Differential forms

(Lecture by Prof. Dr. M.M. Wolf, 23/24 @ TUM)

Lecture on differential forms

Motivation & outlook

Differential forms
- generalize vector calculus to diff. manifolds
- allow to tackle topology be means of analysis
- are also used in physics (e.g. whenever gravity is involved but also in electro- and thermodynamics)

From vector calculus we know (for $U \subseteq \mathbb{R}^3$ open):

$$C^\infty(U) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{rot}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U)$$

Moreover, $(\text{rot grad } v)_i = \sum_{jk} \varepsilon_{ijk} \partial_j \partial_k v = 0$
$= \nabla \times \nabla v$

Schwarz's theorem

and $\text{div rot } v = \sum_{ijk} \partial_i \varepsilon_{ijk} \partial_j v_k = 0$
$= \nabla \cdot \nabla \times v$

This is generalized to m-dim. smooth manifolds by the de Rham complex:

$$\boxed{C^\infty(M) = \Omega^0 M \xrightarrow{d_1} \Omega^1 M \xrightarrow{d_2} \Omega^2 M \xrightarrow{d_3} \dots \xrightarrow{d} \Omega^m M \simeq C^\infty(M)}$$

where d is the exterior derivative for which $\boxed{d \circ d = 0}$

and $\Omega^k M$ is the space of differential k-forms on M.

Since $\mathrm{rot}\,\mathrm{grad} = 0$ and $\mathrm{div}\,\mathrm{rot} = 0$ we know that

$$\mathrm{Im}(\mathrm{grad}) \subseteq \ker(\mathrm{rot}), \quad \mathrm{Im}(\mathrm{rot}) \subseteq \ker(\mathrm{div})$$

are (infinite dimensional) linear subspaces. So we can define the quotient spaces

$$H^1(U) := \frac{\ker(\mathrm{rot})}{\mathrm{Im}(\mathrm{grad})}$$

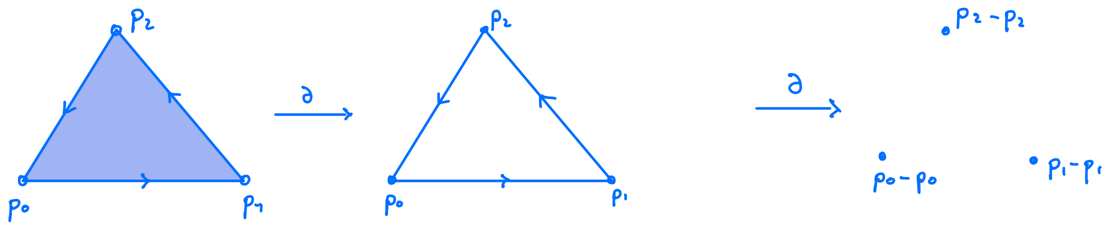
$$H^2(U) := \frac{\ker(\mathrm{div})}{\mathrm{Im}(\mathrm{rot})}$$

If U is starshaped (or, more general, contractible), then the spaces coincide so that $H^1(U) = \{0\} = H^2(U)$.

In general, however, this is not true. E.g. for $U = \mathbb{R}^2 \setminus \{z_1, \dots, z_k\}$ $\dim(H^1(U)) = k$. Somehow, these spaces 'count holes'.

Similarly, for smooth manifolds $\boxed{H^k(M) := \frac{\ker d_k}{\mathrm{Im}\, d_{k-1}}}$ defines the k'th de Rham cohomology group. Remarkably, the k'th Betti number $\dim_{\mathbb{R}}(H^k(M)) =: \beta_k$ is finite (for compact M) and a topological invariant (i.e. it does <u>not</u> depend on the differentiable structure).

Excursion: Consider a 'triangulation' of a manifold to which we apply the boundary operator ∂. This acts as follows:



'2-simplex' 'sum of 1-simplexes' 'sum of 0-simplexes'

$$(p_0\, p_1\, p_2) \xrightarrow{\partial} (p_1 p_2) + (p_2 p_0) + (p_0 p_1) \xrightarrow{\partial} p_2 - p_1 + p_0 - p_2 + p_1 - p_0$$
$$= 0$$

In fact $\partial \circ \partial = 0$ holds in general for the chain complex

$$\dots \xleftarrow{\partial_{r-1}} C_{r-1}(M) \xleftarrow{\partial_r} \underbrace{C_r(M)}_{\text{space of (images of) } r\text{-simplexes}} \xleftarrow{\partial_{r+1}} C_{r+1}(M) \longleftarrow \dots$$

As ∂_r is linear, we can again define $H_r(M) := \dfrac{\ker(\partial_r)}{\operatorname{Im}(\partial_{r+1})}$, the (singular) homology group.

By de Rham's theorem $\boxed{H_r(M) \simeq H^r(M)}$ are dual vector spaces and ∂ and d dual linear maps.

This duality is rooted in Stokes' theorem:

$$\boxed{\int_c d\omega = \int_{\partial c} \omega} \qquad \text{for } \omega \in \Omega^{k-1} M,\ c \in C_k(M)$$

This generalizes the fundamental thm. of calculus, Green's thm., the 2dim. Stokes' theorem and Gauss' divergence theorem from vector calculus.

Manifolds

countable basis　　separation by open sets　　topology

Def.: A second countable Hausdorff space (M, T) is a **topological manifold** of dimension $m \in \mathbb{N}_0$ if it is locally homeomorphic to $\mathbb{R}^m$. That is, $\forall p \in M$ there is an open neighborhood $U \subseteq M$ and a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^m$.

- (U, φ) is called a **chart**, $\varphi_1, \dots, \varphi_n$ **coordinate functions** and φ^{-1} a **parametrization**.
- A collection $\{(U_\lambda, \varphi^{(\lambda)})\}$ of charts is called an **atlas** for M if $\bigcup_\lambda U_\lambda = M$.

examples:

- spheres: $S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \}$ is n-dim. top. manifold.

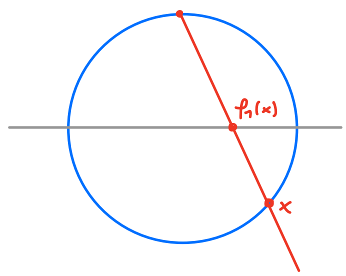
  Two charts are given by the 'stereographic projections'

  $\varphi_1 : S^n \setminus (0, \dots, 0, 1) \longrightarrow \mathbb{R}^n$

  $\varphi_1(x) := \dfrac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$

  $\varphi_2 : S^n \setminus (0, \dots, 0, -1) \longrightarrow \mathbb{R}^n$

  $\varphi_2(x) := \dfrac{1}{1 + x_{n+1}} (x_1, \dots, x_n)$



- open subsets of a top. manifold are again top. manifolds of the same dimension. E.g. $GL(n, \mathbb{R}) := \{ A \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0 \}$ is an open subset of $\mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$ and thus a top. manifold of dim n^2.

<u>remarks:</u>

- Every top. manifold can be 'embedded' into some $\mathbb{R}^N$.
  That is, there is a homeomorphism $\psi: M \to \psi(M) \subseteq \mathbb{R}^N$.
  If $m := \dim(M)$, then $N = 2m+1$ suffices. For 'smooth' manifolds $N = 2m$ is sufficient (Withney's embedding thm.)
  Examples where $N < 2m$ (with $m = 2$) is not possible, are

  klein bottle & projective plane

  where opposite edges are identified ('glued together') according to the arrows.

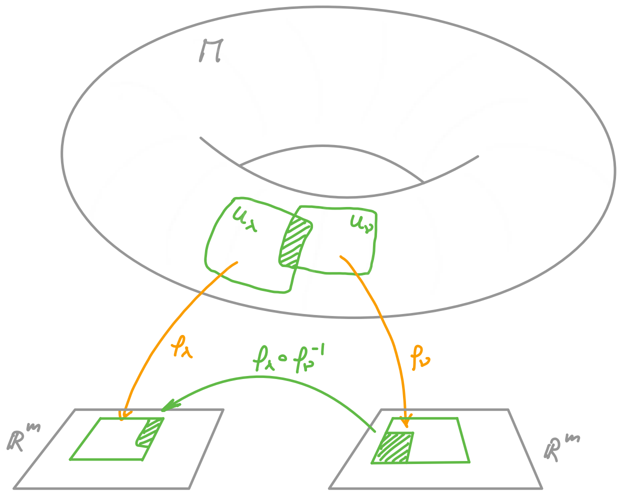
- The Hausdorff assumption guarantees that limits are unique. Second-countability is assumed in order for a 'partition of unity' (more on this later...) and an embedding into a finite-dim. Euclidean space to exist. Not all authors include these two assumptions in the def. of a top. manifold.
- The second-countability assumption implies that there is a countable atlas.

If we want to differentiate or integrate on a manifold, we need extra structures: smooth structure & orientation.

Def.: An atlas $\mathcal{A} = \{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ of a topological m-dim. manifold M is called a C^k-atlas ($k \in \mathbb{N}$) if $\forall \lambda, \nu \in \Lambda$:

$$\varphi_\lambda \circ \varphi_\nu^{-1} : \varphi_\nu(U_\lambda \cap U_\nu) \subseteq \mathbb{R}^m \longrightarrow \varphi_\lambda(U_\lambda \cap U_\nu) \subseteq \mathbb{R}^m$$

is a C^k-diffeomorphism



Remarks: $\mathcal{A}$ and $\mathcal{B}$ are said to be C^k-compatible if $\mathcal{A} \cup \mathcal{B}$ is a C^k atlas. One can always extend an atlas $\mathcal{A}$ to a unique 'maximal atlas' that contains all compatible ones. This max. atlas is called a C^k-structure.

Def.: A pair $(M, \mathcal{A})$ of a manifold M with C^k-structure $\mathcal{A}$ is called C^k-manifold (and smooth manifold if $k = \infty$).

Examples: ∘ S^n with (U_1, φ_1), (U_2, φ_2) stereographic projections.

$$\varphi_2 \circ \varphi_1^{-1}(z) = \frac{z}{\|z\|^2} \quad \text{is a } C^\infty\text{-diff. on } \varphi_1(U_1 \cap U_2) = \mathbb{R}^n \setminus \{0\}.$$

So S^n becomes a smooth manifold.

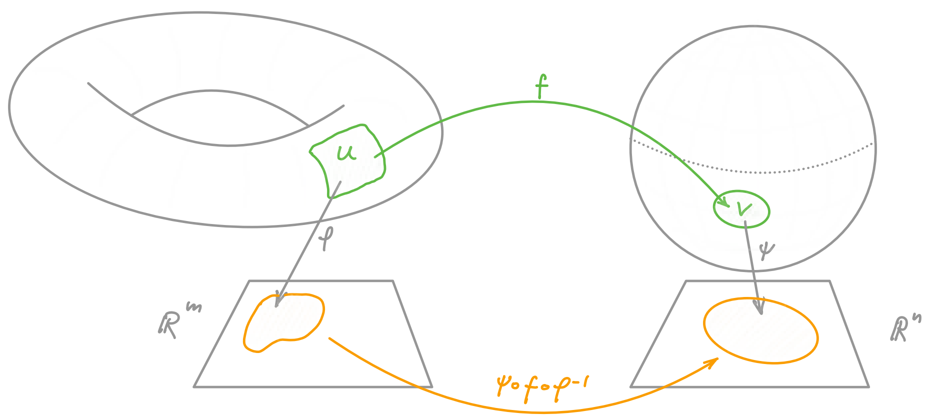
∘ Other standard examples of smooth manifolds:

$SO(n)$, $SU(n)$, $Sp(n)$, $GL(n)$, $T^n := S^1 \times \ldots \times S^1$, $\mathbb{R}P^n$, $\mathbb{C}P^n$, graphs of C^∞-functions, …

Thm. [Whitney]: For $k \geq 1$, every C^k-structure contains a C^∞-structure.

- Motivated by this, we only consider C^∞ manifolds (a.k.a. smooth manifolds)
- There are top. manifolds for which no smooth structure exists.
  (e.g. the 4-dim. E8-manifold discovered by Freedman.)
- From a given smooth structure $\{(U_\lambda, \varphi_\lambda)\}$ we can obtain another one $\{(\psi^{-1}(U_\lambda), \varphi_\lambda \circ \psi)\}$ by acting with a homeomorphism $\psi: M \to M$.
  Such smooth structures are called equivalent.
  For $\mathbb{R}^n$ with $n \in \mathbb{N} \setminus \{4\}$, all smooth structures are equivalent (Smale).
  For $\mathbb{R}^4$ there are uncountable inequivalent ones (Freedman & Donaldson).

Def.: Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. A map $f: M \to N$ is called smooth if for all $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ with $f(U) \subseteq V$ the map $\psi \circ f \circ \varphi^{-1}: \varphi(U) \subseteq \mathbb{R}^m \longrightarrow \psi(V) \subseteq \mathbb{R}^n$ is C^∞.
f is called a diffeomorphism if it is smooth and has smooth inverse. $C^\infty(M, N)$ denotes the space of smooth maps $M \to N$, and $C^\infty(M) := C^\infty(M, \mathbb{R})$.



Thm.: [smooth partition of unity] Let M be a smooth manifold and $\{U_\lambda\}_{\lambda\in\Lambda}$ an open cover of M. Then there exist functions $\{f_\lambda \in C^\infty(M,[0,1])\}_{\lambda\in\Lambda}$ s.t.

(i) $\operatorname{supp}(f_\lambda) := \overline{\{p\in M \mid f_\lambda(p) \neq 0\}} \subseteq U_\lambda$

(ii) Every $p\in M$ has a neighborhood in which only finitely many f_λ are non-zero.

(iii) $\sum_{\lambda\in\Lambda} f_\lambda(p) = 1 \quad \forall p \in M$ (note: finite sum due to (ii))

A related Lemma that we will need:

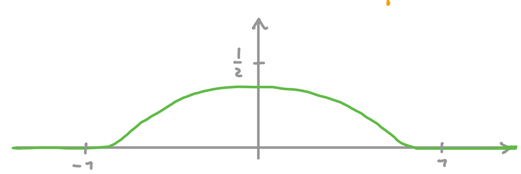
Lemma: Let $V \subseteq U$ be open subsets of a smooth manifold M and $\overline{V} \subseteq U$ compact. Then there is a smooth function $f: M \to [0,1]$ s.t.

$$f(p) = \begin{cases} 1 & , \ p\in V \\ 0 & , \ p \notin U \end{cases}$$

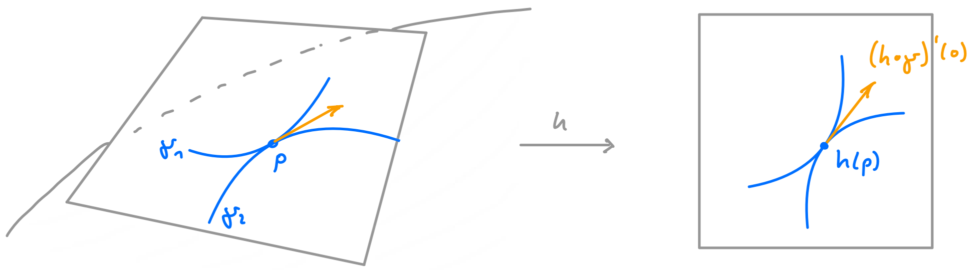
A central ingredient for the proof of both is that $g: \mathbb{R} \to \mathbb{R}$

$$g(t) := \begin{cases} \exp\left[-\frac{1}{1-t^2}\right] & , \ t\in(-1,1) \\ 0 & , \ |t| \geq 1 \end{cases}$$

is a smooth (C^∞) bump function.



Tangent spaces



Def.: Let $(M,\mathcal{A})$ be a smooth manifold and $(U,h)\in\mathcal{A}$ a chart around $p\in M$. On the set of curves $K_pM := \{\gamma\in C^\infty((-1,1),M) \mid \gamma(0)=p\}$ define the equivalence relation $\gamma_1\sim\gamma_2 :\Leftrightarrow (h\circ\gamma_1)'(0) = (h\circ\gamma_2)'(0)$.

The (geometric) tangent space of M at p is then

$$\boxed{T_pM^{geom} := \{\,[\gamma] \mid \gamma\in K_pM\,\}}$$

remarks:

- The relation is independent of the chart since:

$$(h\circ\gamma)'(0) = (h\circ g^{-1}\circ g\circ\gamma)'(0) \underset{\uparrow \text{ chain rule}}{=} \underbrace{d_{g(p)}(h\circ g^{-1})}_{\text{isomorphism, indep. of } \gamma}\ (g\circ\gamma)'(0)$$

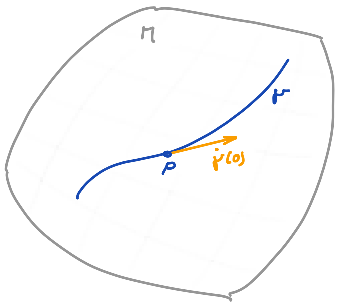
- $T_pM^{geom}\simeq \mathbb{R}^m$ since $T_pM^{geom}\ni[\gamma]\overset{\phi_h}{\mapsto}(h\circ\gamma)'(0)\in\mathbb{R}^m$ is bijective as for any $a\in\mathbb{R}^m$, $\gamma_a(t) := h^{-1}(h(p)+ta)$ satisfies $[\gamma_a]\mapsto a$.

- The linear structure of $\mathbb{R}^m$ then induces one on T_pM^{geom} so that T_pM^{geom} becomes an m-dim. $\mathbb{R}$-vector space (and ϕ_h a a vector space isomorphism).

  Elements of T_pM^{geom} are called tangent vectors.

From tangent vectors to directional derivative operators:

Suppose $M \subseteq \mathbb{R}^n$ is smooth and $\gamma \in C^\infty((-1,1), M)$ s.t. $p = \gamma(0)$. Then $\dot\gamma(0) =: v \in \mathbb{R}^n$ lies in the plane tangent to M at p.



