2)$, the Hamiltonian of the system can be written as

$$H_0 = \sum_{A \in \mathcal{S}(2)} Q_A, \tag{1.72}$$

where Q_A is an interaction with support on A, and with the following properties:

$$Q_A^2 = Q_A, \qquad Q_A Q_B = Q_B Q_A, \qquad \text{for all } A, B \in \mathcal{S}(2). \tag{1.73}$$

On top of that, I require that the ground state of H_0 minimizes the expectation value of each of the Q_A simultaneously, such a Hamiltonian is called *frustration-free*. The commuting property of the Hamiltonian, as stated in (1.73) and the requirement for frustration-free-ness, imposes a strong restriction on it, and even though this holds as a requirement for the following definition, it is possible to describe topological order in more general terms, since the commuting property is neither a requirement, nor a

defining feature for topological order in general. It is just a convenient class of models, for which it is easy and instructive to define the meaning of the term topological order. Defining the Hamiltonian H_0 to have zero ground state energy and a finite spectral gap between the ground state energy and the energy of the first excited state, the projectors onto the ground subspace and the excited subspace of H_0 can be defined as P and Q, respectively, as:

$$P = \prod_{A \in \mathcal{S}(2)} (I - Q_A), \qquad Q = I - P, \tag{1.74}$$

where I is the identity operator. Similarly, the local versions of these operators, for any block $B \in \mathcal{S}(r \geq 2)$ are:

$$P_B = \prod_{\substack{A \in \mathcal{S}(2) \\ A \subseteq B}} (I - Q_A), \qquad Q_B = I - P_B. \tag{1.75}$$

To define the existence of topological order, the following two properties described in detail in [26] (commonly known as *TQO-1* and *TQO-2*) need to hold true, assuming that there exists an integer $L^\star \geq \alpha L$ for some constant $\alpha > 0$ and sufficiently large L:

1. **TQO-1:** For any block $A \in \mathcal{S}(r)$ with $r \leq L^\star$,

$$PO_AP = cP, \qquad c \in \mathbb{C}, \tag{1.76}$$

   for any operator O_A acting on A.

2. **TQO-2:** For blocks $A \in \mathcal{S}(r)$ with $r \leq L^\star$ and $B \in \mathcal{S}(r+2)$, where B is the block that contains A and all nearest neighbors of the sites in A, define reduced density matrices $\rho_A = \mathrm{tr}_{A^c}(P)$ and $\rho_A^{(B)} = \mathrm{tr}_{A^c}(P_B)$, where $A^c = \Lambda \backslash A$. Then

$$\ker \rho_A = \ker \rho_A^{(B)}. \tag{1.77}$$

The integer $L^\star$ is chosen to depend on the size of the lattice, L, in a linear fashion so that it defines a length scale for local operations in the system. Its dependence on L ensures that the stated properties of a system with topological order do not refer only to strictly local subsystems, but describe global properties of the system instead.

The first condition (TQO-1) is commonly thought of as the chief definition of topological order, and it states that it is impossible for a local operator to induce a transition between orthogonal ground states of the system, or to distinguish between two such states. Consequently, it is colloquially known as the condition of local indistinguishability of ground states of a topologically ordered system. From the statement of the condition, it straightforwardly follows that for a system with an orthonormal basis of ground states $\{|\psi_i\rangle\}$, the local operator O_A acts in the following manner:

$$\langle\psi_i| O_A |\psi_j\rangle = \begin{cases} \text{const.} & \text{if } \ i = j; \\ 0 & \text{if } \ i \neq j. \end{cases} \tag{1.78}$$

Therefore, it can be said that any information encoded in the ground state space of a topologically ordered system is not affected by local perturbations.

The second condition defining the existence of topological order (TQO-2) states that the projectors P_B and P must act equivalently on the subset $A \subset B$, certifying that the local ground subspace of the system P_B will be consistent with its global ground subspace P, on subsets which are sufficiently far from the boundary of B. This consistency may be violated in cases where the observed region possesses a non-trivial topology, e.g. a

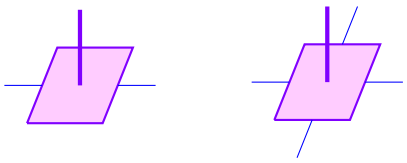


Figure 1.12: Examples of tensor depictions, shown at an angle. a) Tensor with $n = 2$ virtual (shown horizontally) indices and one physical (shown vertically) index. b) Tensor with $n = 4$ virtual (shown horizontally) indices and one physical (shown vertically) index.

hole — then, the local ground subspace may include areas with a non-trivial topological charge inside the hole, whereas this will not be the case for the global ground subspace.

Phase transitions are not exclusive to quantum systems, but also incident to classical statistical mechanical systems. For example, the Ginzburg–Landau theory provides a mechanism for describing phase transitions with local order parameters, and it applies to classical systems just as well as it does to quantum. As a concept, topological order can also be defined for a classical statistical mechanical system [29].

1.2.6 Some Spin System Models and Tensor Network Methods

The AKLT model [30, 31] was originally developed in order to describe a system of SU(2)-symmetric particles in a one-dimensional chain. However, further expansions upon this model have been made, which can now be used to describe particles of any SU(n) group with $n \in \mathbb{N}$. In this section, the principle of implementing a tensor network in such a system will be presented.

To describe the large-scale properties of a lattice system with interacting particles, each individual element in a lattice may be described by a tensor — a mathematical object that holds information on both the properties of the element and the manner in which it interacts with its neighbors. A tensor T is defined as a multidimensional array of values; a $d_1 \times d_2 \times ... \times d_n$-dimensional tensor is an element of $\mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times ... \times \mathbb{C}^{d_n}$. Within the past decade, the quantum information community has adopted the use of tensor networks as the standard method for the description of a multitude of lattice-based systems. If individual elements of a particle lattice and their interactions can be described sufficiently accurately by the simple mathematical construct of a tensor network, its global properties can be obtained in a straightforward (albeit not trivial) fashion, by merely iterating the calculation of a tensor product of neighboring particles.

When set up in the form of a network of tensors, some systems lend themselves to a simple analytical calculation of such global values as the ground state energy. But even if that is not the case, a numerical investigation of their properties can still be conducted.

In lattice analysis, individual tensors are often depicted as blob-like structures with extending tentacles — the tensor T would be shown with n tentacles or *legs*, each depicting a specific tensor index, as seen in Figure 1.12. Each of them is drawn extending toward the neighbor with which the corresponding particle interacts; the touching legs of such two tensors indicate that they may be contracted along the tensor dimensions they represent. Evidently, drawing a complex lattice structure as a multidimensional graph with tensors at its vertices elucidates the problem where its formulaic expression may be lacking. An example of a one-dimensional and a two-dimensional tensor network diagram can be seen in Figure 1.13.

Definition 1.2.13. A *matrix product state* (MPS) is a pure quantum state of a large

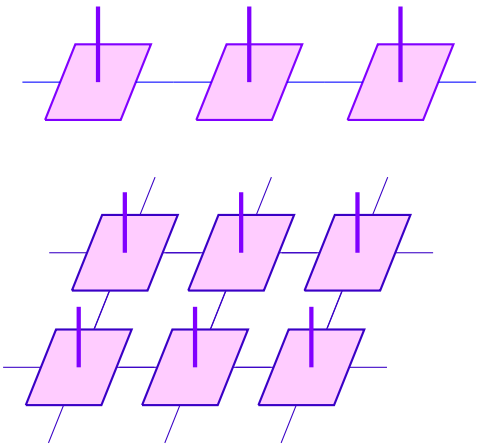


Figure 1.13: Examples of tensor network diagrams — the upper image shows a snippet of a one-dimensional, and the lower one a two-dimensional tensor network.

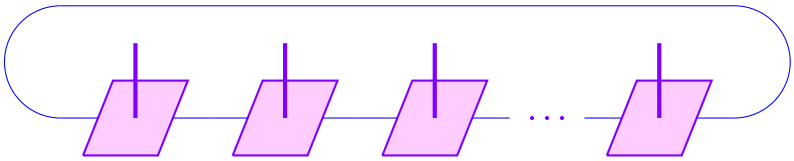


Figure 1.14: Sketch of a matrix product state for a one-dimensional tensor network — the pink squares represent tensors, the lines connecting them their contracted virtual indices, and the thick vertical purple lines depict their physical indices.

number of particles, described by

$$|\psi\rangle = \sum_{\{s\}} \mathrm{Tr}\left[A_1^{(s_1)} A_2^{(s_2)} \dots A_N^{(s_N)}\right] |s_1 s_2 \dots s_N\rangle\,, \tag{1.79}$$

where the indices s_j go through all the states of the computational basis, and $A_j^{(s_j)}$ are complex square matrices with a rank that corresponds to the local virtual dimension of the system called bond dimension, D (very often it is alternatively denoted as χ).

A visual representation of a matrix product state for a finite-sized system with periodic boundary conditions can be seen in Figure 1.14. In addition, an MPS can be defined in a slightly altered way, which is more suitable to describe systems with open boundary conditions. As shown in Figure 1.16, in this adjusted definition for a system with open boundary conditions, the boundary tensors accordingly have one index fewer and lead to $|\psi\rangle = \sum_s A_1^{(s_1)} A_2^{(s_2)} \dots A_n^{(s_n)} |s_1 s_2 \dots s_n\rangle$. This system can be equivalently described by acting on the boundary tensors with appropriate vectors, $\vec{L}$ and $\vec{R}$, which in that way close the loose indices:

$$|\psi\rangle = \sum_s \vec{L}^\dagger A_1^{(s_1)} A_2^{(s_2)} \dots A_n^{(s_n)} \vec{R}\, |s_1 s_2 \dots s_n\rangle\,. \tag{1.80}$$

A *matrix product operator* (MPO) is a *tensor network operator* (TNO) that can be described as an MPS in which each tensor has an additional physical index, as depicted in Figure 1.15. Such an operator can act on an MPS by contracting those added indices with the dangling physical indices of the MPS — creating an altered MPS in the process. This newly created MPS then has a higher total bond dimension, as a result (and product) of the relevant bond dimensions of the MPO and the original state.

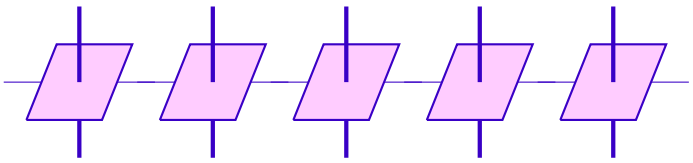


Figure 1.15: Example of a tensor network operator that can be applied to a one-dimensional tensor network state. It is also referred to as a matrix product operator (MPO), as it may act on a so-called matrix product state (MPS).

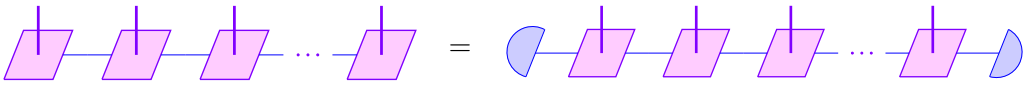


Figure 1.16: MPS with open boundary conditions, where boundary tensors that have one index fewer may be constructed by acting upon the dangling boundary indices with appropriate vectors, shown in blue.

Definition 1.2.14. A *transfer matrix* is a linear operator defined on D^2 that aids to express the coefficients of a wave function defined on one side of a specific index i as those of the wave function defined on the other, as

$$\mathbb{E}_{i\alpha\alpha'\beta\beta'} = \sum_s A^{(s)}_{i\alpha\beta} \overline{A}^{(s)\dagger}_{i\alpha'\beta'}. \tag{1.81}$$

Its depiction in a tensor network can be seen in Figure 1.17. Transfer matrices can then be used to express the norm of the MPS; in the case of an MPS defined with periodic boundary conditions, its norm is the trace of the product of all transfer matrices, $\langle\Psi|\Psi\rangle = \mathrm{tr}\prod_i \mathbb{E}_i$. This relation is obvious from Figure 1.18. The transfer matrix eigenvalues with the largest magnitude can be used to approximate the correlation length of the system, as:

$$\xi = -\frac{1}{\log\left|\frac{\lambda_2}{\lambda_1}\right|}, \tag{1.82}$$

where λ_1 and λ_2 are, in order, the two eigenvalues with the largest magnitude of the transfer matrix [32].

This type of representation can be immensely helpful in the calculation of the correlations of local operators and, subsequently, expectation values of the operators in MPSs. For example, for a local operator Ξ that is located at specific site between two MPSs, as seen in Figure 1.19, the concept of the transfer matrix can be expanded to aid in the calculation of its norm using a simple expression. In this example, the so-called generalized transfer matrix that corresponds to the site which carries the operator Ξ can be expressed as:

$$(\mathbb{E}_\Xi)_{i\alpha\alpha'\beta\beta'} = \sum_{s,s'} A^{(s)}_{i\alpha\beta} \Xi_{s's} \overline{A}^{(s')}_{i\alpha'\beta'}. \tag{1.83}$$

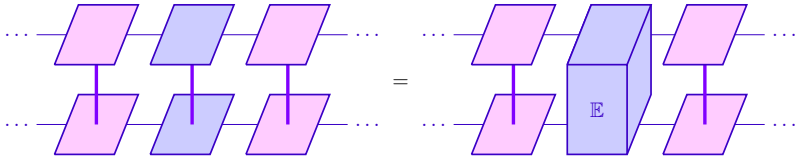


Figure 1.17: Construction of a transfer matrix; the blue tensors in the two MPSs in the depiction on the left are represented by a transfer matrix, shown in blue in the depiction on the right.

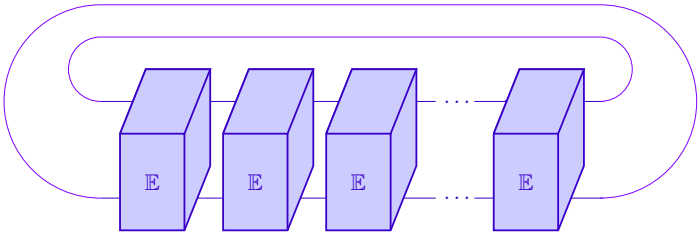


Figure 1.18: Trace of the product of all transfer matrices in an MPS with periodic boundary conditions, shown here, gives the norm of the MPS itself.

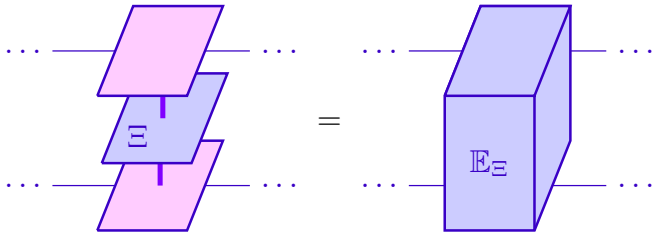


Figure 1.19: A local operator Ξ between two MPSs can be described using a type of transfer operator, $\mathbb{E}_\Xi$.

It is clear that this generalized expression corresponds to (1.81) for $\Xi = \mathbb{1}$.

The definition of an MPS (1.79) can also be examined in the thermodynamic limit, for an infinite chain of particles or subsystems; when restricted to a translationally invariant state (i.e., an MPS comprising a specific tensor A on each site), which restricts the number of parameters required to specify the state, this thermodynamic limit is referred to as *uniform* or *infinite MPS*.

Definition 1.2.15. An infinite MPS (iMPS) is a pure, translationally invariant many-particle state of an infinite system of individual particles, which can be defined for periodic boundary conditions as:

$$|\psi(A)\rangle = \lim_{N\to\infty} \sum_{\{s_n\}=1}^{q} \operatorname{tr}\left(\prod_{n=-N}^{+N} A^{s_n}\right) |\{s_n\}\rangle = \sum_{\{s_n\}=1}^{q} \operatorname{tr}\left(\prod_{n\in\mathbb{Z}} A^{s_n}\right) |\{s_n\}\rangle\,, \tag{1.84}$$

and for open boundary conditions as:

$$|\psi(A)\rangle = \lim_{N\to\infty} \sum_{\{s_n\}}^{q} v_L^\dagger \left(\prod_{n=-N}^{+N} A^{s_n}\right) v_R\, |\{s_n\}\rangle = \sum_{\{s_n\}=1}^{q} v_L^\dagger \left(\prod_{n\in\mathbb{Z}} A^{s_n}\right) v_R\, |\{s_n\}\rangle\,. \tag{1.85}$$

Unless otherwise specified, iMPSs are the states applied for the numerical analyses presented in this work.

For iMPSs to represent physical states, they should be normalized. Since the norm of an iMPS is equivalent to the contraction of an infinite chain of transfer operators, the condition that a state is normalized can be expressed as a condition on the spectrum of the transfer operator. According to its definition, the transfer operator only has positive eigenvalues, so an iMPS is normalized if the largest eigenvalue of the corresponding transfer operator is equal to 1. If an iMPS is not normalized and η is the largest eigenvalue of the not normalized transfer operator, then the iMPS can be normalized by rescaling the tensor entries as

$$\tilde{A}^{s_n} := \frac{1}{\sqrt{\eta}} A^{s_n}. \tag{1.86}$$

Similarly, the transfer operator also allows for the calculation of the expectation values of local operators, i.e., local observables for an iMPS. The local operator gets sandwiched between two instances of the iMPS. As it is a local operator, by definition it has support only on a limited area of physical sites. On the left and the right side of this area, the MPS tensors are directly connected by their physical legs and (on both sides) there are infinitely repeated applications of the transfer operator. This means that those can be replaced with the left and right eigenvectors, corresponding to the largest eigenvalue of the transfer operator, which is, as stated earlier, the fixed point of the transfer operator. After the replacement, calculating the expectation value means contracting a tensor network consisting of a finite number of operators, which can be done efficiently and is easily addressed numerically (as long as the area of the support of the local operator is small enough).

Because it has been shown that the dynamics of one-dimensional quantum systems can be computationally approximated in a way that scales polynomially with the size of the system [33], this computational tensor network approach is often referred to as efficient, as many values relevant for quantum systems scale exponentially with their size. For example, MPS-based variational ground state search algorithms for one-dimensional systems can be performed with computational costs scaling as D^3 (or scaling as D^5 for systems with periodic boundary conditions) [34].

Beyond one-dimensional systems, MPSs can be thought of as a special case of the *projected entangled pair states* (PEPS) model, in which entangled pair states, $|\varphi\rangle$, are set up on a lattice, and a linear map $\mathcal{P}$ is applied to adjacent halves of neighboring pairs:

$$\mathcal{P} = \sum_{i;\alpha,\beta} A_{i;\alpha,\beta} \left|i\right\rangle \left\langle\alpha\beta\right| , \tag{1.87}$$

where $A_{i;\alpha,\beta}$ corresponds to the MPS tensor. It holds that:

$$\mathcal{P}^{(1)} \otimes \mathcal{P}^{(2)} \left|\varphi\right\rangle_{2,3} = \sum_{i_1,i_2;\alpha_1,\beta_1,\alpha_2,\beta_2,j} A^{(1)}_{i_1;\alpha_1,\beta_1} A^{(2)}_{i_2;\alpha_2,\beta_2} \left|i_1 i_2\right\rangle \left\langle\alpha_1\beta_1\alpha_2\beta_2\right| (\mathbb{1} \otimes \left|jj\right\rangle \otimes \mathbb{1}) = \tag{1.88}$$

$$= \sum_{i_1,i_2;\alpha_1,\beta_1,\beta_2} A^{(1)}_{i_1;\alpha_1,\beta_1} A^{(2)}_{i_2;\beta_1,\beta_2} \left|i_1, i_2\right\rangle \left\langle\alpha_1\beta_2\right| . \tag{1.89}$$

For example, if $A_0 = 1$ and $A_1 = 0$, the resultant state is $|00\ldots0\rangle$, and if $A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, with boundary vectors $\vec{L} = (1, 0)$ and $\vec{R} = (0, 1)$, the resultant state is the W-state, $|W\rangle = \sum_{j=1}^{N} \left|00\ldots01_j0\ldots0\right\rangle$, because $A_0A_0 = A_0$, $A_0A_1 = A_1$, and $A_1^2 = 0$.

Due to their versatility and buildability, tensor networks have enjoyed tremendous success in the numerical simulation approximations of problems in low-temperature condensed matter systems. Arguably, the most famous of the numerical techniques used is the *density matrix renormalization group*, or DMRG. This variational technique is commonly used in quantum many-body physics to efficiently discover the MPS wave function of a quantum system that corresponds to its ground state.

The Hilbert space of a quantum system grows exponentially with its size, meaning that a Majumdar–Ghosh chain of N spin-½ particles comes up at a vast 2^N degrees of freedom. By using DMRG, the effective degrees of freedom can be reduced to a point where only those relevant to the state in question are kept track of, while the technique's accuracy is affected only minimally.

Singular Value Decomposition

An important method that begs elaboration in the field of tensor network simulations is singular value decomposition.

Definition 1.2.16. Singular values of $M : A \to B$, where A and B are Hilbert spaces, are square roots of the non-negative self-adjoint operator $M^\dagger M : A \to B$.

Definition 1.2.17. Let M be an $\alpha \times \beta$ matrix with entries $m_{ab} \in \mathbb{C}$. Then, there exists a factorization of M — referred to as the singular value decomposition (SVD) — such that $M = U\Sigma V^\dagger$, where U is an $m \times m$ unitary matrix over $\mathbb{C}$, Σ is a diagonal $m \times n$ matrix with non-negative real numbers on its diagonal, and V is an $n \times n$ unitary matrix over $\mathbb{C}$.

The diagonal values in Σ are then the singular values of M, which are traditionally set in the matrix Σ in descending order. When this convention is respected, Σ is unique for the starting matrix M.

The rank of a matrix is equal to the number of non-zero singular values it has, which can be easily extracted from its SVD decomposition. In some numerical uses of tensor networks, a matrix of a high rank can be efficiently approximated by one of a low rank, which is where the SVD method graciously steps in. A matrix M with rank r is approximated by the matrix $\widetilde{M}$ with rank $\tilde{r}$, such that the norm of the difference between the two matrices is smaller than some cutoff value. Then,

$$\widetilde{M} = U\widetilde{\Sigma}V^\dagger, \tag{1.90}$$

where the $\widetilde{\Sigma}$ matrix contains the same non-zero values as the matrix Σ, up to and including the r-th value — the rest are equal to zero.

Numerically, the SVD of a matrix M can be obtained by first reducing it to a bidiagonal matrix, after which the SVD can be calculated iteratively up to some precision. Alternatively — and most often in tensor network calculations — the SVD is obtained using the eigenvalue decomposition. First, the problem is rephrased by converting the matrix M to a matrix W, which equals $MM^\dagger$, $M^\dagger M$, or $\begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}$. Then, this problem can be address efficiently by the iterative application of the RQ eigenvalue algorithm [35] to those matrices[2]. At step number k of the RQ algorithm, a decomposition procedure is performed to obtain $W_k = Q_k R_k$, where R_k is an upper triangular matrix and Q_k is an orthogonal one. This procedure boils down to the application of the Gram–Schmidt process to the issue matrix. Then,

$$W_{k+1} = R_k Q_k = Q_k^{-1} Q_k R_k Q_k = Q_k^{-1} W_k Q_k = Q_k^T W_k Q_k. \tag{1.91}$$

Therefore, all W_k have the same eigenvalues. The matrix R can be written out using the LQ decomposition, which bears similarity to the RQ algorithm, except that a lower triangular matrix L is used instead of an upper triangular matrix R [36]. Then, $R_{l+1} = Q_l L_l$, which means that $W_{k+1} = Q'_k L'_k Q_k$. In the following step, the matrix W can be updated using the value of L, and the cyclical procedure may continue.

More details on this topic can be found in various introductory and review articles, e.g., [37].

These decompositions (SVD and QR decomposition) are crucial components in many tensor network algorithms, often used to simplify the mathematical forms of tensor

[2]Note that the RQ algorithm is performed in the same manner as the QR algorithm, but the order of the matrices R and Q is reversed in the solution. This change keeps their shape, but leads to alterations in their content.

network states, bringing them to a particular canonical form. For example, for an MPS, as long as it is in its canonical form, certain quantities (like the entanglement spectrum) can be directly read off the state, while others (like the correlation length) can be calculated easily. It also allows for a straightforward truncation of the state, reducing its bond dimension while keeping the most significant contributions by throwing away all but the $D_{\text{new}} < D$ largest values of the entanglement spectrum. This truncation is also essential for many algorithms involving MPS. How much a state can be truncated without steering away from the original state to an unacceptable degree depends on the original state, or, more precisely, on its entanglement spectrum.

For more details on these canonical forms and algorithms used to bring states into their canonical forms, see [38].

An Example of a Tensor Network Algorithm

What is important in any numerical simulation is the cutoff point for an iterative algorithm. A typical tensor network algorithm — such as the one used here — may be based on the following procedure: This is an algorithm which, for a given MPO, approximates the eigenstate corresponding to the dominant eigenvalue of the MPO, i.e., it also finds a fixed point of the MPO. This will work as long as the random state with which the algorithm is initialized has any non-zero overlap with the eigenstate for the dominant eigenvalue. Otherwise, it will still find a fixed point, which will be the eigenvalue corresponding to the eigenstate with the eigenvalue of largest magnitude with which the initial random state has any overlap.

A common protocol in tensor network algorithms may proceed as follows.

1. Pick a random MPS tensor of the correct physical dimension and of a desired bond dimension D.
2. Construct a new MPS by applying the MPO to the original MPS, contracting the MPS tensor with the MPO tensor.
3. Bring the new MPS to the canonical form. This is a unitary transformation on the virtual degrees of freedom represented by the operators U and $U^\dagger$ in the figure.
4. Truncate the MPS to the originally desired bond dimension D by discarding all but the D largest singular values of the SVD in the canonical form.
5. Calculate the overlap (defined as $|\langle\psi_a|\psi_b\rangle|$ for the two states) between the truncated MPS and the MPS before applying the MPO.
6. If the overlap is larger than $1-\epsilon$, where ϵ is some desired cutoff value, stop the algorithm here; otherwise go back to step 2.
7. The MPS now describes the resulting state.

For an illustration, see Figure 1.20.

Some Spin System Models

In the context of spin system models, the most well-known example would likely be the Heisenberg model, which gives a simple quantum-mechanical model of the phenomenon of ferromagnetism, criticality and phase transitions. The Ising model can serve as a simpler and related model to introduce it. The Hamiltonian of a one-dimensional system of N particles that can be described using the Heisenberg model can be written as

$$\hat{H} = -J\sum_{j=1}^{N}\sigma_j\sigma_{j+1} - h\sum_{j=1}^{N}\sigma_j, \tag{1.92}$$

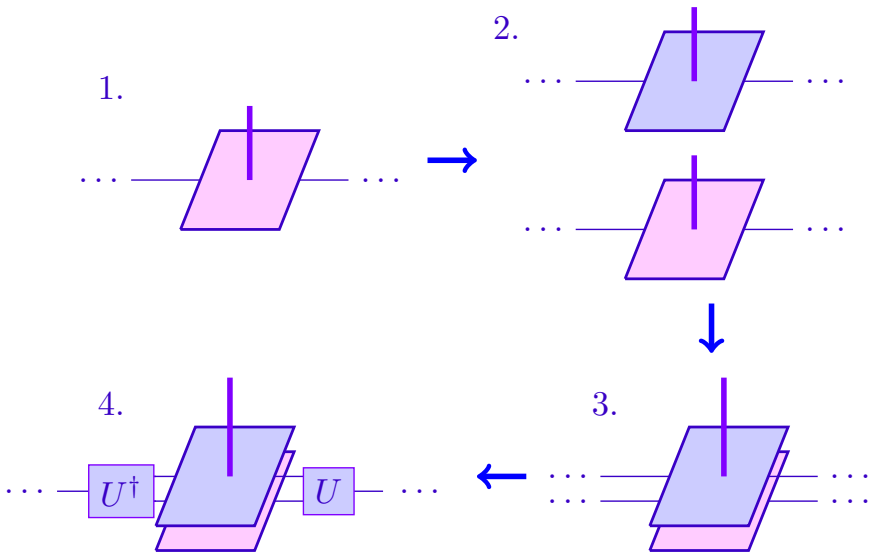


Figure 1.20: Illustration of the initial portion of the described tensor network algorithm.

where each particle is denoted by the index j (with periodic boundary conditions, $\sigma_{N+1} = \sigma_1$), h denotes the external magnetic field acting on the system, and J is its coupling constant. Here, each particle in such a lattice structure is thought to have a binary spin, $\sigma = \pm 1$, acting as a magnetic dipole for which $\sigma = +1$ and $\sigma = -1$ correspond to up and down spins, respectively, and the lowest energy is achieved in a system when nearest neighbor particles in the lattice possess the same spin. The Heisenberg model expands on this setup, allowing all spins to take three-dimensional values corresponding to the Pauli spin-½ matrices, σ^x, σ^y, and σ^z (described in detail in [39]), and assigning three values to the coupling constant, J_x, J_y, J_z. Then, the system can be described using a Hamiltonian of this form:

$$\hat{H} = -\frac{1}{2}\sum_{j=1}^{N}\left(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z + h\sigma_j^z\right). \tag{1.93}$$

The states of the system are described by the tensor product $\left(\mathbb{C}^2\right)^{\otimes N}$, with the expected dimension of 2^N.

The Heisenberg model has subsequently been expanded in a variety of other models, each adapted to a system of a particular nuance. A highlight of these is the Majumdar–Ghosh model, which doesn't consider only the nearest neighbor interaction of the particles in the lattice, but extends the coupling to second-nearest-neighbor particles, too. It is defined by the Hamiltonian

$$\hat{H} = J\sum_{j=1}^{N}\vec{S}_j\cdot\vec{S}_{j+1} + \frac{J}{2}\sum_{j=1}^{N}\vec{S}_j\cdot\vec{S}_{j+2}, \tag{1.94}$$

where the full quantum spin vector of a particle has been gathered as the operator $\vec{S}_j$, with the quantum number $S = \frac{1}{2}$. The Majumdar–Ghosh model [40, 41] has been shown to exhibit two different ground states, both of which comprise only pairs of spins that form singlets — one is shifted by one spin with respect to the other, as can be seen in Figure 1.21. This gapped model is the basis of the more celebrated AKLT model, a quantum spin model named for its authors, Affleck, Kennedy, Lieb, and Tasaki.

The AKLT model describes a one-dimensional lattice with spin-1 particles on its vertices, in which a valence bond would be formed between each pair of neighboring

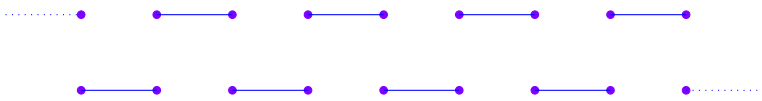


Figure 1.21: Majumdar–Ghosh model. The dots represent particles with spin in the one-dimensional chain, and the lines denote their full entanglement. The two variants of entanglement in the chain are created by merely shifting the entanglement pattern by one spin to the side.

Figure 1.22: AKLT model. The dots represent particles with spin in the one-dimensional chain, and the lines denote their entanglement in the singlet state — their valence bonds. The circled spins show the projection operators that form triplet, spin-1 objects.

particles, as is shown in Figure 1.22. Originally, the idea that led to the construction of the AKLT model [30, 31] was a simple expansion of the Majumdar–Ghosh model — i.e., the question of what would happen if more than two spin-½ particles were bound in the chain. As each of the singlets formed in the original model has a spin of 1, it was unclear how these neighboring, effective-spin-1 particles could form entangled structures of a higher spin. Because of the specific nature of these groups, the group representations of the these symmetries are not trivially arranged. In order to preserve the representation properties of the particles — i.e., the physical indices of the tensors — the representations ascribed to their virtual indices must be determined.

Then, for a one-dimensional chain of spin-½ particles, the symmetries imposed on the virtual indices of the tensors ascribed to these particles cannot be a spin-½ representation, because in that case the spin-½ properties of the particles couldn't be realized. This is because $\frac{1}{2} \notin \frac{1}{2} \otimes \frac{1}{2}$. A translationally-invariant solution would be to impose the vector sum of the spin-0 and the spin-1 representations as the symmetry of the virtual indices, seeing as $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. For a general case of a one-dimensional chain of particles with an SU(n) symmetry of a particular representation, it is possible to determine the minimal number of particles in an entangled set — the minimal length. For example, for the fundamental representation of the SU(2) group — the 2-dimensional spin-½ representation — the minimal length is two, which informs the representational symmetries on the virtual indices of the tensors assigned to the particles.

As each two adjacent spin-1 virtual particles, two of the relevant spin-½ particles assume the total spin-0 state, preventing the two spin-1 particles from forming a larger spin-2 structure. The Hamiltonian that describes the AKLT model setup is the following:

$$\hat{H} = \sum_{j=1}^{N} \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} \left(\vec{S}_j \cdot \vec{S}_{j+1} \right)^2 . \tag{1.95}$$

The simplicity of this model lies in the fact that its ground state can be represented exactly, in a clear form, as a matrix product state [42, 43, 44].

The AKLT model can be described in the MPS formalism by assigning each particle a tensor with one physical and two virtual indices, with the symmetries imposed on the virtual indices of the tensors corresponding to the symmetry group of the particles. Define $|\varphi\rangle = (|01\rangle - |10\rangle)$ as the singlet state and let $\mathcal{P} : \mathbb{C}^{2\times 2} \to \mathbb{C}$ be defined as:

$$\mathcal{P} = |+\rangle \langle 00| + |0\rangle \frac{\langle 01| + \langle 10|}{\sqrt{2}} + |-\rangle \langle 11| \,, . \tag{1.96}$$

where kets $|+\rangle$, $|0\rangle$, and $|-\rangle$ denote the standard spin-1 basis states, such that $|s = 1, s_z = 1\rangle$, $|s = 1, s_z = 0\rangle$, and $|s = 1, s_z = -1\rangle$, respectively, with s_z showing the eigenstates of

the S_z operator. If (S_x, S_y, S_z) is the spin vector of the resultant spin-1 particle and (X_i, Y_i, Z_i) the spin vector of particle i, it follows:

$$S_Z\mathcal{P} = (|11\rangle\langle 11| - |1-1\rangle\langle 1-1|)\left(|11\rangle\langle 00| + |10\rangle\frac{\langle 01| + \langle 10|}{\sqrt{2}} + |1-1\rangle\langle 11|\right) = \tag{1.97}$$

$$= |11\rangle\langle 00| - |1-1\rangle\langle 11| = \tag{1.98}$$

$$= \mathcal{P}\frac{Z_1 + Z_2}{2} \tag{1.99}$$

$$S_X\mathcal{P} = \frac{|10\rangle(\langle 11| + \langle 1-1|) + (|11\rangle + |1-1\rangle)\langle 10|}{\sqrt{2}}$$

$$\left(|11\rangle\langle 00| + |10\rangle\frac{\langle 01| + \langle 10|}{\sqrt{2}} + |1-1\rangle\langle 11|\right) = \tag{1.100}$$

$$= \frac{|10\rangle(\langle 00| + \langle 11|)}{\sqrt{2}} + \frac{(|11\rangle + |1-1\rangle)(\langle 01| + \langle 10|)}{\sqrt{2}} = \tag{1.101}$$

$$= \mathcal{P}\frac{X_1 + X_2}{2} \tag{1.102}$$

$$S_Y\mathcal{P} = \mathcal{P}\frac{Y_1 + Y_2}{2}. \tag{1.103}$$

The motivator for the AKLT research paper [30, 31] was Haldane's prediction [45, 46] that, in the case of an integer-spin model, one-dimensional isotropic antiferromagnets would have an energy gap above the ground state and exponentially decaying correlations, a behavior that was experimentally observed for some systems at that time.

In recent years, PEPS have been established as a tool to study spin liquid wave functions[47]. In particular, they might be used to provide a unified description of the RVB model and the quantum dimer wave function, allowing for an interpolation to be constructed between these two models, potentially identifying the topological phase. PEPS transfer operator techniques also allow for the absence of a conventional order to be certified by the spin degrees of freedom in the system.

The RVB model [48] was first proposed by P. W. Anderson and G. Baskaran in 1987 to serve as a theoretical model for the emergence of high-temperature superconductivity. According to their theory, electrons in copper oxide lattices interact and form valence bonds with one another, but when the lattice is doped, the constituent electrons can form mobile Cooper pairs that support superconductivity. These systems can be described by the Hamiltonian of the repulsive Hubbard model:

$$H = -t\sum_{i,j} c^\dagger_{i\sigma}c_{j\sigma} + \text{h.c.} + U\sum_i n_{i\uparrow}n_{i\downarrow}, \tag{1.104}$$

where h.c. stands for the Hermitian conjugate of the previously listed summand. Such a Hamiltonian can have a non-degenerate ground state when its spins are disordered. The RVB state can, very simply, be denoted as

$$|\text{RVB}\rangle = \sum_C |C\rangle, \tag{1.105}$$

where C is a full lattice covering of the lattice that consists of nearest-neighbor dimers.

The system of two spin-½ particles, which each have two basis states, ↑ and ↓, has four basis states: ↑↑, ↑↓, ↓↑, and ↓↓. In other words,

$$|s, m\rangle = |s_1, m_1\rangle|s_2, m_2\rangle = |s_1, m_1\rangle \otimes |s_2, m_2\rangle, \tag{1.106}$$

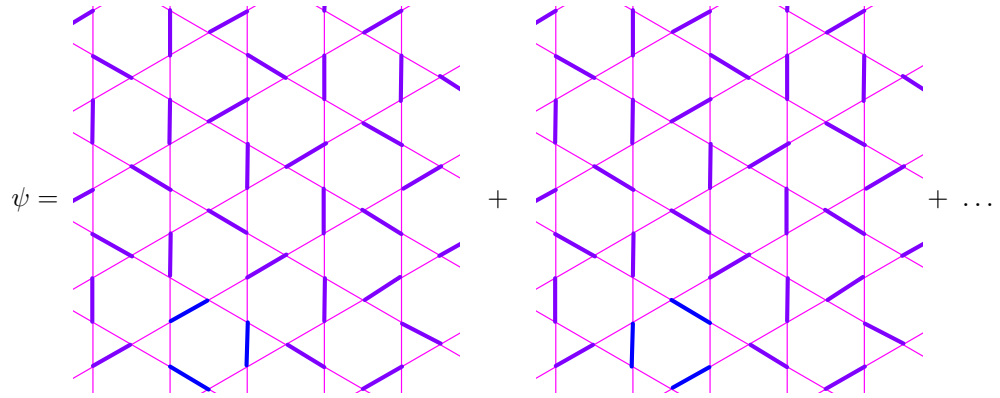


Figure 1.23: Depiction of a resonating valence bond state, where each thick lines show entangled pairs, $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The blue lines show the entanglement changes between the two shown summands.

where the spins of the particles and their projections onto the z-axis are denoted by s_i and m_i, respectively. This expression can be simplified by the use of Clebsch–Gordan coefficients, C:

$$|s, m\rangle = \sum_{m_1+m_2} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 m_1\rangle |s_2 m_2\rangle . \tag{1.107}$$

In the case of spin-½ particles, $s_1 = s_2 = \frac{1}{2}$, and three of the four resultant states have a total spin angular momentum of 1, forming a triplet, whereas the remaining state has a spin angular momentum of 0. The triplet states are symmetric:

$$|1, 1\rangle = \uparrow\uparrow, \tag{1.108}$$

$$|1, 0\rangle = (\uparrow\downarrow + \downarrow\uparrow) \cdot \frac{1}{\sqrt{2}}, \tag{1.109}$$

$$|1, -1\rangle = \downarrow\downarrow, \tag{1.110}$$

whereas the singlet state is antisymmetric:

$$|0, 0\rangle = (\uparrow\downarrow - \downarrow\uparrow) \cdot \frac{1}{\sqrt{2}}. \tag{1.111}$$

The complete result of the calculation is a superposition of these states.

If the valence bond coverings of the RVB model do not produce a perfect set of dimers, or if some of them break apart, free spins may appear in the lattice. Spin liquids also support a form of exotic excitation named a spinon, carrying no charge but a nonzero spin at temperatures close to absolute zero. These excitations can be fractional in nature. The attempts at experimental realization of spin liquids are currently at the cutting edge of solid state physics — the most famous example of a material that behaves as a spin liquid is herbertsmithite [49], a mineral with a Kagome lattice, first discovered by the mineralogist Herbert Smith. Competing experiments mark herbertsmithite as being a gapless $U(1)$ Dirac spin liquid and a $\mathbb{Z}_2$ spin liquid, which may make it an interesting material for research.

Quantum spin liquids are a promising avenue for the creation of topological quantum computers and quantum memories. By using quasiparticles with fractional excitations named anyons and braiding their world lines, it may be possible to form quantum logical gates resistant to decoherence, as the topological nature of the system preserves some of the mathematical operations performed by the braiding. Anyons are quasiparticles found

only in two-dimensional systems, named for their exchange statistics. Unlike bosons and fermions, whose quantum state functions upon an exchange of indistinguishable particles adopt a + or a − sign, respectively, when two indistinguishable anyons are exchanged, the wave function of the system can gain a phase other than π or 2π — it could be of any real angle. I.e.,

$$|\psi_1\psi_2\rangle = e^{i\theta}\,|\psi_2\psi_1\rangle\,, \tag{1.112}$$

where $\theta \in \mathbb{R}$.

Chapter 2

A Perturbative Solution to a Many-Body Localization Hamiltonian

Contents

One particularly interesting kind of order for a quantum many-body system is that of many-body localization (MBL). Systems which exhibit MBL have some form of disorder in their Hamiltonian. Typically, this makes it much harder to study their properties than, for example, for a translationally invariant system, where one might be able to exploit that symmetry to characterize its solutions. While methods to study such systems certainly do exist, it would nonetheless be desirable to have more methods available in the toolbox. That is why I devote this chapter to finding new methods of studying such disordered systems analytically.

The contents of this chapter are divided into two parts. Starting in **Section 2.1: Analytical Solutions for Local Observables in Perturbed Many-Body Systems**, I will first provide an introduction to standard perturbation theory as it is used in quantum physics, and follow it with my derivations of general analytical solutions for time-evolved local observables in a disordered quantum many-body system. The second part of the chapter is **Section 2.2: Stochastic Derivation of Disorder Averages,** which provides an in-depth derivation of a stochastic solution for the expectation values of observables in disordered quantum many-body systems, as well as examples of application to a number of different systems that fall under this moniker. In further subsections, the derived general solutions will be applied to more challenging setups, where they will be conduit to more interesting conclusions about the effects of specific sets of attributes a quantum many-body system may have.

Parts of the work presented in this chapter have been published as part of [1].

This chapter contains a first result in the application of a perturbative approach to a many-body quantum system [50]. The system is a one-dimensional chain of particles, with periodic boundary conditions. It is described by a Hamiltonian with both an interaction (H_0) and a disorder (H_1) part:

$$H = H_0 + H_1 \equiv \sum_j \sigma_j^z \sigma_{j+1}^z + \left(\sum_j h_j \sigma_j^x + J_j \sigma_j^x \sigma_{j+1}^x \right), \tag{2.1}$$

where $h_j, J_j \ll 1$ are random variables from two different Gaussian distributions, with a mean of 0. These variables vary the strength of the disorder acting on the system, and are defined as small enough to warrant the use of a perturbative approach. Using the method presented in [51], a lower bound can be set on the ground state energy of such a system.

The first step of this approach is to obtain the exact expression for the expectation value for an operator in such a system:

$$\overline{M(t)} = \mathbb{E}_{h,J} \left(e^{-it(H_0 + \varepsilon H_1(h,J))} M e^{it(H_0 + \varepsilon H_1(h,J))} \right). \tag{2.2}$$

This can be done by applying the Dyson series to the expression in (2.2). The result of this application is the perturbative expansion in the variables representing the disorder, h_j and J_j. A more general result can be reached by use of the Choi–Jamiołkowski isomorphism.

This analytical approach to a many-body Hamiltonian provides a prime opportunity to examine the locality properties of the summands in the perturbative expansion. It is possible to reach conclusions on the topic of the boundaries of the expectation values for the locality of the different summands. Therefore, the next step in this approach is to tackle the problem of the exact calculation of the expectation value for a strictly

local operator — say, σ_j. Following this, it would prove to be useful to compare the obtained general results to the well-known derivations for many-body Hamiltonians for which analytic solutions exist in the field of condensed matter physics.

Ultimately, the goal of this approach is to be able to express and calculate the values of variables that describe a system that may exhibit the effects of many-body localization (MBL) [52, 53, 50, 54, 55] in a concise way. Then, these calculations may be compared to the numerical or experimentally measured values.

2.1 Analytical Solutions for Local Observables in Perturbed Many-Body Systems

In this section I will present a method that builds upon the well-known mathematical framework of analytical perturbation theory and enhances it, leading to a general perturbative solution to the time evolution of a local operator in a many-body system. To accomplish this goal, I will utilize a diagrammatic framework not completely unlike that used for tensor networks, which allows for a more intuitive understanding of my derivations and the obtained solutions. To achieve these goals, I will conduct perturbative calculations on many-body Hamiltonians that are acted on by a random Gaussian disorder, using the Dyson series and the Choi–Jamiołkowski isomorphism. The completed analytical calculations include the expectation values for time-evolved local operators for both short ($t \approx 0$) and long ($t \to \infty$) temporal evolutions, but also some of the crucial partition functions.

2.1.1 Perturbative Calculations

The calculations I will present in this subsection will be performed on a system that can be described using the Hamiltonian of a general form of $H = H_0 + \lambda V$, where H_0 is some basic Hamiltonian, V represents a perturbative potential acting on the system, whereas $0 < \lambda \ll 1$, such that $\|\lambda V\|/\|H_0\| \ll 1$. Results obtained for more specific Hamiltonian forms will be used to better illustrate the results of the applied methods, but these can be extended in a straightforward manner.

In a system in which part of the Hamiltonian corresponds to a minute perturbation, λV, I can construct an accurate wave function to describe it as:

$$|\psi\rangle = \sum_{\kappa=0}^{\infty} \lambda^{\kappa} |\psi_{\kappa}\rangle , \tag{2.3}$$

with the corresponding ground state energy being equal to:

$$E = \sum_{\tau=0}^{\infty} \lambda^{\tau} E_{\tau}. \tag{2.4}$$

In the zeroth order, the Schrödinger equation gives $(H_0 - E_0)|\psi_0\rangle = 0$, and in the first order the corresponding equation equals:

$$(H_0 - E_0)|\psi_1\rangle = (E_1 - V)|\psi_0\rangle , \tag{2.5}$$

with the solution of:

$$E_1 = \langle\psi_0 | V | \psi_0\rangle ; \tag{2.6}$$

$$|\psi_1\rangle = -\frac{\mathbb{1} - P_0}{H_0 - E_0} V |\psi_0\rangle , \tag{2.7}$$

where E_k and $|\psi_k\rangle$ respectively correspond to the energy and wave function of the system when it is regarded in the k-th order of perturbation. The second order calculation leads to:

$$(H_0 - E_0)\,|\psi_2\rangle = (E_1 - V)\,|\psi_1\rangle + E_2\,|\psi_0\rangle\,, \tag{2.8}$$

with

$$E_2 = \langle\psi_0\,|\,V\,|\,\psi_1\rangle\,. \tag{2.9}$$

The second-order wave function can then be calculated as:

$$|\psi_2\rangle = -\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2 VP_0V\,|\psi_0\rangle + \frac{\mathbb{1}-P_0}{H_0-E_0}V\frac{\mathbb{1}-P_0}{H_0-E_0}V\,|\psi_0\rangle\,. \tag{2.10}$$

Normalizing this expression leads to:

$$|\psi_2\rangle = -\left[\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2 VP_0V + \left(\frac{\mathbb{1}-P_0}{H_0-E_0}V\right)^2 - \frac{1}{2}P_0V\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2 V\right]|\psi_0\rangle\,. \tag{2.11}$$

To adjust these equations to the problem at hand, I will introduce several mathematical concepts and methods throughout the remainder of this subsection.

Dyson Series Expansion

To perform a Dyson series expansion, which can be used to perturbatively expand a time-evolution operator [56], I take an operator

$$W(s) \equiv e^{-sA}e^{s(A+\varepsilon B)}, \tag{2.12}$$

where A and B are local operators in this system, $s \in \mathbb{C}$, and $\varepsilon \ll 1$ is a small parameter, and define

$$B(s) \equiv e^{-sA}Be^{sA}. \tag{2.13}$$

Then, it holds that

$$\frac{\mathrm{d}W(s)}{\mathrm{d}s} = -Ae^{-sA}e^{s(A+\varepsilon B)} + e^{-sA}(A+\varepsilon B)e^{s(A+\varepsilon B)} = \{[A.e^{-sA}] = 0\} = \tag{2.14}$$

$$= \varepsilon e^{-sA}Be^{s(A+\varepsilon B)} = \varepsilon e^{-sA}Be^{sA}e^{-sA}e^{s(A+\varepsilon B)} = \{B(s) \equiv e^{-sA}Be^{sA}\} = \tag{2.15}$$

$$= \varepsilon B(s)W(s). \tag{2.16}$$

It follows that

$$\int_0^p \frac{\mathrm{d}W(s)}{\mathrm{d}s}\mathrm{d}s = \varepsilon\int_0^p B(s)W(s)\mathrm{d}s = W(p) - W(0) \tag{2.17}$$

$$W_0(p) = W(0) + \varepsilon\int_0^p B(s)W(s)\mathrm{d}s, \tag{2.18}$$

and by performing the integration of $\int_0^p \frac{\mathrm{d}W(s)}{\mathrm{d}s}$, I can obtain $W(s)$.

$$\int_0^p \frac{\mathrm{d}W(s)}{\mathrm{d}s}\mathrm{d}s = \varepsilon\int_0^p B(s)W(s)\mathrm{d}s = \tag{2.19}$$

$$= W(p) - W(0) \tag{2.20}$$

This gives the solution of:

$$W_0(p) = W(0) + \varepsilon \int_0^p B(s)W(s)\mathrm{d}s \tag{2.21}$$

in the zeroth order. By inputting the obtained expression into the integral within it, I can obtain further orders of iteration for this solution. As an example, the first iteration gives:

$$W_1(p) = W(0) + \varepsilon \int_0^p B(s) \left(W(0) + \varepsilon \int_0^s B(s_1)W(s_1)\mathrm{d}s_1 \right) \mathrm{d}s = \tag{2.22}$$

$$= W(0) + \varepsilon \int_0^p B(s)W(0)\mathrm{d}s + \varepsilon^2 \int_0^p B(s) \int_0^s B(s_1)W(s_1)\mathrm{d}s_1\mathrm{d}s, \tag{2.23}$$

whereas the second one gives:

$$W_2(p) = W(0) + \varepsilon \int_0^p B(s)W(0)\mathrm{d}s + \varepsilon^2 \int_0^p B(s) \int_0^s B(s_1)$$
$$\left(W(0) + \varepsilon \int_0^{s_1} B(s_2)W(0)\mathrm{d}s_2 + \varepsilon^2 \int_0^{s_1} B(s_2) \int_0^{s_2} B(s_3)W(s_3)\mathrm{d}s_3\mathrm{d}s_2 \right) \mathrm{d}s_1\mathrm{d}s = \tag{2.24}$$

$$= W(0) + \varepsilon \int_0^p B(s)W(0)\mathrm{d}s + \varepsilon^2 \int_0^p B(s) \int_0^s B(s_1)W(0)\mathrm{d}s_1\mathrm{d}s +$$
$$+ \varepsilon^3 \int_0^p B(s) \int_0^s B(s_1) \int_0^{s_1} B(s_2)W(0)\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s +$$
$$+ \varepsilon^4 \int_0^p B(s) \int_0^s B(s_1) \int_0^{s_1} B(s_2) \int_0^{s_2} B(s_3)W(s_3)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s. \tag{2.25}$$

At the n-th iteration of inputting the definition of $W(p)$ into (2.21), a solution emerges that contains factors up to the 2^n-th order in ε:

$$W_n(p) = W(0) + \sum_{k=0}^{2^n-2} \varepsilon^{k+1} \int_0^p \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_k} \mathrm{d}s_{k+1} B(s_1) \ldots B(s_{k+1})W(0)$$
$$+ \varepsilon^{2^n} \int_0^p \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_{2^n-1}} \mathrm{d}s_{2^n} B(s_1) \ldots B(s_{2^n})W(s_{2^n}). \tag{2.26}$$

Notably, when $n \to \infty$, the final summand in the expression above can be discarded. Finally, to obtain the expression for $e^{p(A+\varepsilon B)}$, the following is valid:

$$e^{p(A+\varepsilon B)} = e^{pA}W(p). \tag{2.27}$$

Let the examined system be defined by a Hamiltonian of the type

$$H = H_0 + H_\nu(h, J), \tag{2.28}$$

where $H_\nu(h, J)$ is a small perturbation that depends on parameters $-1 \gg h, J \ll 1$, which take random values in a Gaussian distribution of the form $N(0, \eta^2_{h,J})$, as shown in Figure 2.1 and defined by:

$$N(x, \eta^2) = \frac{e^{-\frac{x^2}{2\eta^2}}}{\eta\sqrt{2\pi}}. \tag{2.29}$$

Specifically, let

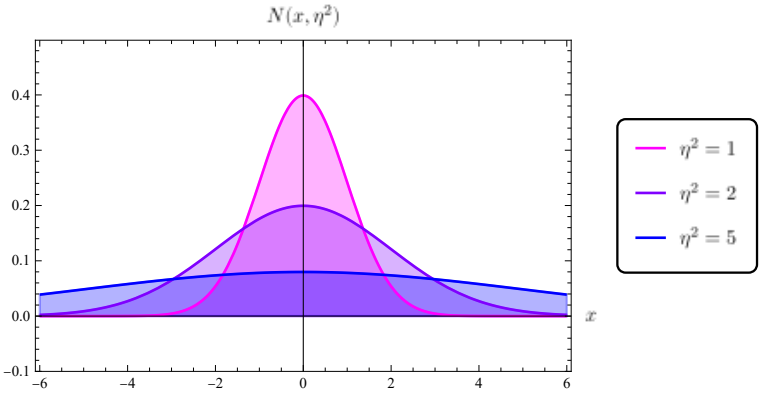


Figure 2.1: Gaussian (normal) distribution (2.29) with various values of variance, η^2.

$$H_0 = \sum_j \sigma_j^z \sigma_{j+1}^z, \qquad H_\nu(h,J) = \sum_j \left(h_j \sigma_j^x + J_j \sigma_j^x \sigma_{j+1}^x\right). \tag{2.30}$$

In the Heisenberg picture, the time evolution of an operator M is determined as $M(t) = e^{-itH} M e^{itH}$, so it follows that

$$\overline{M(t)} = \mathbb{E}_{h,J}\left(e^{-it(H_0+\varepsilon H_\nu(h,J))} M e^{it(H_0+\varepsilon H_\nu(h,J))}\right), \tag{2.31}$$

where $\mathbb{E}_{h,J}$ marks the expectation value with respect to the variables h and J. From (2.26), $W(-it) = e^{itH_0} e^{-it(H_0+\varepsilon H_\nu)}$, so I can define:

$$\xi(\pm it) \equiv e^{\pm it(H_0+\varepsilon H_\nu)} = e^{\pm itH_0} W(\pm it). \tag{2.32}$$

When $n \to \infty$, it holds that $\xi_n(p) \to \xi(p)$, such that:

$$\begin{aligned}\xi(\pm it) =& e^{\pm itH_0}\Bigg(W(0)+ \\ &+ \sum_{k=0}^{\infty} \varepsilon^{k+1} \int_0^{\pm it} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_k} \mathrm{d}s_{k+1} H_\nu(s_1) \ldots H_\nu(s_{k+1}) W(0)\Bigg).\end{aligned} \tag{2.33}$$

From the definition of $B(s)$ in (2.13), I can define:

$$H_\nu(s_k) \equiv e^{-s_k H_0} H_\nu e^{s_k H_0}., \tag{2.34}$$

which gives:

$$\begin{aligned}\overline{M(t)} = \lim_{n\to\infty} \mathbb{E}_{h,J}\Bigg(& e^{-itH_0}\Bigg(W(0)+ \\ &+ \sum_{k=0}^{2^n-2} \varepsilon^{k+1} \int_0^{-it} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_k} \mathrm{d}s_{k+1} H_\nu(s_1) \ldots H_\nu(s_{k+1}) W(0)\Bigg)\cdot \\ & M e^{itH_0}\left(W(0) + \sum_{k=0}^{2^n-2} \varepsilon^{k+1} \int_0^{it} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_k} \mathrm{d}s_{k+1} H_\nu(s_1) \ldots H_\nu(s_{k+1}) W(0)\right)\Bigg).\end{aligned} \tag{2.35}$$

For $x \in \{h, J\}$, it is clear that

$$\mathbb{E}_x(x^\zeta) = \begin{cases} 0, & \text{if } \zeta \text{ is odd;} \\ \eta_x^\zeta (\zeta-1)!!, & \text{if } \zeta \text{ is even,} \end{cases} \tag{2.36}$$

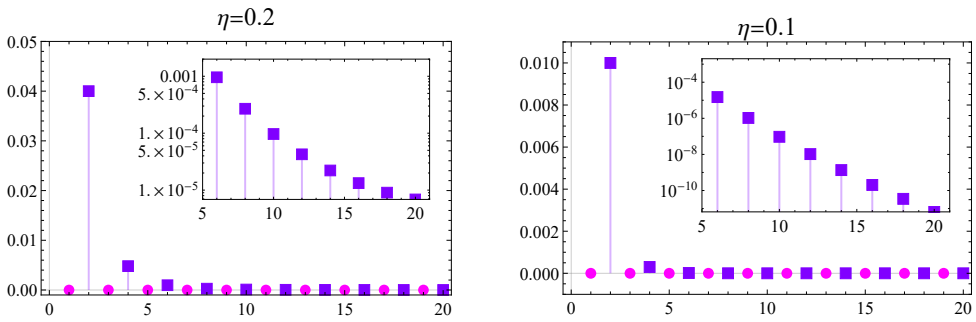


Figure 2.2: Expectation values for x^ζ, where x is a random variable from the Gaussian distribution $N(0,\eta^2)$, where $\zeta \in \mathbb{N}$ and $\eta = \{0.2, 0.1\}$. The figure includes an inset of the higher values of ζ, in the logarithmic scale.

as can be seen in Figure 2.2, so $\mathbb{E}_{h,J}(h^\alpha J^\beta) = \mathbb{E}_h(h^\alpha)\mathbb{E}_J(J^\beta) \neq 0$ only if both α and β are even. Thus, all odd orders of $\overline{M(t)}$ are equal to zero. In the zeroth order, $\overline{M(t)}_0 = e^{-itH_0} M e^{itH_0}$, and in the higher even orders it follows from (2.35).

In the spirit of the tensor network framework, I can represent my solutions diagrammatically to make them easier to grasp intuitively. As such, I can define a visual representation for a local operator acting in the system, M:

$$M \sim \text{-}(M)\text{-}. \tag{2.37}$$

This operator evolves in time in the Heisenberg picture as:

$$\text{-}(M_t)\text{-} = \text{-}(T)\text{-}(M)\text{-}(T^\dagger)\text{-}, \tag{2.38}$$

where

$$\text{-}(T)\text{-} \sim e^{-itH}, \tag{2.39}$$

t represents time, and i is the imaginary unit. The disorder-averaged expectation value of this time-evolved operator can then be expressed as:

$$\mathbb{E}_\varepsilon\left(\text{-}(M)\text{-}\right) = (\psi)\text{-}(T)\text{-}(M)\text{-}(T^\dagger)\text{-}(\psi), \tag{2.40}$$

where ε is the disorder variable. By using the Dyson series expansion, an operator $W(\pm it) = e^{\mp itA} e^{\pm it(A+\varepsilon B)}$, where A and B are local operators, $t \in \mathbb{C}$, and $|\varepsilon| \ll 1$, can be written as a nested series in ε:

$$W_n(p) = W(0) + \varepsilon \int_0^p B(s) W_{n-1}(s) \mathrm{d}s. \tag{2.41}$$

If the temporal evolution operator $\text{-}(T_0)\text{-} \sim e^{-itH_0}$ is defined by

$$\text{-}(T_0)\text{-}(W_n)\text{-} \equiv \text{-}(\xi_n)\text{-} \tag{2.42}$$

up to the order n in ε, then the temporal evolution of the operator $\text{-}(M)\text{-}$ up to the order ξ can be depicted as:

$$\text{-}(M_t)\text{-} = \text{-}(\xi_n^\dagger)\text{-}(W)\text{-}(T^\dagger)\text{-}. \tag{2.43}$$

If the tensor s_k is defined to represent $e^{-s_k H_0} H_\nu e^{s_k H_0}$, then ξ can be written in the n-th order of ε, as follows:

$$\xi = T_0 \left(\mathbb{1} + \sum_{k=0}^{2^n-2} \varepsilon^{k+1} \int_0^{it} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_k} \mathrm{d}s_{k+1} \, s_1 \, s_2 \ldots s_{k+1} + \right.$$
$$\left. + \varepsilon^{2^n} \int_0^{it} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_{2^n-1}} \mathrm{d}s_{2^n} \, s_1 \, s_2 \ldots s_{2^n} \, e^{-s_{2^n} H_0} e^{s_{2^n} H} \right), \tag{2.44}$$

where the final summand may be disregarded for small ε. The expectation value of a local observable $M(t)$ within the system described by the Hamiltonian originally presented in (2.28) can then be calculated as:

$$\overline{M(t)} = \mathbb{E}_{h,E} \left(\xi \, \mathrm{M} \, \xi^\dagger \right). \tag{2.45}$$

Note that, as shown in (2.36), because the elected Gaussian distributions of the random variables h and J are symmetric around zero, the results in all odd orders in ε vanish; for even orders in ζ, it holds that $\mathbb{E}_x(x^\zeta) = \eta_k^\zeta (\zeta - 1)!!$.

Choi–Jamiołkowski Isomorphism

In the Choi–Jamiołkowski isomorphism, an operator M that can be written out in some basis as $M = \sum_{j,k} M_{j,k} |j\rangle \langle k|$ can also be represented in a so-called state form, such that

$$M = \sum_{j,k} M_{j,k} |j\rangle \otimes |k\rangle . \tag{2.46}$$

In this mapping, the procedure of acting on the operator M from the left by an operator A may be written as $L_A(|M\rangle) \equiv |AM\rangle \equiv A \otimes \mathbb{I} \, |M\rangle$, while acting on it from the right with the operator A may be written as $R_A(|M\rangle) \equiv |MA\rangle \equiv \mathbb{I} \otimes A^T |M\rangle$. It holds that

$$(L_A - R_A) |M\rangle = |[A, M]\rangle . \tag{2.47}$$

Defining

$$\Xi A \equiv A \otimes \mathbb{I} - \mathbb{I} \otimes A^\dagger, \tag{2.48}$$

it follows that

$$\left| e^{-itH} M e^{itH} \right\rangle = e^{-it\Xi H} |M\rangle . \tag{2.49}$$

When this isomorphism is applied to the system and problem at this isomorphism is applied to the system and problem at hand, it can be tackled in a much simpler fashion. Then, it follows that

$$e^{-it(\Xi H_0 + \varepsilon \Xi(H_\nu))} |M\rangle = e^{-it\Xi H_0} W(-it) |M\rangle = \tag{2.50}$$
$$= |M_0(t)\rangle + \varepsilon \int_0^{-it} \left(\left| e^{-itH_0} \left[e^{-sH_0} H_\nu e^{sH_0}, M \right] e^{itH_0} \right\rangle \right) \mathrm{d}s + \left(\varepsilon^2 \int_0^{-it} \ldots \right) |M\rangle . \tag{2.51}$$

For a strictly local operator M, the zeroth order of the series is a local object. The Hamiltonian part in the calculated commutator in the first order in ε is a 2-local object, which makes the commutator 3-local, and the whole contribution in the first order a

3-local object. For the second order, the calculation leads to the following result:

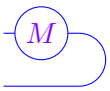
$$\varepsilon^2 \int_0^{-it} e^{-it\Xi H_0} e^{-s\Xi H_0} \Xi H_\nu e^{s\Xi H_0} \int_0^{s} e^{-s_1\Xi H_0} \Xi H_\nu e^{s_1\Xi H_0} \mathrm{d}s_1 \mathrm{d}s \, |M\rangle =$$
$$= e^2 \int_0^{-it} \int_0^{s} \left| e^{-itH_0} \left[H_\nu(s), \left[H_\nu\left(s_1\right), M \right] \right] e^{itH_0} \right\rangle \mathrm{d}s_1 \mathrm{d}s. \tag{2.52}$$

Because the first commutator is 3-local, the whole expression is 5-local. This formulation also allows for the general solution to be constructed in the $(k+1)$-th order, dubbed O_{k+1}:

$$O_{k+1} = e^{-it\Xi H_0} \varepsilon^{k+1} \int_0^{-it} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \ldots \int_0^{s_k} \mathrm{d}s_{k+1} \Xi H_\nu(s_1) \Xi H_\nu(s_2) \ldots \Xi H_\nu(s_{k+1}) \, |M\rangle = \tag{2.53}$$

$$= \varepsilon^{k+1} \int_0^{-it} \int_0^{s_1} \ldots \int_0^{s_k} \left| e^{-itH_0} \left[H_\nu(s_1), \left[H_\nu(s_2), \ldots \left[H_\nu(s_{k+1}), M \right] \right] \right] e^{itH_0} \right\rangle \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_{k+1}. \tag{2.54}$$

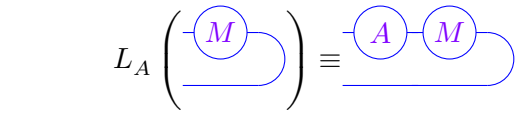
In the $(k+1)$-th order, the correction is $(1+(k+1)\cdot 2) = (2k+3)$-local. In other words, in the k-th order it is $(2k+1)$-local. This result holds for a system where H_0 is a sum of strictly local operators. For a system where H_0 is a sum of 2-local operators, the k-th order correction is $(6k+3)$-local.

In the Choi–Jamiołkowski isomorphism [57, 58, 59], instead of being treated as a tensor vulgaris of solid state physics and defined as $M = \sum_{jk} M_{jk} \, |j\rangle \, \langle k|$, an operator  may be considered as $\sum_{jk} M_{jk} \, |j\rangle \otimes |k\rangle$, or  in the newly constructed visual representation. All further calculations on thus represented constructs may then proceed as is usual for the original mathematical definitions of the forms themselves.

This mapping allows for a number of interesting manipulations, as can be shown in the case of commutators. The product of an operator  and an appropriately defined operator  that acts on it from the left, i.e.,

$$L_A \left(|M\rangle \right) \equiv |AM\rangle \equiv A \otimes \mathbb{1} \, |M\rangle \, , \tag{2.55}$$

can be depicted as:

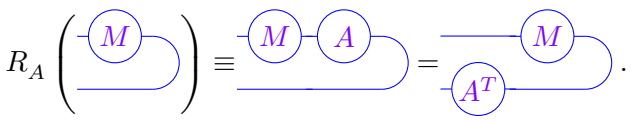


$$L_A \left(\right) \equiv \tag{2.56}$$

Similarly, if this operator is acted on by an appropriate operator  from the right,

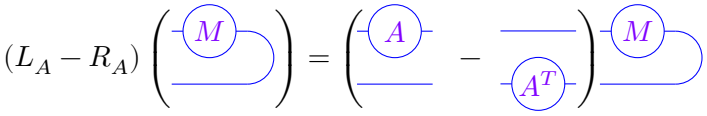
$$R_A \left(|M\rangle \right) \equiv |MA\rangle \equiv \mathbb{1} \otimes A^T \, |M\rangle \, , \tag{2.57}$$

which can be shown as:



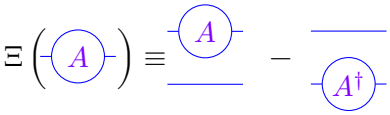
$$R_A \left(\right) \equiv \; = \; . \tag{2.58}$$

It holds that $(L_A - R_A) \, |M\rangle = |[A, M]\rangle$, or



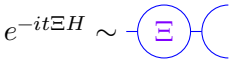
$$(L_A - R_A) \left(\right) = \left(\; - \; \right) \; . \tag{2.59}$$

Thus, if I define that



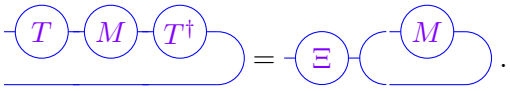
(2.60)

and recognize that it holds that $e^{-itH}Me^{itH}$ can be simplified using



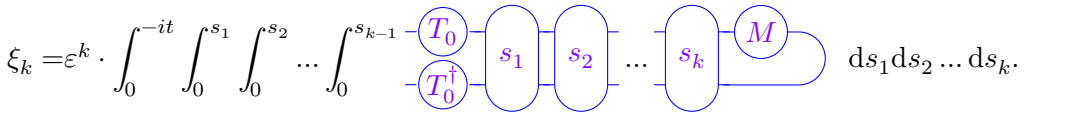
(2.61)

in the Choi–Jamiołkowski mapping, it holds that:



(2.62)

From $e^{-it(\Xi H_0+\varepsilon\Xi H_\nu)}\left|M\right\rangle = e^{-it\Xi H_0}W_0(-it)\left|M\right\rangle$, it follows that the summands of the expansion in the k-th order can be calculated using this expression:



(2.63)

I hope that this short excursion into the creation of diagrammatic notation has provided sufficient illustration of its use, not only for tensor network problems, but also for the efficient management of quite complicated equations in a readable manner. This approach is not shown further than its presentation here, but it may be explored in future applications.

