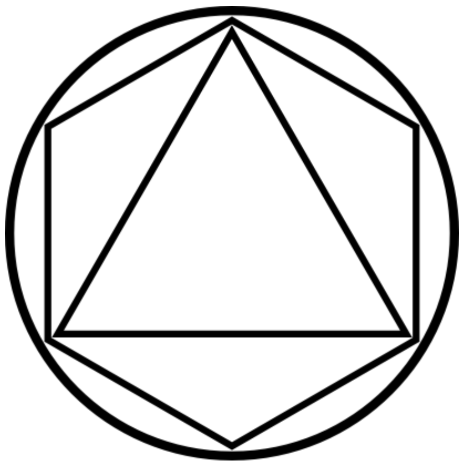

# Variations of the Nerve Theorem

Master's Thesis by
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# Variations of the Nerve Theorem


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I hereby declare that this thesis is my own work and that no other sources have been used except those clearly indicated and referenced.

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# Contents

# 1. Introduction

The nerve theorem, whose early versions are usually attributed to Karol Borsuk and Jean Leray, is a basic result in algebraic topology that relates the nerve of a cover $\mathcal{A} = \{A_i\}_{i\in I}$ of a topological space $X$ to $X$ itself, given suitable hypotheses. Here, the nerve of $\mathcal{A}$ is the simplicial complex whose simplices are the non-empty finite subsets $J \subseteq I$ such that $\bigcap_{i\in J} A_i$ is non-empty. The hypotheses always include the assumption that the cover is *good* in an (algebraic) topological way.

One possible assumption is the contractibility of all non-empty intersections of cover elements. Borsuk [Bor48] was concerned with closed covers such that in addition all these intersections lie in the class of absolute retracts; from a modern point of view this assumption guarantees the solvability of extension problems. Andrzej Nagórko incorporated this idea in his work [Nag07] to give a proof of a similar nerve theorem. Other well known versions of the nerve theorem require an open cover that admits a subordinate partition of unity $\{\varphi_i\}_{i\in I}$ [Wei52; Seg68; Hat02]. This partition of unity can be used to manufacture a homotopy equivalence

$$X \to |N(\mathcal{A})|,\ x \mapsto \sum_{i\in I} \varphi_i(x) \cdot v_i,$$

where $v_i$ is the vertex that corresponds to $A_i$. A close inspection of this construction (compare [Bau+17, Lemma 3]) shows that if $f\colon X \to Y$ is a continuous map, $g\colon I \to J$ is a set map and $\mathcal{B} = \{B_j\}_{j\in J}$ is an open cover for $Y$ such that for all $i \in I$ the inclusion $f(A_i) \subseteq B_{g(i)}$ holds, then there is a square

$$\begin{array}{ccc} X & \longrightarrow & |N(\mathcal{A})| \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & |N(\mathcal{B})| \end{array}$$

that commutes up to homotopy. While this approach is elementary, there are two things one has to be aware of: the maps depend, up to homotopy, on the chosen partitions of unity and the above square does not commute on the nose in general.

A weak nerve theorem, assuming that the non-empty intersections of cover elements have the weak homotopy type of a point, has been proved by Michael McCord [McC67] under the assumption of a point-finite and basis-like open cover by using a union theorem for weak homotopy equivalences.

Weak equivalences have been studied extensively in algebraic topology. One difficulty that arises is that weak equivalences in general cannot be inverted. To cope with this issue, many techniques have been developed to “formally invert” those. One such tool is the *theory of model categories* (see [Hir03] for a definition). In this setting, one can define two spaces $X$ and $Y$ to be *weakly equivalent* if there is a third space $Z$ and weak equivalences

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\simeq}\swarrow & & \searrow{\scriptstyle\simeq} \\ X & & Y. \end{array}$$

In a similar fashion, we aim for a space $Z$ and two *natural* morphisms

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & |N(\mathcal{A})| \end{array}$$

such that we get a *commuting* diagram

$$\begin{array}{ccccc} X & \longleftarrow & Z & \longrightarrow & |N(\mathcal{A})| \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Z' & \longrightarrow & |N(\mathcal{B})|. \end{array}$$

Thus, by introducing a third space, we gain the freedom to give canonical morphisms that will allow us to prove *functorial nerve theorems*. When the covers consist of open subsets this situation has been addressed in [CO08]; [CS18] deals with covers by subcomplexes.

Nerve theorems for finite covers of simplicial complexes by subcomplexes have been investigated by Anders Björner [Bjö03] and Jonathan Barmak [Bar11]. Björner gives a proof of an $n$-connectivity version of the nerve theorem, whereas Barmak uses a slight modification of Quillen's theorem A for partially ordered sets (posets, for short) to establish a simple homotopy version.

Instead of being concerned with homotopy equivalences, Leray's nerve theorem assumes that the non-empty intersections of open cover elements have the homology of a point and gives an isomorphism on the level of homology of the spaces involved. This can be proved by using the Mayer-Vietoris spectral sequence [Bro94, VII Theorem 4.4.].

In recent years persistent nerve theorems have been developed [GS18; CS18]. These relate the persistent homology of a filtration to the persistent homology of the induced nerves of "almost good filtered covers" in terms of the bottleneck distance.

Let us give a short summary of the material we will cover in this thesis.

We start Chapter 2 by recapitulating simplicial complexes to fix the notation. Then we discuss several constructions that carry the name "nerve". After this we examine the relations between several assumptions one can put on the cover of a topological space. This introductory chapter ends with a brief revision of the basics of homological algebra and spectral sequences.

In Chapter 3 it will be our task to understand the definition of the *homotopy colimit* and also to consider its different constructions for topological spaces as well as for chain complexes.

Chapter 4 is dedicated to classical nerve theorems. We will see strong and weak nerve theorems for open covers as well as one for covers by subcomplexes and a homological version. An elementary proof of the nerve theorem for compact and convex sets in $\mathbb{R}^d$ is also presented.

In Chapter 5 the focus lies on persistent nerve theorems. We are mainly interested in the ideas presented in [CS18]. To this end, we will review the technical details and fill some minor and also one more serious gap in the proof.

Finally, in Chapter 6 we will convey some ideas that could help to conceptually understand persistent nerve theorems and that indicate the direction of our future research.

# 2. Preliminaries

We assume that the reader is familiar with the basic notions of combinatorial and algebraic topology as well as category theory.

## 2.1. The Nerve Constructions

**Definition 2.1.1.** An *abstract simplicial complex* $K$ on the set $I$ is a subset of

$$\{\emptyset \neq S \subseteq I \mid |S| < \infty\}$$

such that for each $\sigma \in K$ the implication $\emptyset \neq \tau \subseteq \sigma \Rightarrow \tau \in K$ holds.

The cardinality of $\sigma$ is called the *dimension* $\dim \sigma$ *of* $\sigma$. The *n-skeleton* is defined to be

$$K^n = \{\sigma \in K \mid \dim \sigma \leq n\}.$$

A *simplicial map* $f\colon K \to L$ between abstract simplicial complexes is a set map such that

$$\sigma = \{v_0, \dots, v_n\} \in K \Rightarrow f(\tau) = \{f(v_0), \dots, f(v_n)\} \in L.$$

The collection of abstract simplicial complexes together with simplicial maps forms a category that we will denote by Simp.

The *boundary* of a simplex $\sigma \in K$ is the abstract simplicial complex

$$\partial\sigma = \{S \subseteq I \mid \emptyset \neq S \subsetneq \sigma\}.$$

The *geometric realization* of $K$ is defined to be

$$|K| = \bigcup_{\sigma \in K} \operatorname{conv}\{e_j \mid j \in \sigma\} \subseteq \mathbb{R}^{(I)},$$

where conv is the convex hull. Recall that $\mathbb{R}^{(I)}$ is the space of all $I$-tuples that are almost everywhere zero together with the topology that is coherent with the subspaces

$$\{(c_i)_{i \in I} \in \mathbb{R}^{(I)} \mid \forall i \notin J: \ c_i = 0\},$$

where $J \subseteq I$, $|J| < \infty$. Therefore, $|K|$ is coherent with the subspaces $\operatorname{conv}\{e_j \mid j \in \sigma\}$.

A simplicial map $f\colon K \to L$ induces a continuous map $|f|\colon |K| \to |L|$ that is affine linear on each simplex and hence fully determined by the values of the simplices in $K^0$ under $f$. This manufactures a functor $|\cdot|\colon \text{Simp} \to \text{Top}$.

*Remark* 2.1.2. The geometric realization of an abstract simplicial complex is in fact a CW-complex. We will not distinguish between the realization as a CW-complex and as a topological space.

*Remark* 2.1.3. It follows from the definition that a simplicial map $f\colon K \to L$ is fully determined by its action on $K^0$.

**Definition 2.1.4.** An *ordered simplicial complex* $K$ is an abstract simplicial complex on a partially ordered set $(I, \leq)$ such that for each simplex $\sigma$ the restriction $\leq_{|\sigma}$ is a total order. We write a simplex $\sigma = \{v_0, \dots, v_n\} \in K$ as a sequence $v_0 \to \cdots \to v_n$ according to the total order.

An *ordered simplicial map* $f\colon K \to L$ between ordered simplicial complexes is a simplicial map that respects the partial order on $K^0$.

The category of ordered simplicial complexes is denoted by oSimp.

**Definition 2.1.5.** Let $K$ be an abstract simplicial complex. The *barycentric subdivision* of $K$ is the ordered simplicial complex

$$\operatorname{sd} K = \{\{J_0, \dots, J_n\} \mid n \in \mathbb{N}_0,\ J_0 \supseteq \cdots \supseteq J_n \text{ in } K\}.$$

This forms a functor $\operatorname{sd}\colon \mathrm{Simp} \to \mathrm{oSimp} \subseteq \mathrm{Simp}$, where a morphism $f\colon K \to L$ is sent to $\operatorname{sd} f\colon \operatorname{sd} K \to \operatorname{sd} L,\ \{J_0, \dots, J_n\} \mapsto \{f(J_0), \dots, f(J_n)\}$.

**Lemma 2.1.6** ([Mun84, p. 83 ff.])**.** *Let $K$ be an abstract simplicial complex. There is a piecewise linear homeomorphism $h\colon |\operatorname{sd} K| \xrightarrow{\cong} |K|$, which sends a vertex $\{J\} \in \operatorname{sd} K^0$ to the barycenter of $|J| \subseteq |K|$.*

**Definition 2.1.7.** Let $X$ be a set and let $\{A_i\}_{i \in I}$ be a collection of subsets. We define for $J \subseteq I$

$$A_J = \bigcap_{j \in J} A_j.$$

**Definition 2.1.8.** Let $\mathcal{F} = \{U_i\}_{i \in I}$ be a cover of a topological space $X$. The *Alexandrov nerve* of $\mathcal{F}$ is the abstract simplicial complex

$$N(\mathcal{F}) = \{J \subseteq I \mid |J| < \infty,\ U_J \neq \emptyset\}.$$

To turn this construction into a functor, we introduce the following concepts.

**Definition 2.1.9.** Let $S \subseteq X$ be a subspace of a topological space $X$ and let $f\colon S \to Y$ be a continuous map. Further, let $\{A_i\}_{i \in I}$ be a cover of $X$ and $\{B_j\}_{j \in J}$ a cover of $Y$. We say that $f$ is carried by the map of covers $C\colon I \to J$ if for all $i \in I$ we have

$$f(A_i \cap S) \subseteq B_{C(i)}.$$

**Lemma 2.1.10.** *If $f\colon S \to Y$ is carried by $C$, $f(S) \subseteq T$, and $g\colon T \to Z$ is carried by $D$, then $g \circ f$ is carried by $D \circ C$.*

*Proof.* Follows from the definition. □

*Remark* 2.1.11. A map of covers gives us, in a combinatorial way, control on the continuous map it carries.

**Definition 2.1.12.** Consider as objects pairs of the form $(X, \{U_i\}_{i\in I})$, where $X$ is a topological space and $\{U_i\}_{i\in I}$ is a cover. A morphism $(f, C)\colon (X, \{U_i\}_{i\in I}) \to (Y, \{V_j\}_{j\in J})$ consists of a continuous map $f\colon X \to Y$ and a set map $C\colon I \to J$ such that $f$ is carried by $C$.

This yields a category that we call the *category of covered spaces* and we denote it by Cov. Moreover, we denote the canonical functor that sends a covered space $(X, \{U_i\}_{i\in I})$ to $X$ by $\Phi_S\colon \mathrm{Cov} \to \mathrm{Top}$.

**Lemma 2.1.13.** *The Alexandrov nerve $N$ is a functor*

$$N\colon \mathrm{Cov} \to \mathrm{Simp}\,.$$

*Proof.* A morphism $(f, C)\colon (X, \{U_i\}_{i\in I}) \to (Y, \{V_j\}_{j\in J}))$ induces a map $N(\{U_i\}_{i\in I}) \to \{V_j\}_{j\in J})$ that is given on the 0-simplices by $C$. This is well-defined because for any simplex $\{i_0, \ldots, i_n\} \in N(\{U_i\}_{i\in I})$ we have

$$\emptyset \neq f(U_{\{i_0,\ldots,i_n\}}) \subseteq \bigcap_{j=0,\ldots,n} f(U_{i_j}) \subseteq \bigcap_{j=0,\ldots,n} V_{C(i_j)}$$

and hence $\{C(i_0), \ldots, C(i_n)\}$ is a simplex in $N(\{V_j\}_{j\in J})$. □

More generally, we can define the nerve of an arbitrary category $\mathcal{C}$.

**Definition 2.1.14.** The *nerve* $N(\mathcal{C})$ of $\mathcal{C}$ is the *simplicial set*

$$N(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}(\cdot, \mathcal{C})\colon \Delta^{\mathrm{op}} \to \mathrm{Set},$$

where $\Delta$ is the category of finite ordinals $\{[n] = \{0, \ldots, n\} \mid n \in \mathbb{N}_0\}$ together with order preserving maps. In other words, the nerve $N(\mathcal{C})$ is given by the following datum:

- For each natural number $k \in \mathbb{N}_0$ we have a set $N(\mathcal{C})_k$, whose elements are called *$k$-simplices*, that consists of all strings of composable morphisms

$$C_0 \overset{f_0}{\to} C_1 \overset{f_1}{\to} \cdots \overset{f_{k-1}}{\to} C_k$$

  in $\mathcal{C}$.