The directional derivative of a function $f \in C^\infty(\mathbb{R}^n)$ at p in the direction of v is

$$\frac{d}{dt} f(p+tv)\Big|_{t=0} = \langle \nabla f|_p, v\rangle = \langle \nabla f|_p, \dot\gamma(0)\rangle$$
$$= \underbrace{(f\circ\gamma)'(0)}$$

The r.h.s. is still well-defined if M is an abstract smooth manifold (i.e. not embedded into $\mathbb{R}^n$) and $f \in C^\infty(M)$. In this way, a 'tangent vector' can be identified with a map $C^\infty(M) \to \mathbb{R}$. The fact that a derivative like $f \mapsto (f\circ\gamma)'(0)$ satisfies the Leibniz product rule, motivates the following definition:

Def.: Let M be a smooth manifold. The (algebraic) tangent space T_pM^{alg} of M at $p \in M$ is the space of all linear derivations at p. That is, linear maps $v: C^\infty(M) \to \mathbb{R}$ s.t. for all $f, g \in C^\infty(M)$:

$$\boxed{v(fg) = f(p)v(g) + g(p)v(f)} \qquad \text{'Leibniz product rule'}$$

remarks:

- T_pM^{alg} becomes a vector space with $(v_1 + c\cdot v_2)(f) := v_1(f) + c\cdot v_2(f)$
- The derivation of a constant function is zero, since $\forall f \in C^\infty(M)$:
  $v(f) = v(f\cdot 1) = v(1)f(p) + v(f)$. So $v(1) = 0$.

- I.g. linear derivations are defined on 'algebras' (here $C^\infty(M)$).
  Poisson brackets and commutators are also lin. derivations.
- If (U, h) is a chart around p and $h(q) =: (x_1(q), \dots, x_n(q))$, then $\boxed{\left.\frac{\partial}{\partial x_i}\right|_p : C^\infty(M) \ni f \mapsto \partial_i (f \circ h^{-1})\big|_{h(p)}}$ defines an element of T_pM^{alg}. If there is no confusion in sight, we may omit the "$|_p$".

Thm.: If M is an n-dimensional smooth manifold and $p \in M$, then $\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$ form a basis of T_pM^{alg}.

proof: Linear independence can be seen as follows: let $h = (x_1, \dots, x_n)$ be the coordinate functions of the chart (U, h). Then $\left.\frac{\partial}{\partial x_i}\right|_p x_j = \delta_{ij}$. So $\left.\frac{\partial}{\partial x_i}\right|_p$ cannot be a linear combination of the others.

For $f \in C^\infty(M)$ define $F := f \circ h^{-1}$ in a neighborhood of some $y \in h(U)$ and assume w.l.o.g. $h(p) = 0$ and that $h(U)$ is convex.

Then $F(y) = F(0) + \int_0^1 \frac{d}{dt} F(ty)\, dt = F(0) + \sum_{i=1}^n y_i\, g_i(y)$, where $g_i(y) := \int_0^1 \partial_i F(ty)\, dt$ is a C^∞ function with $g_i(0) = \partial_i F(0) = \left.\frac{\partial}{\partial x_i}\right|_p f$

With $f(q) = (F \circ h)(q) = F(0) + \sum_i h_i(q)\, g_i(h(q))$, we get for an arbitrary derivation $v : C^\infty(M) \to \mathbb{R}$:

$$v(f) = \sum_i \underbrace{h_i(p)}_{=0} v(g_i \circ h) + \underbrace{g_i(h(p))}_{= g_i(0) = \left.\frac{\partial}{\partial x_i}\right|_p f} v(h_i)$$
$$= \sum_i v(h_i) \left.\frac{\partial}{\partial x_i}\right|_p f$$

□

We will use $T_pM := T_pM^{alg}$ as our definition of the tangent space.

<u>remark:</u> For $M = \mathbb{R}^n$ there is a canonical isomorphism $\boxed{T_p\mathbb{R}^n \cong \mathbb{R}^n}$ via $T_p\mathbb{R}^n \ni \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\Big|_p \longmapsto v \in \mathbb{R}^n$. In fact:

<u>Lemma:</u> For every finite-dim. $\mathbb{R}$-vec. space V and $p \in V$ a canonical (i.e., basis-independent) isomorphism $I: V \to T_pV$ is given by: $V \ni v \mapsto \Big(C^\infty(V) \ni f \mapsto \underbrace{\frac{d}{dt}\Big|_{t=0} f(p+tv)}_{= \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}\big|_p} \Big)$.

One often exploits this and 'identifies' T_pV with V. In particular, if $V = \mathbb{R}$.

<u>Lemma:</u> (coordinate change) Let $(U, (x_1, \dots, x_n))$ and $(V, (y_1, \dots, y_n))$ be two charts around a point p on a C^∞-manifold M. Then

$$\frac{\partial}{\partial x_i}\Big|_p = \sum_j \underbrace{\left(\frac{\partial}{\partial x_i}\Big|_p y_j \right)}_{\text{Jacobian of the coordinate change } (y \circ x^{-1}) \text{ at } x(p)} \frac{\partial}{\partial y_j}\Big|_p$$

<u>proof:</u>

$$\frac{\partial}{\partial x_i}\Big|_p f = \partial_i \Big|_{x(p)} f \circ x^{-1} = \partial_i \Big|_{x(p)} \Big[\underbrace{(f \circ y^{-1})}_{} \circ \underbrace{(y \circ x^{-1})}_{} \Big]$$

maps between Euclidean spaces

$$= \sum_j \partial_j \Big|_{y(p)} (f \circ y^{-1}) \; \partial_i \Big|_{x(p)} \underbrace{(y \circ x^{-1})_j}_{(y_j \circ x^{-1})}$$

$$= \sum_j \underbrace{\left(\frac{\partial}{\partial y_j}\Big|_p f \right)}_{} \underbrace{\left(\frac{\partial}{\partial x_i}\Big|_p y_j \right)}_{} \qquad \square$$

Lemma: (equivalence of tangent space definitions)

The map

$$\begin{array}{ccc} T_pM^{geom} & \xrightarrow{\ \Psi\ } & T_pM^{alg} \\ \ni & & \ni \\ [\gamma] & \longmapsto & \Psi([\gamma]) : \boxed{C^\infty(M) \ni f \mapsto (f\circ\gamma)'(0)} \end{array}$$

is a vector space isomorphism s.t. every curve $\gamma \in k_pM$ with

$(h\circ\gamma)'(0) = e_i$ w.r.t. a chart (U,h), $h = (x_1,\dots,x_n)$, is mapped to $\Psi : [\gamma] \mapsto \frac{\partial}{\partial x_i}\Big|_p$.

remark: This is probably the easiest way to understand elements of T_pM^{alg}: as 'directional derivatives along a curve'

proof: $\Psi([\gamma])$ is independent of the representative since

$$(f\circ\gamma)'(0) = d_{h(p)}(f\circ h^{-1}) \underbrace{(h\circ\gamma)'(0)}_{\text{equal for all representatives of } [\gamma]}$$

$\Psi([\gamma])$ is a derivation since it is linear and with $v(f) := \Psi([\gamma])(f)$:

$$v(fg) = \big((f\circ\gamma)(g\circ\gamma)\big)'(0) = (f\circ\gamma)'(0)\,(g\circ\gamma)(0) + (g\circ\gamma)'(0)\,(f\circ\gamma)(0)$$

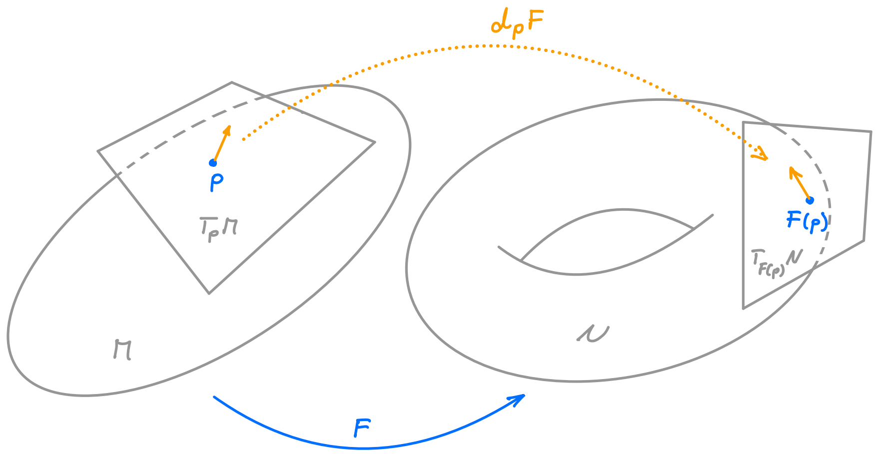
$$= v(f)\cdot g(p) + v(g)\cdot f(p)$$

Ψ is a vector space isomorphism since $\dim(T_pM^{alg}) = \dim(T_pM^{geom})$

and from $(h\circ\gamma)'(0) = e_i$ we obtain

$$v(f) = (f\circ\gamma)'(0) = d_{h(p)}(f\circ h^{-1})\,(h\circ\gamma)'(0) =$$

$$= d_{h(p)}(f\circ h^{-1})\, e_i = \partial_i (f\circ h^{-1})\Big|_{h(p)} = \frac{\partial}{\partial x_i}\Big|_p f . \qquad \square$$



<u>Def.:</u> Let $F: M \to N$ be smooth. The differential (a.k.a. pushforward) of F at $p \in M$ is defined as

$$d_pF \equiv d_pF^{alg} : T_pM^{alg} \longrightarrow T_{F(p)}N^{alg}$$

$$d_pF(v)f := v(f \circ F) \qquad \text{for } v \in T_pM^{alg},\ f \in C^\infty(N)$$

$$d_pF^{geom} : T_pM^{geom} \longrightarrow T_{F(p)}N^{geom}$$

$$d_pF([\gamma]) := [F \circ \gamma] \qquad \text{for } [\gamma] \in T_pM^{geom}$$

<u>remark:</u> ◦ the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "d_{h(p)}(g \circ F \circ h^{-1})"] & \mathbb{R}^n \\
T_pM^{geom} \arrow[u, "\phi_h"] \arrow[r, "d_pF"] \arrow[d, "\psi"] & T_{F(p)}N^{geom} \arrow[u, "\phi_g"] \arrow[d, "\psi"] \\
T_pM^{alg} \arrow[r, "d_pF"] & T_{F(p)}N^{alg}
\end{tikzcd}$$

That is, expressed in local coordinates, d_pF is the usual total / Fréchet derivative represented by the Jacobian matrix.

remarks:

- d_pF is a linear map
- $d_p(id_M) = id_{T_pM}$
- If $[\gamma] \in T_pM^{geom}$ then $d_pF(\psi([\gamma])) : C^\infty(N) \ni f \mapsto (f \circ F \circ \gamma)'(0)$
- If M is connected and $d_pF = 0$, then F is constant.
- For any linear map $F: V \to W$ between finite-dim. $\mathbb{R}$-vector spaces, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{I} & T_pV \\ F\downarrow & & \downarrow d_pF \\ W & \xrightarrow{I} & T_{F(p)}W \end{array}$$

Lemma: For $f \in C^\infty(M)$ and $v \in T_pM$: $I^{-1} \circ d_pf(v) = v(f)$

remark: The isomorphism $I^{-1}: T_{f(p)}\mathbb{R} \to \mathbb{R}$ is usually not written explicitly. In this sense $\boxed{d_pf(v) = v(f)}$.

proof: Note that any element of $T_{f(p)}\mathbb{R}$ is a derivation $C^\infty(\mathbb{R}) \to \mathbb{R}$.

By definition of $d_pf: T_pM \to T_{f(p)}\mathbb{R}$ this derivation maps any $\varphi \in C^\infty(\mathbb{R})$

to $\quad d_pf(v)\,\varphi = v(\varphi \circ f) = (\varphi \circ f \circ \gamma)'(0) = \varphi'(f(p))\underbrace{(f \circ \gamma)'(0)}_{= v(f)}$

This coincides with $I(v(f))\,\varphi = \frac{d}{dt}\Big|_{t=0} \varphi(f(p) + t\,v(f)) = v(f) \cdot \varphi'(f(p))$

$\square$

Lemma: (chain rule) If $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ are smooth, then

$$\boxed{d_p(g \circ f) = d_{f(p)}(g)\, d_pf}$$

Def.: The disjoint union $\bigcup_{p\in M} T_pM =: TM$ is called the tangent bundle of M.

remark: If we consider elements of TM as pairs $(p,X) \in M \times T_pM$ we can define the projection $\pi: TM \to M$, $\pi: (p,X) \mapsto p$.

Thm.: Let M be an m-dimensional manifold with smooth atlas $\{(U_\alpha, x_\alpha)\}$. Then a smooth atlas for TM is given in terms of the charts $\phi_\alpha : \underbrace{\pi^{-1}(U_\alpha)}_{\subseteq TM} \longrightarrow \mathbb{R}^{2m}$

$$\phi_\alpha\left(\sum_{i=1}^m v_i \frac{\partial}{\partial x_{\alpha,i}}\bigg|_p \right) := (x_\alpha(p), v) \quad .$$

Hence, TM is smooth manifold with $\dim(TM) = 2 \cdot \dim(M)$.

Def.: If $f: M \to N$ is smooth, the derivative of f (a.k.a. pushforward) is the map $df : M \ni p \mapsto d_pf$

remark: df induces a smooth map $TM \to TN$ that maps $T_pM \ni v \mapsto d_pf\, v \in T_{f(p)}N$ (and is sometimes also denoted by df).

Alternating multilinear maps

Let V be a finite-dimensional real vector space throughout.

Def.: The space $V^* := \{ f : V \to \mathbb{R} \text{ linear} \}$ is called the dual space of V. The elements of V and V^* are called vectors and covectors, respectively.

remarks: V^* is again a real vector space.

If $\dim(V) = n \in \mathbb{N}$, then $\dim(V^*) = n$ and $(V^*)^* = V$.

For $f \in V^*$, $v \in V$ one often writes $f(v) =: \langle f, v \rangle$. If $(e_i)_{i=1}^n$ is a basis of V, then $(f_j \in V^*)_{j=1}^n$ is called the dual basis if $\langle f_j, e_i \rangle = \delta_{ij}$. This always exists and is unique.

Exp.: (1) If $V = \mathbb{R}^n$ s.t. its elements are column vectors, then V^* can be regarded as the space of row vectors s.t. $\langle f, v \rangle$ is the 'matrix product', i.e. the standard scalar product of v with f^T.

(2) If $V := \{ v : (-1,1) \to \mathbb{R} \mid \exists a \in \mathbb{R}^{d+1} : v(x) = \sum_{i=0}^{d} a_i x^i \}$ for some degree $d \in \mathbb{N}$, then $f(v) := \int_{-1}^{1} v(x)\, dx$ is an element of the dual space $V^* \ni f$.

3 If (U, x) is a chart around $p \in M$ and $x(p) =: (x_1(p), \dots, x_n(p))$,
We define $dx_i : T_pM \to \mathbb{R}$ as the differential of the coordinate func. $x_i : U \to \mathbb{R}$, $x_i = \underset{\text{coordinate proj.}}{\pi_i} \circ x$ at p, composed with the canonical isomorphism $T_{x_i(p)}\mathbb{R} \to \mathbb{R}$. That is,

$$\boxed{dx_i(v) := v(x_i)} .$$

With $V := T_pM$, $(dx_i)_{i=1}^n$ are elements of $V^* =: T_p^*M$ (the cotangent space). Recall that $\frac{\partial}{\partial x_i}\Big|_p : C^\infty(M) \ni f \mapsto \partial_i(f \circ x^{-1})\Big|_{x(p)}$ form a basis of V.

Thm.: $\boxed{(dx_i \in T_p^*M)_{i=1}^n \text{ and } \left(\frac{\partial}{\partial x_i}\Big|_p \in T_pM\right)_{i=1}^n \text{ are dual bases}}$

proof: $dx_i\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \frac{\partial}{\partial x_j}\Big|_p x_i = \partial_j\Big|_{x(p)}(\pi_i \circ x \circ x^{-1}) = \delta_{ij}$ $\square$

remark: dx_i is the paradigm of a 1-form as defined in the following ...

Def.: $f: V \times \ldots \times V =: V^k \to W$ is called multilinear or k-linear if it is linear in each of its k arguments. A k-linear map is called alternating or anti-symmetric if for all $v \in V^k$ and all permutations π:

$$f(v_1, \ldots, v_k) = \operatorname{sgn}(\pi)\, f(v_{\pi(1)}, \ldots, v_{\pi(k)}) .$$

$\mathrm{Alt}^k(V, W)$ denotes the space of all such alternating k-linear maps and $\Lambda^k V^* := \mathrm{Alt}^k(V, \mathbb{R})$ is called the space of k-forms (short for 'k-linear alternating forms') on V (or the k'th exterior power of V^*).

remarks: $\mathrm{Alt}^k(V,W)$ is again a real vector space and $\Lambda^1 V^* = V^*$.

A useful convention is $\Lambda^0 V^* := \mathbb{R}$.

Corollary: For a k-linear map $f: V^k \to W$ the following are equivalent:

(i) $f \in \mathrm{Alt}^k(V,W)$

(ii) $f(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ for some $i \neq j$.

(iii) $f(v_1, \ldots, v_k) = 0$ if $v_1, \ldots, v_k$ are linearly dependent.

proof: $\to$ exercise.