Integrating $H_\nu(s_k)$

A useful expression to calculate in this problem is the integral of $H_\nu(s_k)$ in the setup of the derived expansion.

Under the assumption that ΞH_0 is invertible, the following is valid:

$$\int_0^x H_\nu(s_k)\mathrm{d}s_k = \int_0^x e^{-s_kH_0}H_\nu e^{s_kH_0}\mathrm{d}s_k = \tag{2.64}$$

$$= \int_0^x e^{-s_k\Xi H_0}\left|H_\nu\right\rangle \mathrm{d}s_k = \tag{2.65}$$

$$= \left[(-\Xi H_0)^{-1}e^{-s_k\Xi H_0}\right]_0^x \left|H_\nu\right\rangle \mathrm{d}s_k = \tag{2.66}$$

$$= \left((-\Xi H_0)^{-1}e^{-x\Xi H_0} - (-\Xi H_0)^{-1}\right)\left|H_\nu\right\rangle \mathrm{d}s_k = \tag{2.67}$$

$$= (-\Xi H_0)^{-1}(e^{-x\Xi H_0} - \mathbb{I})\left|H_\nu\right\rangle \mathrm{d}s_k = \tag{2.68}$$

$$= \left((-H_0)\otimes\mathbb{I} - \mathbb{I}\otimes(-H_0)^\dagger\right)^{-1}(e^{-x(H_0\otimes\mathbb{I}-\mathbb{I}\otimes H_0^\dagger)} - \mathbb{I})\left|H_\nu\right\rangle \mathrm{d}s_k, \tag{2.69}$$

which is an expression that can be solved in a straightforward manner. To proceed, the expression to be calculated first must be of the form:

$$\int_0^x e^{-s_k\Xi(\sum_j \sigma_j^z\sigma_{j+1}^z)}\mathrm{d}s_k\left|H_\nu\right\rangle. \tag{2.70}$$

Then,

$$\int_0^x e^{-s_k H_0} H_\nu e^{s_k H_0} \mathrm{d}s_k = \int_0^x e^{-s_k H_0} \sum_{j=1}^N \left(h_j \sigma_j^x + J_j \sigma_j^x \sigma_{j+1}^x\right) e^{s_k H_0} \mathrm{d}s_k = \tag{2.71}$$

$$= \sum_{j=1}^N h_j \int_0^x e^{-s_k H_0} \sigma_j^x e^{s_k H_0} \mathrm{d}s + \sum_{j=1}^N J_j \int_0^x e^{-s_k H_0} \sigma_j^x \sigma_{j+1}^x e^{s_k H_0} \mathrm{d}s_k \equiv \tag{2.72}$$

$$\equiv \mathrm{A} + \mathrm{B}. \tag{2.73}$$

Note that the operators σ_j^z all commute, which gives:

$$e^{-s_k H_0} = e^{-s_k \sum_{j=1}^N \sigma_j^z \sigma_{j+1}^z} = \prod_{j=1}^N e^{-s_k \sigma_j^z \sigma_{j+1}^z}, \tag{2.74}$$

and the following relation holds:

$$\left[\sigma_j^z \sigma_{j+1}^z, \sigma_l^x\right] = 0, \quad \text{for} \quad j \neq l \neq j+1. \tag{2.75}$$

Then, it follows for A:

$$A = \sum_{j=1}^N h_j \int_0^x e^{-s_k H_0} \sigma_j^x e^{s_k H_0} \mathrm{d}s_k = \tag{2.76}$$

$$= \sum_{j=1}^N h_j \sigma_j^x \int_0^x e^{2 s_k \sigma_{j-1}^z \sigma_j^z} e^{2 s_k \sigma_j^z \sigma_{j+1}^z} \mathrm{d}s_k. \tag{2.77}$$

The calculation for B is analogous[1].

$$B = \sum_{j=1}^N J_j \int_0^x e^{-s_k H_0} \sigma_j^x \sigma_{j+1}^x e^{s_k H_0} \mathrm{d}s_k = \tag{2.78}$$

$$= \sum_{j=1}^N J_j \sigma_j^x \sigma_{j+1}^x \int_0^x e^{2 s_k \sigma_{j-1}^z \sigma_j^z} e^{2 s_k \sigma_{j+1}^z \sigma_{j+2}^z} \mathrm{d}s_k \tag{2.79}$$

The matrices in the exponents are not invertible, but they are diagonal. Thus, A and B can be integrated as follows.

$$\mathrm{A} = \sum_{j=1}^N h_j \sigma_j^x \int_0^x e^{2 s_k \sigma_{j-1}^z \sigma_j^z} e^{2 s_k \sigma_j^z \sigma_{j+1}^z} \mathrm{d}s_k = \tag{2.80}$$

$$= \sum_{j=1}^N h_j \sigma_j^x \int_0^x e^{2 s_k (\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)} \mathrm{d}s_k = \tag{2.81}$$

$$= \sum_{j=1}^N h_j \sigma_j^x \int_0^x e^{2 s_k \,\mathrm{diag}(2,0,-2,0,0,-2,0,2)} \mathrm{d}s_k \tag{2.82}$$

$$B = \sum_{j=1}^N J_j \sigma_j^x \sigma_{j+1}^x \int_0^x e^{2 s_k \sigma_{j-1}^z \sigma_j^z} e^{2 s_k \sigma_{j+1}^z \sigma_{j+2}^z} \mathrm{d}s_k = \tag{2.83}$$

$$= \sum_{j=1}^N J_j \sigma_j^x \sigma_{j+1}^x \int_0^x e^{2 s_k (\sigma_{j-1}^z \sigma_j^z + \sigma_{j+1}^z \sigma_{j+2}^z)} \mathrm{d}s_k = \tag{2.84}$$

$$= \sum_{j=1}^N J_j \sigma_j^x \sigma_{j+1}^x \int_0^x e^{2 s_k \,\mathrm{diag}(2,0,0,2,0,-2,-2,0,0,-2,-2,0,2,0,0,2)} \mathrm{d}s_k \tag{2.85}$$

[1] See details in the Appendix here.

This expression can be evaluated for the individual elements of the diagonal matrices:

$$\int_0^x e^0 \mathrm{d}s_k = \int_0^x \mathbb{I} \mathrm{d}s_k = \tag{2.86}$$

$$= [s_k]_0^x = x \tag{2.87}$$

$$\int_0^x e^{4s_k} \mathrm{d}s_k = \left[\frac{e^{4s_k}}{4}\right]_0^x = \tag{2.88}$$

$$= \frac{e^{4x} - 1}{4} \tag{2.89}$$

$$\int_0^x e^{-4s_k} \mathrm{d}s_k = \left[\frac{-e^{-4x}}{4}\right]_0^x = \tag{2.90}$$

$$= \frac{-e^{-4x} + 1}{4} \tag{2.91}$$

The full integral then gives:

$$\int_0^x e^{-s_k H_0} H_\nu e^{s_k H_0} \mathrm{d}s_k = A + B = \tag{2.92}$$

$$= \sum_{j=1}^{N} h_j \sigma_j^x \int_0^x e^{2s_k \,\mathrm{diag}(2,0,-2,0,0,-2,0,2)} \mathrm{d}s_k +$$

$$+ \sum_{j=1}^{N} J_j \sigma_j^x \sigma_{j+1}^x \int_0^x e^{2s_k \,\mathrm{diag}(2,0,0,2,0,-2,-2,0,0,-2,-2,0,2,0,0,2)} \mathrm{d}s_k = \tag{2.93}$$

$$= \sum_{j=1}^{N} h_j \sigma_j^x \,\mathrm{diag}\left(\frac{e^{4x} - 1}{4}, \mathbb{I}, \frac{-e^{-4x} + 1}{4}, \mathbb{I}, \mathbb{I}, \frac{-e^{-4ix} + 1}{4}, \mathbb{I}, \frac{e^{4x} - 1}{4}\right) +$$

$$+ \sum_{j=1}^{N} J_j \sigma_j^x \sigma_{j+1}^x \,\mathrm{diag}\left(\frac{e^{4x} - 1}{4}, \mathbb{I}, \mathbb{I}, \frac{e^{4x} - 1}{4}, \mathbb{I}, \frac{-e^{-4x} + 1}{4}, \frac{-e^{-4x} + 1}{4}, \mathbb{I},\right.$$

$$\left. \mathbb{I}, \frac{-e^{-4x} + 1}{4}, \frac{-e^{-4x} + 1}{4}, \mathbb{I}, \frac{e^{4x} - 1}{4}, \mathbb{I}, \mathbb{I}, \frac{e^{4x} - 1}{4}\right) \tag{2.94}$$

From (2.33), the lowest order contributions in ε can be calculated in a straightforward manner:

$$\epsilon^{k+1} \int_0^{-it} \int_0^{s_1} \ldots \int_0^{s_k} |e^{-itH_0}[H_\nu(s_1), [H_\nu(s_2), \ldots [H_\nu(s_{k+1}, M] \ldots]] e^{itH_0}\rangle \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_{k+1} = \tag{2.95}$$

$$= \epsilon^{k+1} \int_0^{-it} \int_0^{s_1} \ldots \int_0^{s_{k-1}} |e^{-itH_0}[H_\nu(s_1), [H_\nu(s_2), \ldots$$

$$\left[\int_0^{s_k} H_\nu(s_{k+1}) \mathrm{d}s_{k+1}, M\right] \ldots]] e^{itH_0}\rangle \mathrm{d}s_1 \mathrm{d}s_2 \ldots \mathrm{d}s_k. \tag{2.96}$$

As the basic integral has been evaluated, it is possible to use the result to subsequently evaluate the main derived expression for the time-evolved expectation value. However, the result in (2.94) doesn't seem to be practical for further analytical calculations. It would also be possible to write out the solution as a linear combination of projectors in the $\sigma_j \sigma_{j+1}$ vector space.

Application to a System with a Strictly Local Operator

This subsection covers the exact calculation with a strictly local operator M. The examples I have chosen to present are fairly simple and quite demonstrative, but the performed methods can be applied in the same manner to a more complex set of locally-acting operators in a many-body system. In the following I will calculate and describe the temporal evolution of local operators in perturbed a many-body system within several orders of perturbation and demonstrate the effects minute properties of the system's Hamiltonian may have on the derived values.

In the first case, I choose $M = \sigma_j^z$, or

$$M = \mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma_j^z \otimes ... \otimes \mathbb{I} \otimes \mathbb{I} = \sigma_j^z. \tag{2.97}$$

In a second variant, let

$$F = \mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma_j^x \otimes ... \otimes \mathbb{I} \otimes \mathbb{I} = \sigma_j^x. \tag{2.98}$$

The system's Hamiltonian is of the form

$$H = \sum_k \sigma_k^z \sigma_{k+1}^z + h_k \sigma_k^x, \tag{2.99}$$

where the variables h_k take values in a Gaussian distribution stated in (2.29). The expression to be calculated is the following:

$$\overline{M(t)} = \mathbb{E}_h \left(e^{-it(H_0 + \varepsilon H_\nu(h))} M e^{it(H_0 + \varepsilon H_\nu(h))} \right), \tag{2.100}$$

which is easier when broken down into contributions in order of correction.

Henceforth the corrections in the order k of ε are going to be denoted with Ω_k, such that

$$\overline{M(t)} = \sum_k \Omega_k(\varepsilon^k). \tag{2.101}$$

For the zeroth order for the operator $M(t)$, the time-evolved expectation value can be calculated as:

$$\begin{aligned}
\Omega_{0;M} =& \mathbb{E}_h \left(M_0(t) \right) = \mathbb{E}_h \left(e^{-itH_0} M e^{itH_0} \right) = && (2.102)\\
=& \mathbb{E}_h \left(e^{-it \sum_k \sigma_k^z \sigma_{k+1}^z} \left(\mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma_j^z \otimes ... \otimes \mathbb{I} \otimes \mathbb{I} \right) e^{it \sum_l \sigma_l^z \sigma_{l+1}^z} \right) = && (2.103)\\
=& \mathbb{E}_h \left(\mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma_j^z \otimes ... \otimes \mathbb{I} \otimes \mathbb{I} \right) = && (2.104)\\
=& \sigma_j^z, && (2.105)
\end{aligned}$$

and that for $F(t)$ as:

$$\begin{aligned}
\Omega_{0;F} =& \mathbb{E}_h \left(F_0(t) \right) = \mathbb{E}_h \left(e^{-itH_0} M e^{itH_0} \right) = && (2.106)\\
=& \mathbb{E}_h \left(e^{-it \sum_k \sigma_k^z \sigma_{k+1}^z} \left(\mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma_j^x \otimes ... \otimes \mathbb{I} \otimes \mathbb{I} \right) e^{it \sum_l \sigma_l^z \sigma_{l+1}^z} \right) = && (2.107)\\
=& \mathbb{E}_h \left(\mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \sigma_j^x \otimes ... \otimes \mathbb{I} \otimes \mathbb{I} \right) = && (2.108)\\
=& \sigma_j^x. && (2.109)
\end{aligned}$$

The first orders for the time-evolved expectations for $M(t)$ and $F(t)$ can be evaluated in detail[2] and give

$$\Omega_{1;M} = \Omega_{1;F} = 0, \tag{2.110}$$

[2] The normalization calculation is included in the Appendix here.

which corresponds to the expected value from the derived general form of the perturbative solution.

The second-order calculations are more involved. The derivation for the operator $M(t)$ leads[3] to the following expression:

$$\Omega_{2;M} = \mathbb{E}_h\left(\varepsilon^2 \int_0^{-it}\int_0^{s} e^{-itH_0}\left[H_\nu(s),[H_\nu(s_1),M]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s\right) = \tag{2.111}$$

$$\begin{aligned}=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^{s} e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}2h_j\sum_k h_k\sigma_k^x\sigma_j^x\sigma_j^z\cdot\\ &\Big(e^{2s(-1)^{\delta(j,k-1)}(-1)^{\delta(j,k)}\sigma_{k-1}^z\sigma_k^z}e^{2s(-1)^{\delta(j,k)}(-1)^{\delta(j,k+1)}\sigma_k^z\sigma_{k+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-\\ &-(-1)^{\delta(j,k)}e^{2s_1(-1)^{\delta(j-1,k)}(-1)^{\delta(j,k)}\sigma_{j-1}^z\sigma_j^z}e^{2s_1(-1)^{\delta(j,k)}(-1)^{\delta(j+1,k)}\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Big)\cdot\\ &e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)\equiv\end{aligned} \tag{2.112}$$

$$\equiv\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^{s} e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\sum_k h_k\sigma_k^x\sigma_j^x\sigma_j^z\Theta(j,k)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\right) \tag{2.113}$$

Next, it is important to calculate the defined $\theta(j,k)\equiv\theta$:

$$\Theta_{j=k} = e^{-2s\sigma_{j-1}^z\sigma_j^z}e^{-2s\sigma_j^z\sigma_{j+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}+e^{-2s_1\sigma_{j-1}^z\sigma_j^z}e^{-2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}= \tag{2.114}$$

$$=2\cosh\left(2\cdot\left(s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)-s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\right) \tag{2.115}$$

$$\Theta_{j=k-1} = e^{-2s\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{-2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}= \tag{2.116}$$

$$=e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}\left(e^{-2s\sigma_j^z\sigma_{j+1}^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-e^{-2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\right)= \tag{2.117}$$

$$=e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}\cdot\left(-2\sinh\left(2(s-s_1)\sigma_j^z\sigma_{j+1}^z\right)\right) \tag{2.118}$$

$$\Theta_{j=k+1} = e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{-2s\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-e^{-2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s\sigma_{j-1}^z\sigma_j^z}= \tag{2.119}$$

$$=e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\left(e^{-2s\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}-e^{-2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_{j-1}^z\sigma_j^z}\right)= \tag{2.120}$$

$$=e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\sinh\left(2(s-s_1)\sigma_{j-1}^z\sigma_j^z\right)\right) \tag{2.121}$$

$$\begin{aligned}\Theta(j\neq\{k-1,k,k+1\}) = &e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-\\ &-e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}=0\end{aligned} \tag{2.122}$$

For all other relations between j and k, the value of $\theta(j,k)$ is equal to zero. With that in mind, and by considering that $\mathbb{E}_h(h_a h_{b\neq a}) = \mathbb{E}_h(h_a)\cdot\mathbb{E}_h(h_b) = 0$, the expression for

[3]Details of the normalization calculation are included in the Appendix here.

$\Omega_{2;M}$ becomes simpler:

$$\Omega_{2;M} = \mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}2h_j\cdot$$
$$\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta(k=j-1)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta(k=j)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta(k=j+1)\Big)\cdot$$
$$e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg) = \tag{2.123}$$
$$=2\varepsilon^2\eta_h^2\sigma_j^z\int_0^{-it}\int_0^s 2\cosh\big(2(s-s_1)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\big)\,\mathrm{d}s_1\mathrm{d}s \tag{2.124}$$

The integrals of this type have already been evaluated and have the following form:

$$\alpha_j(x;\kappa)\equiv\int_0^x e^{\kappa s_k(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\mathrm{d}s_k = \tag{2.125}$$
$$=\int_0^x e^{\kappa s_k\,\mathrm{diag}(2,0,-2,0,0,-2,0,2)_{\{(j-1),j,(j+1)\}}}\mathrm{d}s_k = \tag{2.126}$$
$$=\mathrm{diag}\Big(\frac{e^{2\kappa x}-1}{2\kappa},\mathbb{I},\frac{-e^{-2\kappa x}+1}{2\kappa},\mathbb{I},\mathbb{I},\frac{-e^{-2\kappa x}+1}{2\kappa},\mathbb{I},\frac{e^{2\kappa x}-1}{2\kappa}\Big)_{\{(j-1),j,(j+1)\}} \tag{2.127}$$

$$\beta_j(x;\kappa)\equiv\int_0^x e^{2s_k\sigma_{j-1}^z\sigma_j^z}e^{2s_k\sigma_{j+1}^z\sigma_{j+2}^z}\mathrm{d}s_k = \tag{2.128}$$
$$=\int_0^x e^{\kappa s_k\,\mathrm{diag}(2,0,0,2,0,-2,-2,0,0,-2,-2,0,2,0,0,2)_{\{(j-1),j,(j+1),(j+2)\}}}\mathrm{d}s_k = \tag{2.129}$$
$$=\mathrm{diag}\Big(\frac{e^{2\kappa x}-1}{2\kappa},\mathbb{I},\mathbb{I},\frac{e^{2\kappa x}-1}{2\kappa},\mathbb{I},\frac{-e^{-2\kappa ix}+1}{2\kappa},\frac{-e^{-2\kappa x}+1}{2\kappa},\mathbb{I},\mathbb{I},$$
$$\frac{-e^{-2\kappa x}+1}{2\kappa},\frac{-e^{-2\kappa x}+1}{2\kappa},\mathbb{I},\frac{e^{2\kappa x}-1}{2\kappa},\mathbb{I},\mathbb{I},\frac{e^{2\kappa x}-1}{2\kappa}\Big)_{\{(j-1),j,(j+1),(j+2)\}} \tag{2.130}$$

By using those expressions, $\Omega_{2;M}$ can be rewritten as:

$$\Omega_{2;M} = 2\varepsilon^2\eta_h^2(\sigma_j^x)^2\sigma_j^z\int_0^{-it}\int_0^s\Big(e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}+$$
$$+e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{-2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\Big)\mathrm{d}s_1\mathrm{d}s = \tag{2.131}$$
$$=2\varepsilon^2\eta_h^2(\sigma_j^x)^2\sigma_j^z\Big(\int_0^{-it}\int_0^s e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\mathrm{d}s_1\mathrm{d}s+$$
$$+\int_0^{-it}\int_0^s e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{-2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\mathrm{d}s_1\mathrm{d}s\Big) = \tag{2.132}$$
$$=2\varepsilon^2\eta_h^2(\sigma_j^x)^2\sigma_j^z\Big(\int_0^{-it}e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\alpha_j(s;2)\mathrm{d}s+$$
$$+\int_0^{-it}e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\alpha_j(s;-2)\mathrm{d}s\Big) = \tag{2.133}$$
$$=2\varepsilon^2\eta_h^2\sigma_j^z\left(\int_0^{-it}e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\alpha_j(s;2)\mathrm{d}s+\int_0^{-it}e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\alpha_j(s;-2)\mathrm{d}s\right) \tag{2.134}$$

To evaluate this expression, it would be useful to find a way to write $\alpha_j(x;\kappa)$ in the Pauli basis. These matrices are diagonal, so they can be written in the $\{\sigma^0,\sigma^z\}$ basis,

as follows.

$$\alpha_j(x;\kappa) = \mathrm{diag}(\frac{e^{2\kappa x}-1}{2\kappa}, \mathbb{I}, \frac{-e^{-2\kappa x}+1}{2\kappa}, \mathbb{I}, \mathbb{I}, \frac{-e^{-2\kappa x}+1}{2\kappa}, \mathbb{I}, \frac{e^{2\kappa x}-1}{2\kappa})_{\{(j-1),j,(j+1)\}} = \tag{2.135}$$

$$= \sum_{\iota=1}^{8} \xi_\iota(x;\kappa) \cdot \left(\sigma^{\tau_1}_{(j-1)}\sigma^{\tau_2}_{j}\sigma^{\tau_3}_{(j+1)}\right)_{\sigma^{\tau_{1,2,3}}=\{\sigma^0,\sigma^z\}} = \tag{2.136}$$

$$= \sum_{\iota=1}^{8} \frac{1}{8}\,\mathrm{Tr}\left(\xi_\iota(x;\kappa) \cdot \left(\sigma^{\tau_1}_{(j-1)}\sigma^{\tau_2}_{j}\sigma^{\tau_3}_{(j+1)}\right)_{\sigma^{\tau_{1,2,3}}=\{\sigma^0,\sigma^z\}}\right) = \tag{2.137}$$

$$\begin{aligned}=&\frac{1}{8}\cdot\left(4+\frac{e^{2\kappa x}-e^{-2\kappa x}}{\kappa}\right)\left(\sigma^0_{(j-1)}\sigma^0_j\sigma^0_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{-2+e^{-2\kappa x}+e^{2\kappa x}}{\kappa}\right)\left(\sigma^0_{(j-1)}\sigma^z_j\sigma^z_{(j+1)}\right)+\\&+\frac{1}{8}\cdot\left(-4+\frac{e^{2\kappa x}-e^{-2\kappa x}}{\kappa}\right)\left(\sigma^z_{(j-1)}\sigma^0_j\sigma^z_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{-2+e^{-2\kappa x}+e^{2\kappa x}}{\kappa}\right)\left(\sigma^z_{(j-1)}\sigma^z_j\sigma^0_{(j+1)}\right)=\end{aligned} \tag{2.138}$$

$$\begin{aligned}=&\frac{1}{8}\cdot\left(4+\frac{2\sinh(2\kappa x)}{\kappa}\right)\left(\sigma^0_{(j-1)}\sigma^0_j\sigma^0_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{-2}{\kappa}+\frac{2\cosh(2\kappa x)}{\kappa}\right)\left(\sigma^0_{(j-1)}\sigma^z_j\sigma^z_{(j+1)}\right)+\\&+\frac{1}{8}\cdot\left(-4+\frac{2\sinh(2\kappa x)}{\kappa}\right)\left(\sigma^z_{(j-1)}\sigma^0_j\sigma^z_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{-2}{\kappa}+\frac{2\cosh(2\kappa x)}{\kappa}\right)\left(\sigma^z_{(j-1)}\sigma^z_j\sigma^0_{(j+1)}\right)\end{aligned} \tag{2.139}$$

This gives the following result for $\alpha_j(x;2)$ and $\alpha_j(x;-2)$:

$$\begin{aligned}\alpha_j(x;2) = &\frac{1}{8}\cdot\left(4+\frac{2\sinh(4x)}{2}\right)\left(\sigma^0_{(j-1)}\sigma^0_j\sigma^0_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{-2}{2}+\frac{2\cosh(4x)}{2}\right)\left(\sigma^0_{(j-1)}\sigma^z_j\sigma^z_{(j+1)}\right)+\\&+\frac{1}{8}\cdot\left(-4+\frac{2\sinh(4x)}{2}\right)\left(\sigma^z_{(j-1)}\sigma^0_j\sigma^z_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{-2}{2}+\frac{2\cosh(4x)}{2}\right)\left(\sigma^z_{(j-1)}\sigma^z_j\sigma^0_{(j+1)}\right)=\end{aligned} \tag{2.140}$$

$$\begin{aligned}=&\frac{1}{8}\cdot(4+\sinh(4x))\left(\sigma^0_{(j-1)}\sigma^0_j\sigma^0_{(j+1)}\right)+\frac{1}{8}\cdot(-1+\cosh(4x))\left(\sigma^0_{(j-1)}\sigma^z_j\sigma^z_{(j+1)}\right)+\\&+\frac{1}{8}\cdot(-4+\sinh(4x))\left(\sigma^z_{(j-1)}\sigma^0_j\sigma^z_{(j+1)}\right)+\frac{1}{8}\cdot(-1+\cosh(4x))\left(\sigma^z_{(j-1)}\sigma^z_j\sigma^0_{(j+1)}\right)\end{aligned} \tag{2.141}$$

$$\begin{aligned}\alpha_j(x;-2) = &\frac{1}{8}\cdot\left(4+\frac{-2\sinh(-4x)}{2}\right)\left(\sigma^0_{(j-1)}\sigma^0_j\sigma^0_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{2}{2}+\frac{-2\cosh(-4x)}{2}\right)\left(\sigma^0_{(j-1)}\sigma^z_j\sigma^z_{(j+1)}\right)+\\&+\frac{1}{8}\cdot\left(-4+\frac{-2\sinh(-4x)}{2}\right)\left(\sigma^z_{(j-1)}\sigma^0_j\sigma^z_{(j+1)}\right)+\frac{1}{8}\cdot\left(\frac{2}{2}+\frac{-2\cosh(-4x)}{2}\right)\left(\sigma^z_{(j-1)}\sigma^z_j\sigma^0_{(j+1)}\right)=\end{aligned} \tag{2.142}$$

$$\begin{aligned}=&\frac{1}{8}\cdot(4+\sinh(4x))\left(\sigma^0_{(j-1)}\sigma^0_j\sigma^0_{(j+1)}\right)+\frac{1}{8}\cdot(1-\cosh(4x))\left(\sigma^0_{(j-1)}\sigma^z_j\sigma^z_{(j+1)}\right)+\\&+\frac{1}{8}\cdot(-4+\sinh(4x))\left(\sigma^z_{(j-1)}\sigma^0_j\sigma^z_{(j+1)}\right)+\frac{1}{8}\cdot(1-\cosh(4x))\left(\sigma^z_{(j-1)}\sigma^z_j\sigma^0_{(j+1)}\right)\end{aligned} \tag{2.143}$$

These expressions allow for a direct way of evaluating the integral for $\Omega_{2;M}$. However, what can be immediately seen is that the object $\Omega_{2;M}$ is 3-local, affecting only the subset of $\{j-1, j, j+1\}$ particles.

On a simple example, where $H_0 = 0$ and $H_\nu(s) = \sum_k h_k \sigma^x_k$, it is not difficult to

evaluate this object analytically[4], giving

$$\Omega_{2;M} = \mathbb{E}_h\left(-2\varepsilon^2 h_j^2 \sigma_j^z t^2\right) = \tag{2.144}$$

$$= -2t^2 \eta_h^2 \sigma_j^z. \tag{2.145}$$

The second-order calculation[5] for the operator F gives the following:

$$\Omega_{2;F} = \mathbb{E}_h\left(\varepsilon^2 \int_0^{-it}\int_0^s e^{-itH_0}\left[H_\nu(s),[H_\nu(s_1),F]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s\right) \equiv \tag{2.146}$$

$$\equiv \mathbb{E}_h\left(\varepsilon^2 \int_0^{-it}\int_0^s e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}\sum_k\left(\Sigma_1+\Sigma_2+\Sigma_3\right)e^{it\sum_b \sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\right) \tag{2.147}$$

By explicitly calculating the values of the summands Σ_1, Σ_2, and Σ_3, it is shown that the expression $\Omega_{2;F}$ is 5-local:

$$\Omega_{2;F;k\neq\{j-2,j-1,j,j+1,j+2\}} = 0. \tag{2.148}$$

The derivation of $\Omega_{2;F}$ is then finalized by considering that $\mathbb{E}_h(h_a h_{b\neq a}) = 0$:

$$\Omega_{2;F} = \mathbb{E}_h\left(\varepsilon^2 \int_0^{-it}\int_0^s e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}\sum_k\left(\Sigma_1+\Sigma_2+\Sigma_3\right)e^{it\sum_b \sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\right) = \tag{2.149}$$

$$= \eta_h^2\varepsilon^2 \int_0^{-it}\int_0^s e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}\sum_k\left(\frac{\Sigma_{1;k=j-1}}{h_{j-1}^2}+\frac{\Sigma_{2;k=j}}{h_j^2}+\frac{\Sigma_{3;k=j+1}}{h_{j+1}^2}\right)e^{it\sum_b \sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s = \tag{2.150}$$

$$\begin{aligned}
&= \eta_h^2\varepsilon^2\sigma_j^x e^{2it(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\cdot\\
&\cdot\int_0^{-it}\Bigg(e^{2s\sigma_{j-1}^z\sigma_j^z}\cdot\int_0^s\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)\left(-2\sinh(2(s-s_1)\sigma_{j-2}^z\sigma_{j-1}^z)\right)+\\
&+2e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\int_0^s\left(-2\cdot\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+\\
&+e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\int_0^s\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\left(-2\sinh(2(s-s_1)\sigma_{j+1}^z\sigma_{j+2}^z)\right)\Bigg)\mathrm{d}s_1\mathrm{d}s
\end{aligned} \tag{2.151}$$

Further, because of the symmetry properties of the chosen Gaussian, with $\mathbb{E}_h\left(h_j\right) = 0$, it holds that:

$$\Omega_3 = \Omega_5 = \Omega_7 = ... = 0, \tag{2.152}$$

so only the even contributors to Ω must be considered.

As an example for the involvedness of the calculation of higher-order perturbative terms, I will summarize the calculation[6] for $\Omega_{4;M}$.

$$\Omega_{4;M} = \mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\left|e^{-itH_0}\left[H_\nu(s_1),[H_\nu(s_2),[H_\nu(s_3),[H_\nu(s_4),M]]]\right]e^{itH_0}\right\rangle$$

$$\cdot\,\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\Bigg) \equiv \tag{2.153}$$

$$\equiv \mathbb{E}_h\left(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\left[H_\nu(s_1),[H_\nu(s_2),\zeta_2(s_2,s_3)]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\right) \tag{2.154}$$

[4]The calculation is included explicitly in the Appendix here.
[5]Details of the normalization calculation are included in the Appendix here.
[6]The full calculation is included in the Appendix here.

This expression was simplified using

$$\zeta_2(s_2, s_3) = \int_0^{s_2} \int_0^{s_3} [H_\nu(s_3), [H_\nu(s_4), M]] \, \mathrm{d}s_3 \mathrm{d}s_4. \tag{2.155}$$

From (2.112) it can be seen that

$$\Omega_2 = \mathbb{E}_h \left(\varepsilon^2 e^{-itH_0} \zeta_2(-it, s) e^{itH_0} \right), \tag{2.156}$$

so

$$\zeta_2(-it, s) = \int_0^{-it} \int_0^{s} 2h_j \Big(h_{j-1} \sigma^x_{j-1} \sigma^x_j \sigma^z_j \Theta(k = j-1) + h_j \sigma^x_j \sigma^x_j \sigma^z_j \Theta(k = j) + \\ + h_{j+1} \sigma^x_{j+1} \sigma^x_j \sigma^z_j \Theta(k = j+1) \Big) \mathrm{d}s_1 \mathrm{d}s. \tag{2.157}$$

After an extensive calculation, the final expression obtained for $\Omega_{4;M}$ can be specified as follows.

$$\Omega_{4;M} = \int_0^{-it} \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \Bigg(2\eta_h^4 \sigma^z_j (-2 \sinh(2(s_3 - s_4) \sigma^z_{j-1} \sigma^z_j)) \cdot \\ \Bigg(-2 \sinh\left(2\left(s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) + s_2(\sigma^z_{j-2}\sigma^z_{j-1} - \sigma^z_{j-1}\sigma^z_j) - s_3\sigma^z_{j-2}\sigma^z_{j-1} - s_4(\sigma^z_j\sigma^z_{j+1})\right)\right) - \\ - 2 \sinh\left(2\left(s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) - s_2(\sigma^z_{j-2}\sigma^z_{j-1} + \sigma^z_{j-1}\sigma^z_j) + s_3\sigma^z_{j-2}\sigma^z_{j-1} - s_4(\sigma^z_j\sigma^z_{j+1})\right)\right) - \\ - 2 \sinh\left(2\left(s_1(\sigma^z_{j-2}\sigma^z_{j-1} + \sigma^z_{j-1}\sigma^z_j) - s_2(\sigma^z_{j-1}\sigma^z_j - \sigma^z_j\sigma^z_{j+1}) - s_3\sigma^z_{j-2}\sigma^z_{j-1} - s_4(\sigma^z_j\sigma^z_{j+1})\right)\right) + \\ + 2 \sinh\left(2\left(s_1(\sigma^z_{j-2}\sigma^z_{j-1} + \sigma^z_{j-1}\sigma^z_j) - s_2(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) - s_3\sigma^z_{j-2}\sigma^z_{j-1} + s_4(\sigma^z_j\sigma^z_{j+1})\right)\right) \Bigg) + \\ + 2\eta_h^4 \sigma^z_j (-2 \sinh(2(s_3 - s_4) \sigma^z_{j+1} \sigma^z_j)) \cdot \\ \Bigg(-2 \sinh\left(2\left(s_1(\sigma^z_{j+1}\sigma^z_j + \sigma^z_j\sigma^z_{j-1}) + s_2(\sigma^z_{j+2}\sigma^z_{j+1} - \sigma^z_{j+1}\sigma^z_j) - s_3\sigma^z_{j+2}\sigma^z_{j+1} - s_4(\sigma^z_j\sigma^z_{j-1})\right)\right) - \\ - 2 \sinh\left(2\left(s_1(\sigma^z_{j+1}\sigma^z_j + \sigma^z_j\sigma^z_{j-1}) - s_2(\sigma^z_{j+2}\sigma^z_{j+1} + \sigma^z_{j+1}\sigma^z_j) + s_3\sigma^z_{j+2}\sigma^z_{j+1} - s_4(\sigma^z_j\sigma^z_{j-1})\right)\right) - \\ - 2 \sinh\left(2\left(s_1(\sigma^z_{j+2}\sigma^z_{j+1} + \sigma^z_{j+1}\sigma^z_j) - s_2(\sigma^z_{j+1}\sigma^z_j - \sigma^z_j\sigma^z_{j-1}) - s_3\sigma^z_{j+2}\sigma^z_{j+1} - s_4\sigma^z_j\sigma^z_{j-1}\right)\right) + \\ + 2 \sinh\left(2\left(s_1(\sigma^z_{j+2}\sigma^z_{j+1} + \sigma^z_{j+1}\sigma^z_j) - s_2(\sigma^z_{j+1}\sigma^z_j + \sigma^z_j\sigma^z_{j-1}) - s_3\sigma^z_{j+2}\sigma^z_{j+1} + s_4\sigma^z_j\sigma^z_{j-1}\right)\right) \Bigg) + \\ + \eta_h^4 \cdot 8\sigma^z_j \Bigg(2 \sinh\left(2(s_1 - s_2)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right) \cdot \cosh\left(2(s_4 - s_3)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right) + \\ + \left(e^{-2(s_1-s_2)(\sigma^z_{j-2}\sigma^z_{j-1} + \sigma^z_{j-1}\sigma^z_j)} + e^{-2(s_1-s_2)(\sigma^z_j\sigma^z_{j+1} + \sigma^z_{j+1}\sigma^z_{j+2})} \right) \cdot \\ \sinh\left(2s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right) \sinh\left(2s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right) - \\ - \left(e^{2(s_1-s_2)(\sigma^z_{j-2}\sigma^z_{j-1} + \sigma^z_{j-1}\sigma^z_j)} + e^{2(s_1-s_2)(\sigma^z_j\sigma^z_{j+1} + \sigma^z_{j+1}\sigma^z_{j+2})} \right) \cdot \\ \sinh\left(2s_4(\sigma^z_{j-1}\sigma^z_j - \sigma^z_j\sigma^z_{j+1})\right) \sinh\left(2s_3(\sigma^z_{j-1}\sigma^z_j - \sigma^z_j\sigma^z_{j+1})\right) \Bigg) \Bigg) \mathrm{d}s_4 \mathrm{d}s_3 \mathrm{d}s_2 \mathrm{d}s_1 \tag{2.158}$$

Correlation Propagation

To investigate the temporal (and spatial) propagation of information — in other words, the effects — of a local operator acting in a perturbed many-body system, it is useful to also examine the correlation function between two such local operators. In the following I will present the calculation of the correlation function between two

local operators and determine how information spreads in such a system.

Unlike for some previously derived solutions that rely on a perturbative approach to treat disorder in a system (e.g., [60]), the solution presented here is analytical and exact. This means that, where the perturbative approach is able to describe the system accurately on small time scales ($t \ll 1$), the approach presented here has, in principle, no such restrictions. From this derivation I will be able to make a statement about the speed of the spread of correlations.

The correlation function of two local operators acting on the same system, σ_j^α and σ_k^β, where $j \neq k$ and $\alpha, \beta \in \{x, y, z, 0\}$, can be expressed as:

$$C_{\sigma_j^\alpha, \sigma_k^\beta}(t) = \langle\Omega| \sigma_j^\alpha(t)\sigma_k^\beta(t) |\Omega\rangle - \langle\Omega| \sigma_j^\alpha(t) |\Omega\rangle \langle\Omega| \sigma_k^\beta(t) |\Omega\rangle . \tag{2.159}$$

Clearly, the expectation of the correlation function up to some order of perturbation $o(h, J)$, $C^{o(h,J)}_{\sigma_j^\alpha, \sigma_k^\beta}$, will be equal to zero for a large enough distance between the particles j and k, $d(j, k)$. For a Hamiltonian of the form

$$H = \sum_j \sigma_j^z \sigma_{j+1}^z + h_j \sigma_j^x + J_j \sigma_j^x \sigma_{j+1}^x, \tag{2.160}$$

and up to o-th order, $\overline{M(t)}$ is $(2o + 3)$-local, i.e.,

$$\mathbb{E}_{h,J}\left(C^{o(h,J)}_{\sigma_j^\alpha, \sigma_k^\beta}(d(j, k) \geq 2(o + 1))\right) = 0. \tag{2.161}$$

Because j and k are fixed, the non-zero terms of this disorder-averaged expectation value can be bounded:

$$\mathcal{O}(\overline{M(t; o(h, J))}) = \mathcal{O}(|M| + \sum_{k=0}^{o} o(h, J)_k) \leq \tag{2.162}$$

$$\leq |M| \cdot \left(1 + \sum_{k=0}^{o} \mathcal{O}\left(\eta_J^{k+1} k!! + \sum_{\zeta=1}^{k} \eta_h^\zeta (\zeta - 1)!! \eta_J^{k+1-\zeta} (k - \zeta)!! + \eta_h^{k+1} k!!\right) \cdot \frac{|t|^{k+1}}{(k + 1)!}\right). \tag{2.163}$$

Then, it is evident that:

$$\mathbb{E}_{h,J}\left(C_{\sigma_j^\alpha, \sigma_k^\beta}\right) \leq \|M\|^2 \cdot \sum_{p+r>d(j,k)}^{\infty} \mathcal{O}\left(\frac{1}{p!!}\right) \cdot \mathcal{O}\left(\frac{1}{r!!}\right) \cdot |t|^{p+r}. \tag{2.164}$$

For some natural number $N > \frac{d(j,k)}{2}$, the right side of this inequality can be written as $\|M\|^2 \cdot \mathcal{O}\left(\frac{1}{(N!!)^2}\right) \cdot |t|^{2N}$. Thus, from $|t|^{2N} \lessapprox (N!!)^2$, it follows that propagation of negligible non-zero correlation terms is bounded by:

$$t \lessapprox \sqrt{\frac{d(j, k)}{2}}, \tag{2.165}$$

as seen in Figure 2.3.

Partition Functions

I will derive the partition function for a system describable by the following, simple

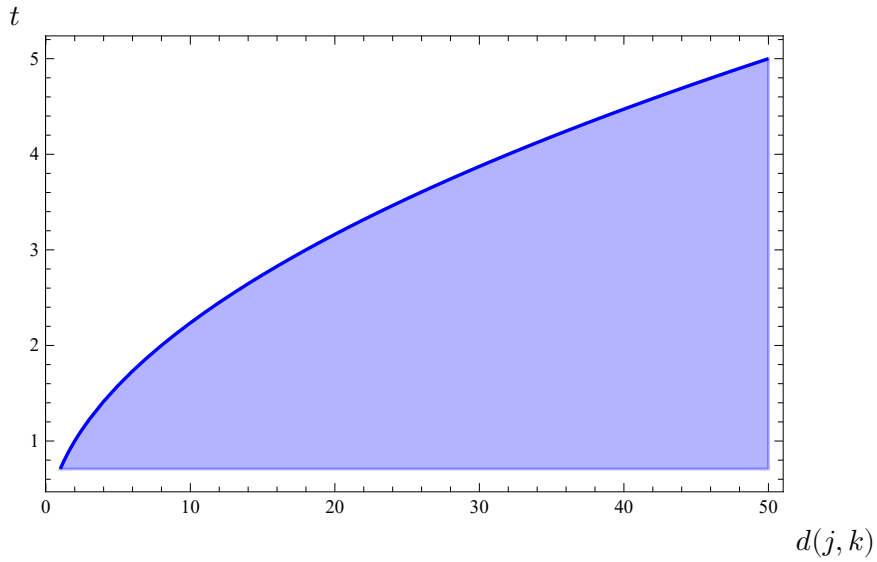


Figure 2.3: Large-time bound for the correlation function in a system described by the many-body Hamiltonian (2.160). On the abscissa the distance $d(j,k)$ is shown and on the ordinate the time t is shown.

Hamiltonian:

$$\begin{aligned} H =& H_0 + H_\nu \\ H_0 =& \sum_{j=1}^{n} \sigma_j^z \\ H_\nu =& \sum_{j=1}^{n} h_j \sigma_j^x \sigma_{j+1}^x \end{aligned} \tag{2.166}$$

The final part of this segment will demonstrate precisely that application.

For this system, the Gibbs measure can be written as:

$$P = \mathbb{E}_h \left(\frac{e^{-\beta(H_0+H_\nu)}}{\mathrm{tr}(e^{-\beta(H_0+H_\nu)})} \right) \approx \mathbb{E}_h \left(\frac{e^{-\beta H_0} + \Delta_1 + \Delta_2}{\mathrm{tr}\,(e^{-\beta H_0} + \Delta_1 + \Delta_2)} \right), \tag{2.167}$$

where $\beta = \frac{1}{k_B T}$ is the thermodynamic parameter.

In this setup, I define Z_0 as:

$$Z_0 = \mathrm{tr}\left(e^{-\beta H_0}\right) = \mathrm{tr}\left(\prod_{j=1}^{n} e^{-\beta \sigma_j^z} \right) = \tag{2.168}$$

$$= \mathrm{tr}\left(e^{-\beta \sigma_j^z}\right)^n, \tag{2.169}$$

where j in the above equation is any $j \in \{1, \dots, n\}$. I also evaluate the denominator:

$$Z \equiv \mathrm{tr}\left(e^{-\beta(H_0+H_\nu)}\right) = \mathrm{tr}\left(e^{-\beta H_0}\sum_{j=0}^{\infty}\frac{(-\beta H_\nu)^j}{j!}\right) = \tag{2.170}$$

$$= \mathrm{tr}\left(e^{-\beta H_0}\left(1-\beta H_\nu+\frac{\beta^2 H_\nu^2}{2!}-\frac{\beta^3 H_\nu^3}{3!}+\dots\right)\right) = \tag{2.171}$$

$$= Z_0 + \sum_{j=1}^{\infty}\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right). \tag{2.172}$$

Following a straightforward calculation[7], it is clear that

$$P = \mathbb{E}_h\left(\frac{e^{-\beta(H_0+H_\nu)}}{\mathrm{tr}(e^{-\beta(H_0+H_\nu)})}\right) = \tag{2.173}$$

$$= \mathbb{E}_h\Bigg(Z_0^{-1}\sum_{l=0}^{\infty}\frac{(-\beta)^l}{l!}e^{-\beta H_0}H_\nu^l + \\ + Z_0^{-1}\sum_{l=0}^{\infty}\frac{(-\beta)^l}{l!}e^{-\beta H_0}H_\nu^l\sum_{k=0}^{\infty}(-1)^k Z_0^{-k}\left(\sum_{j=1}^{\infty}\frac{(-\beta)^j}{j!}\mathrm{tr}\left(e^{-\beta H_0}H_\nu^j\right)\right)^k\Bigg). \tag{2.174}$$

In addition, a general form for the Gibbs measure can be calculated, up to the second order:

$$P = \mathbb{E}_h\Bigg(Z_0^{-1}\left(e^{-\beta H_0}-\beta e^{-\beta H_0}H_\nu+\frac{\beta^2}{2}e^{-\beta H_0}H_\nu^2\right) + \\ + Z_0^{-1}\Big(\Delta(l=0;k=0)+\Delta(l=0;k=1;j=1,2)+\Delta(l=0;k=2;j=1)+ \\ + \Delta(l=1;k=0)+\Delta(l=1;k=1;j=1)+\Delta(l=2;k=0)\Big)\Bigg) = \tag{2.175}$$

$$= \frac{e^{-\beta H_0}}{Z_0}\Bigg(2+\beta^2\eta_h^2\left(\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^x\right)^2 + \frac{\beta}{Z_0}\mathbb{E}_h\left(\mathrm{tr}\left(e^{-\beta H_0}H_\nu\right)\right) - \\ - \frac{\beta^2}{2Z_0}\mathbb{E}_h\left(\mathrm{tr}\left(e^{-\beta H_0}H_\nu^2\right)\right) + \frac{\beta^2}{Z_0^2}\mathbb{E}_h\left(\left(\mathrm{tr}\left(e^{-\beta H_0}H_\nu\right)\right)^2\right) - \frac{\beta^2}{Z_0}\mathbb{E}_h\left(H_\nu\,\mathrm{tr}\left(e^{-\beta H_0}H_\nu\right)\right)\Bigg) \tag{2.176}$$

Any further attempts to evaluate this expression require direct dealings with the system's Hamiltonian.

In the following, I will calculate the specific values for the relevant traces[8]. First, I will derive an expression for the simple case of $\mathrm{tr}\left(e^{-\beta H_0}\right)$, where M is the vector chosen to write the exponential function in the Pauli matrix basis.

$$\mathrm{tr}\left(e^{-\beta H_0}\right) = \mathrm{tr}\left(\prod_{j=1}^{n}e^{-\beta\sigma_j^z}\right) = \tag{2.177}$$

$$= \mathrm{tr}\left(\prod_{j=1}^{n}\sum_{\alpha=0}^{3}M_\alpha\sigma_j^\alpha\right) = \tag{2.178}$$

$$= \sum_{\alpha_1,\alpha_2,\dots,\alpha_n=0}^{3}M_{\alpha_1}M_{\alpha_2}\dots M_{\alpha_n}\,\mathrm{tr}\left(\sigma_1^{\alpha_1}\right)\mathrm{tr}\left(\sigma_2^{\alpha_2}\right)\dots\mathrm{tr}\left(\sigma_n^{\alpha_n}\right) \tag{2.179}$$

[7] The calculation is included explicitly in the Appendix here.
[8] The full calculations are included in the Appendix here.

The vector M can be constructed if the matrix exponential is written as a sum of Pauli matrices. In other words,

$$e^{ib\vec{c}\cdot\vec{\sigma}} = \cos(b)\mathbb{I} + i\sin(b)\vec{c}\cdot\vec{\sigma}, \tag{2.180}$$

where $\vec{\sigma}$ is the Pauli vector. In the case of $e^{-\beta\sigma_j^z}$, $b \to i\beta$ and $\vec{c}\cdot\vec{\sigma} \to \sigma_j^z$, so

$$\begin{aligned} e^{-\beta\sigma_j^z} &= \cos(i\beta)\mathbb{I} + i\sin(i\beta)\sigma_j^z = && (2.181)\\ &= \cosh(\beta)\mathbb{I} - \sinh(\beta)\sigma_j^z. && (2.182) \end{aligned}$$

Evidently, the only non-zero parameters in the vector M are:

$$\begin{aligned} M_0 &= \cosh(\beta) && (2.183)\\ M_3 &= -\sinh(\beta). && (2.184) \end{aligned}$$

The traces of the corresponding matrices are

$$\begin{aligned} \mathrm{tr}\left(\mathbb{I}_j\right) &= 2 && (2.185)\\ \mathrm{tr}\left(\sigma_j^z\right) &= 0. && (2.186) \end{aligned}$$

Then, the product in (2.179) is different than zero only in the case when $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. It follows that

$$\begin{aligned} \mathrm{tr}\left(e^{-\beta H_0}\right) &= \left(\cosh\left(\beta\right)\right)^n \cdot 2^n = && (2.187)\\ &= \left(2\cosh\left(\beta\right)\right)^n && (2.188) \end{aligned}$$

Then, for $\mathrm{tr}\left(e^{-\beta H_0} H_\nu^j\right)$, it follows:

$$\mathrm{tr}\left(e^{-\beta H_0} H_\nu^j\right) = \mathrm{tr}\left(\prod_{j=1}^{n} e^{-\beta\sigma_j^z} \cdot \sum_{k=1}^{n} \left(h_k \sigma_k^x \sigma_{k+1}^x\right)^j\right) = \tag{2.189}$$

$$= \mathrm{tr}\left(\prod_{j=1}^{n} \left(M_0\mathbb{I}_j + M_3\sigma_j^z\right) \cdot \sum_{k=1}^{n} h_k^j \left(\sigma_k^x \sigma_{k+1}^x\right)^j\right) = \tag{2.190}$$

$$\begin{aligned} &= \sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} \cdot \\ &\cdot h_k^j \cdot \ldots \cdot \mathrm{tr}\left(\sigma_{k-1}^{\alpha_{k-1}}\right) \cdot \mathrm{tr}\left(\sigma_k^{\alpha_k} \cdot \left(\sigma_k^x\right)^j\right) \cdot \mathrm{tr}\left(\sigma_{k+1}^{\alpha_{k+1}} \cdot \left(\sigma_{k+1}^x\right)^j\right) \cdot \mathrm{tr}\left(\sigma_{k+2}^{\alpha_{k+2}}\right) \cdot \ldots && (2.191) \end{aligned}$$

The arguments of the traces give:

$$\begin{aligned} \mathbb{I} \cdot \left(\sigma_k^x\right)^j &= \left(\sigma_k^x\right)^j && (2.192)\\ \sigma_k^z \cdot \sigma_k^x &= i\sigma_k^y && (2.193)\\ \sigma_k^z \cdot \left(\sigma_k^x\right)^2 &= \sigma_k^z \cdot \mathbb{I} = \sigma_k^z && (2.194)\\ \sigma_k^z \cdot \left(\sigma_k^x\right)^3 &= \sigma_k^z \cdot \sigma_k^x = i\sigma_k^y && (2.195) \end{aligned}$$

and

$$\begin{aligned} \sigma_k^0 \cdot \left(\sigma_k^x\right)^{2j} &= \mathbb{I} && (2.196)\\ \sigma_k^0 \cdot \left(\sigma_k^x\right)^{2j-1} &= \sigma_k^x && (2.197)\\ \sigma_k^z \cdot \left(\sigma_k^x\right)^{2j} &= \sigma_k^z && (2.198)\\ \sigma_k^z \cdot \left(\sigma_k^x\right)^{2j-1} &= i\sigma_k^y, && (2.199) \end{aligned}$$

which leads to:

$$\operatorname{tr}\left(e^{-\beta H_0} H_\nu^0\right) = \left(2\cosh\left(\beta\right)\right)^n \tag{2.200}$$

and, as an example,

$$\begin{aligned} &M_{\alpha_1} M_{\alpha_2} \dots M_{\alpha_n} \cdot \left(\sum_{k=1}^{n} h_k^j\right) \cdot \Big(i \operatorname{tr}(\sigma_1^y) \cdot i \operatorname{tr}(\sigma_2^y) \cdot \operatorname{tr}(\sigma_3^{\alpha_3}) \cdot \dots \operatorname{tr}(\sigma_n^{\alpha_n}) + \\ &+ \operatorname{tr}(\sigma_1^{\alpha_1}) \cdot i \operatorname{tr}(\sigma_2^y) \cdot i \operatorname{tr}(\sigma_3^y) \cdot \operatorname{tr}(\sigma_4^{\alpha_4}) \cdot \dots \cdot \operatorname{tr}(\sigma_n^{\alpha_n}) + \dots + \\ &+ \operatorname{tr}(\sigma_1^{\alpha_1}) \cdot \dots \cdot i \operatorname{tr}(\sigma_{n-1}^y) \cdot i \operatorname{tr}(\sigma_n^y) + i \operatorname{tr}(\sigma_1^y) \cdot \operatorname{tr}(\sigma_2^{\alpha_2}) \cdot \dots \operatorname{tr}(\sigma_{n-1}^{\alpha_{n-1}}) \cdot i \operatorname{tr}(\sigma_n^y)\Big) = \end{aligned} \tag{2.201}$$

$$\begin{aligned} =&M_{\alpha_1} M_{\alpha_2} \dots M_{\alpha_n} \cdot \left(-\sum_{k=1}^{n} h_k^j\right) \cdot \Big(\operatorname{tr}(\sigma_1^y) \cdot \operatorname{tr}(\sigma_2^y) \cdot \operatorname{tr}(\sigma_3^{\alpha_3}) \cdot \dots \operatorname{tr}(\sigma_n^{\alpha_n}) + \\ &+ \operatorname{tr}(\sigma_1^{\alpha_1}) \cdot \operatorname{tr}(\sigma_2^y) \cdot \operatorname{tr}(\sigma_3^y) \cdot \operatorname{tr}(\sigma_4^{\alpha_4}) \cdot \dots \cdot \operatorname{tr}(\sigma_n^{\alpha_n}) + \dots + \\ &+ \operatorname{tr}(\sigma_1^{\alpha_1}) \cdot \dots \cdot \operatorname{tr}(\sigma_{n-1}^y) \cdot \operatorname{tr}(\sigma_n^y) + \operatorname{tr}(\sigma_1^y) \cdot \operatorname{tr}(\sigma_2^{\alpha_2}) \cdot \dots \operatorname{tr}(\sigma_{n-1}^{\alpha_{n-1}}) \cdot \operatorname{tr}(\sigma_n^y)\Big) = \end{aligned} \tag{2.202}$$

$$=0 \tag{2.203}$$

Thus, for $j = 2$ it can be written:

$$\begin{aligned} &\operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right) = M_0 M_0 \dots M_0 \cdot \Big(h_1^2 \cdot \operatorname{tr}(\mathbb{I}_1) \cdot \operatorname{tr}(\mathbb{I}_2) \cdot \operatorname{tr}(\mathbb{I}_3) \cdot \dots \operatorname{tr}(\mathbb{I}_n) + \\ &+ h_2^2 \cdot \operatorname{tr}(\mathbb{I}_1) \cdot \operatorname{tr}(\mathbb{I}_2) \cdot \operatorname{tr}(\mathbb{I}_3) \cdot \operatorname{tr}(\mathbb{I}_4) \cdot \dots \cdot \operatorname{tr}(\mathbb{I}_n) + \dots + \\ &+ h_{n-1}^2 \cdot \operatorname{tr}(\mathbb{I}_1) \cdot \dots \cdot \operatorname{tr}(\mathbb{I}_{n-1}) \cdot \operatorname{tr}(\mathbb{I}_n) + h_n^2 \cdot \operatorname{tr}(\mathbb{I}_1) \cdot \operatorname{tr}(\mathbb{I}_2) \cdot \dots \operatorname{tr}(\mathbb{I}_{n-1}) \cdot \operatorname{tr}(\mathbb{I}_n)\Big) = \end{aligned} \tag{2.204}$$

$$=M_0^n \cdot 2^n \cdot \sum_{l=1}^{n} h_l^2 = \tag{2.205}$$

$$= \left(2\cosh(\beta)\right)^n \cdot \sum_{l=1}^{n} h_l^2, \tag{2.206}$$

so for all $j \in \mathbb{N}$ it holds that:

$$\operatorname{tr}\left(e^{-\beta H_0}\right) = \left(2\cosh(\beta)\right)^n \tag{2.207}$$

$$\operatorname{tr}\left(e^{-\beta H_0} H_\nu^{2j}\right) = \left(2\cosh(\beta)\right)^n \cdot \sum_{l=1}^{n} h_l^{2j} \tag{2.208}$$

$$\operatorname{tr}\left(e^{-\beta H_0} H_\nu^{2j-1}\right) =0. \tag{2.209}$$

Thus, the expression for the Gibbs measure from (2.176) can be completed[9]:

$$\begin{aligned} P =&\frac{e^{-\beta H_0}}{Z_0}\left(2 + \beta^2 \eta_h^2 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + \frac{\beta}{Z_0}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) - \frac{\beta^2}{2Z_0}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) + \right. \\ &\left. + \frac{\beta^2}{Z_0^2}\mathbb{E}_h\left(\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2\right) - \frac{\beta^2}{Z_0}\mathbb{E}_h\left(H_\nu \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)\right) \end{aligned} \tag{2.210}$$

$$=\frac{e^{-\beta H_0}}{Z_0}\left(2 + \beta^2 \eta_h^2 \left(\sum_{j=1}^{n} \sigma_j^x \sigma_{j+1}^x - \frac{n}{2Z_0} \cdot \left(2\cosh(\beta)\right)^n\right)\right). \tag{2.211}$$

[9] See details in the Appendix here.

The expectation values of other observables can now be calculated fairly easily. For example, for $\mathbb{E}(Z)$ it follows in a straightforward manner that:

$$\mathbb{E}_h(Z) = \mathbb{E}_h(Z_0) + \mathbb{E}_h\left(\sum_{j=1}^{\infty} \mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0} H_\nu^j}{j!}\right)\right) = \tag{2.212}$$

$$= \mathrm{tr}\left(e^{-\beta\sigma_j^z}\right)^n + \sum_{j=1}^{\infty} \frac{(-\beta)^{2j}}{(2j)!} \mathbb{E}_h\left((2\cosh(\beta))^n \cdot \sum_{l=1}^{n} h_l^{2j}\right) = \tag{2.213}$$

$$= (2\cosh(\beta))^n + (2\cosh(\beta))^n \cdot n \cdot (\cosh(\beta\eta_h) - 1). \tag{2.214}$$

However, in the case in which H_0 and H_ν do not commute, it is pertinent to use the Lie product formula to derive the correct expressions:

$$e^{X+Y} = \lim_{\tau\to\infty}\left(e^{\frac{X}{\tau}} e^{\frac{Y}{\tau}}\right)^\tau \tag{2.215}$$

$$e^{H_0+H_\nu} = \lim_{\tau\to\infty}\left(e^{\frac{H_0}{\tau}} e^{\frac{H_\nu}{\tau}}\right)^\tau. \tag{2.216}$$

However, as this general case is much more complicated to resolve and not intrinsically pertinent to this demonstration, it will not be included here.