- For each natural number $k \in \mathbb{N}_0$ we have
  1. *face maps*: If $k > 0$ we have for $i \in \{0, 1, \ldots, k\}$ the map

  $$d_i\colon N(\mathcal{C})_k \to N(\mathcal{C})_{k-1}$$

     given by forgetting the first or last object if $i = 0$ or $i = k$, respectively. Otherwise, we compose $f_{k-1}$ with $f_k$.

2. *degeneracy maps*: For $i \in \{0, 1, \dots, k\}$ the map

$$s_i \colon N(\mathcal{C})_k \to N(\mathcal{C})_{k+1}$$

is given by inserting an identity morphism at $C_i$.

We denote by $N_m$ the set of *non-degenerate* elements of $N(\mathcal{C})_m$, which are those elements that do not lie in the image of some $s_i$.

Further, we define the *geometric realization* of $N(\mathcal{C})$ as the quotient space

$$|N(\mathcal{C})| = \Big( \coprod_{k=0}^{\infty} N(\mathcal{C})_k \times |\Delta^k| \Big) / \sim,$$

where $|\Delta^k| = \operatorname{conv}\{e_1, \dots, e_{k+1}\}$ is the standard topological $k$-simplex in $\mathbb{R}^{k+1}$ and $N(\mathcal{C})_k$ is equipped with the discrete topology. For $\sigma \in N(\mathcal{C})_k$, $p \in |\Delta^{k-1}|$ and any natural number $i \in \{1, \dots, k+1\}$ we identify

$$(d_i(\sigma), p) \sim (\sigma, d^i(p)),$$

where $d^i \colon |\Delta^{k-1}| \hookrightarrow |\Delta^k|$ is the inclusion onto the face opposite to $e_i$. Further, we identify for $q \in |\Delta^{k+1}|$

$$(s_i(\sigma), p) \sim (\sigma, s^i(p)),$$

where $s^i \colon |\Delta^{k+1}| \to |\Delta^k|$ is the affine map

$$e_j \mapsto \begin{cases} e_j & \text{if } j \leq i \\ e_{j-1} & \text{if } j > i. \end{cases}$$

*Remark* 2.1.15. If $N(\mathcal{P})$ is the nerve of some poset $\mathcal{P}$, then the set

$$A(\mathcal{C}) = \{\{C_0, \dots, C_k\} \mid k \in \mathbb{N}_0,\ C_0 \to \dots \to C_k \in N_k\}$$

is an ordered simplicial complex on $\mathcal{P}$. But note that not every ordered simplicial complex arises in this way. The reason is that in general the 1-skeleton of an ordered simplicial complex, which can be interpreted to be a directed graph, can have cycles.

Let us discuss two examples.

*Example* 2.1.16. Let $G$ be a group. We can view $G$ as a category with one object and one morphism for each element of $G$. The composition of morphisms is given by the group operation. The nerve of $G$ is called the *classifying space* of $G$.

*Example* 2.1.17. Let $\mathcal{F} = \{U_i\}_{i \in I}$ be a cover of a topological space $X$. The finite subsets $J$ of $I$ with $U_J \neq \emptyset$ form a category $\mathcal{C}_{\mathcal{F}}$ with a unique morphism

$$J \to J'$$

if $J' \subseteq J$. There is a homeomorphism $|N(\mathcal{C}_{\mathcal{F}})| \to |\operatorname{sd} N(\mathcal{F})|$ induced for an element $\gamma \in N_k$ by

$$|\Delta^k| \overset{\cong}{\to} \operatorname{conv}\{e_j \mid j \in \{\gamma(0), \dots, \gamma(k)\}\}.$$

*Remark* 2.1.18. The last example shows that from a categorical point of view the correct complex to consider is the barycentric subdivision of the Alexandrov nerve. This complex will show up in the proof of various nerve theorems.

### Variations of the Definition

In the literature one can find various objects that can be built from a cover. Our next goal is to discuss some of them and their relation to our definition of the nerve of a cover.

**Definition 2.1.19.** Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. We define

$$\mathcal{N}(\mathcal{F}) = \{\{U_j\}_{j\in J} \,|J \subseteq I,\ |J| < \infty,\ U_J \neq 0\}.$$

*Remark* 2.1.20. At first glance, this definition does look quite similar to what we have seen before. But there is slight difference; this definition does not keep track of the cover elements that appear multiple times.

If no cover element appears multiple times, then $N(\mathcal{F}) = \mathcal{N}(\mathcal{F})$.

We have the following result.

**Proposition 2.1.21.** *Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. Then, $|\mathcal{N}(\mathcal{F})|$ is a strong deformation retract of $|N(\mathcal{F})|$.*

*Proof.* Choose a subset $I' \subseteq I$ and a surjection $\varphi\colon I \to I'$ such that for any $k, j \in I'$ with $k \neq j$ we have $U_k \neq U_j$ and such that for any $U_i$ we have $U_i = U_{\varphi(i)}$. This choice defines an embedding

$$\iota\colon \mathcal{N}(\mathcal{F}) \hookrightarrow N(\mathcal{F}).$$

Moreover, the map $\varphi$ induces a retract

$$r\colon N(\mathcal{F}) \to \mathcal{N}(\mathcal{F}).$$

To see that $|r|$ is a strong deformation retract, take any simplex $\sigma = \{i_0, \dots, i_n\} \in N(\mathcal{F})$. Then, $|\sigma|$ and $|\iota \circ r(\sigma)|$ are contained in the realization of

$$\{i_0, \dots, i_n, \varphi(i_0), \dots, \varphi(i_n)\} \in N(\mathcal{F}),$$

which is a simplex, because $\bigcap U_{i_k} \cap \bigcap U_{\varphi(i_k)} = \bigcap U_{i_k} \neq \emptyset$. Thus, a straight line homotopy proves the claim. ☐

Next, we address the question whether we can assume that the cover is closed under finite intersections instead of barycentrically subdividing the nerve, as is done in some proofs of the nerve theorem.

**Proposition 2.1.22.** *Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. Let*

$$\mathcal{F}' = \{U_J\}_{J\subseteq I, |J|<\infty, U_J\neq\emptyset}.$$

*Then, $|N(\mathcal{F})|$ is a strong deformation retract of $|N(\mathcal{F}')|$.*

*Proof.* The map $\varphi\colon I \to \mathcal{P}(I),\ i \mapsto \{i\}$ induces an embedding

$$\iota\colon \operatorname{sd} N(\mathcal{F}) \hookrightarrow \operatorname{sd} N(\mathcal{F}').$$

We will now define

$$r\colon \operatorname{sd} N(\mathcal{F}') \to \operatorname{sd} N(\mathcal{F}),\ \{\{J_k\}_{k\in L_0}, \dots, \{J_k\}_{k\in L_n}\} \mapsto \{\bigcup_{i\in L_0} J_i, \cdots, \bigcup_{i\in L_n} J_i\},$$

where $L_0 \supseteq \cdots \supseteq L_n$. The induced map of $r$ on the realizations is a retraction. To prove that it is a strong deformation retraction, take any simplex $\sigma = \{\{J_k\}_{k\in L_0}, \dots, \{J_k\}_{k\in L_n}\} \in \operatorname{sd} N(\mathcal{F}')$. Then, $|\sigma|$ and $|\iota \circ r(\sigma)|$ are contained in the realization of

$$\{J_k, \{i\}\}_{k\in L_0, i\in \bigcup_{k\in L_0} J_k} \in N(\mathcal{F}'),$$

which is a simplex, because

$$\bigcap_{k\in L_0} U_{J_k} \cap \bigcap_{i\in \bigcup_{k\in L_0} J_k} U_{\{i\}} = \bigcap_{k\in L_0} U_{J_k} \neq \emptyset.$$

Thus, a straight line homotopy proves the claim. □

Let us define one more complex.

**Definition 2.1.23.** Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. Let $K(\mathcal{F})$ denote the subcomplex of $N(\mathcal{F})$ such that a simplex $\sigma$ is contained in $K(\mathcal{F})$ if and only if $\{U_i\}_{i\in\sigma}$ can be totally ordered with respect to inclusion.

The following result, up to a mistake in the proof, can be found in [McC67, Lemma 3].

**Proposition 2.1.24.** *Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. Assume that the intersection of finitely many cover elements is either empty or a member of $\mathcal{F}$. Then $|K(\mathcal{F})|$ is a strong deformation retract of $|N(\mathcal{F})|$.*

*Proof.* For this proof we need the concept of the *barycentric subdivision holding a subcomplex fixed*; see for example [Mun84, p.89]. By [Mun84, Lemma 16.2.], an $n$-simplex $\tau$ in $\operatorname{sd}(N(\mathcal{F})/K(\mathcal{F}))$ can be described as a set $\{v_0, \dots, v_r, \sigma_1, \dots, \sigma_{n-r}\}$ such that $s = \{v_0, \dots, v_r\} \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_{n-r}$ with $s$ a simplex in $K(\mathcal{F})$ and the $\sigma_i$ simplices in $N(\mathcal{F})$ but not in $K(\mathcal{F})$. Geometrically speaking, the simplex $\tau$ is spanned by the vertices of $s$ and the barycenters of the $\sigma_i$.

By assumption, there is a set map $\varphi\colon N(\mathcal{F}) \to I$ such that for any $J \in N(\mathcal{F})$ and $v \in I$ the equations

$$U_J = U_{\varphi(J)}$$

and $\varphi(\{v\}) = v$ hold. Consider the following map

$$r\colon \operatorname{sd}(N(\mathcal{F})/K(\mathcal{F}))^0 \to K(\mathcal{F})^0,\ \sigma \mapsto \begin{cases} \sigma & \text{if } \sigma \in K(\mathcal{F})^0 \\ \{\varphi(\sigma)\} & \text{else.} \end{cases}$$

This map extends to a simplicial map, because for any simplex $s \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_{n-\dim s}$ in $\mathrm{sd}(N(\mathcal{F})/K(\mathcal{F}))$ we have

$$U_s \supseteq U_{r(\sigma_1)} \supseteq \cdots \supseteq U_{r(\sigma_{n-\dim s})}$$

and hence the image under $r$ is a simplex in $K(\mathcal{F})$.

By definition, $r$ fixes the simplices in $K(\mathcal{F})$. To see that $|r|$ is indeed a strong deformation retract, note that for any simplex $\tau = s \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_{n-\dim s}$ in $\mathrm{sd}(N(\mathcal{F})/K(\mathcal{F}))$ the simplices $|\tau|$ and $|\iota \circ r(\tau)|$, where $\iota\colon K(\mathcal{F}) \hookrightarrow N(\mathcal{F})$ is the canonical inclusion, are both contained in the realization of the simplex

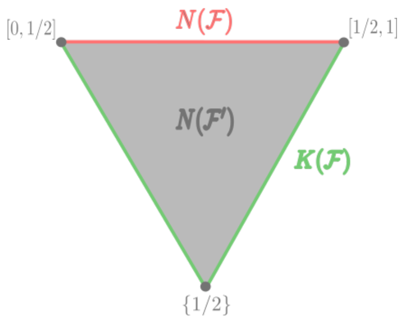
$$\sigma_{n-\dim s} \cup \{\varphi(\sigma_i)\}_{i=1,\ldots,n-\dim s}$$

of $N(\mathcal{F})$. This is indeed a simplex, because

$$U_{\sigma_{n-\dim s}} \cap \bigcap_{i=1,\ldots,n-\dim s} U_{\varphi(\sigma_i)} = U_{\sigma_{n-\dim s}} \cap \bigcap_{i=1,\ldots,n-\dim s} U_{\sigma_i} = U_{\sigma_{n-\dim s}} \neq \emptyset.$$

Thus, a straight line homotopy proves the claim. □

*Example* 2.1.25. Consider the cover of $[0,1]$ by $[0,1/2]$ and $[1/2,1]$. Then, we have the following picture




*Remark* 2.1.26. This last example shows that all three complexes $N(\mathcal{F})$, $N(\mathcal{F}')$ and $K(\mathcal{F})$ are combinatorially different.

*Remark* 2.1.27. Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. Consider the two complexes $\mathrm{sd}\, N(\mathcal{F})$ and $K(\mathcal{F}')$, where $\mathcal{F}'$ is as in Proposition 2.1.22. By Proposition 2.1.22 and Proposition 2.1.24 we have

$$|\,\mathrm{sd}\, N(\mathcal{F})| \cong |N(\mathcal{F})| \simeq |N(\mathcal{F}')| \simeq |K(\mathcal{F}')|.$$

But more can be said. By the proofs of the propositions mentioned above, we know that the homotopy equivalence is induced by the map

$$\mathrm{sd}(N(\mathcal{F})/N(\mathcal{F}) \cap K(\mathcal{F}'))^0 \to K(\mathcal{F}')^0,\ \sigma \mapsto \begin{cases} \sigma & \text{if } \sigma \in K(\mathcal{F}')^0 \\ \{\varphi(\sigma)\} & \text{else,} \end{cases}$$

where $\varphi$ is given as in the proof of Proposition 2.1.24.

One might wonder whether $\operatorname{sd} N(\mathcal{F})$ and $K(\mathcal{F}')$ are isomorphic as simplicial complexes. In general this is not the case: Consider $X = [0, 1]$ and $\mathcal{F} = \{[0, 1/2], [0, 1]\}$. Then, $\operatorname{sd} N(\mathcal{F})$, which is the barycentric subdivision of a one simplex, is not isomorphic to $K(\mathcal{F}')$, which is a two simplex.

## 2.2. Cover Assumptions

As nerve theorems try to recover the homotopy type of a space from the intersection pattern of a cover, it is only natural to discuss several possible cover assumptions.