Exp.: 1 The cross product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $(a \times b)_i := \sum_{j,k} \varepsilon_{ijk}\, a_j b_k$,

where $\varepsilon_{ijk} = \begin{cases} \operatorname{sgn}(\pi), & (1,2,3) = (\pi(i), \pi(j), \pi(k)) \\ 0 \end{cases}$ is the Levi-Civita tensor,

is element of $\mathrm{Alt}^2(\mathbb{R}^3, \mathbb{R}^3)$.

2 For any $(f_i \in V^*)_{i=1}^k$, the map $V^k \ni (v_1, \ldots, v_k) \mapsto \det(\langle f_i, v_j \rangle)_{ij}$ is a k-form.

③ $dx_i : T_pM \to \mathbb{R}$ is a 1-form on T_pM.

remark: recall that the cross product and the determinant both quantify the volume/area while their sign indicates an 'orientation'.

Lemma: Let $(e_1, \dots, e_n)$ be a basis of V and for any $\omega \in \Lambda^k V^*$ define its components w.r.t. that basis as $\omega_{i_1 \dots i_k} := \omega(e_{i_1}, \dots, e_{i_k}) \in \mathbb{R}$.

Then $\Lambda^k V^* \longrightarrow \mathbb{R}^{\binom{n}{k}}$, $\omega \mapsto (\omega_{i_1 \dots i_k})_{i_1 < i_2 < \dots < i_k}$ is a vector space isomorphism.

proof: The map is linear by definition.

Injectivity: if $\omega_{i_1 \dots i_k} = 0$ for all $i_1 < \dots < i_k$, then all components vanish since $\omega_{\pi(i_1), \dots, \pi(i_k)} \overset{(*)}{=} \operatorname{sgn}(\pi)\, \omega_{i_1, \dots, i_k}$. By multilinearity of ω this means $\omega = 0$.

Surjectivity: if $(\omega_{i_1 \dots i_k})_{i_1 < \dots < i_k}$ is given, $(*)$ enables us to define $\omega_{i_1 \dots i_k}$ for all i and from here a corresponding k-form $\hat{\omega}(v_1, \dots, v_k) := \sum_{j_1 \dots j_k} \omega_{j_1 \dots j_k} \langle b_{j_1}, v_1 \rangle \cdot \ldots \cdot \langle b_{j_k}, v_k \rangle$ where $(b_1, \dots, b_n)$ is the dual basis w.r.t. $(e_1, \dots, e_n)$, i.e. $\langle b_i, e_j \rangle = \delta_{ij}$.

By construction, $\hat{\omega}(e_{i_1}, \dots, e_{i_k}) = \omega_{i_1 \dots i_k}$. □

Corollary: If $\dim(V) = n$, then $\boxed{\dim(\Lambda^k V^*) = \binom{n}{k}}$. In particular, $\dim \Lambda^n V^* = 1$ and $k > n \Rightarrow \Lambda^k V^* = \{0\}$.

Def.: For $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$ the exterior product $\omega \wedge \eta \in \Lambda^{k+l} V^*$ is defined as

$$\omega \wedge \eta(v_1, \dots, v_{k+l}) := \frac{1}{k!\,l!} \sum_{\pi \in S_{k+l}} \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, \dots, v_{\pi(k)}) \cdot \eta(v_{\pi(k+1)}, \dots, v_{\pi(k+l)}).$$

remarks:
- An alternative, equivalent definition: let $S(k,l) \subseteq S_{k+l}$ be the set of '(k,l)-shuffles', i.e. permutations satisfying $\pi(1) < \dots < \pi(k) \wedge \pi(k+1) < \dots < \pi(k+l)$. Then
$$\omega \wedge \eta(v_1, \dots, v_{k+l}) = \sum_{\pi \in S(k,l)} \operatorname{sgn}(\pi)\, \omega(v_{\pi(1)}, \dots, v_{\pi(k)})\, \eta(v_{\pi(k+1)}, \dots, v_{\pi(k+l)}).$$
- For $c \in \mathbb{R}$: $c \wedge \omega := c \cdot \omega$.

Exp.: If $\omega_1, \omega_2 \in V^*$, then $\omega_1 \wedge \omega_2(v_1, v_2) = \omega_1(v_1)\omega_2(v_2) - \omega_1(v_2)\omega_2(v_1)$

Prop.: For $\omega, \mu \in \Lambda^k V^*$, $\eta \in \Lambda^l V^*$, $\nu \in \Lambda^m V^*$:

(i) $(\omega + \mu) \wedge \eta = (\omega \wedge \eta) + (\mu \wedge \eta)$ distributivity

(ii) $\omega \wedge \eta = (-1)^{k \cdot l}\, \eta \wedge \omega$ (anti-) commutativity

(iii) $(\omega \wedge \eta) \wedge \nu = \omega \wedge (\eta \wedge \nu)$ associativity

(iv) $(c\,\omega) \wedge \eta = \omega \wedge (c\,\eta) = c(\omega \wedge \eta)$ for any $c \in \mathbb{R}$

The proofs of (ii) and (iii) are a bit longer (see e.g. [do Carmo]).

(i) + (ii) implies that $(\omega, \eta) \mapsto \omega \wedge \eta$ is bilinear.

(iii) implies that $\omega \wedge \eta \wedge \nu$ makes sense without brackets. In fact,

$$(\omega \wedge \eta \wedge \nu)(v_1, \dots, v_{k+l+m}) = \frac{1}{k!\,l!\,m!} \sum_{\pi \in S_{k+l+m}} \omega(v_{\pi(1)}, \dots, w_{\pi(k)}) \cdot \eta(v_{\pi(k+1)}, \dots) \cdot \nu(v_{\pi(k+l+1)}, \dots)$$

Corollary: If k is odd, and $\omega \in \Lambda^k V^*$, then $\omega \wedge \omega = 0$.

proof: $\omega \wedge \omega \overset{(ii)}{=} (-1)^{k^2} \omega \wedge \omega = -\omega \wedge \omega$. □

However, $\omega \wedge \omega$ can be non-zero for forms of even degree (→ Exercise)

Prop.: If $\varphi_1, \dots, \varphi_n$ is a basis of V^*, then $(\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k})_{i_1 < \dots < i_k} =: \phi_I$ form a basis of $\Lambda^k V^*$.

proof: Let $e_1, \dots, e_n \in V$ be the dual basis. Then $\sum_I a_I \phi_I = 0$ implies $0 = \sum_I a_I \phi_I (e_{j_1}, \dots, e_{j_k}) = a_{j_1 \dots j_k}$. So the ϕ_I's are lin. indep. As there are $\binom{n}{k} = \dim(\Lambda^k V^*)$ of them, they form a basis. □

Prop.: For $\varphi_1, \dots, \varphi_k \in V^*$ and $v_1, \dots, v_k \in V$:

$$\boxed{(\varphi_1 \wedge \dots \wedge \varphi_k)(v_1, \dots, v_k) = \det (\langle \varphi_i, v_j \rangle)_{i,j}}$$

proof: by induction on k. We know it for $k=2$. From the definition of the exterior product we get

$$\varphi_1 \wedge (\varphi_2 \wedge \dots \wedge \varphi_k)(v_1, \dots, v_k) = \sum_{j=1}^{k} (-1)^{j+1} \varphi_1(v_j) (\varphi_2 \wedge \dots \wedge \varphi_k)(v_1, \dots, \underset{\text{excluded}}{\overline{v_j}}, \dots, v_k)$$

The statement then follows by expanding the determinant w.r.t. the first row as for any $k \times k$ matrix A:

$$\det(A) = \sum_{j=1}^{k} (-1)^{j+1} A_{1,j} \cdot \det(\hat{A}_{1,j})$$

where $\hat{A}_{1,j}$ is the $(k-1) \times (k-1)$ matrix constructed from A by omitting the first row and j'th column. □

Differential forms on manifolds

Def.: A **k-form** ω on a smooth manifold M is an assignment of a k-form $\omega_p \in \Lambda^k T_p^* M$ to each $p \in M$.

That is, each ω_p is an alternating k-linear map of the form

$$\omega_p : T_pM \times \dots \times T_pM \to \mathbb{R}$$

W.r.t. a chart (U, x) around $p \in M$, we know that the dx_i's form a basis of $T_p^* M$. So we can write

$$\boxed{\omega_p = \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k}(p)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}}$$

where $\omega_{i_1, \dots, i_k}(p) = \omega_p\left(\frac{\partial}{\partial x_{i_1}}\Big|_p, \dots, \frac{\partial}{\partial x_{i_k}}\Big|_p \right)$ are the **components** of ω_p w.r.t. the chart. Changing the chart to (V, y) results in $\tilde{\omega}_{i_1, \dots, i_k}(p) = \omega_p\left(\frac{\partial}{\partial y_{i_1}}\Big|_p, \dots, \frac{\partial}{\partial y_{i_k}}\Big|_p \right)$

$$= \sum_{j_1, \dots, j_k} J_{i_1 j_1}(p) \cdots J_{i_k j_k}(p)\; \omega_{j_1, \dots, j_k}(p)$$

where $J_{ij}(p) := \partial_i \Big|_{y(p)} (x \circ y^{-1})_j$ is the Jacobian of the coordinate change.

Since $J_{ij} \in C^\infty$, the following is chart-independent:

Def.: A k-form on a smooth manifold is called **differentiable** (or of class C^k) if the coordinates $\omega_I(p)$ are as a function of p.

The set of all C^∞-differentiable k-forms on M will be denoted by $\Omega^k M$ and we define

$$\Omega M := \bigoplus_{k=0}^{\dim(M)} \Omega^k M \quad \text{with} \quad \Omega^0 M := C^\infty(M) \, , \; \Omega^{-1} M := \{0\} \, .$$

<u>remark:</u> The def. of ΩM makes sense since each $\Omega^k M$ is a natural vector space. In fact, since there is a scalar multiplication $C^\infty(M) \times \Omega^k(M) \to \Omega^k(M)$

$$(f, \omega) \mapsto (f \cdot \omega) \text{ with } (f \cdot \omega)_p := f(p)\, \omega_p$$

ΩM is a module over the ring $C^\infty(M)$.

<u>examples:</u>

- 0-forms on M are just smooth functions on M.
- If $f \in C^\infty(M)$, then the differential $df : M \ni p \mapsto d_p f$ is a 1-form

$$d_p f : T_p M \to T_{f(p)} \mathbb{R} \simeq \mathbb{R}$$

  W.r.t. to a chart (u, x) around p we have

$$d_p f = \sum_i d_p f \left(\frac{\partial}{\partial x_i} \Big|_p \right) dx_i$$

$$\underset{\substack{\uparrow \\ d_p f(v) = v(f)}}{=} \sum_i \left(\frac{\partial}{\partial x_i} \Big|_p f \right) dx_i$$

  In this sense: $\boxed{df = \sum_i \frac{\partial f}{\partial x_i} dx_i}$

- If $n = \dim(M)$, and (u, x) is a chart around p, then w.r.t. that chart every $\omega \in \Omega^n M$ is of the form $\boxed{\omega_p = f(p) \det}$, where $f \in C^\infty(M)$ and $\det := dx_1 \wedge \ldots \wedge dx_n$.

remark: note that the notation 'dx' for an element of T_p^*M omits the chosen $p \in M$. Then dx should be read as $(dx)_p$ or $d_p x$. In $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$, however, '$dx_i$' mean a map $M \to T^*M$ that assigns to each $p \in M$ an element of T_p^*M.

Def.: Let ω be a k-form on M and η be an l-form. The exterior product $\omega \wedge \eta$ is defined as the $(k+l)$-form determined by $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$.

This inherits the properties of exterior products of forms on vector spaces. That is, associativity, bilinearity, $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$ and if ω and η are smooth, then

$$f \cdot (\omega \wedge \eta) = (f \cdot \omega) \wedge \eta = \omega \wedge (f \cdot \eta) \quad \forall f \in C^\infty(M)$$

$(\underbrace{\Omega M, +, \cdot}_{C^\infty(M)\text{-module}}, \wedge)$ is the Grassmann algebra on M.

bilinear $\wedge : \Omega M \times \Omega M \to \Omega M$ defined by linear continuation.

Note that the constant function $\iota \in C^\infty(M)$ $\iota(p) = 1$ serves as identity, i.e. $\iota \wedge \omega = \omega$.

More generally, for any $f \in C^\infty(M) = \Omega^0 M$:

$$f \wedge \omega = f \cdot \omega$$

Having in mind substitutions and coordinate transformations, we define:

Def.: For a smooth map $f: M \to N$, we define an $\mathbb{R}$-linear map $f^*: \Omega N \to \Omega M$ via: $f^*: \Omega^k N \to \Omega^k M$, $\omega \mapsto (f^*\omega)$

for $k \geq 1$: $\boxed{(f^*\omega)_p (v_1, \dots, v_k) := \omega_{f(p)} (d_p f \, v_1, \dots, d_p f \, v_k)}$

where $p \in M$ and $v_1, \dots, v_k \in T_p M$.

and for $k = 0$ via: $f^*\omega := \omega \circ f$.

$f^*\omega$ is called the pullback (a.k.a. induced form) of ω by f.

remarks:
- by definition:
  - $\mathrm{id}^*(\omega) = \omega$
  - $(f \circ g)^*(\omega) = g^*(f^*(\omega))$
  - $f^*(\omega + \eta) = f^*(\omega) + f^*(\eta)$
- Consider the 'pushforward' $f_* := d_p f : T_p M \longrightarrow T_{f(p)} N$.

  Then the 'pullback' $f^*: T^*_{f(p)} N \longrightarrow T^*_p M$ is the corresponding dual map in the sense that

$$(f^*\omega)(v) \equiv \omega (f_* v) \quad \text{for } \omega \in T^*_{f(p)} N, \; v \in T_p M$$

Lemma: For a smooth map $f: M \to N$:

(i) $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$

(ii) If $\varphi \in C^\infty(N)$, then $f^*(\varphi \cdot \omega) = (\varphi \circ f) \cdot f^*(\omega)$

pointwise product / scalar prod. in ΩM.

(iii) For $\omega \in \Omega^k N$ if (U, x) is a chart around $f(p)$ w.r.t. which

$\omega_{f(p)}$ has components $\omega_{i_1,\dots,i_k}(f(p))$, then

$$(f^*\omega)_p = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}(f(p))\, d_p(x_{i_1}\circ f)\wedge\dots\wedge d_p(x_{i_k}\circ f)$$

<u>proof:</u> (i) $(f^*(\omega\wedge\eta))_p(v_1,\dots,v_{k+l}) = (\omega\wedge\eta)_{f(p)}(d_pf\, v_1,\dots,d_pf\, v_{k+l})$

$$= \sum_{\pi\in S(k,l)} \operatorname{sgn}(\pi)\ \omega_{f(p)}(d_pf\, v_{\pi(1)},\dots,d_pf\, v_{\pi(k)}) \cdot \eta_{f(p)}(d_pf\, v_{\pi(k+1)},\dots,d_pf\, v_{\pi(k+l)})$$

$$= (f^*(\omega)_p \wedge f^*(\eta)_p)(v_1,\dots,v_{k+l})$$

(ii) $f^*(\varphi\omega) = f^*(\varphi\wedge\omega) \overset{(i)}{=} f^*(\varphi)\wedge f^*(\omega)$

$$= (\varphi\circ f)\cdot f^*(\omega)$$

(iii) by linearity, (ii) & (i) we get:

$$(f^*\omega)_p = \sum_{i_1<\dots<i_k} \omega_{i_1\dots i_k}(f(p))\, f^*(dx_{i_1})\wedge\dots\wedge f^*(dx_{i_k})$$

Moreover, $f^*(dx_i)_p(v) = (dx_i)_{f(p)}(d_pf\, v)$

$$\overset{\text{chain rule}}{=} d_p(x_i\circ f)(v)$$

□

<u>example:</u> (polar coordinates) on $\mathbb{R}^2\setminus\{(0,0)\}$ consider the 1-form (w.r.t. the canonical/identity chart):

$$\omega := -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \quad \text{on } \mathbb{R}^2\setminus\{0\}.$$

Let $f(r,\theta) := (r\cos\theta, r\sin\theta)$ on $(0,\infty)\times(0,2\pi)$ map from 'polar' to 'Cartesian' coordinates. Then at $p=(r,\theta)$

$$(f^*\omega)_p = -\frac{r\sin\theta}{r^2}\, d_p(x\circ f) + \frac{r\cos\theta}{r^2}\, d_p(y\circ f)$$

$$= -\frac{r\sin\theta}{r^2}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{r\cos\theta}{r^2}(\sin\theta\, dr + r\cos\theta\, d\theta) = d\theta.$$

Prop.: Let $f: M \to N$ be smooth between two n-dim. manifolds and (U,x) and (V,y) charts around $p \in M$ and $f(p)$, resp. For any $\varphi \in C^\infty(N)$ and with $f_i := y_i \circ f$:

$$\boxed{f^*(\varphi \cdot dy_1 \wedge \ldots \wedge dy_n) = (\varphi \circ f) \cdot \det\left(\frac{\partial}{\partial x_j} f_i\right) dx_1 \wedge \ldots \wedge dx_n}$$

proof: We show that both sides have the same action on the basis $\left(\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\right)$ dual to dx_i:

$$\left(f^*(\varphi \cdot dy_1 \wedge \ldots \wedge dy_n)\right)_p \left(\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\right) \overset{\text{Lemma}}{=}$$

$$(\varphi \circ f)(p) \underbrace{(d_p f_1 \wedge \ldots \wedge d_p f_n)\left(\frac{\partial}{\partial x_1}\Big|_p, \ldots, \frac{\partial}{\partial x_n}\Big|_p\right)}$$

$$= \det\left(d_p f_i \left(\frac{\partial}{\partial x_j}\Big|_p\right)\right) = \det\left(\frac{\partial}{\partial x_j}\Big|_p f_i\right). \qquad \square$$

Application to $f = id$ yields:

Corollary: If (U,x), (V,y) are two charts around $p \in M$ of an n-dim. manifold M, then

$g \cdot dy_1 \wedge \ldots \wedge dy_n = h \cdot dx_1 \wedge \ldots \wedge dx_n$ for $g, h \in C^\infty(M)$

iff $h = g \cdot \det\left(\frac{\partial}{\partial x_i}\Big|_p y_j\right)$.