Further, for a slightly more complicated version of the full Hamiltonian, for which it holds that:

$$H = H_0 + H_\nu \tag{2.217}$$

$$H_0 = \sum_{j=1}^{N} \sigma_j^z \sigma_{j+1}^z \tag{2.218}$$

$$H_\nu = \sum_{j=1}^{N} h_j \sigma_j^z, \tag{2.219}$$

a similar calculation can be completed to obtain the Gibbs measure:

$$P = \frac{e^{-\beta H_0}}{Z_0}\left(2 + \beta^2 (H_\nu)^2 + \frac{\beta}{Z_0}\mathbb{E}_h\left(\mathrm{tr}\left(e^{-\beta H_0} H_\nu\right)\right) - \frac{\beta^2}{2Z_0}\mathbb{E}_h\left(\mathrm{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) + \right.$$
$$\left. + \frac{\beta^2}{Z_0^2}\mathbb{E}_h\left(\left(\mathrm{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2\right) - \frac{\beta^2}{Z_0}\mathbb{E}_h\left(H_\nu \,\mathrm{tr}\left(e^{-\beta H_0} H_\nu\right)\right)\right). \tag{2.220}$$

First, I will consider the case in which $j = 0$.

$$\mathrm{tr}\left(e^{-\beta H_0}\right) = \mathrm{tr}\left(\prod_{j=1}^{n} e^{-\beta\sigma_j^z\sigma_{j+1}^z}\right) = \tag{2.221}$$

$$= \mathrm{tr}\left(\prod_{j=1}^{n}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3} N_{\gamma\delta}\sigma_j^\gamma\sigma_{j+1}^\delta\right) \tag{2.222}$$

By using the already-derived expression (2.180), it is easy to assign correct values to $N_{\gamma\delta}$. Thus, from

$$e^{-\beta\sigma_j^z\sigma_{j+1}^z} = \cosh(\beta)\mathbb{I} - \sinh(\beta)\sigma_j^z\sigma_{j+1}^z \tag{2.223}$$

it follows that all $N_{\gamma\delta}$ are equal to zero, unless $(\gamma,\delta) = \{(0,0),(z,z)\}$. In those cases,

$$N_{00} = \cosh(\beta) \tag{2.224}$$

$$N_{zz} = -\sinh(\beta). \tag{2.225}$$

Thus, the calculation can proceed as shown below.

$$\begin{aligned}\operatorname{tr}\left(\prod_{j=1}^{n} e^{-\beta\sigma_j^z\sigma_{j+1}^z}\right) &= \operatorname{tr}\Big(N_{00}\sigma_1^0\otimes\sigma_2^0\cdot N_{00}\sigma_2^0\otimes\sigma_3^0\cdot\ldots\cdot N_{00}\sigma_n^0\otimes\sigma_1^0+\\ &+N_{00}\sigma_1^0\otimes\sigma_2^0\cdot N_{00}\sigma_2^0\otimes\sigma_3^0\cdot\ldots\cdot N_{0x}\sigma_n^0\otimes\sigma_1^x+\ldots+\\ &+N_{00}\sigma_1^0\otimes\sigma_2^0\cdot N_{00}\sigma_2^0\otimes\sigma_3^0\cdot\ldots\cdot N_{zz}\sigma_n^z\otimes\sigma_1^z+\ldots+\\ &+N_{zz}\sigma_1^z\otimes\sigma_2^z\cdot N_{zz}\sigma_2^z\otimes\sigma_3^z\cdot\ldots\cdot N_{zz}\sigma_n^z\otimes\sigma_1^z\Big)= \end{aligned} \tag{2.226}$$

$$= \sum_{\substack{\gamma_1,\ldots,\gamma_n=0\\ \delta_1,\ldots,\delta_n=0}}^{3} \operatorname{tr}\left(N_{\gamma_1\delta_1}N_{\gamma_2\delta_2}\ldots N_{\gamma_n\delta_n}\left(\sigma_1^{\gamma_1}\otimes\sigma_2^{\delta_1}\right)\cdot\left(\sigma_2^{\gamma_2}\otimes\sigma_3^{\delta_2}\right)\cdot\ldots\cdot\left(\sigma_n^{\gamma_n}\otimes\sigma_1^{\delta_n}\right)\right)= \tag{2.227}$$

$$= \sum_{\substack{\gamma_1,\ldots,\gamma_n=0\\ \delta_1,\ldots,\delta_n=0}}^{3} N_{\gamma_1\delta_1}N_{\gamma_2\delta_2}\ldots N_{\gamma_n\delta_n}\operatorname{tr}\left(\sigma_1^{\gamma_1}\sigma_1^{\delta_n}\otimes\sigma_2^{\delta_1}\sigma_2^{\gamma_2}\otimes\sigma_3^{\delta_2}\sigma_3^{\gamma_3}\otimes\ldots\otimes\sigma_n^{\delta_{n-1}}\sigma_n^{\gamma_n}\right)= \tag{2.228}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}} N_{\alpha_1\alpha_1}N_{\alpha_2\alpha_2}\ldots N_{\alpha_n\alpha_n}\operatorname{tr}\left(\sigma_1^{\alpha_n}\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\sigma_3^{\alpha_3}\otimes\ldots\otimes\sigma^{\alpha_{n-1}}\sigma_n^{\alpha_n}\right)= \tag{2.229}$$

$$\begin{aligned}=&N_{00}N_{00}\ldots N_{00}N_{zz}\operatorname{tr}\left(\sigma_1^z\sigma_1^0\otimes\sigma_2^0\sigma_2^0\otimes\sigma_3^0\sigma_3^0\otimes\ldots\otimes\sigma_n^0\sigma_n^z\right)+\\ &+N_{00}N_{00}\ldots N_{00}N_{zz}N_{00}\operatorname{tr}\left(\sigma_1^0\sigma_1^0\otimes\sigma_2^0\sigma_2^0\otimes\sigma_3^0\sigma_3^0\otimes\ldots\otimes\sigma_{n-1}^0\sigma_{n-1}^z\otimes\sigma_n^0\sigma_n^0\right)+\\ &+N_{00}N_{00}\ldots N_{00}N_{zz}N_{zz}\operatorname{tr}\left(\sigma_1^z\sigma_1^0\otimes\sigma_2^0\sigma_2^0\otimes\sigma_3^0\sigma_3^0\otimes\ldots\otimes\sigma_{n-1}^z\sigma_{n-1}^z\otimes\sigma_n^0\sigma_n^z\right)\end{aligned} \tag{2.230}$$

The traces of the combinations appearing in the expression are:

$$\operatorname{tr}\left(\sigma_j^0\sigma_j^z\right)=0, \tag{2.231}$$

$$\operatorname{tr}\left(\sigma_j^0\sigma_j^0\right)=\operatorname{tr}\left(\sigma_j^z\sigma_j^z\right)=2. \tag{2.232}$$

This means that the only two cases in which these summands do not equal zero are those where all the α_j are equal:

$$\operatorname{tr}\left(\prod_{j=1}^{n} e^{-\beta\sigma_j^z\sigma_{j+1}^z}\right)=$$

$$=N_{00}^n\operatorname{tr}\left(\sigma_1^0\sigma_1^0\otimes\sigma_2^0\sigma_2^0\otimes\ldots\otimes\sigma_n^0\sigma_n^0\right)+N_{zz}^n\operatorname{tr}\left(\sigma_1^z\sigma_1^z\otimes\sigma_2^z\sigma_2^z\otimes\ldots\otimes\sigma_n^z\sigma_n^z\right)= \tag{2.233}$$

$$=N_{00}^n\cdot 2^n+N_{zz}^n\cdot 2^n= \tag{2.234}$$

$$=2^n\cdot\left((\cosh(\beta))^n+(-\sinh(\beta))^n\right). \tag{2.235}$$

Including this expression in the derivation gives the following.

$$\mathrm{tr}\left(e^{-\beta H_0}H_\nu\right) = \mathrm{tr}\left(\prod_{j=1}^{n} e^{-\beta\sigma_j^z\sigma_{j+1}^z}\cdot\sum_{k=1}^{n}h_k\sigma_k^z\right) = \tag{2.236}$$

$$= \mathrm{tr}\Big(\left(N_{00}\mathbb{I}_{1,2}+N_{zz}\sigma_1^z\otimes\sigma_2^z\right)\cdot\left(N_{00}\mathbb{I}_{2,3}+N_{zz}\sigma_2^z\otimes\sigma_3^z\right)\cdot\ldots\cdot\left(N_{00}\mathbb{I}_{n-1,n}+N_{zz}\sigma_{n-1}^z\otimes\sigma_n^z\right)\cdot$$
$$\cdot\left(N_{00}\mathbb{I}_{n,1}+N_{zz}\sigma_n^z\otimes\sigma_1^z\right)\cdot\left(h_1\sigma_1^z+h_2\sigma_2^z+\ldots+h_{n-1}\sigma_{n-1}^z+h_n\sigma_n^z\right)\Big) = \tag{2.237}$$

$$= \mathrm{tr}\Big(\sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}}\Big(N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_n}\left(\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\right)\cdot\left(\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\right)\cdot\ldots\cdot$$
$$\cdot\left(\sigma_{n-1}^{\alpha_{n-1}}\otimes\sigma_n^{\alpha_{n-1}}\right)\cdot\left(\sigma_n^{\alpha_n}\otimes\sigma_1^{\alpha_n}\right)\Big)\cdot\left(h_1\sigma_1^z+h_2\sigma_2^z+\ldots+h_{n-1}\sigma_{n-1}^z+h_n\sigma_n^z\right)\Big) = \tag{2.238}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}}\sum_{j=1}^{n}\mathrm{tr}\Big(N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_n}h_j\cdot$$
$$\cdot\left(\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\right)\cdot\left(\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\right)\cdot\ldots\cdot\left(\sigma_{n-1}^{\alpha_{n-1}}\otimes\sigma_n^{\alpha_{n-1}}\right)\cdot\left(\sigma_n^{\alpha_n}\otimes\sigma_1^{\alpha_n}\right)\cdot\sigma_j^z\Big) = \tag{2.239}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}}\sum_{j=1}^{n}\mathrm{tr}\Big(N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_n}h_j\cdot$$
$$\cdot\left(\sigma_1^{\alpha_n}\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\sigma_3^{\alpha_3}\otimes\ldots\otimes\sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}}\otimes\sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}\right)\cdot\sigma_j^z\Big). \tag{2.240}$$

The only case in which this trace is not equal to zero is when all α's are equal, because the σ_j^z only affects one of the products — in all cases there will be at least one part of the trace product that is going to be equal to $\mathrm{tr}\left(\sigma^z\right)=0$. Thus, for $j=0$ it follows that:

$$\mathrm{tr}\left(e^{-\beta H_0}H_\nu\right) = 0. \tag{2.241}$$

I will now present the calculation for the second order, with $j = 2$.

$$\mathrm{tr}\left(e^{-\beta H_0} H_\nu^2\right) = \mathrm{tr}\Big(\sum_{\alpha_1,...,\alpha_n=\{0,z\}} N_{\alpha_1} N_{\alpha_2} ... N_{\alpha_n} (\sigma_1^{\alpha_n}\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_1}\sigma_2^{\alpha_2} \otimes \sigma_3^{\alpha_2}\sigma_3^{\alpha_3} \otimes ... \otimes$$
$$\otimes \sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}} \otimes \sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}) \cdot \left(\sum_{j=1}^{n} h_j \sigma_j^z\right)^2 \Big) = \tag{2.242}$$

$$= \sum_{\alpha_1,...,\alpha_n=\{0,z\}} \mathrm{tr}\Big(N_{\alpha_1} N_{\alpha_2} ... N_{\alpha_n} (\sigma_1^{\alpha_n}\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_1}\sigma_2^{\alpha_2} \otimes \sigma_3^{\alpha_2}\sigma_3^{\alpha_3} \otimes ... \otimes \sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}} \otimes \sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}) \cdot$$
$$\cdot (h_1\sigma_1^z + h_2\sigma_2^z + ... + h_n\sigma_n^z) \cdot (h_1\sigma_1^z + h_2\sigma_2^z + ... + h_n\sigma_n^z)\Big) = \tag{2.243}$$

$$= \sum_{\alpha_1,...,\alpha_n=\{0,z\}} N_{\alpha_1} N_{\alpha_2} ... N_{\alpha_n} \cdot$$
$$\cdot \mathrm{tr}\left((\sigma_1^{\alpha_n}\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_1}\sigma_2^{\alpha_2} \otimes \sigma_3^{\alpha_2}\sigma_3^{\alpha_3} \otimes ... \otimes \sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}} \otimes \sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}) \cdot \sum_{j,k=1}^{n} h_j h_k \sigma_j^z \sigma_k^z \right) = \tag{2.244}$$

$$= \sum_{j,k=1}^{n} \sum_{\alpha_1,...,\alpha_n=\{0,z\}} N_{\alpha_1} N_{\alpha_2} ... N_{\alpha_n} \cdot$$
$$\cdot \mathrm{tr}\left((\sigma_1^{\alpha_n}\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_1}\sigma_2^{\alpha_2} \otimes \sigma_3^{\alpha_2}\sigma_3^{\alpha_3} \otimes ... \otimes \sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}} \otimes \sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}) \cdot h_j h_k \sigma_j^z \sigma_k^z \right) = \tag{2.245}$$

$$= \sum_{j=1}^{n} \sum_{\alpha_1,...,\alpha_n=\{0,z\}} N_{\alpha_1} N_{\alpha_2} ... N_{\alpha_n} \cdot$$
$$\cdot \mathrm{tr}\Big((\sigma_1^{\alpha_n}\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_1}\sigma_2^{\alpha_2} \otimes \sigma_3^{\alpha_2}\sigma_3^{\alpha_3} \otimes ... \otimes \sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}} \otimes \sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}) \cdot$$
$$\cdot \left(h_j^2 \mathbb{I}_j + h_j h_{j+1} \sigma_j^z \otimes \sigma_{j+1}^z + ... + h_j h_{j-1} \sigma_j^z \otimes \sigma_{j-1}^z\right)\Big) \tag{2.246}$$

Each of the combinations from the brackets gives a non-zero trace just for two specific combinations of α's, which means that the expression can be rewritten as follows.

$$\mathrm{tr}\left(e^{-\beta H_0} H_\nu^2\right) = \sum_{j=1}^{n} \Big(h_j^2 \left(N_{00}^n \cdot 2^n + N_{zz}^n \cdot 2^n\right) + h_j h_{j+1} \cdot 2^n \cdot \left(N_{00}^{n-1} N_{zz} + N_{00} N_{zz}^{n-1}\right) +$$
$$+ 2^n h_j h_{j+2} \left(N_{00}^{n-2} N_{zz}^2 + N_{00}^2 N_{zz}^{n-2}\right) + ... + 2^n h_j h_{j-1} \left(N_{00}^{n-1} N_{zz} + N_{00} N_{zz}^{n-1}\right)\Big) = \tag{2.247}$$

$$= 2^n n \cdot \Big(h_j^2 \left(N_{00}^n + N_{zz}^n\right) + h_j h_{j+1} \cdot \left(N_{00}^{n-1} N_{zz} + N_{00} N_{zz}^{n-1}\right) +$$
$$+ h_j h_{j+2} \left(N_{00}^{n-2} N_{zz}^2 + N_{00}^2 N_{zz}^{n-2}\right) + ... + h_j h_{j-1} \left(N_{00}^{n-1} N_{zz} + N_{00} N_{zz}^{n-1}\right)\Big) \tag{2.248}$$

In conclusion, for $j = 2$, the needed contribution equals:

$$\mathbb{E}_h \left(\mathrm{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) = 2^n n \eta_h^2 \cdot \left(\left(\cosh(\beta)\right)^n + \left(-\sinh(\beta)\right)^n\right). \tag{2.249}$$

It is now evident that the higher orders can also be obtained in a straightforward pro-

cedure. For illustration, I will include the calculation of the contribution for $j = 4$.

$$\begin{aligned}\mathrm{tr}\left(e^{-\beta H_0}H_\nu^4\right) = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}} N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_n}\cdot \\ \mathrm{tr}\left(\left(\sigma_1^{\alpha_n}\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\sigma_3^{\alpha_3}\otimes\ldots\otimes\sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}}\otimes\sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}\right)\cdot\right. \\ \cdot\left(h_j^2\mathbb{I}_j + h_jh_{j+1}\sigma_j^z\otimes\sigma_{j+1}^z + \ldots + h_jh_{j-1}\sigma_j^z\otimes\sigma_{j-1}^z\right)\cdot \\ \left.\cdot\left(h_k^2\mathbb{I}_k + h_kh_{k+1}\sigma_k^z\otimes\sigma_{k+1}^z + \ldots + h_kh_{k-1}\sigma_k^z\otimes\sigma_{k-1}^z\right)\right)\end{aligned} \tag{2.250}$$

Naturally, the calculation flows as the one for the lower orders.

$$\begin{aligned}\mathbb{E}_h\left(e^{-\beta H_0}H_\nu^4\right) = \mathbb{E}_h\left(\sum_{j=1}^{n}\sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}} N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_n}\cdot\right. \\ \cdot\,\mathrm{tr}\left(\left(\sigma_1^{\alpha_n}\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\sigma_3^{\alpha_3}\otimes\ldots\otimes\sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}}\otimes\sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}\right)\cdot\right. \\ \cdot\left(\left(h_j^4\mathbb{I} + h_j^2h_{j+1}^2\mathbb{I} + \ldots + h_j^2h_{j-1}^2\mathbb{I}\right)\right) + \left(h_j^2h_{j+1}^2\mathbb{I} + h_j^2h_{j+1}^2\mathbb{I}\right) + \\ \left.\left.+\left(h_j^2h_{j+2}^2\mathbb{I} + h_j^2h_{j+2}^2\mathbb{I}\right) + \ldots + \left(h_j^2h_{j-1}^2\mathbb{I} + h_j^2h_{j-1}^2\mathbb{I}\right)\right)\right) =\end{aligned} \tag{2.251}$$

$$\begin{aligned}=\mathbb{E}_h\left(\sum_{j=1}^{n}\sum_{\alpha_1,\ldots,\alpha_n=\{0,z\}} N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_n}\left(\left(\left(h_j^4\mathbb{I} + h_j^2h_{j+1}^2\mathbb{I} + \ldots + h_j^2h_{j-1}^2\mathbb{I}\right)\right) + \right.\right. \\ \left.+\left(h_j^2h_{j+1}^2\mathbb{I} + h_j^2h_{j+1}^2\mathbb{I}\right) + \left(h_j^2h_{j+2}^2\mathbb{I} + h_j^2h_{j+2}^2\mathbb{I}\right) + \ldots + \left(h_j^2h_{j-1}^2\mathbb{I} + h_j^2h_{j-1}^2\mathbb{I}\right)\right)\cdot \\ \left.\left.\cdot\,\mathrm{tr}\left(\left(\sigma_1^{\alpha_n}\sigma_1^{\alpha_1}\otimes\sigma_2^{\alpha_1}\sigma_2^{\alpha_2}\otimes\sigma_3^{\alpha_2}\sigma_3^{\alpha_3}\otimes\ldots\otimes\sigma_{n-1}^{\alpha_{n-2}}\sigma_{n-1}^{\alpha_{n-1}}\otimes\sigma_n^{\alpha_{n-1}}\sigma_n^{\alpha_n}\right)\right)\right) =\end{aligned} \tag{2.252}$$

$$\begin{aligned}=\mathbb{E}_h\left(\sum_{j=1}^{n}\left(\left(\left(h_j^4\mathbb{I} + h_j^2h_{j+1}^2\mathbb{I} + \ldots + h_j^2h_{j-1}^2\mathbb{I}\right)\right) + \left(h_j^2h_{j+1}^2\mathbb{I} + h_j^2h_{j+1}^2\mathbb{I}\right) + \right.\right. \\ \left.\left.+\left(h_j^2h_{j+2}^2\mathbb{I} + h_j^2h_{j+2}^2\mathbb{I}\right) + \ldots + \left(h_j^2h_{j-1}^2\mathbb{I} + h_j^2h_{j-1}^2\mathbb{I}\right)\right)\cdot 2^n\cdot\left(N_{00}^n + N_{zz}^n\right)\right) =\end{aligned} \tag{2.253}$$

$$=\sum_{j=1}^{n} 2^n\cdot\left(N_{00}^n + N_{zz}^n\right)\cdot\left(\left(\eta_h^4\cdot 3!! + \eta_h^4 + \eta_h^4 + \ldots + \eta_h^4\right) + 2\eta_h^4 + \ldots 2\eta_h^4\right) = \tag{2.254}$$

$$=\sum_{j=1}^{n} 2^n\cdot\left(N_{00}^n + N_{zz}^n\right)\cdot\left(\eta_h^4\cdot 3!! + (n-1)\eta_h^4 + (n-1)2\eta_h^4\right) = \tag{2.255}$$

$$=\sum_{j=1}^{n} 2^n\cdot\left(N_{00}^n + N_{zz}^n\right)\cdot\left(3\eta_h^4 + 3(n-1)\eta_h^4\right) = \tag{2.256}$$

$$=2^n\cdot 3n\eta_h^4\left(N_{00}^n + N_{zz}^n\right) \tag{2.257}$$

What I have obtained for the Gibbs measure in this case is then:

$$P = \frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\left(H_\nu\right)^2+\frac{\beta}{Z_0}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)\right)-\frac{\beta^2}{2Z_0}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu^2\right)\right)+\right.$$
$$\left.+\frac{\beta^2}{Z_0^2}\mathbb{E}_h\left(\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)\right)^2\right)-\frac{\beta^2}{Z_0}\mathbb{E}_h\left(H_\nu\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)\right)\right)= \tag{2.258}$$
$$=\frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\left(\sum_{j=1}^n h_j\sigma_j^z\right)^2-\frac{\beta^2}{2Z_0}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu^2\right)\right)\right)= \tag{2.259}$$
$$=\frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\sum_{j,k=1}^n h_jh_k\sigma_j^z\sigma_k^z-\frac{\beta^2}{2Z_0}\cdot n\eta_h^2\cdot\left(N_{00}^n\cdot 2^n+N_{zz}^n\cdot 2^n\right)\right)= \tag{2.260}$$
$$=\frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\sum_{j,k=1}^n h_jh_k\sigma_j^z\sigma_k^z-\frac{\beta^2\cdot 2^n n\eta_h^2}{2Z_0}\cdot\left(\cosh(\beta)^n+(-\sinh(\beta))^n\right)\right)= \tag{2.261}$$
$$=2\frac{e^{-\beta H_0}}{Z_0}+\frac{e^{-\beta H_0}\beta}{Z_0}\sum_{j,k=1}^n h_jh_k\sigma_j^z\sigma_k^z-\frac{e^{-\beta H_0}\beta^2\cdot 2^n n\eta_h^2}{2Z_0^2}\cdot\left(\cosh(\beta)^n+(-\sinh(\beta))^n\right) \tag{2.262}$$

As an illustrative example, I can also obtain $\mathbb{E}\left(Z\right)$:

$$\mathbb{E}_h\left(Z\right)=\mathbb{E}_h\left(Z_0+\sum_{j=1}^\infty\operatorname{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)= \tag{2.263}$$
$$=\mathbb{E}_h\left(Z_0\right)+\mathbb{E}_h\left(\sum_{j=1}^\infty\operatorname{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)= \tag{2.264}$$
$$=\operatorname{tr}\left(e^{-\beta H_0}\right)+\sum_{j=1}^\infty\frac{(-\beta)^j}{j!}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu^j\right)\right)= \tag{2.265}$$
$$=\operatorname{tr}\left(e^{-\beta H_0}\right)+\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}\right)\right)-\beta\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)\right)+$$
$$+\frac{\beta^2}{2!}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu^2\right)\right)-\frac{\beta^3}{3!}\mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0}H_\nu^3\right)\right)+...= \tag{2.266}$$
$$=2^n\cdot\left(N_{00}^n+N_{zz}^n\right)+2^n\cdot\left(N_{00}^n+N_{zz}^n\right)+\frac{\beta^2}{2!}\cdot 2^n n\eta_h^2\cdot\left(N_{00}^n+N_{zz}^n\right)+$$
$$+\frac{\beta^4}{4!}\cdot 2^n\cdot 3n\eta_h^4\left(N_{00}^n+N_{zz}^n\right)+... \tag{2.267}$$

2.2 Stochastic Derivation of Disorder Averages

To supplement and improve on the derived solutions, I will provide calculations to describe the dynamics of interacting quantum many-body systems in the presence of disorder. The following derivations give an Itô stochastic integral [61] representation for a number of disorder-averaged quantities, which can then reliably be approximated analytically, but also combined with existing analysis methods in the tensor network framework, giving a strong new insight into the characteristics and behavior of many-body systems that are often too difficult to accurately analyze by only numerical methods. The integral expansion gives a series of approximations, the first of which already includes all diffusive corrections and is (manifestly) completely positive. The addition of fluctuations leads to a convergent series of systematic corrections.

2.2.1 Derivation of a Stochastic Integral Representation

To calculate a general expression for disorder averages of dynamical processes in disordered many-body systems, consider a generalized Hamiltonian for a system of m particles, of the type

$$H(\mathbf{x}) = H_0 + \sum_{j=1}^{m} x_j D_j, \tag{2.268}$$

where H_0 is a fixed Hamiltonian (typically a kinetic energy term), D_j represent disordered terms (e.g., a local magnetic field or potential energy terms), and x_j are random variables drawn from the Gaussian distribution with the probability density function

$$g_\gamma(x) \equiv \frac{e^{-\frac{x^2}{2\gamma^2}}}{\sqrt{2\pi}\gamma},$$

where γ is its standard deviation.

To start with, as a prototype, we consider a system with only a single disorder term

$$H(x) = A + xB, \tag{2.269}$$

where A and B are $n \times n$ matrices and $x \in \mathbb{R}$ is a parameter chosen randomly from the Gaussian distribution with the variance of γ^2. In further calculations, it will prove to be useful to derive an expression for the disorder-averaged propagator,

$$S_\gamma(t) = e^{itA} e^{-\frac{\gamma^2 t^2}{2} B^2}. \tag{2.270}$$

First, I will write the propagator e^{itH} as $\left(e^{\frac{it}{n}H}\right)^n$, such that

$$S_\gamma(t) = \int g_\gamma(x) \left(e^{\frac{it}{n}(A+xB)}\right)^n \mathrm{d}x. \tag{2.271}$$

Then, by substituting x by a vector of n independent variables, $x_1, x_2, \ldots, x_n$, it follows that

$$S_\gamma(t) = \frac{1}{\gamma\sqrt{2\pi}} \int e^{-\frac{\|x\|^2}{2n\gamma^2}} \boldsymbol{\delta}\left(f(\mathbf{x})\right) \prod_{j=1}^{n} e^{\frac{it}{n}(A+x_j B)} \mathrm{d}\mathbf{x}, \tag{2.272}$$

where the equality of all x_j has been ensured using $\boldsymbol{\delta}\left(f(\mathbf{x})\right) = \prod_{j=1}^{n-1} \delta\left(x_{j+1} - x_j\right)$. By using the delta function identity

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \mathrm{d}k, \tag{2.273}$$

the preceding expression can be integrated over x_j by introducing two auxiliary variables $k_0 = k_n = 0$:

$$S_\gamma(t) = \frac{1}{\sqrt{2\pi}\gamma} \int e^{-\frac{\|x\|^2}{2n\gamma^2}} \prod_{r=1}^{n-1} \left(\frac{1}{2\pi} e^{ik_r(x_{r+1}-x_r)} \mathrm{d}k_r\right) \left(\prod_{j=1}^{n} e^{\frac{it}{n}(A+x_j B)}\right) \mathrm{d}\mathbf{x} = \tag{2.274}$$

$$= \frac{1}{(2\pi)^{n-1}} \frac{1}{\sqrt{2\pi}\gamma} \int e^{-\frac{\|x\|^2}{2n\gamma^2}} \left(\prod_{r=1}^{n} e^{ix_r(k_{r-1}-k_r)}\right) \left(\prod_{j=1}^{n} e^{\frac{it}{n}(A+x_j B)}\right) \mathrm{d}k_1 \ldots \mathrm{d}k_{n-1} \mathrm{d}\mathbf{x}. \tag{2.275}$$

To integrate this expression, it would be prudent to complete the square in the exponents of $\exp\left(-\frac{x_j^2}{2n\gamma^2}\right)$ and $\exp\left(ix_j(k_{r-1} - k_r)\right)$, by using the formula:

$$-\alpha x^2 + \beta x = -\left(\sqrt{\alpha}x - \frac{\beta}{2\sqrt{\alpha}}\right)^2 + \frac{\beta^2}{4\alpha}, \tag{2.276}$$

for

$$\alpha = \frac{1}{2n\gamma^2} \quad \text{and} \quad \beta = i\left(k_{r-1} - k_r\right). \tag{2.277}$$

It follows that

$$e^{-\frac{x_j^2}{2n\gamma^2}} e^{ix_j(k_{r-1}-k_r)} = e^{-\left(\frac{1}{\sqrt{2n}\gamma}x_j - i\sqrt{\frac{n}{2}}\gamma(k_{r-1}-k_r)\right)^2} e^{-\frac{n\gamma^2}{2}(k_{r-1}-k_r)^2}, \tag{2.278}$$

so if

$$\kappa_j \equiv n\left(k_{j-1} - k_j\right) \tag{2.279}$$

and

$$\widetilde{F}_j\left(k_{r-1} - k_r\right) = \int_{-\infty}^{\infty} e^{-\left(\frac{1}{\sqrt{2n}\gamma}x_j - i\sqrt{\frac{n}{2}}\gamma(k_{r-1}-k_r)\right)^2} e^{\frac{it}{n}(A+x_jB)}\mathrm{d}x_j = \tag{2.280}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}\left(x_j - i\gamma^2\kappa_j\right)^2} e^{\frac{it}{n}(A+x_jB)}\mathrm{d}x_j = \tag{2.281}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}\left(x_j - i\gamma^2\kappa_j\right)^2} \left(\mathbb{1} + \frac{it}{n}\left(A + x_jB\right) - \frac{t^2}{2n^2}\left(A + x_jB\right)^2 + ...\right)\mathrm{d}x_j, \tag{2.282}$$

the integral (2.275) can be written as:

$$S_\gamma(t) = \frac{1}{(2\pi)^{n-1}}\frac{1}{\sqrt{2\pi}\gamma}\int e^{-\frac{n\gamma^2}{2}\sum_{j=1}^{n}(k_{j-1}-k_j)}\left(\prod_{j=1}^{n}\widetilde{F}_j\left(k_{j-1}-k_j\right)^2\right)\mathrm{d}k_1 ... \mathrm{d}k_{n-1}\mathrm{d}\mathbf{x}. \tag{2.283}$$

To evaluate the integral, it is first necessary to solve (2.282), which I will now do up to the second order in x_j within the integral. The integration that includes the zeroth order term of the infinite sum ($\mathbb{1}$) equals:

$$\mathcal{I}nt_0 = \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}\left(x_j - i\gamma^2\kappa_j\right)^2}\mathrm{d}x_j = \tag{2.284}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}x_j^2}\mathrm{d}x_j = \tag{2.285}$$

$$= \sqrt{2\pi n}\gamma. \tag{2.286}$$

Further calculations require the integration of terms that include x_j and x_j^2. The former equals

$$\mathcal{I}nt_{x_j} = \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}\left(x_j - i\gamma^2\kappa_j\right)^2} x_j\mathrm{d}x_j = \tag{2.287}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}x_j^2}\left(x_j + i\gamma^2\kappa_j\right)\mathrm{d}x_j = \tag{2.288}$$

$$= i\gamma^2\kappa_j\sqrt{2\pi n}\gamma, \tag{2.289}$$

and the latter equals:

$$\mathcal{I}nt_{x_j^2} = \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}\left(x_j - i\gamma^2\kappa_j\right)^2} x_j^2\mathrm{d}x_j = \tag{2.290}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}x_j^2}\left(x_j + i\gamma^2\kappa_j\right)^2\mathrm{d}x_j = \tag{2.291}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2n\gamma^2}x_j^2} x_j^2\mathrm{d}x_j - \gamma^4\kappa_j^2\sqrt{2\pi n}\gamma = \tag{2.292}$$

$$= \left(n\gamma^2 - \gamma^4\kappa_j^2\right)\sqrt{2\pi n}\gamma. \tag{2.293}$$

Thus, the integral in (2.282) can be evaluated as:

$$\widetilde{F}_j\left(k_{r-1}-k_r\right)=\sqrt{2\pi n}\gamma\Big(\mathbb{1}+\frac{it}{n}\left(A+i\gamma^2\kappa_j B\right)- \tag{2.294}$$

$$-\frac{t^2}{2n^2}\left(A^2+\{A,B\}i\gamma^2\kappa_j+B^2\left(n\gamma^2-\gamma^4\kappa_j^2\right)\right)+\mathcal{O}\left(k^3\right)\Big)= \tag{2.295}$$

$$=\sqrt{2\pi n}\gamma\Big(\mathbb{1}+\frac{it}{n}\left(A+i\gamma^2 n\left(k_{j-1}-k_j\right)B\right)- \tag{2.296}$$

$$-\frac{t^2}{2n^2}\left(\{A,B\}i\gamma^2 n\left(k_{j-1}-k_j\right)+B^2\left(n\gamma^2-\gamma^4 n^2\left(k_{j-1}-k_j\right)^2\right)\right)+\mathcal{O}\left(k^3\right)\Big).$$

For convenience, I will define

$$F_j\left(k_{j-1}-k_j\right)\equiv\frac{\widetilde{F}_j\left(k_{j-1}-k_j\right)}{\sqrt{2\pi n}\gamma}, \tag{2.297}$$

which leaves (2.283) as

$$S_\gamma(t)=\frac{1}{(2\pi)^{n-1}}\frac{\sqrt{2\pi n\gamma^2}}{\sqrt{2\pi}\gamma}\int e^{-\frac{n\gamma^2}{2}\sum_{j=1}^{n}\left(k_{j-1}-k_j\right)^2}\left(\prod_{j=1}^{n}F_j\left(k_{j-1}-k_j\right)\right)\mathrm{d}k_1\ldots\mathrm{d}k_{n-1}\mathrm{d}\mathbf{x}. \tag{2.298}$$

At this stage, it is possible to normalize the probability measure to 1. For easier calculation, I use the vector $\mathbf{k}=(k_1,\ldots,k_n)$ and introduce

$$\mathbf{M}\equiv\begin{pmatrix}2&-1&0&0&\cdots&0\\-1&2&-1&0&\cdots&0\\0&-1&2&-1&&0\\\vdots&&&&\ddots&\vdots\\0&\cdots&&0&-1&2\end{pmatrix}=2\mathbb{I}-P_{n-1}, \tag{2.299}$$

where $\mathbf{M}$ is an $(n-1)\times(n-1)$ matrix due to the properties of $\mathbf{k}$ and P_{n-1} is the adjacency matrix of the path graph. Its elements that indicate whether pairs of vertices of this graph are adjacent or not — they equal 1 if there exists a relevant edge, and 0 if there is none — i.e., $p_{ij}=\delta_{i,j+1}+\delta_{i,j-1}$. This implies that

$$e^{-\frac{n\gamma^2}{2}\sum_{j=1}^{n}(k_{j-1}-k_j)^2}=e^{-\frac{n\gamma^2}{2}\mathbf{k}^T\mathbf{M}\mathbf{k}}. \tag{2.300}$$

The eigenvalues of $\mathbf{M}$ can be obtained as

$$\lambda_j=2-2\cos\left(\frac{\pi j}{n}\right),\quad j=1,2,\ldots,n-1 \tag{2.301}$$

and their corresponding (unnormalized) eigenvectors equal

$$\mathbf{v}_j=\left(\frac{1}{2}\sin\left(\frac{\pi j}{n}\right),\frac{1}{2}\sin\left(\frac{2\pi j}{n}\right),\ldots,\frac{1}{2}\sin\left(\frac{(n-1)\pi j}{n}\right)\right). \tag{2.302}$$

From $\det\mathbf{M}=2\pi n\left(n\gamma^2\right)^{n-1}$, it follows that

$$\int e^{-\frac{n\gamma^2}{2}\mathbf{k}^T\mathbf{M}\mathbf{k}}\,\mathrm{d}k_1\ldots\mathrm{d}k_{n-1}=\sqrt{\frac{(2\pi)^{n-1}}{n(n\gamma^2)^{n-1}}}, \tag{2.303}$$

which gives the probability measure of

$$\mathrm{d}\mu\equiv\sqrt{\frac{n(n\gamma^2)^{n-1}}{(2\pi)^{n-1}}}e^{-\frac{n\gamma^2}{2}\mathbf{k}^T\mathbf{M}\mathbf{k}}\,\mathrm{d}k_1\ldots\mathrm{d}k_{n-1}. \tag{2.304}$$

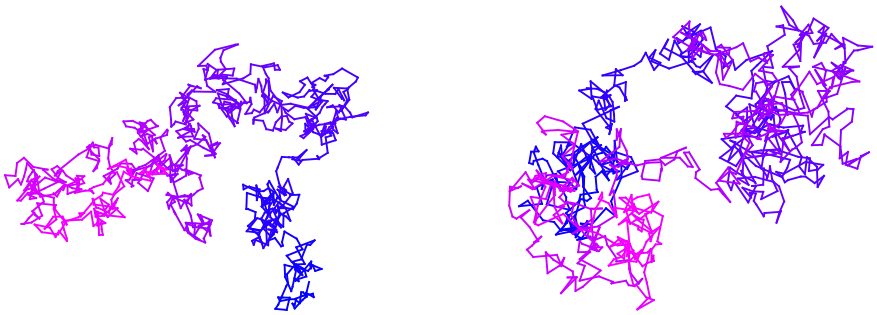


Figure 2.4: One thousand steps of a two-dimensional Wiener process with drift $\mu = 0$ and volatility $\sigma = 1$ (left), and a two-dimensional Brownian bridge process with one thousand steps, with volatility $\sigma = 1$ (right). The starts of these processes are marked in blue, and their paths are then gradated toward magenta.

As a quick consistency check, look at the edge cases where either $B = 0$ or $A = 0$. For the case of $B = 0$ it follows that $F \approx e^{i\frac{t}{n}A}$ and $S_\gamma(t) = e^{itA}$, as one would expect. For the case of $A = 0$, by inserting (2.304) into (2.283), it follows:

$$S_\gamma(t) = \int \left(\prod_{j=1}^{n} F_j \left(k_{j-1} - k_j \right) \right) \mathrm{d}\mu = \int \prod_{j=1}^{n} \left(e^{-t\gamma^2 (k_{j-1} - k_j) B - \frac{t^2}{2n}\gamma^2 B^2} \right) d\mu. \tag{2.305}$$

Because the sum in this exponential function collapses to zero, what remains is:

$$S_\gamma(t) = e^{-\frac{\gamma^2 t^2}{2} B^2}, \tag{2.306}$$

which is, again, expected for $A = 0$.

Brownian Bridge

To further analyze this expression, I can think of its constituents in terms of a Brownian bridge — a stochastic process in continuous time, with the property that its probability distribution equals the conditional probability distribution of a Wiener process. A Wiener process is merely a mathematical model usually used to describe the random motion of particles suspended in a fluid.

Definition 2.2.1. A *Wiener process* W_t satisfies the following four properties:

1. $W_0 = 0$.
2. $\forall u \geq 0, t > 0$ the increments $W_{t+u} - W_t$ are independent of all values W_s where $s \leq t$.
3. W_t is continuous.
4. For $t \geq s \geq 0$, $W_t - W_s$ is distributed according to a Gaussian distribution with a mean of zero and a variance of $t - s$.

Figure 2.4 shows examples of a two-dimensional Wiener process and a two-dimensional Brownian bridge process.

To reach a useful approximation of W_t, I will first discretize the interval $[0, t]$ as:

$$[0, t] = [0, \tfrac{t}{n}) \cup [\tfrac{t}{n}, \tfrac{2t}{n}) \cup \cdots \cup [\tfrac{(n-1)t}{n}, t] \tag{2.307}$$

and define:

$$\Delta w_j \equiv W_{\frac{jt}{n}} - W_{\frac{(j-1)t}{n}}, \tag{2.308}$$

which must be distributed according to the Gaussian distribution with the following probability distribution function:

$$p_{t,n}(x) \equiv \sqrt{\frac{n}{2\pi t}} e^{-\frac{n}{t}\frac{x^2}{2}}. \tag{2.309}$$

Because

$$W_t = \sum_{j=1}^{n} \Delta w_j, \tag{2.310}$$

the probability distribution function can be written as:

$$\sqrt{\frac{n^n}{(2\pi t)^n}} e^{-\frac{n}{t}\sum_{j=1}^{n}\frac{(\Delta w_j)^2}{2}}. \tag{2.311}$$

If I define

$$w_j \equiv \sum_{k=1}^{j} \Delta w_k, \tag{2.312}$$

I can write the Jacobian for the change of variables from Δw_j to w_j as:

$$[\mathbf{J}]_{jk} \equiv \frac{\partial w_j}{\partial \Delta w_k} = \begin{cases} 1, & k \leq j; \\ 0, & \text{otherwise}, \end{cases} \tag{2.313}$$

which means that $\mathbf{J}$ is a lower-triangular matrix with ones in all non-zero entries:

$$\mathbf{J} \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}. \tag{2.314}$$

The inverse matrix of this Jacobian, $\mathbf{D} \equiv \mathbf{J}^{-1}$, equals

$$\mathbf{D} \equiv \mathbf{J}^{-1} \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \tag{2.315}$$

and the matrix $\mathbf{D}^T\mathbf{D}$ has the following form:

$$\mathbf{D}^T\mathbf{D} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}. \tag{2.316}$$

Then, the probability density function for w_j can then be expressed as:

$$\sqrt{\frac{n^n}{(2\pi t)^n}} e^{-\frac{n}{t}\frac{\mathbf{w}^T\mathbf{D}^T\mathbf{D}\mathbf{w}}{2}}. \tag{2.317}$$

According to Donsker's theorem [62], also referred to as the functional central limit theorem, the continuum limit of this expression tends to W_t in its distribution. However, the path measure for the variables k_j exhibits a crucial difference from this case. To illustrate it, I will discretize the Brownian bridge

$$B_t = W_t - tW_{t=1} \tag{2.318}$$

using the variables b_j, which set $B_1 = 0$:

$$b_j \equiv w_j - \frac{j}{n} w_n, \quad j = 0, 1, \dots, n. \tag{2.319}$$

It is important to note here that the increments of B_t are not independent. These variables can also be represented using Δw_j as:

$$b_j = \sum_{k=1}^{j} \Delta w_k - \frac{j}{n} \sum_{k=1}^{n} \Delta w_k, \quad j = 0, 1, \dots, n, \tag{2.320}$$

or by using a Jacobian:

$$[\Gamma^{-1}]_{jk} \equiv \frac{\partial b_j}{\partial \Delta w_k} = \begin{cases} 1 - \frac{j}{n}, & k \leq j; \\ -\frac{j}{n}, & \text{otherwise.} \end{cases} \tag{2.321}$$

In a more instructive form, the Jacobian can be written as:

$$\Gamma^{-1} = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ 1-\frac{2}{n} & 1-\frac{2}{n} & -\frac{2}{n} & \cdots & -\frac{2}{n} \\ 1-\frac{3}{n} & 1-\frac{3}{n} & 1-\frac{3}{n} & \cdots & -\frac{3}{n} \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & -1+\frac{1}{n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.322}$$

which is the partial left inverse of the $n \times n$ matrix:

$$\Gamma \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.323}$$

Because the matrix $\mathbf{M}$ can be determined using the $(n-1) \times (n-1)$ submatrix of $\Gamma^T \Gamma$, its inverse is then given by the corresponding $(n-1) \times (n-1)$ submatrix of $\Gamma^{-1}(\Gamma^{-1})^T$:

$$[\mathbf{M}^{-1}]_{j,k} = \min\left\{ j\left(1 - \tfrac{k}{n}\right), k\left(1 - \tfrac{j}{n}\right) \right\}. \tag{2.324}$$

Thus, the relevant probability distribution moments can be calculated using the generating function:

$$\ell \sqrt{\frac{n(n\gamma^2)^{n-1}}{(2\pi)^{n-1}}} \int e^{-\frac{n\gamma^2}{2}\mathbf{k}^T\mathbf{M}\mathbf{k}} e^{\boldsymbol{\ell}^T\mathbf{k}}\, dk_1 \cdots dk_{n-1} = e^{\frac{1}{2n\gamma^2}\boldsymbol{\ell}^T\mathbf{M}^{-1}\boldsymbol{\ell}}. \tag{2.325}$$

For $j \leq j'$, it holds for these moments that:

$$\langle k_j k_{j'} \rangle = \frac{\partial}{\partial \ell_j} \frac{\partial}{\partial \ell_k} e^{\frac{1}{2n\gamma^2}\boldsymbol{\ell}^T\mathbf{M}^{-1}\boldsymbol{\ell}} \Big|_{\boldsymbol{\ell}=\mathbf{0}} = \frac{1}{n\gamma^2} [\mathbf{M}^{-1}]_{j,j'} = \frac{1}{\gamma^2} \frac{j}{n} \left(1 - \frac{j'}{n}\right). \tag{2.326}$$

Now, from the expression for $S_\gamma(t)$,

$$S_\gamma(t) = \sqrt{\frac{n(n\gamma^2)^{n-1}}{(2\pi)^{n-1}}} \int e^{-\frac{n\gamma^2}{2}\mathbf{k}^T\mathbf{M}\mathbf{k}} \left(\prod_{j=1}^{n} F_j(k_{j-1} - k_j) \right) dk_1 \cdots dk_{n-1}, \tag{2.327}$$

it is possible to obtain a coupled set of stochastic differential equations (SDE). If I define $l_j \equiv \gamma k_j$, it follows that:

$$S_\gamma(t) = \sqrt{\frac{n^n}{(2\pi)^{n-1}}} \int e^{-\frac{n}{2}\mathbf{l}^T\mathbf{M}\mathbf{l}} \left(\prod_{j=1}^{n} F_j \left(\frac{1}{\gamma}(l_{j-1} - l_j) \right) \right) dl_1 \cdots dl_{n-1}, \tag{2.328}$$

and by substituting for F_j, the starting expression can be written as:

$$\begin{aligned} S_\gamma(t) =& \sqrt{\frac{n^n}{(2\pi)^{n-1}}} \int e^{-\frac{n}{2}\mathbf{l}^T\mathbf{M}\mathbf{l}} \cdot \\ & \cdot \prod_{j=1}^{n} \left(\mathbb{I} + \frac{it}{n}A + t\gamma(\Delta l_j)B - \frac{t^2}{2n^2}(n\gamma^2 - n^2\gamma^2(\Delta l_j)^2)B^2 \right) dl_1 \cdots dl_{n-1}, \end{aligned} \tag{2.329}$$

where

$$\Delta l_j = l_j - l_{j-1}. \tag{2.330}$$

As shown above, it is possible to identify this variable with a discretization of the standard Brownian bridge.

If I define

$$X_k \equiv \prod_{j=1}^{k} \left(\mathbb{I} + \frac{it}{n}A + t\gamma(\Delta l_j)B - \frac{t^2}{2n^2}(n\gamma^2 - n^2\gamma^2(\Delta l_j)^2)B^2 \right), \tag{2.331}$$

it holds that:

$$X_{k+1} = \left(\mathbb{I} + \frac{it}{n}A + t\gamma(\Delta l_k)B - \frac{t^2}{2n^2}(n\gamma^2 - n^2\gamma^2(\Delta l_k)^2)B^2 \right) X_k, \tag{2.332}$$

which makes the difference between $X_{k+1} - X_k$ equal the following:

$$\Delta X_k \equiv X_{k+1} - X_k = \left(\frac{it}{n}A + t\gamma(\Delta l_k)B - \frac{t^2}{2n^2}(n\gamma^2 - n^2\gamma^2(\Delta l_k)^2)B^2 \right) X_k. \tag{2.333}$$

Note that, in this distribution, it holds that:

$$(\Delta l_k)^2 = \frac{1}{n} - \frac{1}{n^2}, \tag{2.334}$$

where the second term is negligible in the limit. By defining

$$dz_{s=k/n} \equiv \Delta l_k \frac{t}{n}, \quad X_{s=k/n} \equiv \Delta X_k, \tag{2.335}$$

the increment ΔX_k can be more compactly written as an infinitesimal component in the limit of $n \to \infty$, which builds a system of stochastic differential equations:

$$\boxed{\begin{aligned} dX_s &= itAX_s\, ds + t\gamma BX_s\, dz_s, \\ dz_s &= -\frac{z_s}{1-s}\, ds + dW_s. \end{aligned}} \tag{2.336}$$

This allows me to take the continuum limit:

$$S(t) = \int \mathcal{T} e^{\int_0^1 K\,ds + \gamma t \int_0^1 B\,dz}\,d\mu, \tag{2.337}$$

where $\mathcal{T}$ is the time-ordering operation,

$$K = itA - \frac{\gamma^2 t^2}{2} B^2, \tag{2.338}$$

and the increment dz obeys the stochastic differential equation:

$$dz = -\frac{z}{1-s}\,ds + dW. \tag{2.339}$$

It is important to note that (2.337) is an *equality* — this formula is not an approximation. In this way we have obtained a representation of the operator S via the operator SDE:

$$dS = itAS\,ds + \gamma t BS\,dz. \tag{2.340}$$

This representation may be subjected to a variety of solution and approximation techniques, from direct sampling, moment expansions, and the Dyson series. These will all be the subject of future studies.

By following the derivation described above we can immediately write down the stochastic integral representation for Hamiltonians H of the form (2.268):

$$S(t) = \int \mathcal{T} e^{\int_0^1 K\,ds + \gamma t \sum_{j=1}^m \int_0^1 D_j\,dz_j}\,d\mu(z), \tag{2.341}$$

where

$$K = itH_0 - \frac{\gamma^2 t^2}{2} \sum_{j=1}^m D_j^2, \tag{2.342}$$

with

$$dz_j = -\frac{z_j}{1-s}\,ds + dW_j, \tag{2.343}$$

and W_j are m independent Brownian motions.

An alternative derivation of this representation can be found using the Lie–Trotter formula, $e^{A+B} \approx e^A e^B + O(\|[A,B]\|^2)$, and an operator Hubbard–Stratonovich transformation.

2.2.2 Stochastic Dyson Series Expansion

In this subsection I will show the expansion of the derived integral representation in powers of the disorder parameter γ, by utilizing the indispensable Dyson series expansion, the first term of which already explicitly includes the disorder corrections. It also describes a completely positive evolution with diffusive behavior. The terms of higher orders stand for the quantum fluctuation corrections around the created diffusion values. If the propagator were to be approached directly using the Dyson series expansion, its disorder average would subsequently not exhibit tidy behavior for large values of t — however, that is not the case for the following procedure.

In the simplified case, where the system is described using the Hamiltonian $H = A + xB$, the integral (2.337) can be expanded in powers of the exponentiated stochastic term, $\gamma t \int_0^1 B\,\mathrm{d}z$. To use the Dyson series expansion, I define:

$$B(s) \equiv e^{sK} B e^{-sK}, \tag{2.344}$$

which allows me to write:

$$e^{-K}S(t) = \int \mathcal{T} e^{\gamma t \int_0^1 B(s)\,dz}\,\mathrm{d}\mu(z) = \tag{2.345}$$

$$= \mathbb{E}\left[\mathbb{I} + \gamma t \int_0^1 B(s)\,\mathrm{d}z + \frac{\gamma^2 t^2}{2}\int_0^1\int_0^1 \mathcal{T}[B(s_1)B(s_2)]\,dz_1 dz_2 + \cdots\right] \tag{2.346}$$

From the covariance of the Brownian bridge,

$$\mathbb{E}[z(s)z(t)] = \min\{s(1-t), t(1-s)\} = C_{st}, \tag{2.347}$$

it follows that:

$$\mathbb{E}[dz(s)dz(t)] = (\delta(s-t)-1)\mathrm{d}s\mathrm{d}, t \tag{2.348}$$

which gives:

$$e^{-K}S(t) = \mathbb{I} - \gamma^2 t^2 \int_0^1 ds_1 \int_0^{s_1} ds_2\, B(s_2)B(s_1) + \frac{\gamma^2 t^2}{2}\int_0^1 B(s)^2\,\mathrm{d}s + \mathcal{O}\left(\gamma^3\right). \tag{2.349}$$

This results may now serve as a starting point for the calculation of higher-order terms, using the classical Wick's theorem:

$$\mathbb{E}[z(s_1)z(s_2)z(s_3)z(s_4)] = C_{s_1s_2}C_{s_3s_4} + C_{s_1s_3}C_{s_2s_4} + C_{s_1s_4}C_{s_2s_3}. \tag{2.350}$$

From the expansion (2.349) it is evident that the $\mathcal{O}(1)$ term, $S_1(t) = e^{itA - \frac{\gamma^2 t^2}{2}B^2}$, as already mentioned before, explicitly incorporates the effects of disorder in the form of diffusive corrections, $e^{-\frac{\gamma^2 t^2}{2}B^2}$. In that vein, the disorder-averaged density operator can be written as:

$$\rho(t) \approx e^{\mathcal{L}}[\rho(0)], \tag{2.351}$$

where

$$\mathcal{L}(X) \equiv it[A, X] - \frac{\gamma^2 t^2}{2}\{B, X\} + \gamma^2 t^2 BXB \tag{2.352}$$

is a generator of the Lindblad form. This implies that the observed evolution $e^{\mathcal{L}}$ is completely positive — in other words, physical.

All subsequent fluctuation corrections are exponentially suppressed, as the diffusive solution incorporates an exponential suppression in t. In addition, and in contrast with some field-theoretic approaches [63, 64], the resulting series in $\mathcal{O}(\gamma)$ appears to be convergent.

2.2.3 Density of States for the Anderson Model

To show the utility of this form for the disorder-averaged propagator, in this subsection I will use it to calculate the density of states for the Anderson model.

By employing the derived stochastic integral representation for the Anderson model, it holds that

$$X(t) \equiv \frac{1}{n}\mathbb{E}[\mathrm{tr}(e^{itH(\mathbf{x})})] \tag{2.353}$$

and

$$X(t) = \frac{1}{n}\int \mathrm{tr}\left(\mathcal{T} e^{\int_0^1 itT - \frac{\gamma^2 t^2}{2}\mathbb{I}\,ds + \gamma t\sum_{j=1}^n \int_0^1 D_j\,dz_j}\right)\mathrm{d}\mu(z), \tag{2.354}$$

the Fourier transform of which directly gives the density of states. As already shown in this section, this representation of $X(t)$ can also be written in terms of a Dyson series in powers of γ:

$$X(t) = X_0(t) + X_1(t) + X_2(t) + \ldots, \tag{2.355}$$

where

$$X_0(t) = \frac{1}{N}\,\mathrm{tr}(e^K), \tag{2.356}$$

and

$$K \equiv itT - \frac{\gamma^2 t^2}{2}\mathbb{I}. \tag{2.357}$$

It holds that

$$X_1(t) = 0, \tag{2.358}$$

but the term $X_2(t)$ is already a complicated expression. To make calculation easier, I can express using two terms:

$$\begin{aligned} X_2(t) =& \frac{\gamma^2 t^2}{2}\frac{1}{n}\sum_{j=1}^{n}\int_0^1 \mathrm{tr}\left(e^{(1-s)K}D_j e^{sK}\right)\,ds - \\ &- \gamma^2 t^2 \frac{1}{n}\sum_{j=1}^{n}\int_0^1 \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2\ \mathrm{tr}\left(e^{(1-s_1)K}D_j e^{(s_1-s_2)K}D_j e^{s_2 K}\right), \end{aligned} \tag{2.359}$$

where I choose:

$$(\mathrm{i}) \equiv \frac{\gamma^2 t^2}{2}\frac{1}{n}\sum_{j=1}^{n}\int_0^1 \mathrm{tr}\left(e^{(1-s)K}D_j e^{sK}\right)\,\mathrm{d}s = \frac{\gamma^2 t^2}{2}X_0(t) \tag{2.360}$$

and

$$(\mathrm{ii}) \equiv \gamma^2 t^2 \frac{1}{n}\sum_{j=1}^{n}\int_0^1 \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2\ \mathrm{tr}\left(e^{(1-s_1)K}D_j e^{(s_1-s_2)K}D_j e^{s_2 K}\right) = \tag{2.361}$$

$$= \gamma^2 t^2 \frac{1}{n}\sum_{j=1}^{n}\int_0^1 ds_1 \int_0^{s_1} ds_2\ \langle j|e^{(1-s_1+s_2)K}|j\rangle\langle j|e^{(s_1-s_2)K}|j\rangle, \tag{2.362}$$

so that:

$$X_2(t) = (\mathrm{i}) - (\mathrm{ii}). \tag{2.363}$$

To solve this integral, K should be diagonalized. The eigenvectors of this matrix can be shown to correspond to the following:

$$|W_l\rangle \equiv \frac{1}{\sqrt{n}}\sum_{k=1}^{n} e^{\frac{2\pi i}{n}kl}|k\rangle, \tag{2.364}$$

and its eigenvalues equal:

$$\omega_l = i\left(2 - 2\cos\left(\frac{2\pi}{n}l\right)\right)t - \frac{\gamma^2 t^2}{2}. \tag{2.365}$$

In this setup, it holds that

$$\langle j|W_l\rangle = \frac{1}{\sqrt{n}}e^{\frac{2\pi i}{n}kl} \tag{2.366}$$

and

$$\langle j|W_l\rangle\langle W_l|j\rangle = \frac{1}{n}. \tag{2.367}$$

Then, (2.362) gives:

$$(\mathrm{ii}) = \gamma^2 t^2 \frac{1}{n^2}\sum_{k_1,k_2=1}^{n}\int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{(1-s_1+s_2)\omega_{k_1}} e^{(s_1-s_2)\omega_{k_2}} = \tag{2.368}$$

$$= \gamma^2 t^2 e^{2it-\frac{\gamma^2 t^2}{2}} \frac{1}{n^2}\sum_{k_1,k_2=1}^{n}\int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{2it(1-s_1+s_2)\cos(\frac{2\pi}{n}k_1)} e^{2it(s_1-s_2)\cos(\frac{2\pi}{n}k_2)}. \tag{2.369}$$

After the introduction of the variables

$$z_1 \equiv \frac{2\pi}{n} k_1, \quad z_2 \equiv \frac{2\pi}{n} k_2, \tag{2.370}$$

where

$$dz_2 = dz_1 \approx \frac{2\pi}{n}; \quad \frac{1}{n} \sum_{k_1=1}^{n} \approx \frac{1}{2\pi} \int_0^{2\pi} dz_1, \tag{2.371}$$

the limit of $n \to \infty$ gives:

$$\begin{aligned} &\text{(ii)} = \gamma^2 t^2 e^{2it - \frac{\gamma^2 t^2}{2}} \cdot \\ &\cdot \int_0^1 ds_1 \int_0^{s_1} ds_2 \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{2it(1-s_1+s_2)\cos(z_1)}\, dz_1 \times \frac{1}{2\pi} \int_0^{2\pi} e^{2it(s_1-s_2)\cos(z_2)}\, dz_2 \right\}. \end{aligned} \tag{2.372}$$

These integrals correspond to the Bessel functions of the first kind:

$$\text{(ii)} = \gamma^2 t^2 e^{2it - \frac{\gamma^2 t^2}{2}} \int_0^1 ds_1 \int_0^{s_1} ds_2\, J_0(2t(1 - s_1 + s_2)) J_0(2t(s_1 - s_2)). \tag{2.373}$$

Because $J_0(x)$ can be represented as:

$$J_0(x) = \sum_{l=0} \frac{(-1)^l}{2^{2l}(l!)^2} x^{2l}, \tag{2.374}$$

the integration of this double integral finally gives:

$$\int_0^1 ds_1 \int_0^{s_1} ds_2\, J_0(2t(1 - s_1 + s_2)) J_0(2t(s_1 - s_2)) = \frac{\sin(2t)}{4t}, \tag{2.375}$$

so (2.362) resolves to:

$$\text{(ii)} = \gamma^2 t e^{2it - \frac{\gamma^2 t^2}{2}} \frac{\sin(2t)}{4}, \tag{2.376}$$

which gives an analytical solution for $X_2(t)$. In a similar (yet much simpler) fashion, the calculation of $X_0(t)$ gives:

$$X_0(t) = \frac{1}{n} e^{2it - \frac{\gamma^2 t^2}{2}} \sum_{k=1}^{n} e^{2it\cos(\frac{2\pi}{n}k)} = \tag{2.377}$$

$$= e^{2it - \frac{\gamma^2 t^2}{2}} J_0(2t). \tag{2.378}$$

In conclusion, up to second order in γ, $X(t)$ can be written as:

$$X(t) = \left(1 + \frac{\gamma^2 t^2}{2}\right) e^{2it - \frac{\gamma^2 t^2}{2}} J_0(2t) - \frac{\gamma^2 t}{4} e^{2it - \frac{\gamma^2 t^2}{2}} \sin(2t). \tag{2.379}$$

The density of states is then derived by a Fourier transform of $X(t)$:

$$\begin{aligned} \widehat{X}(k) = &\left(1 + \left(\frac{i}{2\pi}\right)^2 \frac{\gamma^2}{2} \frac{d^2}{dk^2}\right) \sqrt{2\pi}\gamma^2 e^{-\frac{\pi^2\gamma^2 k^2}{2}} \times \frac{\mathrm{rect}(\frac{k}{2} - \frac{1}{2\pi})}{\sqrt{1 - \pi^2(k - \frac{1}{\pi})^2}} - \\ &- \left(\frac{i}{2\pi} \frac{d}{dk}\right) \frac{\gamma^2}{8i} \sqrt{2\pi}\gamma^2 e^{-\frac{\pi^2\gamma^2 k^2}{2}} \times (\delta(k - \tfrac{2}{\pi}) - \delta(k)). \end{aligned} \tag{2.380}$$

As can be inferred from Figure 2.5, which shows the calculated function $X(t)$ set against a numerical solution, the diffusion correction for this tight-binding Anderson model gives a simple convolution of the density of states, with a Gaussian of width γ, and the calculated second-order corrections already include the effects of level repulsion in this solution.

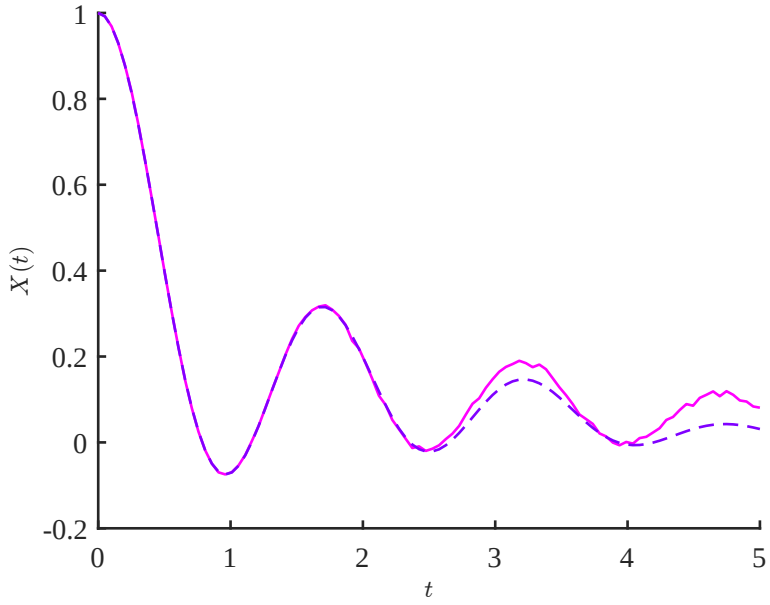


Figure 2.5: The quantity $X(t) \equiv \frac{1}{N}\mathbb{E}[\mathrm{tr}(e^{itH})]$ for the Anderson model on 30 sites (the x axis is time in units where $\hbar = 1$). (It turns out that the results for 30 sites are already indistinguishable from the $N \to \infty$ limit.) Shown pink is the result of numerical sampling with 100 samples. Shown purple (dashed) is $X(t) = X_0(t) + X_2(t)$, the second order result calculated via the stochastic Dyson series Eq. (2.379).

2.2.4 Spectral Form Factors and Out-of-Time-Order Correlation Function

In this subsection I demonstrate the application of the derived framework to approximate the $k = 1$ spectral form factor and OTOCs for the Anderson model.

From the completed derivations, it is clear that the spectral form factor can be calculated from the expression for $S_{(2)}(t)$, which is the average propagator for two copies of the Hamiltonian: $H_{(2)} = H_1 \otimes \mathbb{I}_2 - \mathbb{I}_1 \otimes H_2$. In this setup, the OTOCs equal:

$$\mathrm{tr}\left(A \otimes B \otimes C \otimes D S_{(4)}(t) \mathrm{SWAP}_{1234}\right), \tag{2.381}$$

where $S_{(4)}(t)$ is then the average propagator for four copies of the Hamiltonian:

$$H_{(4)} = H_1 \otimes \mathbb{I}_{234} - \mathbb{I}_1 \otimes H_2 \otimes \mathbb{I}_{34} + \mathbb{I}_{12} \otimes H_3 \otimes \mathbb{I}_4 - \mathbb{I}_{123} \otimes H_4, \tag{2.382}$$

and the corresponding SWAP_{1234} is the unitary permutation that cycles through all four copies. The integral representations can then be used to calculate both $S_{(2)}(t)$ and $S_{(4)}(t)$ as:

$$S_{(2k)}(t) = e^{it(H_0)_{(2k)} - \frac{\gamma^2 t^2}{2} \sum_{j=1}^{N} (D_j)^2_{(2k)}}. \tag{2.383}$$

If the initial local perturbation equals $A = C = |1\rangle\langle 1|$, and the observation site corresponds to $B = D = |\ell\rangle\langle \ell|$, the OTOCs give:

$$\langle 0\ell 0\ell | S_{(4)}(t) | \ell 0 \ell 0 \rangle, \tag{2.384}$$

as can also be seen in Figure 2.6.

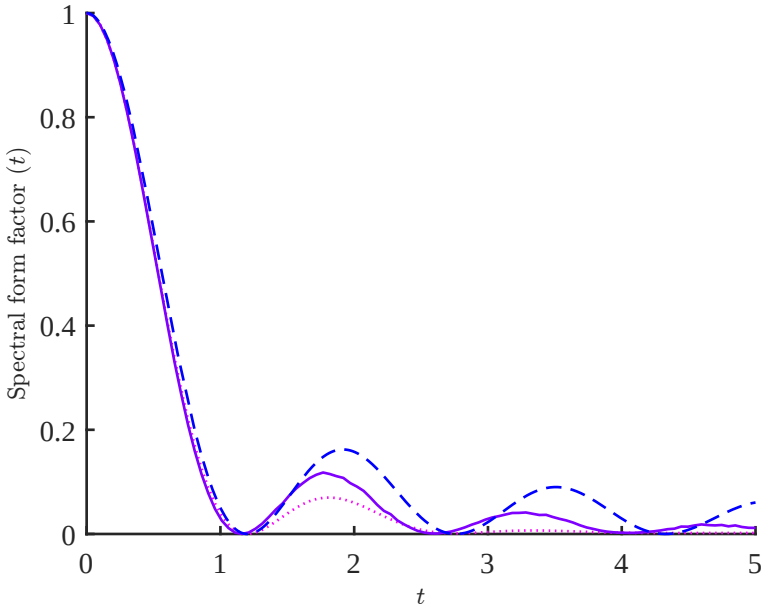


Figure 2.6: The spectral form factor $\frac{1}{N^2}\mathbb{E}[|\operatorname{tr}(e^{itH})|^2]$ for the Anderson model on 30 sites (the x axis is time in units where $\hbar = 1$). Again, the results for 30 sites are indistinguishable from the $N \to \infty$ limit. Shown purple is the result of numerical sampling with 100 samples. Shown pink (dotted) is the zeroth order diffusion-corrected term from the stochastic Dyson series. Shown for comparison, in blue (dashed), is the zeroth order result from ordinary time-dependent perturbation theory. Note that the zeroth-order stochastic Dyson series result already incorporates the dephasing decay resulting from the disorder average.

2.2.5 Derivation Aided by Visual Representation

Calculating the full expressions for the stochastic integral representation of disordered quantum many-body systems could be made simpler by using graphic representations, in the tradition of Feynman diagrams. This section demonstrates the creation and usage of these nifty tools, but stops short of completing the full derivation and halts at the setup of the solution, as the full calculation stays beyond the scope of this dissertation.

From looking at $S(t)$, which is equal to the following:

$$S(t) \approx \int \prod_{j=1}^{n} e^{\frac{it}{n}A - t\gamma^2(k_{j-1}k_j)B - \frac{t^2}{2n}\gamma^2 B^2} \mathrm{d}\mu + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \tag{2.385}$$

I can define

$$\mathcal{U} \equiv M_n \cdot M_{n-1} \cdot M_{n-2} \cdot \ldots \cdot M_1, \tag{2.386}$$

where M_j are linear 2×2 matrices:

$$M_j \equiv e^{\frac{it}{n}A - t\gamma^2(k_{j-1}-k_j)B - \frac{t^2}{2n}\gamma^2 B^2}. \tag{2.387}$$

Then, I can write:

$$S(t) = \mathbb{E}_{k_s}(\mathcal{U}) + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right), \tag{2.388}$$

which can then be illustrated using graphical methods.

Take a simple example, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, t is fixed as $t = 1$, $n \in \mathbb{N}$ is an arbitrarily large integer, and $\gamma \ll 1$ is fixed. The expression for M_j can be

expanded for $\gamma^2 \ll 1$, up to $\mathcal{O}(\gamma^2)$:

$$M_j = e^{\frac{it}{n}A} e^{-t\gamma^2(k_{j-1}-k_j)B} e^{-\frac{t^2}{2n}\gamma^2 B^2} = \tag{2.389}$$

$$= e^{\frac{it}{n}A}\left(\mathbb{1} - t\gamma^2\left(k_{j-1}-k_j\right)B + \frac{1}{2!}\left(t\gamma^2\left(k_{j-1}k_j\right)B\right)^2 + ...\right)\cdot$$

$$\cdot\left(\mathbb{1} - \frac{t^2}{2n}\gamma^2 B^2 + \frac{1}{2!}\left(\frac{t^2}{2n}\gamma^2 B^2\right)^2 + ...\right). \tag{2.390}$$

Because of the dependency of k on γ, all the relevant summands taken from the first series will be at most of the n-th order in γ, since:

$$\mathcal{O}\left(k_{j-1}-k_j\right) \sim \mathcal{O}\left(\frac{1}{\gamma\sqrt{n}}\right). \tag{2.391}$$

It follows that:

$$M_j = e^{\frac{it}{n}A}\Big(\mathbb{1} - t\gamma^2\left(k_{j-1}-k_j\right)B + \frac{1}{2}\left(t\gamma^2\left(k_{j-1}-k_j\right)B\right)^2 -$$

$$-\frac{1}{6}\left(t\gamma^2\left(k_{j-1}-k_j\right)B\right)^3 - \frac{t^2}{2n}\gamma^2 B^2 + \mathcal{O}\left(\gamma^4\right)\Big) = \tag{2.392}$$

$$= e^{\frac{it}{n}A}\Big(\mathbb{1} - t\gamma^2\left(k_{j-1}-k_j\right)B + \frac{1}{2}\left(t\gamma^2\left(k_{j-1}-k_j\right)B\right)^2 - \frac{t^2}{2n}\gamma^2 B^2 + \mathcal{O}\left(\gamma^3\right)\Big) \tag{2.393}$$

Diagrammatic Representation

The end goal of this procedure is the ability to diagrammatically represent products of operators. First, I construct a setup in which I use the following instead of $\mathcal{U}$ up to the second order in γ:

$$\mathcal{V} = e^{\frac{\mathcal{K}t}{n}} \prod_{j=1}^{n}\left(\mathbb{1} + \mathcal{A}_j x_j + \frac{\mathcal{A}_j^2}{2}x_j^2\right), \tag{2.394}$$

where

$$\mathcal{A}_j = -t\gamma^2 B, \tag{2.395}$$

$$x_j = k_{j-1} - k_j, \tag{2.396}$$

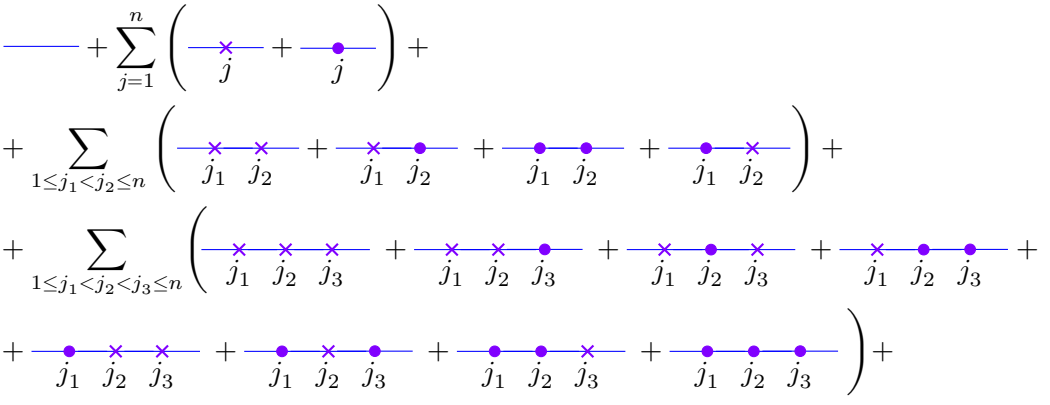
$$\mathcal{K} = itA - \frac{t^2\gamma^2}{2}B^2. \tag{2.397}$$

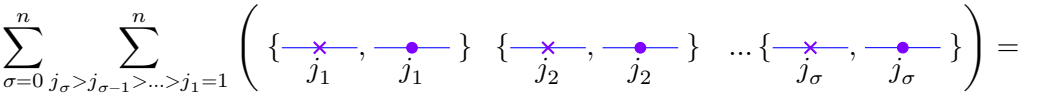
The expansion may then be represented diagrammatically as follows, if $t_j \equiv \frac{t}{n}\cdot j$ and $t = 1$.

$\equiv e^{\mathcal{K}}$ (2.398)


$\equiv e^{(\frac{1}{n}-t_j)\mathcal{K}} A_j x_j e^{t_j\mathcal{K}}$ (2.399)


$\equiv e^{(\frac{1}{n}-t_j)\mathcal{K}} \frac{A_j^2}{2} x_j^2 e^{t_j\mathcal{K}}$ (2.400)

$\mathcal{V} =$  (2.401)

$=\sum_{\sigma=0}^{n}\sum_{j_\sigma>j_{\sigma-1}>\ldots>j_1=1}^{n}$  (2.402)

$=\sum_{\sigma=0}^{n}\sum_{j_\sigma>j_{\sigma-1}>\ldots>j_1=1}^{n}\sum_{\xi_\upsilon\in\{\times_{j_\upsilon},\bullet_{j_\upsilon}\}}(\xi_1\xi_2\ldots\xi_\sigma)$ (2.403)

In the above expression, the curly brackets in (2.402) denote the n-ary Cartesian power of the set $\{\times_{j_\upsilon}, \bullet_{j_\upsilon}\}$. The lack of a cross or a dot on the line for a particular $j_{k>\sigma}$ assumes that it is filled by a ———, even for $\sigma = 0$. Now consider $\mathbb{E}\left(x_j\right) = 0$ and $\mathbb{E}\left(s_j\right) = s_j$ for $s_j \equiv x_j^2$. Then,

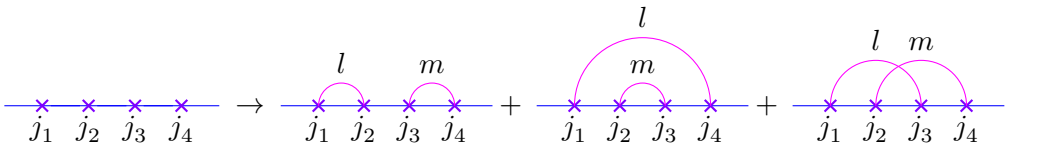
$$\mathbb{E}\left(x_j x_k\right) \equiv \lambda \tag{2.404}$$

$$\mathbb{E}\left(x_j s_k\right) = 0 \tag{2.405}$$

$$\mathbb{E}\left(x_{j_1} x_{j_2} x_{j_3} x_{j_4}\right) = ? \tag{2.406}$$

To divine the last expression, remember that $x_j = k_{j-1} - k_j$ and consider the following calculations.