In the subsequent, let $\mathcal{F} = \{A_i\}_{i\in I}$ be a cover of a topological space $X$.

**Definition 2.2.1.** We call $\mathcal{F}$

- *point finite*, if every point $x \in X$ is contained in only finitely many cover elements.
- *locally finite*, if for every point $x \in X$ there exists an open neighborhood that intersects only finitely many cover elements.
- *locally finite dimensional* if for each $A_i$ there exists a natural number $k_i \in \mathbb{N}$ with

$$A_i \cap A_J = \emptyset$$

for any subset $J \subseteq I$, $|J| \geq k_i$.

*Remark* 2.2.2. The collection $\mathcal{F}$ is locally finite dimensional if and only if for every simplex $\sigma$ in the nerve $N(\mathcal{F})$ the dimensions of the simplices that have $\sigma$ as a face is bounded from above.

**Definition 2.2.3.** We say that $\mathcal{F}$ satisfies the *ascending chain condition (a.c.c)* if in the nerve $N(\mathcal{F})$ there does not exist and infinite ascending chain

$$J_0 \subsetneq J_1 \subsetneq \cdots .$$

**Proposition 2.2.4.** *If $\mathcal{F}$ is locally finite, then it it point finite.*

*Proof.* By definition of locally finiteness, every point is contained in only finitely many cover elements. □

**Proposition 2.2.5.** *If $\mathcal{F}$ satisfies the a.c.c., then it is point finite.*

*Proof.* If $\mathcal{F}$ was not point finite, then there would exist a point $x \in X$ and infinitely many cover elements that contain $x$. In particular there would exist an infinite chain in $N(\mathcal{F})$, contradicting the definition of the a.c.c. . □

**Proposition 2.2.6.** *If $\mathcal{F}$ is an open cover that satisfies the a.c.c, then it is locally finite.*

*Proof.* For any point $x \in X$, take a maximal (finite) subset $J \subseteq I$ with $x \in A_J$. Then, $A_J$ is an open neighborhood of $x$, such that

$$A_j \cap A_J \neq \emptyset$$

if and only if $j \in J$. Hence, the cover is locally finite. □

**Proposition 2.2.7.** *If $\mathcal{F}$ is locally finite dimensional, then it satisfies the a.c.c. .*

*Proof.* This follows immediately from Remark 2.2.2. □

**Proposition 2.2.8.** *Let $K$ be an abstract simplicial complex, $X = |K|$ and $\mathcal{F} = \{|A_i|\}_{i \in I}$ a cover by subcomplexes. If $\mathcal{F}$ point finite, then it is locally finite.*

*Proof.* Note, that we only have to check the locally finiteness condition for every vertex $v$ of $K$ and as open neighborhood the *open vertex star*

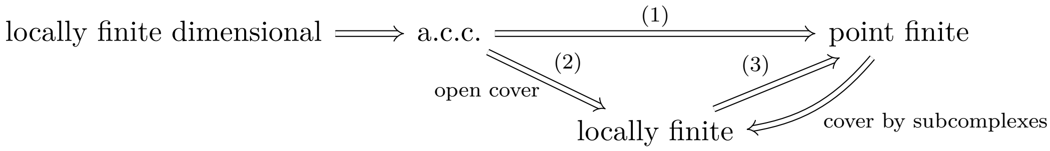
$$\operatorname{st} v = \operatorname{int} |\{\tau \in K \mid v \in \tau\}|.$$

Then, it is easy to see that a cover element $|A_i|$ intersects $\operatorname{st} v$ if and only if $v \in A_i$. Hence, $\mathcal{F}$ is locally finite if and only if for every vertex $v \in K$ the set

$$\{i \in I \mid v \in A_i\}$$

is finite. This is the case if and only if $\mathcal{F}$ is point finite. □

In summary, we have the following relations among the conditions on the cover:




*Remark* 2.2.9. The converses of (1) and (2) do not hold. A counterexample is given by $X = \mathbb{R}_{>0}$ and

$$\mathcal{F} = \{(n, \infty) \mid n \in \mathbb{N}_0\}.$$

*Remark* 2.2.10. To see that (2) does not hold for closed covers, consider $X = [0, 1]$ and the cover

$$\mathcal{F} = \{0\} \cup \{[\frac{1}{n+1}, \frac{1}{n}] \mid n \in \mathbb{N}\}.$$

Every open neighborhood of 0 intersects infinitely many cover elements.

This example also shows that the converse of (3) does not hold in general.

*Remark* 2.2.11. The a.c.c. does not imply locally finite dimensionality. Consider for example the cover

$$\mathcal{F} = \{\mathbb{R}_{\geq 0} \times \{0\}\} \cup \bigcup_{n \in \mathbb{N}_0} \bigcup_{i=0}^{n} \{\{n\} \times [0, i+1]\}$$

of the space $X = \bigcup_{A \in \mathcal{F}} A \subseteq \mathbb{R}^2$. This cover satisfies the a.c.c., but it is not locally finite dimensional.

*Remark* 2.2.12. Local finite dimensionality is an essential assumption in Nagórko's nerve theorem [Nag07]. It allows him to employ induction over what we call the *depth of a simplex* in the nerve $N(\mathcal{F})$

$$\operatorname{depth}(\sigma) = \sup\{n \mid \exists \sigma = \tau_0 \subsetneq \cdots \subsetneq \tau_n \text{ in } N(\mathcal{F})\} < \infty$$

to establish a homotopy equivalence between $X$ and the geometric realization of the nerve $|N(\mathcal{F})|$ of the cover.

## 2.3. Homological Algebra

### 2.3.1. Basics

We briefly recapitulate some basics of homological algebra to fix the notation.

**Definition 2.3.1.** Let $R$ be a commutative ring with unit. A *chain complex* $C = (\{C_i\}_{i\in\mathbb{N}_0}, \{\partial_i\}_{i\in\mathbb{N}_0})$[1] is a sequence of $R$-modules together with *boundary morphisms*

$$0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} \cdots$$

such that $\partial_i \circ \partial_{i+1} = 0$ holds for all $i \in \mathbb{N}_0$. For every natural number $D \in \mathbb{N}_0$ define the *$D$-truncated complex* $C_{\leq D}$ by

$$(C_{\leq D})_i = \begin{cases} C_i & i \leq D \\ 0 & i > D \end{cases}$$

together with the same boundary morphisms. The *$n$-th homology* is defined to be

$$H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1},$$

and we call elements of $\ker \partial_n$ *cycles* and elements of $\operatorname{im} \partial_{n+1}$ *boundaries*.

The *homology* of $\mathcal{C}$ is the graded object

$$H_*(C) = \bigoplus_{n\in\mathbb{N}_0} H_n(C).$$

Let $D = (\{D_i\}_{i\in\mathbb{N}_0}, \{\overline{\partial}\}_{i\in\mathbb{N}_0})$ be another chain complex. A *morphism of chain complexes* $f\colon C \to D$ is a family of $R$-module morphisms $\{f_i\colon C_i \to D_i\}_{i\in\mathbb{N}_0}$ such that $f_i \circ \partial_{i+1} = \overline{\partial}_{i+1} \circ f_{i+1}$ holds for all $i \in \mathbb{N}_0$. The *induced map on homology* is defined to be

$$f_*\colon H_*(C) \to H_*(D),\ [c] \mapsto [f(c)].$$

The category of chain complexes of $R$-modules is denoted by $\mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$.

[1] After this section, we will drop the index of $\partial_i$.

**Definition 2.3.2.** Let $f, g\colon C \to D$ be chain maps. A *chain homotopy* is a family of maps $\{h_i\}_{i\in\mathbb{N}_0}$ such that

$$\overline{\partial}_{i+1}h_i + h_{i-1}\partial_i = f_i - g_i.^2$$

holds for all $i \in \mathbb{N}_0$. We call $f$ and $g$ *chain homotopic* and write $f \simeq g$.

If the family $\{h_i\}_{i\leq D}$ is only given up to some natural number $D \in \mathbb{N}_0$ such that the just mentioned relations hold up to degree $D$, we call $f$ and $g$ *chain homotopic up to dimension* $D$ and write $f \simeq_{\leq D} g$. The collection $\{h_i\}_{i\leq D}$ is then called a *$D$-chain homotopy.*

**Lemma 2.3.3.** *Let $f, g\colon C \to D$ be $D$-chain homotopic chain maps. Then, they induce the same map in homology up to degree $D$.*

*Proof.* Let $n \leq D$ be a natural number and let $c \in C_n$ be such that $\partial_n c = 0$. Then, $f_n - g_n = \overline{\partial}_{n+1}h_n(c)$ and hence $[f_n(c)] = [g_n(c)] \in H_n(D)$. □

**Definition 2.3.4.** We call a chain map $f\colon C \to D$ a *weak equivalence* (or *quasi-isomorphism*) if the induced map in homology $f_*\colon H_*(C) \to H_*(D)$ is an isomorphism. We call it a *chain equivalence* if there exists a chain map $g\colon D \to C$ such that $f \circ g \simeq \mathrm{id}_D$ and $g \circ f \simeq \mathrm{id}_C$ hold.

*Remark* 2.3.5. The relation between strong and weak equivalence is analogous to the related notions in topology. A strong equivalence is always a weak equivalence, but the converse only holds for "nice chain complexes". For example, this holds for a morphism between chain complexes of free $R$-modules. See [Wei94, Theorem 10.4.8] for this general statement. For a more elementary proof in the case $R = \mathbb{Z}$ see [Mun84, Theorem 46.2].

Hence, for such chain complexes we do not need to distinguish between the notions of weak equivalence and chain equivalence.

**Definition 2.3.6.** Let $C = (\{C_i\}_{i\in\mathbb{N}_0}, \{\partial_i\}_{i\in\mathbb{N}_0})$ and $D = (\{D_i\}_{i\in\mathbb{N}_0}, \{\tilde{\partial}_i\}_{i\in\mathbb{N}_0})$ be chain complexes. The *tensor product* $C \otimes D$ is the chain complex with

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$$

and the boundary morphism is given for $x \otimes y \in C_k \otimes C_l$ by

$$\overline{\partial}(x \otimes y) = (\partial x) \otimes y + (-1)^k x \otimes (\tilde{\partial} y).$$

**Proposition 2.3.7.** *Let $X$ and $Y$ be CW-complexes. Then, the cellular chain complex of the product CW-complex $X \times Y$ is given by the tensor product*

$$C_\bullet(X \times Y) \cong C_\bullet(X) \otimes C_\bullet(Y).$$

*Proof.* See [Mas91, p. 280 ff.]. □

[2] $h_{-1} = 0$

*Remark* 2.3.8. There is another binary operation on chain complexes that is obtained through an equivalence of categories but not isomorphic to the standard tensor product. In particular this shows that equivalences of categories need not preserve tensor products. This operation is induced by the *Dold-Kan correspondence* (compare [Wei94, Section 8.4. The Dold-Kan Correspondence]), which establishes an equivalence of categories

$$\mathrm{Nor}\colon \mathrm{sRmod} \rightleftarrows \mathrm{Ch}_{\geq 0}(\mathrm{Rmod})\colon \Gamma \ ,$$

where sRmod is the category of *simplicial R-modules*. This category is the category of functors $\Delta^{op} \to \mathrm{Rmod}$, where $\Delta$ is as in Definition 2.1.14, and the tensor product $\otimes$ is given level-wise by the tensor product of $R$-modules. For any two chain complexes $C$ and $D$ define

$$C \boxtimes D = \mathrm{Nor}(\Gamma(C) \otimes \Gamma(D)).$$

For an explicit formula for $\boxtimes$ see [Law]. It is a consequence of the *Eilenberg-Zilber theorem* [GJ09, Theorem 2.4] that the two chain complexes $C \otimes D$ and $C \boxtimes D$ are weakly equivalent.

Let $K$ and $L$ be ordered simplicial complexes. The product $K \times L$ exists and is an ordered simplicial complex. Further, there exists a homeomorphism $|K \times L| \cong |K| \times |L|$, where $|K| \times |L|$ is the product CW-complex (compare [Rüs, Section 1.3 Products] and [Hat02, Simplicial CW Structures]).

Now, the chain complex $C_*^{simp}(K) \boxtimes C_*^{simp}(L)$ is isomorphic to $C_*^{simp}(K \times L)$ and $C_*^{simp}(K) \otimes C_*^{simp}(L)$ is isomorphic to the cellular chain complex $C_*^{cell}(|K| \times |L|)$.

This illustrates that the phenomena we observe is best understood as the two sides of the same coin. On the one side there is the category of ordered simplicial complexes and the category of simplicial $R$-modules and on the other side there is the category of CW-complexes and the category of chain complexes.

### 2.3.2. Spectral Sequences

The main results of this sections are Proposition 2.3.20 and Corollary 2.3.21. If the reader feels uncomfortable with spectral sequences he or she is invited to use these results as a black box in the subsequent. Spectral sequences will not appear after this section.

"A spectral sequence is an algebraic gadget like an exact sequence, but more complicated." (John Frank Adams [Ada72])

"The subject of spectral sequences is elementary, but the notion of the spectral sequence of a double complex involves so many objects and indices that it seems at first repulsive. " (David Eisenbud [Eis95])

In this section, we will recall the basics of the theory of spectral sequences. Further, we will see in an important application how they first provide a coarse view on the homology of a chain complex that gets more refined in each step.

The main sources of the following material are [Bro94], [MP12], [Wei94] and [Cho06].

**Definition 2.3.9.** Let $R$ be a commutative ring. A *spectral sequence* $E = \{E^r\}_{r\in\mathbb{N}_0}$ consists of a sequence of $\mathbb{Z}$-bigraded $R$-modules $E^r = \{E^r_{p,q}\}_{p,q\in\mathbb{Z}}$ together with differentials

$$d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1},\ d^r \circ d^r = 0$$

such that $E^{r+1} = H_*(E^r)$. A morphism $f\colon E \to E'$ of spectral sequences is a family of morphisms of complexes $f^r\colon E^r \to E'^r$ (a morphism in each degree that commute with the differentials) such that $f^{r+1}$ is the morphism $H_*(f^r)$. We call $E^r$ the *r-th page of the spectral sequence.*

We will always visualize the $r$-th page of a spectral sequence as being aligned in the grid $\mathbb{Z}^2$ as in Example 2.3.15.