Similarly: $$dy_{j_1} \wedge \ldots \wedge dy_{j_k} = \sum_{i_1 < \ldots < i_k} \det\left(\frac{\partial y_{j_s}}{\partial x_{i_t}}\right)_{s,t=1..k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

Thm.:

For any smooth manifold M there is a unique map $d : \Omega M \to \Omega M$ s.t. $\boxed{d(\Omega^k M) \subseteq \Omega^{k+1} M}$ and

(i) $\forall\, \omega, \eta \in \Omega M$: $d(\omega + \eta) = d\omega + d\eta$

(ii) $\forall\, \omega \in \Omega^k M, \eta \in \Omega M$: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

(iii) $\forall\, f \in C^\infty(M) \equiv \Omega^0 M$: df is the differential of f

(iv) $\forall\, \omega \in \Omega M$: $d^2\omega := d(d\omega) = 0$

This map is called exterior derivative and w.r.t. a chart (U, x) around $p \in M$: $(d\omega)_p = \sum_{i_1 < \dots < i_k} (d_p \underbrace{\omega_{i_1 \dots i_k}}_{\omega_I}) \wedge \underbrace{dx_{i_1} \wedge \dots \wedge dx_{i_k}}_{dx_I}$ for $\omega \in \Omega^k M$.

Hence, using multiindex notation: $\boxed{d\left(\sum_I \omega_I dx_I\right) = \sum_I d\omega_I \wedge dx_I}$

proof: Suppose $\omega_1, \omega_2 \in \Omega M$ coincide on an arbitrary open subset $U \subseteq M$. We first show that then $d\omega_1|_U = d\omega_2|_U$, i.e., that d is 'local'.

To this end, for $p \in V \subseteq \overline{V} \subseteq U$ let $f \in C^\infty(M)$ be s.t.
$f(q) = \begin{cases} 1, & q \in V \\ 0, & q \notin U \end{cases}$. Then $0 = f \cdot (\omega_1 - \omega_2) \in \Omega M$

and therefore
$$0 \overset{(iii)}{=} d(0) = d\left(f \wedge (\omega_1 - \omega_2)\right)$$
$$\overset{(ii)}{=} df \wedge (\omega_1 - \omega_2) + f \wedge d(\omega_1 - \omega_2)$$
$$\overset{(iii),(i)}{=} 0 + f \wedge d\omega_1 - f \wedge d\omega_2$$

So $(d\omega_1)|_V = (d\omega_2)|_V$ and since this applies to an arbitrary $p \in U$ it holds on all of U.

Consider $\omega \in \Omega^k M$ that within U is of the form $\omega = \sum_I \omega_I \, dx_I$.

We can always extend ω_I smoothly to all of M so that the resulting ω coincides with the initial one. Since d is local this does not affect $d\omega$. We get:

$$
\begin{aligned}
& d\Big(\sum_I \omega_I \, dx_I\Big) \\
\overset{(i)}{=}\ & \sum_I d\big(\underbrace{\omega_I \, dx_I}_{= \omega_I \wedge dx_I \text{ since } \omega_I \in C^\infty(M)}\big) \\
\overset{(ii)}{=}\ & \sum_I d\omega_I \wedge dx_I + \omega_I \wedge \underbrace{d(dx_I)}_{=0 \text{ by (ii) and (iii)}} \\
=\ & \sum_I d\omega_I \wedge dx_I
\end{aligned}
$$

This proves that $d\omega$ is of the claimed form and thus unique.

It remains to show that this fullfills (i)-(iv). (i) and (iii) are obvious.

Due to linearity it suffices to prove (ii) for $\omega = f \, dx_I \in \Omega^k M$ and $\eta \in g \, dx_J$:

$$
\begin{aligned}
d(\omega \wedge \eta) &= d(f g \, dx_I \wedge dx_J) \\
&= (g\,df + f\,dg) \wedge dx_I \wedge dx_J \\
&= \underbrace{(df \wedge dx_I)}_{d\omega} \wedge \underbrace{(g\,dx_J)}_{\eta} + (-1)^k \underbrace{(f\,dx_I)}_{\omega} \wedge \underbrace{(dg \wedge dx_J)}_{d\eta} \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta .
\end{aligned}
$$

To show (iv) consider again $\omega = f \, dx_I$ so that

$$d\omega = df \wedge dx_I = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

Then

$$
\begin{aligned}
d^2\omega &= \sum_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k} dx_k \wedge dx_j \wedge dx_I \\
&\underset{\uparrow}{=} \sum_{j<k} \frac{\partial^2 f}{\partial x_j \partial x_k} (dx_k \wedge dx_j + dx_j \wedge dx_k) \wedge dx_I = 0
\end{aligned}
$$

Schwarz's thm. i.e. $\dfrac{\partial^2 f}{\partial x_j \partial x_k} = \dfrac{\partial^2 f}{\partial x_k \partial x_j}$ for $f \in C^\infty$

$\square$

Lemma: If $F : M \to N$ is smooth and $\omega \in \Omega^k N$, then

$$\boxed{F^*(d\omega) = d(F^*\omega)}$$

proof: Due to locality and linearity it suffices to consider

$$F^* d\left(f \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right) = F^*\left(df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right)$$

$$= d(f \circ F) \wedge d(x_{i_1} \circ F) \wedge \ldots \wedge d(x_{i_k} \circ F)$$

$$= d\left(f \circ F \wedge d(x_{i_1} \circ F) \wedge \ldots \wedge d(x_{i_k} \circ F)\right)$$

$$= d\left(F^*\left(f \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right)\right) .$$

$\square$

Def.: $\omega \in \Omega^k M$ is called

- closed if $d\omega = 0$,
- exact if $\exists \eta \in \Omega^{k-1} : d\eta = \omega$.

remarks:

- Being 'closed' is a local property. Being 'exact' a global one. Since $d^2 = 0$, every exact form is closed. Whether the converse holds depends on the topology of M and will lead us to 'De Rham cohomology' ...

- For $M = \mathbb{R}^3$ with $\omega^0 \in \Omega^0 M$,

$$\omega^1 := f_1^1 \, dx + f_2^1 \, dy + f_3^1 \, dy \in \Omega^1 M$$

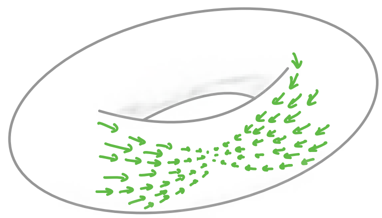
$$\omega^2 := f_1^2 \, dy \wedge dz + f_2^2 \, dz \wedge dx + f_3^2 \, dx \wedge dy \in \Omega^2 M$$

$$\omega^3 := f^3 \, dx \wedge dy \wedge dz$$

we have $\omega^0 \xmapsto{d} \omega^1 \xmapsto{d} \omega^2 \xmapsto{d} \omega^3$ is equal to

$$\omega^0 \xmapsto{\text{grad}} f^1 \xmapsto{\text{rot}} f^2 \xmapsto{\text{div}} f^3$$ (see exercise)

Vector fields



Def.: A vector field X on a smooth manifold M is a map

$$X : M \to TM, \quad M \ni p \mapsto X_p \in T_pM$$

The set of smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

remarks: • If (U,x) is a chart around p, we can write any vector field X locally as $X_p = \sum_i X_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$ where the X_i's are the component functions of X w.r.t. the chart.

Lemma: For a vector field X on a smooth M the following are equivalent:

(i) X is smooth.

(ii) The component functions of X are smooth (w.r.t. any chart).

(iii) For any $f \in C^\infty(M)$, the function $Xf : M \to \mathbb{R}$ defined by $M \ni p \mapsto X_pf$ is smooth.

remarks: • By (iii) any $X \in \mathfrak{X}(M)$ induces a linear operator $X : C^\infty(M) \to C^\infty(M)$. In fact, it is a linear derivation since $X(f \cdot g) = f \cdot Xg + g \cdot Xf$. Moreover, for $X, Y \in \mathfrak{X}(M)$:

$$X = Y \iff \forall f \in C^\infty(M) : Xf = Yf.$$

• By (ii) $\mathfrak{X}(M)$ is a $C^\infty(M)$-module.

Prop.: For $X, Y \in \mathfrak{X}(M)$ there exists a unique $Z \in \mathfrak{X}(M)$ satisfying $Zf = (X \circ Y - Y \circ X)f$ for any $f \in C^\infty(M)$.

Z is called the Lie bracket of X and Y, denoted by $Z =: [X, Y]$.

proof (sketch): $Zf = (X \circ Y - Y \circ X) f$ already defines Z. It remains to show that $Z \in \mathfrak{X}(M)$. This follows from observing that $Z_p f := (Zf)(p)$ is of the form

$$Z_p = \sum_i (X Y_i - Y X_i)(p) \left.\frac{\partial}{\partial x_i}\right|_p \quad \text{w.r.t. a chart } (U, x).$$

(see exercise for details) $\square$

remarks:
- I.g., $X \circ Y$ and $Y \circ X$ are not in $\mathfrak{X}(M)$.
- The Lie bracket $[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ makes $\mathfrak{X}(M)$ a Lie algebra.

A differential form $\omega \in \Omega^k M$ can now be regarded as a map

$$\omega : \mathfrak{X}(M)^k = \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

$$\omega(X_1, \dots, X_k) \mapsto \left(M \ni p \mapsto \omega_p(X_{1,p}, \dots, X_{k,p}) \right)$$

This leads to a chart-independent formula for the exterior derivative:

Prop.: If $\omega \in \Omega^k M$ and $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$, then:

omitted

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \left(\omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) \right)$$
$$+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1} \right)$$

In particular, for $\omega \in \Omega^1 M$: $\boxed{d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])}$

<u>proof (sketch):</u> First, one verifies that the r.h.s. is a $k+1$-form: it is alternating and C^∞-linear (the latter requires the second summand).

Then it suffices to show that it acts correctly on $\omega = f\, dx_1 \wedge \dots \wedge dx_k$ with $X_i = \frac{\partial}{\partial x_{\alpha_i}} =: \partial_{\alpha_i}$. Using $[\partial_i, \partial_j] = 0$, we get

$$\sum_{i=1}^{k+1} (-1)^{i+1} X_i\, \omega(\dots \hat{X}_i \dots) + \dots = \sum_{i=1}^{k+1} (-1)^{i+1} \partial_{\alpha_i}\, \omega(\partial_{\alpha_1}, \dots, \hat{\partial}_{\alpha_i}, \dots \partial_{\alpha_{k+1}}).$$

For $\alpha_1 < \dots < \alpha_{k+1}$ this vanishes except for $(\alpha_1, \dots, \alpha_k) = (1, \dots, k)$ and $i = k+1$ and thus $\alpha_i \geq k+1$. So we can write

$$d\omega = \sum_{\alpha_1 < \dots < \alpha_{k+1}} d\omega(\partial_{\alpha_1}, \dots, \partial_{\alpha_{k+1}})\, dx_{\alpha_1} \wedge \dots \wedge dx_{\alpha_k}$$

$$\overset{\text{assumption}}{=} \sum_{j > k} (-1)^k \partial_j f\, dx_1 \wedge \dots \wedge dx_k \wedge dx_j$$

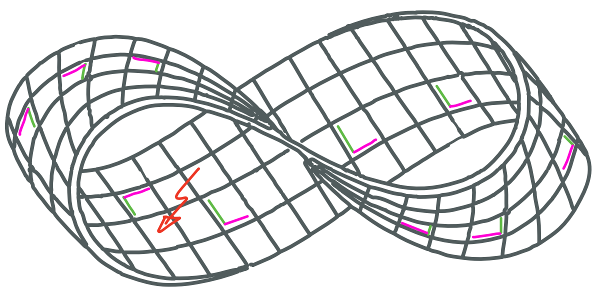
$$= \sum_j \frac{\partial}{\partial x_j} f\, dx_j \wedge dx_1 \wedge \dots \wedge dx_k$$, which is the correct form.

□

Orientation

Def.: Two ordered bases $b_1, \dots, b_n$ and $c_1, \dots, c_n$ of a real vector space V are said to have the same orientation if the automorphism $A: V \to V$ defined by $A b_i = c_i$ satisfies $\det(A) > 0$. Each of the two equivalence classes under this relation is called an orientation of V.

The two orientations are sometimes called right-/lefthanded and the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ is referred to as right-handed.



Consistent definition of an orentation on a manifold is not always possible (e.g. the Moebius strip is not orientable).

Def.: A smooth manifold M of dim. $n \geq 1$ is called orientable if one (and then both) of the following equivalent statements hold(s):

(i) There is an atlas $\mathcal{A} = \{(U_\lambda, \varphi_\lambda)\}_\lambda$ whose charts are orientation compatible in the sense that $\det\left(d_p(\varphi_\lambda \circ \varphi_\mu^{-1})\right) > 0 \quad \forall p \in \varphi_\lambda(U_\lambda) \cap \varphi_\mu(U_\mu)$.

(ii) There is a nowhere vanishing $\omega \in \Omega^n M$ (i.e., $\omega_p \neq 0 \ \forall p \in M$). ω is then called an orientation form.

remarks: • two orient. forms $\omega, \tilde{\omega} \in \Omega^n M$ must be related via $\tilde{\omega} = f \cdot \omega$ by a nowhere vanishing $f = C^\infty(M)$. If $f > 0$, we set $\tilde{\omega} \sim \omega$.

The resulting equivalence class $[\omega]$ is then called an orientation of M.

A connected, orientable manifold then has two orientations.

- Using homology, (i) can be extended to a definition of orientability of topological manifolds.

<u>proof</u>: (of the equivalence)

(ii) $\Rightarrow$ (i) Let $\omega \in \Omega^n M$ be an orient. form. Then w.r.t. a chart (U, x) around p: $\omega_p = f(p)\, dx_1 \wedge \dots \wedge dx_n$ for some $f \in C^\infty(U)$ that satisfies $\omega_p\left(\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p\right) = f(p) \neq 0$.

W.l.o.g. $f(p) > 0$ (otherwise replace x_1 by $-x_1$).

If (V, y) is another chart around p with $\omega_p = g(p)\, dy_1 \wedge \dots \wedge dy_n$ and $g(p) > 0$, then, in the intersection $U \cap V$:

$$f\, dx_1 \wedge \dots \wedge dx_n = g\, dy_1 \wedge \dots \wedge dy_n = g \det\left(\frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_n$$

so that $\det\left(\frac{\partial y_i}{\partial x_j}\right) = \frac{f}{g} > 0$. In this way, we can construct an atlas with orient. compatible charts.

(i) $\Rightarrow$ (ii) For each chart $(U_\lambda, x^\lambda) \in \mathcal{A}$ define $\omega^\lambda := dx_1^\lambda \wedge \dots \wedge dx_n^\lambda$.

Let $\{ \varphi_\lambda \in C^\infty(M, [0,1]) \}$ be a partition of unity subordinate to $\{U_\lambda\}$ and define $\omega := \sum_\lambda \varphi_\lambda \omega^\lambda$.

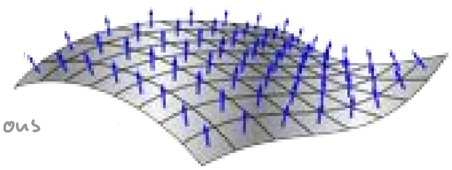
Every $p \in M$ has a neighborhood in which this sum is finite and using coordinate transformations we can express

$$\omega = \sum_\lambda \varphi_\lambda\, dx_1^\lambda \wedge \dots \wedge dx_n^\lambda = \underbrace{\sum_\lambda \varphi_\lambda \det\left(\frac{\partial x_i^\lambda}{\partial x_j^1}\right)}_{> 0 \text{ near } p}\, dx_1^1 \wedge \dots \wedge dx_n^1 .$$

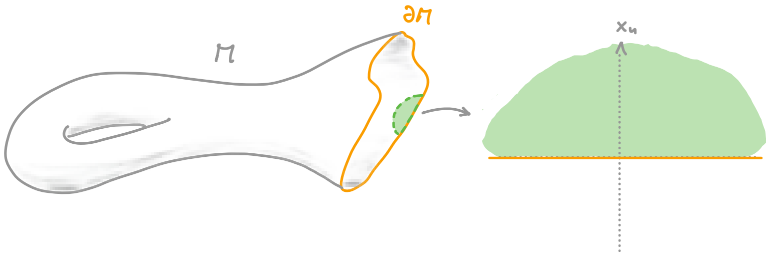
□

<u>remarks:</u>

- W.r.t. a given orientation form ω we call an ordered basis $(b_1, \dots, b_n)$ of T_pM 'positively oriented' if $\omega(b_1, \dots, b_n) > 0$.
- A smooth map between oriented manifolds is called orientation preserving if it maps positively oriented bases to positively oriented bases.
- To every point of a zero-dim. manifold we also assign two orientations, denoted $+1$ and -1.
- $\mathbb{R}P^n$ is orientable iff n is odd.
- An n-dim submanifold of $\mathbb{R}^{n+1}$ is orientable if there is a continuous vector field of 'unit normal vectors'. E.g. S^n is orientable.



Def.: A topological manifold with boundary M is a second-countable Hausdorff space that is locally homeomorphic to a half space $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Its boundary ∂M is the set of all points in M that are mapped onto $\partial \mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$. $\mathrm{Int}(M) := M \setminus \partial M$.