This diagrammatic rule can be used to see whether there is any 'doubling up' of the variables in the product:

 (2.407)

To achieve

$$\sqrt{\frac{n\left(n\gamma^2\right)^{n-1}}{(2\pi)^{n-1}}}\int \mathbb{E}\left(x_{j_1}x_{j_2}x_{j_3}x_{j_4}\right)e^{-\frac{n\gamma^2}{2}k^T M k}\mathrm{d}k_1\ldots\mathrm{d}k_{n-1}, \tag{2.408}$$

with

$$\mathbb{E}\left(x_{j_1}x_{j_2}x_{j_3}x_{j_4}\right) \equiv \mathbb{E}\left[\left(k_{j_1-1}-k_{j_1}\right)\left(k_{j_2-1}-k_{j_2}\right)\left(k_{j_3-1}-k_{j_3}\right)\left(k_{j_4-1}-k_{j_4}\right)\right], \tag{2.409}$$

use the generating function and complete the square in the integral.

$$\mathbb{Z}\left[l_1,\dots,l_{n-1}\right]=\sqrt{\frac{n\left(n\gamma^2\right)^{n-1}}{\left(2\pi\right)^{n-1}}}\int e^{-\frac{n\gamma^2}{2}k^TMk+\sum_{j=1}^{n-1}l_jk_j}\mathrm{d}k_1\dots\mathrm{d}k_n\sim \tag{2.410}$$

$$\sim e^{\frac{1}{2n\gamma^2}l^TM^{-1}l} \tag{2.411}$$

Then, the following can be calculated.

$$\mathbb{E}\left(x_{j_1}x_{j_2}x_{j_3}x_{j_4}\right)=\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\frac{\partial}{\partial l_3}\frac{\partial}{\partial l_4}e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.412}$$

$$=\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\frac{\partial}{\partial l_3}\left(\frac{\partial}{\partial l_4}\frac{-1}{2n\gamma^2}l^TM^{-1}l\right)e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.413}$$

$$=\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\frac{\partial}{\partial l_3}\left(\frac{\partial}{\partial l_4}\frac{-1}{2n\gamma^2}\sum_{a,b}l_a\left[M^{-1}\right]_{ab}l_b\right)e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.414}$$

$$=\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\frac{\partial}{\partial l_3}\frac{-1}{2n\gamma^2}\sum_{a,b}\left(\left[M^{-1}\right]_{j_4b}l_b+\left[M^{-1}\right]_{aj_4}l_4\right)e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.415}$$

$$=\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\frac{\partial}{\partial l_3}\frac{-1}{2n\gamma^2}\sum_{a}2\left[M^{-1}\right]_{j_4a}l_ae^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.416}$$

$$=\frac{\partial^3}{\partial l_1\partial l_2\partial l_3}\frac{-1}{2n\gamma^2}2\left[M^{-1}l\right]_{j_4}e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.417}$$

$$=\frac{\partial^2}{\partial l_1\partial l_2}\left(\frac{-1}{n\gamma^2}\left[M^{-1}\right]_{j_4j_3}+\frac{-1}{n\gamma^2}\left[M^{-1}\right]_{j_4}l\frac{-1}{n\gamma^2}\left[M^{-1}\right]_{j_3}l\right)e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}= \tag{2.418}$$

$$=\text{const.}\cdot\frac{-1}{\left(2n\gamma^2\right)}\left[M^{-1}\right]_{j_1j_2}\left[M^{-1}\right]_{j_3j_4}e^{-\frac{1}{2n\gamma^2}l^TM^{-1}l}\sim\text{const.} \tag{2.419}$$

The actual calculation that should be done is the one that corresponds to the Anderson model:

$$\int\mathrm{d}\mu\prod_{j=1}^{n}e^{\frac{it}{n}H_0-t\gamma^2\left(k_{j-1}^{\alpha}-k_j^{\alpha}\right)D_{\alpha}-\frac{t^2}{2n}\gamma^2\mathbb{1}}=\int\mathrm{d}\mu\mathcal{U}. \tag{2.420}$$

With $\mathcal{K}\equiv itH_0-t^2\gamma^2\mathbb{1}$, it is:

$$\int\mathrm{d}\mu\mathcal{U}=\int\mathrm{d}\mu\prod_{j=1}^{n}e^{\mathcal{K}-t\gamma^2\left(k_{j-1}^{\alpha}-k_j^{\alpha}\right)D_{\alpha}}. \tag{2.421}$$

Therefore, to calculate through the orders of γ, if $d_j=j/n$, these designations can be used:

$$\equiv e^{\mathcal{K}} \tag{2.422}$$

$$\equiv e^{\mathcal{K}d_j}\left(-t\gamma^2\left(k_{j-1}^{\alpha}-k_j^{\alpha}\right)D_{\alpha}\right)e^{\mathcal{K}(1-d_j)} \tag{2.423}$$

$$\equiv e^{\mathcal{K}d_j}\left(t^2\gamma^4\frac{\left(k_{j-1}^{\alpha}-k_j^{\alpha}\right)^2}{2}D_{\alpha}^2\right)e^{\mathcal{K}(1-d_j)}. \tag{2.424}$$

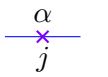
Here,

$$\mathcal{A}_{\alpha}\equiv-t\gamma^2\left(k_{j-1}^{\alpha}-k_j^{\alpha}\right)D_{\alpha} \tag{2.425}$$

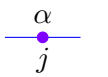
$$\mathcal{B}_{\alpha}\equiv t^2\gamma^4\frac{\left(k_{j-1}^{\alpha}-k_j^{\alpha}\right)^2}{2}D_{\alpha}^2=\frac{1}{2}\mathcal{A}_{\alpha}^2, \tag{2.426}$$

so

$$= e^{\mathcal{K}} \tag{2.427}$$

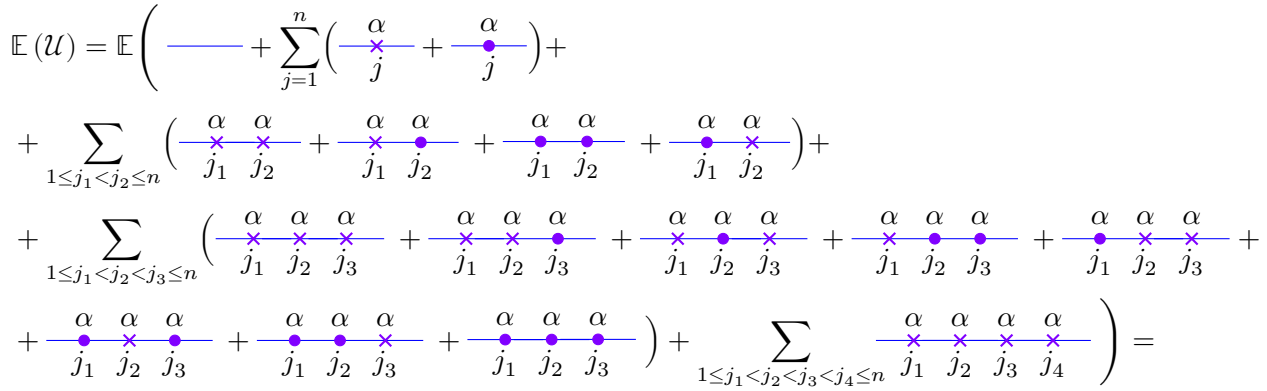


$$= e^{\mathcal{K}d_j}\mathcal{A}_\alpha e^{\mathcal{K}(1-d_j)} \tag{2.428}$$

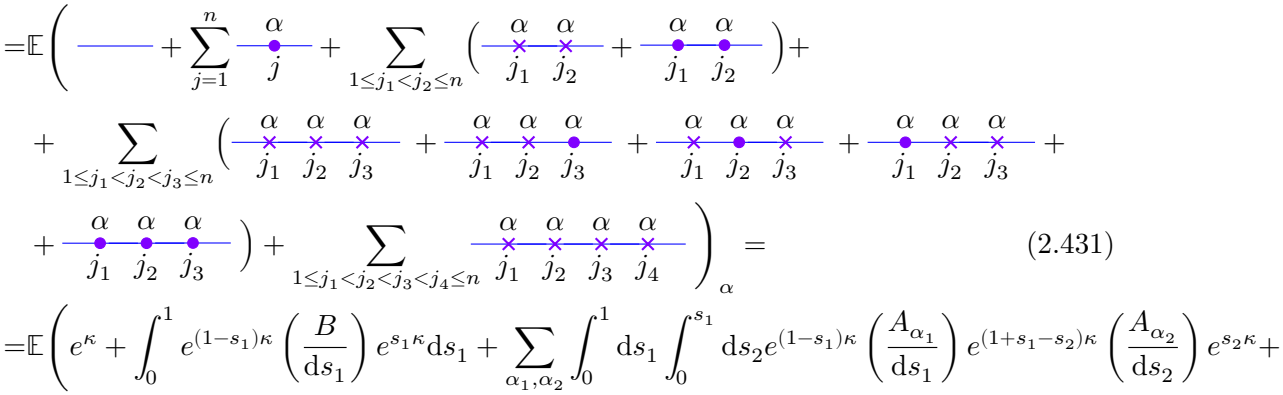


$$= e^{\mathcal{K}d_j}\mathcal{B}_\alpha e^{\mathcal{K}(1-d_j)}. \tag{2.429}$$

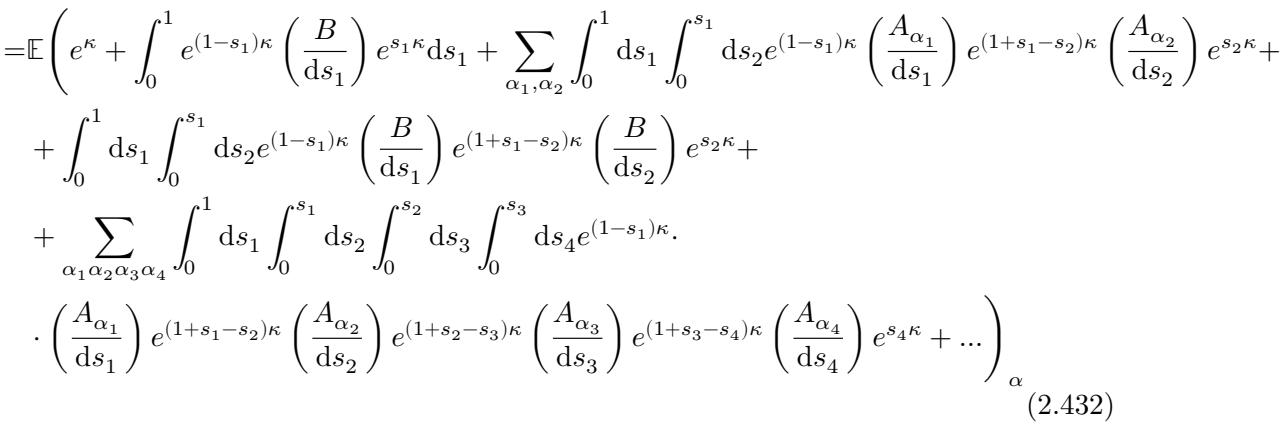
Then, up to the fourth order in γ, $\mathcal{U}$ can be expressed as follows. Note that, in the following equation, certain summands can be neglected because their expectation value comes out as zero:



$$= \tag{2.430}$$



$$= \tag{2.431}$$



$$= \tag{2.432}$$

To calculate the full expectation value, $(\langle(k^\alpha_{j-1}-k^\alpha_j)^2\rangle = \langle(k^\alpha_{j-1})^2\rangle + \langle(k^\alpha_j)^2\rangle - 2\langle k^\alpha_{j-1}k^\alpha_j\rangle)$, as defined in (2.326), certain snippets must be resolved. For example:

$$\mathbb{E}\left(\left(k^\alpha_{j-1}-k^\alpha_j\right)^2\right) \approx \frac{1}{\gamma^2}\left(\frac{1}{n}-\frac{1}{n^2}\right) \approx \frac{1}{n\gamma^2} = \frac{1}{\gamma^2}\mathrm{d}s \tag{2.433}$$

and

$$\mathbb{E}\left(\left(k^\alpha_{j-1}-k^\alpha_j\right)\left(k^\beta_{k-1}-k^\beta_k\right)\right) = \delta^{\alpha\beta}\left(-\frac{1}{n^2\gamma^2}\right) = -\frac{1}{\gamma^2}\delta^{\alpha\beta}\left(\mathrm{d}s\right)^2. \tag{2.434}$$

This leads to:

$$\begin{aligned}
&\mathrm{tr}\left(\mathcal{U}\right) = \mathrm{tr}\left(e^{\mathcal{K}}\right) + \mathrm{tr}\left(e^{\mathcal{K}}\frac{n}{2\gamma^2}\left(\frac{1}{n}-\frac{1}{n^2}\right)t^2\gamma^4\right) - \\
&- t^2\gamma^2\sum_{\alpha}\int_0^1 \mathrm{d}s_1\int_0^{s_1}\mathrm{tr}\left(e^{(1-s_1)\mathcal{K}}D_\alpha e^{(1+s_1-s_2)\mathcal{K}}D_\alpha e^{s_2\mathcal{K}}\right) = \qquad (2.435)\\
&= \mathrm{tr}\left(e^{\mathcal{K}}\right) + \frac{t^2\gamma^2}{2}\mathrm{tr}\left(e^{\mathcal{K}}\right) - t^2\gamma^2\sum_{\alpha}\int_0^1 \mathrm{d}s_1\int_0^{s_1}\mathrm{tr}\left(e^{(1-s_1)\mathcal{K}}D_\alpha e^{(1+s_1-s_2)\mathcal{K}}D_\alpha e^{s_2\mathcal{K}}\right) \qquad (2.436)
\end{aligned}$$

The higher order terms then lead to a more involved computation. However, for each setup, the expectation values can be calculated given this consideration:

$$\begin{aligned}
&\mathbb{E}\left(\underset{\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4}{\times\!\!-\!\!\times\!\!-\!\!\times\!\!-\!\!\times}\right) = \\
&= t^4\gamma^8\left(-\frac{1}{\gamma^2}\right)^2\sum_{\alpha_1\alpha_2\alpha_3\alpha_4}\left(\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4}+\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}+\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3}\right)e^{(1-s_1)\mathcal{K}}D_{\alpha_1}\ldots = \qquad (2.437)\\
&= t^4\gamma^4\sum_{\alpha\beta}\int\int\int\int\mathrm{tr}\left(e^{(1-s_1)\mathcal{K}}D_\alpha e^{(1+s_1-s_2)\mathcal{K}}D_\alpha e^{(1+s_2-s_3)\mathcal{K}}D_\beta e\ldots\right) \qquad (2.438)
\end{aligned}$$

The calculation can then be concluded from the following equality.

$$\langle\alpha|\,e^{\omega\mathcal{K}}\,|\beta\rangle = \sum_l \langle\alpha|\,e^{\omega\left(it\lambda_l-\frac{t^2\gamma^2}{2}\right)}\,|\omega(l)\rangle\,\langle\omega(l)\,|\,\beta\rangle \qquad (2.439)$$

As the continuation of this line of investigation steers away from the main purpose of this work, further calculation will not be presented here.

2.2.6 Summary of the Results for Disordered Systems

In this section I demonstrated the derivation of an exact representation for the disorder-averaged propagator of a quantum system by using stochastic calculus, after reaching some analytical solutions for local observables in perturbed many-body systems. The derivation was performed by solving a stochastic integral over a time-ordered operator expression that included a Lindblad generator and a temporally random external field — with a Brownian bridge in place of the path measure. By expanding this stochastic expression into a stochastic Dyson series, I obtained a power series in the disorder parameter, which includes the diffusive disorder effects in term $\mathcal{O}(1)$. The higher-order terms then explicitly include the fluctuations around the diffusive solution.

By further using the derived representation, I showed the calculation of the density of states, the spectral form factor, and OTOCs for the tight-binding Anderson model. Further work may see these results used to investigate higher-order corrections, exploring moments, and researching many-body localized models — perhaps using the tensor network framework.

Chapter 3

Creating a Tensor Bridge for Perturbative Many-Body Localization Solutions

Contents

As was the case in the previous chapter, this one is about methods for the study of disordered quantum many-body systems. However, in this chapter, the focus is going to be put on tensor-network approaches. I dedicate tensor-network-based approaches their own chapter, as they are a huge topic in and on itself. In the following chapter the focus is toward finding solutions that are analytically exact albeit still sometimes intractable to evaluate; as a complementary view, in this chapter I will try to find approaches that are more practical for getting yielding concrete results, even if those results are but rough approximations. Taken together, the two chapters will hopefully provide a holistic approach to this problem area.

Perturbation theory has been proven as reliable in the investigation of the robust part of topological phases, but its application fails to deliver accurate results when approaching a phase transition. Different quantum phases naturally require differing perturbative approximations, which makes the description of critical properties of such phases quite difficult to obtain using perturbation theory. However, unlike perturbation theory, the variational approach can still be used at those critical points. If the perturbative expansions that represent ground states of certain quantum phases can be described in the tensor network formalism, the perturbative coefficients could be lifted to variational parameters, after which tensor network methods can be used to variationally optimize the mathematical description of the system [65]. Thus, separate quantum phases can be merged in a unique mathematical description by bridging their corresponding perturbative expansions by using the tensor network formalism.

Conveniently, the formalism of tensor network relies on the sequential application of ever more operators onto an existing state — allowing for the potential replication of a perturbative expansion order-by-order. Thus, if a quantum state can be represented as a tensor network state, this job is already half done. As mentioned, the computational complexity of a tensor network state is highly reliant on the bond dimension of the network, which increases linearly in the size of the network, i.e., the number of clusters of local operators used. Note that this number tends to scale exponentially in the order of perturbation in perturbative expansions. This approach can be demonstrated on the ferromagnetic transverse-field Ising model.

It is important to note that the disorder-perturbed model comes with some challenges in terms of the numerical evaluation of its tensor-network description. With that in mind, this chapter will include an approach to addressing those challenges as well as some results that by no means present a comprehensive analysis benchmarked against other results, but a step toward further investigation.

My results will separately concern the ferromagnetic state and the polarized state, deriving the relevant exponentiated forms of the wave functions in the first and second orders of perturbation, but also their corresponding density operators. Through the success of these derivations, I will have the tools to calculate expressions for a number of observables important for quantum many-body systems.

This chapter consists of several sections. In **Section 3.1: Exponentiated Wave Functions**, I will show the application of perturbation theory for the ordinary transverse-field Ising model around either of the two phases, using an exponentiated form of the perturbation. After that, I will introduce a modified system in **Section 3.2: Varying Perturbative Parameters in Exponentiated Wave Functions**, in which the perturbation contains disorder, in the form of random-variable parameters. I will show how this will affect the calculation of the perturbed states and calculate some expectation values with respect to them. Based on the results of the previous chapters, I will then

define a tensor network state which allows to build a bridge from one phase to the other. This is done for the normal model in **Section 3.3: Bridging between Variational Solutions using Tensor Networks**, and I will show how it is to be adapted for the disorder-perturbed model in **Section 3.4: Combining Tensor Bridging with the Derived Perturbative Solution**. Finally, in this section I will also attempt to study properties of the disordered model by evaluating the tensor-network description of its ground state numerically.

In 2017, Laurens Vanderstraeten et al. published a work showing that perturbative expansions for quantum many-body systems can be addressed in terms of tensor networks [65] and constructing the first steps of a natural framework for the interpolation of perturbative expansions across separate quantum phase transitions. In their study, tensor networks are parametrized by a small number of parameters, but they still provide exceptionally accurate variational results.

As the construction of my method is based on the tensor network construction shown in this paper, I will summarize the relevant content and central definitions here.

3.1 Exponentiated Wave Functions

There exists a quaint method of writing out the wave function of a quantum system in an exponentiated form, which makes it easier to manage in various calculations — not only analytical ones, but also when it comes to the numerical analyses of large systems. Suitable exponentiated expansions of linear perturbative expressions can simplify some calculations and lead to accurate perturbative results in the lowest orders. Specifically, here I will show how the exponentiation of wave functions can be used in certain circumstances to reach reliable results in the study of disordered quantum many-body systems. In fact, the obtained results are better than expected for the standard form of perturbative wave functions, already providing strictly positive contributions of higher-order terms within lower-order expressions derived for a small disorder.

In this section I will investigate the system described by the Hamiltonian:

$$H = -\sum_{\langle i,j\rangle} Z_i Z_j - \sum_k X_k = H_1 + H_2, \tag{3.1}$$

$$H_1 \equiv -\sum_{\langle i,j\rangle} Z_i Z_j, \quad H_2 \equiv -\sum_k X_k, \tag{3.2}$$

and examine the two phases in which either the first summand group or the second summand group contributes to the Hamiltonian only in the form of disorder, i.e., the $\|H_1\|/\|H_2\| \ll 1$ and the $\|H_2\|/\|H_1\| \ll 1$ phases. The results presented here will prove to be immensely useful in the derivation of so-called *bridging* between the perturbative solutions for these two phases — ferromagnetic and polarized (paramagnetic, which I am calling polarized here, since the ground state of H_2 is polarized in the $+X$ direction).

In this sense, perturbative solutions for these two chief phases can be expressed in the framework of tensor networks, which allows for the derivation of an interpolation between these solutions, even separated by a critical point of a quantum phase transition. This construction is conducive to the creation of tensor network states that are defined by just a handful of parameters, but that do still provide a reliable setup for the calculation of the variational energies of the system. Interestingly, the parameters of the tensor network states constructed using this method are not abstract and unrelated to an

intuitive understanding of the characteristics of the investigated system — instead, they correspond (to some measure) to the elements of derived perturbative solutions for this system, giving an unexpected immediate understanding of their properties, general role, and behavior.

My results in this section will separately concern the ferromagnetic state and the polarized state, deriving the relevant exponentiated forms of the wave functions in the first and second orders of perturbation.

3.1.1 Exponentiation of the Ferromagnetic State

In the ferromagnetic state, H_1 is the chief Hamiltonian that affects the base state of the system, whereas H_2 is treated as perturbation onto it, i.e. $\|H_2\|/\|H_1\| \ll 1$.

To illustrate the following results better, I will choose

$$|\varphi_0\rangle = |+Z\rangle = \otimes_j \left|+Z_j\right\rangle , \tag{3.3}$$

where $\left|+Z_j\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j$, as the exemplary ground state of the non-perturbed system with $H_0 = H_1$. The projector to this state is then defined as $P_0 = |\varphi_0\rangle\langle\varphi_0| = |+Z\rangle\langle+Z|$ and the projection operator $(\mathbb{1} - P_0)$ projects onto all states orthogonal to the ground state of H_0. Here I will define the perturbation as H_2 and present results for both one-dimensional and two-dimensional systems on an orthogonal and equidistant square lattice with periodic boundary conditions and a total of N particles.

Two-Dimensional System

The calculation of the correction to the wave function in the first order of perturbation gives:

$$\begin{aligned} |\varphi_{\Delta 1}\rangle &= -\frac{(\mathbb{1} - P_0)}{H_0 - E_0} H_2 |\varphi_0\rangle = \\ &= \sum_i \frac{1}{H_0 - E_0} \left(X_i |\varphi_0\rangle - |\varphi_0\rangle \langle\varphi_0| X_i |\varphi_0\rangle \right) , \end{aligned} \tag{3.4}$$

where E_0 is the relevant ground-state energy for the ferromagnetic phase.

It is clear that $|\varphi_0\rangle\langle\varphi_0| X_i |\varphi_0\rangle = 0$ because $\langle\varphi_0| X_i |\varphi_0\rangle = 0$, so the energy difference between the states $|\varphi_0\rangle$ and $X_i |\varphi_0\rangle$ must be calculated. In a two-dimensional system, this difference is $4 \cdot 2 = 8$, as there are four nearest neighbor couplings in the base part of the Hamiltonian for each of the lattice sites. It follows that:

$$|\varphi_1\rangle = \sum_i \frac{1}{8} X_i |\varphi_0\rangle \tag{3.5}$$

In the second order, the calculation grows slightly more complex. What follows is the

calculation of the correction in the second order:

$$|\Delta\varphi_2\rangle = -\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2 H_2 P_0 H_2 |\varphi_0\rangle + \frac{\mathbb{1}-P_0}{H_0-E_0} H_2 \frac{\mathbb{1}-P_0}{H_0-E_0} H_2 |\varphi_0\rangle = \tag{3.6}$$

$$= -\frac{\mathbb{1}-P_0}{H_0-E_0} H_2 \sum_i \frac{X_i}{8} |\varphi_0\rangle = \tag{3.7}$$

$$= \frac{\mathbb{1}-P_0}{H_0-E_0} \sum_j X_j \sum_i \frac{X_i}{8} |\varphi_0\rangle = \tag{3.8}$$

$$= \sum_{i\neq j} \frac{\mathbb{1}-P_0}{H_0-E_0} X_i X_j \cdot \frac{1}{8} |\varphi_0\rangle - \frac{\mathbb{1}-P_0}{H_0-E_0} \frac{X_i^2}{8} |\varphi_0\rangle = \tag{3.9}$$

$$= \sum_{i\neq j} \frac{1}{H_0-E_0} \cdot \left(\frac{X_i X_j}{8} |\varphi_0\rangle - |\varphi_0\rangle \langle\varphi_0| X_i X_j |\varphi_0\rangle\right) = \tag{3.10}$$

$$= \sum_{i\neq j} \frac{1}{H_0-E_0} \cdot \frac{X_i X_j}{8} |\varphi_0\rangle . \tag{3.11}$$

The energy difference depends on whether the positions i and j are neighboring or not. If they are not nearest neighbors, the fraction acts on the newly created state with $\frac{1}{16}$ (or $\frac{1}{8}$ if each pair of indices is counted only once), whereas if they are neighboring, it acts on the state with $\frac{1}{12}$ (or $\frac{1}{6}$ if each pair is counted only once), because applying $Z_i Z_j$ onto $X_i X_j |\varphi_0\rangle$ gives the result of $(-1)\cdot(-1) = 1$. Thus, the full correction to the wave function in the second order equals:

$$|\varphi_{\Delta 2}\rangle = \sum_{(ij)_d} \frac{X_i X_j}{64} |\varphi_0\rangle + \sum_{(ij)_n} \frac{X_i X_j}{48} |\varphi_0\rangle , \tag{3.12}$$

where the indices $(ij)_d$ and $(ij)_n$ indicate that the respective pairs of sites with distant (d) and neighboring (n) indices were counted only once in the sums.

To make these results mathematically more convenient, these expressions can be represented in an exponentiated form. I will use the expression in (3.5) to treat the state up to the first order of perturbation as part of an exponential expansion by including the perturbation parameter λ as the magnitude ratio of the perturbative summand and the basic Hamiltonian, so that the wave function up to the first order of correction equals:

$$|\varphi_1\rangle = \left(\mathbb{1} + \lambda \cdot \frac{1}{8} \sum_i X_i\right) |\varphi_0\rangle . \tag{3.13}$$

It holds that

$$e^{\lambda \cdot \frac{1}{8} \sum_i X_i} = \mathbb{1} + \lambda \cdot \frac{1}{8} \sum_i X_i + \frac{\lambda^2}{2} \cdot \left(\frac{1}{64} \sum_{ij} X_i X_j\right) + \mathcal{O}(\lambda^3) = \tag{3.14}$$

$$= \mathbb{1} + \lambda \cdot \frac{1}{8} \sum_i X_i + \lambda^2 \cdot \left(\frac{N}{128} + \frac{1}{64} \sum_{(ij)} X_i X_j\right) + \mathcal{O}(\lambda^3) , \tag{3.15}$$

where the index (ij) indicates pairs of spins where $i \neq j$ and each of the pairs is counted only once, which shows that, in the first order in λ, the exponential function (3.14) has the same form as the wave function prefactor in (3.13). Thus, it can be used interchangeably for up to the first order of perturbation.

In the second order of perturbation, the wave function has been calculated as:

$$|\varphi_2\rangle = \left(\mathbb{1} + \lambda \cdot \frac{1}{8} \sum_i X_i + \lambda^2 \sum_{(ij)_d} \frac{X_i X_j}{64} + \lambda^2 \sum_{(ij)_n} \frac{X_i X_j}{48}\right) |\varphi_0\rangle , \tag{3.16}$$

and the easiest way to construct a sum of up to the k-th order is to use a k-th order expression in λ, marked as ν, in an exponential function that creates a product with the already derived expression (3.14) that is valid up to the first order of perturbation:

$$e^{\lambda^2\nu}e^{\lambda\cdot\frac{1}{8}\sum_i X_i} = (\mathbb{1}+\lambda^2\nu)\cdot\left(\mathbb{1}+\lambda\cdot\frac{1}{8}\sum_i X_i+\frac{\lambda^2}{2}\cdot\left(\frac{1}{64}\sum_{ij}X_iX_j\right)\right)+\mathcal{O}(\lambda^3)= \tag{3.17}$$

$$=\mathbb{1}+\lambda^2\nu+\lambda\cdot\frac{1}{8}\sum_i X_i+\frac{\lambda^2}{2}\cdot\left(\frac{N}{64}+\frac{1}{32}\sum_{(ij)}X_iX_j\right)+\mathcal{O}(\lambda^3)= \tag{3.18}$$

$$=\mathbb{1}+\lambda^2\nu+\lambda\frac{1}{8}\sum_i X_i+\lambda^2\frac{N}{128}+\lambda^2\frac{1}{64}\sum_{(ij)_d}X_iX_j+\lambda^2\frac{1}{64}\sum_{(ij)_n}X_iX_j+\mathcal{O}(\lambda^3). \tag{3.19}$$

From this and (3.13), it follows that:

$$\lambda^2\nu+\lambda^2\cdot\frac{N}{128}+\lambda^2\cdot\frac{1}{64}\sum_{(ij)_n}X_iX_j=\lambda^2\cdot\frac{1}{48}\sum_{(ij)_n}X_iX_j \tag{3.20}$$

$$\lambda^2\nu=\lambda^2\cdot\frac{1}{192}\sum_{(ij)_n}X_iX_j-\lambda^2\cdot\frac{N}{128}. \tag{3.21}$$

Thus,

$$\nu=\frac{1}{192}\sum_{(ij)_n}X_iX_j-\frac{N}{128}, \tag{3.22}$$

and the wave function that corresponds to the calculated wave function up to the second order of perturbation can be written as:

$$|\psi_2\rangle\equiv\exp\left(\frac{\lambda^2}{192}\sum_{(ij)_n}X_iX_j-\frac{\lambda^2N}{128}\right)\exp\left(\frac{\lambda^2}{8}\sum_i X_i\right)|\varphi_0\rangle\,. \tag{3.23}$$

However, if the second order wave function is to be normalized, (3.13) contains an additional summand:

$$\frac{1}{2}P_0H_2\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2H_2|\varphi_0\rangle=-\frac{1}{2}P_0H_2\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)\frac{1}{8}\sum_i X_i|\varphi_0\rangle= \tag{3.24}$$

$$=\frac{1}{128}\cdot P_0\cdot\left(\sum_i X_i^2+\sum_{i\neq j}X_iX_j\right)|\varphi_0\rangle= \tag{3.25}$$

$$=\frac{N}{128}|\varphi_0\rangle\,, \tag{3.26}$$

which allows for the wave function to be used in a cleaner form:

$$|\psi_{2n}\rangle\equiv\exp\left(\frac{\lambda^2}{192}\sum_{(ij)_n}X_iX_j\right)\exp\left(\frac{\lambda^2}{8}\sum_i X_i\right)|\varphi_0\rangle\,. \tag{3.27}$$

One-Dimensional System

For the correction in the first order of perturbation, the only difference of the one-dimensional from the two-dimensional system is that in the energy difference operator, which gives $2\cdot 2=4$, i.e.,

$$|\varphi_{\Delta 1}\rangle=\frac{1}{4}\sum_i X_i|\varphi_0\rangle\,. \tag{3.28}$$

Similarly, the only difference for the second order correction to the wave function

regards the energy operator; so,

$$|\varphi_{\Delta 2}\rangle = \sum_{(ij)_d} \frac{X_i X_j}{32} |\varphi_0\rangle + \sum_{(ij)_n} \frac{X_i X_j}{16} |\varphi_0\rangle . \tag{3.29}$$

Thus, in the first order, the system is in the state:

$$|\varphi_1\rangle = \left(\mathbb{1} + \lambda \cdot \frac{1}{4} \sum_i X_i \right) |\varphi_0\rangle , \tag{3.30}$$

and

$$e^{\lambda \cdot \frac{1}{4} \sum_i X_i} = \mathbb{1} + \lambda \cdot \frac{1}{4} \sum_i X_i + \lambda^2 \cdot \left(\frac{N}{32} + \frac{1}{16} \sum_{(ij)} X_i X_j \right) + \mathcal{O}\left(\lambda^3\right), \tag{3.31}$$

which clearly corresponds to the wave function calculated up to the first order of perturbation.

In the second order of perturbation, the wave function can be written as:

$$|\varphi_2\rangle = \left(\mathbb{1} + \lambda \cdot \frac{1}{4} \sum_i X_i + \lambda^2 \cdot \sum_{(ij)_d} \frac{X_i X_j}{32} |\varphi_0\rangle + \lambda^2 \cdot \sum_{(ij)_n} \frac{X_i X_j}{16} \right) |\varphi_0\rangle , \tag{3.32}$$

so to construct a product of exponential functions that corresponds to it, I will calculate the following:

$$e^{\lambda^2 \nu} e^{\frac{\lambda}{4} \sum_i X_i} |\varphi_0\rangle = \left((\mathbb{1} + \lambda^2 \nu) \left(\mathbb{1} + \lambda \frac{1}{4} \sum_i X_i + \frac{\lambda^2}{2} \left(\frac{1}{16} \sum_{ij} X_i X_j \right) \right) + \mathcal{O}(\lambda^3) \right) |\varphi_0\rangle = \tag{3.33}$$

$$= \left(\mathbb{1} + \lambda \frac{1}{4} \sum_i X_i + \frac{\lambda^2 N}{32} + \frac{\lambda^2}{16} \sum_{(ij)_d} X_i X_j + \frac{\lambda^2}{16} \sum_{(ij)_d} X_i X_j + \lambda^2 \nu + \mathcal{O}(\lambda^3) \right) |\varphi_0\rangle . \tag{3.34}$$

For this expression to be valid, it must hold that

$$\frac{\lambda^2 N}{32} + \lambda^2 \nu = \frac{\lambda^2}{16} \cdot \sum_{(ij)_d} X_i X_j, \tag{3.35}$$

or

$$\nu = \frac{1}{16} \cdot \sum_{(ij)_d} X_i X_j - \frac{N}{32}. \tag{3.36}$$

If the wave function calculated up the second order of perturbation is then normalized, the corresponding exponentiated wave function can be constructed more cleanly.

$$\frac{1}{2} P_0 H_2 \left(\frac{\mathbb{1} - P_0}{H_0 - E_0} \right)^2 H_2 |\varphi_0\rangle = \frac{1}{8} P_0 \frac{1}{4} \sum_{ij} X_i X_j |\varphi_0\rangle = \tag{3.37}$$

$$= \frac{N}{32} \tag{3.38}$$

It follows that the normalized wave function up to the second order in λ can be written as:

$$|\psi_{2n}\rangle \equiv \exp\left(\frac{1}{16} \sum_{(ij)_d} X_i X_j \right) \exp\left(\frac{\lambda}{4} \sum_i X_i \right) |\varphi_0\rangle . \tag{3.39}$$

3.1.2 Exponentiation of the Polarized State

In the polarized state, H_2 acts as the chief Hamiltonian to describe the system, while H_1 can be treated as perturbation onto it. In a non-perturbed system, a polarized state is characterized by all of its spins having the same orientation.

The ground state of the non-perturbed system with $H = H_2$ can be chosen as

$$|\varphi_0\rangle = |+X\rangle = \otimes_j \left|+X_j\right\rangle, \tag{3.40}$$

where $\left|+X_j\right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_j = \frac{1}{\sqrt{2}} \left(\left|+Z_j\right\rangle + \left|-Z_j\right\rangle\right)$. The projector onto this state is defined as $P_0 = |\varphi_0\rangle\langle\varphi_0| = |+X\rangle\langle+X|$ and the operator $(\mathbb{1} - P_0)$ projects onto all states orthogonal to the ground state of H_0. In the remainder of this subsection I will present results for both a one-dimensional and a two-dimensional systems on an orthogonal and equidistant lattice with periodic boundary conditions and a total of N particles.

Two-dimensional System

The calculation of the correction in the first order of perturbation can be completed as follows:

$$|\varphi_{\Delta 1}\rangle = \frac{\mathbb{1} - P_0}{H_0 - E_0} \sum_{\langle ij \rangle} Z_i Z_j |\varphi_0\rangle = \tag{3.41}$$

$$= \sum_{\langle ij \rangle} \frac{1}{H_0 - E_0} \left(Z_i Z_j |\varphi_0\rangle - |\varphi_0\rangle\langle\varphi_0| Z_i Z_j |\varphi_0\rangle\right) = \tag{3.42}$$

$$= \sum_{\langle ij \rangle} \frac{1}{H_0 - E_0} Z_i Z_j |\varphi_0\rangle, \tag{3.43}$$

where the second summand equals zero because $i \neq j$.

The energy difference encoded in the fraction equals $2 \cdot 2$, which gives the correction in the first order:

$$|\varphi_{\Delta 1}\rangle = \sum_{\langle ij \rangle} \frac{Z_i Z_j}{4} |\varphi_0\rangle. \tag{3.44}$$

In the second order, the correction calculation gives:

$$|\varphi_{\Delta 2}\rangle = \frac{\mathbb{1} - P_0}{H_0 - E_0} \sum_{\langle kl \rangle} Z_k Z_l \sum_{\langle ij \rangle} \frac{Z_i Z_j}{4} |\varphi_0\rangle. \tag{3.45}$$

If $(k, l) \in \{(i, j), (j, i)\}$, the total of that summand is equal to zero, as the product of the

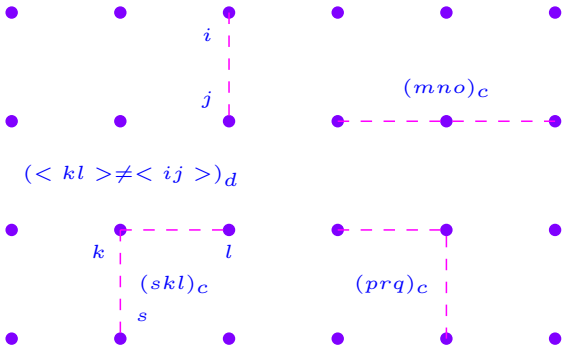


Figure 3.1: An example of the definition of indices in a two-dimensional square lattice used in this work. Note that all combinations of indices are addressed to with a unique label, but only once.

Pauli operators gives $\mathbb{1}$. Then,

$$|\varphi_{\Delta 2}\rangle = \sum_{\langle kl\rangle\neq\langle ij\rangle} \frac{\mathbb{1}-P_0}{H_0-E_0} Z_k Z_l \frac{Z_i Z_j}{4} |\varphi_0\rangle = \tag{3.46}$$

$$= \sum_{\langle kl\rangle\neq\langle ij\rangle} \frac{1}{H_0-E_0} \cdot \frac{1}{4} \left(Z_k Z_l Z_i Z_j |\varphi_0\rangle - |\varphi_0\rangle \langle\varphi_0| Z_k Z_l Z_i Z_j |\varphi_0\rangle \right) = \tag{3.47}$$

$$= \sum_{\langle kl\rangle\neq\langle ij\rangle} \frac{1}{H_0-E_0} \cdot \frac{1}{4} Z_k Z_l Z_i Z_j |\varphi_0\rangle = \tag{3.48}$$

$$= \sum_{(\langle kl\rangle\neq\langle ij\rangle)_d} \frac{1}{4\cdot 4\cdot 2\cdot \frac{1}{2}} Z_k Z_l Z_i Z_j |\varphi_0\rangle + \sum_{(ijk)_c} \frac{1}{4\cdot 2\cdot 2\cdot \frac{1}{2}} Z_i Z_j^2 Z_k |\varphi_0\rangle = \tag{3.49}$$

$$= \sum_{(\langle kl\rangle\neq\langle ij\rangle)_d} \frac{1}{16} Z_k Z_l Z_i Z_j |\varphi_0\rangle + \sum_{(ijk)_c} \frac{1}{8} Z_i Z_k |\varphi_0\rangle , \tag{3.50}$$

where the indices $(\langle kl\rangle \neq \langle ij\rangle)_d$ and $(ijk)_c$ denote the non-overlapping and clustering combinations of indices, respectively. An example of the thus defined index combinations can be seen in Figure 3.1. All combinations have been counted only once and the central index of the clustered index combination is the nearest neighbor of both the other ones.

If λ now denotes the magnitude ratio of the perturbative and the main part of the Hamiltonian, up to the first order of perturbation the wave function of the system equals:

$$|\varphi_1\rangle = \left(\mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} \right) |\varphi_0\rangle . \tag{3.51}$$

By mimicking the procedure already completed for the ferromagnetic case in Subsection 3.1.1, I can simplify the following expression:

$$e^{\frac{\lambda}{4}\cdot\sum_{\langle ij\rangle} Z_i Z_j} = \mathbb{1} + \lambda \cdot \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{2} \cdot \left(\frac{1}{16} \sum_{\langle ij\rangle} Z_i Z_j \sum_{\langle kl\rangle} Z_k Z_l \right) + \mathcal{O}(\lambda^3) = \tag{3.52}$$

$$= \mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{32} \left(\sum_{\langle ij\rangle} \mathbb{1} + 2 \sum_{(ijk)_c} Z_i Z_j^2 Z_k + 2 \sum_{(\langle ij\rangle,\langle kl\rangle)_d} Z_i Z_j Z_k Z_l \right) + \mathcal{O}(\lambda^3) = \tag{3.53}$$

$$= \mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{32} N + \frac{\lambda^2}{16} \sum_{(ijk)_c} Z_i Z_k + \frac{\lambda^2}{16} \sum_{(\langle ij\rangle,\langle kl\rangle)_d} Z_i Z_j Z_k Z_l + \mathcal{O}(\lambda^3). \tag{3.54}$$

Up to the first order of perturbation, this expression corresponds to the calculated perturbative expression.

In the second order, the (non-normalized) wave function:

$$|\psi_2\rangle = \left(\mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \lambda^2 \cdot \sum_{(\langle kl\rangle \neq \langle ij\rangle)_d} \frac{1}{16} Z_k Z_l Z_i Z_j + \lambda^2 \cdot \sum_{(ijk)_c} \frac{1}{8} Z_i Z_k\right) |\varphi_0\rangle \quad (3.55)$$

can be expressed as a product of exponential functions, in a procedure similar to that in the ferromagnetic case in the second order. The calculation of the ansatz follows.

$$e^{\lambda^2\nu} e^{\lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4}} = (\mathbb{1} + \lambda^2\nu)\left(\mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{2}\left(\frac{1}{16} \sum_{\langle ij\rangle} Z_i Z_j \sum_{\langle kl\rangle} Z_k Z_l\right)\right) + \mathcal{O}(\lambda^3) = \quad (3.56)$$

$$= \mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{32}\left(\sum_{\langle ij\rangle} \mathbb{1} + 2 \sum_{(ijk)_c} Z_i Z_j^2 Z_k + \sum_{(\langle ij\rangle, 2\langle kl\rangle)_d} Z_i Z_j Z_k Z_l\right) + \lambda^2\nu + \mathcal{O}(\lambda^3) = \quad (3.57)$$

$$= \mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \lambda^2 \frac{N}{32} + \frac{\lambda^2}{16} \sum_{(ijk)_c} Z_i Z_k + \frac{\lambda^2}{16} \sum_{(\langle ij\rangle, \langle kl\rangle)_d} Z_i Z_j Z_k Z_l + \lambda^2\nu + \mathcal{O}(\lambda^3) \quad (3.58)$$

Therefore,

$$\lambda^2 \cdot \frac{N}{32} + \frac{\lambda^2}{16} \sum_{(ijk)_c} Z_i Z_k + \lambda^2\nu = \lambda^2 \cdot \sum_{(ijk)_c} \frac{1}{8} Z_i Z_k, \quad (3.59)$$

and from this,

$$\nu = \sum_{(ijk)_c} \frac{1}{16} Z_i Z_k - \frac{N}{32}. \quad (3.60)$$

Again, similarly to the ferromagnetic case, I can calculate the normalization for the derived wave function of the system up to the second order of perturbation:

$$\frac{1}{2} P_0 H_1 \left(\frac{\mathbb{1} - P_0}{H_0 - E_0}\right)^2 H_1 |\varphi_0\rangle = \frac{N}{32} |\varphi_0\rangle . \quad (3.61)$$

Thus, it holds that the following expression can be used to determine the wave function up to the second order in λ:

$$|\psi_2\rangle \equiv \exp\left(\lambda^2 \sum_{(ijk)_c} \frac{1}{16} Z_i Z_k\right) \exp\left(\lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4}\right) |\varphi_0\rangle . \quad (3.62)$$

One-dimensional System

The only difference from the two-dimensional system in the case of the first order correction for the wave function in a one-dimensional system may exist in the energy difference operator. However, in this case it gives the same result:

$$|\varphi_{\Delta 1}\rangle = \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} |\varphi_0\rangle . \quad (3.63)$$

In the second order, the calculation is, again, as follows:

$$|\varphi_{\Delta 2}\rangle = \sum_{(\langle kl\rangle \neq \langle ij\rangle)_d} \frac{1}{4\cdot 4\cdot 2\cdot \frac{1}{2}} Z_k Z_l Z_i Z_j |\varphi_0\rangle + \sum_{(ijk)_c} \frac{1}{4\cdot 2\cdot 2\cdot \frac{1}{2}} Z_i Z_j^2 Z_k |\varphi_0\rangle = \tag{3.64}$$

$$= \sum_{(\langle kl\rangle \neq \langle ij\rangle)_d} \frac{1}{16} Z_k Z_l Z_i Z_j |\varphi_0\rangle + \sum_{(ijk)_c} \frac{1}{8} Z_i Z_k |\varphi_0\rangle , \tag{3.65}$$

i.e., the result is equal to the one obtained for the two-dimensional system.

As shown previously, the expression in (3.44) can be used to treat the state up to its order as part of an exponential expansion. If λ denotes the magnitude ratio of the perturbative summand and the basic Hamiltonian, the wave function of the one-dimensional system up to the first order of perturbation equals:

$$|\varphi_1\rangle = \left(\mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4}\right) |\varphi_0\rangle . \tag{3.66}$$

The test exponential function that would correspond to this result is the following:

$$e^{\lambda\cdot\sum_{\langle ij\rangle} \frac{Z_i Z_j}{4}} = \mathbb{1} + \lambda \cdot \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{2} \cdot \left(\frac{1}{16} \sum_{\langle ij\rangle} Z_i Z_j \sum_{\langle kl\rangle} Z_k Z_l\right) + \mathcal{O}(\lambda^3) = \tag{3.67}$$

$$= \mathbb{1} + \lambda \cdot \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{32} \cdot N + \frac{\lambda^2}{16} \cdot \sum_{(ijk)_c} Z_i Z_k + \frac{\lambda^2}{16} \cdot \sum_{(\langle ij\rangle, \langle kl\rangle)_d} Z_i Z_j Z_k Z_l + \mathcal{O}(\lambda^3). \tag{3.68}$$

Evidently, in the first order of λ, the exponentiated form of the perturbation can be used as a mathematically more convenient form of delivery of (3.66).

In the second order, the non-normalized wave function has been calculated as:

$$|\varphi_2\rangle = \left(\mathbb{1} + \lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \lambda^2 \cdot \sum_{(\langle kl\rangle \neq \langle ij\rangle)_d} \frac{1}{16} Z_k Z_l Z_i Z_j + \lambda^2 \cdot \sum_{(ijk)_c} \frac{1}{8} Z_i Z_k\right) |\varphi_0\rangle , \tag{3.69}$$

which can also be expressed as a product of exponential functions. I will now calculate the ansatz.

$$e^{\lambda^2\nu} e^{\lambda \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4}} = \left((\mathbb{1} + \lambda^2\nu)\left(\mathbb{1} + \lambda \cdot \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{2} \cdot \left(\frac{1}{16} \sum_{\langle ij\rangle} Z_i Z_j \sum_{\langle kl\rangle} Z_k Z_l\right)\right) + \mathcal{O}(\lambda^3)\right) |\varphi_0\rangle = \tag{3.70}$$

$$= \left(\mathbb{1} + \lambda \cdot \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \frac{\lambda^2}{32} \cdot \left(\sum_{\langle ij\rangle} \mathbb{1} + 2 \sum_{(ijk)_c} Z_i Z_j^2 Z_k + \sum_{(\langle ij\rangle, 2\langle kl\rangle)_d} Z_i Z_j Z_k Z_l\right) + \lambda^2\nu + \mathcal{O}(\lambda^3)\right) |\varphi_0\rangle = \tag{3.71}$$

$$= \left(\mathbb{1} + \lambda \cdot \sum_{\langle ij\rangle} \frac{Z_i Z_j}{4} + \lambda^2 \cdot \frac{N}{32} + \frac{\lambda^2}{16} \sum_{(ijk)_c} Z_i Z_k + \frac{\lambda^2}{16} \sum_{(\langle ij\rangle, \langle kl\rangle)_d} Z_i Z_j Z_k Z_l + \lambda^2\nu + \mathcal{O}(\lambda^3)\right) |\varphi_0\rangle \tag{3.72}$$

Therefore, it must hold that

$$\lambda^2 \cdot \frac{N}{32} + \frac{\lambda^2}{16} \sum_{(ijk)_c} Z_i Z_k + \lambda^2\nu = \lambda^2 \cdot \sum_{(ijk)_c} \frac{1}{8} Z_i Z_k, \tag{3.73}$$

which means that

$$\nu = \sum_{(ijk)_c} \frac{1}{16} Z_i Z_k - \frac{N}{32}. \tag{3.74}$$

To normalize the perturbative addition for the second order calculation, I need to consider the summand:

$$\frac{1}{2} P_0 H_1 \left(\frac{\mathbb{1} - P_0}{H_0 - E_0} \right)^2 H_1 |\varphi_0\rangle = \frac{1}{32} P_0 \sum_{\langle kl \rangle} Z_k Z_l \sum_{\langle ij \rangle} Z_i Z_j |\varphi_0\rangle = \tag{3.75}$$

$$= \frac{1}{32} P_0 \sum_{\langle ij \rangle} Z_i^2 Z_j^2 |\varphi_0\rangle = \tag{3.76}$$

$$= \frac{N}{32} |\varphi_0\rangle . \tag{3.77}$$

This means that the following expression can be used to determine the wave function up to the second order in λ:

$$|\psi_2\rangle \equiv \exp\left(\lambda^2 \sum_{(ijk)_c} \frac{1}{16} Z_i Z_k \right) \exp\left(\lambda \sum_{\langle ij \rangle} \frac{Z_i Z_j}{4} \right) |\varphi_0\rangle . \tag{3.78}$$

3.2 Varying Perturbative Parameters in Exponentiated Wave Functions

After deriving the exponentiated forms of the wave functions calculated through perturbative expansion for both the ferromagnetic and polarized phase of the many-body Hamiltonian (3.2), I will proceed to calculate the density operator in the relevant orders of magnitude of the disorder parameter. By doing so, I will open up a pathway to a straightforward calculation of other interesting observables for this system. In this section I will only present the more interesting two-dimensional case.

3.2.1 Calculation for a Ferromagnetic State

In the following calculations I will use the same setup as in Subsection 3.1.1 and derive the relevant expressions for a two-dimensional system. However, the perturbation parameters will not be constant, but will instead vary for each site of the lattice, chosen from a Gaussian distribution centered around zero. In other words, in this subsection, the full Hamiltonian will have the form $H = H_0 + V$, where $H_0 = -\sum_{\langle jk \rangle} Z_j Z_k$ and $V = -\sum_j v_j X_j$, with parameters $v_j \in \mathbb{R}$, for which it holds that $\|v_j\| \ll 1$.

In the first order of perturbation, the correction to the wave function of the system equals:

$$|\varphi_{\Delta 1}\rangle = \sum_j v_j \frac{1}{8} X_j |\varphi_0\rangle \tag{3.79}$$

and the exponentiated form of the full wave function can be written as:

$$e^{\sum_j v_j \frac{1}{8} X_j} = \mathbb{1} + \sum_j v_j \frac{1}{8} X_j + \frac{1}{2} \cdot \left(\sum_j v_j^2 \frac{1}{64} + \sum_{j \neq k} v_j v_k \frac{1}{64} X_j X_k \right) + \mathcal{O}\left((v_j)^3 \right). \tag{3.80}$$

In the second order of perturbation, the correction to the wave function equals:

$$|\varphi_{\Delta 2}\rangle = \sum_{(jk)_d} v_j v_k \frac{X_j X_k}{64} |\varphi_0\rangle + \sum_{(jk)_n} v_j v_k \frac{X_j X_k}{48} |\varphi_0\rangle , \tag{3.81}$$

where the indices $(ij)_d$ and $(ij)_n$ indicate that the respective pairs of distant (d) and neighboring (n) indices were counted only once in the sums.

Then, the exponentiated wave function for the full system can be derived using a procedure similar to that of Section 3.1, by setting up its structure as follows.

$$e^{\nu\sum_k\sum_l v_k v_l}e^{\sum_j v_j\frac{1}{8}X_j}=\left(\mathbb{1}+\nu\sum_{k,l}v_k v_l\right)\left(\mathbb{1}+\sum_j v_j\frac{1}{8}X_j+\frac{1}{128}\sum_{m,n}v_m v_n X_m X_n\right)+\mathcal{O}(v_j^3)= \tag{3.82}$$

$$=\mathbb{1}+\nu\sum_{k,l}v_k v_l+\sum_j v_j\frac{1}{8}X_j+\frac{1}{128}\sum_{m,n}v_m v_n X_m X_n+\mathcal{O}(v_j^3)= \tag{3.83}$$

$$=\mathbb{1}+\frac{1}{8}\sum_j v_j X_j+\nu\sum_{j,k}v_j v_k+\frac{N}{128}\sum_j v_j^2+ \tag{3.84}$$

$$+\frac{1}{64}\sum_{(jk)_n}v_j v_k X_j X_k+\frac{1}{64}\sum_{(jk)_d}v_j v_k X_j X_k+\mathcal{O}(v_j^3) \tag{3.85}$$

From the equivalence requirement for the perturbatively calculated second order wave function, it holds that:

$$\nu\sum_{j,k}v_j v_k+\sum_j\frac{v_j^2}{128}+\sum_{(jk)_n}\frac{v_j v_k}{64}X_j X_k+\sum_{(jk)_d}\frac{v_j v_k}{64}X_j X_k=\sum_{(jk)_n}\frac{v_j v_k}{48}X_j X_k+\sum_{(jk)_d}\frac{v_j v_k}{64}X_j X_k, \tag{3.86}$$

$$\nu\sum_{j,k}v_j v_k=-\frac{1}{128}\sum_j v_j^2+\frac{1}{192}\sum_{(jk)_n}v_j v_k X_j X_k. \tag{3.87}$$

If the form of the exponential function to be added in the second order is chosen as $e^{\sum_j\nu_{jj}v_j^2+\sum_{(jk)_n}\nu_{jk}v_j v_k+\sum_{(jk)_d}\nu_{jk}v_j v_k}$, the equality takes the following form:

$$\sum_j\nu_{jj}v_j^2+\sum_{(jk)_n}\nu_{jk}v_j v_k+\sum_{(jk)_d}\nu_{jk}v_j v_k=\frac{1}{192}\sum_{(jk)_n}v_j v_k X_j X_k-\frac{1}{128}\sum_j v_j^2, \tag{3.88}$$

which gives:

$$\nu_{jk}=\begin{cases}-\frac{1}{128} & \text{if } j=k;\\ \frac{1}{192}X_j X_k & \text{if } j,k \text{ are nearest neighbors;}\\ 0 & \text{otherwise.}\end{cases} \tag{3.89}$$

However, if the wave function in the second order of perturbation is normalized using

$$\frac{1}{2}P_0 V\left(\frac{\mathbb{1}-P_0}{H_0-P_0}\right)^2 V|\varphi_0\rangle=-\frac{1}{2}P_0 V\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)\sum_j v_j\frac{1}{8}X_j|\varphi_0\rangle= \tag{3.90}$$

$$=\frac{1}{128}P_0\left(\sum_k v_k^2+\sum_{j\neq k}v_k X_k v_j X_j\right)|\varphi_0\rangle= \tag{3.91}$$

$$=\frac{1}{128}\sum_j v_j^2|\varphi_0\rangle, \tag{3.92}$$

it holds that:

$$e^{\sum_{(jk)_n} \nu_{jk} v_j v_k} e^{\sum_j v_j \frac{1}{8} X_j} = \left(\mathbb{1} + \sum_{(jk)_n} \nu_{jk} v_j v_k \right) \cdot$$

$$\cdot \left(\mathbb{1} + \sum_j v_j \frac{1}{8} X_j + \frac{1}{128} \sum_{j,k} v_j v_k X_j X_k \right) + \mathcal{O}(v_j^3) = \tag{3.93}$$

$$= \mathbb{1} + \sum_j v_j \frac{1}{8} X_j + \frac{1}{64} \sum_{(jk)_n} v_j v_k X_j X_k + \frac{1}{64} \sum_{(jk)_d} v_j v_k X_j X_k + \frac{1}{192} \sum_{(jk)_n} v_j v_k X_j X_k + \mathcal{O}(v_j^3) = \tag{3.94}$$

$$= \mathbb{1} + \sum_j v_j \frac{1}{8} X_j + \frac{1}{48} \sum_{(jk)_n} v_j v_k X_j X_k + \frac{1}{64} \sum_{(jk)_d} v_j v_k X_j X_k + \mathcal{O}(v_j^3). \tag{3.95}$$

Thus, the correct exponentiated wave function valid up to the second order is:

$$|\psi_2\rangle = e^{\sum_{(jk)_n} \frac{1}{192} v_j v_k X_j X_k} e^{\sum_j v_j \frac{1}{8} X_j} |\varphi_0\rangle . \tag{3.96}$$

The expectation value for the density operator of this wave function can be calculated as:

$$\rho = \mathbb{E}_v \left[|\varphi(v)\rangle \langle \varphi(v)| \right], \tag{3.97}$$

where $|\varphi(v)\rangle$ is normalized. The main part of the Hamiltonian alone would support a wave function of the form $|\varphi_0\rangle = \{|+Z\rangle, |-Z\rangle\}$; for these calculations, as an example, I will use the form $|\varphi_0\rangle = \frac{1}{\sqrt{2}} (|+Z\rangle + |-Z\rangle)$, which is normalized. The density operator is, then:

$$\rho_0 = |\varphi_0\rangle \langle \varphi_0| = \tag{3.98}$$

$$= \frac{1}{2} \left(\otimes_j |+Z\rangle_j \langle +Z|_j + \otimes_j |+Z\rangle_j \langle -Z|_j + \otimes_j |-Z\rangle_j \langle +Z|_j + \otimes_j |-Z\rangle_j \langle -Z|_j \right). \tag{3.99}$$

In the first order of perturbation, the wave function $|\varphi_1\rangle = e^{\sum_j v_j \frac{1}{8} X_j} |\varphi_0\rangle$ can be

normalized using $N = \prod_j \cosh\left(\frac{1}{4}v_j\right)$[1]:

$$|\varphi_{1n}\rangle = \frac{1}{\sqrt{N}}|\varphi_1\rangle = \tag{3.100}$$

$$= \prod_j \frac{e^{\frac{1}{8}v_j X_j}}{\left(\cosh\left(\frac{1}{4}v_j\right)\right)^{\frac{1}{2}}}|\varphi_0\rangle = \tag{3.101}$$

$$= \prod_j \frac{\cosh\left(\frac{1}{8}v_j\right) + \sinh\left(\frac{1}{8}v_j\right) X_j}{\left(\left(\cosh\left(\frac{1}{8}v_j\right)\right)^2 + \left(\sinh\left(\frac{1}{8}v_j\right)\right)^2\right)^{\frac{1}{2}}}|\varphi_0\rangle = \tag{3.102}$$

$$= \frac{1}{\sqrt{2}}\left(\otimes_j \frac{\cosh\left(\frac{1}{8}v_j\right)|+Z_j\rangle + \sinh\left(\frac{1}{8}v_j\right)|-Z_j\rangle}{\left(\cosh\left(\frac{1}{8}v_j\right)^2 + \left(\sinh\left(\frac{1}{8}v_j\right)\right)^2\right)^{\frac{1}{2}}} + \otimes_j \frac{\cosh\left(\frac{1}{8}v_j\right)|-Z_j\rangle + \sinh\left(\frac{1}{8}v_j\right)|+Z_j\rangle}{\left(\cosh\left(\frac{1}{8}v_j\right)^2 + \left(\sinh\left(\frac{1}{8}v_j\right)\right)^2\right)^{\frac{1}{2}}}\right) = \tag{3.103}$$

$$= \left(2\prod_j \left(\cosh^2\left(\frac{1}{8}v_j\right) + \sinh^2\left(\frac{1}{8}v_j\right)\right)\right)^{-\frac{1}{2}} \cdot$$
$$\cdot \left(\otimes_j \left(\cosh\left(\frac{1}{8}v_j\right)|+Z_j\rangle + \sinh\left(\frac{1}{8}v_j\right)|-Z_j\rangle\right) + \right.$$
$$\left. + \otimes_j \left(\cosh\left(\frac{1}{8}v_j\right)|-Z_j\rangle + \sinh\left(\frac{1}{8}v_j\right)|+Z_j\rangle\right)\right). \tag{3.104}$$

Further, to calculate the corresponding density operator, I will first define ζ_+ and ζ_-.

$$\zeta_+ \equiv \mathbb{E}_v\left[\cosh^2\left(\frac{1}{8}v_j\right)\right] = \tag{3.105}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \frac{1}{\cosh^2\left(\frac{1}{8}v_j\right) + \sinh^2\left(\frac{1}{8}v_j\right)} \cosh^2\left(\frac{1}{8}v_j\right) = \tag{3.106}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \frac{\cosh^2\left(\frac{1}{8}v_j\right)}{\cosh^2\left(\frac{1}{8}v_j\right) + \sinh^2\left(\frac{1}{8}v_j\right)} = \tag{3.107}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \frac{\cosh\left(\frac{1}{4}v_j\right) + 1}{2\cosh\left(\frac{1}{4}v_j\right)} = \tag{3.108}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \left(\frac{1}{2} + \frac{1}{2\cosh\left(\frac{1}{4}v_j\right)}\right) \tag{3.109}$$

$$\zeta_- \equiv \mathbb{E}_v\left[\sinh^2\left(\frac{1}{8}v_j\right)\right] = \tag{3.110}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \frac{1}{\cosh^2\left(\frac{1}{8}v_j\right) + \sinh^2\left(\frac{1}{8}v_j\right)} \sinh^2\left(\frac{1}{8}v_j\right) = \tag{3.111}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \frac{\sinh^2\left(\frac{1}{8}v_j\right)}{\cosh^2\left(\frac{1}{8}v_j\right) + \sinh^2\left(\frac{1}{8}v_j\right)} = \tag{3.112}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \frac{\sinh\left(\frac{1}{4}v_j\right) - 1}{2\cosh\left(\frac{1}{4}v_j\right)} = \tag{3.113}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}}\mathrm{d}v_j e^{-\frac{1}{2\sigma^2}v_j^2} \left(-\frac{1}{2} + \frac{1}{2\cosh\left(\frac{1}{4}v_j\right)}\right) \tag{3.114}$$

[1]The normalization calculation is included in the Appendix here.

Then, the density operator can be calculated as follows:

$$\rho = \mathbb{E}_v\left[|\varphi_{1n}\rangle\langle\varphi_{1n}|\right] = \tag{3.115}$$

$$= \mathbb{E}_v\Bigg[\frac{1}{2}\prod_j \frac{1}{\cosh^2\left(\frac{1}{8}v_j\right)+\sinh^2\left(\frac{1}{8}v_j\right)}\cdot$$
$$\cdot\left(\otimes_j\left(\cosh\left(\frac{1}{8}v_j\right)|+Z_j\rangle+\sinh\left(\frac{1}{8}v_j\right)|-Z_j\rangle\right)+\otimes_j\left(\cosh\left(\frac{1}{8}v_j\right)|-Z_j\rangle+\sinh\left(\frac{1}{8}v_j\right)|+Z_j\rangle\right)\right)\cdot$$
$$\cdot\left(\otimes_j\left(\cosh\left(\frac{1}{8}v_j\right)\langle+Z_j|+\sinh\left(\frac{1}{8}v_j\right)\langle-Z_j|\right)+\otimes_j\left(\cosh\left(\frac{1}{8}v_j\right)\langle-Z_j|+\sinh\left(\frac{1}{8}v_j\right)\langle+Z_j|\right)\right)\Bigg] = \tag{3.116}$$

$$= \frac{1}{2}\Bigg(\otimes_j\left(\zeta_+|+Z_j\rangle\langle-Z_j|+\zeta_-|-Z_j\rangle\langle-Z_j|\right)+\otimes_j\left(\zeta_+|+Z_j\rangle\langle-Z_j|+\zeta_-|-Z_j\rangle\langle+Z_j|\right)+$$
$$+\otimes_j\left(\zeta_+|-Z_j\rangle\langle+Z_j|+\zeta_-|+Z_j\rangle\langle-Z_j|\right)+\otimes_j\left(\zeta_+|-Z_j\rangle\langle-Z_j|+\zeta_-|+Z_j\rangle\langle+Z_j|\right)\Bigg), \tag{3.117}$$

with the mixed terms that contain both sinh and cosh having been integrated to zero.