**Definition 2.3.10.** Let $E = \{E^r\}_{r\in\mathbb{N}_0}$ be a spectral sequence. We define $Z^r, B^r \subseteq E^0$ such that with $Z^0 = E^0,\ B^0 = 0$ we have

$$Z^{r+1}/B^r = \ker\left(d^r\colon Z^r/B^r \cong E^r \to E^r \cong Z^r/B^r\right),$$
$$B^{r+1}/B^r = \operatorname{im}\left(d^r\colon Z^r/B^r \cong E^r \to E^r \cong Z^r/B^r\right).$$

Then, we have the following chain of inclusions

$$0 = B^0 \subseteq B^1 \subseteq \cdots \subseteq Z^1 \subseteq Z^0 = E^0$$

and we can define

$$Z^\infty = \bigcap_{r=0}^{\infty} Z^r,\ B^\infty = \bigcup_{r=0}^{\infty} B^r.$$

We call

$$E^\infty = Z^\infty/B^\infty$$

the *infinity page of the spectral sequence.*

To breathe life into this definition, we will discuss the important example of a spectral sequence that arises from the filtration of a chain complex.

**Proposition 2.3.11.** *Let $(C,\partial)$ be a non-negative chain complex and let*

$$0 = F_0C \subseteq F_1C \subseteq \cdots \subseteq F_kC \subseteq \cdots \subseteq C$$

*be a filtration (nested family of subcomplexes) that collapses to a finite filtration in each degree of the chain complex and such that*

$$\bigcup_{i=0}^{\infty} F_iC = C.$$

*Then, there is a spectral sequence*

$$E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q},\ d^0 = \partial$$

*that* converges *to* $H_*(C)$*; this means that*

$$E^\infty_{pq} \cong F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C),$$

*where* $F_pH_n(C)$ *is the image of* $H_n(F_pC) \to H_n(C)$*.*

*Explicitly the spectral sequence is constructed as follows.*[3] *Let* $\eta_p$ *denote the surjection* $F_pC \to F_pC/F_{p-1}C = E^0_p$ *and let*

$$A^r_p = \{c \in F_pC \colon \partial(c) \in F_{p-r}C\}.$$

*Further, we define*

$$Z^r_p = \eta_p(A^r_p) \subseteq E^0_p \text{ and } B^{r+1}_{p-r} = \eta_{p-r}(\partial(A^r_p)) \subseteq E^0_{p-r}.$$

*Note, that* $A^r_p \cap F_{p-1}C = A^{r-1}_{p-1}$*. Then, the spectral sequence is given by*

$$E^r_p := \frac{Z^r_p}{B^r_p} \cong \frac{A^r_p + F_{p-1}(C)}{\partial(A^{r-1}_{p+r-1}) + F_{p-1}(C)} \cong \frac{A^r_p}{\partial(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}} \tag{2.3.1}$$

*and the boundary operator* $\partial$ *induces the maps*

$$d^r_p \colon E^r_p \to E^r_{p-r},\ [x] \mapsto [\partial x].$$

*Proof.* See the discussion in [Wei94, Section 5: Spectral Sequences] □

*Remark* 2.3.12. Let us lose some words on the intuition behind this construction. The chains in $A^r_p$ should be seen as "approximate cycles" as their boundaries need not be zero but must only be contained in an early filtration step. Further, $\partial(A^r_p)$ should be seen as consisting of those boundaries that are obtained from chains that come from a later filtration step. Hence, $E^r_p$ consists of "approximate cycles" modulo a portion of all possible boundaries, relative to $F_{p-1}(C)$.

When $r$ increases, the elements of $A^r_p$ will get closer to being actual cycles and $\partial(A^r_p)$ will truncate the set of boundaries. This was meant at the beginning of the section when saying that the coarse view on the homology gets refined in each step.

*Remark* 2.3.13. The notion of convergence of a spectral sequence is subtle. Even though the above spectral sequence converges, it may not be possible to recover the homology $H_*(C)$ from the infinity page of the spectral sequence as one might encounter *extension problems*. This means that it is not always possible to recover merely from the knowledge of the outer objects of a short exact sequence the object in the middle; an example for this is

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

[3] For ease of notation we drop the second index.

where another valid object in the middle is $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ that is not isomorphic to $\mathbb{Z}$.

In our case, one needs to solve for every natural number $n \in \mathbb{N}_0$ iteratively the extension problem

$$0 \to F_{p-1}H_n(C) \to F_pH_n(C) \to E^\infty_{p(n-p)} \to 0$$

for every $p$.

Fortunately, we will not use spectral sequences to compute homology but we will use them to detect weak equivalences.

Let $C$ and $C'$ be filtered chain complexes as in Proposition 2.3.11. Any chain map $\tau\colon C \to C'$ that respects this filtration induces a morphism $\{E^r(\tau)\colon E^r(C) \to E^r(C')\}_{r\in\mathbb{N}_0}$ between the induced spectral sequences that is given by

$$E^r(\tau)\colon E^r(C) \to E^r(C'),\ [x] \mapsto [\tau(x)],$$

where we use the explicit description in Eq. (2.3.1). This gives indeed a morphism of spectral sequences because the differentials on the pages of the spectral sequences are induced by the boundary operators and hence they commute with the morphisms $E^r(\tau)$.

**Lemma 2.3.14.** *Let $C, C'$ be filtered chain complexes and let $\tau\colon C \to C'$ be a filtration-preserving chain map. If the induced map*

$$E^r(\tau)\colon E^r(C) \to E^r(C')$$

*of spectral sequences is an isomorphism for some $r$, then*

$$H(\tau)\colon H(C) \to H(C')$$

*is an isomorphism, as well.*

*Proof.* The assumptions guarantee that $\tau$ induces an isomorphism on the infinity pages of the spectral sequences. Thus, $\tau$ induces an isomorphism

$$\bigoplus_{n=p+q} F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C) \to \bigoplus_{n=p+q} F_pH_{p+q}(C')/F_{p-1}H_{p+q}(C')$$

for every $n$. From this it follows in an elementary way that $H(\tau)$ is an isomorphism.

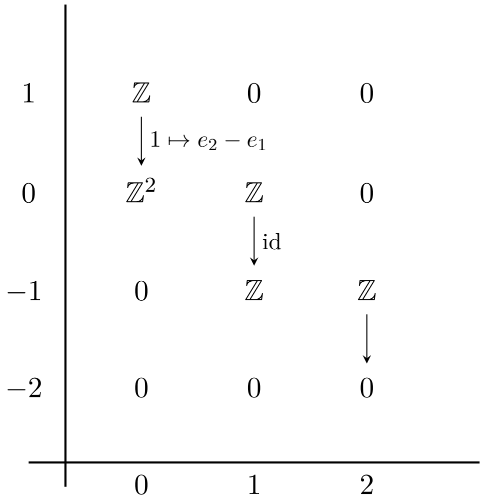
If $x \in \ker H(\tau)$, then the isomorphism above implies that if $x \in F_pH_{p+q}(C)$, then it must already be contained in $F_{p-1}H_{p+q}(C)$ and hence in $F_0H_{p+q}(C) = 0$.

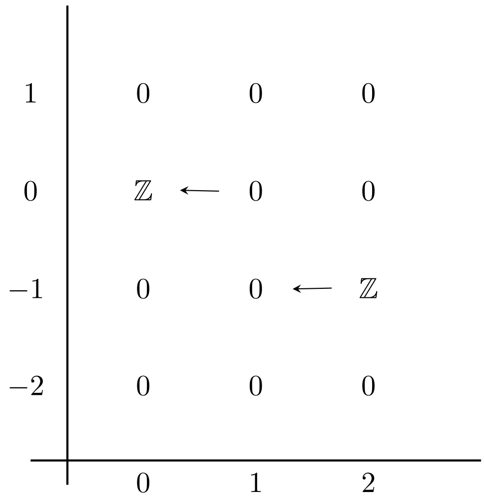
Take any $y \in H_n(C')$. There exists a $p$ with $y \in F_pH_{p+q}(C') \setminus F_{p-1}H_{p+q}(C)$. Again, by the isomorphism above, we know that there exists an element $x_p \in F_pH_{p+q}(C)$ such that $y - \tau(x_p)$ is contained in $F_{p-1}H_{p+q}(C')$. By iterating this construction, we get

$$y = \tau(x_p + \cdots + x_0).$$

□

*Example* 2.3.15. Consider the filtration of the boundary of the standard 2-simplex by adding one facet in each step. The zeroth and first page of the spectral associated to the induced filtration of the simplicial chain complex are given by



By summing the antidiagonal terms of the first page, which is also the infinity page, and because there are no extension problems to solve we compute

$$H_0 = \mathbb{Z},\ H_1 = \mathbb{Z},$$

which is what we expected.

We will now turn our attention to first quadrant double complex, which will become important in the next section.

**Definition 2.3.16.** A *first quadrant double complex* $D = (\{A_{p,q}\}_{p,q\in\mathbb{N}_0}, d, \delta)$ is a collection $\{A_{p,q}\}_{p,q\in\mathbb{N}_0}$ of $R$-modules together with *differentials*

$$d\colon A_{p,q} \to A_{p-1,q},\ \delta\colon A_{p,q} \to A_{p,q-1}$$

such that

$$d^2 = \delta^2 = 0 \text{ and } \delta \circ d = -d \circ \delta.$$

A *morphism of double complexes* $f\colon D \to D'$ is a collection of maps $\{f_{p,q}\colon A_{p,q} \to A'_{p,q}\}_{p,q\in\mathbb{N}_0}$ that commute with the differentials.

For any $p, q \in \mathbb{N}_0$ one can consider the chain complexes $(\{A_{p,q'}\}, \delta)_{q'\in\mathbb{N}_0}$ and $(\{A_{p',q}\}, d)_{p'\in\mathbb{N}_0}$. Their homology is called the *vertical* and *horizontal homology*, respectively.

**Definition 2.3.17.** Let $D$ be a first quadrant double complex. The *associated total complex* $TD$ is the chain complex $(TD, \tilde{\delta})$ with

$$(TD)_n = \bigoplus_{n=p+q} A_{p,q},\ \tilde{\delta} = d + \delta.$$

A morphism of double complexes induces, in a functorial way, a morphism between the associated double complexes.

*Remark* 2.3.18. Sometimes, one assumes that $\delta \circ d = d \circ \delta$ holds and defines $\tilde{\delta} = d + (-1)^p \delta$. This is just convention and we can transform any such double complex into a double complex as in Definition 2.3.16, and vice versa, if we replace $\delta_{pq}$ by $(-1)^p \delta_{pq}$. This is usually called the "sign trick".

**Definition 2.3.19.** Let $D$ be a first quadrant double complex. There are two filtrations

$$(F_k^l(TD))_n = \bigoplus_{p \leq k} A_{p,n-p}$$

and

$$(F_k^r(TD))_n = \bigoplus_{q \leq k} A_{n-q,q}$$

of $TD$.

By Proposition 2.3.11, the associated spectral sequences converge to $H_*(TD)$, but they may have very different infinity pages.

**Proposition 2.3.20.** *Let $f : D \to D'$ be a morphism of first quadrant double complexes. If $f$ induces an isomorphisms on the horizontal homology, then $f$ induces a weak equivalence*

$$f_* : TD \to TD'.$$

*Proof.* Consider the filtrations $F^r(TD)$ and $F^r(TD')$. Then, $f$ is a filtration preserving chain map such that the induced map of spectral sequences is an isomorphism on the $E^1$-level, so $f$ induces an isomorphism $H_*(TD) \to H_*(TD')$ by Lemma 2.3.14. □

**Corollary 2.3.21.** *Let $D = (\{A_{p,q}\}_{p,q \in \mathbb{N}_0}, d, \delta)$ be a first quadrant double complex such that the vertical homology of $D$ is concentrated on the $x$-axis, i.e. it vanishes for $q \neq 0$. Let $C$ be the resulting chain complex with the boundary map induced by $\delta$. Then, we have a weak equivalence*

$$\tau : TD \to C,$$

*where $\tau$ is the canonical surjection that comes from the fact that $C$ is the vertical 0-dimensional homology of $D$.*

*Proof.* Consider $\tau$ as being induced by a morphism of double complexes $D \to C$ by interpreting $C$ as a double complex that is concentrated on the $x$-axis. As $TC = C$, the claim then follows from the analogous statement of Proposition 2.3.20, which is obtained by mirroring the double complexes along the diagonal. □

# 3. The Homotopy Colimit

In the context of nerve theorems, the homotopy colimit plays the role of a synthetic object that relates the cover with its nerve.

We will first discuss some constructions that are used in the literature. Then, we will take a detour through homotopy theory and look at different ways of axiomatizing the concept of a homotopy colimit. Unfortunately, sometimes the literature does not distinguish between the concrete constructions and the abstract definition. As a result these constructions are also called homotopy colimit. We chose to adopt this naming in order to be consistent with the literature.

## 3.1. Concrete Constructions for the Homotopy Colimit

### 3.1.1. For Topological Spaces

We begin with a construction that can be found in slightly greater generality in [Koz08, Chapter 15 Homotopy Colimits].