M is a smooth manifold with boundary if it is additionally equipped with a smooth structure. (In this context, a map on a subset $U \subseteq \mathbb{H}^n$ is called smooth if it has a smooth extension to a neighborhood of U that is open in $\mathbb{R}^n$.)

examples:

- Every (smooth) manifold is a (smooth) manifold with boundary, albeit $\partial M = \emptyset$. A compact manifold with empty boundary is called closed manifold.
- $M := \{ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1 \}$ with $\partial M = S^{n-1}$
- If $f: N \to \mathbb{R}$ is smooth with regular value $y \in \mathbb{R}$, then $\{ x \in N \mid f(x) \leq y \} =: M$ is a smooth manifold with boundary $\partial M = f^{-1}(\{y\})$.

remark: If M, N are two smooth manifolds with boundary and $f: M \to N$ is a diffeomorphism, then $f(\partial M) = \partial N$ and $f|_{\partial M} : \partial M \longrightarrow \partial N$ is again a diffeomorphism.

Prop.: If M is a smooth manifold with boundary $\partial M \neq \emptyset$, then:

(i) ∂M is a smooth manifold with $\dim(\partial M) = \dim(M) - 1$ and $\partial(\partial M) = \emptyset$.

(ii) ∂M is orientable if M is.

proof: (i) (sketch): If $(U, (x_1, \dots, x_n))$ is a chart around $p \in \partial M$ s.t. U is homeomorphic to an open subset of $\mathbb{H}^n$, then

$$U \cap \partial M = \{ p \in U \mid x_n(p) = 0 \}$$

and $(U \cap \partial M, (x_1, \dots, x_{n-1}))$ is a chart of ∂M ...

(ii) Let (U, x) and (V, y) be two orientation compatible charts of M around $p \in \partial M$ s.t. $x_n \geq 0$ in U and $y_n \geq 0$ in V. Since the coordinate change $\varphi := y \circ x^{-1}$ has to preserve the boundary, we have:

$$\varphi_n(x_1, \dots, x_n) \begin{cases} = 0 & \text{if } x_n = 0, \\ > 0 & \text{if } x_n > 0. \end{cases}$$

So $$\partial_i \varphi_n(x_1, \dots, x_{n-1}, 0) \begin{cases} = 0 & \text{for } i < n \\ \geq 0 & \text{for } i = n \end{cases}.$$

Hence, evaluated at a boundary point, we get:

$$\underset{\substack{\uparrow \\ \text{orientation} \\ \text{comp. charts}}}{0 <} \det\left(\partial_i \varphi_j\right)_{i,j=1}^{n} = \det\begin{pmatrix} \left(\partial_i \varphi_j\right)_{i,j=1}^{n-1} & 0 \\ * & \partial_n \varphi_n \end{pmatrix}$$

$$= \underbrace{\left(\partial_n \varphi_n\right)}_{\geq 0} \cdot \det\left(\partial_i \varphi_j\right)_{i,j=1}^{n-1}$$

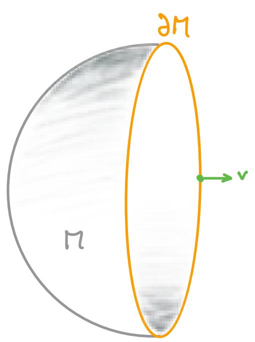
Consequently, the coordinate change $(\varphi_1, \dots, \varphi_{n-1})$ between the boundary charts is orientation-preserving as well. $\square$

<u>Def.:</u> Let $[\omega]$ be an orientation of a smooth manifold M with boundary $\partial M \neq \emptyset$. If w.r.t. a chart (U, x) of M around $p \in \partial M$ we have $\omega = f \, dx_1 \wedge \dots \wedge dx_n$ for some $f > 0$, then the induced orientation $[\eta]$ of ∂M is defined locally via

$$\eta := (-1)^n \, dx_1 \wedge \dots \wedge dx_{n-1}$$

<u>remarks:</u>

- These locally defined η's can then be glued together to a $(n-1)$-form η that is an orientation form on all of ∂M.
- According to ω, the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \in T_pM$ is positively oriented. At $p \in \partial M$ we can regard $v := -\frac{\partial}{\partial x_n}$ as outward pointing vector. An ordered basis $v_1, \dots, v_{n-1}$ of $T_p\partial M$ is then positively oriented w.r.t. η if $v, v_1, \dots, v_{n-1}$ is positively oriented w.r.t ω since



$$d(-x_n) \wedge \eta = (-1)^n \cdot d(-x_n) \wedge dx_1 \wedge \dots \wedge dx_{n-1}$$

$$= dx_1 \wedge \dots \wedge dx_n \, .$$

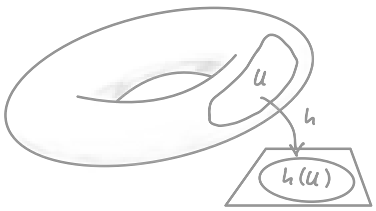
Integration of n-forms on n-dim. manifolds

Def.: • The support of $\omega \in \Omega^n M$ is $\mathrm{supp}(\omega) := \overline{\{ p \in M \mid \omega_p \neq 0 \}}$

(i.e. its complement is the largest open subset of M on which $\omega = 0$)

• Let (U, h) be a chart of an n-dim. smooth manifold (possibly with boundary), and $\omega \in \Omega^n M$.

For $p \in U$ let $f(p) := \omega\left(\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p \right) \in \mathbb{R}$ define the component function of ω, i.e. $\omega_p = f(p)\, dx_1 \wedge \dots \wedge dx_n$. Then



$$\boxed{\int_U \omega := \underbrace{\int_{h(U)} f \circ h^{-1}(x)\, dx}_{\text{Lebesgue integral in } \mathbb{R}^n}}$$

no 1-form! merely a symbol in Lebesgue-integral

if the Lebesgue integral on the r.h.s. exists.

Lemma: Two orientation-compatible charts (U, h) and $(U, \tilde{h})$ lead to the same value of $\int_U \omega$.

proof: If $\omega_p = \tilde{f}(p)\, dy_1 \wedge \dots \wedge dy_n$ then $\tilde{f}(p) = f(p)\, \det \underbrace{\left(\frac{\partial}{\partial y_i}\Big|_p x_j \right)}_{=: J_\varphi}$

where J_φ is the Jacobian of the coordinate change $\varphi := h \circ \tilde{h}^{-1}$ at $\tilde{h}(p)$.

$$\int_{\tilde{h}(U)} \tilde{f} \circ \tilde{h}^{-1}(y)\, dy \underset{\varphi := h \circ \tilde{h}^{-1}}{=} \int_{\tilde{h}(U)} f \circ h^{-1} \circ \varphi(y)\, \underbrace{|\det(J_\varphi(y))|}_{>0 \text{ due to orientation compatibility}}\, dy$$

$$= \int_{h(U) = \varphi \circ \tilde{h}(U)} f \circ h^{-1}(x)\, dx$$

change of variable formula for Lebesgue integral

$\square$

Now suppose $\{U_\lambda\}_\lambda$ is a finite open covering of M with orientation compatible charts and $\{\varphi_\lambda \in C^\infty(U_\lambda, [0,1])\}_\lambda$ is a smooth partition of unity subordinate to it. Then

$$\boxed{\int_M \omega := \sum_\lambda \int_{U_\lambda} \varphi_\lambda \cdot \omega}$$

Lemma: The integral $\int_M \omega$ is independent of the chosen covering and partition of unity.

(as long as it is a finite covering with orient. comp. charts.)

proof: Let $\{\tilde{U}_\mu\}_\mu$ be another such covering and $\{\tilde{\varphi}_\mu\}$ a corresponding partition of unity. Then

$$\sum_\lambda \int_{U_\lambda} \varphi_\lambda \cdot \omega = \sum_\lambda \int_{U_\lambda} \sum_\mu \tilde{\varphi}_\mu \cdot \varphi_\lambda \cdot \omega$$

$$= \sum_\mu \sum_\lambda \int_{U_\lambda \cap \tilde{U}_\mu} \tilde{\varphi}_\mu \cdot \varphi_\lambda \cdot \omega = \sum_\mu \int_{\tilde{U}_\mu} \sum_\lambda \varphi_\lambda \cdot \tilde{\varphi}_\mu \cdot \omega \quad \Big| \text{ using finiteness}$$

$$= \sum_\mu \int_{\tilde{U}_\mu} \tilde{\varphi}_\mu \cdot \omega . \qquad \square$$

To summarize, we have defined integrals of n-forms on n-dim. manifolds under the assumption that the manifold is oriented (i.e. we chose an atlas with orient. comp. charts) and the n-form has compact support (wich is automatically satisfied if M is compact). The latter could be relaxed in principle, but the central theorem (Stokes' thm.) would still require compact support.

Elementary properties:

Linearity: $$\boxed{\int_M (a\omega + b\eta) = a\int_M \omega + b\int_M \eta}$$ for $a,b \in \mathbb{R}$, $\omega, \eta \in \Omega^n M$

Orientation dependence: $$\boxed{\int_{-M} \omega = -\int_M \omega}$$ if "$-M$" is M with opposite orientation

<u>Prop.:</u> If $\varphi: M \to N$ is an orientation preserving diffeomorphism, $A \subseteq M$, $n := \dim(M)$, and $\omega \in \Omega^n N$, then:

$$\boxed{\int_A \varphi^* \omega = \int_{\varphi(A)} \omega}$$

(meaning that one side is well-defined iff the other side is, in which case they are equal)

The proof follows again by realizing that the change of variables formula for the Lebesgue integral corresponds to

$$\varphi^*(f \cdot dy_1 \wedge \ldots \wedge dy_n) = (f \circ \varphi) \cdot \det\left(\frac{\partial}{\partial x_j} y_i \circ \varphi\right) dx_1 \wedge \ldots \wedge dx_n .$$

All this extends to the case of 0-forms (i.e. functions) over an oriented 0-dim. manifold M, when we define $\boxed{\int_M f := \sum_{p \in M} \sigma(p)\, f(p)}$, where $\sigma(p) \in \{\pm 1\}$ is the orientation at p.

This sum is finite if f is compactly supported.

Stokes' theorem

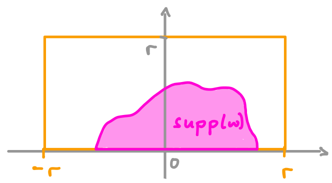
Thm.: [Stokes] Let M be an n-dim. oriented smooth manifold with boundary ∂M and $\omega \in \Omega^{n-1}M$ have compact support. Then

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

explanation concerning the r.h.s.: ∂M is supposed to be equipped with the 'induced' orientation and ω is understood as $\iota^*\omega$ with $\iota : \partial M \to M$ the inclusion map. If $\partial M = \emptyset$, the r.h.s. is zero.

proof: We will consider three increasingly general cases that are based on each other:

(i) $M = \mathbb{H}^n$. There is an $r > 0$ s.t.



$\operatorname{supp}(\omega) \subseteq [-r,r]^{n-1} \times [0,r]$ and we can write $\omega = \sum_{i=1}^{n} f_i \, dx_1 \wedge \ldots \wedge \underbrace{\widehat{dx_i}}_{\text{omitted}} \wedge \ldots \wedge dx_n$.

Then
$$d\omega = \sum_{i=1}^{n} \underbrace{df_i}_{\sum_j \frac{\partial}{\partial x_j} f_i \, dx_j} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots dx_n$$
$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n$$

So
$$\int_M d\omega = \sum_{i=1}^{n} (-1)^{i-1} \int_0^r \int_{-r}^r \cdots \int_{-r}^r \frac{\partial f_i}{\partial x_i} dx_1 \cdots dx_n$$

For $i \neq n$ we have $\int_{-r}^{r} \frac{\partial f_i}{\partial x_i} dx_i \underset{\substack{\uparrow \\ \text{fund. thm. calc.}}}{=} f_i \Big|_{x_i=-r}^{x_i=r} = 0$ since f_i vanishes if $x_i = \pm r$. Hence,

$$\int_M d\omega = (-1)^{n-1} \int_{-r}^{r} \cdots \int_{-r}^{r} f_n \Big|_{x_n=0}^{x_n=r} dx_1 \cdots dx_{n-1}$$
$$= (-1)^n \int_{-r}^{r} \cdots \int_{-r}^{r} f_n(x_1, \ldots, x_{n-1}, 0) \, dx_1 \cdots dx_{n-1}$$

This has to be compared with $\int_{\partial M} \omega = \int_{\partial M} \iota^* \omega$

Since every $(n-1)$-form on $\partial M = \partial \mathbb{H}^n$ is a C^∞-multiple of

$dx_1 \wedge \ldots \wedge dx_{n-1}$, we have $\iota^* \omega = f_n(x_1, \ldots, x_{n-1}, 0) \; dx_1 \wedge \ldots \wedge dx_{n-1}$

so that
$$\int_{\partial M} \omega = \int_{\partial M} f_n(x_1, \ldots, x_{n-1}, 0) \; dx_1 \wedge \ldots \wedge dx_{n-1}$$

$$= (-1)^n \int_{-r}^{r} \cdots \int_{-r}^{r} f_n(x_1, \ldots, x_{n-1}, 0) \; dx_1 \cdots dx_{n-1}$$

$(-1)^n dx_1 \wedge \ldots \wedge dx_n$ is the induced orientation.

Consequently, $\int_M d\omega = \int_{\partial M} \omega$ for $M = \mathbb{H}^n$.

(ii) Suppose ω is supported in the domain U of a single chart (U, φ) where φ is orientation preserving. Then

more details below

$$\int_M d\omega = \int_{\mathbb{H}^n} (\varphi^{-1})^* d\omega = \int_{\mathbb{H}^n} d\left((\varphi^{-1})^* \omega\right) \overset{(i)}{=} \int_{\partial \mathbb{H}^n} (\varphi^{-1})^* \omega = \int_{\partial M} \omega$$

ext. der. commutes with pullback $\quad (\varphi^{-1})^* d\omega$ has compact supp.

(iii) Suppose $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ is an atlas of orientation compatible charts that define the orientation of M. If $\{\psi_\lambda \in C^\infty(U_\lambda, [0,1])\}_{\lambda \in \Lambda}$ is a corresponding smooth partition of unity, then:

$$\int_{\partial M} \omega = \sum_\lambda \int_{\partial M} \psi_\lambda \omega \overset{(ii)}{=} \sum_\lambda \int_M d(\psi_\lambda \omega)$$

$$= \sum_\lambda \int_M d\psi_\lambda \wedge \omega + \psi_\lambda d\omega$$

$$\overset{\text{linearity}}{=} \underbrace{\int_M d\Big(\underbrace{\sum_\lambda \psi_\lambda}_{=1}\Big) \wedge \omega}_{=0} + \int_M \underbrace{\sum_\lambda \psi_\lambda}_{=1} d\omega = \int_M d\omega .$$

$\square$

remark: for a more detailed discussion suppose (U,φ) with $\varphi = (\varphi_1, \dots, \varphi_n)$ is the considered chart of M, $(U \cap \partial M, \tilde{\varphi})$ with $\tilde{\varphi} = (\varphi_1, \dots, \varphi_{n-1})$ the boundary chart of ∂M and $\iota : \partial M \to M$, $\tilde{\iota} : \partial \mathbb{H}^n \to \mathbb{H}^n$ the inclusion maps. Then with $\varphi^{-1} \circ \tilde{\iota} = \iota \circ \tilde{\varphi}^{-1}$ we get:

$$\int_{\mathbb{H}^n} d(\varphi^{-1})^* \omega \overset{(i)}{=} \int_{\partial \mathbb{H}^n} \tilde{\iota}^* (\varphi^{-1})^* \omega = \int_{\partial \mathbb{H}^n} (\tilde{\varphi}^{-1})^* \iota^* \omega = \int_M \iota^* \omega .$$

Corollary: If M is a closed (= compact & boundary less), orientable smooth n-dim. manifold and $\omega \in \Omega^n M$ is exact, then $\int_M \omega = 0$.

proof:

$$\int_M \omega \underset{\omega = d\eta}{=} \int_M d\eta \underset{\text{Stokes}}{=} \int_{\partial M} \eta = 0 \quad \text{since } \partial M = \emptyset . \qquad \square$$

Corollary: If M is a compact, orientable smooth n-dim manifold and $\omega \in \Omega^{n-1} M$ is closed, then $\int_{\partial M} \omega = 0$.

proof:

$$\int_{\partial M} \omega \underset{\text{Stokes}}{=} \int_M d\omega \underset{d\omega = 0}{=} 0 . \qquad \square$$

Corollary: [Fund. thm. for line integrals] Let $\gamma : [a,b] \to N$ be a smooth curve s.t. $M := \gamma([a,b])$ is a 1-dim. submanifold of N and $\gamma : [a,b] \to M$ is an orientation preserving diffeomorphism. Then for any $f \in C^\infty(N)$: $\int_M df = f(\gamma(b)) - f(\gamma(a))$

proof:

$$\int_M df = \int_{\partial M} f \quad \text{with } \partial M = \{ \underset{\text{negative}}{\gamma(a)}, \underset{\text{positive}}{\gamma(b)} \} \qquad \square$$

negative / positive orientation

<u>Thm.:</u> [No retraction thm.]

Let M be a compact, oriented smooth manifold with boundary $\partial M \neq \emptyset$. There is no smooth map $f: M \to \partial M$ s.t. $f|_{\partial M} = id$.