In the second order of perturbation, the exponentiated wave function is of the form $|\varphi_2\rangle = e^{\sum_{(jk)_n}\frac{1}{192}v_jv_kX_jX_k}e^{\sum_j v_j\frac{1}{8}X_j}|\varphi_0\rangle$. It can be normalized using the following:

$$N = \langle\varphi_2\,|\,\varphi_2\rangle = \tag{3.118}$$
$$= \langle+Z_0|\,e^{\sum_{(jk)_n}\frac{1}{192}v_jv_kX_jX_k}e^{\sum_j v_j\frac{1}{8}X_j}e^{\sum_{(lm)_n}\frac{1}{192}v_lv_mX_lX_m}e^{\sum_l v_l\frac{1}{8}X_l}|+Z_0\rangle = \tag{3.119}$$
$$= \langle+Z_0|\prod_{(jk)_n}e^{\frac{1}{192}v_jv_kX_jX_k}\prod_j e^{\frac{v_j}{8}X_j}\prod_{(lm)_n}e^{\frac{1}{192}v_lv_mX_lX_m}\prod_l e^{\frac{v_l}{8}X_l}|+Z_0\rangle = \tag{3.120}$$
$$= \langle+Z_0|\prod_j\left(\cosh\left(\frac{v_j}{8}\right)\mathbb{1}+\sinh\left(\frac{v_j}{8}\right)X_j\right)\prod_{(jk)_n}\left(\cosh\left(\frac{1}{192}v_jv_k\right)\mathbb{1}+\sinh\left(\frac{1}{192}v_jv_k\right)X_jX_k\right)\cdot$$
$$\cdot\prod_{(lm)_n}\left(\cosh\left(\frac{1}{192}v_lv_m\right)\mathbb{1}+\sinh\left(\frac{1}{192}v_lv_m\right)X_lX_m\right)\prod_l\left(\cosh\left(\frac{v_l}{8}\right)\mathbb{1}+\sinh\left(\frac{v_l}{8}\right)X_l\right)|+Z_0\rangle = \tag{3.121}$$
$$= \langle+Z_0|\prod_j\left(\cosh\left(\frac{v_j}{8}\right)\mathbb{1}+\sinh\left(\frac{v_j}{8}\right)X_j\right)\prod_{(jk)_n}\left(\cosh\left(\frac{v_jv_k}{192}\right)\mathbb{1}+\sinh\left(\frac{v_jv_k}{192}\right)X_jX_k\right)\cdot$$
$$\cdot\prod_{(lm)_n}\left(\cosh\left(\frac{v_lv_m}{192}\right)\mathbb{1}+\sinh\left(\frac{v_lv_m}{192}\right)X_lX_m\right)\prod_l\left(\cosh\left(\frac{v_l}{8}\right)\mathbb{1}+\sinh\left(\frac{v_l}{8}\right)X_l\right)|+Z_0\rangle = \tag{3.122}$$
$$= \langle+Z_0|\left(\prod_j\left(\cosh\left(\frac{v_j}{8}\right)\mathbb{1}+\sinh\left(\frac{v_j}{8}\right)X_j\right)\right)^2\left(\prod_{(jk)_n}\left(\cosh\left(\frac{v_jv_k}{192}\right)\mathbb{1}+\sinh\left(\frac{v_jv_k}{192}\right)X_jX_k\right)\right)^2|+Z_0\rangle \tag{3.123}$$

The first product gives:

$$\left(\prod_j\left(\cosh\left(\frac{v_j}{8}\right)\mathbb{1}+\sinh\left(\frac{v_j}{8}\right)X_j\right)\right)^2 =$$
$$= \prod_j\left(\cosh^2\left(\frac{v_j}{8}\right)\mathbb{1}+\sinh^2\left(\frac{v_j}{8}\right)X_j^2+2\cosh\left(\frac{v_j}{8}\right)\sinh\left(\frac{v_j}{8}\right)X_j\right) = \tag{3.124}$$
$$= \prod_j\left(\mathbb{1}+2\sinh\left(\frac{v_j}{8}\right)\mathbb{1}+\sinh\left(\frac{v_j}{4}\right)X_j\right) \tag{3.125}$$

The second product gives:

$$\Big(\prod_{(jk)_n}\Big(\cosh\Big(\frac{v_jv_k}{192}\Big)\mathbb{1}+\sinh\Big(\frac{v_jv_k}{192}\Big)X_jX_k\Big)\Big)^2=$$
$$=\prod_{(jk)_n}\Big(\cosh^2\Big(\frac{v_jv_k}{192}\Big)\mathbb{1}+\sinh^2\Big(\frac{v_jv_k}{192}\Big)\mathbb{1}+2\cosh\Big(\frac{v_jv_k}{192}\Big)\sinh\Big(\frac{v_jv_k}{192}\Big)X_jX_k\Big)= \tag{3.126}$$
$$=\prod_{(jk)_n}\Big(\mathbb{1}+2\sinh^2\Big(\frac{v_jv_k}{192}\Big)\mathbb{1}+\sinh\Big(\frac{v_jv_k}{96}\Big)X_jX_k\Big) \tag{3.127}$$

Obviously, the final sum in the first product leads to nothing on its own, so it can be disregarded except in combination with the second product. The only combination that works for the final part of the second product on its own is the 'full circle', which gives:

$$\prod_{(jk)_{n;all}}\sinh\Big(\frac{v_jv_k}{96}\Big)=\sinh^N\Big(\frac{v_jv_k}{96}\Big) \tag{3.128}$$

The aforementioned product of the sums within the two products that contain non-identity operators and results in a non-zero expectation value has to contain an even number of X_j operators for every j. This means that the X_j operators from the relevant part of the first product delimit a 'boundary' for the possible sequence of X_jX_k factors from the second product, which refers to a set of $\{j,k\}$ that may but are not necessarily neighboring. Where s is a subset of the full set of links between nearest-neighbor lattice points and S is the set of all s, the boundary ∂s will define the set of lattice points that are considered exactly an odd number of times. The complement of s will be marked as $\bar{s}$.

Thus, the sought after result can be written as follows.

$$N=\langle+Z|\prod_j\Big(\mathbb{1}+2\sinh\Big(\frac{v_j}{8}\Big)\mathbb{1}+\sinh\Big(\frac{v_j}{4}\Big)X_j\Big)\cdot$$
$$\cdot\prod_{(jk)_n}\Big(\mathbb{1}+2\sinh^2\Big(\frac{v_jv_k}{192}\Big)\mathbb{1}+\sinh\Big(\frac{v_jv_k}{96}\Big)X_jX_k\Big)|+Z\rangle= \tag{3.129}$$
$$=\langle+Z|\,\mathbb{1}\sum_{s\in S}\prod_{(jk)_n\notin s}\Big(1+2\sinh^2\Big(\frac{v_jv_k}{192}\Big)\Big)\prod_{(jk)_n\in s}\sinh\Big(\frac{v_jv_k}{96}\Big)X_jX_k\cdot$$
$$\cdot\prod_{j\notin\partial s}\Big(1+2\sinh\Big(\frac{v_j}{8}\Big)\Big)\prod_{j\in\partial s}\sinh\Big(\frac{v_j}{4}\Big)X_j|+Z\rangle= \tag{3.130}$$
$$=\sum_{s\in S}\prod_{(jk)_n\in\bar{s}}\Big(1+2\sinh^2\Big(\frac{v_jv_k}{192}\Big)\Big)\prod_{(jk)_n\in s}\sinh\Big(\frac{v_jv_k}{96}\Big)\prod_{j\notin\partial s}\Big(1+2\sinh\Big(\frac{v_j}{8}\Big)\Big)\prod_{j\in\partial s}\sinh\Big(\frac{v_j}{4}\Big) \tag{3.131}$$

In the second order of perturbation, the exponentiated wave function is of the form

$|\varphi_2\rangle = e^{\sum_{(jk)_n} \frac{1}{192} v_j v_k X_j X_k} e^{\sum_j v_j \frac{1}{8} X_j} |\varphi_0\rangle$. It can be normalized using the following:

$$N = \langle \varphi_2 \,|\, \varphi_2 \rangle = \tag{3.132}$$

$$= \langle +Z_0 | \, e^{\sum_{(jk)_n} \frac{1}{192} v_j v_k X_j X_k} e^{\sum_j v_j \frac{1}{8} X_j} e^{\sum_{(lm)_n} \frac{1}{192} v_l v_m X_l X_m} e^{\sum_l v_l \frac{1}{8} X_l} \, |+Z_0\rangle = \tag{3.133}$$

$$= \langle +Z_0 | \prod_{(jk)_n} e^{\frac{1}{192} v_j v_k X_j X_k} \prod_j e^{\frac{v_j}{8} X_j} \prod_{(lm)_n} e^{\frac{1}{192} v_l v_m X_l X_m} \prod_l e^{\frac{v_l}{8} X_l} \, |+Z_0\rangle = \tag{3.134}$$

$$= \langle +Z_0 | \prod_j \Big(\cosh\Big(\frac{v_j}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_j}{8}\Big) X_j \Big) \prod_{(jk)_n} \Big(\cosh\Big(\frac{1}{192} v_j v_k\Big) \mathbb{1} + \sinh\Big(\frac{1}{192} v_j v_k\Big) X_j X_k \Big) \cdot$$
$$\cdot \prod_{(lm)_n} \Big(\cosh\Big(\frac{1}{192} v_l v_m\Big) \mathbb{1} + \sinh\Big(\frac{1}{192} v_l v_m\Big) X_l X_m \Big) \prod_l \Big(\cosh\Big(\frac{v_l}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_l}{8}\Big) X_l \Big) |+Z_0\rangle = \tag{3.135}$$

$$= \langle +Z_0 | \prod_j \Big(\cosh\Big(\frac{v_j}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_j}{8}\Big) X_j \Big) \prod_{(jk)_n} \Big(\cosh\Big(\frac{v_j v_k}{192}\Big) \mathbb{1} + \sinh\Big(\frac{v_j v_k}{192}\Big) X_j X_k \Big) \cdot$$
$$\cdot \prod_{(lm)_n} \Big(\cosh\Big(\frac{v_l v_m}{192}\Big) \mathbb{1} + \sinh\Big(\frac{v_l v_m}{192}\Big) X_l X_m \Big) \prod_l \Big(\cosh\Big(\frac{v_l}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_l}{8}\Big) X_l \Big) |+Z_0\rangle = \tag{3.136}$$

$$= \langle +Z_0 | \Big(\prod_j \Big(\cosh\Big(\frac{v_j}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_j}{8}\Big) X_j \Big) \Big)^2 \Big(\prod_{(jk)_n} \Big(\cosh\Big(\frac{v_j v_k}{192}\Big) \mathbb{1} + \sinh\Big(\frac{v_j v_k}{192}\Big) X_j X_k \Big) \Big)^2 |+Z_0\rangle \tag{3.137}$$

The first product gives

$$\Big(\prod_j \Big(\cosh\Big(\frac{v_j}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_j}{8}\Big) X_j \Big) \Big)^2 =$$
$$= \prod_j \Big(\cosh^2\Big(\frac{v_j}{8}\Big) \mathbb{1} + \sinh^2\Big(\frac{v_j}{8}\Big) X_j^2 + 2 \cosh\Big(\frac{v_j}{8}\Big) \sinh\Big(\frac{v_j}{8}\Big) X_j \Big) = \tag{3.138}$$
$$= \prod_j \Big(\mathbb{1} + 2 \sinh\Big(\frac{v_j}{8}\Big) \mathbb{1} + \sinh\Big(\frac{v_j}{4}\Big) X_j \Big) \tag{3.139}$$

The second product gives

$$\Big(\prod_{(jk)_n} \Big(\cosh\Big(\frac{v_j v_k}{192}\Big) \mathbb{1} + \sinh\Big(\frac{v_j v_k}{192}\Big) X_j X_k \Big) \Big)^2 =$$
$$= \prod_{(jk)_n} \Big(\cosh^2\Big(\frac{v_j v_k}{192}\Big) \mathbb{1} + \sinh^2\Big(\frac{v_j v_k}{192}\Big) \mathbb{1} + 2 \cosh\Big(\frac{v_j v_k}{192}\Big) \sinh\Big(\frac{v_j v_k}{192}\Big) X_j X_k \Big) = \tag{3.140}$$
$$= \prod_{(jk)_n} \Big(\mathbb{1} + 2 \sinh^2\Big(\frac{v_j v_k}{192}\Big) \mathbb{1} + \sinh\Big(\frac{v_j v_k}{96}\Big) X_j X_k \Big) \tag{3.141}$$

Obviously, the final sum in the first product leads to nothing on its own, so it can be disregarded except in combination with the second product. The only combination that works for the final part of the second product on its own is the 'full circle', which gives

$$\prod_{(jk)_{n;all}} \sinh\Big(\frac{v_j v_k}{96}\Big) = \sinh^N\Big(\frac{v_j v_k}{96}\Big) \tag{3.142}$$

The aforementioned product of the sums within the two products that contain non-identity operators and results in a non-zero expectation value has to contain an even

number of X_j operators for every j. This means that the X_j operators from the relevant part of the first product delimit a 'boundary' for the possible sequence of X_jX_k factors from the second product, which refers to a set of $\{j,k\}$ that may but are not necessarily neighboring. Where s is a subset of the full set of links between nearest-neighbor lattice points and S is the set of all s, the boundary ∂s will define the set of lattice points that are considered exactly an odd number of times. The complement of s will be marked as $\bar{s}$.

Thus, the sought after result can be written as follows.

$$N = \langle +Z| \prod_j \Big(\mathbb{1} + 2\sinh\Big(\frac{v_j}{8}\Big)\mathbb{1} + \sinh\Big(\frac{v_j}{4}\Big) X_j \Big) \cdot$$
$$\cdot \prod_{(jk)_n} \Big(\mathbb{1} + 2\sinh^2\Big(\frac{v_j v_k}{192}\Big)\mathbb{1} + \sinh\Big(\frac{v_j v_k}{96}\Big) X_j X_k \Big) |+Z\rangle = \tag{3.143}$$
$$= \langle +Z| \mathbb{1} \sum_{s\in S} \prod_{(jk)_n \notin s} \Big(1 + 2\sinh^2\Big(\frac{v_j v_k}{192}\Big)\Big) \prod_{(jk)_n \in s} \sinh\Big(\frac{v_j v_k}{96}\Big) X_j X_k \cdot$$
$$\cdot \prod_{j\notin\partial s} \Big(1 + 2\sinh\Big(\frac{v_j}{8}\Big)\Big) \prod_{j\in\partial s} \sinh\Big(\frac{v_j}{4}\Big) X_j |+Z\rangle = \tag{3.144}$$
$$= \sum_{s\in S} \prod_{(jk)_n \in \bar{s}} \Big(1 + 2\sinh^2\Big(\frac{v_j v_k}{192}\Big)\Big) \prod_{(jk)_n \in s} \sinh\Big(\frac{v_j v_k}{96}\Big) \prod_{j\notin\partial s} \Big(1 + 2\sinh\Big(\frac{v_j}{8}\Big)\Big) \prod_{j\in\partial s} \sinh\Big(\frac{v_j}{4}\Big) \tag{3.145}$$

The calculation of the density operator can be performed in the following fashion.

$$\rho_2 = \mathbb{E}_v \left[\frac{1}{N} e^{\sum_{(jk)_n} \frac{1}{192} v_j v_k X_j X_k} e^{\sum_j v_j \frac{1}{8} X_j} |\varphi_0\rangle \langle\varphi_0| e^{\sum_{(jk)_n} \frac{1}{192} v_j v_k X_j X_k} e^{\sum_j v_j \frac{1}{8} X_j} \right] = \tag{3.146}$$
$$= \frac{1}{\big(\sigma\sqrt{2\pi}\big)^n} \int \frac{e^{-\frac{1}{2\sigma^2}v_1^2} e^{-\frac{1}{2\sigma^2}v_2^2} \dots e^{-\frac{1}{2\sigma^2}v_n^2}}{N} \mathrm{d}v_1 \mathrm{d}v_2 \dots \mathrm{d}v_n \cdot$$
$$\cdot \prod_{(jk)_n} e^{\frac{1}{192} v_j v_k X_j X_k} \prod_j e^{\frac{1}{8} v_j X_j} |+Z\rangle \langle +Z| \prod_m e^{\frac{1}{8} v_m X_m} \prod_{(lm)_n} e^{\frac{1}{192} v_l v_m X_l X_m} = \tag{3.147}$$
$$= \frac{1}{\big(\sigma\sqrt{2\pi}\big)^n} \int \frac{e^{-\frac{1}{2\sigma^2}v_1^2} e^{-\frac{1}{2\sigma^2}v_2^2} \dots e^{-\frac{1}{2\sigma^2}v_n^2}}{N} \mathrm{d}v_1 \mathrm{d}v_2 \dots \mathrm{d}v_n \tag{3.148}$$
$$\prod_{(jk)_n} \Big(\cosh\Big(\frac{v_j v_k}{192}\Big)\mathbb{1} + \sinh\Big(\frac{v_j v_k}{192}\Big) X_j X_k \Big) \prod_j \Big(\cosh\Big(\frac{v_j}{8}\Big)\mathbb{1} + \sinh\Big(\frac{v_j}{8}\Big) X_j \Big) |+Z\rangle$$
$$\langle +Z| \prod_m \Big(\cosh\Big(\frac{v_m}{8}\Big)\mathbb{1} + \sinh\Big(\frac{v_m}{8}\Big) X_m \Big) \prod_{(lm)_n} \Big(\cosh\Big(\frac{v_l v_m}{192}\Big)\mathbb{1} + \sinh\Big(\frac{v_l v_m}{192}\Big) X_l X_m \Big) = \tag{3.149}$$

$$=\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n}\int\frac{e^{-\frac{1}{2\sigma^2}v_1^2}e^{-\frac{1}{2\sigma^2}v_2^2}...\,e^{-\frac{1}{2\sigma^2}v_n^2}}{N}\mathrm{d}v_1\mathrm{d}v_2...\,\mathrm{d}v_n\cdot$$
$$\cdot\prod_{(jk)_n}\left(\cosh\left(\frac{v_jv_k}{192}\right)\mathbb{1}+\sinh\left(\frac{v_jv_k}{192}\right)X_jX_k\right)\otimes_j\left(\cosh\left(\frac{v_j}{8}\right)+\sinh\left(\frac{v_j}{8}\right)X_j\right)|+Z_j\rangle$$
$$\otimes_m\langle+Z_m|\left(\cosh\left(\frac{v_m}{8}\right)+\sinh\left(\frac{v_m}{8}\right)X_m\right)\prod_{(lm)_n}\left(\cosh\left(\frac{v_lv_m}{192}\right)\mathbb{1}+\sinh\left(\frac{v_lv_m}{192}\right)X_lX_m\right)= \tag{3.150}$$

$$=\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n}\int\frac{e^{-\frac{1}{2\sigma^2}v_1^2}e^{-\frac{1}{2\sigma^2}v_2^2}...\,e^{-\frac{1}{2\sigma^2}v_n^2}}{N}\mathrm{d}v_1\mathrm{d}v_2...\,\mathrm{d}v_n\cdot$$
$$\cdot\prod_{(jk)_n}\left(\cosh\left(\frac{v_jv_k}{192}\right)\mathbb{1}+\sinh\left(\frac{v_jv_k}{192}\right)X_jX_k\right)\otimes_j\left(\cosh\left(\frac{v_j}{8}\right)|+Z_j\rangle+\sinh\left(\frac{v_j}{8}\right)|-Z_j\rangle\right)\cdot$$
$$\cdot\otimes_m\left(\langle+Z_m|\cosh\left(\frac{v_m}{8}\right)+\langle-Z_m|\sinh\left(\frac{v_m}{8}\right)\right)\prod_{(lm)_n}\left(\cosh\left(\frac{v_lv_m}{192}\right)\mathbb{1}+\sinh\left(\frac{v_lv_m}{192}\right)X_lX_m\right) \tag{3.151}$$

A more detailed evaluation of the form of the density operator can be performed in conjunction with the calculation of, e.g., the expectation value of a specific local operator defined in this system, and is not included in this work.

Using the density operator, it is now possible to calculate expectation values for some interesting observables. As a quick demonstrative example, for the wave function of the system equal to $|+Z\rangle$ (and in the zeroth order of perturbation), it is straightforward to show that:

$$\langle Z_j\rangle=\int\mathrm{d}p\,(v)\,\mathrm{Tr}\,(Z_j\rho)= \tag{3.152}$$
$$=\frac{1}{\left(\sigma\sqrt{2\pi}\right)^n}\int_{v_1}...\int_{v_n}\mathrm{d}v_1...\,\mathrm{d}v_ne^{-\frac{1}{2\sigma^2}v_1^2}...\,e^{-\frac{1}{2\sigma^2}v_n}\,\mathrm{Tr}(Z_j\,|\varphi_0\rangle\,\langle\varphi_0|)= \tag{3.153}$$
$$=1. \tag{3.154}$$

Similarly, for $|-Z\rangle$ it is $\langle Z_j\rangle=-1$. On the other hand, for $\langle X_j\rangle$ in $|+Z\rangle$, it holds that $\langle X_j\rangle=0$.

In the first order for $|+Z\rangle$, $|\varphi_{1n}\rangle=\otimes_j\frac{\cosh(\frac{1}{8}v_j)|+Z_j\rangle+\sinh(\frac{1}{8}v_j)|-Z_j\rangle}{\left(\cosh(\frac{1}{8}v_j)^2+(\sinh(\frac{1}{8}v_j))^2\right)^{\frac{1}{2}}}$, so these expectation

values give the following results.

$$\langle Z_j \rangle = \int \mathrm{d}p(v)\, \mathrm{Tr}\left(Z_j \rho\right) = \tag{3.155}$$

$$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \sum_k \langle \xi_k |\, Z_j \otimes_l \left(\int_v \mathrm{d}v \frac{e^{-\frac{1}{2\sigma^2}v^2}}{\sigma\sqrt{2\pi}} \frac{\cosh^2\left(\frac{v}{8}\right)}{\cosh\left(\frac{v}{8}\right)} |+Z_l\rangle \langle +Z_l| + \right. \tag{3.156}$$

$$\left. + \int_v \mathrm{d}v \frac{e^{-\frac{1}{2\sigma^2}v^2}}{\sigma\sqrt{2\pi}} \frac{\sinh^2\left(\frac{v}{8}\right)}{\cosh\left(\frac{v}{8}\right)} |-Z_l\rangle \langle -Z_l| \right) |\xi_k\rangle \equiv$$

$$\equiv \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \sum_k \langle \xi_k |\, Z_j \otimes_l \left(\alpha\, |+Z_l\rangle \langle +Z_l| + \beta\, |-Z_l\rangle \langle -Z_l|\right) |\xi_k\rangle = \tag{3.157}$$

$$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \cdot$$

$$\cdot \sum_k \left(\langle \xi_k |\, Z_j \otimes_l \alpha\, |+Z_l\rangle \langle +Z_l\, |\, \xi_k\rangle + \langle \xi_k |\, Z_j \otimes_l \beta\, |-Z_l\rangle \langle -Z_l\, |\, \xi_k\rangle \right) = \tag{3.158}$$

$$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \left(\langle +Z |\, Z_j \otimes_l \alpha\, |+Z_l\rangle + \langle -Z |\, Z_j \otimes_l \beta\, |-Z_l\rangle \right) = \tag{3.159}$$

$$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \left(\alpha - \beta\right) = \tag{3.160}$$

$$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \prod_m \int_{v_m} \mathrm{d}v_m \frac{e^{-\frac{1}{2\sigma}v_m^2}}{\sigma\sqrt{2\pi}} \frac{1}{\cosh\left(\frac{v_m}{4}\right)} = \tag{3.161}$$

$$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^{2n}} \prod_l \int_{v_l} \mathrm{d}v_l e^{-\frac{1}{2\sigma^2}v_l^2} \prod_m \int_{v_m} \mathrm{d}v_m \frac{e^{-\frac{1}{2\sigma}v_m^2}}{\cosh\left(\frac{v_m}{4}\right)}, \tag{3.162}$$

where

$$\alpha \equiv \int \mathrm{d}v \frac{e^{-\frac{1}{2\sigma^2}v^2}}{\sigma\sqrt{2\pi}} \frac{\cosh^2\left(\frac{v}{8}\right)}{\cosh\left(\frac{v}{8}\right)}, \quad \beta \equiv \int \mathrm{d}v \frac{e^{-\frac{1}{2\sigma^2}v^2}}{\sigma\sqrt{2\pi}} \frac{\sinh^2\left(\frac{v}{8}\right)}{\cosh\left(\frac{v}{8}\right)}. \tag{3.163}$$

In the first order, it remains that $\langle X_j \rangle = 0$.

3.2.2 Calculation for a Polarized State

In the ferromagnetic phase, the Hamiltonian of the system is structured as $H = H_0 + V$, where $H_0 = -\sum_j v_j X_j$ and $V = -\sum_{\langle jk \rangle} Z_j Z_k$, with the parameter $v_j \in \mathbb{R}$ sampled from a Gaussian distribution centered around zero and $\|V\| \ll \|H_0\|$. The ground state of the system is

$$|\varphi_0\rangle = \otimes_j \left|\mathrm{sgn}\left(v_j\right) X_j\right\rangle, \tag{3.164}$$

as the eigenstate for each site depends on the sign of the corresponding parameter in the Hamiltonian.

To derive the exponentiated wave function in the first order of perturbation, I first need to calculate several energy equalities:

$$H_0 |\varphi_0\rangle = \left(-\sum_j v_j X_j \right) \otimes_k |\mathrm{sgn}(v_k) X_k\rangle = \tag{3.165}$$

$$= \left(-\sum_j \mathrm{sgn}(v_j) v_j \right) |\varphi_0\rangle = E_0 |\varphi_0\rangle \tag{3.166}$$

and

$$H_0 Z_j Z_k \left|\varphi_0\right\rangle = \left(-\sum_j v_j X_j\right) Z_k Z_l \otimes_m \left|\mathrm{sgn}(v_m) X_m\right\rangle = \tag{3.167}$$

$$= \sum_j \left(-v_j X_j Z_k Z_l \otimes_m \left|\mathrm{sgn}(v_m) X_m\right\rangle\right) = \tag{3.168}$$

$$= \left(-\sum_{j\neq\{k,l\}} \mathrm{sgn}(v_j) v_j + \mathrm{sgn}(v_k) v_k + \mathrm{sgn}(v_l) v_l\right) Z_k Z_l \left|\varphi_0\right\rangle . \tag{3.169}$$

Thus, the difference in energies needed to complete the perturbative calculation is $2\,\mathrm{sgn}(v_k)v_k + 2\,\mathrm{sgn}(v_l)v_l$. It follows that:

$$\left|\varphi_1\right\rangle = -\frac{\mathbb{1} - P_0}{H_0 - E_0} V \left|\varphi_0\right\rangle = \tag{3.170}$$

$$= \frac{\mathbb{1} - P_0}{H_0 - E_0} \sum_{\langle ij\rangle} Z_i Z_j \left|\varphi_0\right\rangle = \tag{3.171}$$

$$= \sum_{\langle jk\rangle} \frac{Z_j Z_k}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)} \left|\varphi_0\right\rangle , \tag{3.172}$$

and I can check if the exponentiated form of the system's wave function up to the first order corresponds to the following:

$$e^{\sum_{\langle jk\rangle} \frac{Z_j Z_k}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)}} = \mathbb{1} + \sum_{\langle jk\rangle} \frac{1}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)} Z_j Z_k +$$
$$+ \frac{1}{2} \sum_{\langle jk\rangle} \sum_{\langle mn\rangle} \frac{1}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)\left(2\,\mathrm{sgn}(v_m)v_m + 2\,\mathrm{sgn}(v_n)v_n\right)} Z_j Z_k Z_m Z_n + \mathcal{O}(v_j^3) = \tag{3.173}$$

$$= \mathbb{1} + \sum_{\langle jk\rangle} \frac{1}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)} Z_j Z_k +$$
$$+ \sum_{\langle jk\rangle} \frac{1}{2} \frac{1}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)^2} +$$
$$+ \frac{1}{2} \sum_{(jkl)_c} \frac{2}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)\left(2\,\mathrm{sgn}(v_k)v_k + 2\,\mathrm{sgn}(v_l)v_l\right)} Z_j Z_l +$$
$$+ \frac{1}{2} \sum_{(\langle jk\rangle,\langle mn\rangle)_d} \frac{2}{\left(2\,\mathrm{sgn}(v_j)v_j + 2\,\mathrm{sgn}(v_k)v_k\right)\left(2\,\mathrm{sgn}(v_m)v_m + 2\,\mathrm{sgn}(v_n)v_n\right)} Z_j Z_k Z_m Z_n + \mathcal{O}(v_j^3) = \tag{3.174}$$

$$= \mathbb{1} + \sum_{\langle jk\rangle} \frac{1}{2\left(\mathrm{sgn}(v_j)v_j + \mathrm{sgn}(v_k)v_k\right)} Z_j Z_k +$$
$$+ \sum_{\langle jk\rangle} \frac{1}{8\left(\mathrm{sgn}(v_j)v_j + \mathrm{sgn}(v_k)v_k\right)^2} +$$
$$+ \sum_{(jkl)_c} \frac{1}{4\left(\mathrm{sgn}(v_j)v_j + \mathrm{sgn}(v_k)v_k\right)\left(\mathrm{sgn}(v_k)v_k + \mathrm{sgn}(v_l)v_l\right)} Z_j Z_l +$$
$$+ \sum_{(\langle jk\rangle,\langle mn\rangle)_d} \frac{1}{4\left(\mathrm{sgn}(v_j)v_j + \mathrm{sgn}(v_k)v_k\right)\left(\mathrm{sgn}(v_m)v_m + \mathrm{sgn}(v_n)v_n\right)} Z_j Z_k Z_m Z_n + \mathcal{O}(v_j^3), \tag{3.175}$$

where the indices $(\langle jk\rangle, \langle mn\rangle)_d$ and $(jkl)_c$ denote the non-overlapping and clustering combinations of indices, respectively; all combinations have been counted only once.

In the second order of perturbation, the correction to the base wave function of the system can be calculated as follows:

$$|\varphi_{\Delta 2}\rangle = -\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2 V P_0 V |\varphi_0\rangle + \frac{\mathbb{1}-P_0}{H_0-E_0} V \frac{\mathbb{1}-P_0}{H_0-E_0} V |\varphi_0\rangle = \tag{3.176}$$

$$= \frac{\mathbb{1}-P_0}{H_0-E_0} \sum_{\langle mn\rangle} Z_m Z_n \sum_{\langle jk\rangle} \frac{Z_j Z_k}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_k)v_k\right)} |\varphi_0\rangle . \tag{3.177}$$

If $\langle mn\rangle = \langle jk\rangle$, the total of that summand equals zero, as the Pauli operators give $\mathbb{1}$. This implies the following.

$$|\varphi_{\Delta 2}\rangle = \sum_{\langle jk\rangle \neq \langle mn\rangle} \frac{\mathbb{1}}{H_0-E_0} \frac{1}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_k)v_k\right)} Z_m Z_n Z_j Z_k |\varphi_0\rangle = \tag{3.178}$$

$$= \sum_{(\langle jk\rangle \neq \langle mn\rangle)_d} \frac{\mathbb{1}}{2\left(\operatorname{sgn}(v_m)v_m + \operatorname{sgn}(v_n)v_n\right)} \frac{2}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_k)v_k\right)} Z_m Z_n Z_j Z_k |\varphi_0\rangle +$$

$$+ \sum_{(jkl)_c} \frac{1}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_l)v_l\right)} \frac{\mathbb{1}}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_k)v_k\right)} Z_j Z_l |\varphi_0\rangle + \tag{3.179}$$

$$+ \sum_{(jkl)_c} \frac{1}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_l)v_l\right)} \frac{\mathbb{1}}{2\left(\operatorname{sgn}(v_k)v_k + \operatorname{sgn}(v_l)v_l\right)} Z_j Z_l |\varphi_0\rangle = \tag{3.180}$$

$$= \sum_{(\langle jk\rangle \neq \langle mn\rangle)_d} \frac{\mathbb{1}}{2\left(\operatorname{sgn}(v_m)v_m + \operatorname{sgn}(v_n)v_n\right)} \frac{1}{2\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_k)v_k\right)} Z_m Z_n Z_j Z_k |\varphi_0\rangle +$$

$$+ \sum_{(jkl)_c} \frac{\mathbb{1}\cdot\left(\operatorname{sgn}(v_j)v_j + 2\operatorname{sgn}(v_k)v_k + \operatorname{sgn}(v_l)v_l\right)}{4\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_l)v_l\right)\left(\operatorname{sgn}(v_j)v_j + \operatorname{sgn}(v_k)v_k\right)\left(\operatorname{sgn}(v_k)v_k + \operatorname{sgn}(v_l)v_l\right)} Z_j Z_l |\varphi_0\rangle \tag{3.181}$$

The expressions in (3.179) and (3.180) cannot be fully unified as they are not symmetric. If I use the following definition:

$$\iota_{jk} \equiv \frac{1}{\left(2\operatorname{sgn}(v_j)v_j + 2\operatorname{sgn}(v_k)v_k\right)}, \tag{3.182}$$

the full second order wave function is:

$$|\varphi_2\rangle = \left(\mathbb{1} + \sum_{\langle jk\rangle} Z_j Z_k \iota_{jk} + \sum_{\langle jk\rangle \neq \langle mn\rangle} \iota_{jk}\iota_{mn} Z_j Z_k Z_m Z_n + \right. \tag{3.183}$$

$$\left. + \sum_{(jkl)_c} 2\iota_{jk}\iota_{kl}\iota_{jl} \cdot \left(\operatorname{sgn}(v_j)v_j + 2\operatorname{sgn}(v_k)v_k + \operatorname{sgn}(v_l)v_l\right) Z_j Z_l \right) |\varphi_0\rangle \tag{3.184}$$

If I calculate the trial exponentiated wave function, I can obtain the following:

$$e^{\sum_{(jkl)_c}\nu\iota_{jk}\iota_{kl}}e^{\sum_{\langle jk\rangle}Z_jZ_k\iota_{jk}}=\left(\mathbb{1}+\nu\sum_{(jkl)}\iota_{jk}\iota_{kl}\right)\cdot$$

$$\cdot\left(\mathbb{1}+\sum_{\langle jk\rangle}Z_jZ_k\iota_{jk}+\frac{1}{2}\sum_{\langle jk\rangle}\sum_{\langle mn\rangle}Z_jZ_kZ_mZ_n\iota_{jk}\iota_{mn}\right)+\mathcal{O}(\iota^3)= \tag{3.185}$$

$$=\mathbb{1}+\nu\sum_{(jkl)_c}\iota_{jk}\iota_{kl}+\sum_{\langle jk\rangle}Z_jZ_k\iota_{jk}+\frac{1}{2}\sum_{\langle jk\rangle}\iota_{jk}^2+\sum_{\langle jk\rangle\neq\langle mn\rangle}Z_jZ_kZ_mZ_n\iota_{jk}\iota_{mn}+$$

$$+\sum_{(jkl)_c}\iota_{jk}\iota_{kl}Z_jZ_l+\mathcal{O}(\iota^3). \tag{3.186}$$

For ν, this gives:

$$\nu\sum_{(jkl)_c}\iota_{jk}\iota_{kl}=\sum_{(jkl)_c}2\iota_{jk}\iota_{kl}\left(\iota_{jl}\cdot(\mathrm{sgn}(v_j)v_j+2\,\mathrm{sgn}(v_k)v_k+\mathrm{sgn}(v_l)v_l)-1\right)Z_jZ_l-\frac{1}{2}\sum_{\langle jk\rangle}\iota_{jk}^2. \tag{3.187}$$

If the derived wave function $|\varphi_2\rangle$ is normalized using

$$\frac{1}{2}P_0V\left(\frac{\mathbb{1}-P_0}{H_0-E_0}\right)^2V|\varphi_0\rangle=\frac{1}{2}P_0\sum_{\langle mn\rangle}\sum_{\langle jk\rangle}\iota_{jk}^2Z_jZ_kZ_mZ_n= \tag{3.188}$$

$$=\frac{1}{2}\sum_{\langle jk\rangle}\iota_{jk}^2, \tag{3.189}$$

it holds that:

$$\nu\sum_{(jkl)_c}\iota_{jk}\iota_{kl}=\sum_{(jkl)_c}2\iota_{jk}\iota_{kl}\left(\iota_{jl}\cdot(\mathrm{sgn}(v_j)v_j+2\,\mathrm{sgn}(v_k)v_k+\mathrm{sgn}(v_l)v_l)-1\right)Z_jZ_l. \tag{3.190}$$

This means that the parameter ν must depend on the components of this exponent, i.e.,

$$\nu_{jkl}=2\left(\iota_{jl}\cdot(\mathrm{sgn}(v_j)v_j+2\,\mathrm{sgn}(v_k)v_k+\mathrm{sgn}(v_l)v_l)-1\right)Z_jZ_l. \tag{3.191}$$

The full exponentiated wave function valid up to the second order of perturbation then has the following form:

$$e^{\sum_{(jkl)_c}\nu_{jkl}\iota_{jk}\iota_{kl}}e^{\sum_{\langle jk\rangle}Z_jZ_k\iota_{jk}}=$$

$$=e^{\sum_{(jkl)_c}2(\iota_{jl}\cdot(\mathrm{sgn}(v_j)v_j+2\,\mathrm{sgn}(v_k)v_k+\mathrm{sgn}(v_l)v_l)-1)Z_jZ_l\iota_{jk}\iota_{kl}}e^{\sum_{\langle jk\rangle}Z_jZ_k\iota_{jk}}. \tag{3.192}$$

To calculate the density operator for this system in higher orders of perturbation, I will first approach that for the zeroth order. The wave function in question, $|\varphi_0\rangle=\otimes_j\left|\mathrm{sgn}(v_j)X_j\right\rangle$, is normalized, and leads to the density operator:

$$\rho_0=\otimes_j\left|\mathrm{sgn}(v_j)X_j\right\rangle\otimes_k\left\langle\mathrm{sgn}(v_k)X_k\right|= \tag{3.193}$$

$$=\otimes_j\frac{1}{2}\left(\mathbb{1}+\mathrm{sgn}(v_j)X_j\right). \tag{3.194}$$

In the first order, however, the calculation already becomes more complicated. To

normalize $|\varphi_1\rangle = e^{\sum_{\langle jk\rangle} \iota_{jk} Z_j Z_k} |\varphi_0\rangle$, the first step is to find its norm.

$$N = \langle \varphi_1 | \varphi_1 \rangle = \tag{3.195}$$

$$= \langle \varphi_0 | e^{\sum_{\langle jk\rangle} \iota_{jk} Z_j Z_k} e^{\sum_{\langle jk\rangle} \iota_{jk} Z_j Z_k} |\varphi_0\rangle = \tag{3.196}$$

$$= \langle \varphi_0 | \prod_{\langle jk\rangle} e^{\frac{1}{\mathrm{sgn}(v_j)v_j + \mathrm{sgn}(v_k)v_k} Z_j Z_k} |\varphi_0\rangle = \tag{3.197}$$

$$= \prod_{\langle jk\rangle} \left(\otimes_l \langle \mathrm{sgn}(v_l) X_l |\right) \left(\mathbb{1} + 2\iota_{jk} Z_j Z_k + \frac{1}{2}\left(2\iota_{jk} Z_j Z_k\right)^2 + \frac{1}{3!}\left(2\iota_{jk} Z_j Z_k\right)^3 + ...\right) \cdot$$

$$\cdot \left(\otimes_m |\mathrm{sgn}(v_m) X_m \rangle\right) = \tag{3.198}$$

$$= \prod_{\langle jk\rangle} \cosh\left(2\iota_{jk}\right) = \prod_{\langle jk\rangle} \cosh\left(\frac{1}{\mathrm{sgn}(v_j)v_j + \mathrm{sgn}(v_k)v_k}\right) \tag{3.199}$$

The normalized state can then be calculated. The calculation is laborious and does not help further elucidate the topic, so it has been omitted from this work.

3.3 Bridging between Variational Solutions using Tensor Networks

In this section I will show how the calculated exponentiated wave functions can be used to bridge between two variational solutions for the two robust phases of the system described in (3.2). Because of the perturbative structure of the system, these equalities can be used to construct tensors, which allows for an investigation of this system using the tensor network framework. In addition, I will compare the achieved results with a standard, perturbative numerical analysis of this system.

Take the transverse-field Ising model (TFIM) defined on a square lattice:

$$H_{TFIM} = -\lambda_A \sum_i X_i - \lambda_B \sum_{\langle ij\rangle} Z_i Z_j, \tag{3.200}$$

where X_i and Z_i are X and Z Pauli operators, respectively, acting on the site i. This model has a phase transition between a polarized phase and a symmetry-broken phase, and the ground state of $H_B = -\lambda_A \sum_{\langle ij\rangle} Z_i Z_j$ is doubly degenerate. The method I will describe and build upon in this section is presented by Vanderstraeten et al., in their article titled *Bridging Perturbative Expansions with Tensor Networks* [65]. The authors choose to address the problem of interpolating between the two known ground states at either ends, $|\psi_A\rangle = |+X\rangle$ at $\lambda_B = 0$ and $|\psi_B\rangle = |\pm Z\rangle$ at $\lambda_A = 0$ by building a tensor network state based on the perturbative expansion around each of those two points.

The perturbative expansion for each of the constituents of the Hamiltonian can be described in an exponential form. The one corresponding to the ferromagnetic state, where λ_A is suppressed, is calculated for the chief Hamiltonian H_B and from the state $|\psi_B\rangle$. In the first order, $E_B = \langle \psi_B | X_i | \psi_B \rangle$ and the system's wave function can be written as:

$$|\psi\rangle = \left(\mathbb{1} + \frac{\lambda_A}{8\lambda_B} \sum_i X_i\right) |\psi_B\rangle, \tag{3.201}$$

which I have shown in Section 3.1. This equation can then be exponentiated, leading to:

$$|\varphi\rangle = \exp\left(\frac{\lambda_A}{8\lambda_B}\sum_i X_i\right)|\psi_B\rangle = \tag{3.202}$$

$$= \left(\mathbb{1} + \frac{\lambda_A}{8\lambda_B}\sum_i X_i + \frac{\lambda_A^2}{64\lambda_B^2}\sum_{(ij)} X_iX_j + \mathcal{O}(\lambda_A^3)\right)|\psi_B\rangle\,, \tag{3.203}$$

where $\sum_{(ij)}$ denotes the sum over all pairs of sites for which $i \neq j$. This expression doesn't only reproduce the first order of perturbation, but it also contains a summand from the next higher order. In the second order, the perturbative wave function has the following form:

$$|\psi\rangle = \left(\mathbb{1} + \frac{\lambda_A}{8\lambda_B}\sum_i X_i + \frac{\lambda_A^2}{64\lambda_B^2}\sum_{(ij)_d} X_iX_j + \frac{\lambda_A^2}{48\lambda_B^2}\sum_{(ij)_n} X_iX_j\right)|\psi_B\rangle\,, \tag{3.204}$$

where $\sum_{(ij)_n}$ denotes a sum over all nearest neighbor pairs of sites, whereas $\sum_{(ij)_d}$ denotes a sum over all non-neighboring (i.e., distant) ones. The exponentiated version of this state equals:

$$|\varphi\rangle = \exp\left(\frac{\lambda_A^2}{192\lambda_B^2}\sum_{(ij)_n} X_iX_j\right)\exp\left(\frac{\lambda_A}{8\lambda_B}\sum_i X_i\right)|\psi_B\rangle\,, \tag{3.205}$$

where the second order of the perturbative wave function is replicated, but the result also contains summands of higher orders. In a one-dimensional system, the use of this exponentiation technique results in the following wave function:

$$|\varphi\rangle = \exp\left(\frac{\lambda_A^2}{16\lambda_B^2}\sum_{(ij)_d} X_iX_j\right)\exp\left(\frac{\lambda_A}{4\lambda_B}\sum_i X_i\right)|\psi_B\rangle\,. \tag{3.206}$$

In the case where $\lambda_B = 0$ and the chief Hamiltonian is H_A, the first order of perturbation brings forth this ground state wave function:

$$|\psi\rangle = \left(\mathbb{1} + \frac{\lambda_B}{4\lambda_A}\sum_{\langle ij\rangle} Z_iZ_j\right)|\psi_A\rangle\,, \tag{3.207}$$

which upon exponentiation becomes:

$$|\varphi\rangle = \exp\left(\sum_{\langle ij\rangle}\frac{\lambda_B}{4\lambda_A}Z_iZ_j\right)|\psi_A\rangle \sim \tag{3.208}$$

$$\sim \left(\mathbb{1} + \frac{\lambda_B}{4\lambda_A}\sum_{\langle ij\rangle} Z_iZ_j + \frac{\lambda_B^2}{16\lambda_A^2}\sum_{(ijk)_s} Z_iZ_j^2Z_l + \frac{\lambda_B^2}{16\lambda_A^2}\sum_{(\langle ij\rangle,\langle kl\rangle)_d} Z_iZ_jZ_kZ_l + \mathcal{O}\left(\lambda_B^3\right)\right)|\psi_A\rangle\,, \tag{3.209}$$

where $(\langle ij\rangle, \langle kl\rangle)_d$ are pairs of nearest-neighbor pairs that do not possess overlapping indices and $(ijk)_s$ represents three-site clusters of indices, where j is the index of the central site of the cluster. In a similar vein, the second order wave function is:

$$|\psi\rangle = \left(\mathbb{1} + \frac{\lambda_B}{4\lambda_A}\sum_{\langle ij\rangle} Z_iZ_j + \frac{\lambda_B^2}{16\lambda_A^2}\sum_{(\langle ij\rangle,\langle kl\rangle)_d} Z_iZ_jZ_kZ_l + \frac{\lambda_B^2}{8\lambda_A^2}\sum_{(ijk)_s} Z_iZ_j^2Z_k\right)|\psi_A\rangle\,. \tag{3.210}$$

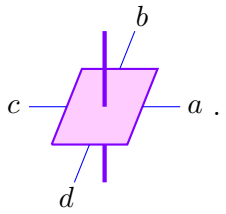
Its exponentiated form is the following:

$$|\varphi\rangle = \exp\left(\frac{\lambda_B^2}{16\lambda_A^2}\sum_{(ijk)_s} Z_i Z_j^2 Z_k\right)\exp\left(\frac{\lambda_B}{4\lambda_A}\sum_{\langle ij\rangle} Z_i Z_j\right)|\psi_A\rangle\,, \tag{3.211}$$

which clearly shows the advantages of the exponentiated forms of the wave functions obtained by perturbation theory calculations. In a one-dimensional system, the resultant exponentiated wave function maintains the same form.

Construction of a Tensor Network Depiction

For the construction of tensor network equivalents for these perturbative calculations, it is crucial to clearly define the use of tensor network operators in the construction of physical wave functions. The elementary building block of this system is a tensor operator with four virtual indices (labeled with a, b, c, and d) and two physical ones to represent action on one of the spins in the system (shown in bold):



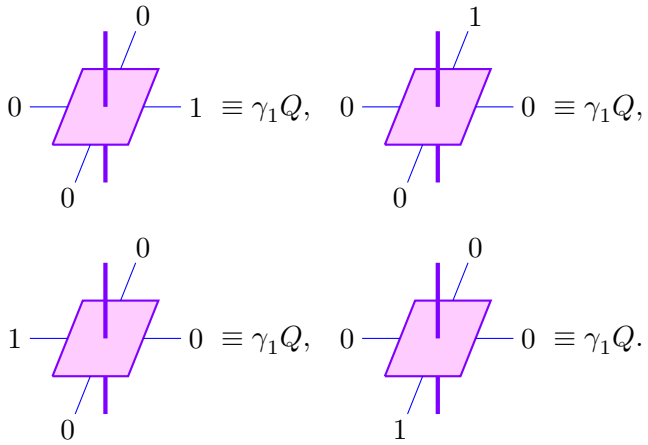
(3.212)

The bond dimension of this tensor network operator is the dimension of its virtual legs, D, and its virtual indices can be contracted with the virtual indices of its neighboring tensors of the lattice that corresponds to the full system, giving rise to a network that can describe it.

The 'zeroth' dimension of the bond dimension of the virtual indices can be assigned so that it equals the perturbative sum, $\mathbb{1} + \beta Q$. Then, the tensor network operator (TNO) can be expanded in the powers of β:

$$O(T) = \mathbb{1} + \beta\sum_i Q_i + \beta^2\sum_{i\neq j} Q_i Q_j + \mathcal{O}(\beta^3). \tag{3.213}$$

Two-site clusters of Q_i can be constructed if a new definition is made, such that the non-zero tensor coefficients are given by



(3.214)

Then, from a single tensor being assigned the expression $\mathbb{1} + \gamma_1 Q_i Q_j$, where i and j denote neighboring tensor sites, it follows that

$$O(T) = \mathbb{1} + \gamma_1 \sum_{\langle ij \rangle} Q_i Q_j + \gamma_1^2 \sum_{(\langle ij \rangle, \langle kl \rangle)_d} Q_i Q_j Q_k Q_l + \mathcal{O}(\gamma_1^3). \tag{3.215}$$

Three-site (and larger) clusters can be defined in a similar manner, and if more levels in the virtual indices are opened, the clusters of different operators can also be included in the TNO build. The index $(ijk)_l$ here denotes all three-site clusters found on a line, and $(ijk)_c$ those that comprise a corner. One tensor cluster can be assigned an expression that encompasses the contracting tensor combinations that can create three-site clusters:

$$\mathbb{1} + \gamma_1 \gamma_2 \gamma_1 Q_i Q_j Q_k + \gamma_1 \gamma_3 \gamma_1 Q_\iota Q_\theta Q_\kappa, \tag{3.216}$$

where sites i, j, and k form a grouping in a line, and sites ι, θ, and κ form a grouping with a corner. Then, the tensor can be expanded as:

$$O(T) = \mathbb{1} + \gamma_1^2 \gamma_2 \sum_{\langle ijk \rangle_l} Q_i Q_j Q_k + \gamma_1^2 \gamma_3 \sum_{\langle \iota\theta\kappa \rangle_c} Q_\iota Q_\theta Q_\kappa, \tag{3.217}$$

where three neighboring sites that form a line are denoted by $\langle ijk \rangle_l$ and three neighboring sites that form a corner by $\langle \iota\theta\kappa \rangle_c$. A line here is defined as depicted in Figure 3.1 for the cluster (mno) and a corner cluster corresponds to the depictions of (skl) and (prq) within the same figure.

For example, the perturbative expansion of the TFIM can be built using TNOs in the following manner. Let $D = 3$ and define the single and two-site TNO as above. Then, around the ferromagnetic state of the model, the TNO is expanded as follows:

$$O(T) = \mathbb{1} + \beta \sum_i X_i + \beta^2 \sum_{(ij)} X_i X_j + \gamma_1^2 \sum_{(ij)_n} X_i X_j + \mathcal{O}(\beta^3) + \gamma_1^4. \tag{3.218}$$

If the parameters are chosen so that this result corresponds to the perturbative solution,

$$\beta = \frac{\lambda_0}{8\lambda_1}, \ \gamma_1 = \sqrt{\frac{\lambda_0^2}{192\lambda_1^2}}. \tag{3.219}$$

Higher orders of the perturbative expansion can be represented by the use of other γ parameters that create larger clusters of X operators.

Around the polarized state, the 'second' level in the bond dimension of the virtual indices is used to assign the δ_ι parameter TNOs. Then, it holds that

$$\begin{aligned} O(T) =& \mathbb{1} + \delta_1^2 \sum_{(ij)_n} Z_i Z_j + \delta_1^4 \sum_{(\langle ij \rangle, \langle kl \rangle)_d} Z_i Z_j Z_k Z_l + \delta_1^2 \delta_2 \sum_{(ijk)_c} Z_i Z_k + \delta_1^2 \delta_3 \sum_{(ijk)_l} Z_i Z_k + \\ &+ \mathcal{O}(\delta_1^6) + \mathcal{O}\left(\delta_1^2 \delta_2^2 + \delta_1^2 \delta_3^2\right). \end{aligned} \tag{3.220}$$

The perturbative wave function can thus be recreated if the parameter values are chosen as:

$$\delta_1 = \sqrt{\frac{\lambda_1}{4\lambda_0}}, \ \delta_2 = \delta_3 = \frac{\lambda_1}{2\lambda_0}. \tag{3.221}$$

Note that the mean-field result is obtained by using only α as the parameter, the first order by keeping $\{\alpha, \beta, \delta_1\}$, and the second by keeping $\{\alpha, \beta, \delta_1, \gamma_1, \delta_2 = \delta_3\}$.

When the TFIM defined on a square lattice is described by a tensor network in its orders of perturbation, by using tensors with $D = 2$, the perturbative wave function can be reproduced. Because of the extensivity of the network formulation, the TNO also creates a summand with disconnected pairs of Z_iZ_j clusters in the second order. The first and second order tensor networks can both be described for $D = 2$, but if they are to be implemented together, $D = 3$ can be chosen. Then, the ansatz wave function can be created by using five variational parameters: (β_0, β_1) for the first order, (γ_0, γ_1) for the second order, and α for the $|\psi_\alpha\rangle$ reference state.

Then, the energy of the system can be computationally minimized and the optimal values for the used parameters can be obtained as a function of λ_A/λ_B. In [65], the authors report that in the first order, a critical transition is found at $\lambda_A/\lambda_B \approx 3.35$, and in the second order at $\lambda_A/\lambda_B = 3.1$, which is fairly close to the purely perturbative result (using Monte Carlo simulations), at $\lambda_A/\lambda_B = 3.04438(2)$.

If a particular phase of a quantum many-body system can be accurately analytically described by a perturbative expansion, the employment of this bridging method would vastly aid the numerical investigation of the phase transition between the starting phase and its neighboring one — given that a wave function can be found that can act as a parametrized problem setter, resulting in one of the phases in one of its extremes, and the other in the other one. Thus, by considering a system for which I have already found an analytical solution for two phases born of varying the impact of a parameter, and creating a tensor network bridge for the wave functions that describe them, I am in the position to efficiently variationally calculate the exact behavior of the system close to the transition point — but also efficiently find the transition point by using a virtual parameter.

To analyze the applicability of the exponentiation to different types of systems, I examined a system with the following Hamiltonian:

$$H = H_{TFIM} + \nu Z_j, \tag{3.222}$$

where ν are variables of a Gaussian distribution with a mean at 0, starting with a mere one-dimensional chain of particles.

3.3.1 Perturbation Theory Applied to Single Impurity Models

Using exponentiated perturbation theory, an efficient ansatz for many-body localized systems can be constructed. Notably, a single impurity model can be solved in an analytical fashion, producing intuitively solid results.

Let the examined Hamiltonian be of the form $H = H_0 + H_1$, where $H_0 = -\sum_{j=1}^{N} x_j X_j$ and $H_1 = -\sum_{\langle j,k\rangle} Z_j Z_k$. In this system, the variables x_j are randomly chosen from a Gaussian distribution. The wave function associated with this system can be calculated perturbatively, as:

$$|\Omega\rangle = |\Omega_1\rangle - \frac{\mathbb{1} - P_1}{H_1 - E_1} V |\Omega_1\rangle + \dots, \tag{3.223}$$

where $|\Omega_1\rangle$ is the wave function of the H_1 Hamiltonian. Assume that the state of the system can be described as $|\Omega\rangle = e^{\kappa \sum_j x_j X_j} |\Omega_1\rangle$ in the first order of perturbation. Then,

it follows that:

$$\overline{\rho} = \mathbb{E}_x \left(|\Omega\rangle \langle\Omega| \right) = \tag{3.224}$$

$$= \frac{1}{\left(\gamma\sqrt{2\pi}\right)^N} \int \prod_{j=1}^{N} e^{=\frac{x_j^2}{2\gamma^2}} e^{\kappa \sum_{j=1}^{N} x_j X_j} |\Omega_1\rangle \langle\Omega_1| e^{\kappa \sum_{k=1}^{N} x_k X_k} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N \sim \tag{3.225}$$

$$\sim \frac{1}{\left(\gamma\sqrt{2\pi}\right)^N} \int e^{-\frac{\|x\|^2}{2\gamma^2}} \left(e^{\kappa \sum_{j=1}^{N} x_j X_j} \right) \otimes \left(e^{\kappa \sum_{k=1}^{N} x_k X_k} \right)^T |\Omega_1\rangle |\Omega_1\rangle \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N = \tag{3.226}$$

$$= \frac{1}{\left(\gamma\sqrt{2\pi}\right)^N} \int e^{-\frac{\|x\|^2}{2\gamma^2}} \left(e^{\kappa \sum_{j=1}^{N} x_j (X_j \otimes \mathbb{1} + \mathbb{1} \otimes X_j)} \right) |\Omega_1\rangle |\Omega_1\rangle \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N, \tag{3.227}$$

by using the Choi–Jamiołkowski isomorphism. The integral of this type can be resolved to a simpler form using the operator Hubbard–Stratonovich transformation.

$$\int e^{-\frac{x^2}{2\gamma^2} + \lambda x} \mathrm{d}x \overset{!}{=} S e^{\lambda^2 \frac{\gamma^2}{2}} \tag{3.228}$$

$$-\frac{x^2}{2\gamma^2} + \lambda x = -\left(\frac{x}{\sqrt{2}\gamma} + \alpha \right)^2 + \text{const.} \tag{3.229}$$

$$\lambda x = -\alpha^2 - \frac{2\alpha x}{\sqrt{2}\gamma} + \text{const.} \tag{3.230}$$

$$\alpha = -\frac{\lambda}{\sqrt{2}} \gamma \tag{3.231}$$

I.e., const. $= \frac{\lambda^2}{\sqrt{2}} \gamma^2$. In this case, the linear correction in the exponential function to be integrated is an operator: $x\mu \left(X_j \otimes \mathbb{1} + \mathbb{1} \otimes X_j \right) \equiv x\lambda$. Thus,

$$\int e^{-\frac{x^2}{2\gamma^2}} + x\lambda = \sqrt{2\pi}\gamma e^{\frac{\gamma^2}{2}\mu^2 \left(X_j \otimes \mathbb{1} + \mathbb{1} \otimes X_j \right)^2}. \tag{3.232}$$

Then, the exponential function can be written as:

$$e^{\gamma^2 \mu^2 \sum_{j=1}^{N} X_j \otimes X_j} = \left(\mathbb{1} + \gamma^2 \mu^2 \sum_{j=1}^{N} X_j \otimes X_j + \ldots \right). \tag{3.233}$$

Thus,

$$|\overline{\rho}\rangle = |\uparrow\uparrow \ldots \uparrow\rangle |\uparrow\uparrow \ldots \uparrow\rangle + \gamma^2 \mu^2 \left| \uparrow\uparrow \ldots \uparrow\downarrow_j \uparrow \ldots \uparrow \right\rangle \left| \uparrow\uparrow \ldots \uparrow\downarrow_j \uparrow \ldots \uparrow \right\rangle + \ldots . \tag{3.234}$$

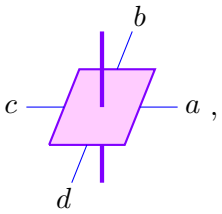
The case in which $\|H_0\| \ll \|H_1\|$ can be addressed using the same method, albeit through a more complicated calculation.

3.3.2 Tensor Construction

In this section I will demonstrate how the derived exponentiated wave functions can be used to build tensors — and then tensor networks — which allow for a speedy and reliable framework to analyze a many-body system, both analytically and numerically.

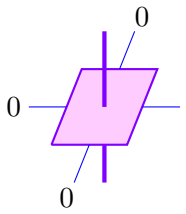
The derived wave function that unifies both the variational solutions for a two-phase quantum many-body system can be represented using tensor networks, as a low-depth quantum circuit, which then provides reliable means for an effective calculation of the expectation values for both these phases. From the linked-cluster theorem [66], it can be deduced that, in the tensor network formalism, the increasing orders of perturbative

solutions can be represented using clusters of local operators of increasing size. The constituent building blocks of these clusters are local operators that act on physical spins, such that $T_{a,b,c,d}$ corresponds to the visual representation:



(3.235)

and the non-named tensor legs stand for the action of tensor onto a physical spin. As an example, say that I define:

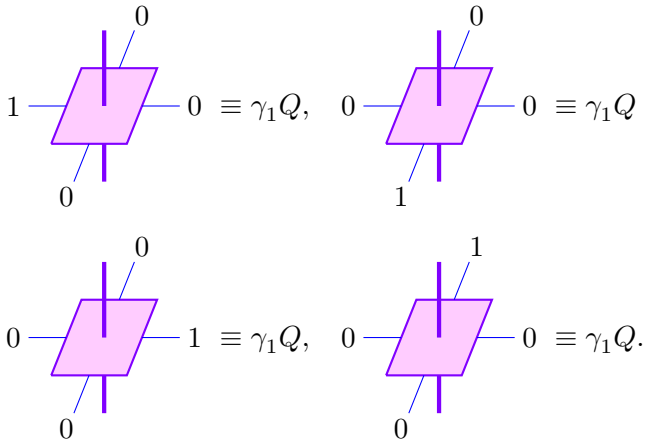


$\equiv \mathbb{1} + \alpha Q,$ (3.236)

where $\alpha \ll 1$, so that there exists a tensor network operator that can be expanded in powers of α:

$$M(T) = \mathbb{1} + \alpha \sum_j Q_i + \alpha^2 \sum_{j \neq k} Q_j Q_k + \mathcal{O}\left(\alpha^3\right), \tag{3.237}$$

where Q_j are the clusters of local operators of the tensor representation. The next step in the cluster construction is the definition of tensors that would allow operations with products of two Q_j-s:



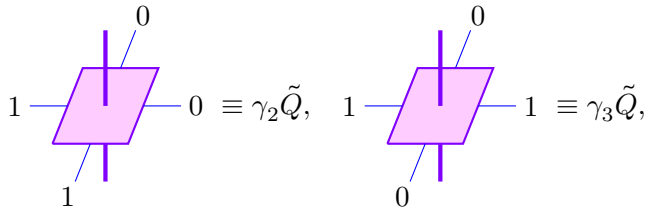
(3.238)

Obviously, these four tensor forms correspond to one another under rotation. It is then evident that the corresponding tensor network operator expansion has the following form:

$$M(T) = \mathbb{1} + \gamma_1 \sum_{\langle jk \rangle} Q_j Q_k + \gamma_1^2 \sum_{\left(\langle jk \rangle, \langle ml \rangle\right)_d} Q_j Q_k Q_l Q_m + \mathcal{O}\left(\gamma_1^3\right), \tag{3.239}$$

where $\langle jk \rangle$ stand for nearest-neighbor pairs of sites and $\left(\langle jk \rangle, \langle lm \rangle\right)_d$ denote such pairs that do not have common elements. Then, to go to further relevant orders of the tensor

network operators, it is necessary to construct tensor clusters that connect three sites. Then, by defining rotationally equivalent:



(3.240)

where these 'middle' cluster tensors correspond to a different operator, $\tilde{Q}$, it is possible to construct the following expansion for the tensor network operator:

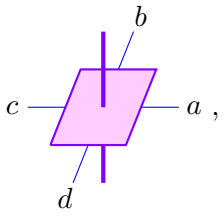
$$M(T) = \mathbb{1} + \gamma_1^2 \gamma_2 \sum_{(jkl)_l} Q_j \tilde{Q}_k Q_l + \gamma_1^2 \gamma_3 \sum_{(jkl)_c} Q_j \tilde{Q}_k Q_l + \mathcal{O}\left(\gamma_1^2 \gamma_2^2\right) + \mathcal{O}\left(\gamma_1^2 \gamma_3^2\right), \quad (3.241)$$

where $(jkl)_l$ show linear three-site clusters, and $(jkl)_c$ those that are connected with a bend, or corner. In a similar manner it is possible to create ever-larger clusters that correspond to higher orders of magnitude for the relevant tensor network operator.

In a system described by the transverse-field Ising model on a square lattice, the construction of these operators proceeds as follows. The usable wave function associated to this state is one that interpolates between the wave functions describing the ground state for each of the relevant phases — $|+X\rangle$ for $\lambda_1 = 0$ and $|+Z\rangle$ for $\lambda_0 = 0$, as one of the symmetry-broken ground states,

$$|\psi_\alpha\rangle = \prod_j \left(\mathbb{1} + \alpha Z_j\right) |+X\rangle . \quad (3.242)$$

The wave function that corresponds to the perturbative solutions in this system can then be represented using a PEPS system with a bond dimension of three:



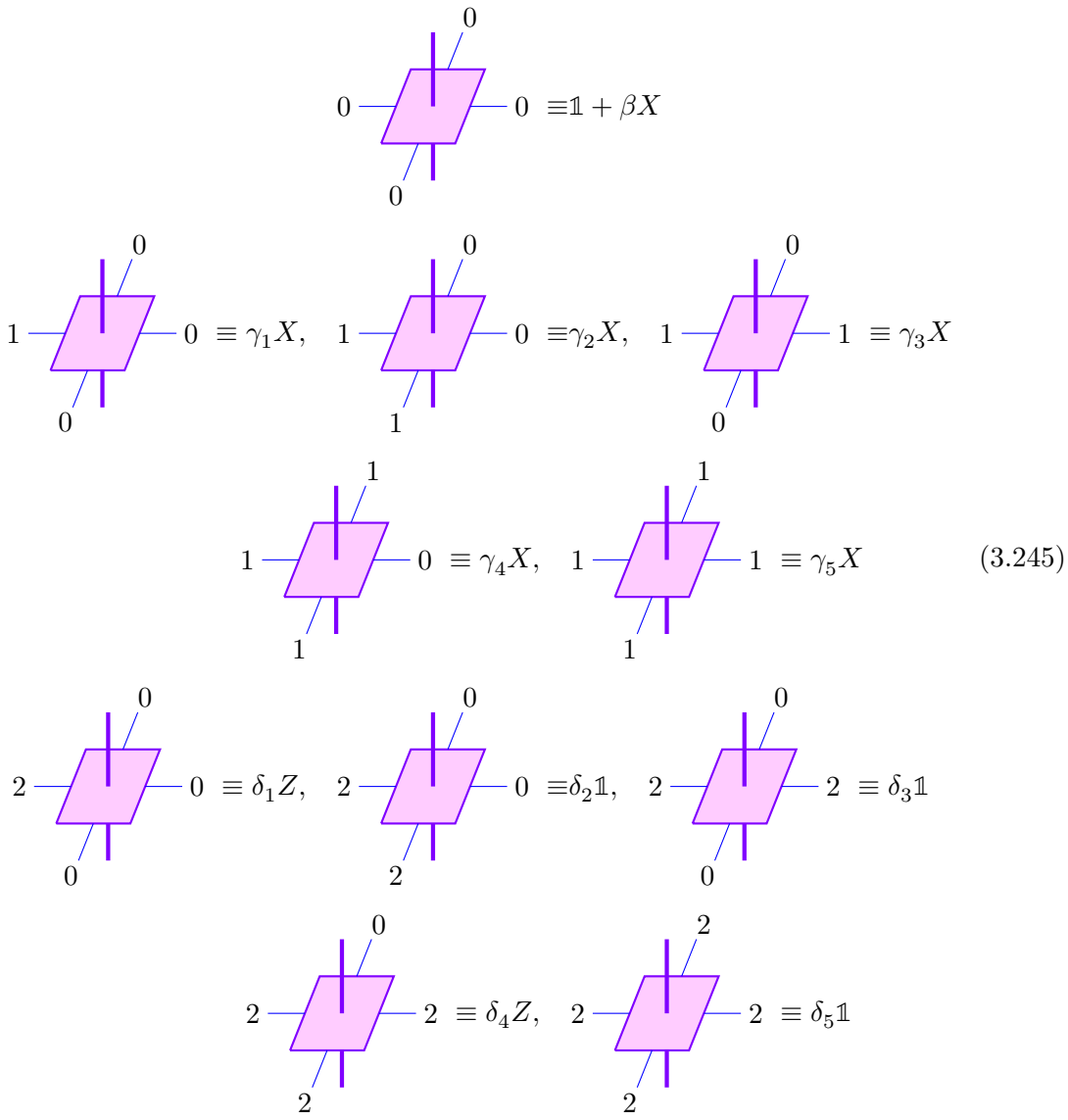
(3.243)

where each virtual leg of the tensor (indicated by the indices a, b, c, and d) can assume values in $\{0, 1, 2\}$. Then, the tensor clusters have to be constructed such that they correspond to the tensor network operator of the form:

$$M(T) = \mathbb{1} + \beta \sum_j X_j + \beta^2 \sum_{(jk)} X_j X_k + \gamma_1^2 \sum_{(jk)_n} X_j X_k + \mathcal{O}\left(\beta^3\right) + \mathcal{O}\left(\gamma_1^4\right), \quad (3.244)$$

where (jk) denote a pair of indices for which $j \neq k$.

Thus, I can define the necessary tensors that are equivalent under rotation as follows:



(3.245)

The parameters $\beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ then define this tensor network operator.