**Definition 3.1.1.** Let $K$ be an ordered simplicial complex. A *diagram of spaces $\mathcal{D}$ over $K$* assigns to each vertex $v \in K^0$ a topological space $\mathcal{D}(v)$ and to each edge $v \to w \in K^1$ a continuous map

$$\mathcal{D}(v \to w) \colon \mathcal{D}(v) \to \mathcal{D}(w).$$

Moreover, we assume that the diagram commutes over every triangle in $K$; this means that for any triangle $v \to w \to z \in K^2$ we have

$$\mathcal{D}(w \to z) \circ \mathcal{D}(v \to w) = \mathcal{D}(v \to z).$$

**Definition 3.1.2.** The *homotopy colimit* of a diagram of spaces $\mathcal{D}$ over $K$ is defined to be

$$\operatorname{hocolim} \mathcal{D} = \Big( \coprod_{\sigma = v_0 \to \cdots \to v_n} \mathcal{D}(v_0) \times |\sigma| \Big) / \sim,$$

where the disjoint union is taken over all simplices in $K$. For a face

$$d_i \sigma := \tau_i = v_1 \to \cdots \to \hat{v}_i \to \cdots \to v_n,$$

let $d^i \colon |\tau_i| \hookrightarrow |\sigma|$ be the inclusion. The relation $\sim$ identifies points in the following way:

- for $i > 0$, $\mathcal{D}(v_0) \times |\tau_i|$ is identified with the subset of $\mathcal{D}(v_0) \times |\sigma|$ via the map $d^i$.

- for $\tau_0 = v_1 \to \cdots \to v_n$, we identify

$$(x, d^0(\alpha)) \sim (\mathcal{D}(v_0 \to v_1)(x), \alpha),$$

for any $x \in \mathcal{D}(v_0)$ and $\alpha \in |\tau_0|$.

A *map of diagrams of spaces* $F\colon \mathcal{D}_1 \to \mathcal{D}_2$ over $K$ is a collection of maps

$$\{F(v)\colon \mathcal{D}_1(v) \to \mathcal{D}_2(v) \mid v \in K^0\},$$

such that for each edge $v \to w$ in $K^1$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}_1(v) & \xrightarrow{F(v)} & \mathcal{D}_2(v) \\ {\scriptstyle \mathcal{D}_1(v\to w)}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_2(v\to w)} \\ \mathcal{D}_1(w) & \xrightarrow{F(w)} & \mathcal{D}_2(w). \end{array}$$

*Example* 3.1.3. Let $\mathcal{D}$ be a diagram of spaces over $K = \{\{0\}, \{1\}, \{0, 1\}\}$. Then, the homotopy colimit $\operatorname{hocolim} \mathcal{D}$ is just the mapping cylinder of $\mathcal{D}(\{0\} \to \{1\})$.

**Definition 3.1.4.** Let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover of a topological space $X$. The diagram of spaces $\mathcal{D}_\mathcal{F}$ over the barycentric subdivision $\operatorname{sd} N(\mathcal{F})$ of the Alexandrov nerve, where the poset structure on the vertices is given by the opposite face poset of $N(\mathcal{F})$, that assigns to each vertex $\{J\} \in \operatorname{sd} N(\mathcal{F})$ the non-empty intersection

$$\mathcal{D}_\mathcal{F}(\{J\}) = U_J$$

and to each edge $J \to J' \in \operatorname{sd} N(\mathcal{F})$, $J \supseteq J'$, the inclusion $U_J \hookrightarrow U_{J'}$ is called the *nerve diagram* of $\mathcal{F}$.

*Example* 3.1.5. Consider the cover of the circle by three open arcs. Then, the homotopy colimit of the nerve diagram can be visualized as the colored part in the following right picture.

Figure 3.1.: The homotopy colimit of a cover by three open arcs.

Let us introduce two more maps.

**Definition 3.1.6.** Let $\mathcal{D}$ be a diagram of spaces over $K$. We have the following maps

$$\operatorname{colim}\mathcal{D} \xleftarrow{\;p_f\;} \operatorname{hocolim}\mathcal{D} \xrightarrow{\;p_b\;} |K|,$$

where the *fiber projection* $p_f$ is induced by the projection of $\mathcal{D}(v_0) \times |\sigma|$ onto the first coordinate and the *base projection* $p_b$ is induced by the projection onto the second coordinate.

In the subsequent we will silently assume that one of the following conditions hold.

**Lemma 3.1.7.** *If $\mathcal{D} = \mathcal{D}_{\mathcal{F}}$ is the nerve diagram of a cover $\mathcal{F}$ of a topological space $X$, then*

$$\operatorname{colim}\mathcal{D}_{\mathcal{F}} \cong X$$

*if one of the following holds*

*(i) $\mathcal{F}$ is an open cover*

*(ii) $\mathcal{F}$ is a locally finite closed cover*

*(iii) $X$ is a simplicial complex, or more generally a CW complex, and $\mathcal{F}$ is a cover by subcomplexes.*

*Proof.* This is elementary point-set topology. □

**Definition 3.1.8.** Let $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ be diagrams of spaces over $K$ and let $F_1 \colon \mathcal{D}_1 \to \mathcal{D}_2$ be a map of diagrams of spaces. The morphism

$$\operatorname{hocolim} F_1 \colon \operatorname{hocolim}\mathcal{D}_1 \to \operatorname{hocolim}\mathcal{D}_2$$

is induced by the maps $\mathcal{D}_1(v) \times |\sigma| \overset{F_1(v)\times \mathrm{id}}{\to} \mathcal{D}_2(v) \times |\sigma|$.

This construction is functorial; in other words, if $F_2 \colon \mathcal{D}_2 \to \mathcal{D}_3$ is another map of diagrams of spaces, then we have $\operatorname{hocolim}\mathrm{id}_{\mathcal{D}_1} = \mathrm{id}_{\operatorname{hocolim}\mathcal{D}_1}$ and $\operatorname{hocolim} F_2 \circ F_1 = \operatorname{hocolim} F_2 \circ \operatorname{hocolim} F_1$.

The following proposition is a key property of the homotopy colimit.

**Proposition 3.1.9** ([Koz08, Theorem 15.12 (homotopy lemma)])**.** *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be diagrams of spaces over $K$ and let $F \colon \mathcal{D}_1 \to \mathcal{D}_2$ be a map of diagrams of spaces. Assume that for each vertex $v \in K^0$ the map $F(v)$ is a homotopy equivalence. Then so is*

$$\operatorname{hocolim} F \colon \operatorname{hocolim}\mathcal{D}_1 \overset{\simeq}{\to} \operatorname{hocolim}\mathcal{D}_2.$$

Now, let us give a key proposition that is needed to establish *functorial nerve theorems.* In Chapter 4 we will discuss several assumptions that guarantee that the horizontal maps in the following proposition are (weak) homotopy equivalences.

**Proposition 3.1.10.** *Let $(X, \mathcal{A} = \{A_i\}_{i\in I})$ and $(Y, \mathcal{B} = \{B_j\}_{j\in J})$ be covered spaces. Further, let $(f, C)\colon (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a morphism. Then, $C$ induces a simplicial map*

$$C'\colon \operatorname{sd} N(\mathcal{A}) \to \operatorname{sd} N(\mathcal{B}),\ \{J_0, \dots, J_n\} \mapsto \{\{C(i)\}_{i\in J_0}, \dots, \{C(i)\}_{i\in J_n}\}$$

*and together with $f$, this induces the map*

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{A}} \to \operatorname{hocolim} \mathcal{D}_{\mathcal{B}},\ \mathcal{D}_{\mathcal{A}}(v_0) \times |\sigma| \ni (x, p) \mapsto (f(x), |C'|(p)),$$

*where $\mathcal{D}_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}}$ are the nerve diagrams of $\mathcal{A}$ and $\mathcal{B}$, respectively.*

*Moreover, the following diagram commutes*

$$\begin{array}{ccccc} X & \xleftarrow{p_f} & \operatorname{hocolim} \mathcal{D}_{\mathcal{A}} & \xrightarrow{p_b} & |\operatorname{sd} N(\mathcal{A})| \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle |C'|} \\ Y & \xleftarrow[p_f]{} & \operatorname{hocolim} \mathcal{D}_{\mathcal{B}} & \xrightarrow[p_b]{} & |\operatorname{sd} N(\mathcal{B})|. \end{array}$$

*Proof.* To see that $C$ induces the desired simplicial map, first recall that $C$ induces the simplicial map

$$N(\mathcal{A}) \to N(\mathcal{B}),\ \{i_1, \dots, i_n\} \mapsto \{C(i_1), \dots, C(i_n)\}.$$

As sd is a functor, this induces the desired morphism on the barycentric subdivision.

Moreover, if $p$ is a point in the interior of $|\sigma|$ and $x \in \mathcal{D}_{\mathcal{A}}(J_0) = A_{J_0}$, where $\sigma = \{J_0, \dots, J_n\}$ is a simplex with $J_0 \supseteq \cdots \supseteq J_n$, then $f(x) \in f(A_{J_0}) \subseteq B_{C'(\{J_0\})}$. Hence, $(f(x), |C'|(p)) \in \mathcal{D}_{\mathcal{B}}(C'(\{J_0\})) \times |C'(\sigma)|$ is a point in $\operatorname{hocolim} \mathcal{D}_{\mathcal{B}}$ and the map is well-defined. The commutativity of the diagram follows easily from the fact that the horizontal maps are induced by the canonical projections onto the first and second coordinate, respectively. □

*Remark* 3.1.11. Note that, in essence, Proposition 3.1.10 shows that hocolim is a functor $\operatorname{hocolim}\colon \mathrm{Cov} \to \mathrm{Top}$ and that there are natural transformations $\operatorname{hocolim} \Rightarrow \Phi_S$ and $\operatorname{hocolim} \Rightarrow |\cdot| \circ \operatorname{sd} \circ N$.

*Remark* 3.1.12. Allen Hatcher [Hat02, 4.G Gluing Constructions] gives a different definition of the homotopy colimit. Let $\mathcal{D}$ be a diagram of spaces over $K$ and let $v_0 \to v_1 \to \cdots v_n$ be a simplex of $K$. For the corresponding diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

of topological spaces, define the *iterated mapping cylinder* $M(f_1, \dots, f_n)$ as follows. For $n = 0$, this is simply the space $X_0$ and for $n = 1$, this is just the ordinary mapping cylinder $M(f_1)$. For $n > 1$, define the iterated mapping cylinder as the mapping cylinder of

$$g\colon M(f_1, \dots, f_{n-1}) \to X_{n-1} \xrightarrow{f_n} X_n,$$

where the first map is the canonical projection of the mapping cylinder onto its' target space. Hatcher then claims, that "all these iterated mapping cylinders over the various simplices [...] fit together to form a space" [Hat02, p. 457]. At this point we will not go into details as this construction is not used in the subsequent. The interested reader is referred to Appendix A.1, where we have included a precise definition and a proof of the fact that this definition agrees with the one we gave before.

In [ZC08], the *blowup complex* was introduced.

**Definition 3.1.13.** Let $X$ be a topological space and let $\mathcal{F} = \{U_i\}_{i\in\{0,\dots,n\}}$ be a finite cover. The *blowup complex* is defined to be

$$X^{\mathcal{F}} = \bigcup_{\emptyset\neq J\subseteq\{0,\dots,n\}} U_J \times |J| \subseteq X \times |N(\mathcal{F})|.$$

The *fiber projection* $p_f\colon X^{\mathcal{F}} \to X$ is the projection onto the first coordinate and the *base projection* $p_b\colon X^{\mathcal{F}} \to |N(\mathcal{F})|$ is the projection onto the second coordinate.

To relate this construction to the homotopy colimit of the nerve diagram we introduce the following.

**Definition 3.1.14.** Let $X$ be a topological space and let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover. We define

$$\mathrm{B}(\mathcal{F}) := \Big(\coprod_{\sigma\in N(\mathcal{F})} U_\sigma \times |\sigma|\Big)/\sim,$$

where $\sim$ is induced by the relation

$$(x, d^i(p)) \sim ((U_\sigma \hookrightarrow U_{\tau^i})(x), p),$$

with $x \in U_\sigma$, $p \in |\tau^i|$ and $d^i\colon |\tau^i| \hookrightarrow |\sigma|$ is the inclusion onto the side opposite to the $i$-th vertex of $\sigma$.

Further, we write $p_1\colon \mathrm{B}(\mathcal{F}) \to X$ for the map that is induced by the projection onto the first coordinate and $p_2\colon \mathrm{B}(\mathcal{F}) \to |N(\mathcal{F})|$ for the map that is induced by the projection onto the second coordinate.

**Proposition 3.1.15.** *Let $X$ be a topological space and let $\mathcal{F} = \{U_i\}_{i\in I}$ be a cover. Let $h$ be as in Lemma 2.1.6 for $N(\mathcal{F})$. Then, $h$ induces a homeomorphism*

$$\tilde{h}\colon \operatorname{hocolim} \mathcal{D}_{\mathcal{F}} \xrightarrow{\cong} \mathrm{B}(\mathcal{F})$$

*such that*

$$p_1 \circ \tilde{h} = p_f,\ p_2 \circ \tilde{h} = h \circ p_b$$

*holds.*

*Proof.* The homeomorphism $\tilde{h}$ is given for any $\sigma \in N(\mathcal{F})$ by the homeomorphism

$$U_\sigma \times |\operatorname{sd}\sigma| \xrightarrow{\mathrm{id}\times h} U_\sigma \times |\sigma|.$$

These respect all identifications and hence they glue together to give $\tilde{h}$. □

**Proposition 3.1.16.** *Let $X$ be a topological space and let $\mathcal{F} = \{X_i\}_{i\in[n-1]}$ be a finite cover. Then, there exists a homeomorphism*

$$X^{\mathcal{F}} \cong \operatorname{hocolim} \mathcal{D}_{\mathcal{F}}.$$

*Proof.* By Proposition 3.1.15, we need to show:

$$X^{\mathcal{F}} \cong \mathrm{B}(\mathcal{F}).$$

The homeomorphism $\mathrm{B}(\mathcal{F}) \to X^{\mathcal{F}}$ is induced for any simplex $\sigma \in N(\mathcal{F})$ by the inclusion

$$U_\sigma \times |\sigma| \hookrightarrow X^{\mathcal{F}}.$$

□

*Remark* 3.1.17. If the cover $\mathcal{U}$ is not finite, then Proposition 3.1.16 might not hold as the subspace topology inherited from the product topology could be coarser than the topology on the homotopy colimit. Unfortunately the author is not aware of an easy counterexample. But we see from the last proof that there is always a continuous bijection $\mathrm{B}(\mathcal{F}) \to X^{\mathcal{F}}$.