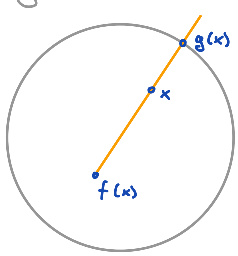
<u>proof:</u> Let $n := \dim(M)$ and $\eta \in \Omega^{n-1}\partial M$ be s.t. $\int_{\partial M} \eta \neq 0$ (e.g. an orientation form on ∂M). Then with the inclusion $\iota: \partial M \to M$ and an assumed retraction $f: M \to \partial M$ s.t. $f \circ \iota = id$:

$$\int_{\partial M} \eta = \int_{\partial M} \iota^* f^* \eta \underset{\text{Stokes}}{=} \int_M d(f^*\eta) = \int_M f^* d\eta \underset{d\eta \in \Omega^n \partial M = \{0\}}{=} 0 \quad \lightning \qquad \square$$

<u>Corollary:</u> [Brouwer's fixed point thm - smooth version]

Consider $M := \{ x \in \mathbb{R}^n \mid \|x\|_2 \leq 1 \}$ with $\partial M = S^{n-1}$ and a smooth map $f: M \to M$. f has a fixed point (i.e. $\exists x \in M: f(x) = x$).

<u>proof:</u> Suppose there is no fixed point. Then define $g: M \to \partial M$ s.t. $g(x) := x + t(x - f(x))$ for a suitable $t > 0$ depending on x.



Then g would be a smooth retraction. $\lightning$

$\square$

<u>remark:</u> using Weierstrass approximation this can be extended to continuous functions $f: M \to M$ on any top. space M that is homeomorphic to a closed ball.

Vector analysis in $\mathbb{R}^3$

To recover theorems of vector analysis in $\mathbb{R}^3$ from the generalized Stokes' thm. we can use the following definitions & conventions:

Let $U \subseteq \mathbb{R}^3$ be open and $\mathcal{V} := C^\infty(U, \mathbb{R}^3)$. On U define the vector-valued forms $d\vec{s} := \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$ $\quad d\vec{F} := \begin{pmatrix} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{pmatrix}$

and $dV := dx_1 \wedge dx_2 \wedge dx_3$. These lead to the following isomorphisms:

$$\mathcal{V} \xrightarrow{\simeq} \Omega^1 U, \qquad \vec{a} \mapsto \vec{a} \cdot d\vec{s}$$

$$\mathcal{V} \xrightarrow{\simeq} \Omega^2 U, \qquad \vec{b} \mapsto \vec{b} \cdot d\vec{F}$$

$$C^\infty(U) \longrightarrow \Omega^3 U, \qquad c \mapsto c\, dV$$

Then Stokes' thm. for differential forms translates to:

Gauss' divergence thm.: For any $\vec{b} \in \mathcal{V}$ and any compact 3-dim. submanifold M of U with boundary ∂M:

$$\boxed{\int_M \operatorname{div} \vec{b}\; dV = \int_{\partial M} \vec{b} \cdot d\vec{F}}$$

Kelvin-Stokes thm.: For any $\vec{a} \in \mathcal{V}$ and any compact, oriented 2-dim. submanifolds $M \subseteq U$ with boundary ∂M:

$$\boxed{\int_M \operatorname{rot} \vec{a} \cdot d\vec{F} = \int_{\partial M} \vec{a} \cdot d\vec{s}}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\Omega^0 U \arrow[r, "d"] & \Omega^1 U \arrow[r, "d"] & \Omega^2 U \arrow[r, "d"] & \Omega^3 U \\
C^\infty(U) \arrow[u, "="] \arrow[r, "\mathrm{grad}"] & \mathcal{V} \arrow[u, "\simeq"] \arrow[r, "\mathrm{rot}"] & \mathcal{V} \arrow[u, "\simeq"] \arrow[r, "\mathrm{div}"] & C^\infty(U) \arrow[u, "\simeq"]
\end{tikzcd}$$

In particular, $d^2 = 0$ translates to $\mathrm{rot}\,\mathrm{grad}\, f = 0$ and $\mathrm{div}\,\mathrm{rot}\, \vec{a} = 0$.

Riemannian & Lorentzian manifolds

Recall from Linear Algebra: If $g: V\times V \to \mathbb{R}$ is a symmetric, non-degenerate* bilinear form on a finite dim. real vector space V with basis $b_1, \dots, b_n \in V$, then $(g(b_i, b_j))_{i,j=1}^n$ is an invertible matrix. By Sylvester's law of inertia the number $s \in \{0, \dots, n\}$ of negative eigenvalues is independent of the basis. We call s the index of g. Note that g is an inner product iff $s=0$.

*this means: $g(x,y)=0 \ \forall x \Rightarrow y=0$

Def.: Let M be a smooth manifold and $s \in \{0, \dots, \dim(M)\}$.

A pseudo-Riemannian metric of index s on M is an assignment of a symmetric, nondegenerate, bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ of index s to every point $p \in M$, s.t. in any chart

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p \right) \text{ depends smoothly on } p.$$

(M, g) is then called pseudo-Riemannian manifold of index s and for

$$s = \begin{cases} 1 < \dim(M) : & \text{Lorentzian manifold} \\ 0 : & \text{Riemannian manifold} \end{cases}$$

remarks: ◦ Note that if $X_p = \sum_i x_i \frac{\partial}{\partial x_i}\Big|_p$ and $Y_p = \sum_i y_i \frac{\partial}{\partial x_i}\Big|_p$, then

$$g_p(X_p, Y_p) = \sum_{ij} x_i \, g_{ij}(p)\, y_j = \langle x, g(p) y \rangle .$$

◦ A common notation is ds^2 for the bilinear form g_p. This, in turn, leads to expressions of the form "$ds^2 = \sum_{ij} g_{ij}\, dx_i\, dx_j$".

examples: • The Minkowski space $M = \mathbb{R}^4$ with constant Minkowski metric $(g_{ij}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ w.r.t. the canonical basis of $\mathbb{R}^4$ is a simple Lorentzian manifold.

• $\mathbb{R}^n$ with the standard inner product is a Riemannian manifold.

Lemma: Let $F: M \to N$ be smooth and s.t. d_pF is injective for all $p \in M$. If (N, g) is Riemannian, then so is (M, F^*g).

remarks: • The pullback for symmetric bilinear forms is defined in the same way as for anti-symmetric — " — " .

• Injectivity of d_pF holds in particular for embeddings.

proof: $(F^*g)_p(v,v) = g_{F(p)}(d_pFv, d_pFv) \geq 0$

and $\ldots = 0 \overset{g \text{ R. metric}}{\Longleftrightarrow} d_pFv = 0 \underset{+\text{ linear}}{\overset{d_pF \text{ inj.}}{\Longleftrightarrow}} v = 0$ $\square$

Corollary: For every smooth manifold there exists a Riemannian metric.

proof: By Whitney's embedding thm. there is an embedding $F: M \to \mathbb{R}^{2n}$. If g is the standard inner product on $\mathbb{R}^{2n}$, then F^*g is a Riemannian metric on M. $\square$

remark: an alternative proof would construct a Riem. metric locally within any single chart of an atlas and then exploit a partition of unity together with convexity of the space of inner products.

Having a manifold equipped with a Riemannian metric has two immediate benefits:

1. We can talk about distances
2. We can identify T_pM with T_p^*M and thus $\mathfrak{X}(M)$ with $\Omega^1 M$.

1: **Def.:** Let (M,g) be a Riemannian manifold.

- The length of a curve $\gamma \in C^1([a,b],M)$ is defined as

$$L(\gamma) := \int_a^b \underbrace{\left[g_{\gamma(t)}\left(\dot\gamma(t), \dot\gamma(t)\right)\right]^{\frac{1}{2}}}_{} \, dt$$

$= \|\dot\gamma(t)\|$ where $\dot\gamma(t) \in T_{\gamma(t)}M$ is s.t.
$\dot\gamma(t) f := (f \circ \gamma)'(t)$ for $f \in C^\infty(M)$

This extends to piecewise-C^1 curves by summing up the lengths of the pieces.

- The distance between $x, y \in M$ is defined as

$$d_g(x,y) := \inf\left\{ L(\gamma) \;\middle|\; \gamma \text{ is piecewise } C^1 \text{ \& connects } x \text{ and } y \right\}$$

remark: $L(\gamma)$ is independent of the parametrization of γ and given in local coordinates by $\int_a^b \left[\sum_{ij} g_{ij}(\gamma(t)) \, (x_i \circ \gamma)'(t) \, (x_j \circ \gamma)'(t) \right]^{1/2} dt$

Thm.: If (M,g) is a connected Riemannian manifold, then (M, d_g) is a metric space whose metric topology coincides with the manifold topology of M.

2: Any pseudo-Riemannian metric g induces an isomorphism

$$\psi : T_pM \longrightarrow T_p^*M \,, \quad v \longmapsto g_p(v, \cdot)$$

(note that ψ is a linear map that is injective since $\psi(v) = 0 \Rightarrow g_p(v,x) = 0$ for all x $\Rightarrow v = 0$. As $\dim(T_pM) = \dim(T_p^*M)$, ψ is an isomorphism.)

Applying this pointwise we get an isomorphism between $\mathfrak{X}(M)$ and $\Omega^1(M)$.

E.g. if $f \in C^\infty(M)$ we can assign a vector field to $df \in \Omega^1(M)$, which then defines the gradient $\boxed{\operatorname{grad}(f) := \psi^{-1} df \in \mathfrak{X}(M)}$.

ψ also allows us to define a (pseudo-) inner product on T_p^*M via

$$T_p^*M \times T_p^*M \ni (\omega, \eta) \mapsto g_p\left(\psi^{-1}(\omega), \psi^{-1}(\eta)\right)$$

Pointwise application yields: $\langle \cdot , \cdot \rangle : \Omega^1 M \times \Omega^1 M \to C^\infty(M)$

$$\langle \omega, \eta \rangle := \left(p \mapsto g_p\left(\psi^{-1}(\omega_p), \psi^{-1}(\eta_p)\right)\right)$$

This can be extended to k-forms:

<u>Def.:</u> For a pseudo-Riemannian manifold (M,g) we define $\langle \cdot , \cdot \rangle : \Omega^k M \times \Omega^k M \longrightarrow C^\infty(M)$ pointwise by bilinear extension of

$$\boxed{\langle \alpha_1 \wedge \ldots \wedge \alpha_k , \beta_1 \wedge \ldots \wedge \beta_k \rangle := \det\left(g_p\left(\psi^{-1}\alpha_i , \psi^{-1}\beta_j\right)\right)}$$

for $\alpha_i, \beta_j \in T_p^*M$.

Prop.: Let (M,g) be an oriented Riemannian manifold. There is a unique orientation form ν s.t. for any positively oriented ONB $v_1,\dots,v_n \in T_pM$: $\boxed{\nu_p(v_1,\dots,v_n) = 1}$

In local coordinates this Riemannian volume form has the form

$$\boxed{\nu_p = \sqrt{\det\left((g_{ij}(p))_{i,j}\right)}\; dx_1 \wedge \dots \wedge dx_n}$$

remark: In the literature this is often written $\nu \equiv dV$ or $dVol_M$. This should not mislead you to think that it is an exact form.

proof: In a positively oriented chart we can write $v_i = \sum_k B_{ik} \frac{\partial}{\partial x_k}\Big|_p$ where orthogonality means $\delta_{ij} = g_p(v_i,v_j) = \sum_{kl} B_{ik}\, g_{kl}(p)\, B_{jl}$ and thus $\mathbb{1} = BGB^T$ with $G := (g_{kl}(p))_{k,l=1}^n$.

Consequently, $\det(B) = \frac{1}{\sqrt{\det(G)}}$ and this holds for any positively oriented ONB since these are related like $\tilde{B} = O \cdot B$ via $O \in SO(n)$.

Every orientation form has the form $\nu_p = f(p)\, dx_1 \wedge \dots \wedge dx_n$ in local coordinates. So $\nu_p(v_1,\dots,v_n) = f(p) \det\Big(\underbrace{(dx_i(v_j))}_{=B}\Big)$ s.t. $f(p) = \sqrt{\det(G)}$ is necessary for the claim.

To show that this gives a globally well-defined orientation form we have to show consistency of the definition over different charts. So consider a different chart given by $\tilde{x}$ at p. Then $G = S^T \tilde{G} S$ where $S_{kl} := \frac{\partial \tilde{x}_k}{\partial x_l}\Big|_p$ and $\sqrt{\det(\tilde{G})}\; d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_n =$

$= \sqrt{\det(\tilde{G})}\, \det(S)\; dx_1 \wedge \dots \wedge dx_n = \sqrt{\det(G)}\; dx_1 \wedge \dots \wedge dx_n$. $\square$

Thm.: A smooth manifold M admits a Lorentzian metric iff there exist a nowhere vanishing vector field $X \in \mathfrak{X}(M)$.

proof: $\rightarrow$ exercise class ...

Corollary: For $n \in \mathbb{N}$ even, there is no Lorentzian metric on S^n.

proof: According to the 'hairy ball thm.' S^n does not admit a non-vanishing smooth vector field if $n \in 2\mathbb{N}$. $\square$

Hodge theory

If $\dim(M) = n$, then $\dim\left(\Lambda^k T_p^* M\right) = \binom{n}{k} = \binom{n}{n-k} = \dim\left(\Lambda^{n-k} T_p^* M\right)$ so that the spaces are isomorphic vectorspaces. If (M,g) is an oriented Riemannian manifold, there is a natural isomorphism given by the Hodge star operator $* : \Omega^k M \longrightarrow \Omega^{n-k} M$ that is defined pointwise as follows: Let $\theta_1, \dots, \theta_k, \theta_{k+1}, \dots, \theta_n$ a positively oriented ONB (w.r.t. the inner product induced by g) of $T_p^* M$. Then a linear map $* : \Lambda^k(T_p^* M) \longrightarrow \Lambda^{n-k}(T_p^* M)$ is defined by setting

$$\boxed{*\left(\theta_1 \wedge \dots \wedge \theta_k\right) = \theta_{k+1} \wedge \dots \wedge \theta_n}$$

So if $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} \, \theta_{i_1} \wedge \dots \wedge \theta_{i_k}$ then

$$*\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} \operatorname{sgn}(I,J) \, \theta_{j_1} \wedge \dots \wedge \theta_{j_{n-k}}$$

where $j_1 < \dots < j_{n-k}$ is the complement of $i_1 < \dots < i_k$ in $\{1, \dots, n\}$ and $\operatorname{sgn}(I,J)$ the sign of the permutation $(1, \dots, n) \mapsto (i_1, \dots, i_k, j_1, \dots, j_{n-k})$.

In this way, $*1 = \nu \in \Omega^n M$ is the Riemannian volume form.

Prop.: For any $f, g \in C^\infty(M)$ and $\omega, \eta \in \Omega^k M$ on an oriented Riem. M:

i) $*(f\omega + g\eta) = f(*\omega) + g(*\eta)$

ii) $**\omega = (-1)^{k(n-k)} \omega$

iii) $\eta \wedge *\omega = \boxed{\omega \wedge *\eta = \langle \omega, \eta \rangle \nu}$

Since both sides are non-degenerate bilinear, this uniquely characterizes (or defines) the Hodge-$*$ (in a basis-independent way)

iv) $*(\omega \wedge *\eta) = *(\eta \wedge *\omega) = \langle \omega, \eta \rangle$

v) $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$

<u>proof:</u> We can consider all identities pointwise (i.e. at a $p \in M$)

i) linearity holds by definition.

ii) If $\theta_1, \dots, \theta_n$ is a pos. oriented ONB of $T_p^* M$, then

$\omega_p = \theta_1 \wedge \dots \wedge \theta_k \Rightarrow * \omega_p = \theta_{k+1} \wedge \dots \wedge \theta_n$ and

$** \omega_p = \sigma \; \theta_1 \wedge \dots \wedge \theta_k$ where σ is the sign of the

permutation $(k+1, \dots, n, 1, \dots, k)$. So $\sigma = (-1)^{k(n-k)}$

iii) Due to linearity it suffices to consider $\eta_p = \theta_{i_1} \wedge \dots \wedge \theta_{i_k}$.

Then $* \eta_p = \operatorname{sgn}(I, J) \; \theta_{j_1} \wedge \dots \wedge \theta_{j_{n-k}}$ so that

$(\theta_1 \wedge \dots \wedge \theta_k) \wedge * \eta_p \neq 0$ only if $\{ i_1, \dots, i_k \} = \{ 1, \dots, k \}$ for which

$$\underbrace{\omega_p}_{} \wedge * \eta_p = \underbrace{\operatorname{sgn}(I, J)}_{= \operatorname{sgn}(I)} \; \underbrace{\theta_1 \wedge \dots \wedge \theta_k \wedge \theta_{k+1} \wedge \dots \wedge \theta_n}_{= \nu_p}$$

$J = (k+1, \dots, n)$ is not permuted

Here, $\operatorname{sgn}(I)$ is the sign of the permutation $(i_1, \dots, i_k)$.