In the ferromagnetic state, $|\psi_1\rangle$, the two-site operator clusters correspond to the usage of the tensor that contains the γ_1 parameter, such that the first two tensors in the above definition already determine:

$$M(T) = \mathbb{1} + \beta \sum_j X_j + \beta^2 \sum_{(jk)} X_j X_k + \gamma_1^2 \sum_{(jk)_n} X_j X_k + \mathcal{O}\left(\beta^3\right) + \mathcal{O}\left(\gamma_1^4\right). \tag{3.246}$$

This expression, of course, holds if:

$$\beta = \frac{\lambda_0}{8\lambda_1}, \quad \gamma_1 = \sqrt{\frac{\lambda_0^2}{192\lambda_1^2}}, \tag{3.247}$$

and the higher orders can be constructed in a straightforward manner from the tensors corresponding to higher order variables γ_ξ. Around the ferromagnetic state, however, it

is necessary to also include three-site clusters to obtain the following expression:

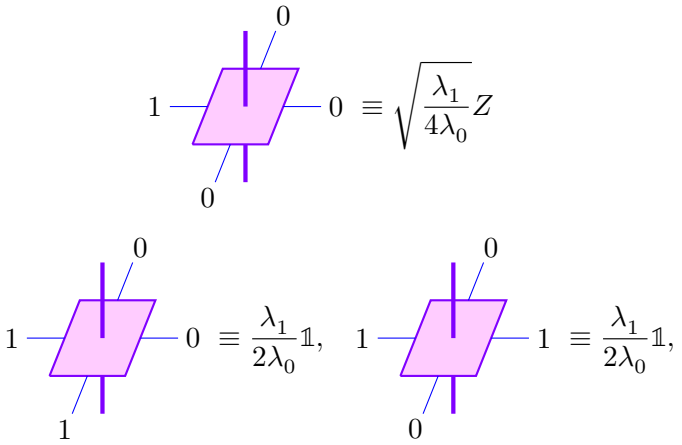
$$M(T) = \mathbb{1} + \delta_1^2 \sum_{(jk)_n} Z_j Z_k + \delta_1^4 \sum_{(\langle jk\rangle,\langle lm\rangle)_d} Z_j Z_k Z_l Z_m + \\ + \delta_1^2\delta_2 \sum_{(jkl)_c} Z_j Z_l + \delta_1^2\delta_3 \sum_{(jkl)_l} Z_j Z_l + \mathcal{O}\left(\delta_1^6\right) + \mathcal{O}\left(\delta_1^2\delta_2^2 + \delta_1^2\delta_3^2\right), \tag{3.248}$$

where $(jkl)_c$ and $(jkl)_l$ are corner and line clusters, respectively. This equation can then be reproduced using the appropriate predefined tensors, with the following parameter values:

$$\delta_1 = \sqrt{\frac{\lambda_1}{4\lambda_0}}, \quad \delta_2 = \delta_3 = \frac{\lambda_1}{2\lambda_0}. \tag{3.249}$$

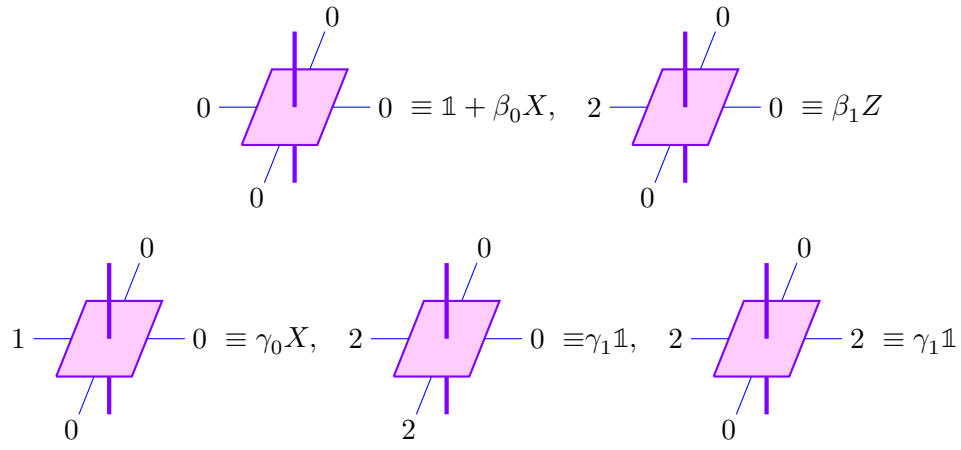
In contrast, the mean-field result for this system can be obtained by only holding onto the parameter α and setting all others to zero.

So, in the transverse-field Ising model, the constructed wave function that bridges between the polarized and the symmetry-broken phase, $|\psi_\alpha\rangle$, already gives the critical point of $\frac{\lambda_0}{\lambda_1} \equiv 4$ for the used α in the zeroth order, which is close to the numerically obtained solution For the tensor cluster, I define the following:



(3.250)

which allows me to construct clusters which correspond to the first-order and second-order perturbative expansions of the used wave function. This PEPS network then has a bond dimension of two. Then, these tensor network clusters can be implemented in a single tensor network, with a bond dimension of three. This allows for the used perturbative coefficients to be used as variational parameters, where (β_0, β_1) correspond to the first-order solution, and (γ_0, γ_1) show that of the second order:



(3.251)

By using the tensor network constructed here, the authors [65] obtain a critical transition for $\frac{\lambda_0}{\lambda_1} = 3.35$ for the first order solution, and $\frac{\lambda_0}{\lambda_1} = 3.1$ for the second-order solution.

3.4 Combining Tensor Bridging with the Derived Perturbative Solution

To now combine the tensor network bridging tactic with the derived perturbative solutions for many-body systems, the procedure shall be as follows:

1. Construct the tensors to be used for the tensor network clusters up to the k-th order.
2. Simplify the analytical integral forms for the perturbative calculations in each identifiable phase of the many-body system.
3. Build an appropriately formulated joint wave function.
4. Numerically explore the system using variational tensor network methods and the built joint wave function.

From the perturbation calculation for the disordered system it can be seen that, to the lowest order, the tensor coefficients have to change in comparison to the tensor coefficients described above, as follows. The β coefficient is now dependent on the random value which corresponds to the site of the tensor:

$$\beta_j = \frac{v_j \lambda_0}{8\lambda_1}, \tag{3.252}$$

and the same thing occurs to γ_1:

$$\gamma_{1;j} = \frac{v_j \lambda_0}{\sqrt{192}\lambda_1}. \tag{3.253}$$

The evaluation is more complicated for the δ coefficient, where the rotational symmetry of the tensor described in (3.245) gets broken. The multiplicative factor for the $\sum_{(jk)_n} Z_j Z_k$ term of the tensor network operator can be read off from (3.172) as $\frac{\lambda_1}{\lambda_0(|v_j|+|v_k|)}$, which doesn't factorize as a hypothetical $v_{1;j}v_{1;k}$. Instead, the following δ_1 parameter can be defined for the part of the tensor at site i, making the connection with the tensor at site j and vice-versa:

$$\delta_{1;jk} = \sqrt{\frac{\lambda_1}{\lambda_0(|v_j| + |v_k|)}}. \tag{3.254}$$

In other words, each of the different rotations of the δ_1-contributions for each site has a different coefficient that depends on the sites which surround the tensor.

3.4.1 Numerical Evaluations via the Tensor Network

With the tensor parameters defined as above, the TNO tensors now depend on random values. To sample properties of the TNO correctly, tensor network operators (or states) may be added to one another, giving the resultant tensor network operator (or state) again. However, building the expectation value of an individual tensor in that way is, in general, difficult. Even though it is possible to add tensor networks to one another, this will, without any further assumptions, come at the high cost of increasing the bond dimension, although it may be feasible for single discrete random variables that come with a handful of possible values. In general, when dealing with probability distributions of tensor network states, they could be encoded into a quantum states on some ancillary

systems. Then, it would be possible to calculate properties of the system by, for example, obtaining the expectation values of local operators.

Unfortunately, for the system of interest here, this would require an additional continuous-variable degree of freedom to be added to each site, which again puts this approach outside of the realm of things that can analyzed numerically. As before, this may be more feasible with a system that is based on discrete random variables, which may even be amenable to numerical study.

One simulation approach that can be performed rather easily with this type of system is to calculate an effective MPS representation of the environment when contracting a PEPS expectation value (for more details of the method itself see Section 4.3). In a straightforward approach, the transfer operator would be applied as an MPO to the MPS environment tensor until a sufficient convergence is reached. However, as mentioned above, the tensors that describe this system contain random variables, and the same will be true for the transfer operators. This means that, in general, one cannot hope for the MPS environment tensor to ever converge.

Instead, the MPO can simply be applied a larger number of times, so the input becomes sufficiently independent from the resulting MPS — then the resulting distribution can be used for sampling.

In contrast to some more accurate methods of sampling, this method comes with the benefit of a more feasible implementation of the necessary calculations in practice.

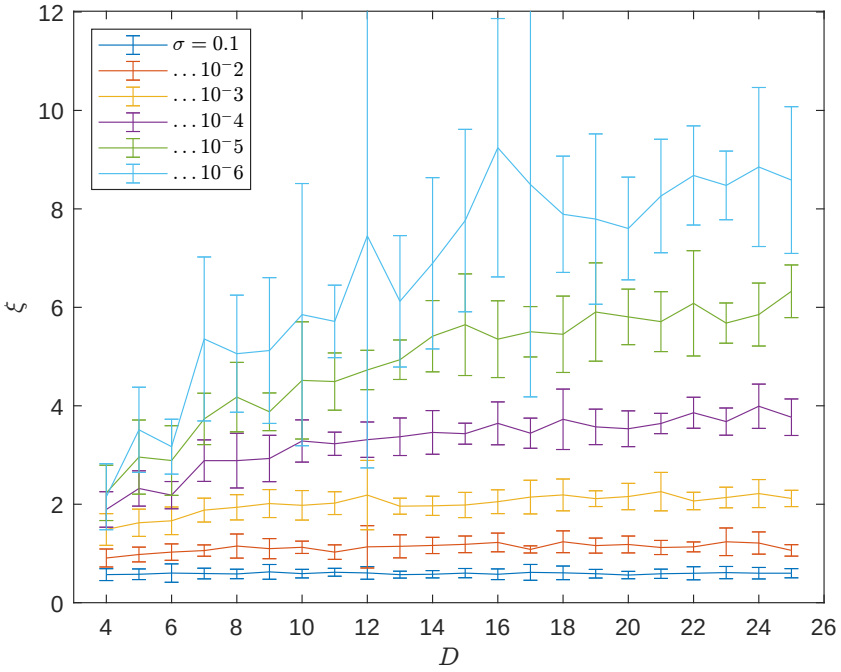
By performing the evaluation described above, I obtained the results shown in Figures 3.2 to 3.5. The results are obtained by implementing the algorithm in MATLAB and running it on a regular home computer.

This was done at the point $\frac{\lambda_0}{\lambda_1} = 10$, i.e., a point far into the polarized phase in the normal TFIM. More points in the phase space were not considered, since the point of these numerics was to investigate the viability of the results in the first place. For the random parameters, the v_i were taken to be normally distributed random variables, with a standard deviation σ. By varying the parameter σ, multiple scenarios from strong to weak influences of randomness were studied. For each point, the MPO was applied for 50 iterations, after which the correlation length was sampled over 50 further iterations. In the figures, each of the shown points is the mean of the correlation length over those 50 iterations, with the standard deviation as error bars.

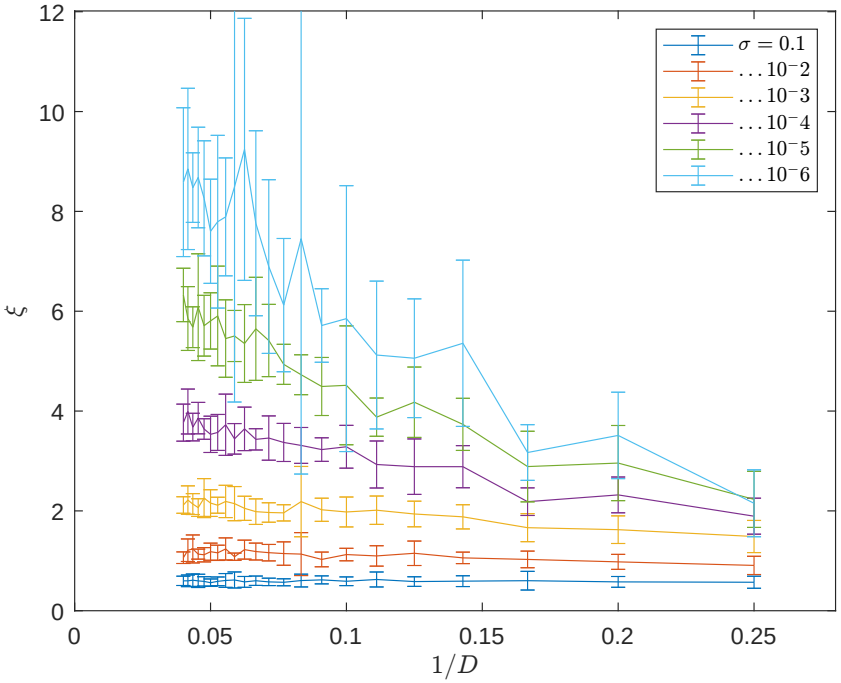
It can be seen that the correlation length drops off drastically for larger σ, which makes sense intuitively, as a more strongly disordered system will exhibit fewer spatial correlations.

Since the algorithm I've used here is based on uniform MPS, it is only really applicable to translationally invariant states. To address this, for one of the spatial dimensions I had sampled an intermediate distribution instead of trying to find a fixed point. For the other spatial dimension, I looked at blocking groups of sites together, so that the MPS needs to only be translationally invariant with respect to the translation of a group of sites. The dependence on the block size can be seen in Figure 3.5.

Again, it can be seen that the correlation length drops off for larger block sizes, which, the larger the block size, should be closer to the *true* value. A more detailed analysis of the scaling behavior has not been performed yet.

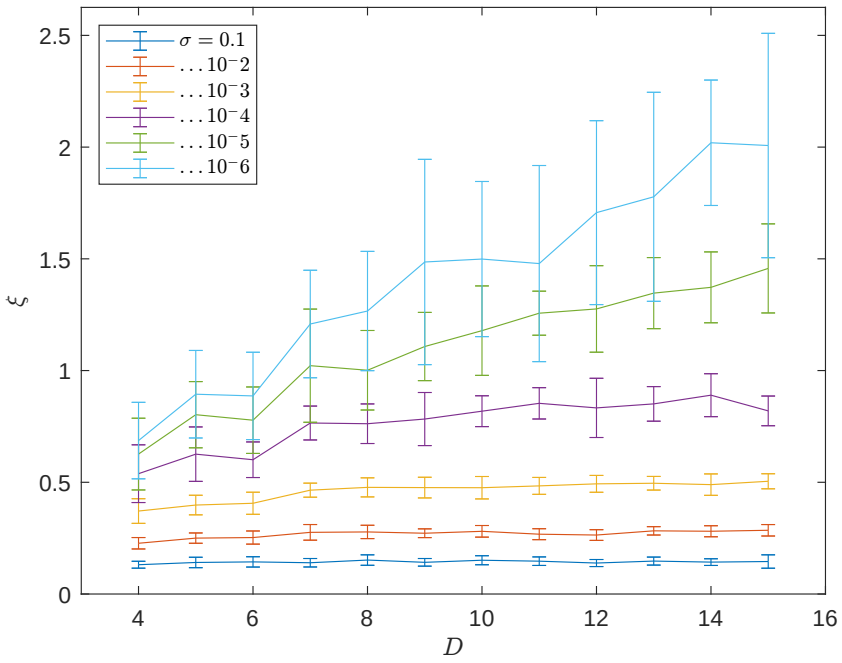


(a) over bond dimension

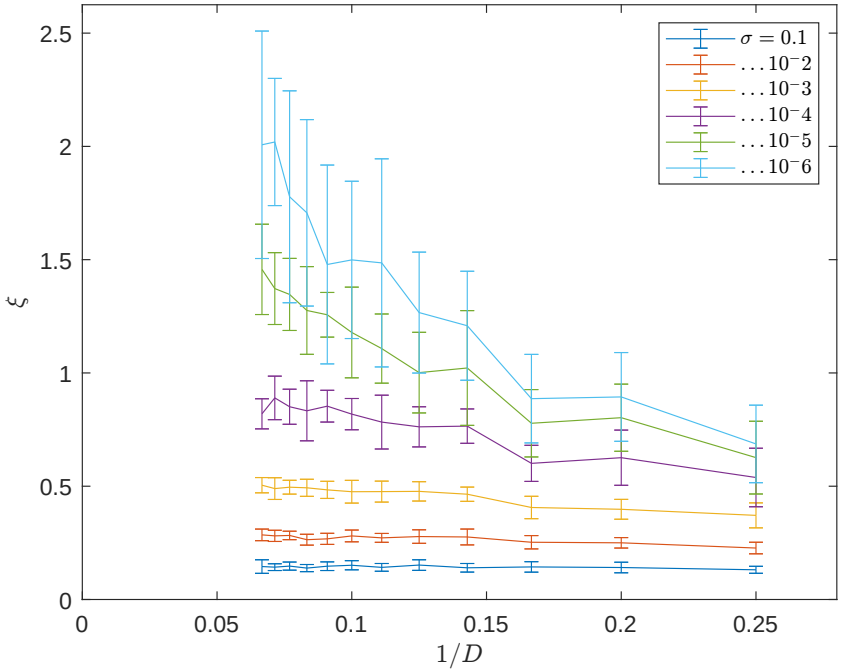


(b) over inverse bond dimension

Figure 3.2: Correlation length for the disorder-perturbed TFIM model around the polarized state, as obtained by the applied tensor network algorithm. Shown are example results for different bond dimensions for a block size of a single site (i.e. no blocking).

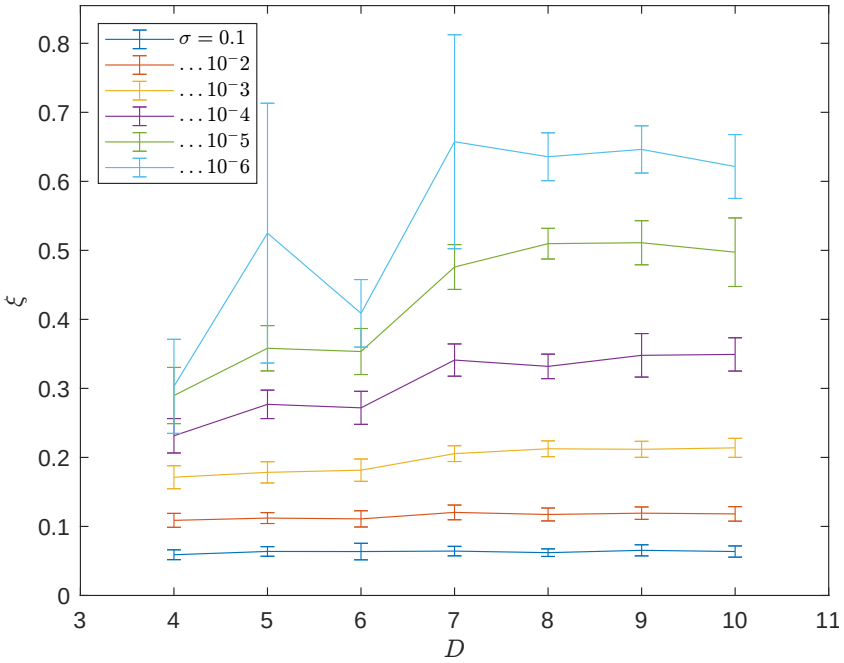


(a) over bond dimension

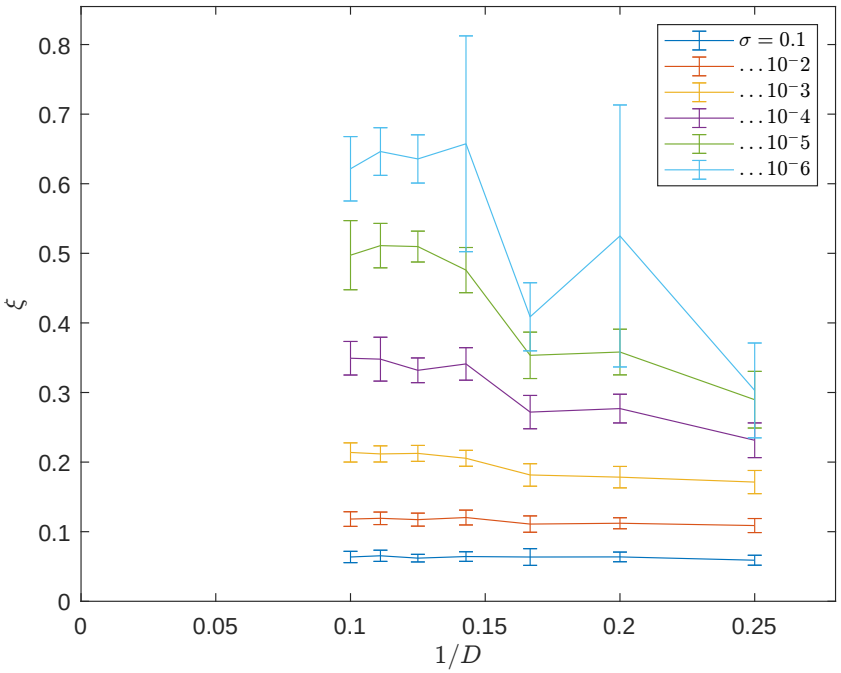


(b) over inverse bond dimension

Figure 3.3: Correlation length for the disorder-perturbed TFIM model around the polarized state, as obtained by the applied tensor network algorithm. Shown are example results for different bond dimensions for a block size of two sites.

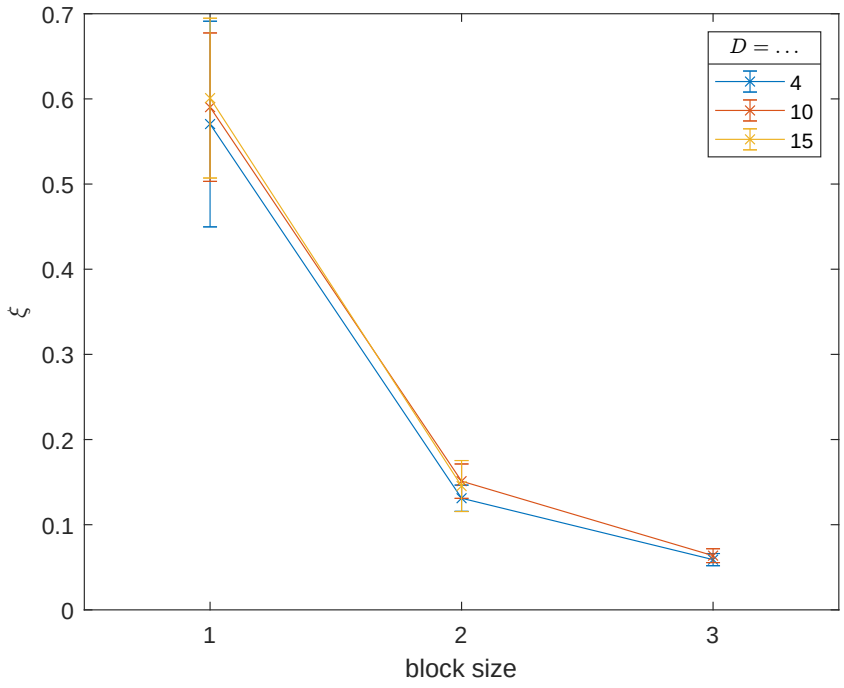


(a) over bond dimension

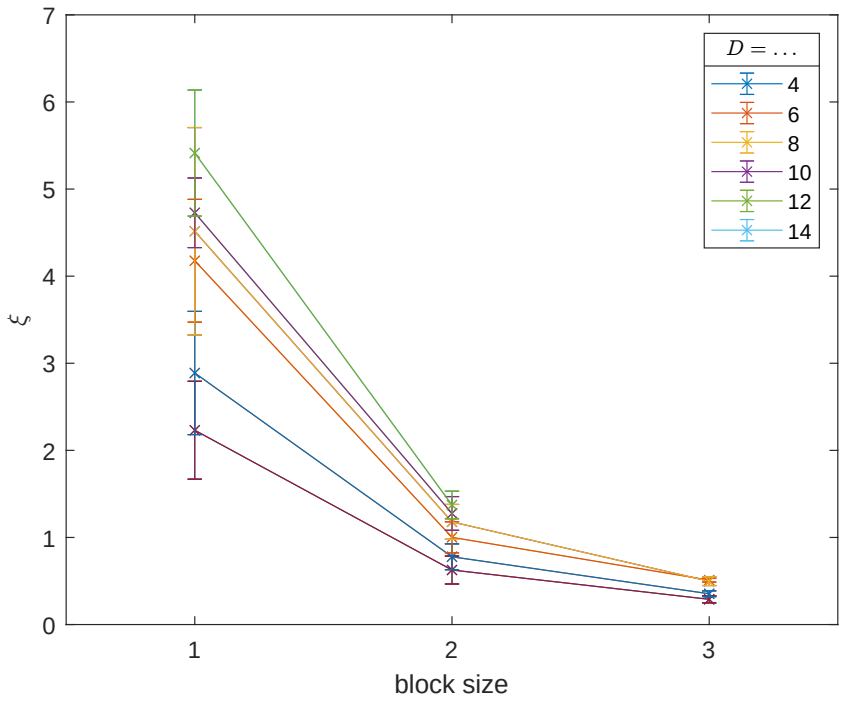


(b) over inverse bond dimension

Figure 3.4: Correlation length for the disorder-perturbed TFIM model around the polarized state, as obtained by the applied tensor network algorithm. Shown are example results for different bond dimensions for a block size of three sites.



(a) $\sigma = 0.1$



(b) $\sigma = 10^{-5}$

Figure 3.5: Correlation length for the disorder-perturbed TFIM model around the polarized state, as obtained by the tensor network algorithm. Shown are example results for different bond dimensions for block sizes of 1, 2 and 3 sites, respectively.

Chapter 4

$\mathbb{Z}_3$ Topologically Ordered Systems

Contents

In addition to MBL, another intriguing type of order for quantum many-body systems is that of *topological order*. In order to better understand the emergence of topological order, I will show how this global topological order of a certain class can be constructed on the basis of local symmetries. For this construction, tensor-network methods will again play a central role, as they make the calculation of certain system properties tractable, allowing to verify their topological nature.

This chapter shows my work on the applications of group theoretic approaches to the tensor network formalism. I have investigated the properties and behavior of a system describable by the two-dimensional Kagome lattice created with a local SU(3) symmetry. I generalized the characteristics of the constituents of this lattice, investigating the emergent global properties of the system both analytically and numerically. The first few sections of the chapter provide an introduction into the mathematical foundations of group theory and the tensor network formalism, after which I present my findings and conclusions.

The contents of this chapter are as follows. In **Section 4.1: Entangled Structures in Locally-SU(n)-Symmetric Systems on the Kagome Lattice** I will provide an overview of prior work on the emergence of topological order from local symmetries, to provide the context for my results. I will introduce the setup for the model in **Section 4.2: Construction of an SU(3)-Symmetric System on the Kagome Lattice**, where I will elaborate on the lattice structure and how the local SU(3) group structure is represented on it and show how the model maps to a string-net model. In **Section 4.3: Numerical Evaluation of the Tensor Network** I will present the results of the numerical study of the system using tensor-network methods and I will give an explanation of the algorithm used for that. This section will also present the results for the properties of the system, showing that it exhibits topological order.

Parts of the work I present in this chapter have been published as part of [2].

4.1 Entangled Structures in Locally-SU(n)-Symmetric Systems on the Kagome Lattice

In this chapter I will present work on the entanglement structures in locally SU(n)-symmetric systems that has already been accomplished by other authors and expand on their conclusions with my own work. The study I will focus on is that which expands on the resonating valence bond state interpretation of the locally SU(2)-symmetric system on a Kagome lattice, and instead investigates it under the condition of SU(3) symmetry. Group structures of higher dimensionality will also be briefly considered, in addition to a string-net interpretation of the derived structures. Further, the correlation length of the investigated system will be divined, and its ground state wave function will be prepared to be inspected for topological properties using the (both analytical and numerical) tensor network methodology.

Particles that exhibit an SU(n) symmetry form specific classes of entangled structures based on the properties of their representations. I will demonstrate this on the example of the **2** representation of the SU(2) symmetry group, which corresponds to its fundamental, two-dimensional representation. For a system that consists of a one-dimensional chain of spin-$\frac{1}{2}$ particles, the entangled state with the lowest energy is the one with pair-wise entanglement. In the case of a one-dimensional chain of particles of the fundamental representation of the SU(3) group, **3**, the entangled state with the lowest energy, is the one in which the formed entangled structures contain exactly three particles each.

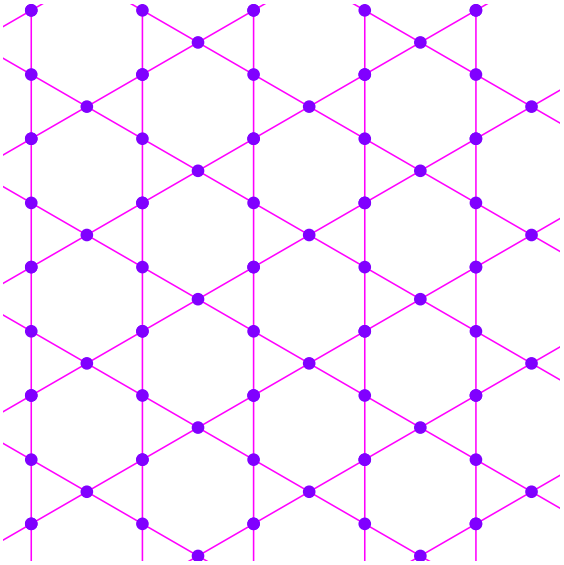


Figure 4.1: A Kagome lattice section; each intersection contains precisely one particle, denoted by purple points, and the pattern of straight pink lines defines its symmetry.

However, the particles may form structures with pair-wise entanglement because of the symmetry properties of the **8** representation of the SU(3) group — the conjugate of this representation is equivalent to itself, $\overline{\mathbf{8}} = \mathbf{8}$. A step-by-step introduction to the influence of the particles' symmetry group to the entangled structures they form, see [67].

In recent years, the Kagome lattice, as depicted in Figure 4.1, had taken center stage for the discussions of the possible existence of the emergence of topological order [68, 26] at its ground state. Previous work suggests that a resonating valence bond (RVB) state of a lattice with a local SU(2) symmetry may present a topologically ordered state [47]. In an RVB state, all neighboring particles, located on the lattice vertices, interact with one another and form valence bonds. As shown in [69, 70, 47], a bijective function can be constructed between a single RVB state in a system with a local SU(2) symmetry and a unique distribution of closed loops on its dual lattice, the honeycomb. According to the theory of string-net condensation by M. A. Levin and X.-G. Wen [71] (and other works, e.g., [72]), such a loop structure corresponds to a $\mathbb{Z}_2$ topologically ordered system.

This work expands on the conclusions reached for the SU(2)-symmetric system [73] and explores a similar equivalency for a system with a local SU(3) symmetry. However, instead of to the states of individual particles, the focus is given to the representation groups that describe the states of various elements in the system. To elucidate the effects of a local symmetry of a system of interacting particles to its global properties, I use the formalism of tensor networks. This versatile tool allows for a straightforward approach to the examination of the characteristics of an interconnected system.

4.2 Construction of an SU(3)-Symmetric System on the Kagome Lattice

By using the expressions derived in the previous section of this chapter, the locally SU(3)-symmetric system of particles on a Kagome lattice can be fully analyzed for its properties. This section describes the exact actions taken to proceed with this analysis and presents the conclusive results of the full investigation.

The examined system is a Kagome lattice where a point-like particle which exhibits an SU(3) local fundamental symmetry is set at each of its vertices, as can be seen in

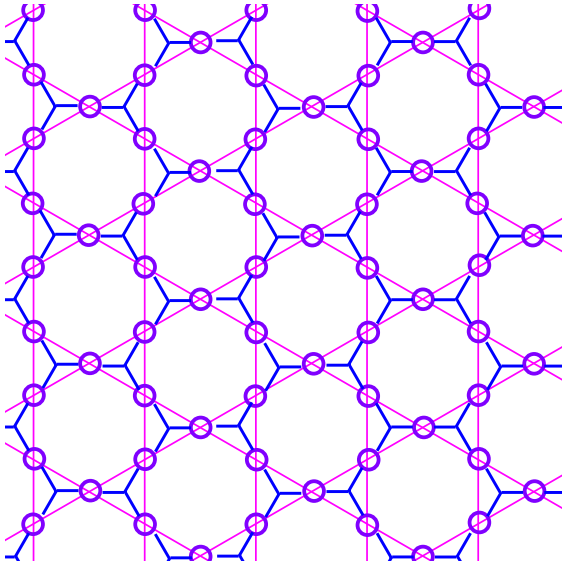


Figure 4.2: Model constructed from SU(3) trimers around each of the triangles on the Kagome lattice, forming a singlet state with the representation $\mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$ shown by blue lines. The circles (purple) are projectors from the virtual degrees of freedom (blue lines) to the physical ones, parametrized by α, β.

Figure 4.1. Details of the phase structure of one-dimensional chains with an SU(3) structure can be found in [74].

4.2.1 Definition of the Model

An SU(3) spin liquid wave function with $\mathbb{Z}(3)$ topological order can be constructed on the Kagome lattice, as I will show in this subsection. I will construct a PEPS wave function with the following properties:

1. On each site of the lattice, there is an SU(3) symmetry with the fundamental representation. Such a function is invariant under translation and rotation of the lattice, transforming as $|\psi\rangle \to |\overline{\psi}\rangle$ under reflection.

2. The model is a spin liquid with an absence of a conventional long-range order.

3. The function describes a topologically ordered system and its anyons correspond to the $\mathbb{Z}(3)$ quantum double model. This effect stems from the conservation of the SU(3) color charge.

4. The obtained function will be the ground state of a local Hamiltonian. Both the Hamiltonian and the function can be smoothly connected to a fixed point model with $\mathbb{Z}_3$ topological order.

5. The model will have trivial charge in each unit cell of the lattice, corresponding to an unbiased mapping to the topological model.

To keep the setup comparable to the one used to explore the properties of the Kagome lattice with a local SU(2) symmetry, the particles are chosen to exhibit an SU(3) symmetry corresponding to the **3** representation — analogously to the 2-dimensional representation of spin-½ in the SU(2) case. The tensors used to stand in for the particles, P, are constructed such that each has one physical index — that carrying the **3** representation symmetry — and two virtual indices with an SU(3) symmetry. The virtual index-legs connect them to the two neighboring triangle motifs of the lattice, as seen in

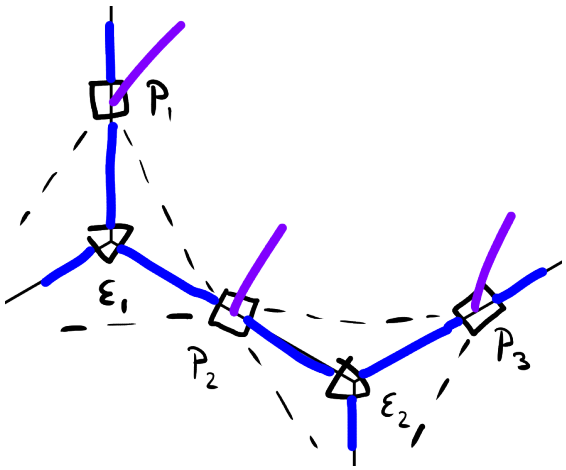


Figure 4.3: Tensor network model applied to the Kagome lattice, showing the ϵ and P tensors.

Figure 4.3. Another set of tensors, ϵ, is distributed on the vertices of the dual lattice, in the centers of the triangle motifs. These tensors consist of three virtual indices which are subsequently contracted with the corresponding virtual indices of the P tensors that pertain to the triangle.

The constraints put on the virtual indices of the P and ε tensors are twofold:

1. Representations that correspond to the symmetry properties of the two virtual indices of P need to allow for the initially defined $\mathbf{3}$ representation symmetry to be achieved.

2. Representations that correspond to the symmetry properties of the three virtual indices of ϵ need to allow for the tensors to exhibit a trivial symmetry.

The first condition restricts the representations assigned to the virtual indices of P to the combinations of $\{(\mathbf{1},\mathbf{3}),(\overline{\mathbf{3}},\overline{\mathbf{3}})\}$, where $\mathbf{1}$ is the trivial representation, and $\overline{\mathbf{3}}$ conjugate to $\mathbf{3}$. This is valid because $\mathbf{1}\otimes\mathbf{3}=\mathbf{3}$ and $\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}=\mathbf{3}\oplus\overline{\mathbf{6}}$. The second condition imposed on the network describes the interaction of the particles on the vertices of a triangle motif. In order for ϵ to exhibit a trivial symmetry, the vector product of the representations that correspond to the symmetries imposed on its virtual indices must offer the trivial representation as a result. Therefore, the possible combinations of these representations are $\{(\mathbf{1},\mathbf{1},\mathbf{1}),(\mathbf{3},\mathbf{3},\mathbf{3}),(\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}}),(\mathbf{1},\mathbf{3},\overline{\mathbf{3}})\}$, since $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}=\mathbf{1}$, $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}$, $\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\overline{\mathbf{10}}$, and $\mathbf{1}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{1}\oplus\mathbf{8}$.

The model is constructed as in Figure 4.2, where trimers $|\tau\rangle$ that meet in each of the vertices of the honeycomb lattice and form triangle motifs in the corresponding Kagome lattice. These trimers are in a singlet state, with the representation $\mathcal{H}=\mathbb{1}\oplus\mathbf{3}\oplus\overline{\mathbf{3}}$. The map $\mathcal{P}$ is then applied to the connecting edges of the weight lines of neighboring triangles, selecting only the physical degrees of freedom from the representations of the two triangle motifs. The trimers are built of three 'virtual' SU(3) particles, such that $|\tau\rangle\in\mathcal{H}_v^{\otimes 3}$, thus, its state space is $\mathbb{C}^1\oplus\mathbb{C}^3\oplus\mathbb{C}^3$. In line with that, the state space contains a total of nine singlets, three of them in the $\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$, $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$, and $\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}$ spaces, and six of them in the spatial permutations of the $\mathbb{1}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}$ state space. The singlet $|\tau\rangle$ is defined as the equal-weight superposition of all the singlets. The six states of $\mathbb{1}\otimes\mathbf{3}\otimes\overline{\mathbf{3}}$ and the $\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$ state are combined with the amplitudes ±1 to form a fully symmetric state, $|S\rangle$. The remaining two states, formed exclusively by the fundamental and the fundamental conjugate representation, are combined with the amplitude $+1$, forming a fully antisymmetric state, $|A\rangle$. Then, $|\tau\rangle=|S\rangle+i\,|A\rangle$, and the state has

chiral symmetry, transforming trivially under translation and rotation, and as $|\tau\rangle \to |\bar{\tau}\rangle$ under reflection.

4.2.2 Mapping the Kagome SU(3) Model to the $\mathbb{Z}_3$ Honeycomb Lattice String-Net Model

There exists a way to map a state on the Kagome lattice with a local SU(3) symmetry in the 3 representation a honeycomb lattice string-net with $\mathbb{Z}_3$ topological order

Accounting for these constraints on the tensor network, the representations pertaining to the symmetry properties of the virtual indices may be distributed. For each P that represents a particle on a Kagome vertex, three distinct setups are observed. In the case where the virtual indices of the tensor respectively exhibit a trivial and a **3** representation, a directed line is drawn through the vertex. The direction of the arrow is chosen to point toward where the virtual index with the trivial representation extends. If both virtual indices correspond to the $\overline{\mathbf{3}}$ representation, no line is drawn. When the lines are extended to the midpoints of the triangle motifs and connected there, the constructed network becomes a honeycomb lattice string-net model with a $\mathbb{Z}_3$ topological order. For each vertex of the honeycomb lattice, let the number of arrows entering the vertex be denoted as ν_{in} and the number of arrows exiting the vertex as ν_{out}. Then, the following is valid:

$$\nu_{in} - \nu_{out} = 0 \pmod 3. \tag{4.1}$$

A visualization of what such a string-net configuration may look like can be seen in Figure 4.4.

Here I show a primer to the topic of string-nets and discusses how the obtained results for the locally SU(3)-symmetric system can be expanded — and even generalized — for a system that exhibits a local SU(n) symmetry, by using this mathematical structure deeply investigated by Michael A. Levin and Xiao–Gang Wen [71].

As already explained in Section 1.2.5, topological order is a special phase of matter appearing in zero-temperature conditions, and quite interesting for quantum information research and the design of quantum error correction algorithms.

It is characterized by a robust ground state degeneracy and a long-range pattern of quantum entanglement — states with different patterns cannot be changed into one another without the system going through a phase transition.

In contrast to the axiomatic definition of topological order presented in the introduction, the focus here is on a particular class of microscopical models exhibiting topological order.

It has been shown that a large class of $(2 + 1)$-dimensional topologically ordered phases is formed through the process of string-net condensation. A string-net liquid is a type of system proposed by Michael A. Levin and Xiao–Gang Wen [71] — it involves a graph-like structure of directed or undirected interconnected strings with certain gauge properties. For example, the toric code, a two-dimensional spin-½ lattice stretched on the surface of a torus, can be faithfully described by a string-net model. The ground state of the standard toric code corresponds to an equal-weight superposition of closed string states of the $\mathbb{Z}_2$ gauge group, otherwise known as the $\mathbb{Z}_2$ spin liquid.

A quantum spin liquid is a state that may exist in a system of interacting quantum spins; unlike ordered states such as the ferromagnetic spin state, a quantum spin liquid is disordered, even to low temperatures. For example, when triangular lattices such as the Kagome lattice are formed by spin-½ particles, their respective spins become frustrated, unable to uniquely choose and maintain a lowest-energy state of the whole system, as can be seen for a simple triangle motif in Figure 4.5. However, if valence bond states

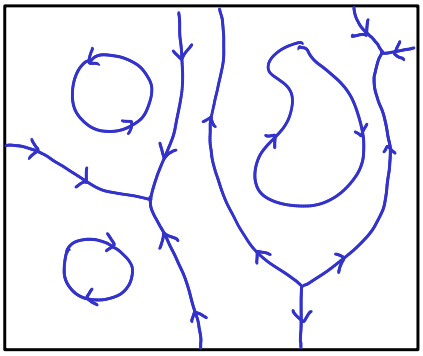


Figure 4.4: Example of a string-net. Shown is an example configuration of a string-net following the rules for the $\mathbb{Z}_3$ model described in section 4.2.2, where the strings must obey a Gaussian law $(mod\,3)$.

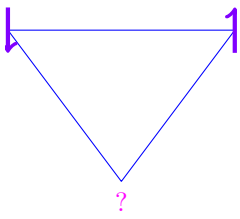


Figure 4.5: Triangular motif with vertices that contain particles with binary spins, exhibiting a frustrated state, in which the lower spin cannot fall into a unique lowest-energy ground state with respect to the upper two spins.

are implemented in the system, in which spin pairs interact antiferromagnetically and form singlets, the systems as a whole maintains a spin of 0. The spin pairs here are maximally entangled, and if all particles of the lattice belong to a pair, the state of the quantum system is referred to as a valence bond solid. Unlike a spin liquid, a valence bond solid does not exhibit a case of unbroken lattice symmetry, and due to the pair-wise entanglement, its entanglement pattern is not of the long-range type.

The toric code is a topological quantum error correction code defined on a two-dimensional lattice of spins; its periodic boundary conditions give its lattice the shape of a torus, as in Figure 4.7. Most commonly, this lattice is of the square type and one spin-½ particle is positioned on each of its edges. On the edges around each of the vertices and plaquettes of the lattice, vertex and plaquette stabilizer operators are defined, respectively. A vertex stabilizer operator for the vertex v, A_v, and a plaquette stabilizer operator for the plaquette p, B_p, are defined as:

$$A_v = \prod_{i \in v} \sigma_i^x, \;\; B_p = \prod_{i \in p} \sigma_i^z, \tag{4.2}$$

where σ_i^ξ are Pauli operators acting on spins i that belong to the corresponding vertex or plaquette. The space on which all stabilizers act in a trivial manner, such that $A_v \left|\psi\right\rangle = \left|\psi\right\rangle, \;\forall v$, $B_p \left|\psi\right\rangle = \left|\psi\right\rangle, \;\forall p$, is named the stabilizer space. In the toric code, the stabilizer space is four-dimensional and able to store two qubits of information. Famously, such a code can be used for fault-tolerant quantum computation, as well as for the creation of quantum memories [75, 76].

Those states which do not belong to the stabilizer space may be mathematically described as containing quasiparticles. When a Pauli operator acts on a spin located at an edge of the lattice, depending on the type of operator, the two adjacent vertices or plaquettes spawn a quasiparticle excitation named anyon, as demonstrated in [77]. If

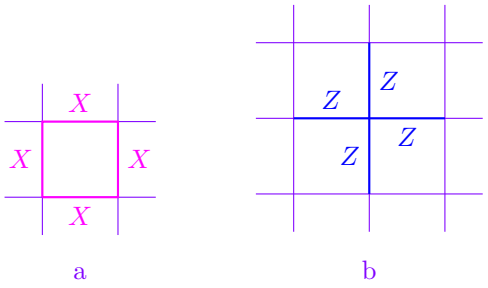


Figure 4.6: Stabilizer operators of the standard toric code. Pauli-X and Pauli-Z operators acting on spins set on particular edges of the toric code lattice are indicated by red X and blue Z, respectively. a) Vertex operator; b) plaquette operator.

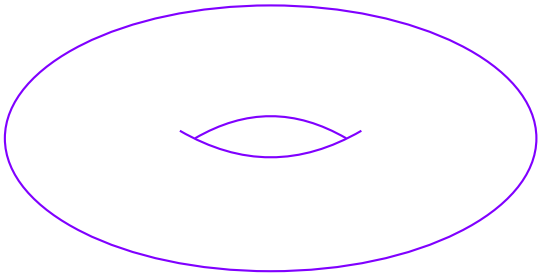


Figure 4.7: Two-dimensional torus, with genus 2.

the same type of Pauli operator then acts on an edge adjacent to the initially chosen one, the corresponding vertices or plaquettes adjacent to it either spring an anyon or lose the originally spawned anyon — it is annihilated. By acting on adjacent edges with the same type of Pauli operator in such a manner, the pair of created anyons can be moved further away from one another and the imaginary string between them can be stretched long. This string corresponds to the connected edges on which Pauli operators acted to create and *move* the anyons on the lattice.

Interestingly, if the anyons are moved around on the surface of the torus until they are adjacent to each other again, and the edge between them is acted on by the corresponding Pauli operator, they will both be annihilated by it and the created string will be closed. If these anyons were created on the two-dimensional surface of a sphere and an imaginary string were pulled from one to the other, this string could then always be uniformly deformed until it reaches a point — and vanishes. The same is true for a two-dimensional toroidal surface, but only in some cases. If this string is stretched around the non-trivial topology of the torus in one of two specific ways, as shown in Figure 4.8, and the string closed, it would not be possible to uniformly deform it into a point. Instead, the best that could be done is to stretch it taut around the torus. As the toric code can be acted on by two independent Pauli operators — one creating anyons on the vertices and the other on the plaquettes — these strings can be created by $2 \cdot 2 = 4$ types of strings, two Pauli-X and two Pauli-Z types.

This property is specific to topologically nontrivial surfaces and is intimately related to the genus of the surface on which the toric code is defined.

The Hamiltonian of a system with a $\mathbb{Z}_2$ gauge can be expressed as:

$$H_{\mathbb{Z}_2} = -U \sum_i \sigma_i^x + t \sum_p \prod_{\text{p-edge}} \sigma_j^z, \tag{4.3}$$

and its Hilbert space is defined by states $|\Phi\rangle$ that satisfy:

$$\prod_{\text{I-edge}} \sigma_i^x |\Phi\rangle = |\Phi\rangle \tag{4.4}$$

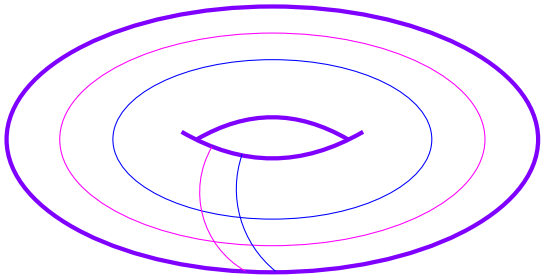


Figure 4.8: Nontrivial string operators of the standard toric code, shown on a square lattice set on a two-dimensional torus with genus 2. The operators created by the repeated use of the Pauli-X and Pauli-Z site operators are shown in pink and blue, respectively.

for all sites **I** of the lattice. To avoid the very dual thinking related to the created strings, they can be thought of as belonging to a string network or a string-net.

A particular string-net is commonly defined by three types of information: the types of strings in the network, the branching rules of the strings, and the orientation of the strings. Different string types can be simply labeled by a numeral index, i. The rules of branching define the types of strings that can all meet in one point. This in turn depends on the orientation the strings have: two separate directions or lacking one.

The RVB state on the Kagome lattice has been shown to have a bijective relation with the $\mathbb{Z}_2$ topologically ordered honeycomb system [69] — all full dimer coverings correspond to a particular set of strings in the honeycomb string-net in which all strings are directionless and branch so that each vertex contains either 0 or 2 connecting strings, i.e., it is either empty or a string passed through it. Thus, the full RVB state corresponds to the sum of all such string-net coverings on the honeycomb lattice. As has already been shown, such a construction is equivalent to the ground state of the toric code, confirming the $\mathbb{Z}_2$ topological order of the constructed system. In a Kagome lattice which contains SU(3) particles in each of its vertices, the resultant string-net is different.

Because the only representations used are the trivial, the fundamental, and the fundamental conjugate, each vertex of the resultant honeycomb string-net has either zero or three strings reaching it, and all the strings adopt an orientation — necessary to distinguish between the fundamental and the fundamental conjugate. However, unlike the complicated bijection developed for the RVB model, the SU(3) string-net can be reached in a straightforward manner.

String-nets can be naturally defined in the context of gauge theories and all deconfined gauge theories can be described as string-net condensates — with electric flux lines acting as strings. For example, the Hamiltonian of the $\mathbb{Z}_2$ lattice gauge theory defined on the honeycomb lattice can be written as:

$$H_{\mathbb{Z}_2} = -U \sum_i \sigma_i^x + t \sum_p \prod_{\text{edges of } p} \sigma_j^z, \tag{4.5}$$

where the links, plaquettes, and sites of the lattice are marked by i, p, and I, respectively. The Hilbert space of this system is formed by quantum states that satisfy:

$$\prod_{\text{legs of } I} \sigma_i^x \left|\Phi\right\rangle = \left|\Phi\right\rangle \tag{4.6}$$

for every site I. It has been proven that the $\mathbb{Z}_2$ lattice gauge theory is dual to the Ising model in $(2+1)$ dimensions, which can be seen by the following setup. If links

with $\sigma^x = -1$ are thought of as being occupied by a string and those with $\sigma^x = +1$ as being unoccupied. Then, only closed strings are allowed in this Hilbert space — the $\mathbb{Z}_2$ gauge theory can be described as a closed string theory. In this dual picture the electric part of the Hamiltonian, $-U\sum_i \sigma_i^x$, acts as string tension, whereas the magnetic part, $t\sum_p \prod_{\text{edges of } p} \sigma_j^z$, acts as the string kinetic energy. The confining phase corresponds to a large electric energy, i.e., $U \gg t$, and the deconfined phase to a large magnetic energy. The ground state of the system is a superposition of numerous large string configurations — the deconfined phase is a quantum liquid with large strings, a string condensate.

4.2.3 Interpolation between the Kagome SU(3) Model and the Model with Locally Orthogonal States

The described tensor network model for a state with a local SU(3) symmetry does not uniquely describe it. It is not immediately clear if this ambiguity affects the result of the mapping — is it bijective and reversible? This issue can be examined by means of a numerical interpolation between its description pertaining to the representation and that pertaining to the exact state of the system.

The precise issue with the mapping pertains to its bijectivity — is the mapping reversible if the only information that is kept is the one on the representations, and not on the exact states of the Kagome system? For this purpose, the problem is investigated numerically, by interpolation between the mapping that contains the full information on the exact state that may be constructed from the representations assigned to the virtual indices of the tensors in the network, and the mapping that only considers the representations.

If the virtual indices of P have the $\overline{\mathbf{3}}$ representation symmetry, a possible combination of specific states from these representations could be expressed as $|\overline{u}\rangle\,|\overline{u}\rangle$, which is a state in the $\overline{\mathbf{6}}$ representation, but it has no projection in the $\mathbf{3}$ representation of the SU(3) group. Another example is a combination of states of the type $|\overline{u}\rangle\,|\overline{d}\rangle$, which may be thought of as a linear combination of both a state in the $\mathbf{6}$ representation (the state $\frac{1}{\sqrt{2}}(|\overline{d}\rangle\,|\overline{u}\rangle + |\overline{u}\rangle\,|\overline{d}\rangle)$) and a state in the $\mathbf{3}$ representation (the state $\frac{1}{\sqrt{2}}(|\overline{d}\rangle\,|\overline{u}\rangle - |\overline{u}\rangle\,|\overline{d}\rangle)$). In both cases, a loss of information is evident in the representation-based mapping scheme.

To combat this loss, two parameters are introduced to be used as the amplitudes of projectors that map the state of full information into that with a possible partial information loss. The first of these, α, varies the proportion of inclusion for the information on the exact states described by the $\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}$ representations of the virtual indices of P. The full information is carried through by $\alpha = 1$, and $\alpha = 0$ loses it completely, keeping only the information on the representations themselves. Analogously, the parameter β varies the kept amount of information on the states that overlap with the $\overline{\mathbf{6}}$ representation, as exemplified above.

The setup used to describe this interpolation requires the state of each of the virtual indices to be defined. The virtual indices used in the model may exhibit the trivial symmetry ($\mathbf{1}$), the symmetry properties of the fundamental group ($\mathbf{3}$) or of its conjugate ($\overline{\mathbf{3}}$). The vector sum of these spaces gives a seven-dimensional basis: $\{\mathbb{1}, u, d, s, \overline{u}, \overline{d}, \overline{s}\}$, where u, d, and s are the shorthand for the up, down, and strange state, commonly used to describe the eponymous quarks, describable by the SU(3) symmetry group. The listed constraints are applied to these conditions and each P is fitted with a projector parametrized by the variables α and β, as described above.

The projectors $\mathcal{P}_{\alpha,\beta}$ are used with the parameters $(\alpha, \beta) \in [0, 1]$ described earlier, to allow for an investigation of the interpolation properties of the two relevant models. The map $\mathcal{P}_\perp \equiv \mathcal{P}_{(1,1)}$ projects the two adjacent triangle sites, $\mathcal{H}_v^{\otimes 2} = \left(\mathbb{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}\right)^{\otimes 2}$,

onto the union of the following three types of spaces: $\mathcal{H}_{\omega} = \mathbb{1} \otimes \mathbf{3}$, $\mathcal{H}_{\overline{\omega}} = \mathbf{3} \otimes \mathbb{1}$, and $\mathcal{H}_1 = \overline{\mathbf{3}} \otimes \overline{\mathbf{3}}$. In this setup, the order of the representations in the tensor product implies the directionality of the strings to be constructed.

The interpolation in α is obtained by adiabatically projecting out the $\overline{\mathbf{6}}$ component of $\mathcal{H}_1$, as follows:

$$\mathcal{P}_{(\alpha,1)} \left[\mathbb{1}_{\mathcal{H}_{\omega}} \oplus \mathbb{1}_{\mathcal{H}_{\overline{\omega}}} \oplus \left(\alpha \mathbb{1}_{\mathcal{H}_1} + (1-\alpha) \prod_{\mathcal{H}_1^{\mathbf{3}}} \right) \right] \mathcal{P}_{\perp}. \tag{4.7}$$

In this expression, $\mathcal{H}_1^{\mathbf{3}}$ and $\mathcal{H}_1^{\overline{\mathbf{6}}}$ correspond to the spaces of the $\mathbf{3}$ and $\overline{\mathbf{6}}$ representations, respectively.

When $\alpha = 0$, then $\mathcal{P}_{\mathbf{333}} = \mathcal{P}_{(0,1)}$, so $\mathcal{H}_{\mathbf{333}} = \mathcal{H}_{\omega} \oplus \mathcal{H}_{\overline{\omega}} \oplus \mathcal{H}_1^{\mathbf{3}} \approx\cong \mathbf{3} \otimes \mathbb{C}^3$. The first of the tensor components transforms as $\mathbf{3}$, whereas the second one labels the representation under consideration, transforming trivially under SU(3). Now, $\mathbb{C}^3$ can be projected out adiabatically:

$$\mathcal{P}_{0,\beta} = \left[\mathbb{1}_{\mathbf{3}} \otimes \left(\beta \mathbb{1}_{\mathbb{C}^{\mathbf{3}}} + (1-\beta) \left| e \right\rangle \left\langle e \right|_{\mathbb{C}^{\mathbf{3}}} \right) \right] \mathcal{P}_{(0,1)}. \tag{4.8}$$

Then, the label is projected onto the equal weight superposition of the three components, $|e\rangle$. For $(\alpha, \beta) = (0, 0)$, the superposition can be factored out. Then, each site is associated with an SU(3)-invariant wave function with the fundamental representation.

4.3 Numerical Evaluation of the Tensor Network

In this section I show the details of the tensor network algorithms used in the numerical investigations presented in this chapter.

4.3.1 Locally SU(3)-Symmetric Kagome Lattice

In this section I am using iMPS to approximate properties of a PEPS, namely by rewriting the PEPS transfer operator in terms of an MPO, and then performing an approximation for its fixed point. This method enables access to properties such as the correlation length of the investigated state.

Building upon the short elaboration of this method, which has already been presented in Section 3.4, here I will explain the reasoning behind this approach in greater detail. The algorithm which I am using here is described in [78], and the details of the MPO fixed point calculation can be found in [79], under the designation of imaginary-time evolution.

Effective Environments for the PEPS Expectation Value Calculation

The standard procedure used to calculate the expectation value of some observable that is localized to a single site (or, equivalently, a number of sites) is to sandwich this operator between two copies of the PEPS and connecting the free legs of each corresponding site of both of the PEPS copies with the physical legs of the observable — one with the ingoing and one with the outgoing legs. Because of the locality of the operator, the free leg of the PEPS is connected directly to the corresponding one of its copy at almost all sites. These direct physical index connections can now be contracted, resulting in a two-dimensional tensor network lattice that on each site other than the one(s) of the observable has a single tensor connected to its neighbors with indices that are the doubled virtual indices of the original PEPS. The network for this lattice can then be contracted, so that only the site with the observable and its

direct neighbors remain, and the remainder of the network can be referred to as the environment. Given the environment tensor, the calculation of the expectation value of the observable is a relatively simple task, as there remains only a small number of tensors to be contracted.

However, the calculation of the environment tensor based on the PEPS tensors should be performed in a computationally feasible way. The naïve approach of contracting the tensor network over the two-dimensional lattice would lead to the bond dimension growing exponentially. Instead, the techniques use for one-dimensional tensor networks (MPS) can be repurposed for use in two dimensions.

Assuming the reduced tensors make a square lattice (a flat two-dimensional lattice can always be turned into a square lattice by grouping its tensors), we can split it into two halves. For each of the halves, starting from the two outside edges away from the seam, each row of reduced tensors can be contracted one at a time. This operation is mathematically identical to repeatedly acting on an MPS with an MPO that consists of the transfer operators of the original PEPS.

In the MPS picture, applying lots of MPOs to a state is computationally feasible as long as the growth in bond dimension is suppressed by truncating the resultant state. This can be done by transforming the state tensors into their canonical form using singular value decompositions and then discarding all but the D (bond dimension) largest singular values.

Following this procedure to the lattice seam leads to environment tensors that are not exact, but which constitute an *effective environment*.

On top of being able to calculate expectation values of local observables using this effective environment, by the virtue of how expectation values are calculated using the two environment MPS, some properties like the correlation length directly correspond to their counterparts for the MPS and can be obtained directly using this procedure.

This approach also works in the infinite case, where uniform MPS can be used to calculate effective environments for iPEPS. In that case it would be necessary to calculate the fixed point of the (also uniform) MPO to find the effective environment, as that is the only component that survives an infinite number of repeated applications of the operator.

It is then straightforward to numerically approximate the fixed point of an MPO, using the algorithm described at the end of Section 1.2.6.

Application to the SU(3)-Symmetric Kagome Lattice

In order to perform these calculations for the tensors representing an RVB state of SU(3)-symmetrical particles on the Kagome lattice, the first step is to block together neighboring tensors in pairs. By performing this grouping, we look at the unit cell of the honeycomb lattice (see Figure 4.9). With this, the resulting PEPS has the structure of a square lattice. This is necessary to obtain transfer operators for the PEPS, which can be written as an uniform MPO. It would be possible to write a transfer operator as an MPO even without grouping the tensors, but this MPO would not be uniform, which is necessary for the simple version of the algorithm used here, which considers uniform MPS and stores only a single tensor to describe the state.

Now the correlation length of the model can be studied at different points of the parameter space spanned by the two parameters α and β. In addition, the bond dimension can be chosen freely, which means that the change in the correlation length of the resulting MPS can be investigated with respect to an increasing bond dimension.

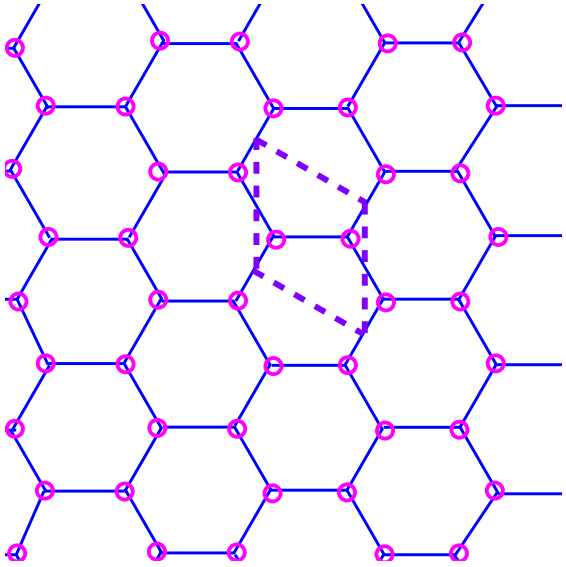


Figure 4.9: Structure of the tensor network, with the unit cell of the honeycomb lattice demarcated by a purple dashed line.

As described before, the parameters α and β are not varied freely, but only along a path where first α is varied between 0 and 1 while $\beta = 0$, and then β is varied between 0 and 1, while $\alpha = 1$.

The algorithm described above can now be applied for each point on this path, starting with a random MPS of a certain bond dimension D to act as a starting state. This can then be repeated for different bond dimensions. Higher bond dimensions will give more accurate results, but will require more resources and be slower to calculate. Therefore, the scaling of the correlation length with increasing bond dimension will be used to aid in the extrapolation of the results. These calculations are shown in Figure 4.10.

The correlation length ξ is not shown directly in this figure; instead, it shows the ratio in magnitude λ between the largest and the subleading eigenvalue of the transfer operator. I.e., $\lambda = \left|\frac{\lambda_1}{\lambda_2}\right|$ where λ_i are the eigenvalues of the transfer operator sorted by descending magnitude. These can be directly converted into one another, since $\lambda = \exp(-\frac{1}{\xi})$.

The results for different bond dimensions are obtained by implementing the algorithm in MATLAB and running it on a regular home computer. Because of the computational capabilities available, the simulated bond dimensions are small, to keep the runtime manageable. However, from the spread of the results it can already be seen that useful results for this parameter region can already be obtained with low bond dimensions. For more detailed results calculated with much higher bond dimensions, see Section 4.3.2 and Figure 4.13.

The results of the simulations for the whole parameter range of the interpolation can be seen in Figure 4.11 and Figure 4.12.

There was a range of parameters for which the algorithm did not converge to a result with a sufficiently low cut-off error ϵ. (Sufficiently low in this case means essentially indistinguishable from zero within the numerical precision of the simulation, which in this situation roughly corresponds to $\epsilon \approx 1 \times 10^{-15}$.) For clarity, these points have been marked in subfigures 4.11a and 4.12a. The most likely explanation of this behavior is that the low bond dimension used for the calculation is insufficient to represent the fixed point of the transfer operator. This conclusion is bolstered by the results from Figure 4.13, which were obtained for a system with a much higher bond dimension.

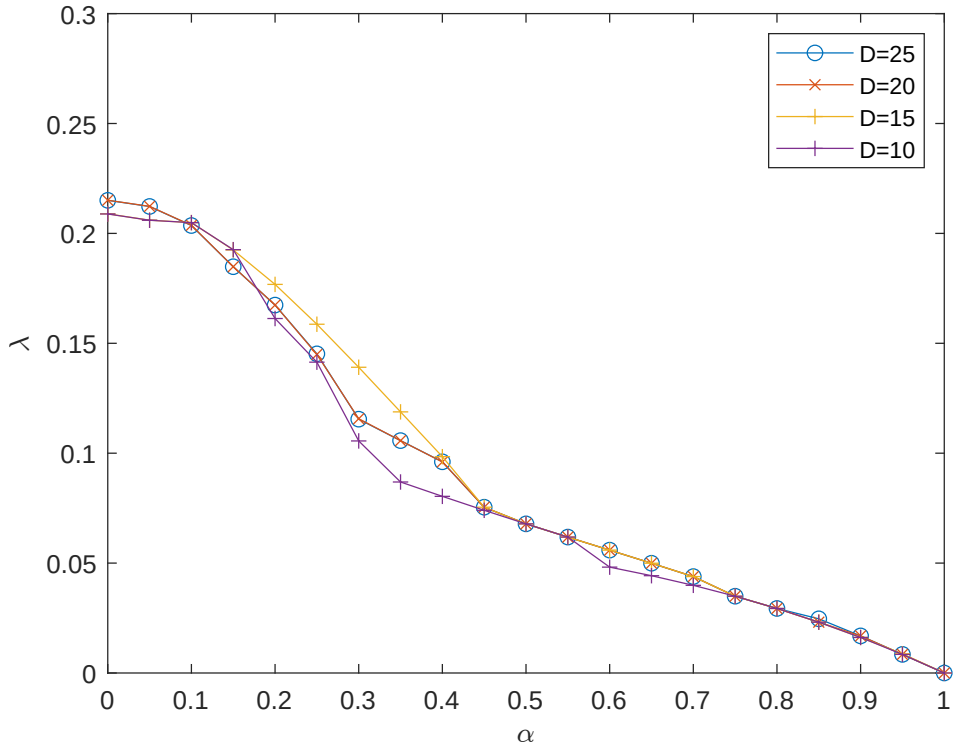
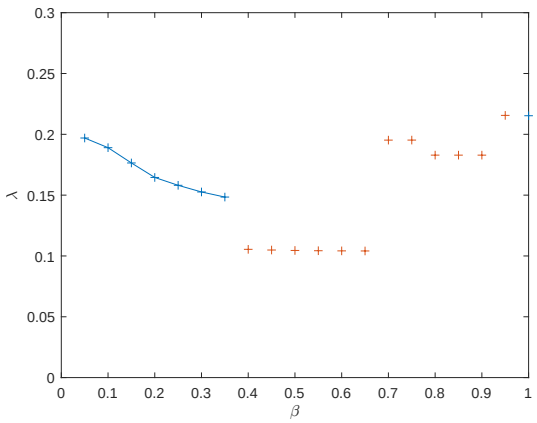
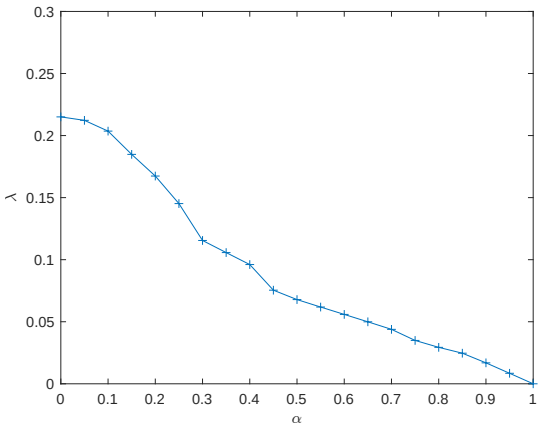


Figure 4.10: Ratio λ of the magnitude of the subleading eigenvalues of the transfer operator along the path of the interpolation for α, comparing results for different bond dimensions. As can be seen from the spread of the results, in this region, relatively good results can already be achieved with small bond dimensions.

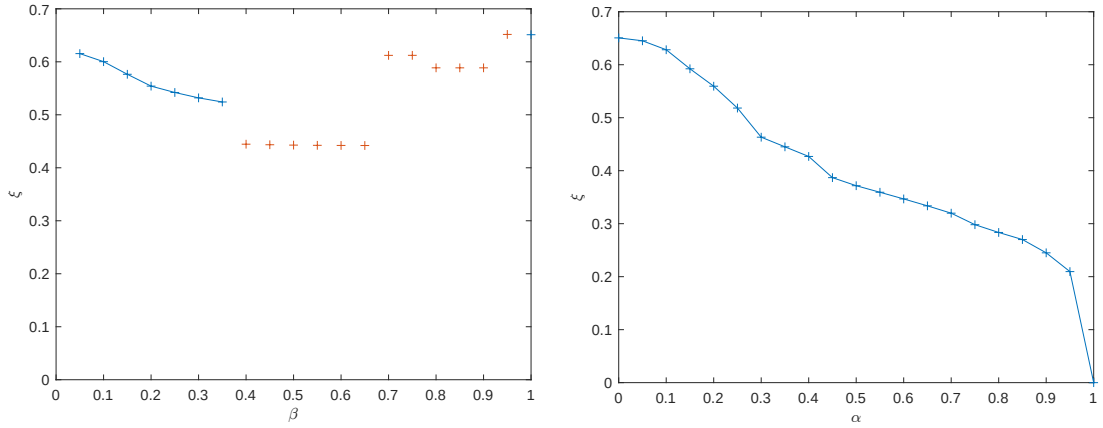


(a) Interpolation for β from 0 to 1, with $\alpha = 0$. Results for bond dimension $D = 20$.