We prove one last proposition that will come in handy shortly.

**Proposition 3.1.18.** *Let $X$ be a CW-complex and assume that $\mathcal{F} = \{S_i\}_{i\in I}$ is a cover by subcomplexes. Then* $\operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$ *is a CW-complex as well.*

*Proof.* By Definition 3.1.2,

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{F}} = \Big( \coprod_{\sigma=\{J_0,\dots,J_n\}\in \operatorname{sd} N(\mathcal{F})} S_{J_0} \times |\sigma| \Big) / \sim .$$

The spaces $S_{J_0} \times |\sigma|$ canonically carry a CW-structure, where the cells are the products of cells of $S_{J_0}$ and $|\sigma|$. It follows from [Mun84, Theorem 20.4] that the product topology on $S_{J_0} \times |\sigma|$ is coherent with those. The relation $\sim$ identifies cells and hence $\operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$ is a CW-complex. □

### 3.1.2. For Chain Complexes

Guided by topology, we will deduce a definition of the *homotopy colimit for chain complexes.*

Let $X$ be a CW-complex and assume that $\mathcal{F} = \{S_i\}_{i\in I}$ is a cover by subcomplexes. We compute the cellular chain complex of $\operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$, which is a CW-complex by Proposition 3.1.18. Similar to the computation of the cellular chain complex of a product space [Mas91, p. 280 ff.], the group $C_n(\operatorname{hocolim} \mathcal{D}_{\mathcal{F}})$ is given by

$$\bigoplus_{k=0}^{n} \bigoplus_{\sigma=\{J_0,\dots,J_k\}\in \operatorname{sd} N(\mathcal{F})} C_{n-k}(J_0) \otimes \sigma \subseteq \bigoplus_{p+q=n} C_p(X) \otimes C_q(\operatorname{sd} N(\mathcal{F})) \cong C_n(X \times |\operatorname{sd} N(\mathcal{F})|)$$

and the boundary operator is given for $u \in C_p(X)$ and $v \in C_q(\operatorname{sd} N(\mathcal{F}))$ by

$$\partial(u \otimes v) = (\partial u) \otimes v + (-1)^p u \otimes (\partial u).$$

Said differently, the cellular chain complex of $\operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$ is the total complex of the double complex (in the sense of Remark 2.3.18)

$$\begin{array}{ccccc}
\vdots & & \vdots & & \iddots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\bigoplus_{\sigma=\{J_0,J_1\}\in \operatorname{sd} N(\mathcal{F})} C_0(J_0)\otimes\sigma & \xleftarrow{d} & \bigoplus_{\sigma=\{J_0,J_1\}\in \operatorname{sd} N(\mathcal{F})} C_1(J_0)\otimes\sigma & \xleftarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\bigoplus_{\sigma=\{J_0\}\in \operatorname{sd} N(\mathcal{F})} C_0(J_0)\otimes\sigma & \xleftarrow{d} & \bigoplus_{\sigma=\{J_0\}\in \operatorname{sd} N(\mathcal{F})} C_1(J_0)\otimes\sigma & \xleftarrow{d} & \cdots
\end{array},$$

where $d = \bigoplus \partial \otimes \mathrm{id}$ is induced by the boundary operator on $C_\bullet(X)$ and $\delta = \bigoplus \mathrm{id} \otimes \tilde{\partial}$ by the one on $C_\bullet(\operatorname{sd} N(\mathcal{F}))$.

This is a form that can be generalized. Let $\mathcal{D}$ be a small category and let $F \colon \mathcal{D} \to \mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$ be any functor. Denote by $N_m$ the set of non-degenerate $m$-simplices in the nerve $N(\mathcal{D})$ (see Definition 2.1.14). Let

$$\delta \colon \bigoplus_{\gamma \in N_m} F(\gamma(0)) \to \bigoplus_{\psi \in N_{m-1}} F(\psi(0))$$

be the map

$$\delta = \sum_{i=0}^{n} (-1)^i \tilde{d}_i,$$

where $\tilde{d}_i$ is the identity

$$F(\gamma(0)) \to F(\gamma(0)) = F((d_i\gamma)(0))$$

if $i \neq 0$ and the morphism

$$F(\gamma(0)) \to F(\gamma(1)) = F((d_i\gamma)(0))$$

induced by the functoriality of $F$ if $i = 0$.

This forges the double complex (in the sense of Remark 2.3.18)

$$D_F := \begin{array}{ccccc}
\vdots & & \vdots & & \iddots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\bigoplus_{\gamma\in N_1} F(\gamma(0))_0 & \xleftarrow{d} & \bigoplus_{\gamma\in N_1} F(\gamma(0))_1 & \xleftarrow{d} & \cdots \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
\bigoplus_{D\in\mathcal{D}} F(D)_0 & \xleftarrow{d} & \bigoplus_{D\in\mathcal{D}} F(D)_1 & \xleftarrow{d} & \cdots
\end{array}, \tag{3.1.1}$$

where $d$ is induced by the boundary maps of the chain complexes $F(\gamma(0))$.

**Definition 3.1.19.** The *homotopy colimit* $\operatorname{hocolim} F$ of a functor $F\colon \mathcal{D} \to \mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$ is defined to be the total complex $T(D_F)$, where the double complex $D_F$ is as in Eq. (3.1.1).

Let $F, F'\colon \mathcal{D} \to \mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$ be functors and let there be a natural transformation $Q\colon F \Rightarrow F'$. Then, this induces a map of double complexes $D_F \to D_{F'}$ given for $\gamma \in N_k$ by

$$F(\gamma(0))_m \overset{Q_{\gamma(0)}}{\to} F'(\gamma(0))_m.$$

This in turn, induces a morphism

$$\operatorname{hocolim} F = T(D_F) \to T(D_{F'}) = \operatorname{hocolim} F'.$$

**Lemma 3.1.20** (homotopy lemma)**.** *Assume that the natural transformation $Q$ is a pointwise weak equivalence. Then, the induced map*

$$\operatorname{hocolim} F \overset{\simeq}{\to} \operatorname{hocolim} F'$$

*is a weak equivalence as well.*

*Proof.* The natural transformation induces a map of double complexes, that in turn induces an isomorphism on the horizontal homology. By Proposition 2.3.20, the induced map on the total complexes is a weak equivalence. □

We close this section by recreating some of the constructions we have seen for topological spaces in the realm of homological algebra.

**Definition 3.1.21.** Let $C$ be a chain complex. A *cover of* $C$ is a family of subcomplexes $\{A_i\}_{i \in I}$ such that

$$C = \sum A_i$$

holds. For $J \subseteq I$ we introduce the shorthand notation

$$A_J := \bigcap_{j \in J} A_j,$$

where the intersection is to be understood degree-wise. The cover is called *good* if for $J \subseteq I, |J| < \infty$ we have $A_J = 0$ or if there exists a weak equivalence

$$A_J \to *,$$

where $*$ is the chain complex with $R$ concentrated in degree zero.

**Definition 3.1.22.** Let $\mathcal{U} = \{A_i\}_{i \in I}$ be a cover of $C$. The *nerve* $N(\mathcal{U})$ of the cover is the abstract simplicial complex

$$N(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty,\ A_J \neq 0\}.$$

**Definition 3.1.23.** Let $C$ be a chain complex and let $\mathcal{U} = \{A_i\}_{i \in I}$ be a cover. We define the *nerve diagram* $\mathcal{D}_{\mathcal{U}}$ to be the functor

$$\mathcal{D}_{\mathcal{U}} \colon \mathcal{D} \to \mathrm{Ch}_{\geq 0}(\mathrm{Rmod}),\ J \mapsto A_J,\ \mathcal{D}_{\mathcal{U}}(J \to J') = A_J \hookrightarrow A_{J'},$$

where $\mathcal{D}$ is the opposite face poset of $N(\mathcal{U})$.

There is a canonical chain map

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \to C$$

that is induced by a map of double complexes $\mathcal{D}_{\mathcal{D}_{\mathcal{U}}} \to C$ given on the horizontal axis by

$$\bigoplus_{D \in \mathcal{D}} \mathcal{D}_{\mathcal{U}}(D)_n \to C_n,$$

which is defined on each summand to be the inclusion $\mathcal{D}_{\mathcal{U}}(D)_n \hookrightarrow C_n$, and zero otherwise.

**Definition 3.1.24.** We call this canonical chain map $p_f \colon \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \to C$ the *fiber projection.*

*Remark* 3.1.25. If one starts with a topological setting, as in the beginning of this section, then this fiber projection for chain complexes is obtained by applying the cellular chain complex functor to the topological fiber projection $\operatorname{hocolim} \mathcal{D}_{\mathcal{F}} \to X$ as in Definition 3.1.6.

## 3.2. The Theory of Homotopy Colimits

For simplicity, in this section we will mostly stick to the category of topological spaces, even though the notion of a homotopy colimit makes sense in a much more general setting. For example, we could as well consider *model categories* (see [Hir03] for a definition), which are fundamental in the study of modern homotopy theory as they axiomatize the notions of *weak equivalence, cofibration* and *fibration* and give a tool to study *localizations of categories* and *derived functors.* A lot of the material we will cover is taken from [Rie14] and [Dug].

Before we get into the details, let us take a step back and ask why it is necessary to have the notion of a "homotopy colimit". The ordinary colimit makes precise the idea of "gluing spaces together". From the point of view of homotopy theory, it is thus desirable to have a similar notion that respects homotopy equivalences, i.e. homotopic objects get assigned homotopic homotopy colimits. To see that the ordinary colimit does not have this property, consider the pushout square

$$\begin{array}{ccc} S^1 & \hookrightarrow & D^2 \\ \downarrow & & \\ D^2 & & \end{array} \quad ,$$

which consists of CW-complexes and inclusions of CW-complexes. These are well behaved objects and thus, we might expect that a good notion of homotopy colimit agrees with the ordinary colimit, namely $S^2$. If we now replace $D^2$ by the homotopy equivalent one point space $*$, the colimit is computed to be $*$, which is not homotopy equivalent to $S^2$. Hence, the ordinary colimit lacks this desirable property and we need to adjust the definition.

There is a *global* and a *local definition* of the homotopy colimit. The global definition asks for the existence of a functor whereas the local definition only asks for the existence of an universal object for a specific diagram.

We will start with the global definition. For this we need the following concepts.

**Definition 3.2.1.** The *homotopy category* hTop of Top is defined to be the category where the objects are all topological spaces and the morphisms are homotopy classes of continuous maps.

We denote the canonical quotient functor by

$$\pi\colon \mathrm{Top} \to \mathrm{hTop}\,.$$

**Definition 3.2.2.** Let $\mathcal{C}$ be a category and let $\mathrm{Top}^{\mathcal{C}}$ be the diagram category. We call a functor $F\colon \mathrm{Top}^{\mathcal{C}} \to \mathrm{hTop}$ *homotopical* if it carries pointwise homotopy equivalences between diagrams to isomorphisms in hTop.

*Remark* 3.2.3. In [Str72] it shown that there exists a model structure on Top such that the weak equivalence are the homotopy equivalences. Hence, the *homotopy category in the sense of model categories* is isomorphic to hTop.

**Definition 3.2.4.** The *homotopy colimit* is a homotopical functor

$$\mathrm{hocolim}\colon \mathrm{Top}^{\mathcal{C}} \to \mathrm{hTop}$$

together with a natural transformation $\mathrm{hocolim} \overset{\epsilon}{\Rightarrow} \pi \circ \mathrm{colim}$ that is final among all such functor. That means if $(F, \delta)$ is another such pair, then there exists a unique natural transformation

$$\mathrm{Top}^{\mathcal{C}} \underset{\mathrm{hocolim}}{\overset{F}{\rightrightarrows}} \mathrm{hTop} \quad (\Downarrow \phi)$$

such that $\epsilon \circ \phi = \delta$.

*Remark* 3.2.5. As always with universal properties, the homotopy colimit is unique up to natural isomorphism. Hence, we will always talk about “the” homotopy colimit.

We will now turn our attention to the local definition. Recall that the ordinary colimit of a diagram is by definition a universal cocone. The following allows us to incorporate homotopy theory into this concept.

**Definition 3.2.6.** Let $F\colon \mathcal{C} \to \mathrm{Top}$ be a diagram. A *homotopy coherent cocone* $\mathfrak{C}$ *on* $F$ *with tip* $X \in \mathrm{Top}$ is given by the following datum

(i) For each object $c \in \mathcal{C}$ consider the non-degenerate simplices $N_k$ in the nerve $N(c/\mathcal{C})$ of the under category. To each simplex $\sigma \in N_k$ we assign a continuous map

$$f_\sigma \colon F(c) \times |\Delta^k| \to X$$

such that

$$f_{d_i\sigma} = F(c) \times |\Delta^{k-1}| \overset{\mathrm{id} \times d^i}{\to} F(c) \times |\Delta^k| \overset{f_\sigma}{\to} X.$$

(ii) For any morphism $\gamma \colon c \to c' \in \mathcal{C}$ and any simplex $c' \to v_1 \to \cdots \to v_n$ in $N(c'/\mathcal{C})$ we have

$$f_{c' \to v_1 \to \cdots \to v_n} \circ (F(\gamma) \times \mathrm{id}) = f_{(c' \to v_1 \circ c \to c') \to \cdots \to v_n}.$$

*Remark* 3.2.7. We have deliberately chosen not to use the abstract formalism of modern homotopy theory in order to simplify the exposition and to illustrate why we can think of a coherent cocone as a collection of morphisms that are linked by "higher homotopies". If the reader is familiar with abstract homotopy theory, note that one obtains this definition by unwinding the local definition that can be found in the literature [Rie14, Section 7.7. Homotopy (co)limits as weighted (co)limits]. Recall that one defines for a functor $F \colon \mathcal{C} \to \mathcal{M}$ into a *simplicial model category* a *homotopy coherent cocone with tip* $m$ to be a natural transformation

$$N(-/\mathcal{C}) \Rightarrow \mathrm{Hom}(F(-), m).$$

**Definition 3.2.8.** Let $F \colon \mathcal{C} \to \mathrm{Top}$ be a diagram. The homotopy colimit hocolim$F$ of $F$, if it exists, is a *representing object* of the functor

$$\mathrm{HoCones}_F \colon \mathrm{Top} \to \mathrm{Set},\ X \mapsto \{\text{homotopy coherent cocones on } F \text{ with tip } X\},$$

which means that there exists a homotopy coherent cocone $\mathfrak{C}$ on $F$ with tip $\operatorname{hocolim} F$ such that for any homotopy coherent cocone $\mathfrak{D}$ on $F$ with tip $X$, there exists a unique map $f \colon \operatorname{hocolim} F \to X$ such that $f \circ \mathfrak{C} = \mathfrak{D}$.