On the other hand, $\langle \omega_p, \eta_p \rangle = \langle \theta_1 \wedge \dots \wedge \theta_k, \theta_{i_1} \wedge \dots \wedge \theta_{i_k} \rangle$

$$= \det \left(\langle \theta_i, \theta_{i_j} \rangle \right)_{i,j=1}^{k} = \operatorname{sgn}(I).$$

So, indeed, $\omega \wedge * \eta = \langle \omega, \eta \rangle \nu$ and using $\langle \omega, \eta \rangle = \langle \eta, \omega \rangle$

gives the second identity.

iv) $$* (\omega \wedge * \eta) \overset{(iii)}{=} * (\langle \omega, \eta \rangle \nu) \overset{(i)}{=} \langle \omega, \eta \rangle * \nu \overset{*\nu = *(*1) = 1}{=} \langle \omega, \eta \rangle = \langle \eta, \omega \rangle = \dots$$

v) $$\langle * \omega, * \eta \rangle \overset{(iv)}{=} * (* \omega \wedge * * \eta) \underset{(i)}{\overset{(ii)}{=}} (-1)^{k(n-k)} * (* \omega \wedge \eta)$$

$$= * (\eta \wedge * \omega) = \langle \eta, \omega \rangle .$$

□

Def.: For any $X \in \mathfrak{X}(M)$ on an oriented Riemannian manifold (M,g), the divergence is defined as $\boxed{\operatorname{div} X := * d * \psi(X)}$ where $\psi(X) \in \Omega^1 M$ is the 1-form associated to X by g.

remarks:

- $\operatorname{div}: \mathfrak{X}(M) \longrightarrow C^\infty(M)$
- On standard $\mathbb{R}^n$ we get for $X = \sum_i f_i(p) \frac{\partial}{\partial x_i}\Big|_p$

$\psi(X) = \sum_i f_i(p)\, dx_i$ so that

$$\begin{aligned}
\operatorname{div} X &= * d \sum_i f_i(p)\, (-1)^{i+1}\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\
&= * \sum_i \frac{\partial}{\partial x_i}\Big|_p f(p)\quad dx_1 \wedge \ldots \wedge dx_n \\
&= \sum_i \frac{\partial}{\partial x_i}\Big|_p f_i \quad \text{as expected.}
\end{aligned}$$

- On standard $\mathbb{R}^3$ we have $*(dx_j \wedge dx_k) = \sum_i \varepsilon_{ijk}\, dx_i$

Hence, $\omega = \sum_{i=1}^{3} f_i\, dx_i$ leads to

$$\begin{aligned}
* d\, \omega &= * \sum_{jk=1}^{3} \frac{\partial}{\partial x_j}\Big|_p f_k \; dx_j \wedge dx_k \\
&= \sum_{ijk=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_j}\Big|_p f_k \; dx_i \\
&= \sum_{i=1}^{3} (\operatorname{curl} f)_i \; dx_i
\end{aligned}$$

Alternative notations are $\operatorname{curl} f \equiv \operatorname{rot} f \equiv \nabla \times f$.

Note that for an n-dim. M we have $* d: \Omega^1 M \to \Omega^{n-2} M$

Def.: Let M be an oriented Riemannian manifold.

- If M is compact and $\nu \in \Omega^n M$ denotes the Riem. volume form, we define the inner product $(\cdot,\cdot) : \Omega^k M \times \Omega^k M \to \mathbb{R}$

$$\boxed{(\omega,\eta) := \int_M \langle \omega,\eta \rangle\, \nu = \int_M \omega \wedge *\eta} = \int_M \eta \wedge *\omega \quad \text{and}$$

  extend it to ΩM by setting $(\omega,\eta) := 0$ for forms of different degree.

- We define the adjoint exterior derivative $d^\dagger : \Omega^k M \to \Omega^{k-1} M$ as

$$\boxed{d^\dagger := (-1)^k *^{-1} d\, * = (-1)^{n(k+1)+1} * d\, *}$$

remarks:
- we write $(\omega,\eta) \in \mathbb{R}$ to distinguish from $\langle \omega,\eta \rangle \in C^\infty(M)$.
- Note that (ω,η) requires compact M or at least that the supports of ω and η have compact overlap.
- For a Lorentz manifold, $(\cdot,\cdot)$ would not be an inner product.
- The Hodge-$*$ is an isometry w.r.t. $(\cdot,\cdot)$ since $(*\omega, *\eta) = (\omega,\eta)$
- By definition the following diagram commutes:

$$\begin{tikzcd}
\Omega^k M \arrow[r, "*"] \arrow[d, "d^\dagger"'] & \Omega^{n-k} M \arrow[d, "d"] \\
\Omega^{k-1} M \arrow[r, "(-1)^k *"'] & \Omega^{n-k+1} M
\end{tikzcd}$$

- This implies $* d^\dagger = (-1)^k d\, *$, and $d^\dagger d^\dagger = 0$

- The name 'adjoint' is justified due to:

Prop.: d and $d^\dagger$ are mutual adjoints w.r.t. $(\cdot,\cdot)$. That is, $\forall \omega,\eta \in \Omega M$:

$$\boxed{(d\omega,\eta) = (\omega, d^\dagger \eta)}\,.$$

proof: Suppose $\omega \in \Omega^k M$, $\eta \in \Omega^{k+1} M$. Then

$$d\omega \wedge *\eta = d(\omega \wedge *\eta) - (-1)^k \omega \wedge d*\eta = d(\omega \wedge *\eta) + \omega \wedge *d^\dagger \eta$$

So $$\int_M (d\omega, \eta) = \underbrace{\int_M d(\omega \wedge *\eta)}_{=0 \text{ by Stokes as } \partial M = \emptyset} + \int_M \omega \wedge *d^\dagger \eta = (\omega, d^\dagger \eta).$$

$\square$

remarks:

- $(d_k)^\dagger : \Omega^{k+1} M \to \Omega^k M$ is adjoint to $d_k : \Omega^k M \to \Omega^{k+1} M$ and similar to $\pm d_{n-k-1}$.
- We can now formulate the remaining/inhomogeneous Maxwell equation(s) simply as $d^\dagger F = j$. In ordinary components this is $\nabla \cdot \vec{E} = \rho$ and $\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$.

Def.: For an oriented Riemannian manifold M the Laplace-Beltrami operator $\Delta : \Omega^k M \to \Omega^k M$ is defined as

$$\boxed{\Delta := (d + d^\dagger)^2 = d d^\dagger + d^\dagger d} = d_{k-1} d_{k-1}^\dagger + d_k^\dagger d_k$$

remarks:

- For $k=0$ we have $\Delta : C^\infty(M) \to C^\infty(M)$:

$$\Delta f = (\underbrace{d d^\dagger}_{\Omega^0 M \to \{0\}} + d^\dagger d) f = \underbrace{d^\dagger}_{\Omega^1 M \to C^\infty(M)} d f = \underbrace{- * d * \varphi}_{\text{div}} \underbrace{\varphi^{-1}(d f)}_{\text{grad} f} = - \text{div} \, \text{grad}(f)$$

So $\boxed{\Delta = - \text{div} \circ \text{grad}}$ on $C^\infty(M)$.

- For standard $\mathbb{R}^n$ this gives:

$$\Delta f = - \text{div} \sum_{i=1}^n \frac{\partial}{\partial x_i}\Big|_p f \, dx_i = - \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}\Big|_p$$

(note that there are different conventions concerning the sign in the definition of Δ. We chose Δ positive.)

- On compact M (where $(\cdot,\cdot)$ is defined) Δ is selfadjoint ($(\Delta\omega, \eta) = (\omega, \Delta\eta)$) and positive ($(\omega, \Delta\omega) \geq 0$).

Def.: The space of harmonic k-forms on an oriented Riem. manifold is defined as $\mathcal{H}^k M := \{ \omega \in \Omega^k M \mid \Delta\omega = 0 \}$.

Thm.: Let M be a compact, oriented Riemannian manifold and $\omega \in \Omega^k M$.

Then $\boxed{\Delta\omega = 0 \iff (d_k\omega = 0 \text{ and } d_{k-1}^{+}\omega = 0)}$

(In words: a differential form is harmonic iff it is closed and 'co-closed'.)

proof: '$\Leftarrow$' is obvious from the definition.

'$\Rightarrow$': $\Delta\omega = 0 \Rightarrow 0 = (\omega, \Delta\omega) = (\omega, dd^{+}\omega) + (\omega, d^{+}d\,\omega)$

$$= \underbrace{(d^{+}\omega, d^{+}\omega) + (d\omega, d\omega)}_{\text{positiv definite !}} .$$

$\square$

Lemma: $\Delta * = * \Delta$. In particular, $\omega \in \mathcal{H}^k M \Rightarrow *\omega \in \mathcal{H}^{n-k} M$.

proof: $\to$ exercise. $\square$

With $\Omega^k M \underset{d_k^{+}}{\overset{d_k}{\rightleftarrows}} \Omega^{k+1} M$ the adjointness leads within $\Omega^k M$ to:

$$\ker(d_k) = \operatorname{Im}(d_k^{+})^{\perp} \text{ and } \ker(d_{k-1}^{+}) = \operatorname{Im}(d_{k-1})^{\perp}$$

Would $\Omega^k M$ be finite-dim., we could argue that

$$\Omega^k M = \ker d_k \oplus \operatorname{Im} d_k^{+} = \ker(d_{k-1}^{+}) \oplus \operatorname{Im}(d_{k-1})$$

and since $\operatorname{Im}(d_{k-1}) \subseteq \ker(d_k)$ also that

$$\ker d_k = \operatorname{Im}(d_{k-1}) \oplus \underbrace{\ker(d_k) \cap \ker(d_{k-1}^{+})}_{= \mathcal{H}^k M}$$

In fact, the following is true:

Thm.: [Hodge decomposition] For an oriented, compact Riemannian manifold, $\dim(\mathcal{H}^k M) < \infty$ and

$$\boxed{\Omega^k M = \mathrm{Im}(d_{k-1}) \oplus \mathrm{Im}(d_k^+) \oplus \mathcal{H}^k M},$$

i.e., $\Omega^k M$ decomposes into subspaces $d\Omega^{k-1}M \oplus d^+\Omega^{k+1}M \oplus \mathcal{H}^k M$ that are orthogonal w.r.t. $(\omega,\eta) = \int_M \omega \wedge *\eta$.

proof: I.g. the above argument only shows that

$$\Omega^k M \supseteq d\Omega^{k-1}M \oplus d^+\Omega^{k+1}M \oplus \mathcal{H}^k M.$$

'=' is much harder to prove and requires some theory on 'elliptic PDEs'. ... $\square$

remark: $\Omega^k M = d\Omega^{k-1}M \oplus d^+\Omega^{k+1}M \oplus \mathcal{H}^k M$ means that every k-form has a unique decomposition into an exact form, a dual exact form and a harmonic form.

For 3-dim. manifolds this becomes the Helmholtz decomposition by which each vector field is the sum of a gradient field, a curl field and a harmonic field. In particular, there exists a decomposition into a 'divergence-free' and a 'curl-free' part.

de Rham cohomology

Def.: Let M be an n-dim. smooth manifold and $p \in \{0, \dots, n\}$. We define the p'th de Rham cohomology group of M as the quotient vector space

$$\boxed{H^p_\Omega(M) := \frac{\ker(d_p)}{\operatorname{Im}(d_{p-1})}} = \frac{\{\text{closed } p\text{-forms}\}}{\{\text{exact } p\text{-forms}\}}$$

and $H^p_\Omega(M) := \{0\}$ for $p \in \mathbb{Z} \setminus \{0, \dots, n\}$. For any closed form $\omega \in \Omega^p M$, we denote $[\omega]$ the corresponding equivalence class, called cohomology class of ω. That is, $[\omega] = [\tilde{\omega}] \Leftrightarrow \omega - \tilde{\omega}$ is exact.

If M is compact, we define the p'th Betti number as

$$\boxed{\beta_p := \dim H^p_\Omega(M)}$$

Examples:

- $H^0_\Omega(M) = \dfrac{\{f \in C^\infty(M) \mid df = 0\}}{\{0\}} = \{\text{locally const. func.s on } M\}$

  So $\boxed{\beta_0 = \#\text{ connected components}}$.

- For $M = \mathbb{R}^2 \setminus \{0\}$ or $M = S^1$ the 1-form $\omega := \dfrac{x\,dy - y\,dx}{x^2 + y^2} \equiv d\theta$ is closed but not exact (since $\omega = d\eta$ would imply $\int_{S^1} \omega = 0 \neq 2\pi$). So $H^1_\Omega(M) \neq \{0\}$.

- More generally, if M is closed and orientable, then there is an orientation form that is closed but not exact. So $H^n_\Omega(M) \neq \{0\}$ for $n := \dim(M)$. Note that its cohomology class $[\omega]$ is all that is 'seen' by the integral $\int_M \omega$ since if $\omega' = \omega + d\eta$, then $\int_M \omega' = \int_M \omega + \underbrace{\int_M d\eta}_{=0 \text{ by Stokes}}$.

Def.: If $F: M \to N$ is smooth, then the pullback $F^*: \Omega^k N \to \Omega^k M$ induces a map $\boxed{F^*: H^k_\Omega(N) \to H^k_\Omega(M)}$ defined as $\boxed{F^*[\omega] := [F^*\omega]}$.

remarks:

- recall that the pullback commutes with the exterior derivative and thus preserves closedness/exactness of forms. So if $\omega' = \omega + d\eta$, then $[F^*(\omega + d\eta)] = [F^*\omega + F^* d\eta] = [F^*\omega + d F^*\eta] = [F^*\omega]$ is well-defined between cohomology classes.
- The assignment $(M, F) \mapsto (H^k_\Omega(M), F^*)$ is a contravariant functor from the category of smooth manifolds and smooth maps to the category of real vector spaces and linear maps.
- The 'contra' (as opposed to 'co'-) refers to a reversal of direction of composition, namely: $\boxed{(F \circ G)^* = G^* \circ F^*}$

  This is also the distinction between 'cohomology' (contravariant) and 'homology' (covariant).

Thm.: Let M be smooth, $\pi: M \times \mathbb{R} \to M$, $(p,t) \mapsto p$ and $i: M \to M \times \mathbb{R}$, $p \mapsto (p,0)$. Then

(i) There are linear maps $\phi_k: \Omega^k(M \times \mathbb{R}) \to \Omega^{k-1}(M \times \mathbb{R})$ s.t.

$$\mathrm{id} - \pi^* \circ i^* = d \circ \phi_k + \phi_{k+1} \circ d \text{ on } \Omega^k(M \times \mathbb{R}).$$

(ii) $\boxed{\pi^*: H^k_\Omega(M) \to H^k_\Omega(M \times \mathbb{R}) \text{ is an isomorphism}}$ with inverse i^*.

proof: (ii) $\pi \circ i = id_M$ implies $i^* \circ \pi^* = id$ so that it remains to show that $\pi^* \circ i^* = id$ on $H^k_\Omega(M\times\mathbb{R})$. Since $d\circ\phi + \phi\circ d$ maps closed forms to exact forms it maps $H^k_\Omega(M\times\mathbb{R}) \ni [\omega] \mapsto [0]$. Due to (i) this implies $id = \pi^* \circ i^*$.

(i) [Sketch]

We can write $\omega \in \Omega^k(M\times\mathbb{R})$ in local coordinates as

$$\omega_p = \tilde{\omega}_p + \sum_{i_1<\dots<i_{k-1}} m_{i_1,\dots,i_k}(p)\, dt\wedge dx_{i_1}\wedge\dots\wedge dx_{i_{k-1}}$$

where t is the coordinate corresponding to $\mathbb{R}$, $p=(x,t)$ and $\tilde{\omega}$ does not depend on dt. Then

$$(\phi_k\omega)_p := \sum_{i_1<\dots<i_{k-1}} \int_0^t m_{i_1\dots i_k}(x,\tau)\,d\tau \;\; dx_{i_1}\wedge\dots\wedge dx_{i_{k-1}}$$

can be shown to have the desired properties. $\square$

Def.: • $f,g \in C(X,Y)$ between top. spaces X,Y are called *homotopic* ($f\simeq g$) if there is $F\in C(X\times[0,1],Y)$ s.t. $F(\cdot,0)=f$, $F(\cdot,1)=g$.

• Two top. spaces X,Y are called *homotopy equivalent* ($X\simeq Y$) if there are continuous maps $X \underset{G}{\overset{F}{\rightleftarrows}} Y$ s.t. $F\circ G \simeq id_Y$ and $G\circ F\simeq id_X$.

remarks: • If X,Y are homeomorphic, then they are homotopy eqiv. However, $S^1 \simeq \mathbb{R}^2\setminus\{0\}$ (using $F(x)=\frac{x}{\|x\|}$ and $G: S^1\ni x\mapsto x\in\mathbb{R}^2\setminus\{0\}$)

• By *Whitney's approximation thm.* every cont. map between smooth manifolds is homotopic to a smooth map. Moreover, homotopic smooth maps are 'smoothly homotopic' (i.e. $F\in C^\infty$).

Thm.: [Homotopy invariance of de Rham cohomology] For any $k \in \mathbb{N}_0$:

1) If $f, g: M \to N$ are homotopic smooth maps, then the induced maps $\boxed{f^* = g^* : H^k_\Omega(N) \longrightarrow H^k_\Omega(M)}$ are identical.

2) If M, N are homotopy equivalent smooth manifolds, then $\boxed{H^k_\Omega(M) \cong H^k_\Omega(N)}$ are isomorphic.

proof: 1) By Whitney's approx. thm. there is a smooth map $F: M \times \mathbb{R} \to N$ s.t. $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. With $i_t: M \to M \times \mathbb{R}$, $i_t(p) := (p, t)$ we have $f = F \circ i_0$, $g = F \circ i_1$ and $i_0^* = \pi^{*-1} = i_1^*$. So
$f^* = i_0^* \circ F^* = i_0^* \circ \pi^{*-1} \circ i_1^* \circ F^* = i_1^* \circ F^* = g^*$.

2) There are smooth maps $M \underset{G}{\overset{F}{\rightleftarrows}} N$ s.t. $F \circ G \simeq id_N$ and $G \circ F \simeq id_M$. According to 1) the induced maps satisfy $F^* \circ G^* = id$ and $G^* \circ F^* = id$. So $F^*: H^k_\Omega(N) \to H^k_\Omega(M)$ is an isomorphism. □

Example: • By induction on n we get:

$$H^k_\Omega(\mathbb{R}^n) = H^k_\Omega(\{0\}) \simeq \begin{cases} \mathbb{R}, & k = 0 \\ \{0\}, & k > 0 \end{cases}$$

Corollary: [Poincaré Lemma] If M is a smooth manifold that is contractable (i.e. homotopy equivalent to a point, e.g. star-shaped in $\mathbb{R}^n$), then $\beta_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$.