(b) Interpolation for α from 0 to 1, with $\beta = 1$. Results for bond dimension $D = 25$.

Figure 4.11: Difference λ of the magnitude of the subleading eigenvalues of the transfer operator along the path of the interpolation. The interpolations for α and β can be continuously connected, and therefore the plots are intentionally placed side-by-side. The individual points marked in orange in (a) are points where the cut-off error did not converge to zero, i.e. the bond dimension was insufficient to represent the state accurately.



(a) Interpolation for β from 0 to 1, with $\alpha = 0$. Results for bond dimension $D = 20$.

(b) Interpolation for α from 0 to 1, with $\beta = 1$. Results for bond dimension $D = 25$.

Figure 4.12: Correlation length along the path of the interpolation. The interpolations for α and β can be continuously connected, and therefore the plots are intentionally placed side-by-side. The individual points marked in orange in (a) are points where the cut-off error did not converge to zero, i.e. the bond dimension was insufficient to represent the state accurately.

When only the well-converging points (marked in blue) are considered, the results in Figure 4.11 do reproduce the lowest non-zero values from Figure 4.13.

4.3.2 Numerical Analysis including Virtual Symmetries

This section summarizes the work I have accomplished in collaboration with Laurens Vanderstraeten and Norbert Schuch [2], and includes the succinct results on the spectrum and wave function of a locally SU(3)-symmetric system of particles on the Kagome lattice, derived numerically by the use of tensor network algorithms. In addition, it shows an interpretation and elaboration of these results, finishing off with a conclusion on the topological properties of this system.

Like in the previous section, the behavior of the wave function along the interpolation was studied numerically, by use of iMPS with a tunable bond dimension, χ, which helped approximate the fixed point of the PEPS transfer operator. By exploiting the virtual symmetry of the state, it is possible to introduce dressed [80] iMPS transfer operators, $\lambda_{g,\eta}$. This method then provides the correlation lengths for general anyonic correlations, $\xi_{g,\eta}$, by using the subleading eigenvalues of each dressed iMPS transfer operator, $\lambda_{g,\eta}$. The anyon fluxes are labeled by their group elements, g, and the anyon charges by their corresponding irreps, η. Flux-type excitations are different from charges; they are modeled by strings of symmetry actions, whereas the charges are modeled by strings of irrep actions on the virtual level. The former are also named visons, whereas the latter are named spinons. Their composites have parafermionic $e^{\pm 2\pi i/3}$ statistics. As Figure 4.13 shows, during the first half of the interpolation, $(1,1) \to (0,1)$, only vison excitations acquire a finite correlation length. This follows from the fact that the effective amplitude of singlets with $\overline{\mathbf{3}}$ representations is reduced when $\overline{\mathbf{6}}$ is projected out (because $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \mathbf{3} \oplus \overline{\mathbf{6}}$). For the most part, these correlations can be suppressed by considering a modified model (named the $\overline{\mathbf{3}}$-enhanced model), in which $\mathcal{P}_{(\alpha,1)}$ contains an additional factor in front of its $(1-\alpha)$ term, ensuring the total weight of the $\mathcal{H}_1$ subspace is fixed.

Throughout the second part of the interpolation, i.e. $\beta = 1$, $0 \leq \alpha < 1$, the vison length does not grow, but ultimately decreases, while the length scale toward the SU(3) point is dominated by one of the parafermions. At that point, the extrapolation of

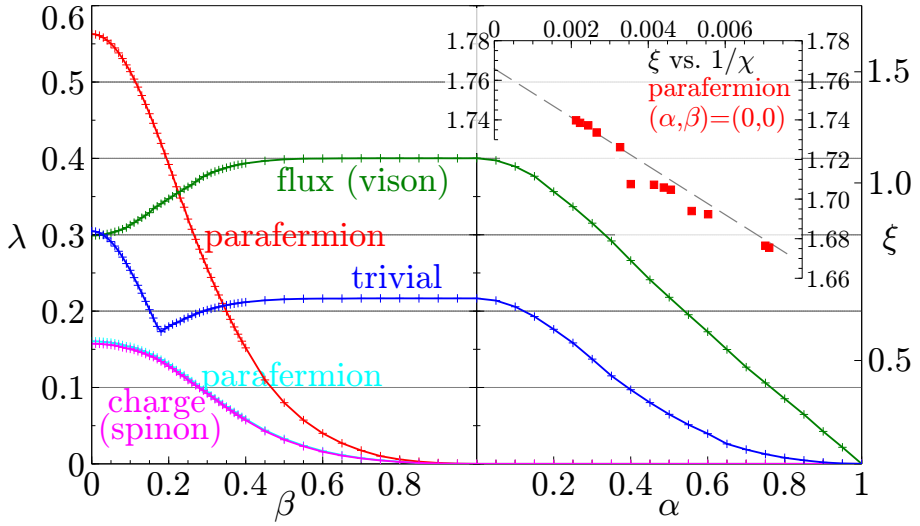


Figure 4.13: Plot from [2]. Further results on the correlation length (ξ) and λ, extending the results from figure 4.11 by differentiating them by excitation type and to higher bond dimension. The calculations here were done with much larger bond dimensions, up to 500. However it can be seen, that the well-converged values from 4.11 reproduce the smallest non-zero values from this plot to a large degree. The inset shows the scaling of ξ at $\alpha = \beta = 0$.

$1/\chi$ shows that $\xi \approx 1.77$. This extrapolation also shows that the correlation length for $\chi \to \infty$ is finite. Because no divergence can be found in any correlation, the system proves to be gapped and in the same phase as the $\mathbb{Z}_3$ quantum double model, with no condensed anyons. The convergence analysis in Figure 4.13 confirms this conclusion.

The presented PEPS model is a topological spin liquid that exhibits a $\mathbb{Z}_3$ topological order.

Chapter 5

Summary of the Results

In this work I have used a medley of analytical and numerical methods to analyze properties of many-body quantum systems with certain symmetries and those with disorder. Here I will concisely review the central results from each of the previous chapters, put them into context and consider potential future approaches.

In the scope of my dissertation, I had announced to complete the following goals.

- **Derivation of a stochastic integral representation for interacting quantum many-body systems with disordered terms**

  I have developed a novel way to express the disorder-averaged propagator of a quantum many-body system with disorder by use of stochastic integral calculus methods.

- **Calculation of a general analytical solution to the time evolution of a local observable in a perturbed many-body system.**

  I have calculated the expectation values of certain local observables for a particular quantum many-body systems using perturbation theory with a disordered perturbation term up to the second order.

- **Construction of a general system of efficient bridging between different variational solutions for a many-body system with disorder, using tensor network formalism and the already obtained perturbative solution.**

  I have applied the mechanism of tensor bridging to a system perturbed by means of random local interactions, and then demonstrated the evaluation of properties of the resulting state numerically.

- **Construction of a tensor network state with local** $\mathrm{SU}(3)$ **symmetry with topological order.**

  I have constructed a tensor network state built to be invariant under local $\mathrm{SU}(3)$ symmetry, which I have shown to exhibit $\mathbb{Z}_3$ topological order and the properties of whose phase I have studied.

In Chapter 2, I looked at methods for studying interacting quantum many-body systems with disorder, using multiple approaches based on perturbation theory. The most successful of these approaches was the stochastic integral representation of the disorder-averaged propagator, a method that allows for the study the dynamical properties of systems with disorder by employing the mechanism of stochastic integral calculus. I have shown how this method can predict dynamical properties for the one-dimensional

Anderson model. Further, I also presented a diagrammatic approach for calculating the disorder-averaged propagator for a specified system. The methods presented in this chapter might prove useful in studying many-body localized systems, an exotic type of order for quantum many-body systems.

With Chapter 3, I continued the theme of quantum many-body systems with disorder and perturbation theory from the first chapter by focusing on a particular two-dimensional system. In this chapter I was able to generalize the tensor-bridging approach from Vanderstraeten et. al. to a system with disorder. By giving a tensor network description of the system using PEPS, I was able to numerically calculate some interesting properties. The work presented in this chapter could be helpful in giving an approach for studying other quantum many-body systems which are disordered.

Chapter 4 connects the themes of exotic order and tensor network methods. Herein I show the use SU(3) symmetries in the construction of a spin liquid wave function with $\mathbb{Z}_3$ topological order via tensor networks, corresponding to the $\mathbb{Z}_3$ quantum double model. I also show how properties of the model can be obtained numerically along an interpolation between the spin liquid and the topologically ordered fixed point. These results are yet another step towards potentially understanding the behavior of quantum many-body systems with local SU(n) symmetry in general.

All in all, I have presented an analysis for the study of symmetries and unusual types of order in interacting quantum many-body systems using an array of analytical and numerical tools, and shown how characteristics of topologically ordered phases can be connected to fundamental symmetries.

In conclusion, in the scope of this work I have presented new methods for the study of quantum many-body systems, especially those exhibiting exotic phases of matter, and thus complemented and advanced upon the previously acknowledged state-of-the-art approach to the analytical investigation of quantum many-body systems. The methods and solutions I have constructed can be applied to a large variety of problems that involve many-particle systems in realistic environments, with perturbation. Thus, my completed dissertation presents an advancement in the scientific research of quantum many-body systems.

Appendix A

Additional Calculations

This appendix chapter includes additional calculations that elaborate on the crucial steps and conclusions presented in the main part of the dissertation. Each calculation is connected to its relevant mention via hyperlink.

Perturbative Solution to a Many-body Localization Hamiltonian

Integrating $H_\nu(s_k)$

As described in (2.73), the calculation for A follows.

$$A = \sum_{j=1}^{N} h_j \int_0^x e^{-s_k H_0} \sigma_j^x e^{s_k H_0} \mathrm{d}s_k = \tag{A.1}$$

$$= \sum_{j=1}^{N} h_j \int_0^x e^{-s_k \sum_{l=1}^{N} \sigma_l^z \sigma_{l+1}^z} \sigma_j^x e^{s_k \sum_{m=1}^{N} \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.2}$$

$$= \sum_{j=1}^{N} h_j \int_0^x \prod_{l=1}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \sigma_j^x \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.3}$$

$$= \sum_{j=1}^{N} h_j \int_0^x e^{-s_k \sigma_{j-1}^z \sigma_j^z} e^{-s_k \sigma_j^z \sigma_{j+1}^z} \sigma_j^x \prod_{\substack{l=1 \\ l \neq \{j-1, j\}}}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.4}$$

$$= \sum_{j=1}^{N} h_j \int_0^x e^{-s_k \sigma_{j-1}^z \sigma_j^z} e^{-s_k \sigma_j^z \sigma_{j+1}^z} \sigma_j^x e^{s_k \sigma_{j+1}^z \sigma_j^z} e^{s_k \sigma_j^z \sigma_{j+1}^z} \mathrm{d}s_k = \tag{A.5}$$

$$= \sum_{j=1}^{N} h_j \int_0^x \sigma_j^x e^{s_k \sigma_{j-1}^z \sigma_j^z} e^{s_k \sigma_j^z \sigma_{j+1}^z} e^{s_k \sigma_{j+1}^z \sigma_j^z} e^{s_k \sigma_j^z \sigma_{j+1}^z} \mathrm{d}s_k = \tag{A.6}$$

$$= \sum_{j=1}^{N} h_j \sigma_j^x \int_0^x e^{2 s_k \sigma_{j-1}^z \sigma_j^z} e^{2 s_k \sigma_j^z \sigma_{j+1}^z} \mathrm{d}s_k \tag{A.7}$$

And the calculation for B is analogous.

$$B = \sum_{j=1}^{N} J_j \int_0^x e^{-s_k H_0} \sigma_j^x \sigma_{j+1}^x e^{s_k H_0} \mathrm{d}s_k = \tag{A.8}$$

$$= \sum_{j=1}^{N} J_j \int_0^x e^{-s_k \sum_{l=1}^{N} \sigma_l^z \sigma_{l+1}^z} \sigma_j^x \sigma_{j+1}^x e^{s_k \sum_{m=1}^{N} \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.9}$$

$$= \sum_{j=1}^{N} J_j \int_0^x \prod_{l=1}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \sigma_j^x \sigma_{j+1}^x \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.10}$$

$$= \sum_{j=1}^{N} J_j \int_0^x e^{-s_k \sigma_{j-1}^z \sigma_j^z} e^{-s_k \sigma_j^z \sigma_{j+1}^z} e^{-s_k \sigma_{j+1}^z \sigma_{j+2}^z} \sigma_j^x \sigma_{j+1}^x \prod_{\substack{l=1 \\ l \neq \{j-1,j,j+1\}}}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.11}$$

$$= \sum_{j=1}^{N} J_j \int_0^x e^{-s_k \sigma_{j-1}^z \sigma_j^z} e^{-s_k \sigma_j^z \sigma_{j+1}^z} \sigma_j^x \sigma_{j+1}^x e^{s_k \sigma_{j+1}^z \sigma_{j+2}^z} \prod_{\substack{l=1 \\ l \neq \{j-1,j,j+1\}}}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.12}$$

$$= \sum_{j=1}^{N} J_j \int_0^x e^{-s_k \sigma_{j-1}^z \sigma_j^z} \sigma_j^x \sigma_{j+1}^x e^{-s_k \sigma_j^z \sigma_{j+1}^z} e^{s_k \sigma_{j+1}^z \sigma_{j+2}^z} \prod_{\substack{l=1 \\ l \neq \{j-1,j,j+1\}}}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.13}$$

$$= \sum_{j=1}^{N} J_j \int_0^x \sigma_j^x \sigma_{j+1}^x e^{s_k \sigma_{j-1}^z \sigma_j^z} e^{-s_k \sigma_j^z \sigma_{j+1}^z} e^{s_k \sigma_{j+1}^z \sigma_{j+2}^z} \prod_{\substack{l=1 \\ l \neq \{j-1,j,j+1\}}}^{N} e^{-s_k \sigma_l^z \sigma_{l+1}^z} \prod_{m=1}^{N} e^{s_k \sigma_m^z \sigma_{m+1}^z} \mathrm{d}s_k = \tag{A.14}$$

$$= \sum_{j=1}^{N} J_j \sigma_j^x \sigma_{j+1}^x \int_0^x e^{2s_k \sigma_{j-1}^z \sigma_j^z} e^{2s_k \sigma_{j+1}^z \sigma_{j+2}^z} \mathrm{d}s_k \tag{A.15}$$

This result is used here.

First orders for the time-evolved expectations for $M(t)$ and $F(t)$:

$$\Omega_{1;M} = \mathbb{E}_h \left(\varepsilon \int_0^{-it} \left(e^{-itH_0} \left[e^{-sH_0} \sum_k h_k \sigma_k^x e^{sH_0}, M \right] e^{itH_0} \right) \mathrm{d}s \right) = \tag{A.16}$$

$$= \mathbb{E}_h \Bigg(\varepsilon \int_0^{-it} \Bigg(e^{-it \sum_a \sigma_a^z \sigma_{a+1}^z} \cdot \left[e^{-s \sum_l \sigma_l^z \sigma_{l+1}^z} \sum_k h_k \sigma_k^x e^{s \sum_m \sigma_m^z \sigma_{m+1}^z}, \left(\mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma_j^z \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I} \right) \right] e^{it \sum_b \sigma_b^z \sigma_{b+1}^z} \Bigg) \mathrm{d}s \Bigg) = \tag{A.17}$$

$$= \mathbb{E}_h \Bigg(\varepsilon \int_0^{-it} \Bigg(e^{-it \sum_a \sigma_a^z \sigma_{a+1}^z} \cdot \left[\sum_k \prod_l e^{-s \sigma_l^z \sigma_{l+1}^z} h_k \sigma_k^x \prod_m e^{s \sigma_m^z \sigma_{m+1}^z}, \left(\mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma_j^z \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I} \right) \right] e^{it \sum_b \sigma_b^z \sigma_{b+1}^z} \Bigg) \mathrm{d}s \Bigg) = \tag{A.18}$$

$$= \mathbb{E}_h \Bigg(\varepsilon \int_0^{-it} \Bigg(e^{-it \sum_a \sigma_a^z \sigma_{a+1}^z} \cdot \Bigg[\sum_k h_k \sigma_k^x \prod_{l<k} e^{-s \sigma_l^z \sigma_{l+1}^z} e^{s \sigma_{k-1}^z \sigma_k^z} e^{s \sigma_k^z \sigma_{k+1}^z} \cdot \prod_{l'>k+1} e^{-s \sigma_{l'}^z \sigma_{l'+1}^z} \prod_m e^{s \sigma_m^z \sigma_{m+1}^z}, \left(\mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma_j^z \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I} \right) \Bigg] e^{it \sum_b \sigma_b^z \sigma_{b+1}^z} \Bigg) \mathrm{d}s \Bigg) = \tag{A.19}$$

$$= \mathbb{E}_h \Bigg(\varepsilon \int_0^{-it} \Bigg(e^{-it \sum_a \sigma_a^z \sigma_{a+1}^z} \cdot \sum_k \left[h_k \sigma_k^x e^{2s \sigma_{k-1}^z \sigma_k^z} e^{2s \sigma_k^z \sigma_{k+1}^z}, \left(\mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma_j^z \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I} \right) \right] e^{it \sum_b \sigma_b^z \sigma_{b+1}^z} \Bigg) \mathrm{d}s \Bigg) = \tag{A.20}$$

$$= \mathbb{E}_h \Bigg(\varepsilon \int_0^{-it} \Bigg(e^{-it \sum_a \sigma_a^z \sigma_{a+1}^z} \cdot \sum_k \Bigg(h_k \sigma_k^x e^{2s \sigma_{k-1}^z \sigma_k^z} e^{2s \sigma_k^z \sigma_{k+1}^z} \left(\mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma_j^z \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I} \right) -$$

$$- \left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^z\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right) h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Bigg) e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.21}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\sum_k\Bigg(h_k\sigma_k^x\left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^z\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}-$$

$$- h_k\sigma_k^x\left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes(-1)^{\delta_{j,k}}\sigma_j^z\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Bigg)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.22}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\sigma_j^x\sigma_j^z e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.23}$$

$$=\mathbb{E}_h\left(h_j\cdot\text{const.}\right) = \tag{A.24}$$

$$=0 \tag{A.25}$$

$$\Omega_{1;F} = \mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-itH_0}\left[e^{-sH_0}\sum_k h_k\sigma_k^x e^{sH_0}, M\right]e^{itH_0}\Bigg)\mathrm{d}s\Bigg) = \tag{A.26}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot$$

$$\left[e^{-s\sum_l\sigma_l^z\sigma_{l+1}^z}\sum_k h_k\sigma_k^x e^{s\sum_m\sigma_m^z\sigma_{m+1}^z}, \left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.27}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot$$

$$\left[\sum_k\prod_l e^{-s\sigma_l^z\sigma_{l+1}^z}h_k\sigma_k^x\prod_m e^{s\sigma_m^z\sigma_{m+1}^z}, \left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.28}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\left[\sum_k h_k\sigma_k^x\prod_{l<k}e^{-s\sigma_l^z\sigma_{l+1}^z}e^{s\sigma_{k-1}^z\sigma_k^z}e^{s\sigma_k^z\sigma_{k+1}^z}\cdot\right.$$

$$\left.\prod_{l'>k+1}e^{-s\sigma_{l'}^z\sigma_{l'+1}^z}\prod_m e^{s\sigma_m^z\sigma_{m+1}^z}, \left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.29}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot$$

$$\sum_k\left[h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}, \left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.30}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\sum_k\Bigg(h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)-$$

$$- \left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Bigg)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.31}$$

$$=\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot$$

$$\sum_k\Bigg(h_k\sigma_k^x\left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)e^{(-1)^{\delta(k,j+1)+\delta(k,j)}2s\sigma_{k-1}^z\sigma_k^z}e^{(-1)^{\delta(k,j)+\delta(k,j-1)}2s\sigma_k^z\sigma_{k+1}^z}-$$

$$- h_k\sigma_k^x\left(\mathbb{I}\otimes\mathbb{I}\otimes ...\otimes\sigma_j^x\otimes ...\otimes\mathbb{I}\otimes\mathbb{I}\right)e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Bigg)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg) = \tag{A.32}$$

$$
\begin{aligned}
=&\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\left(e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{-2s\sigma_{j-1}^z\sigma_j^z}-e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s\sigma_{j-1}^z\sigma_j^z}\right)+\\
&+h_j\sigma_j^x\sigma_j^x\left(e^{-2s\sigma_{j-1}^z\sigma_j^z}e^{-2s\sigma_j^z\sigma_{j+1}^z}-e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\right)+\\
&+h_{j+1}\sigma_{j+1}^x\sigma_j^x\left(e^{-2s\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}-e^{2s\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}\right)\Big)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg)=
\end{aligned}
\tag{A.33}
$$

$$
\begin{aligned}
=&\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}\left(e^{-2s\sigma_{j-1}^z\sigma_j^z}-e^{2s\sigma_{j-1}^z\sigma_j^z}\right)+\\
&+h_j(\sigma_j^x)^2\left(e^{-2s\sigma_{j-1}^z\sigma_j^z}e^{-2s\sigma_j^z\sigma_{j+1}^z}-e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\right)+\\
&+h_{j+1}\sigma_{j+1}^x\sigma_j^x e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}\left(e^{-2s\sigma_j^z\sigma_{j+1}^z}-e^{2s\sigma_j^z\sigma_{j+1}^z}\right)\Big)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg)=
\end{aligned}
\tag{A.34}
$$

$$
\begin{aligned}
=&\mathbb{E}_h\Bigg(\varepsilon\int_0^{-it}\Bigg(e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s\sigma_{j-1}^z\sigma_j^z)\right)+\\
&h_j(\sigma_j^x)^2\cdot\left(-2\sinh(2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+\\
&+h_{j+1}\sigma_{j+1}^x\sigma_j^x e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}\cdot\left(-2\sinh(2s\sigma_j^z\sigma_{j+1}^z)\right)\Big)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\Bigg)\mathrm{d}s\Bigg)=
\end{aligned}
\tag{A.35}
$$

$$
=\mathbb{E}_h\left(\sum_{k\in\{j-1,j,j+1\}}h_k\cdot\text{const.}\right)=
\tag{A.36}
$$

$$
=0,
\tag{A.37}
$$

This result is used here.

Second order for the time-evolved expectation of $M(t)$:

$$
\Omega_{2;M}=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s e^{-itH_0}\left[H_\nu(s),[H_\nu(s_1),M]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s\right)=
\tag{A.38}
$$

$$
=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\left[H_\nu(s),2h_j\sigma_j^x\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\right)=
\tag{A.39}
$$

$$
\begin{aligned}
=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\\
&\left[e^{-s\sum_l\sigma_l^z\sigma_{l+1}^z}\sum_k h_k\sigma_k^x e^{s\sum_m\sigma_m^z\sigma_{m+1}^z},2h_j\sigma_j^x\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)=
\end{aligned}
\tag{A.40}
$$

$$
\begin{aligned}
=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\\
&\sum_k\left[h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z},2h_j\sigma_j^x\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\right]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)=
\end{aligned}
\tag{A.41}
$$

$$
\begin{aligned}
=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\sum_k\Big(h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}2h_j\sigma_j^x\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-\\
&-2h_j\sigma_j^x\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Big)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)=
\end{aligned}
\tag{A.42}
$$

$$=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\cdot$$
$$\sum_k h_k\Big(\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}\sigma_j^x e^{2s(-1)^{\delta(j,k)}(-1)^{\delta(j,k+1)}\sigma_k^z\sigma_{k+1}^z}\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-$$
$$-\sigma_j^x\sigma_j^z e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Big)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)= \tag{A.43}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\cdot$$
$$\sum_k h_k\Big(\sigma_k^x\sigma_j^x\sigma_j^z e^{2s(-1)^{\delta(j,k-1)}(-1)^{\delta(j,k)}\sigma_{k-1}^z\sigma_k^z}e^{2s(-1)^{\delta(j,k)}(-1)^{\delta(j,k+1)}\sigma_k^z\sigma_{k+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-$$
$$-\sigma_j^x\sigma_j^z\sigma_k^x e^{2s_1(-1)^{\delta(j-1,k)}(-1)^{\delta(j,k)}\sigma_{j-1}^z\sigma_j^z}e^{2s_1(-1)^{\delta(j,k)}(-1)^{\delta(j+1,k)}\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Big)\cdot$$
$$e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)= \tag{A.44}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\cdot$$
$$\sum_k h_k\Big(\sigma_k^x\sigma_j^x\sigma_j^z e^{2s(-1)^{\delta(j,k-1)}(-1)^{\delta(j,k)}\sigma_{k-1}^z\sigma_k^z}e^{2s(-1)^{\delta(j,k)}(-1)^{\delta(j,k+1)}\sigma_k^z\sigma_{k+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-$$
$$-(-1)^{\delta(j,k)}\sigma_k^x\sigma_j^x\sigma_j^z e^{2s_1(-1)^{\delta(j-1,k)}(-1)^{\delta(j,k)}\sigma_{j-1}^z\sigma_j^z}e^{2s_1(-1)^{\delta(j,k)}(-1)^{\delta(j+1,k)}\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Big)\cdot$$
$$e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)= \tag{A.45}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\cdot$$
$$\sum_k h_k\sigma_k^x\sigma_j^x\sigma_j^z\Big(e^{2s(-1)^{\delta(j,k-1)}(-1)^{\delta(j,k)}\sigma_{k-1}^z\sigma_k^z}e^{2s(-1)^{\delta(j,k)}(-1)^{\delta(j,k+1)}\sigma_k^z\sigma_{k+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}-$$
$$-(-1)^{\delta(j,k)}e^{2s_1(-1)^{\delta(j-1,k)}(-1)^{\delta(j,k)}\sigma_{j-1}^z\sigma_j^z}e^{2s_1(-1)^{\delta(j,k)}(-1)^{\delta(j+1,k)}\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}\Big)\cdot$$
$$e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)\equiv \tag{A.46}$$

$$\equiv\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\sum_k h_k\sigma_k^x\sigma_j^x\sigma_j^z\Theta(j,k)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg) \tag{A.47}$$

The expression can be further simplified, using the defined functions Θ.

$$\Omega_{2;M}=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_j\cdot$$
$$\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta(k=j-1)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta(k=j)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta(k=j+1)\Big)\cdot$$
$$e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)= \tag{A.48}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}2h_jh_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta(k=j)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)= \tag{A.49}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}2h_j^2(\sigma_j^x)^2\sigma_j^z\cdot$$
$$\left(e^{-2s\sigma_{j-1}^z\sigma_j^z}e^{-2s\sigma_j^z\sigma_{j+1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}+e^{-2s_1\sigma_{j-1}^z\sigma_j^z}e^{-2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\right)\cdot$$
$$e^{it\sum_b \sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)= \tag{A.50}$$

$$=2\varepsilon^2\eta_h^2\int_0^{-it}\int_0^s e^{-it\sum_a \sigma_a^z\sigma_{a+1}^z}(\sigma_j^x)^2\sigma_j^z\cdot$$
$$\left(e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}+e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{-2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\right)\cdot$$
$$e^{it\sum_b \sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s= \tag{A.51}$$

$$=2\varepsilon^2\eta_h^2\int_0^{-it}\int_0^s \prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}(\sigma_j^x)^2\sigma_j^z\cdot$$
$$\left(e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}+e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{-2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\right)\cdot$$
$$e^{it\sum_b \sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s= \tag{A.52}$$

$$=2\varepsilon^2\eta_h^2(\sigma_j^x)^2\sigma_j^z\int_0^{-it}\int_0^s\Big(e^{-2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}+$$
$$+e^{2s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}e^{-2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\Big)\mathrm{d}s_1\mathrm{d}s= \tag{A.53}$$

$$=2\varepsilon^2\eta_h^2\sigma_j^z\int_0^{-it}\int_0^s 2\cosh\left(2(s-s_1)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\mathrm{d}s_1\mathrm{d}s \tag{A.54}$$

This result is used here.

Example calculation of $\Omega_{2;M}$:

$$\Omega_{2;M}=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s e^{-itH_0}\left[H_\nu(s),[H_\nu(s_1),M]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.55}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s\left[\sum_l h_l\sigma_l^x,\left[\sum_k h_k\sigma_k^x,\sigma_j^z\right]\right]\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.56}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s\left[\sum_l h_l\sigma_l^x,\left(\sum_k h_k\sigma_k^x\sigma_j^z-\sigma_j^z\sum_{k_1}h_{k_1}\sigma_k^x\right)\right]\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.57}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s\left[\sum_l h_l\sigma_l^x,\left(\sum_k(-1)^{\delta(k,j)}h_k\sigma_j^z\sigma_k^x-\sigma_j^z\sum_{k_1}h_{k_1}\sigma_k^x\right)\right]\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.58}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s\left[\sum_l h_l\sigma_l^x,\sigma_j^z\left(-2h_j\sigma_j^x\right)\right]\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.59}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s\left(-2\sum_l h_l\sigma_l^x\cdot\sigma_j^z h_j\sigma_j^x+2\sigma_j^z h_j\sigma_j^x\cdot\sum_{l_1}h_{l_1}\sigma_{l_1}^x\right)\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.60}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s\left(-2\sum_l(-1)^{\delta(j,l)}h_l\sigma_j^z\sigma_l^x\cdot h_j\sigma_j^x+2\sigma_j^z h_j\sigma_j^x\cdot\sum_{l_1}h_{l_1}\sigma_{l_1}^x\right)\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.61}$$

$$=\mathbb{E}_h\left(-2\varepsilon^2 h_j\sigma_j^z\sigma_j^x\int_0^{-it}\int_0^s\left(\sum_l(-1)^{\delta(j,l)}h_l\sigma_l^x-\sum_{l_1}h_{l_1}\sigma_{l_1}^x\right)\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.62}$$

$$=\mathbb{E}_h\left(-2\varepsilon^2 h_j\sigma_j^z\sigma_j^x\int_0^{-it}\int_0^s\left(-2h_j\sigma_j^x\right)\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.63}$$

$$=\mathbb{E}_h\left(4\varepsilon^2 h_j^2\sigma_j^z\int_0^{-it}\int_0^s \mathrm{d}s_1\mathrm{d}s\right)= \tag{A.64}$$

$$=\mathbb{E}_h\left(4\varepsilon^2 h_j^2\sigma_j^z\int_0^{-it} s\mathrm{d}s\right)= \tag{A.65}$$

$$=\mathbb{E}_h\left(4\varepsilon^2 h_j^2\sigma_j^z\left[\frac{s^2}{2}\right]_0^{-it}\right)= \tag{A.66}$$

$$=\mathbb{E}_h\left(4\varepsilon^2 h_j^2\sigma_j^z\frac{-t^2}{2}\right)= \tag{A.67}$$

$$=\mathbb{E}_h\left(-2\varepsilon^2 h_j^2\sigma_j^z t^2\right)= \tag{A.68}$$

$$=-2t^2\eta_h^2\sigma_j^z \tag{A.69}$$

This result is used here.

Fourth order for the time-evolved expectation of $M(t)$:

$$\Omega_{2;F}=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s e^{-itH_0}\left[H_\nu(s),[H_\nu(s_1),F]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s\right)= \tag{A.70}$$

$$\begin{aligned}=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\Bigg[H_\nu(s),\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)+\\&+h_j(\sigma_j^x)^2\cdot\left(-2\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+\\&+h_{j+1}\sigma_{j+1}^x\sigma_j^x e^{2s_1\sigma_{j+1}^z\sigma_{j+2}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\Big)\Bigg]\cdot e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)=\end{aligned} \tag{A.71}$$

$$\begin{aligned}=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\Bigg[e^{-s\sum_l\sigma_l^z\sigma_{l+1}^z}\sum_k h_k\sigma_k^x e^{s\sum_m\sigma_m^z\sigma_{m+1}^z},\\&\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)+h_j(\sigma_j^x)^2\cdot\left(-2\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+\\&+h_{j+1}\sigma_{j+1}^x\sigma_j^x e^{2s_1\sigma_{j+1}^z\sigma_{j+2}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\Big)\Bigg]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)=\end{aligned} \tag{A.72}$$

$$\begin{aligned}=&\mathbb{E}_h\Bigg(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\cdot\sum_k\Bigg[h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z},\\&\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2i\sin(2s_1\sigma_{j-1}^z\sigma_j^z)\right)+h_j(\sigma_j^x)^2\cdot\left(-2\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+\\&+h_{j+1}\sigma_{j+1}^x\sigma_j^x e^{2s_1\sigma_{j+1}^z\sigma_{j+2}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\Big)\Bigg]e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\Bigg)=\end{aligned} \tag{A.73}$$

$$=\mathbb{E}_h\left(\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\sum_k\Big(\Sigma_1+\Sigma_2+\Sigma_3\Big)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s\right) \tag{A.74}$$

The commutator for the first summand gives:

$$\begin{aligned}&\Sigma_1\equiv h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)-\\&-h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)\cdot h_k\sigma_k^x e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}=\end{aligned} \tag{A.75}$$

$$\begin{aligned}=&h_k h_{j-1}\sigma_k^x\sigma_{j-1}^x e^{(-1)^{\delta(k,j-1)+\delta(k,j)}2s\sigma_{k-1}^z\sigma_k^z}e^{(-1)^{\delta(k,j-2)+\delta(k,j-1)}2s\sigma_k^z\sigma_{k+1}^z}\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)-\\&-h_k h_{j-1}\sigma_{j-1}^x\sigma_j^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\sigma_k^x\cdot\left(-2\sinh((-1)^{\delta(k,j-1)+\delta(k,j)}2s_1\sigma_{j-1}^z\sigma_j^z)\right)\cdot e^{2s\sigma_{k-1}^z\sigma_k^z}e^{2s\sigma_k^z\sigma_{k+1}^z}=\end{aligned} \tag{A.76}$$

$$=h_k h_{j-1}\sigma_k^x\sigma_{j-1}^x\sigma_j^x e^{(-1)^{\delta(k,j-1)+2\delta(k,j)+\delta(k,j+1)}2s\sigma_{k-1}^z\sigma_k^z}e^{(-1)^{\delta(k,j-2)+2\delta(k,j-1)+\delta(k,j)}2s\sigma_k^z\sigma_{k+1}^z}e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)-$$

$$- h_k h_{j-1} \sigma^x_{j-1} \sigma^x_j \sigma^x_k e^{(-1)^{\delta(k,j-2)+\delta(k,j-1)} 2 s_1 \sigma^z_{j-2} \sigma^z_{j-1}} \cdot \left(-2 \sinh((-1)^{\delta(k,j-1)+\delta(k,j)} 2 s_1 \sigma^z_{j-1} \sigma^z_j)\right) \cdot e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} = \tag{A.77}$$

$$= h_k h_{j-1} \sigma^x_k \sigma^x_{j-1} \sigma^x_j e^{(-1)^{\delta(k,j-1)+\delta(k,j+1)} 2 s \sigma^z_{k-1} \sigma^z_k} e^{(-1)^{\delta(k,j-2)+\delta(k,j)} 2 s \sigma^z_k \sigma^z_{k+1}} e^{2 s_1 \sigma^z_{j-2} \sigma^z_{j-1}} \cdot \left(-2 \sinh(2 s_1 \sigma^z_{j-1} \sigma^z_j)\right) -$$
$$- h_k h_{j-1} \sigma^x_k \sigma^x_{j-1} \sigma^x_j \cdot e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} \cdot e^{(-1)^{\delta(k,j-2)+\delta(k,j-1)} 2 s_1 \sigma^z_{j-2} \sigma^z_{j-1}} \cdot \left(-2 \sinh((-1)^{\delta(k,j-1)+\delta(k,j)} 2 s_1 \sigma^z_{j-1} \sigma^z_j)\right) = \tag{A.78}$$

$$= h_k h_{j-1} \sigma^x_k \sigma^x_{j-1} \sigma^x_j \Bigg(e^{(-1)^{\delta(k,j-1)+\delta(k,j+1)} 2 s \sigma^z_{k-1} \sigma^z_k} e^{(-1)^{\delta(k,j-2)+\delta(k,j)} 2 s \sigma^z_k \sigma^z_{k+1}} e^{2 s_1 \sigma^z_{j-2} \sigma^z_{j-1}} \cdot \left(-2 \sinh(2 s_1 \sigma^z_{j-1} \sigma^z_j)\right) -$$
$$- (-1)^{\delta(k,j-1)+\delta(k,j)} e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} \cdot e^{(-1)^{\delta(k,j-2)+\delta(k,j-1)} 2 s_1 \sigma^z_{j-2} \sigma^z_{j-1}} \cdot \left(-2 \sinh(2 s_1 \sigma^z_{j-1} \sigma^z_j)\right) \Bigg), \tag{A.79}$$

and it holds that $\Sigma_{1;k \neq \{j-2,j-1,j,j+1\}} = 0$. For the second summand, it holds that:

$$\Sigma_2 \equiv h_k \sigma^x_k e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} \cdot h_j (\sigma^x_j)^2 \cdot \left(-2 \sinh(2 s_1 (\sigma^z_{j-1} \sigma^z_j + \sigma^z_j \sigma^z_{j+1}))\right) -$$
$$- h_j (\sigma^x_j)^2 \cdot \left(-2 \sinh(2 s_1 (\sigma^z_{j-1} \sigma^z_j + \sigma^z_j \sigma^z_{j+1}))\right) \cdot h_k \sigma^x_k e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} = \tag{A.80}$$

$$= h_k h_j \sigma^x_k e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} \cdot \left(-2 \sinh(2 s_1 (\sigma^z_{j-1} \sigma^z_j + \sigma^z_j \sigma^z_{j+1}))\right) -$$
$$- h_k h_j \cdot \left(-2 \sinh(2 s_1 (\sigma^z_{j-1} \sigma^z_j + \sigma^z_j \sigma^z_{j+1}))\right) \cdot \sigma^x_k e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} \tag{A.81}$$

The commutators of the trigonometric functions of σ^x_k can be further simplified as:

$$\sinh(\xi \sigma^z_a \sigma^z_b) \cdot \sigma^x_k = \sigma^x_k \cdot \sinh((-1)^{\delta(k,a)+\delta(k,b)} \xi \sigma^z_a \sigma^z_b) = \tag{A.82}$$

$$= (-1)^{\delta(k,a)+\delta(k,b)} \sigma^x_k \cdot \sinh(\xi \sigma^z_a \sigma^z_b) \cosh(\xi \sigma^z_a \sigma^z_b) \cdot \sigma^x_k = \sigma^x_k \cdot \cosh((-1)^{\delta(k,a)+\delta(k,b)} \xi \sigma^z_a \sigma^z_b) = \tag{A.83}$$

$$= \sigma^x_k \cdot \cosh(\xi \sigma^z_a \sigma^z_b) \sinh(\xi(\sigma^z_a \sigma^z_b \pm \sigma^z_c \sigma^z_d)) \cdot \sigma^x_k = \tag{A.84}$$

$$= (\sinh(\xi(\sigma^z_a \sigma^z_b)) \cosh(\xi(\sigma^z_c \sigma^z_d)) \pm \cosh(\xi(\sigma^z_a \sigma^z_b)) \sinh(\xi(\sigma^z_c \sigma^z_d))) \cdot \sigma^x_k = \tag{A.85}$$

$$= \sinh(\xi(\sigma^z_a \sigma^z_b)) \sigma^x_k \cosh((-1)^{\delta(k,c)+\delta(k,d)} \xi(\sigma^z_c \sigma^z_d)) \pm$$
$$\pm \cosh(\xi(\sigma^z_a \sigma^z_b)) \sigma^x_k \sinh((-1)^{\delta(k,c)+\delta(k,d)} \xi(\sigma^z_c \sigma^z_d)) = \tag{A.86}$$

$$= \sigma^x_k \Big(\sinh((-1)^{\delta(k,a)+\delta(k,b)} \xi(\sigma^z_a \sigma^z_b)) \cosh((-1)^{\delta(k,c)+\delta(k,d)} \xi(\sigma^z_c \sigma^z_d)) \pm$$
$$\pm \cosh((-1)^{\delta(k,a)+\delta(k,b)} \xi(\sigma^z_a \sigma^z_b)) \sinh((-1)^{\delta(k,c)+\delta(k,d)} \xi(\sigma^z_c \sigma^z_d)) \Big) = \tag{A.87}$$

$$= \sigma^x_k \Big((-1)^{\delta(k,a)+\delta(k,b)} \sinh(\xi(\sigma^z_a \sigma^z_b)) \cosh(\xi(\sigma^z_c \sigma^z_d)) \pm$$
$$\pm (-1)^{\delta(k,c)+\delta(k,d)} \cosh(\xi(\sigma^z_a \sigma^z_b)) \sinh(\xi(\sigma^z_c \sigma^z_d)) \Big) = \tag{A.88}$$

$$= (-1)^{\delta(k,a)+\delta(k,b)} \sigma^x_k \Big(\sinh(\xi(\sigma^z_a \sigma^z_b)) \cosh(\xi(\sigma^z_c \sigma^z_d)) \pm$$
$$\pm (-1)^{\delta(k,a)+\delta(k,b)+\delta(k,c)+\delta(k,d)} \cosh(\xi(\sigma^z_a \sigma^z_b)) \sinh(\xi(\sigma^z_c \sigma^z_d)) \Big) = \tag{A.89}$$

$$= (-1)^{\delta(k,a)+\delta(k,b)} \sigma^x_k \sinh(\xi(\sigma^z_a \sigma^z_b \pm$$
$$\pm (-1)^{\delta(k,a)+\delta(k,b)+\delta(k,c)+\delta(k,d)} \sigma^z_c \sigma^z_d)) \cosh(\xi(\sigma^z_a \sigma^z_b \pm \sigma^z_c \sigma^z_d)) \cdot \sigma^x_k = \tag{A.90}$$

$$= (\cosh(\xi(\sigma^z_a \sigma^z_b)) \cosh(\xi(\sigma^z_c \sigma^z_d)) \pm \sinh(\xi(\sigma^z_a \sigma^z_b)) \sinh(\xi(\sigma^z_c \sigma^z_d))) \cdot \sigma^x_k = \tag{A.91}$$

$$= \sigma^x_k \Big(\cosh(\xi(\sigma^z_a \sigma^z_b)) \cosh(\xi(\sigma^z_c \sigma^z_d)) \pm$$
$$\pm (-1)^{\delta(k,a)+\delta(k,b)+\delta(k,c)+\delta(k,d)} \sinh(\xi(\sigma^z_a \sigma^z_b)) \sinh(\xi(\sigma^z_c \sigma^z_d)) \Big) = \tag{A.92}$$

$$= \sigma^x_k \cdot \cosh(\xi(\sigma^z_a \sigma^z_b \pm (-1)^{\delta(k,a)+\delta(k,b)+\delta(k,c)+\delta(k,d)} \sigma^z_c \sigma^z_d)) \tag{A.93}$$

Then, Σ_2 gives:

$$\Sigma_2 = h_k h_j \sigma^x_k e^{2 s \sigma^z_{k-1} \sigma^z_k} e^{2 s \sigma^z_k \sigma^z_{k+1}} \cdot \left(-2 \sinh(2 s_1 (\sigma^z_{j-1} \sigma^z_j + \sigma^z_j \sigma^z_{j+1}))\right) -$$

$$- h_k h_j \cdot \left(-2\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\right) \cdot \sigma^x_k e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} = \tag{A.94}$$

$$= h_k h_j \sigma^x_k e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} \cdot \left(-2\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\right) -$$
$$- h_k h_j \sigma^x_k \cdot \left(-2\cdot(-1)^{\delta(k,j-1)+\delta(k,j)}\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + (-1)^{\delta(k,j-1)+2\delta(k,j)+\delta(k,j+1)}\sigma^z_j\sigma^z_{j+1}))\right) \cdot$$
$$e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} = \tag{A.95}$$

$$= h_k h_j \sigma^x_k e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} \cdot \left(-2\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\right) -$$
$$- h_k h_j \sigma^x_k \cdot e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} \cdot$$
$$\left(-2\cdot(-1)^{\delta(k,j-1)+\delta(k,j)}\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + (-1)^{\delta(k,j-1)+\delta(k,j+1)}\sigma^z_j\sigma^z_{j+1}))\right) = \tag{A.96}$$

$$= h_k h_j \sigma^x_k e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} \cdot (-2) \cdot$$
$$\left(\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})) - (-1)^{\delta(k,j-1)+\delta(k,j)}\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + (-1)^{\delta(k,j-1)+\delta(k,j+1)}\sigma^z_j\sigma^z_{j+1}))\right) \tag{A.97}$$

and it holds that

$$\Sigma_{2;k\neq\{j-1,j,j+1\}} = 0. \tag{A.98}$$

Finally, the commutator with the third summand can be simplified to:

$$\Sigma_3 = h_k \sigma^x_k e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} \cdot h_{j+1}\sigma^x_{j+1}\sigma^x_j e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) -$$
$$- h_{j+1}\sigma^x_{j+1}\sigma^x_j e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) \cdot h_k\sigma^x_k e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} = \tag{A.99}$$

$$= h_k h_{j+1}\sigma^x_k\sigma^x_{j+1}\sigma^x_j e^{(-1)^{\delta(k,j)+\delta(k,j+2)}2s\sigma^z_{k-1}\sigma^z_k} e^{(-1)^{\delta(k,j-1)+\delta(k,j+1)}2s\sigma^z_k\sigma^z_{k+1}} \cdot e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) -$$
$$- h_k h_{j+1}\sigma^x_{j+1}\sigma^x_j e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \sigma^x_k \left(-(-1)^{\delta(k,j)+\delta(k,j+1)}2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) \cdot e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} = \tag{A.100}$$

$$= h_k h_{j+1}\sigma^x_k\sigma^x_{j+1}\sigma^x_j e^{(-1)^{\delta(k,j)+\delta(k,j+2)}2s\sigma^z_{k-1}\sigma^z_k} e^{(-1)^{\delta(k,j-1)+\delta(k,j+1)}2s\sigma^z_k\sigma^z_{k+1}} \cdot e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot$$
$$\left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) - \tag{A.101}$$
$$- h_k h_{j+1}\sigma^x_k\sigma^x_{j+1}\sigma^x_j e^{(-1)^{\delta(k,j+1)+\delta(k,j+2)}2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-(-1)^{\delta(k,j)+\delta(k,j+1)}2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) \cdot$$
$$e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} = \tag{A.102}$$

$$= h_k h_{j+1}\sigma^x_k\sigma^x_{j+1}\sigma^x_j \Big(e^{(-1)^{\delta(k,j)+\delta(k,j+2)}2s\sigma^z_{k-1}\sigma^z_k} e^{(-1)^{\delta(k,j-1)+\delta(k,j+1)}2s\sigma^z_k\sigma^z_{k+1}} \cdot e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) -$$
$$- (-1)^{\delta(k,j)+\delta(k,j+1)} e^{2s\sigma^z_{k-1}\sigma^z_k} e^{2s\sigma^z_k\sigma^z_{k+1}} e^{(-1)^{\delta(k,j+1)+\delta(k,j+2)}2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) \Big), \tag{A.103}$$

with

$$\Sigma_{3;k\neq\{j-1,j,j+1,j+2\}} = 0 \tag{A.104}$$

The derivation of $\Omega_{2;F}$ is then finalized by considering that $\mathbb{E}_h(h_a h_{b\neq a}) = 0$

$$\Omega_{2;F} = \mathbb{E}_h\left(\varepsilon^2 \int_0^{-it}\int_0^s e^{-it\sum_a \sigma^z_a\sigma^z_{a+1}} \sum_k \left(\Sigma_1 + \Sigma_2 + \Sigma_3\right) e^{it\sum_b \sigma^z_b\sigma^z_{b+1}} \mathrm{d}s_1 \mathrm{d}s\right) = \tag{A.105}$$

$$= \eta_h^2\varepsilon^2 \int_0^{-it}\int_0^s e^{-it\sum_a \sigma^z_a\sigma^z_{a+1}} \sum_k \left(\frac{\Sigma_{1;k=j-1}}{h^2_{j-1}} + \frac{\Sigma_{2;k=j}}{h^2_j} + \frac{\Sigma_{3;k=j+1}}{h^2_{j+1}}\right) e^{it\sum_b \sigma^z_b\sigma^z_{b+1}} \mathrm{d}s_1 \mathrm{d}s \tag{A.106}$$

$$= \eta_h^2\varepsilon^2 \int_0^{-it}\int_0^s e^{-it\sum_a \sigma^z_a\sigma^z_{a+1}} \Bigg(\sigma^x_{j-1}\sigma^x_{j-1}\sigma^x_j \cdot$$
$$\Bigg(e^{-2s\sigma^z_{j-2}\sigma^z_{j-1}} e^{2s\sigma^z_{j-1}\sigma^z_j} e^{2s_1\sigma^z_{j-2}\sigma^z_{j-1}} \cdot \left(-2\sinh(2s_1\sigma^z_{j-1}\sigma^z_j)\right) +$$
$$+ e^{2s\sigma^z_{j-2}\sigma^z_{j-1}} e^{2s\sigma^z_{j-1}\sigma^z_j} \cdot e^{-2s_1\sigma^z_{j-2}\sigma^z_{j-1}} \cdot \left(-2\sinh(2s_1\sigma^z_{j-1}\sigma^z_j)\right) \Bigg) +$$
$$+ \sigma^x_j e^{2s\sigma^z_{j-1}\sigma^z_j} e^{2s\sigma^z_j\sigma^z_{j+1}} \cdot (-2) \cdot \left(\sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})) + \sinh(2s_1(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\right) +$$
$$+ \sigma^x_{j+1}\sigma^x_{j+1}\sigma^x_j \Big(e^{2s\sigma^z_j\sigma^z_{j+1}} e^{-2s\sigma^z_{j+1}\sigma^z_{j+2}} \cdot e^{2s_1\sigma^z_{j+1}\sigma^z_{j+2}} \cdot \left(-2\sinh(2s_1\sigma^z_j\sigma^z_{j+1})\right) +$$

$$+ e^{2s\sigma_j^z\sigma_{j+1}^z}e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}e^{-2s_1\sigma_{j+1}^z\sigma_{j+2}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\Bigg)e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s = \tag{A.107}$$

$$=\eta_h^2\varepsilon^2\int_0^{-it}\int_0^s e^{-it\sum_a\sigma_a^z\sigma_{a+1}^z}\Bigg(\sigma_j^x e^{2s\sigma_{j-1}^z\sigma_j^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)\cdot$$
$$\left(e^{-2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}+e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{-2s_1\sigma_{j-2}^z\sigma_{j-1}^z}\right)+$$
$$+2\sigma_j^x e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\cdot\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+$$
$$+\sigma_j^x e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\left(e^{-2s\sigma_{j+1}^z\sigma_{j+2}^z}\cdot e^{2s_1\sigma_{j+1}^z\sigma_{j+2}^z}+e^{2s\sigma_{j+1}^z\sigma_{j+2}^z}e^{-2s_1\sigma_{j+1}^z\sigma_{j+2}^z}\right)\Bigg)\cdot$$
$$e^{it\sum_b\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s = \tag{A.108}$$

$$=\eta_h^2\varepsilon^2\int_0^{-it}\int_0^s\prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}\Bigg(\sigma_j^x e^{2s\sigma_{j-1}^z\sigma_j^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)\cdot$$
$$\left(e^{-2(s-s_1)\sigma_{j-2}^z\sigma_{j-1}^z}+e^{2(s-s_1)\sigma_{j-2}^z\sigma_{j-1}^z}\right)+$$
$$+2\sigma_j^x e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\cdot\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+$$
$$+\sigma_j^x e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\left(e^{-2(s-s_1)\sigma_{j+1}^z\sigma_{j+2}^z}+e^{2(s-s_1)\sigma_{j+1}^z\sigma_{j+2}^z}\right)\Bigg)\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s = \tag{A.109}$$

$$=\eta_h^2\varepsilon^2\int_0^{-it}\int_0^s\prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}\Bigg(\sigma_j^x e^{2s\sigma_{j-1}^z\sigma_j^z}\cdot\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)\left(-2\sinh(2(s-s_1)\sigma_{j-2}^z\sigma_{j-1}^z)\right)+$$
$$+2\sigma_j^x e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\cdot\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+$$
$$+\sigma_j^x e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\left(-2\sinh(2(s-s_1)\sigma_{j+1}^z\sigma_{j+2}^z)\right)\Bigg)\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s = \tag{A.110}$$

$$=\eta_h^2\varepsilon^2\sigma_j^x e^{2it(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\int_0^{-it}\Bigg(e^{2s\sigma_{j-1}^z\sigma_j^z}\cdot\int_0^s\left(-2\sinh(2s_1\sigma_{j-1}^z\sigma_j^z)\right)\left(-2\sinh(2(s-s_1)\sigma_{j-2}^z\sigma_{j-1}^z)\right)+$$
$$+2e^{2s\sigma_{j-1}^z\sigma_j^z}e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\int_0^s\left(-2\cdot\sinh(2s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z))\right)+$$
$$+e^{2s\sigma_j^z\sigma_{j+1}^z}\cdot\int_0^s\left(-2\sinh(2s_1\sigma_j^z\sigma_{j+1}^z)\right)\left(-2\sinh(2(s-s_1)\sigma_{j+1}^z\sigma_{j+2}^z)\right)\Bigg)\mathrm{d}s_1\mathrm{d}s \tag{A.111}$$

This result is used here.

As an example for the calculation of a higher order perturbative term, I will present the full calculation for $\Omega_{4;M}$.

$$\Omega_{4;M}=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}$$
$$\left|e^{-itH_0}\left[H_\nu(s_1),[H_\nu(s_2),[H_\nu(s_3),[H_\nu(s_4),M]]]\right]e^{itH_0}\right\rangle\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\Bigg) = \tag{A.112}$$
$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\int_0^{s_2}e^{-itH_0}.$$

$$\left[H_\nu(s_1),\left[H_\nu(s_2),\left[H_\nu(s_3),\left[\int_0^{s_3}H_\nu(s_4),M\right]\right]\right]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\Bigg) = \tag{A.113}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\int_0^{s_2}e^{-itH_0}\cdot$$

$$\left[H_\nu(s_1),\left[H_\nu(s_2),\left[H_\nu(s_3),\int_0^{s_3}H_\nu(s_4)M-M\int_0^{s_3}H_\nu(s_4)\right]\right]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\Bigg) = \tag{A.114}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\int_0^{s_2}e^{-itH_0}\left[H_\nu(s_1),\left[H_\nu(s_2),\left(H_\nu(s_3)\cdot\right.\right.\right.$$

$$\left.\left.\left.\left(\int_0^{s_3}H_\nu(s_4)M-M\int_0^{s_3}H_\nu(s_4)\right)-\left(\int_0^{s_3}H_\nu(s_4)M-M\int_0^{s_3}H_\nu(s_4)\right)H_\nu(s_3)\right)\right]\right]\cdot$$

$$e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\Bigg) = \tag{A.115}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\left[H_\nu(s_1),\left[H_\nu(s_2),\right.\right.$$

$$\left.\left.\int_0^{s_2}\int_0^{s_3}\left(H_\nu(s_3)\left(H_\nu(s_4)M-MH_\nu(s_4)\right)-\left(H_\nu(s_4)M-MH_\nu(s_4)\right)H_\nu(s_3)\right)\right]\right]\cdot$$

$$e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4\Bigg) = \tag{A.116}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\cdot$$

$$\left[H_\nu(s_1),\left[H_\nu(s_2),\int_0^{s_2}\int_0^{s_3}\left[H_\nu(s_3),\left[H_\nu(s_4),M\right]\right]\mathrm{d}s_3\mathrm{d}s_4\right]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\Bigg) \equiv \tag{A.117}$$

$$\equiv\mathbb{E}_h\left(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\left[H_\nu(s_1),\left[H_\nu(s_2),\zeta_2(s_2,s_3)\right]\right]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\right), \tag{A.118}$$

where

$$\zeta_2(s_2,s_3)=\int_0^{s_2}\int_0^{s_3}\left[H_\nu(s_3),\left[H_\nu(s_4),M\right]\right]\mathrm{d}s_3\mathrm{d}s_4 \tag{A.119}$$

From (2.112) it can be seen that

$$\Omega_2=\mathbb{E}_h\left(\varepsilon^2e^{-itH_0}\zeta_2(-it,s)e^{itH_0}\right), \tag{A.120}$$

so

$$\zeta_2(-it,s)=\int_0^{-it}\int_0^{s}2h_j\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta(k=j-1)+$$
$$+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta(k=j)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta(k=j+1)\Big)\mathrm{d}s_1\mathrm{d}s. \tag{A.121}$$

Back to the calculation.

$$\Omega_{4;M}=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\left[H_\nu(s_1),\left[H_\nu(s_2),\int_0^{s_2}\int_0^{s_3}2h_j\cdot\right.\right.$$

$$\left.\left.\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)\Big)\mathrm{d}s_4\mathrm{d}s_3\right]\right]\cdot$$

$$e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\Big)= \tag{A.122}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\Bigg[H_\nu(s_1),\Bigg[e^{-s_2\sum_l\sigma_l^z\sigma_{l+1}^z}\sum_k h_k\sigma_k^x e^{s_2\sum_m\sigma_m^z\sigma_{m+1}^z},$$
$$\int_0^{s_2}\int_0^{s_3}2h_j\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)\Big)\cdot$$
$$\mathrm{d}s_4\mathrm{d}s_3\Bigg]\Bigg]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\Bigg)= \tag{A.123}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\Bigg[H_\nu(s_1),\Bigg[\sum_k h_k\sigma_k^x\prod_l e^{-(-1)^{\delta(k,l)+\delta(k,l+1)}s_2 l\sigma_l^z\sigma_{l+1}^z}\prod_m e^{s_2\sigma_m^z\sigma_{m+1}^z},$$
$$\int_0^{s_2}\int_0^{s_3}2h_j\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)\Big)\cdot$$
$$\mathrm{d}s_4\mathrm{d}s_3\Bigg]\Bigg]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\Bigg)= \tag{A.124}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}e^{-itH_0}\Bigg[H_\nu(s_1),\Bigg[\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z},$$
$$\int_0^{s_2}\int_0^{s_3}2h_j\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)\Big)\cdot$$
$$\mathrm{d}s_4\mathrm{d}s_3\Bigg]\Bigg]e^{itH_0}\mathrm{d}s_1\mathrm{d}s_2\Bigg)= \tag{A.125}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}\Bigg[\sum_c h_c\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z},\Bigg[\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z},$$
$$\int_0^{s_2}\int_0^{s_3}2h_j\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)\Big)\cdot$$
$$\mathrm{d}s_4\mathrm{d}s_3\Bigg]\Bigg]\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s_2\Bigg)= \tag{A.126}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}\Bigg[\sum_c h_c\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z},\Bigg[\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z},$$
$$\int_0^{s_2}\int_0^{s_3}2h_j\Big(h_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)+h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)+h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)\Big)\cdot$$
$$\mathrm{d}s_4\mathrm{d}s_3\Bigg]\Bigg]\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_1\mathrm{d}s_2\Bigg)= \tag{A.127}$$

The first summand in the first commutator in detail:

$$\Pi_1=\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot 2h_jh_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)-$$
$$-\sum_l h_l 2h_jh_{j-1}\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}= \tag{A.128}$$

$$=2h_jh_{j-1}\Bigg(\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)-$$
$$-\sum_l h_l\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}\Bigg)= \tag{A.129}$$

$$=2h_jh_{j-1}\left(\sum_k h_k\sigma_k^x\sigma_{j-1}^x e^{(-1)^{\delta(k,j)+\delta(k,j-1)}2s_2\sigma_{k-1}^z\sigma_k^z}e^{(-1)^{\delta(k,j-1)+\delta(k,j-2)}2s_2\sigma_k^z\sigma_{k+1}^z}\sigma_j^x\sigma_j^z\Theta_{k=j-1}(s_3,s_4)-\right.$$

$$\left.-\sum_l h_l\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}\right)= \tag{A.130}$$

$$=2h_jh_{j-1}\left(\sum_k h_k\sigma_k^x\sigma_{j-1}^x\sigma_j^x\sigma_j^z e^{(-1)^{\delta(k,j+1)+2\delta(k,j)+\delta(k,j-1)}2s_2\sigma_{k-1}^z\sigma_k^z}e^{(-1)^{\delta(k,j)+2\delta(k,j-1)+\delta(k,j-2)}2s_2\sigma_k^z\sigma_{k+1}^z}\Theta_{k=j-1}(s_3,s_4)-\right.$$

$$\left.-\sum_l h_l\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}\right)= \tag{A.131}$$

$$=2h_jh_{j-1}\sum_l\left(h_l\sigma_l^x\sigma_{j-1}^x\sigma_j^x\sigma_j^z e^{(-1)^{\delta(l,j+1)+\delta(l,j-1)}2s_2\sigma_{l-1}^z\sigma_l^z}e^{(-1)^{\delta(l,j)+\delta(l,j-2)}2s_2\sigma_l^z\sigma_{l+1}^z}\Theta_{k=j-1}(s_3,s_4)-\right.$$

$$\left.-h_l\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}\right)= \tag{A.132}$$

$$=2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_jh_{j-1}\sum_l h_l\left((-1)^{\delta(l,j)}\sigma_l^x e^{(-1)^{\delta(l,j+1)+\delta(l,j-1)}2s_2\sigma_{l-1}^z\sigma_l^z}e^{(-1)^{\delta(l,j)+\delta(l,j-2)}2s_2\sigma_l^z\sigma_{l+1}^z}\Theta_{k=j-1}(s_3,s_4)-\right.$$

$$\left.-\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}\right) \tag{A.133}$$

The second summand in (A.133) depends on the relation between l and j. For the cases where $l\neq\{j-1,j,j+1,j+2\}$ the following is true.

$$\Pi_{1;l\neq\{j-1,j,j+1,j+2\}}=2h_jh_{j-1}\left(\sum_k h_k\sigma_k^x\sigma_{j-1}^x\sigma_j^x\sigma_j^z e^{(-1)^{\delta(k,j+1)+\delta(k,j-1)}2s_2\sigma_{k-1}^z\sigma_k^z}e^{(-1)^{\delta(k,j)+\delta(k,j-2)}2s_2\sigma_k^z\sigma_{k+1}^z}\Theta_{k=j-1}(s_3,s_4)-\right.$$

$$\left.-\sum_{l\neq\{j-1,j,j+1,j+2\}} h_l\sigma_l^x\sigma_{j-1}^x\sigma_j^x\sigma_j^z\cdot e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}\Theta_{l=j-1}(s_3,s_4)\right) \tag{A.134}$$

Furthermore,

$$\Pi_{1;l\neq\{j-2,j-1,j,j+1,j+2\}}=0, \tag{A.135}$$

and the summands for the other values of l are non-zero, i.e.

$$\Pi_{1;l=\{j-2,j-1,j,j+1,j+2\}}\neq 0. \tag{A.136}$$

The commutator for the third summand in (A.127) is as follows.

$$\Pi_3=\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)-$$
$$-\sum_l h_lh_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{l=j+1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z} \tag{A.137}$$

It might make sense to examine the commutation relations between Θ_ξ and σ_l^x. Beginning with $\Theta_{k=j-1}\equiv\Theta_{j-1}$.

$$\Theta_{j-1}\cdot\sigma_l^x=e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\sinh\left(2(s-s_1)\sigma_{j-1}^z\sigma_j^z\right)\right)\cdot\sigma_l^x= \tag{A.138}$$
$$=(-1)^{\delta(l,j-1)+\delta(l,j)}e^{2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\sigma_l^x\left(-2\sinh\left(2(s-s_1)\sigma_{j-1}^z\sigma_j^z\right)\right)= \tag{A.139}$$
$$=(-1)^{\delta(l,j-1)+\delta(l,j)}\sigma_l^x e^{(-1)^{\delta(l,j-2)+\delta(j-1)}2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{(-1)^{\delta(l,j)+\delta(l,j+1)}2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\left(-2\sinh\left(2(s-s_1)\sigma_{j-1}^z\sigma_j^z\right)\right)= \tag{A.140}$$
$$=\sigma_l^x(-1)^{\delta(l,j-1)+\delta(l,j)}e^{(-1)^{\delta(l,j-2)+\delta(j-1)}2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{(-1)^{\delta(l,j)+\delta(l,j+1)}2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\Theta_{j-1}\cdot e^{-2s\sigma_{j-2}^z\sigma_{j-1}^z}e^{-2s_1\sigma_j^z\sigma_{j+1}^z}= \tag{A.141}$$

$$=\sigma_l^x \cdot \Theta_{j-1} \cdot (-1)^{\delta(l,j-1)+\delta(l,j)} e^{((-1)^{\delta(l,j-2)+\delta(j-1)}-1)\cdot 2s\sigma_{j-2}^z\sigma_{j-1}^z} e^{((-1)^{\delta(l,j)+\delta(l,j+1)}-1)2s_1\sigma_j^z\sigma_{j+1}^z} \tag{A.142}$$

$$\left[\Theta_{j-1}, \sigma^x_{l\neq\{j-2,j-1,j,j+1\}}\right] = 0 \tag{A.143}$$

$$\Theta_{j+1}\cdot\sigma_l^x = e^{2s_1\sigma_{j-1}^z\sigma_j^z} e^{2s\sigma_{j+1}^z\sigma_{j+2}^z} \cdot \left(-2\sinh\left(2(s-s_1)\sigma_j^z\sigma_{j+1}^z\right)\right)\cdot\sigma_l^x = \tag{A.144}$$

$$=(-1)^{\delta(l,j)+\delta(l,j+1)}\sigma_l^x e^{(-1)^{\delta(l,j-1)+\delta(l,j)}2s_1\sigma_{j-1}^z\sigma_j^z} e^{(-1)^{\delta(l,j+1)+\delta(l,j+2)}2s\sigma_{j+1}^z\sigma_{j+2}^z} \cdot \left(-2\sinh\left(2(s-s_1)\sigma_j^z\sigma_{j+1}^z\right)\right) = \tag{A.145}$$

$$=\sigma_l^x (-1)^{\delta(l,j)+\delta(l,j+1)} e^{(-1)^{\delta(l,j-1)+\delta(l,j)}2s_1\sigma_{j-1}^z\sigma_j^z} e^{(-1)^{\delta(l,j+1)+\delta(l,j+2)}2s\sigma_{j+1}^z\sigma_{j+2}^z} \cdot \Theta_{j+1} \cdot e^{-2s_1\sigma_{j-1}^z\sigma_j^z} e^{-2s\sigma_{j+1}^z\sigma_{j+2}^z} = \tag{A.146}$$

$$=\sigma_l^x \cdot \Theta_{j+1} (-1)^{\delta(l,j)+\delta(l,j+1)} e^{((-1)^{\delta(l,j+1)+\delta(l,j+2)}-1)\cdot 2s\sigma_{j+1}^z\sigma_{j+2}^z} e^{((-1)^{\delta(l,j-1)+\delta(l,j)}-1)\cdot 2s_1\sigma_{j-1}^z\sigma_j^z} \tag{A.147}$$

$$\left[\Theta_{j+1}, \sigma^x_{l\neq\{j-1,j,j+1,j+2\}}\right] = 0 \tag{A.148}$$

$$\Theta_j\cdot\sigma_l^x = 2\cosh\left(2\cdot\left(s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z) - s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\right)\cdot\sigma_l^x = \tag{A.149}$$

$$=\sigma_l^x \cdot 2\cosh\left(2\cdot\left(s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z) - (-1)^{\delta(l,j-1)+\delta(l,j+1)}s(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\right) \tag{A.150}$$

$$\left[\Theta_j\cdot\sigma^x_{l\neq\{j-1,j+1\}}\right] = 0 \tag{A.151}$$

Back to the first summand, Π_1.