*Example* 3.2.9. Take a diagram of the form

$$X \xrightarrow{f} Y.$$

Then, a homotopy coherent cocone on this diagram with tip $Z$ is specified by two continuous maps $g_1 \colon X \to Z$, $g_2 \colon Y \to Z$ and a homotopy $H \colon X \times [0,1] \to Z$ such that

$$H(\cdot, 0) = g_1,\ H(\cdot, 1) = g_2 \circ f.$$

The homotopy colimit of this diagram is easily seen to be the mapping cylinder $M_f$ as $g_1, g_2$ and $H$ glue together to a map $M_f \to Z$.

*Remark* 3.2.10. Our definition of the homotopy colimit implies that, if it exists, it is unique up to unique isomorphism. In the spirit of modern homotopy theory, we could add more coherence and ask it to be unique only "up to contractible choice" (compare [Lur09, Section 1.2.13.]). But we will not need this generality and hence stick with our "light version".

From now on, we will assume for simplicity that $\mathcal{C} = \mathcal{P}$ is a poset. We will now show that for all diagrams $F\colon \mathcal{P} \to \mathrm{Top}$ the homotopy colimit, in the sense of Definition 3.2.8, exists.

The construction makes precise the following intuition.

“ A mathematician blessed with sufficient intuition might define a particular homotopy colimit directly by ‘fattening up’ the ordinary colimit to produce a new object with room to ‘hang a homotopy’. ” (Emily Riehl [Rie])

Let $F\colon \mathcal{P} \to \mathrm{Top}$ be a functor. Then, $F$ can be interpreted as a diagram of spaces $\mathcal{D}_F$ over $A(\mathcal{P})$ (see Remark 2.1.15) by defining

$$\mathcal{D}_F(\{p\}) = F(p),\ \mathcal{D}_F(\{q,r\}) = F(q \to r)$$

for any $p, q, r \in \mathcal{P}$ such that $q \leq r$. Moreover, a natural transformation $F \Rightarrow G$ induces, in a functorial way, a morphism between the diagrams of spaces $\mathcal{D}_F \to \mathcal{D}_G$.

**Definition 3.2.11.** The *bar construction* is defined to be

$$\mathrm{Bar}(F) = \operatorname{hocolim} \mathcal{D}_F,$$

where in this case hocolim is the concrete construction in Definition 3.1.2.

*Remark* 3.2.12. We chose to introduce this naming as the bar construction makes sense in a much more general setting and is common in the literature. See for example [Rie14, Chapter 4: The unreasonably effective (co)bar construction].

**Proposition 3.2.13.** *For any diagram $F\colon \mathcal{P} \to \mathcal{D}$, the bar construction* $\mathrm{Bar}(F)$ *satisfies the universal property in Definition 3.2.8.*

*Proof.* The following datum specifies a homotopy coherent cocone with tip $\mathrm{Bar}(F)$. For every object $c \in \mathcal{P}$ and any non-degenerate $k$-simplex $\sigma = c \to v_0 \to \cdots \to v_k \in N(c/\mathcal{P})$ we take

$$f_\sigma\colon F(c) \times |\Delta^k| \overset{F(c \to v_0)\times \mathrm{id}}{\to} F(v_0) \times |\Delta^k| \to \mathrm{Bar}(F),$$

where the last map is the canonical map into the quotient. Let $\{g_\sigma\}$ be any homotopy coherent cocone with tip $X$. We will show that there exists a unique morphism

$$h\colon \mathrm{Bar}(F) \to X$$

such that

$$h \circ f_\sigma = g_\sigma.$$

Let $v_0 \to \cdots \to v_k$ be a non-degenerate simplex in $N(\mathcal{P})$. Then, for the simplex $\sigma = v_0 \overset{\mathrm{id}}{\to} v_0 \to \cdots \to v_k \in N(v_0/\mathcal{P})$ there exists a map

$$g_\sigma\colon F(v_0) \times |\Delta^k| \to X.$$

By the axioms of a homotopy coherent cocone, these glue together to a map

$$h\colon \operatorname{Bar}(F) \to X.$$

Moreover, we have

$$h \circ f_\sigma = g_\sigma$$

for any $\sigma$ by construction. These equations also prove uniqueness. □

To show that the bar construction also satisfies the global definition is more involved. The proof of this fact goes along the lines of the theory of *deformations* [Rie14].

**Definition 3.2.14.** Let $F\colon \mathcal{P} \to \mathrm{Top}$ be any functor. For every object $c \in \mathcal{P}$ we define the functor

$$F_c\colon \mathcal{P}/c \to \mathrm{Top},\ (z \to c) \mapsto F(z).$$

Then, for every morphism $c \to c'$ in $\mathcal{P}$ there is a canonical map

$$\operatorname{Bar}(F_c) \hookrightarrow \operatorname{Bar}(F_{c'})$$

induced by the obvious inclusion

$$A(\mathcal{P}/c) \hookrightarrow A(\mathcal{P}/c').$$

This assembles to the functor $Q\colon\ \mathrm{Top}^{\mathcal{P}} \to \mathrm{Top}^{\mathcal{P}}$

$$F \mapsto \Big(c \mapsto \operatorname{Bar}(F_c)\Big).$$

For every object $c \in \mathcal{P}$ we know that $c \overset{\mathrm{id}}{\to} c$ is a terminal object in $\mathcal{P}/c$. Hence, $A(\mathcal{P}/c)$ is a cone over the vertex $c \overset{\mathrm{id}}{\to} c$ and we immediately obtain the following lemma.

**Lemma 3.2.15.** *There is a natural transformation* $q\colon Q \Rightarrow \mathrm{id}$ *that is induced on every functor* $F\colon \mathcal{P} \to \mathrm{Top}$ *by contracting* $\operatorname{Bar}(F_c)$ *onto* $F(c)$*. Hence,* $q_F$ *is a pointwise homotopy equivalence.*

**Lemma 3.2.16.** *We have*

$$\operatorname{Bar}(-) = \operatorname{colim} \circ Q$$

*Proof.* This follows directly from the definitions. □

**Corollary 3.2.17.** *The functor*

$$\operatorname{colim} \circ Q\colon\ \mathrm{Top}^{\mathcal{P}} \to \mathrm{Top}$$

*sends pointwise homotopy equivalences to homotopy equivalences.*

*Proof.* This follows from the homotopy lemma together with the previous lemma. □

**Lemma 3.2.18.** *Let* $F\colon \mathcal{P} \to \mathrm{Top}$ *be any diagram. The natural map*

$$\mathrm{colim} \circ Q(Q(F)) \to \mathrm{colim}(Q(F))$$

*that is induced by* $\mathrm{colim} \circ q_Q$ *is homotopic to*

$$\mathrm{Bar}(QF) \to \mathrm{Bar}(F)$$

*induced by* $\mathrm{colim} \circ Q \circ q$*. By the homotopy lemma and Lemma 3.2.15, both are homotopy equivalences.*

*Proof.* The space $\mathrm{Bar}(Q(F))$ is built from pieces of the form

$$F(c) \times |c \to v_1 \to \cdots \to v_n| \times |v_n \to \cdots \to v_{n+k}| \cong F(c) \times |\Delta^n| \times |\Delta^k|$$

for some $n$ and $k$. Hence, a point in this space can be described as a triple $(x, p_1, p_2)$ and the first map is induced by the projection onto $(x, p_1)$, whereas the second map is induced by the projection onto $(x, p_2)$. As $c \to v_1 \to \cdots \to v_n$ and $v_n \to \cdots \to v_{n+k}$ are both faces of $c \to v_1 \to \cdots \to v_{n+k}$, a straight line homotopy in

$$F(c) \times |c \to v_1 \to \cdots \to v_{n+k}| \cong F(c) \times |\Delta^{n+k}|$$

proves the claim. □

**Proposition 3.2.19.** *The functor*

$$(\pi \circ \mathrm{colim}) \circ Q$$

*together with the natural transformation*

$$(\pi \circ \mathrm{colim}) \circ Q \overset{(\pi\circ\mathrm{colim})q}{\to} \pi \circ \mathrm{colim}$$

*satisfies the universal property in Definition 3.2.4.*

*Proof.* We have seen in Corollary 3.2.17 that $\pi \circ (\mathrm{colim} \circ Q)$ is homotopical. Let $F\colon \mathrm{Top}^{\mathcal{P}} \to \mathrm{hTop}$ be any homotopical functor together with a natural transformation $\delta\colon F \Rightarrow \pi \circ \mathrm{colim}$.

As $q$ is a pointwise homotopy equivalence when evaluated at any functor, the morphism $Fq\colon FQ \Rightarrow F$ is a natural isomorphism. From the commutative diagram

$$\begin{array}{ccc} FQ & \overset{\delta_Q}{\Longrightarrow} & \pi \circ \mathrm{colim} \circ Q \\ \Downarrow{\scriptstyle Fq} & & \Downarrow{\scriptstyle (\pi\circ\mathrm{colim})q} \\ F & \overset{\delta}{\Longrightarrow} & \pi \circ \mathrm{colim} \end{array}$$

we deduce that we can factor $\delta$ through $(\pi \circ \mathrm{colim}) \circ Q$ as follows

$$F \overset{(Fq)^{-1}}{\Rightarrow} FQ \overset{\delta_Q}{\Rightarrow} (\pi \circ \mathrm{colim}) \circ Q \overset{(\pi\circ\mathrm{colim})q}{\Rightarrow} \pi \circ \mathrm{colim}\,.$$

To see that this factorization is unique, suppose that $\delta$ factors as

$$F \overset{\delta'}{\Rightarrow} (\pi \circ \operatorname{colim}) \circ Q \overset{(\pi\circ\operatorname{colim})q}{\Rightarrow} \pi \circ \operatorname{colim}.$$

Consider the commutative diagram

$$\begin{array}{ccc} FQ & \overset{\delta'_Q}{\Longrightarrow} & (\pi \circ \operatorname{colim} \circ Q) \circ Q \\ \Downarrow Fq & & \Downarrow (\pi\circ\operatorname{colim}\circ Q)q \\ F & \overset{\delta'}{\Longrightarrow} & \pi \circ \operatorname{colim} \circ Q. \end{array}$$

The vertical arrows are isomorphisms, because $F$ and $\pi \circ (\operatorname{colim} \circ Q)$ are homotopical.

Thus, it suffices to prove that $\delta'_Q$ is uniquely determined. By Lemma 3.2.18, the natural transformation $(\pi \circ \operatorname{colim})q_Q$ is an isomorphism. Hence, $\delta'_Q$ is given by

$$((\pi \circ \operatorname{colim})q_Q)^{-1}\delta_Q. \qquad \square$$

*Remark* 3.2.20. We can also ask ourselves what happens if we replace "homotopy equivalence" by "weak homotopy equivalence" (isomorphism on all homotopy groups) and hTop by the category $\mathrm{Top}[W^{-1}]$, where we formally invert all weak homotopy equivalences (see for example [Rie14] for a definition). Surprisingly, not much will change. The bar construction still computes the homotopy colimit in the new sense.

The attentive reader will have realized that the only ingredient that needs to be adapted is the homotopy lemma. The rest of the proof will go through verbatim.

**Proposition 3.2.21** (weak homotopy lemma)**.** *Let $F, G\colon \mathcal{P} \to \mathrm{Top}$ be any diagrams. Suppose there is a natural transformation $F \Rightarrow G$ that is a pointwise weak equivalence. Then the induced map*

$$\mathrm{Bar}(F) \to \mathrm{Bar}(G)$$

*is a weak equivalence as well.*

*Proof.* This follows from [DI04, Corollary A.6.], whose language we freely use. This corollary states that if we are given a natural transformation $X \Rightarrow Y$ between two functors $X, Y\colon \Delta^{op} \to \mathrm{Top}$ with *free degeneracies* that is a pointwise weak equivalence, then the induced map $|X| \to |Y|$ on the *realizations* is a weak equivalence, as well.

The natural transformation $F \Rightarrow G$ induces a morphism $\tilde{F} \to \tilde{G}$ on the *simplicial replacements* that are given on objects by

$$\tilde{F}\colon \Delta^{op} \to \mathrm{Top},\ [n] \mapsto \coprod_{C_0 \to \cdots \to C_n \in N(\mathcal{P})} F(C_0)$$

and similar for $\tilde{G}$. Moreover, $\tilde{F}$ and $\tilde{G}$ have *free degeneracies* as it is argued in the proof of [DI04, Theorem A.8]. The result now follows from the commuting diagram

$$\begin{array}{ccc} \mathrm{Bar}(F) & \xrightarrow{\cong} & |\tilde{F}| \\ \downarrow & & \downarrow \\ \mathrm{Bar}(G) & \xrightarrow{\cong} & |\tilde{G}| \end{array}$$

together with [DI04, Corollary A.6.]. □

*Remark* 3.2.22. Further, one can also replace Top by $\mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$, where $R$ is any commutative ring with unit, and formally invert all weak chain equivalences. Then, according to [Rod14; Ric] and [Dug, Section 19.8. Abelian categories], the homotopy colimit of a diagram $F\colon \mathcal{P} \to \mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$ can be computed as in Definition 3.1.19.

# 4. Classical Nerve Theorems

In this chapter, we will give conditions under which the horizontal maps in Proposition 3.1.10 are (weak) homotopy equivalences; the functoriality of the nerve theorems will not be addressed here as this was already clarified in the just mentioned proposition. We will also see a homological nerve theorem and discuss an analogy to Proposition 3.1.10.