$\longrightarrow$ $\boxed{\text{Every closed form is exact on any contractable domain.}}$

Thm.: [Hodge thm.] For a compact, oriented smooth manifold M: $\boxed{H^p_\Omega(M) \simeq \mathcal{H}^p M}$ are isomorphic vector spaces. In particular, $\beta_p < \infty$. (this holds for any Riem. metric underlying $\mathcal{H}^p M$)

proof: This follows from the Hodge decomposition: Consider the linear map $\mathcal{H}^p M \ni \omega_H \longmapsto [\omega_H] \in H^p_\Omega(M)$. This is injective since $[\omega_H] = [\tilde{\omega}_H] \Leftrightarrow \omega_H = \tilde{\omega}_H + d\eta$, by uniqueness of the Hodge decomposition, implies $d\eta = 0$ (alternatively: $0 = d^+(\omega - \tilde{\omega}) = d^+ d\eta \Rightarrow \|d\eta\|^2 = 0$)

It is also surjective since for any closed $\omega = \omega_H + d\eta + d^+\theta$ we have $0 = d\omega = dd^+\theta$ so that $(\theta, dd^+\theta) = \|d^+\theta\|^2 = 0$ and thus $d^+\theta = 0$. Hence, $[\omega] = [\omega_H]$. $\square$

Thm.: [Poincaré duality] Let M be a compact, oriented smooth manifold of dimension n. Then for any $k \in \{0, \dots, n\}$ $([\omega], [\eta]) \longmapsto \int_M \omega \wedge \eta$ defines a non-degenerate bilinear map $H^k_\Omega(M) \times H^{n-k}_\Omega(M) \longrightarrow \mathbb{R}$ and thus an isomorphism $\boxed{H^{n-k}_\Omega(M) \simeq H^k_\Omega(M)^*}$. In particular, $\boxed{\beta_{n-k} = \beta_k}$.

proof: First note that $\int_M \omega \wedge \eta$ does only depend on the cohomology classes $[\omega]$ and $[\eta]$ since

$$\int_M (\omega + d\alpha) \wedge (\eta + d\beta) = \int_M \omega \wedge \eta + d\alpha \wedge \eta + \omega \wedge d\beta + d\alpha \wedge d\beta$$

$$\overset{d\omega,\, d\eta = 0}{=} \int_M \omega \wedge \eta + \underbrace{\int_M d(\alpha \wedge \eta + (-1)^k \omega \wedge \beta + \alpha \wedge d\beta)}_{= 0 \text{ by Stokes as } \partial M = \emptyset}.$$

Next, we show that it is non-degenerate, i.e., that for every $[\omega] \neq 0$ there is a closed η s.t. $\int_M \omega \wedge \eta \neq 0$. By the Hodge thm. we can choose $\omega \neq 0$ harmonic (w.r.t. any Riem. metric). Then $\eta := *\omega$ is closed since $\Delta \eta = \Delta * \omega = * \Delta \omega = 0$ and $\int \omega \wedge \eta = \|\omega\|^2 \neq 0$.

Consequently, the dim. of $H_{\Omega}^{n-k}(M)$ is at least as large as the one of the dual space $(H_{\Omega}^{k}(M))^*$. As the same argument also works in the other direction, the spaces are isomorphic. □

example: For $M = S^1$ we obtain $\beta_1 \underset{\substack{\uparrow \\ \text{Poincaré duality}}}{=} \beta_{1-1} = \beta_0 \underset{\substack{\uparrow \\ \text{connected}}}{=} 1$.

Corollary: If $m > n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

proof: If $f: \mathbb{R}^m \to \mathbb{R}^n$ were a homeomorphism, then $\mathbb{R}^m \setminus \{0\} \simeq S^{m-1}$ and $\mathbb{R}^n \setminus \{f(0)\} \simeq S^{n-1}$ would be homotopy equivalent. However, $\beta_{m-1}(S^{m-1}) \underset{\substack{\uparrow \\ \text{Poincaré duality}}}{=} \beta_0(S^{m-1}) = 1 \neq \beta_{m-1}(S^{n-1}) = 0$. □

Corollary: Let M be a closed smooth n-dim. manifold, $\beta_k := \dim\left[H_{\Omega}^k(M)\right]$ and $\boxed{\chi(M) := \sum_{k=0}^{n} (-1)^k \beta_k}$ its Euler characteristic.

If n is odd, then $\chi(M) = 0$.

proof: (for orientable manifolds. The non-orientable case can be reduced to the orientable one by considering a double cover. See e.g. [Morita].)

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k = \frac{1}{2} \sum_k \Big((-1)^k \beta_k + \underbrace{(-1)^{n-k}}_{-(-1)^k} \underbrace{\beta_{n-k}}_{\beta_k} \Big) = 0$$ □

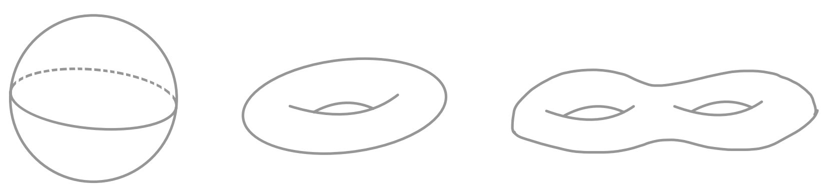
Corollary: If M is an orientable, connected closed smooth 2-dim. manifold, there is a $g \in \mathbb{N}_0$ (called the genus of the surface) s.t. $\boxed{\dim H^1_\Omega(M) = 2g}$ and

$$\boxed{\chi(M) = 2 - 2g}$$

proof: $H^1_\Omega(M) \times H^1_\Omega(M) \longrightarrow \mathbb{R}$, $([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta$ is a non-degenerate bilinear form that is anti-symmetric. W.r.t. any basis of $H^1_\Omega(M)$ we can represent it by a matrix $A = -A^T \in \mathbb{R}^{\beta_1 \times \beta_1}$ that has to be invertible. So $0 \neq \det(A) = (-1)^{\beta_1} \det(A)$, which implies $\beta_1 \in 2 \cdot \mathbb{N}_0$.

Connectedness implies $\beta_0 = 1$ and Poincaré duality $\beta_2 = 1$. So $\chi(M) = 1 - 2g + 1$. $\square$

remarks: ◦ Connected, orientable closed 2-dim. manifolds are completely characterized (up to homeomorphisms) by their genus:



$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$ $\qquad$ ...

Lemma: For any smooth manifold M and $\omega \in \Omega^1 M$

$$\boxed{\omega \text{ exact} \iff \int_{S^1} \gamma^*\omega = 0 \quad \forall \gamma \in C(S^1, M) \text{ piecewise } C^\infty}$$

remark: this means that a vector field is a 'gradient field' iff it is 'conservative'.

proof: (sketch) '$\Rightarrow$': If $\omega = df$, then $\int_{S^1} \gamma^* df = \int_{S^1} d\gamma^* f \underset{\text{Stokes}}{=} \int_{\partial S^1} \gamma^* f = 0$

'$\Leftarrow$': For $p_0, p \in M$, $\gamma \in C^\infty([0,1], M)$ with $\gamma(0) = p_0$, $\gamma(1) = p$

define $f(p) := \int_{[0,1]} \gamma^*\omega$. This does not depend on the specific curve γ between p_0 and p since

$\int_{\gamma_1[0,1]} \omega - \int_{\gamma_2[0,1]} \omega = 0$ by assumption.

f turns out to be smooth and s.t. $df = \omega$.

□

Lemma: Let S be an n-dim. oriented closed manifold and M a smooth manifold. Then

$$\left.\begin{array}{l} \gamma_0, \gamma_1 \in C^\infty(S, M) \text{ homotopic} \\ \text{and } \omega \in \Omega^n M \text{ closed} \end{array}\right\} \Rightarrow \int_S \gamma_0^*\omega = \int_S \gamma_1^*\omega \, .$$

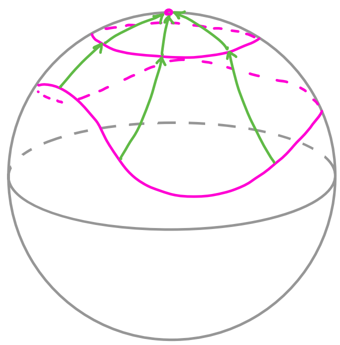
proof: If $F \in C^\infty(S \times [0,1], M)$, $F(\cdot, t) = \gamma_t$ is the homotopy and we choose the orientation s.t. "$\partial(S \times [0,1]) = S \times \{0\} - S \times \{1\}$", then

$$0 \overset{d\omega = 0}{=} \int_{S\times[0,1]} F^* d\omega = \int_{S \times [0,1]} dF^*\omega \underset{\text{Stokes}}{=} \int_S \gamma_0^*\omega - \int_S \gamma_1^*\omega \, .$$

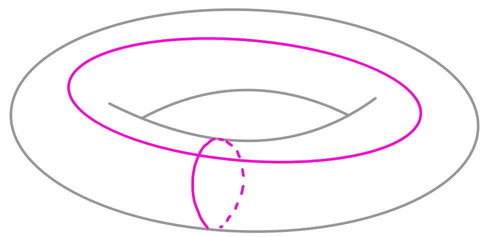
□

Def.: A topological space X is called simply connected if it is path-connected and every $f \in C(S^1, X)$ is homotopic to a constant map $S^1 \ni x \mapsto p_0 \in X$.

remark: for a smooth manifold we can w.l.o.g. assume $f \in C^\infty$.



simply connected



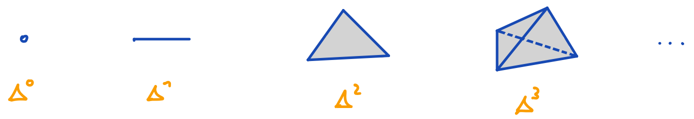
not simply connected

Thm.: $H^1_\Omega(M) = \{0\}$ for any simply connected smooth manifold M.

proof: For any $p \in M$, every (piecewise) smooth loop $\gamma : S^1 \to M$ is homotopic to $S^1 \ni x \mapsto p$. By the second Lemma, $\int_{S^1} \gamma^* \omega = 0$ if $\omega \in \Omega^1 M$ is closed. By the first Lemma, this implies that ω is exact. $\square$

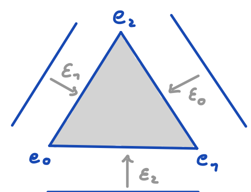
Singular homology

Def.: The convex hull of $n+1$ affinely independent points $v_0, \dots, v_n$ is called an n-simplex, notated as $\sigma = (v_0, \dots, v_n)$. The standard n-simplex is $\Delta^n := \left\{ \sum_{i=0}^{n} x_i e_i \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \right\}$ with $\{e_i\}_{i=0}^{n} \subseteq \mathbb{R}^{n+1}$ the standard basis.



- The n-1 simplex $(v_0, \dots, \hat{v}_i, \dots, v_n)$ obtained from an n-simplex $(v_0, \dots, v_n)$ by omitting the i'th vertex is called its i'th face.
- We define $\varepsilon_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ as the linear map that maps Δ^{n-1} onto the i'th face of Δ^n.

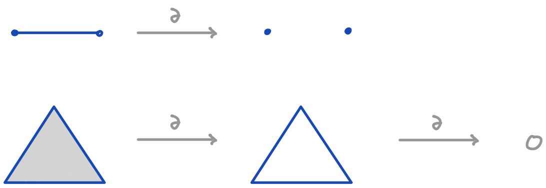
for $n=2$:



Def.:
- Let X be a topological space. A singular n-simplex is a cont. map $\sigma : \Delta^n \longrightarrow X$. A singular n-chain is a formal linear combination $c = \sum_{\sigma} c_\sigma \sigma$ of singular n-simplices with coefficients c_σ in an abelian group G.
- If M is smooth manifold, we denote by $C_n(M)$ the real vector space ('free $\mathbb{R}$-module') of smooth singular n-chains with $G = \mathbb{R}$ and by $\partial_n : C_n(M) \to C_{n-1}(M)$ the boundary operator defined on a singular n-simplex as $\boxed{\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i \, \sigma \circ \varepsilon_i^n}$

examples: ∘ every triangulation corresponds to a singular n-chain, where each 'triangle'/simplex corresponds to one summand in $\sum_\sigma c_\sigma \sigma$ with $c_\sigma = 1$.

∘



Lemma: $\boxed{\partial_{k-1} \circ \partial_k = 0 .}$

proof: $\partial_{k-1} \partial_k \sigma = \partial \left(\sum_i (-1)^i \sigma \circ \varepsilon_i^k \right) = \sum_{i,j} (-1)^{i+j} \sigma \circ \varepsilon_i^k \circ \varepsilon_j^{k-1}$

$$= \sum_{i \leq j} (-1)^{i+j} \sigma \circ \varepsilon_i^k \circ \varepsilon_j^{k-1} + \sum_{j<i} (-1)^{i+j} \sigma \circ \varepsilon_i^k \circ \varepsilon_j^{k-1}$$

In the second sum we can use that $\varepsilon_i^k \circ \varepsilon_j^{k-1} = \varepsilon_j^k \circ \varepsilon_{i-1}^{k-1}$ if $j<i$ and thus replace it by $\sum_{j<i} (-1)^{i+j} \sigma \circ \varepsilon_j^k \circ \varepsilon_{i-1}^{k-1}$

$$= - \sum_{i \leq j} (-1)^{i+j} \sigma \circ \varepsilon_i^k \circ \varepsilon_j^{k-1} .$$

replace j by i and i by $j+1$ □

Def.: A singular k-chain $\sigma \in C_k(M)$ is called

∘ a *cycle* if $\partial \sigma = 0$,

(think of 'loops' for $k=1$ and deformed spheres S^k in general)

∘ a *boundary* if $\exists \tilde{\sigma} \in C_{k+1}(M) : \partial \tilde{\sigma} = \sigma$

- For $\omega \in \Omega^k(M)$ and $c = \sum_\sigma c_\sigma \sigma \in C_k(M)$ we define:

$$\boxed{\int_c \omega := \sum_\sigma c_\sigma \int_{\Delta^k} \sigma^*(\omega)}$$

<u>Thm.:</u> (Stokes' theorem on chains) If M is a smooth manifold, $c \in C_k(M)$, and $\omega \in \Omega^{k-1}(M)$ then $\boxed{\int_{\partial c} \omega = \int_c d\omega}$.

<u>Def.:</u> For the chain complex $C_n(M) \xrightarrow{\partial_n} C_{n-1}(M) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0(M) \xrightarrow{\partial_0} 0$ we define the k-th singular homology group:

$$\boxed{H_k(M,\mathbb{R}) := \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}}} = \text{'cycles mod boundaries'}$$

For a cycle $c \in C_k(M)$ the equivalence class $[c] \in H_k(M,\mathbb{R})$ is called its homology class and $c \sim c' :\Leftrightarrow c = c' + \partial \tilde{c}$.

<u>remark:</u> I.g. a chain complex is a sequence of homomorphisms between abelian groups (or moduls) s.t. $\partial_k \circ \partial_{k+1} = 0$.

Note that for a cycle $c \in C_k(M)$ and a closed form $\omega \in \Omega^k(M)$ the integral $\int_c \omega$ only depends on $[c] \in H_k(M,\mathbb{R})$ and $[\omega] \in H^k_\Omega(M)$ since

$$\int_{c+\partial\tilde{c}} (\omega + d\eta) = \int_c \omega + \underbrace{\int_{\partial\tilde{c}} (\omega + d\eta)}_{\int_{\tilde{c}} d(\omega+d\eta) = 0} + \underbrace{\int_c d\eta}_{\int_{\partial c} \eta = 0}.$$

Consequently, there is a bilinear form $H_k(M,\mathbb{R}) \times H^k_\Omega(M) \longrightarrow \mathbb{R}$ given by $([c],[\omega]) \mapsto \int_c \omega$. With quite some effort this can be shown to be non-degenerate, which then proves:

Thm.: (de Rham's thm.) The map $H^k_\Omega(M) \to H_k(M,\mathbb{R})^*$ given by $[\omega] \mapsto \left([c] \mapsto \int_c \omega \right)$ is a vectorspace isomorphism:

$$\boxed{H^k_\Omega(M) \cong H_k(M,\mathbb{R})^*}$$

remark: due to the duality, closed forms are also called cocycles and exact forms are called coboundaries.

Corollary:

1) $\omega \in \Omega^k(M)$ is closed $\iff$ $\forall c \in C^{k+1}(M) : \int_{\partial c} \omega = 0$

2) $\omega \in \Omega^k(M)$ is exact $\iff$ $\forall$ k-cycles c: $\int_c \omega = 0$

proof: 1) If $d\omega = 0$, then $\int_{\partial c} \omega = \int_c d\omega = 0$.

If $d\omega = \eta \neq 0$, then there is a $p \in M$ and $v_1, \dots, v_{k+1} \in T_pM$ s.t. $\eta_p(v_1, \dots, v_{k+1}) > 0$. Hence, there is a chart (U, x) around p in which $\eta_q\left(\frac{\partial}{\partial x_1}\Big|_q, \dots, \frac{\partial}{\partial x_{k+1}}\Big|_q \right) > 0 \;\; \forall q \in U$. So if $\sigma: \Delta^{k+1} \to U$ is chosen s.t. $x \circ \sigma$ embeds Δ^{k+1} appropriately into the coordinate plane $\{ y \in \mathbb{R}^{\dim(M)} \mid y_i = 0 \; \forall i > k+1 \}$, then

$$\int_{\partial c} \omega = \int_c d\omega = \int_{\Delta^{k+1}} \sigma^*(\eta) \neq 0.$$

2) If $\omega = d\eta$ then $\int_c d\eta = \int_{\partial c} \eta = 0$ since $\partial c = 0$.

Conversely, if $[\omega] \neq 0$, then by de Rham's thm.

there must be a $[c] \in H_k(M, \mathbb{R})$ s.t. $\int_c \omega \neq 0$. $\square$