$$\Pi_1 = 2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_j h_{j-1}\cdot$$
$$\sum_l h_l\Big((-1)^{\delta(l,j)}\sigma_l^x e^{(-1)^{\delta(l,j+1)+\delta(l,j-1)}2s_2\sigma_{l-1}^z\sigma_l^z} e^{(-1)^{\delta(l,j)+\delta(l,j-2)}2s_2\sigma_l^z\sigma_{l+1}^z}\Theta_{k=j-1}(s_3,s_4) -$$
$$-\Theta_{l=j-1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z} e^{2s_2\sigma_l^z\sigma_{l+1}^z}\Big) = \tag{A.152}$$

$$=2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_j h_{j-1}\sum_l h_l\Big((-1)^{\delta(l,j)}\sigma_l^x e^{(-1)^{\delta(l,j+1)+\delta(l,j-1)}2s_2\sigma_{l-1}^z\sigma_l^z} e^{(-1)^{\delta(l,j)+\delta(l,j-2)}2s_2\sigma_l^z\sigma_{l+1}^z}\Theta_{k=j-1}(s_3,s_4) -$$
$$-\sigma_l^x\cdot\Theta_{j-1}\cdot(-1)^{\delta(l,j-1)+\delta(l,j)} e^{((-1)^{\delta(l,j-2)+\delta(j-1)}-1)\cdot 2s_3\sigma_{j-2}^z\sigma_{j-1}^z} e^{((-1)^{\delta(l,j)+\delta(l,j+1)}-1)2s_4\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{l-1}^z\sigma_l^z} e^{2s_2\sigma_l^z\sigma_{l+1}^z}\Big) = \tag{A.153}$$

$$=2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_j h_{j-1}\sum_l (-1)^{\delta(l,j)} h_l\sigma_l^x\Big(e^{(-1)^{\delta(l,j+1)+\delta(l,j-1)}2s_2\sigma_{l-1}^z\sigma_l^z} e^{(-1)^{\delta(l,j)+\delta(l,j-2)}2s_2\sigma_l^z\sigma_{l+1}^z} -$$
$$-(-1)^{\delta(l,j-1)} e^{((-1)^{\delta(l,j-2)+\delta(j-1)}-1)\cdot 2s_3\sigma_{j-2}^z\sigma_{j-1}^z} e^{((-1)^{\delta(l,j)+\delta(l,j+1)}-1)2s_4\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{l-1}^z\sigma_l^z} e^{2s_2\sigma_l^z\sigma_{l+1}^z}\Big)\cdot\Theta_{j-1}(s_3,s_4) \tag{A.154}$$

Because Π_1 contains $h_j h_{j-1}$ as a prefactor, the only cases in which it will give a result different than zero when looking at the full expectation value is when $l=\{j-1,j\}$, given $j\neq j-1$. I call this value P_1.

$$\mathrm{P}_1 = 2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_j h_{j-1}\Big(h_{j-1}\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} + e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big) -$$
$$-h_j\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{-2s_2\sigma_j^z\sigma_{j+1}^z} - e^{-4s_4\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\Big)\cdot\Theta_{j-1}(s_3,s_4) \tag{A.155}$$

I'll try calculating the outer commutator of P_1.

$$\rho_1 = \sum_c \Big(h_c\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c} e^{2s_1\sigma_c^z\sigma_{c+1}^z}\cdot 2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_j h_{j-1}\cdot$$
$$\Big(h_{j-1}\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} + e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big) -$$
$$-h_j\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{-2s_2\sigma_j^z\sigma_{j+1}^z} - e^{-4s_4\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\Big)\cdot\Theta_{j-1}(s_3,s_4) -$$
$$-2\sigma_{j-1}^x\sigma_j^x\sigma_j^z h_j h_{j-1}\Big(h_{j-1}\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} + e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big) -$$
$$-h_j\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{-2s_2\sigma_j^z\sigma_{j+1}^z} - e^{-4s_4\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\Big)\cdot\Theta_{j-1}(s_3,s_4)\cdot h_c\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c} e^{2s_1\sigma_c^z\sigma_{c+1}^z}\Big) = \tag{A.156}$$

$$
\begin{aligned}
=&2h_jh_{j-1}\sum_c h_c\Big(\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z}\cdot\sigma_{j-1}^x\sigma_j^x\sigma_j^z\cdot\\
&\Big(h_{j-1}\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)-\\
&-h_j\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&-\sigma_{j-1}^x\sigma_j^x\sigma_j^z\Big(h_{j-1}\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)-\\
&-h_j\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\Big)\cdot\Theta_{j-1}(s_3,s_4)\cdot\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z}\Big)
\end{aligned}
\tag{A.157}
$$

$$
\begin{aligned}
&\mathbb{E}_h(\rho_1)=2\eta_h^4\cdot\Bigg(\delta(c,j)\cdot\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z}\cdot\\
&\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&-\delta(c,j-1)\cdot\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z}\cdot\\
&\sigma_{j-1}^x\sigma_j^x\sigma_j^z\cdot\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&-\delta(j,c)\cdot\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)\cdot\\
&\Theta_{j-1}(s_3,s_4)\cdot\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z}+\\
&+\delta(c,j-1)\cdot\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\cdot\\
&\Theta_{j-1}(s_3,s_4)\cdot\sigma_c^x e^{2s_1\sigma_{c-1}^z\sigma_k^c}e^{2s_1\sigma_c^z\sigma_{c+1}^z}\Bigg)=
\end{aligned}
\tag{A.158}
$$

$$
\begin{aligned}
=&2\eta_h^4\cdot\Bigg(\sigma_j^x e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\\
&\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&-\sigma_{j-1}^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}\\
&\quad\cdot\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_j^x\cdot\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&-\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_{j-1}^x\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)\\
&\quad\cdot\Theta_{j-1}(s_3,s_4)\cdot\sigma_j^x e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}+\\
&+\sigma_{j-1}^x\sigma_j^x\sigma_j^z\sigma_j^x\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\cdot\\
&\Theta_{j-1}(s_3,s_4)\cdot\sigma_{j-1}^x e^{2s_1\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_1\sigma_{j-1}^z\sigma_j^z}\Bigg)=
\end{aligned}
\tag{A.159}
$$

$$
\begin{aligned}
=&2\eta_h^4\cdot\Bigg(e^{-2s_1\sigma_{j-1}^z\sigma_j^z}e^{-2s_1\sigma_j^z\sigma_{j+1}^z}\cdot\sigma_j^z\\
&\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&+e^{-2s_1\sigma_{j-2}^z\sigma_{j-1}^z}e^{-2s_1\sigma_{j-1}^z\sigma_j^z}\cdot\sigma_j^z\cdot\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\cdot\Theta_{j-1}(s_3,s_4)-\\
&-\sigma_j^x\sigma_j^z\Big(e^{-2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}+e^{-4s_3\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}\Big)\sigma_j^x\cdot\\
&\Theta_{j-1}(s_3,s_4)\cdot(-e^{-4s_4\sigma_j^z\sigma_{j+1}^z})e^{2s_1\sigma_{j-1}^z\sigma_j^z}e^{2s_1\sigma_j^z\sigma_{j+1}^z}+\\
&-\sigma_{j-1}^x\sigma_j^z\Big(e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{-2s_2\sigma_j^z\sigma_{j+1}^z}-e^{-4s_4\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j-1}^z\sigma_j^z}e^{2s_2\sigma_j^z\sigma_{j+1}^z}\Big)\sigma_{j-1}^x\cdot
\end{aligned}
$$

$$\Theta_{j-1}(s_3,s_4)\cdot(-e^{-4s_3\sigma^z_{j-2}\sigma^z_{j-1}})e^{2s_1\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_1\sigma^z_{j-1}\sigma^z_j}\Big)= \tag{A.160}$$

$$\begin{aligned}
=&2\eta_h^4\sigma^z_j\cdot\Big(e^{-2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\cdot\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}+e^{-4s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)+\\
&+e^{-2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\cdot\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}-e^{-4s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)-\\
&-\sigma^x_j\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}+e^{-4s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)\sigma^x_j\cdot e^{-4s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}+\\
&+\sigma^x_{j-1}\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}-e^{-4s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)\cdot\\
&\sigma^x_{j-1}\cdot e^{-4s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)\cdot\Theta_{j-1}(s_3,s_4)=
\end{aligned} \tag{A.161}$$

$$\begin{aligned}
=&2\eta_h^4\sigma^z_j\left(-2\sinh\left(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j\right)\right)\cdot\Big(e^{-2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\cdot\\
&\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)e^{2s_4\sigma^z_j\sigma^z_{j+1}}+\\
&+e^{-2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\cdot\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}-\\
&-\sigma^x_j\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)\sigma^x_j\cdot e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}+\\
&+\sigma^x_{j-1}\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)\sigma^x_{j-1}\cdot e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)=
\end{aligned} \tag{A.162}$$

$$\begin{aligned}
=&2\eta_h^4\sigma^z_j\left(-2\sinh\left(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j\right)\right)\cdot\Big(e^{-2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\cdot\\
&\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)e^{2s_4\sigma^z_j\sigma^z_{j+1}}+\\
&+e^{-2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\cdot\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}-\\
&-\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}\sigma^x_j+e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}\sigma^x_j\Big)\sigma^x_j\cdot e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}+\\
&+\Big(e^{-2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}\sigma^x_{j-1}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(-\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\sigma^x_{j-1}\Big)\sigma^x_{j-1}\cdot\\
&\cdot e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)=
\end{aligned} \tag{A.163}$$

$$\begin{aligned}
=&2\eta_h^4\sigma^z_j\left(-2\sinh\left(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j\right)\right)\cdot\Big(e^{-2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\cdot\\
&\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)e^{2s_4\sigma^z_j\sigma^z_{j+1}}+\\
&+e^{-2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\cdot\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}-\\
&-\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}\Big)\cdot e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}+\\
&+\Big(e^{-2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(-\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)\cdot e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)=
\end{aligned} \tag{A.164}$$

$$=2\eta_h^4\sigma^z_j\left(-2\sinh\left(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j\right)\right)\cdot$$

$$\cdot\Big(e^{-2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\cdot\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}\Big)e^{2s_4\sigma^z_j\sigma^z_{j+1}}+$$
$$+e^{-2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\cdot\Big(e^{2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big)e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}-$$
$$-e^{2s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}\Big(e^{-2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}e^{2s_3\sigma^z_{j-2}\sigma^z_{j-1}}+e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)}e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}\Big)\cdot e^{-2s_4\sigma^z_j\sigma^z_{j+1}}+$$
$$+\Big(e^{-2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})}e^{2s_4\sigma^z_j\sigma^z_{j+1}}-e^{-2s_4\sigma^z_j\sigma^z_{j+1}}e^{-2s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})}\Big)\cdot e^{-2s_3\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}\Big)= \tag{A.165}$$

$$=2\eta_h^4\sigma^z_j(-2\sinh(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j))\cdot$$
$$\cdot\Big(-2\sinh\big(2\big(s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})+s_2(\sigma^z_{j-2}\sigma^z_{j-1}-\sigma^z_{j-1}\sigma^z_j)-s_3\sigma^z_{j-2}\sigma^z_{j-1}-s_4(\sigma^z_j\sigma^z_{j+1})\big)\big)-$$
$$-2\sinh\big(2\big(s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})-s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)+s_3\sigma^z_{j-2}\sigma^z_{j-1}-s_4(\sigma^z_j\sigma^z_{j+1})\big)\big)-$$
$$-2\sinh\big(2\big(s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)-s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})-s_3\sigma^z_{j-2}\sigma^z_{j-1}-s_4(\sigma^z_j\sigma^z_{j+1})\big)\big)+$$
$$+2\sinh\big(2\big(s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)-s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})-s_3\sigma^z_{j-2}\sigma^z_{j-1}+s_4(\sigma^z_j\sigma^z_{j+1})\big)\big)\Big)= \tag{A.166}$$

$$=2\eta_h^4\sigma^z_j(-2\sinh(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j))\cdot$$
$$\cdot\Big(-4\sinh\big(2\big(s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})-s_2\sigma^z_{j-1}\sigma^z_j-s_4\sigma^z_j\sigma^z_{j+1}\big)\big)\cosh\big(2\big(s_2\sigma^z_{j-2}\sigma^z_{j-1}-s_3\sigma^z_{j-2}\sigma^z_{j-1}\big)\big)+$$
$$+4\cosh\big(2\big(s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)-s_2\sigma^z_{j-1}\sigma^z_j-s_3\sigma^z_{j-2}\sigma^z_{j-1}\big)\big)\sinh\big(2\big(-s_2\sigma^z_j\sigma^z_{j+1}+s_4\sigma^z_j\sigma^z_{j+1}\big)\big)\Big)= \tag{A.167}$$

$$=2\eta_h^4\sigma^z_j(-2\sinh(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j))\cdot$$
$$\cdot\Big(-4\sinh\big(2\big((s_1-s_2)\sigma^z_{j-1}\sigma^z_j+(s_1-s_4)\sigma^z_j\sigma^z_{j+1}\big)\big)\cosh\big(2\,(s_2-s_3)\,\sigma^z_{j-2}\sigma^z_{j-1}\big)-$$
$$-4\cosh\big(2\big((s_1-s_3)\sigma^z_{j-2}\sigma^z_{j-1}+(s_1-s_2)\sigma^z_{j-1}\sigma^z_j\big)\big)\sinh\big(2\,(s_2-s_4)\,\sigma^z_j\sigma^z_{j+1}\big)\Big) \tag{A.168}$$

$$\mathbb{E}_h(\rho_1)=2\eta_h^4\sigma^z_j(-2\sinh(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j))\cdot\Big(-4\Big(\sinh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\cosh(2(s_1-s_4)\sigma^z_j\sigma^z_{j+1})+$$
$$+\cosh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\sinh(2(s_1-s_4)\sigma^z_j\sigma^z_{j+1})\Big)\cosh\big(2\,(s_2-s_3)\,\sigma^z_{j-2}\sigma^z_{j-1}\big)-$$
$$-4\Big(\cosh(2(s_1-s_3)\sigma^z_{j-2}\sigma^z_{j-1})\cosh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)$$
$$+\sinh(2(s_1-s_3)\sigma^z_{j-2}\sigma^z_{j-1})\sinh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\Big)\sinh\big(2\,(s_2-s_4)\,\sigma^z_j\sigma^z_{j+1}\big)\Big)= \tag{A.169}$$

$$=16\eta_h^4\sigma^z_j(\sinh(2(s_3-s_4)\sigma^z_{j-1}\sigma^z_j))\cdot$$
$$\Big(\sinh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\cosh(2(s_1-s_4)\sigma^z_j\sigma^z_{j+1})\cosh\big(2\,(s_2-s_3)\,\sigma^z_{j-2}\sigma^z_{j-1}\big)+$$
$$+\cosh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\sinh(2(s_1-s_4)\sigma^z_j\sigma^z_{j+1})\cosh\big(2\,(s_2-s_3)\,\sigma^z_{j-2}\sigma^z_{j-1}\big)-$$
$$-\cosh(2(s_1-s_3)\sigma^z_{j-2}\sigma^z_{j-1})\cosh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\sinh\big(2\,(s_2-s_4)\,\sigma^z_j\sigma^z_{j+1}\big)+$$
$$+\sinh(2(s_1-s_3)\sigma^z_{j-2}\sigma^z_{j-1})\sinh(2(s_1-s_2)\sigma^z_{j-1}\sigma^z_j)\sinh\big(2\,(s_2-s_4)\,\sigma^z_j\sigma^z_{j+1}\big)\Big)= \tag{A.170}$$

$$
\begin{aligned}
=&16\eta_h^4\sigma_j^z(\sinh(2(s_3-s_4)\sigma_{j-1}^z\sigma_j^z))\cdot\Bigg(\\
&\cosh\left(2\left(s_2-s_3\right)\sigma_{j-2}^z\sigma_{j-1}^z\right)\sinh(2(s_1-s_2)\sigma_{j-1}^z\sigma_j^z)\cosh(2(s_1-s_4)\sigma_j^z\sigma_{j+1}^z)+\\
&+\cosh\left(2\left(s_2-s_3\right)\sigma_{j-2}^z\sigma_{j-1}^z\right)\cosh(2(s_1-s_2)\sigma_{j-1}^z\sigma_j^z)\sinh(2(s_1-s_4)\sigma_j^z\sigma_{j+1}^z)-\\
&-\cosh(2(s_1-s_3)\sigma_{j-2}^z\sigma_{j-1}^z)\cosh(2(s_1-s_2)\sigma_{j-1}^z\sigma_j^z)\sinh\left(2\left(s_2-s_4\right)\sigma_j^z\sigma_{j+1}^z\right)+\\
&+\sinh(2(s_1-s_3)\sigma_{j-2}^z\sigma_{j-1}^z)\sinh(2(s_1-s_2)\sigma_{j-1}^z\sigma_j^z)\sinh\left(2\left(s_2-s_4\right)\sigma_j^z\sigma_{j+1}^z\right)\Bigg)
\end{aligned}
\tag{A.171}
$$

$$
\begin{aligned}
\Pi_3=&\sum_k h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)-\\
&-\sum_l h_l h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{l=j+1}(s_3,s_4)\cdot\sigma_l^x e^{2s_2\sigma_{l-1}^z\sigma_l^z}e^{2s_2\sigma_l^z\sigma_{l+1}^z}=
\end{aligned}
\tag{A.172}
$$

$$
\begin{aligned}
=\sum_k h_k\Bigg(&\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\Theta_{k=j+1}(s_3,s_4)-\\
&-h_{j+1}\sigma_{j+1}^x\sigma_j^x\sigma_j^z\sigma_k^x\cdot\Theta_{j+1}(-1)^{\delta(k,j)+\delta(k,j+1)}e^{((-1)^{\delta(k,j-1)+\delta(k,j)}-1)\cdot 2s_1\sigma_{j-1}^z\sigma_j^z}\cdot\\
&e^{((-1)^{\delta(k,j+1)+\delta(k,j+2)}-1)\cdot 2s\sigma_{j+1}^z\sigma_{j+2}^z}e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\Bigg)=\ldots
\end{aligned}
\tag{A.173}
$$

Using simple logic with symmetry conditions and (A.167), $\mathbb{E}_h(\rho_3)$ should be easy to write out.

$$
\begin{aligned}
&\mathbb{E}_h(\rho_3)=2\eta_h^4\sigma_j^z(-2\sinh(2(s_3-s_4)\sigma_{j+1}^z\sigma_j^z))\cdot\\
&\Bigg(-2\sinh\left(2\left(s_1(\sigma_{j+1}^z\sigma_j^z+\sigma_j^z\sigma_{j-1}^z)+s_2(\sigma_{j+2}^z\sigma_{j+1}^z-\sigma_{j+1}^z\sigma_j^z)-s_3\sigma_{j+2}^z\sigma_{j+1}^z-s_4(\sigma_j^z\sigma_{j-1}^z)\right)\right)-\\
&-2\sinh\left(2\left(s_1(\sigma_{j+1}^z\sigma_j^z+\sigma_j^z\sigma_{j-1}^z)-s_2(\sigma_{j+2}^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_j^z)+s_3\sigma_{j+2}^z\sigma_{j+1}^z-s_4(\sigma_j^z\sigma_{j-1}^z)\right)\right)-\\
&-2\sinh\left(2\left(s_1(\sigma_{j+2}^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_j^z)-s_2(\sigma_{j+1}^z\sigma_j^z-\sigma_j^z\sigma_{j-1}^z)-s_3\sigma_{j+2}^z\sigma_{j+1}^z-s_4\sigma_j^z\sigma_{j-1}^z\right)\right)+\\
&+2\sinh\left(2\left(s_1(\sigma_{j+2}^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_j^z)-s_2(\sigma_{j+1}^z\sigma_j^z+\sigma_j^z\sigma_{j-1}^z)-s_3\sigma_{j+2}^z\sigma_{j+1}^z+s_4\sigma_j^z\sigma_{j-1}^z\right)\right)\Bigg)
\end{aligned}
\tag{A.174}
$$

I'm left with $\mathbb{E}_h(\rho_2)$ to calculate. I start with finding P_2.

$$
\mathrm{P}_2=\sum_k\left[h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z},\int_0^{s_2}\int_0^{s_3}2h_j h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)\mathrm{d}s_4\mathrm{d}s_3\right]=
\tag{A.175}
$$

$$
\begin{aligned}
=\sum_k\Bigg(&h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot\int_0^{s_2}\int_0^{s_3}2h_j h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)\mathrm{d}s_4\mathrm{d}s_3-\\
&-\int_0^{s_2}\int_0^{s_3}2h_j h_j\sigma_j^x\sigma_j^x\sigma_j^z\Theta_{k=j}(s_3,s_4)\mathrm{d}s_4\mathrm{d}s_3\cdot h_k\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\Bigg)=
\end{aligned}
\tag{A.176}
$$

$$
\begin{aligned}
=&2h_j^2\int_0^{s_2}\int_0^{s_3}\sum_k h_k\cdot\\
&\left(\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot\sigma_j^z\Theta_{k=j}(s_3,s_4)-\sigma_j^z\Theta_{k=j}(s_3,s_4)\cdot\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\right)\mathrm{d}s_4\mathrm{d}s_3=
\end{aligned}
\tag{A.177}
$$

$$
\begin{aligned}
=&2h_j^2\sigma_j^z\int_0^{s_2}\int_0^{s_3}\sum_k h_k\Bigg((-1)^{\delta(k,j)}\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z}e^{2s_2\sigma_k^z\sigma_{k+1}^z}\cdot\\
&2\cosh\left(2\cdot\left(s_4(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)-s_3(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\right)-
\end{aligned}
$$

$$- \sigma_l^x \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - (-1)^{\delta(l,j-1)+\delta(l,j+1)} s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) \cdot$$
$$e^{2s_2\sigma_{k-1}^z\sigma_k^z} e^{2s_2\sigma_k^z\sigma_{k+1}^z}\Bigg) \mathrm{d}s_4\mathrm{d}s_3 = \tag{A.178}$$

$$= 2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3} \sum_k h_k \cdot$$
$$\Bigg((-1)^{\delta(k,j)} \sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z} e^{2s_2\sigma_k^z\sigma_{k+1}^z} \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) -$$
$$- \sigma_l^x \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - (-1)^{\delta(l,j-1)+\delta(l,j+1)} s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) \cdot$$
$$e^{2s_2\sigma_{k-1}^z\sigma_k^z} e^{2s_2\sigma_k^z\sigma_{k+1}^z}\Bigg) \mathrm{d}s_4\mathrm{d}s_3 \tag{A.179}$$

$$\mathrm{P}_{2,k\neq\{j-1,j,j+1\}} = 2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3} \sum_k h_k \cdot$$
$$\Bigg(\sigma_k^x e^{2s_2\sigma_{k-1}^z\sigma_k^z} e^{2s_2\sigma_k^z\sigma_{k+1}^z} \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) -$$
$$- \sigma_l^x \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) e^{2s_2\sigma_{k-1}^z\sigma_k^z} e^{2s_2\sigma_k^z\sigma_{k+1}^z}\Bigg) \mathrm{d}s_4\mathrm{d}s_3 = \tag{A.180}$$

$$= 0 \tag{A.181}$$

$$\mathrm{P}_2 = \mathrm{P}_{2,k=\{j-1,j,j+1\}} = \tag{A.182}$$

$$= 2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3} \Bigg(h_{j-1}\sigma_{j-1}^x e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} \cdot$$
$$2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) -$$
$$- h_{j-1}\sigma_{j-1}^x \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) + s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} -$$
$$- h_j\sigma_j^x e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{2s_2\sigma_j^z\sigma_{j+1}^z} \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) +$$
$$+ h_j\sigma_j^x \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{2s_2\sigma_j^z\sigma_{j+1}^z} +$$
$$+ h_{j+1}\sigma_{j+1}^x e^{2s_2\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{j+1}^z\sigma_{j+2}^z} \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) -$$
$$- h_{j+1}\sigma_{j+1}^x \cdot 2\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) + s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) e^{2s_2\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{j+1}^z\sigma_{j+2}^z}\Bigg) \mathrm{d}s_4\mathrm{d}s_3 = \tag{A.183}$$

$$= 2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3} \Bigg(2h_{j-1}\sigma_{j-1}^x e^{2s_2\sigma_{j-2}^z\sigma_{j-1}^z} e^{2s_2\sigma_{j-1}^z\sigma_j^z} \cdot$$
$$\Big(\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) -$$
$$\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) + s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right)\Big) -$$
$$- 4h_j\sigma_j^x e^{2s_2\sigma_{j-1}^z\sigma_j^z} e^{2s_2\sigma_j^z\sigma_{j+1}^z}$$
$$\cdot \cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) +$$
$$+ 2h_{j+1}\sigma_{j+1}^x e^{2s_2\sigma_j^z\sigma_{j+1}^z} e^{2s_2\sigma_{j+1}^z\sigma_{j+2}^z}$$
$$\cdot \Big(\cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z) - s_3(\sigma_{j-1}^z\sigma_j^z + \sigma_j^z\sigma_{j+1}^z)\right)\right) -$$

$$- \cosh\left(2 \cdot \left(s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) + s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\right)\Bigg) \mathrm{d}s_4\mathrm{d}s_3 = \tag{A.184}$$

$$\begin{aligned}
=&2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3}\Bigg(\left(h_{j-1}\sigma^x_{j-1}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)} + h_{j+1}\sigma^x_{j+1}e^{2s_2(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\\
&\cdot 4\sinh\left(2s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\sinh\left(-2s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right) -\\
&- 4h_j\sigma^x_j e^{2s_2\sigma^z_{j-1}\sigma^z_j}e^{2s_2\sigma^z_j\sigma^z_{j+1}}\cdot\Big(\cosh(2s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\cosh(2s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\\
&- \sinh(2s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\sinh(2s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\Big)\Bigg)\mathrm{d}s_4\mathrm{d}s_3 =
\end{aligned} \tag{A.185}$$

$$\begin{aligned}
=&2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3}\Bigg(\left(h_{j-1}\sigma^x_{j-1}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)} + h_j\sigma^x_j e^{2s_2\sigma^z_{j-1}\sigma^z_j}e^{2s_2\sigma^z_j\sigma^z_{j+1}} + h_{j+1}\sigma^x_{j+1}e^{2s_2(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\\
&\cdot 4\sinh\left(2s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\sinh\left(-2s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right) -\\
&- h_j\sigma^x_j e^{2s_2\sigma^z_{j-1}\sigma^z_j}e^{2s_2\sigma^z_j\sigma^z_{j+1}}\cdot 4\cosh(2s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\cosh(2s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\Bigg)\mathrm{d}s_4\mathrm{d}s_3 =
\end{aligned} \tag{A.186}$$

$$\begin{aligned}
=&2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3}\Bigg(\left(h_{j-1}\sigma^x_{j-1}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)} + h_j\sigma^x_j e^{2s_2\sigma^z_{j-1}\sigma^z_j}e^{2s_2\sigma^z_j\sigma^z_{j+1}} + h_{j+1}\sigma^x_{j+1}e^{2s_2(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\\
&\cdot 2\left(\cosh(2(s_4 - s_3)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})) - \cosh(2(s_4 + s_3)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\right) -\\
&- h_j\sigma^x_j e^{2s_2\sigma^z_{j-1}\sigma^z_j}e^{2s_2\sigma^z_j\sigma^z_{j+1}}\cdot 2\left(\cosh(2(s_4 - s_3)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})) + \cosh(2(s_4 + s_3)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\right)\Bigg)\mathrm{d}s_4\mathrm{d}s_3 =
\end{aligned} \tag{A.187}$$

$$\begin{aligned}
=&2h_j^2\sigma_j^z \int_0^{s_2}\int_0^{s_3}\Bigg(\left(h_{j-1}\sigma^x_{j-1}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)} + h_{j+1}\sigma^x_{j+1}e^{2s_2(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\\
&\cdot 2\cosh(2(s_3 - s_4)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})) -\\
&- \left(h_{j-1}\sigma^x_{j-1}e^{2s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)} + 2h_j\sigma^x_j e^{2s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})} + h_{j+1}\sigma^x_{j+1}e^{2s_2(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\\
&\cdot 2\cosh(2(s_3 + s_4)(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}))\Bigg)\mathrm{d}s_4\mathrm{d}s_3
\end{aligned} \tag{A.188}$$

$$\begin{aligned}
&\mathbb{E}_h(\rho_2) = \mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\prod_a e^{-it\sigma^z_a\sigma^z_{a+1}}\Bigg[\sum_c h_c\sigma^x_c e^{2s_1\sigma^z_{c-1}\sigma^c_k}e^{2s_1\sigma^z_c\sigma^z_{c+1}}, 2h_j^2\sigma_j^z\cdot\\
&\int_0^{s_2}\int_0^{s_3}\Bigg(2h_{j-1}\sigma^x_{j-1}e^{2s_2\sigma^z_{j-2}\sigma^z_{j-1}}e^{2s_2\sigma^z_{j-1}\sigma^z_j}\cdot\Big(\cosh\left(2\cdot\left(s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) - s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\right) -\\
&- \cosh\left(2\cdot\left(s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) + s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\right)\Big) -\\
&- 4h_j\sigma^x_j e^{2s_2\sigma^z_{j-1}\sigma^z_j}e^{2s_2\sigma^z_j\sigma^z_{j+1}}\cdot\cosh\left(2\cdot\left(s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) - s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\right) +\\
&+ 2h_{j+1}\sigma^x_{j+1}e^{2s_2\sigma^z_j\sigma^z_{j+1}}e^{2s_2\sigma^z_{j+1}\sigma^z_{j+2}}\cdot\Big(\cosh\left(2\cdot\left(s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) - s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\right) -\\
&- \cosh\left(2\cdot\left(s_4(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1}) + s_3(\sigma^z_{j-1}\sigma^z_j + \sigma^z_j\sigma^z_{j+1})\right)\right)\Bigg)\mathrm{d}s_4\mathrm{d}s_3\Bigg]\prod_b e^{it\sigma^z_b\sigma^z_{b+1}}\mathrm{d}s_2\mathrm{d}s_1\Bigg) =
\end{aligned} \tag{A.189}$$

$$=\mathbb{E}_h\Bigg(\varepsilon^4\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\prod_a e^{-it\sigma^z_a\sigma^z_{a+1}}\cdot$$

$$\cdot \left[\sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z}, 2h_j^2 \sigma_j^z \left(2h_{j-1} \sigma_{j-1}^x e^{2s_2 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_2 \sigma_{j-1}^z \sigma_j^z} \cdot \right.\right.$$
$$\left(2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right) -$$
$$- 4h_j \sigma_j^x e^{2s_2 \sigma_{j-1}^z \sigma_j^z} e^{2s_2 \sigma_j^z \sigma_{j+1}^z} \cdot \cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z) - s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right) +$$
$$+ 2h_{j+1} \sigma_{j+1}^x e^{2s_2 \sigma_j^z \sigma_{j+1}^z} e^{2s_2 \sigma_{j+1}^z \sigma_{j+2}^z} \cdot$$
$$\left.\left(2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right) \mathrm{d}s_4 \mathrm{d}s_3 \right] \prod_b e^{it\sigma_b^z \sigma_{b+1}^z} \mathrm{d}s_2 \mathrm{d}s_1 \Bigg) = \tag{A.190}$$

$$= \mathbb{E}_h \Bigg(\varepsilon^4 \int_0^{-it} \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \prod_a e^{-it\sigma_a^z \sigma_{a+1}^z} \cdot$$
$$\cdot \Bigg(\sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z} \cdot 2h_j^2 \sigma_j^z 2h_{j-1} \sigma_{j-1}^x e^{2s_2 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_2 \sigma_{j-1}^z \sigma_j^z}$$
$$\cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) -$$
$$- 2h_j^2 \sigma_j^z 2h_{j-1} \sigma_{j-1}^x e^{2s_2 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_2 \sigma_{j-1}^z \sigma_j^z} \cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)$$
$$\cdot \sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z} -$$
$$- \sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z} \cdot 2h_j^2 \sigma_j^z 4h_j \sigma_j^x e^{2s_2 \sigma_{j-1}^z \sigma_j^z} e^{2s_2 \sigma_j^z \sigma_{j+1}^z}$$
$$\cdot \cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z) - s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right) +$$
$$+ 2h_j^2 \sigma_j^z 4h_j \sigma_j^x e^{2s_2 \sigma_{j-1}^z \sigma_j^z} e^{2s_2 \sigma_j^z \sigma_{j+1}^z} \cdot \cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z) - s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right)$$
$$\cdot \sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z} +$$
$$+ \sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z} \cdot 2h_j^2 \sigma_j^z 2h_{j+1} \sigma_{j+1}^x e^{2s_2 \sigma_j^z \sigma_{j+1}^z} e^{2s_2 \sigma_{j+1}^z \sigma_{j+2}^z}$$
$$\cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) -$$
$$- 2h_j^2 \sigma_j^z 2h_{j+1} \sigma_{j+1}^x e^{2s_2 \sigma_j^z \sigma_{j+1}^z} e^{2s_2 \sigma_{j+1}^z \sigma_{j+2}^z} \cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)$$
$$\cdot \sum_c h_c \sigma_c^x e^{2s_1 \sigma_{c-1}^z \sigma_k^c} e^{2s_1 \sigma_c^z \sigma_{c+1}^z} \mathrm{d}s_4 \mathrm{d}s_3 \Bigg) \prod_b e^{it\sigma_b^z \sigma_{b+1}^z} \mathrm{d}s_2 \mathrm{d}s_1 \Bigg) = \tag{A.191}$$

$$= \mathbb{E}_h \Bigg(\varepsilon^4 \int_0^{-it} \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \prod_a e^{-it\sigma_a^z \sigma_{a+1}^z} \cdot$$
$$\cdot \Bigg(\sigma_{j-1}^x e^{2s_1 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_1 \sigma_{j-1}^z \sigma_j^z} \cdot 2h_j^2 \sigma_j^z 2h_{j-1}^2 \sigma_{j-1}^x e^{2s_2 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_2 \sigma_{j-1}^z \sigma_j^z}$$
$$\cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) -$$
$$- 2h_j^2 \sigma_j^z 2h_{j-1}^2 \sigma_{j-1}^x e^{2s_2 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_2 \sigma_{j-1}^z \sigma_j^z} \cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)$$
$$\cdot \sigma_{j-1}^x e^{2s_1 \sigma_{j-2}^z \sigma_{j-1}^z} e^{2s_1 \sigma_{j-1}^z \sigma_j^z} -$$
$$- \sigma_j^x e^{2s_1 \sigma_{j-1}^z \sigma_j^z} e^{2s_1 \sigma_j^z \sigma_{j+1}^z} \cdot 2h_j^2 \sigma_j^z 4h_j^2 \sigma_j^x e^{2s_2 \sigma_{j-1}^z \sigma_j^z} e^{2s_2 \sigma_j^z \sigma_{j+1}^z}$$
$$\cdot \cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z) - s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right) +$$
$$+ 2h_j^2 \sigma_j^z 4h_j^2 \sigma_j^x e^{2s_2 \sigma_{j-1}^z \sigma_j^z} e^{2s_2 \sigma_j^z \sigma_{j+1}^z} \cdot \cosh\left(2 \cdot \left(s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z) - s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right)\right)$$
$$\cdot \sigma_j^x e^{2s_1 \sigma_{j-1}^z \sigma_j^z} e^{2s_1 \sigma_j^z \sigma_{j+1}^z} +$$
$$+ \sigma_{j+1}^x e^{2s_1 \sigma_j^z \sigma_{j+1}^z} e^{2s_1 \sigma_{j+1}^z \sigma_{j+2}^z} \cdot 2h_j^2 \sigma_j^z 2h_{j+1}^2 \sigma_{j+1}^x e^{2s_2 \sigma_j^z \sigma_{j+1}^z} e^{2s_2 \sigma_{j+1}^z \sigma_{j+2}^z}$$
$$\cdot 2 \sinh\left(2s_4(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) \sinh\left(2s_3(\sigma_{j-1}^z \sigma_j^z + \sigma_j^z \sigma_{j+1}^z)\right) -$$

$$- 2h_j^2\sigma_j^z 2h_{j+1}^2\sigma_{j+1}^x e^{2s_2\sigma_j^z\sigma_{j+1}^z}e^{2s_2\sigma_{j+1}^z\sigma_{j+2}^z}\cdot 2\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)$$
$$\cdot\,\sigma_{j+1}^x e^{2s_1\sigma_j^z\sigma_{j+1}^z}e^{2s_1\sigma_{j+1}^z\sigma_{j+2}^z}\mathrm{d}s_4\mathrm{d}s_3\Bigg)\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_2\mathrm{d}s_1\Bigg) = \tag{A.192}$$

$$=\eta_h^4\cdot\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}\cdot$$
$$\cdot\Bigg(8\sigma_j^z e^{-2(s_1-s_2)(\sigma_{j-2}^z\sigma_{j-1}^z+\sigma_{j-1}^z\sigma_j^z)}\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)-$$
$$-\,8\sigma_j^z e^{2(s_1-s_2)(\sigma_{j-2}^z\sigma_{j-1}^z+\sigma_{j-1}^z\sigma_j^z)}\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)-$$
$$-\,8\sigma_j^z e^{-2(s_1-s_2)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\cdot\cosh\left(2(s_4-s_3)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)+$$
$$+\,8\sigma_j^z e^{2(s_1-s_2)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)}\cdot\cosh\left(2(s_4-s_3)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)+$$
$$+\,8\sigma_j^z e^{-2(s_1-s_2)(\sigma_j^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_{j+2}^z)}\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)-$$
$$-\,8\sigma_j^z e^{2(s_1-s_2)(\sigma_j^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_{j+2}^z)}$$
$$\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)\mathrm{d}s_4\mathrm{d}s_3\Bigg)\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_2\mathrm{d}s_1= \tag{A.193}$$

$$=\eta_h^4\cdot 8\sigma_j^z\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\prod_a e^{-it\sigma_a^z\sigma_{a+1}^z}\cdot$$
$$\cdot\Bigg(\left(e^{-2(s_1-s_2)(\sigma_{j-2}^z\sigma_{j-1}^z+\sigma_{j-1}^z\sigma_j^z)}+e^{-2(s_1-s_2)(\sigma_j^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_{j+2}^z)}\right)$$
$$\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)-$$
$$-\left(e^{2(s_1-s_2)(\sigma_{j-2}^z\sigma_{j-1}^z+\sigma_{j-1}^z\sigma_j^z)}+e^{2(s_1-s_2)(\sigma_j^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_{j+2}^z)}\right)$$
$$\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)+$$
$$+\,2\sinh\left(2(s_1-s_2)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)$$
$$\cdot\cosh\left(2(s_4-s_3)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\Bigg)\prod_b e^{it\sigma_b^z\sigma_{b+1}^z}\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \tag{A.194}$$

$$\mathbb{E}_h(\rho_2)=\eta_h^4\cdot 8\sigma_j^z\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}$$
$$\Bigg(2\sinh\left(2(s_1-s_2)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\cdot\cosh\left(2(s_4-s_3)(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)+$$
$$+\left(e^{-2(s_1-s_2)(\sigma_{j-2}^z\sigma_{j-1}^z+\sigma_{j-1}^z\sigma_j^z)}+e^{-2(s_1-s_2)(\sigma_j^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_{j+2}^z)}\right)$$
$$\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)\right)-$$
$$-\left(e^{2(s_1-s_2)(\sigma_{j-2}^z\sigma_{j-1}^z+\sigma_{j-1}^z\sigma_j^z)}+e^{2(s_1-s_2)(\sigma_j^z\sigma_{j+1}^z+\sigma_{j+1}^z\sigma_{j+2}^z)}\right)$$
$$\cdot\sinh\left(2s_4(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)\sinh\left(2s_3(\sigma_{j-1}^z\sigma_j^z-\sigma_j^z\sigma_{j+1}^z)\right)\Bigg)\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \tag{A.195}$$

Then, I'll combine the three parts to get the final result for the fourth order.

$$\Omega_{4;M}=\int_0^{-it}\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\Bigg(2\eta_h^4\sigma_j^z(-2\sinh(2(s_3-s_4)\sigma_{j-1}^z\sigma_j^z))\cdot$$
$$\cdot\Bigg(-2\sinh\left(2\left(s_1(\sigma_{j-1}^z\sigma_j^z+\sigma_j^z\sigma_{j+1}^z)+s_2(\sigma_{j-2}^z\sigma_{j-1}^z-\sigma_{j-1}^z\sigma_j^z)-s_3\sigma_{j-2}^z\sigma_{j-1}^z-s_4(\sigma_j^z\sigma_{j+1}^z)\right)\right)-$$

$$-2\sinh\left(2\left(s_1(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})-s_2(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)+s_3\sigma^z_{j-2}\sigma^z_{j-1}-s_4(\sigma^z_j\sigma^z_{j+1})\right)\right)-$$
$$-2\sinh\left(2\left(s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)-s_2(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})-s_3\sigma^z_{j-2}\sigma^z_{j-1}-s_4(\sigma^z_j\sigma^z_{j+1})\right)\right)+$$
$$\left.+2\sinh\left(2\left(s_1(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)-s_2(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})-s_3\sigma^z_{j-2}\sigma^z_{j-1}+s_4(\sigma^z_j\sigma^z_{j+1})\right)\right)\right)+$$
$$+2\eta_h^4\sigma^z_j(-2\sinh(2(s_3-s_4)\sigma^z_{j+1}\sigma^z_j))\cdot$$
$$\cdot\left(-2\sinh\left(2\left(s_1(\sigma^z_{j+1}\sigma^z_j+\sigma^z_j\sigma^z_{j-1})+s_2(\sigma^z_{j+2}\sigma^z_{j+1}-\sigma^z_{j+1}\sigma^z_j)-s_3\sigma^z_{j+2}\sigma^z_{j+1}-s_4(\sigma^z_j\sigma^z_{j-1})\right)\right)-\right.$$
$$-2\sinh\left(2\left(s_1(\sigma^z_{j+1}\sigma^z_j+\sigma^z_j\sigma^z_{j-1})-s_2(\sigma^z_{j+2}\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_j)+s_3\sigma^z_{j+2}\sigma^z_{j+1}-s_4(\sigma^z_j\sigma^z_{j-1})\right)\right)-$$
$$-2\sinh\left(2\left(s_1(\sigma^z_{j+2}\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_j)-s_2(\sigma^z_{j+1}\sigma^z_j-\sigma^z_j\sigma^z_{j-1})-s_3\sigma^z_{j+2}\sigma^z_{j+1}-s_4\sigma^z_j\sigma^z_{j-1}\right)\right)+$$
$$\left.+2\sinh\left(2\left(s_1(\sigma^z_{j+2}\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_j)-s_2(\sigma^z_{j+1}\sigma^z_j+\sigma^z_j\sigma^z_{j-1})-s_3\sigma^z_{j+2}\sigma^z_{j+1}+s_4\sigma^z_j\sigma^z_{j-1}\right)\right)\right)+$$
$$+\eta_h^4\cdot 8\sigma^z_j\left(2\sinh\left(2(s_1-s_2)(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})\right)\cdot\cosh\left(2(s_4-s_3)(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})\right)+\right.$$
$$+\left(e^{-2(s_1-s_2)(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}+e^{-2(s_1-s_2)(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\cdot\sinh\left(2s_4(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})\right)\sinh\left(2s_3(\sigma^z_{j-1}\sigma^z_j+\sigma^z_j\sigma^z_{j+1})\right)-$$
$$\left.\left.-\left(e^{2(s_1-s_2)(\sigma^z_{j-2}\sigma^z_{j-1}+\sigma^z_{j-1}\sigma^z_j)}+e^{2(s_1-s_2)(\sigma^z_j\sigma^z_{j+1}+\sigma^z_{j+1}\sigma^z_{j+2})}\right)\cdot\sinh\left(2s_4(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})\right)\sinh\left(2s_3(\sigma^z_{j-1}\sigma^z_j-\sigma^z_j\sigma^z_{j+1})\right)\right)\right)$$
$$\mathrm{d}s_4\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 \quad (A.196)$$

This result is used here.

Partition function for a simple perturbed system

For a calculation of the Gibbs measure for simple perturbed system, as described by (2.166), it would be prudent to first obtain the expectation of Z:

$$\mathbb{E}_h(Z)=\mathbb{E}_h\left(Z_0+\sum_{j=1}^{\infty}\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)= \quad (A.197)$$
$$=Z_0+\sum_{j=1}^{\infty}\mathbb{E}_h\left(\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)= \quad (A.198)$$
$$=Z_0+\sum_{j=1}^{\infty}\frac{(-\beta)^j}{j!}\mathbb{E}_h\left(\mathrm{tr}\left(e^{-\beta H_0}H_\nu^j\right)\right) \quad (A.199)$$

Then, the inverse of the denominator can be calculated as:

$$Z^{-1}=\left(Z_0+\sum_{j=1}^{\infty}\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)^{-1}= \quad (A.200)$$
$$=Z_0^{-1}\left(1+Z_0^{-1}\cdot\sum_{j=1}^{\infty}\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)^{-1}= \quad (A.201)$$
$$=Z_0^{-1}\left(1+Z_0^{-1}\cdot\left(\mathrm{tr}\left(-\beta e^{-\beta H_0}H_\nu\right)+\mathrm{tr}\left(\frac{\beta^2 e^{-\beta H_0}H_\nu^2}{2!}\right)-\mathrm{tr}\left(\frac{\beta^3 e^{-\beta H_0}H_\nu^3}{3!}\right)+\ldots\right)\right)^{-1}= \quad (A.202)$$
$$=Z_0^{-1}\left(1+\sum_{k=0}^{\infty}(-1)^k\left(Z_0^{-1}\cdot\left(\mathrm{tr}\left(-\beta e^{-\beta H_0}H_\nu\right)+\mathrm{tr}\left(\frac{\beta^2 e^{-\beta H_0}H_\nu^2}{2!}\right)-\mathrm{tr}\left(\frac{\beta^3 e^{-\beta H_0}H_\nu^3}{3!}\right)+\ldots\right)\right)^k\right)= \quad (A.203)$$
$$=Z_0^{-1}\left(1+\sum_{k=0}^{\infty}(-1)^k Z_0^{-k}\cdot\left(\mathrm{tr}\left(-\beta e^{-\beta H_0}H_\nu\right)+\mathrm{tr}\left(\frac{\beta^2 e^{-\beta H_0}H_\nu^2}{2!}\right)-\mathrm{tr}\left(\frac{\beta^3 e^{-\beta H_0}H_\nu^3}{3!}\right)+\ldots\right)^k\right)= \quad (A.204)$$

$$=Z_0^{-1}\left(1+\sum_{k=0}^{\infty}(-1)^k Z_0^{-k}\left(\sum_{j=1}^{\infty}\operatorname{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)^k\right)= \tag{A.205}$$

$$=Z_0^{-1}\left(1+\sum_{k=0}^{\infty}(-1)^k Z_0^{-k}\left(\sum_{j=1}^{\infty}\frac{(-\beta)^j}{j!}\operatorname{tr}\left(e^{-\beta H_0}H_\nu^j\right)\right)^k\right)= \tag{A.206}$$

Now, I can calculate the numerator. Here, I choose H_0 and H_ν that commute:

$$e^{-\beta(H_0+H_\nu)}=e^{-\beta H_0}e^{-\beta H_\nu}= \tag{A.207}$$

$$=e^{-\beta H_0}\sum_{l=0}^{\infty}\frac{(-\beta)^l H_\nu^l}{l!}= \tag{A.208}$$

$$=e^{-\beta H_0}\left(1-\beta H_\nu+\frac{\beta^2 H_\nu^2}{2!}-\frac{\beta^3 H_\nu^3}{3!}+...\right)= \tag{A.209}$$

$$=e^{-\beta H_0}-\beta e^{-\beta H_0}H_\nu+\frac{\beta^2 e^{-\beta H_0}H_\nu^2}{2!}-\frac{\beta^3 e^{-\beta H_0}H_\nu^3}{3!}+...= \tag{A.210}$$

$$=\sum_{l=0}^{\infty}\frac{(-\beta)^l e^{-\beta H_0}H_\nu^l}{l!} \tag{A.211}$$

Putting the numerator and denominator together, it follows that:

$$P=\mathbb{E}_h\left(\frac{e^{-\beta(H_0+H_\nu)}}{\operatorname{tr}(e^{-\beta(H_0+H_\nu)})}\right)= \tag{A.212}$$

$$=\mathbb{E}_h\left(\sum_{l=0}^{\infty}\frac{(-\beta)^l}{l!}e^{-\beta H_0}H_\nu^l\cdot Z_0^{-1}\left(1+\sum_{k=0}^{\infty}(-1)^k Z_0^{-k}\left(\sum_{j=1}^{\infty}\frac{(-\beta)^j}{j!}\operatorname{tr}\left(e^{-\beta H_0}H_\nu^j\right)\right)^k\right)\right)= \tag{A.213}$$

$$=\mathbb{E}_h\left(Z_0^{-1}\sum_{l=0}^{\infty}\frac{(-\beta)^l}{l!}e^{-\beta H_0}H_\nu^l+Z_0^{-1}\sum_{l=0}^{\infty}\frac{(-\beta)^l}{l!}e^{-\beta H_0}H_\nu^l\sum_{k=0}^{\infty}(-1)^k Z_0^{-k}\left(\sum_{j=1}^{\infty}\frac{(-\beta)^j}{j!}\operatorname{tr}\left(e^{-\beta H_0}H_\nu^j\right)\right)^k\right) \tag{A.214}$$

I can also derive a general form for the Gibbs measure, up to the second order:

$$\begin{aligned}P=\mathbb{E}_h\Bigg(&Z_0^{-1}\left(e^{-\beta H_0}-\beta e^{-\beta H_0}H_\nu+\frac{\beta^2}{2}e^{-\beta H_0}H_\nu^2\right)+\\&+Z_0^{-1}\Big(\Delta(l=0;k=0)+\Delta(l=0;k=1;j=1,2)+\Delta(l=0;k=2;j=1)+\\&+\Delta(l=1;k=0)+\Delta(l=1;k=1;j=1)+\Delta(l=2;k=0)\Big)\Bigg)=\end{aligned} \tag{A.215}$$

$$\begin{aligned}=\mathbb{E}_h\Bigg(&Z_0^{-1}\left(e^{-\beta H_0}-\beta e^{-\beta H_0}H_\nu+\frac{\beta^2}{2}e^{-\beta H_0}H_\nu^2\right)+\\&+Z_0^{-1}\Big(e^{-\beta H_0}+e^{-\beta H_0}(-1)Z_0^{-1}\left(-\beta\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)+\frac{\beta^2}{2}\operatorname{tr}\left(e^{-\beta H_0}H_\nu^2\right)\right)+\\&+e^{-\beta H_0}Z_0^{-2}\left(-\beta\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)\right)^2-\\&-\beta e^{-\beta H_0}H_\nu-\beta e^{-\beta H_0}H_\nu Z_0^{-1}\beta\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)+\frac{\beta^2}{2}e^{-\beta H_0}H_\nu^2\Big)\Bigg)=\end{aligned} \tag{A.216}$$

$$\begin{aligned}=\frac{e^{-\beta H_0}}{Z_0}\mathbb{E}_h\Bigg(&1-\beta H_\nu+\frac{\beta^2}{2}H_\nu^2+1-Z_0^{-1}\left(-\beta\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)+\frac{\beta^2}{2}\operatorname{tr}\left(e^{-\beta H_0}H_\nu^2\right)\right)+\\&+Z_0^{-2}\left(-\beta\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)\right)^2-\beta H_\nu-\beta H_\nu Z_0^{-1}\beta\operatorname{tr}\left(e^{-\beta H_0}H_\nu\right)+\frac{\beta^2}{2}H_\nu^2\Bigg)=\end{aligned} \tag{A.217}$$

$$= \frac{e^{-\beta H_0}}{Z_0} \mathbb{E}_h \Bigg(2 - 2\beta H_\nu + \beta^2 H_\nu^2 + Z_0^{-1} \beta \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right) - Z_0^{-1} \frac{\beta^2}{2} \operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right) +$$

$$+ Z_0^{-2} \beta^2 \left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2 - \beta H_\nu Z_0^{-1} \beta \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right) \Bigg) = \tag{A.218}$$

$$= \frac{e^{-\beta H_0}}{Z_0} \Bigg(2 - \mathbb{E}_h\left(2\beta H_\nu\right) + \mathbb{E}_h\left(\beta^2 H_\nu^2\right) + \mathbb{E}_h\left(Z_0^{-1} \beta \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) - \mathbb{E}_h\left(Z_0^{-1} \frac{\beta^2}{2} \operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) +$$

$$+ \mathbb{E}_h\left(Z_0^{-2} \beta^2 \left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2\right) - \mathbb{E}_h\left(\beta H_\nu Z_0^{-1} \beta \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) \Bigg) = \tag{A.219}$$

$$= \frac{e^{-\beta H_0}}{Z_0} \Bigg(2 - 2\beta \mathbb{E}_h\left(H_\nu\right) + \beta^2 \mathbb{E}_h\left(H_\nu^2\right) + \frac{\beta}{Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) -$$

$$- \frac{\beta^2}{2Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) + \frac{\beta^2}{Z_0^2} \mathbb{E}_h\left(\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2\right) - \frac{\beta^2}{Z_0} \mathbb{E}_h\left(H_\nu \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) \Bigg) = \tag{A.220}$$

$$= \frac{e^{-\beta H_0}}{Z_0} \Bigg(2 + \beta^2 \eta_h^2 \left(\sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x \right)^2 + \frac{\beta}{Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) -$$

$$- \frac{\beta^2}{2Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) + \frac{\beta^2}{Z_0^2} \mathbb{E}_h\left(\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2\right) - \frac{\beta^2}{Z_0} \mathbb{E}_h\left(H_\nu \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) \Bigg) \tag{A.221}$$

This result is used here.

Specific values for Gibbs measure traces

First, here is the simple case.

$$\operatorname{tr}\left(e^{-\beta H_0}\right) = \operatorname{tr}\left(\prod_{j=1}^{n} e^{-\beta \sigma_j^z} \right) = \tag{A.222}$$

$$= \operatorname{tr}\left(\prod_{j=1}^{n} \sum_{\alpha=0}^{3} M_\alpha \sigma_j^\alpha \right) = \tag{A.223}$$

$$= \operatorname{tr}\left(\left(\sum_{\alpha=0}^{3} M_\alpha \sigma_1^\alpha \right) \cdot \ldots \cdot \left(\sum_{\zeta=0}^{3} M_\zeta \sigma_n^\zeta \right) \right) = \tag{A.224}$$

$$= \operatorname{tr}\left(\left(M_0\sigma_1^0 + M_1\sigma_1^x + M_2\sigma_2^y + M_3\sigma_3^z\right) \cdot \ldots \cdot \left(M_0\sigma_n^0 + M_1\sigma_n^x + M_2\sigma_n^y + M_3\sigma_n^z\right) \right) = \tag{A.225}$$

$$= \operatorname{tr}\left(M_0\sigma_1^0 M_0\sigma_2^0 \cdot \ldots \cdot M_0\sigma_n^0\right) + \operatorname{tr}\left(M_0\sigma_1^0 M_0\sigma_2^0 \cdot \ldots \cdot M_1\sigma_n^x\right) +$$
$$+ \ldots + \operatorname{tr}\left(M_3\sigma_1^z M_3\sigma_2^z \cdot \ldots \cdot M_3\sigma_n^z\right) = \tag{A.226}$$

$$= \operatorname{tr}\left(M_0\sigma_1^0 M_0\sigma_2^0 \cdot \ldots \cdot M_0\sigma_n^0\right) + \operatorname{tr}\left(M_0\sigma_1^0 M_0\sigma_2^0 \cdot \ldots \cdot M_1\sigma_n^x\right) +$$
$$+ \ldots + \operatorname{tr}\left(M_3\sigma_1^z M_3\sigma_2^z \cdot \ldots \cdot M_3\sigma_n^z\right) = \tag{A.227}$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=0}^{3} \operatorname{tr}\left(M_{\alpha_1}\sigma_1^{\alpha_1} M_{\alpha_2}\sigma_2^{\alpha_2} \cdot \ldots \cdot M_{\alpha_n}\sigma_n^{\alpha_n}\right) = \tag{A.228}$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=0}^{3} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} \operatorname{tr}\left(\sigma_1^{\alpha_1}\sigma_2^{\alpha_2} \ldots \sigma_n^{\alpha_n}\right) = \tag{A.229}$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=0}^{3} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} \operatorname{tr}\Big(\left(\sigma_1^{\alpha_1} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) \cdot \left(\mathbb{I} \otimes \sigma_2^{\alpha_2} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) \cdot$$
$$\cdot \ldots \cdot \left(\mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes \sigma_n^{\alpha_n}\right) \Big) = \tag{A.230}$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=0}^{3} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} \operatorname{tr}\left(\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_2} \otimes \ldots \otimes \sigma_n^{\alpha_n}\right) = \tag{A.231}$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n=0}^{3} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} \operatorname{tr}\left(\sigma_1^{\alpha_1}\right) \operatorname{tr}\left(\sigma_2^{\alpha_2}\right) \ldots \operatorname{tr}\left(\sigma_n^{\alpha_n}\right) \tag{A.232}$$

Now follows the next more complicated form.

$$\operatorname{tr}\left(e^{-\beta H_0} H_\nu^j\right) = \operatorname{tr}\left(\prod_{j=1}^{n} e^{-\beta\sigma_j^z} \cdot \sum_{k=1}^{n} \left(h_k \sigma_k^x \sigma_{k+1}^x\right)^j\right) = \tag{A.233}$$

$$= \operatorname{tr}\left(\prod_{j=1}^{n} \left(M_0 \mathbb{I}_j + M_3 \sigma_j^z\right) \cdot \sum_{k=1}^{n} h_k^j \left(\sigma_k^x \sigma_{k+1}^x\right)^j\right) = \tag{A.234}$$

$$= \operatorname{tr}\left(\prod_{j=1}^{n} \left(M_0 \mathbb{I}_j + M_3 \sigma_j^z\right) \cdot \sum_{k=1}^{n} h_k^j \left(\sigma_k^x \otimes \sigma_{k+1}^x\right)^j\right) = \tag{A.235}$$

$$= \operatorname{tr}\left(\left(\sum_{\alpha_1,\ldots,\alpha_n=0}^{3} M_{\alpha_1} \sigma_1^{\alpha_1} M_{\alpha_2} \sigma_2^{\alpha_2} \ldots M_{\alpha_n} \sigma_n^{\alpha_n}\right) \cdot \left(\sum_{k=1}^{n} h_k^j \left(\left(\sigma_k^x\right)^j \otimes \left(\sigma_{k+1}^x\right)^j\right)\right)\right) = \tag{A.236}$$

$$= \operatorname{tr}\left(\sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} M_{\alpha_1} \sigma_1^{\alpha_1} M_{\alpha_2} \sigma_2^{\alpha_2} \ldots M_{\alpha_n} \sigma_n^{\alpha_n} \cdot h_k^j \left(\left(\sigma_k^x\right)^j \otimes \left(\sigma_{k+1}^x\right)^j\right)\right) = \tag{A.237}$$

$$\begin{aligned} = \operatorname{tr}\Bigg(\sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} & M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} \cdot h_k^j \left(\sigma_1^{\alpha_1} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) \cdot \left(\mathbb{I} \otimes \sigma_2^{\alpha_2} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}\right) \cdot \\ & \cdot \ldots \cdot \left(\mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes \sigma_n^{\alpha_n}\right) \cdot \left(\left(\sigma_k^x\right)^j \otimes \left(\sigma_{k+1}^x\right)^j\right)\Bigg) = \end{aligned} \tag{A.238}$$

$$= \operatorname{tr}\left(\sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} h_k^j \left(\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_2} \otimes \ldots \otimes \sigma_n^{\alpha_n}\right) \cdot \left(\left(\sigma_k^x\right)^j \otimes \left(\sigma_{k+1}^x\right)^j\right)\right) = \tag{A.239}$$

$$= \operatorname{tr}\left(\sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} h_k^j \left(\ldots \otimes \sigma_{k-1}^{\alpha_{k-1}} \otimes \sigma_k^{\alpha_k} \cdot \left(\sigma_k^x\right)^j \otimes \sigma_{k+1}^{\alpha_{k+1}} \cdot \left(\sigma_{k+1}^x\right)^j \otimes \sigma_{k+2}^{\alpha_{k+2}} \otimes \ldots\right)\right) = \tag{A.240}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} h_k^j \cdot \operatorname{tr}\left(\ldots \otimes \sigma_{k-1}^{\alpha_{k-1}} \otimes \sigma_k^{\alpha_k} \cdot \left(\sigma_k^x\right)^j \otimes \sigma_{k+1}^{\alpha_{k+1}} \cdot \left(\sigma_{k+1}^x\right)^j \otimes \sigma_{k+2}^{\alpha_{k+2}} \otimes \ldots\right) = \tag{A.241}$$

$$= \sum_{\alpha_1,\ldots,\alpha_n=0}^{3} \sum_{k=1}^{n} M_{\alpha_1} M_{\alpha_2} \ldots M_{\alpha_n} h_k^j \cdot \ldots \cdot \operatorname{tr}\left(\sigma_{k-1}^{\alpha_{k-1}}\right) \cdot \operatorname{tr}\left(\sigma_k^{\alpha_k} \cdot \left(\sigma_k^x\right)^j\right) \cdot \operatorname{tr}\left(\sigma_{k+1}^{\alpha_{k+1}} \cdot \left(\sigma_{k+1}^x\right)^j\right) \cdot \operatorname{tr}\left(\sigma_{k+2}^{\alpha_{k+2}}\right) \cdot \ldots \tag{A.242}$$

This result is used here.

Gibbs measure and expectation

The Gibbs measure expression calculation proceeds as follows.

$$\begin{aligned} P = & \frac{e^{-\beta H_0}}{Z_0} \Bigg(2 + \beta^2 \eta_h^2 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x + \frac{\beta}{Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right) - \frac{\beta^2}{2Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right) + \\ & + \frac{\beta^2}{Z_0^2} \mathbb{E}_h\left(\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)^2\right) - \frac{\beta^2}{Z_0} \mathbb{E}_h\left(H_\nu \operatorname{tr}\left(e^{-\beta H_0} H_\nu\right)\right)\Bigg) \end{aligned} \tag{A.243}$$

$$= \frac{e^{-\beta H_0}}{Z_0} \left(2 + \beta^2 \eta_h^2 \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - \frac{\beta^2}{2Z_0} \mathbb{E}_h\left(\operatorname{tr}\left(e^{-\beta H_0} H_\nu^2\right)\right)\right) = \tag{A.244}$$

$$= \frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\eta_h^2\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^x-\frac{\beta^2}{2Z_0}\mathbb{E}_h\left(\left(2\cosh(\beta)\right)^n\cdot\sum_{l=1}^{n}h_l^2\right)\right)= \tag{A.245}$$

$$= \frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\eta_h^2\sum_{j=1}^{N}\sigma_j^x\sigma_{j+1}^x-\frac{\beta^2}{2Z_0}\cdot\left(2\cosh(\beta)\right)^n\cdot n\eta_h^2\right)= \tag{A.246}$$

$$= \frac{e^{-\beta H_0}}{Z_0}\left(2+\beta^2\eta_h^2\left(\sum_{j=1}^{n}\sigma_j^x\sigma_{j+1}^x-\frac{n}{2Z_0}\cdot\left(2\cosh(\beta)\right)^n\right)\right) \tag{A.247}$$

The expectation value for $\mathbb{E}(Z)$ is next.

$$\mathbb{E}_h(Z) = \mathbb{E}_h(Z_0)+\mathbb{E}_h\left(\sum_{j=1}^{\infty}\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)= \tag{A.248}$$

$$= \mathrm{tr}\left(e^{-\beta\sigma_j^z}\right)^n+\sum_{j=1}^{\infty}\mathbb{E}_h\left(\mathrm{tr}\left(\frac{(-\beta)^j e^{-\beta H_0}H_\nu^j}{j!}\right)\right)= \tag{A.249}$$

$$= \mathrm{tr}\left(e^{-\beta\sigma_j^z}\right)^n+\sum_{j=1}^{\infty}\frac{(-\beta)^{2j}}{(2j)!}\mathbb{E}_h\left(\mathrm{tr}\left(e^{-\beta H_0}H_\nu^{2j}\right)\right)= \tag{A.250}$$

$$= \mathrm{tr}\left(e^{-\beta\sigma_j^z}\right)^n+\sum_{j=1}^{\infty}\frac{(-\beta)^{2j}}{(2j)!}\mathbb{E}_h\left(\left(2\cosh(\beta)\right)^n\cdot\sum_{l=1}^{n}h_l^{2j}\right)= \tag{A.251}$$

$$= Z_0+\sum_{j=1}^{\infty}\frac{(-\beta)^{2j}}{(2j)!}\left(2\cosh(\beta)\right)^n\cdot n\eta_h^{2j}= \tag{A.252}$$

$$= Z_0+\left(2\cosh(\beta)\right)^n\cdot n\cdot\sum_{j=1}^{\infty}\frac{(\beta\eta_h)^{2j}}{(2j)!}= \tag{A.253}$$

$$= Z_0+\left(2\cosh(\beta)\right)^n\cdot n\cdot\left(\sum_{j=0}^{\infty}\frac{(\beta\eta_h)^{2j}}{(2j)!}-1\right)= \tag{A.254}$$

$$= Z_0+\left(2\cosh(\beta)\right)^n\cdot n\cdot\left(\cosh(\beta\eta_h)-1\right)= \tag{A.255}$$

$$= \left(2\cosh(\beta)\right)^n+\left(2\cosh(\beta)\right)^n\cdot n\cdot\left(\cosh(\beta\eta_h)-1\right). \tag{A.256}$$

Creating a Tensor Bridge for Perturbative Many-body Localization Solutions

Norm of $|\varphi_1\rangle = e^{\sum_j v_j\frac{1}{8}X_j}|\varphi_0\rangle$:

$$N = \langle\varphi_1|\varphi_1\rangle= \tag{A.257}$$

$$= \langle\varphi_0|e^{\sum_j v_j\frac{1}{8}X_j}\cdot e^{\sum_j v_j\frac{1}{8}X_j}|\varphi_0\rangle= \tag{A.258}$$

$$= \frac{1}{\sqrt{2}}\left(\langle+Z|+\langle-Z|\right)e^{\sum_j v_j\frac{1}{8}X_j}\cdot e^{\sum_j v_j\frac{1}{8}X_j}\frac{1}{\sqrt{2}}\left(|+Z\rangle+|-Z\rangle\right)= \tag{A.259}$$

$$= \frac{1}{2}\left(\langle+Z|+\langle-Z|\right)e^{\sum_j v_j\frac{1}{4}X_j}\left(|+Z\rangle+|-Z\rangle\right)= \tag{A.260}$$

$$= \frac{1}{2}\left(\langle+Z|+\langle-Z|\right)\prod_j e^{\frac{1}{4}v_jX_j}\left(|+Z\rangle+|-Z\rangle\right)= \tag{A.261}$$

$$= \frac{1}{2}\prod_j\left(\langle+Z|+\langle-Z|\right)\left(\mathbb{1}+\frac{1}{4}v_jX_j\mathbb{1}+\frac{1}{2}\left(\frac{1}{4}v_jX_j\right)^2\mathbb{1}+\ldots\right)\left(|+Z\rangle+|-Z\rangle\right)= \tag{A.262}$$

$$= \frac{1}{2}\prod_j \left(\langle +Z| + \langle -Z|\right)\left(\mathbb{1} + \frac{1}{2}\left(\frac{1}{4}v_j X_j\right)^2 \mathbb{1} + \frac{1}{4!}\left(\frac{1}{4}v_j X_j\right)^4 + ...\right)\left(|+Z\rangle + |-Z\rangle\right) = \tag{A.263}$$

$$= \frac{1}{2}\prod_j \left(\langle +Z| + \langle -Z|\right)\left(\mathbb{1} + \frac{1}{2}\left(\frac{1}{4}v_j\right)^2 \mathbb{1} + \frac{1}{4!}\left(\frac{1}{4}v_j\right)^4 + ...\right)\left(|+Z\rangle + |-Z\rangle\right) = \tag{A.264}$$

$$= \frac{1}{2}\prod_j \left(\langle +Z| + \langle -Z|\right)\cosh\left(\frac{1}{4}v_j\right)\left(|+Z\rangle + |-Z\rangle\right) = \tag{A.265}$$

$$= \prod_j \cosh\left(\frac{1}{4}v_j\right) \tag{A.266}$$

This result is used here.

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