## 4.1. Nerve Theorems for Open Covers

### 4.1.1. A strong Nerve Theorem

We prove the following classical nerve theorem that can be found for example in [Hat02] or [Seg68].

**Theorem 4.1.1**

Let $X$ be a paracompact space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover. Assume that any finite and non-empty intersection of cover elements is contractible (weakly equivalent to a point). Then, we have the following zigzag of (weak) homotopy equivalences.

$$X \xleftarrow[\simeq]{p_f} \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \xrightarrow[\simeq]{p_b} |\operatorname{sd} N(\mathcal{U})|.$$

*Proof.* It follows from the (weak) homotopy lemma, that the right hand side morphism is a (weak) homotopy equivalence.

It remains to prove that the left hand side morphism is a homotopy equivalence. Choose a partition of unity, which exists due to paracompactness, $\{g_i\}_{i \in I}$ subordinate to $\mathcal{U}$. Define $h\colon X \to \operatorname{hocolim} \mathcal{D}_{\mathcal{U}}$ by

$$h(x) = (x, \sum g_i(x) v_i) \in U_\sigma \times |\sigma| \subseteq \mathrm{B}(\mathcal{U}) \cong \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \text{ (Proposition 3.1.15)},$$

where $v_i$ is the vertex that corresponds to $U_i$ and $\sigma$ is spanned by all vertices $v_j$ with $g_j(x) > 0$. Then $p_f \circ h = \mathrm{id}_X$.

Further, $h \circ p_f$ and $\mathrm{id}_{\operatorname{hocolim} \mathcal{D}_{\mathcal{U}}}$ are homotopic via a straight line homotopy: Take any point $z \in \mathrm{B}(\mathcal{U})$. Then $z = (x, p) \in U_\tau \times |\tau|$ for some $\tau \supseteq \sigma$, where $\sigma$ is as above. Hence,

$$h \circ p_f(z) = h(x) = (x, \sum g_i(x) v_i) \in U_\tau \times |\tau|$$

and a straight line homotopy in the second coordinate proves the claim. □

### 4.1.2. A weak Nerve Theorem

In this subsection, we will strengthen McCord's nerve theorem [McC67] by dropping the assumptions of point-finiteness and basis-likeness (pairwise intersections of cover elements are contained in the cover). We also drop the assumption of paracompactness of the previous subsection.

For this, we need one more ingredient.

**Proposition 4.1.2.** *Let $X$ be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover. Then, the canonical map*

$$\mathrm{B}(\mathcal{U}) \to X$$

*is a weak equivalence, where* $\mathrm{B}(\mathcal{U})$ *is as in Definition 3.1.14.*

*Proof.* This follows from [DI04, Theorem 2.1.]. This theorem says that the *canonical map* $\pi\colon |\check{\mathrm{C}}(\mathcal{U})_*| \to X$, where $|\check{\mathrm{C}}(\mathcal{U})_*|$ is the *realization of the simplicial Čech complex*, is a weak equivalence. Now it suffices to notice that this is just $\mathrm{B}(\mathcal{U}) \to X$. □

**Theorem 4.1.3**

Let $X$ be a topological space and let $\mathcal{U}$ be an open cover. Assume that any finite and non-empty intersection of cover elements is weakly equivalent to a point. Then, we have the following zigzag of weak homotopy equivalences.

$$X \xleftarrow{\;p_f\;} \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \xrightarrow{\;p_b\;} |\operatorname{sd} N(\mathcal{U})|.$$

*Proof.* It follows from the weak homotopy lemma that the right hand side is a weak equivalence. The left hand side is a weak equivalence by Proposition 4.1.2. □

*Remark* 4.1.4. Note that Theorem 4.1.1 implies Theorem 4.1.3 if $X$ is a CW-complex. This follows from the interesting fact that CW-complexes are paracompact (see [Miy52] and [Mun84, Theorem 38.3]).

## 4.2. A Nerve Theorem for Compact Convex Sets

In this section[1], we will present an elementary proof of the nerve theorem for compact convex sets in euclidean space that does not make use of the homotopy colimit.

Let $\mathcal{F} = \{C_i\}_{i=1\dots n}$ be a set of compact and convex subsets of $\mathbb{R}^d$. We construct a continuous map $\Gamma\colon |N(\mathcal{F})| \cong |\operatorname{sd} N(\mathcal{F})| \to X$ that establishes a homotopy equivalence. A similar construction can be found in the literature. See for example the proof of [BT82, Theorem 13.4].

[1] I want to thank Ulrich Bauer again, whose advice helped a lot to shape this section and to make the arguments clear and easily understandable.

**Proposition 4.2.1.** *Each vertex v in the barycentric subdivision* $\operatorname{sd} N(\mathcal{F})$ *represents a finite and non-empty intersection*

$$\bigcap_{j \in \{i_1, \ldots, i_l\}} C_j,$$

*from which we choose a point $p_v$. This choice extends uniquely to a map*

$$\Gamma \colon |\operatorname{sd} N(\mathcal{F})| \to X$$

*that is affine linear on each simplex.*

*Proof.* As the map is uniquely determined by its values on each vertex, uniqueness follows. Convexity of the cover elements in $\mathcal{F}$ guarantees that the affine linear extensions map into $X$ and hence existence follows. □

We prove the following theorem:

**Theorem 4.2.2**

Let $C_1, \ldots, C_n \subseteq \mathbb{R}^d$ be compact and convex subsets. Then the nerve of $\mathcal{F} = \{C_i\}_{i=1,\ldots,n}$ is homotopy equivalent to

$$X = \bigcup_{i=1}^{n} C_i$$

and the homotopy equivalence is induced by $\Gamma$.

We prove the theorem by constructing a homotopy inverse to $\Gamma$. For the construction it will be necessary to work with open covers, because as in the proof of the nerve theorem for open covers (Theorem 4.1.1) we want to make use of a partition of unity. Therefore, we thicken the subsets $C_i$ slightly and consider the collection of open $\epsilon$-neighborhoods of the $C_i$,

$$\mathcal{G} = \{U_i^\epsilon = d(-, C_i)^{-1}([0, \epsilon))\}_{i=1,\ldots,n},$$

where $d(-, C_i)$ is the distance function to $C_i$. The parameter $\epsilon > 0$ will be chosen according to the following lemma.

**Lemma 4.2.3.** *There exists an $\epsilon > 0$ such that the simplicial map $N(\mathcal{F}) \to N(\mathcal{G})$ induced by inclusion is an isomorphism.*

*Proof.* Let $S \subseteq \mathcal{F}$ be a collection with empty intersection. Consider the continuous function $x \mapsto \max_{C \in S} d(x, C)$. This function is strictly positive by the assumption that $S$ has an empty intersection. Choosing $\epsilon_S$ as the global minimum of this function, which exists and is positive due to compactness of the elements of $S$, the corresponding $\epsilon_S$-thickenings have an empty intersection. Finally, choosing $\epsilon$ as the minimum of the $\epsilon_S$ over all $S$ yields the desired result. □

For each $i \in \{1, \dots, n\}$, there exists a function $\varphi_i \colon \mathbb{R}^d \to [0,1]$, openly supported in $U_i^\epsilon$ and equal to 1 on $C_i$.

For example, we may take

$$x \mapsto \frac{d(x, \mathbb{R}^d \setminus U_i^\epsilon)}{d(x, C_i) + d(x, \mathbb{R}^d \setminus U_i^\epsilon)}.$$

These induce a partition of unity on $X$ subordinate to the cover $\{U_i^\epsilon \cap X\}_i$ of $X$ by the following standard construction:

$$\psi_i \colon X \to [0,1], \;\; \psi_i = \frac{\varphi_i}{\sum_{j=1}^n \varphi_j}.$$

Using the choice of $\epsilon$ given by Lemma 4.2.3, we define a map $\Psi \colon X \to |N(\mathcal{F})|$ by

$$\Psi \colon x \mapsto \sum_{i=1}^{n} \psi_i(x) \cdot e_{v_i},$$

where $v_i$ is the vertex in $N(\mathcal{F})$ corresponding to $C_i$.

The following two lemmata are straightforward calculations (compare [ES52, p.62]).

**Lemma 4.2.4.** *Let $K$ be a simplicial complex and let $x \in |K|$. Write $x$ in barycentric coordinates of $K$ as*

$$x = \sum_{j=0}^{m} \nu_j e_{w_j}$$

*with $w_i \in K^0$, $\nu_i > 0$ and $\sum\limits_{j=0}^{m} \nu_j = 1$ as well as $\nu_0 \geq \nu_1 \geq \dots \geq \nu_m$. Then, by using the (geometric) simplices*

$$\tau_i = \operatorname{conv}\{e_{w_0}, \dots, e_{w_i}\} \quad \textit{for all } i \in \{0, \dots, m\} \tag{4.2.1}$$

*in the realization $|K|$ and writing $z(\tau_i)$ for the barycenter of $\tau_i$, we have*

$$x \in \operatorname{conv}\{z(\tau_0), \dots, z(\tau_m)\}.$$

*Specifically, writing $x$ in barycentric coordinates of* $\operatorname{sd} K$ *as*

$$x = \sum_{j=0}^{m} \mu_j z(\tau_j),$$

*we have*

$$\begin{aligned} \mu_i &= (i+1)\Big(\nu_i(x) - \nu_{i+1}(x)\Big) \textit{ for } i = 0, \dots, m-1 \\ \mu_m &= (m+1)\nu_m(x). \end{aligned}$$

**Lemma 4.2.5.** *Let $x \in |\operatorname{sd} K|$, written in barycentric coordinates as*

$$x = \sum_{j=0}^{m} \mu_j z(\tau_j)$$

*for some flag of (geometric) simplices $\tau_0 \subseteq \cdots \subseteq \tau_m$, where*

$$\tau_i = \operatorname{conv}\{e_{w_0}, \ldots, e_{w_i}\} \quad \textit{for all } i \in \{1, \ldots, m\}$$

*and $w_i \in K^0$. Then we have*

$$x \in \tau_m = \operatorname{conv}\{e_{w_0}, \ldots, e_{w_m}\}.$$

*Specifically, the barycentric coordinates $\nu_i$ of $x$ in $K$ with respect to $w_0, \ldots, w_m$ take the form*

$$\nu_i = \sum_{j=i}^{m} \frac{1}{j+1} \mu_j. \tag{4.2.2}$$

**Definition 4.2.6.** For every vertex $v$ of $K$, define the *closed barycentric star* $\operatorname{bst} v$ as the space

$$|\{\sigma \in \operatorname{sd} K \mid v \in \sigma\}| \subseteq |\operatorname{sd} K|.$$

We can determine whether a closed barycentric star contains $x$ by looking at the maximal barycentric coordinates of $x$.

**Proposition 4.2.7.** *Let $K$ be a simplicial complex and let $v$ be a vertex of $K$. A point $x \in |K|$ satisfies*

$$b_v(x) \geq b_w(x) \quad \textit{for all } w \in K^0 \tag{4.2.3}$$

*if and only if $x$ is contained in the closed barycentric star $\operatorname{bst}(v)$. Here, $b_v$ denotes the barycentric coordinate with respect to the vertex $v \in K^0$.*

*Proof.* Let $x \in |K|$ be a point such that Eq. (4.2.3) holds. It suffices to show that $x$ is contained in a simplex $\operatorname{sd} K$ having $v$ as a vertex. Let $\{w_i\}_i$ be the vertices in $K$ with $b_{w_i}(x) > 0$ in descending order of barycentric coordinates. By Eq. (4.2.3) we may choose $w_1 = v$. Now by Lemma 4.2.4, we know that the point $x$ is contained in $\operatorname{conv}\{e_w = z(\tau_1), \ldots, z(\tau_n)\}$, where the $\tau_i$ are as in Eq. (4.2.1). Hence, by definition the point $x$ is contained in $\operatorname{bst} v$.

Conversely, let $x \in \operatorname{bst} v$ for some vertex $v \in K^0$. Let $\sigma \in \operatorname{sd} K$ be the simplex with $x \in \operatorname{relint} |\sigma|$, where relint is the *relative interior* which means the interior in the affine hull. Then $\sigma$ has a coface $\tau \in \operatorname{sd} K$ with $v$ as a vertex such that $\tau$ corresponds to a flag $\{v\} = \tau_1 \subset \tau_2 \subset \cdots \subset \tau_n$ of simplices in $K$. From Lemma 4.2.5, or more specifically Eq. (4.2.2), we can deduce that the barycentric coordinate $\nu_0 = b_v(x)$ of $x$ in $K$ with respect to $v$ is maximal. □

**Lemma 4.2.8.** *Let $\sigma = \{w_0, \ldots, w_k\} \in K$ be a simplex and consider $L = \{\{\tau_0, \ldots, \tau_m\} \mid \sigma \subseteq \tau_m\} \subseteq \operatorname{sd} K$. Then*

$$\bigcap_{i=0}^{k} \operatorname{bst} w_i = |L|.$$

*Proof.* First, let $\phi = \{\tau_0, \dots, \tau_m\} \in L$. Then, $\phi$ is contained in a simplex that contains $\{w_i\}$. Thus, the realization of $\phi$ is contained in $\operatorname{bst} w_i$ for all $i$, and so we have $|L| \subseteq \bigcap_{i=0}^{k} \operatorname{bst} w_i$.

Now let $|\phi = \{\tau_0, \dots, \tau_m\}| \in \bigcap_{i=0}^{k} \operatorname{bst} v_i$. Since for any $i$ we have $|\{\tau_0, \dots, \tau_m\}| \subseteq \operatorname{bst} w_i$, the simplex $\phi$ is contained in one that contains $\{w_i\}$. In particular, $w_i \in \tau_m$ for all $i$, and thus $\sigma \subseteq \tau_m$. Therefore $\phi \in L$ and so we have $\bigcap_{i=0}^{k} \operatorname{bst} w_i \subseteq |L|$. □

**Corollary 4.2.9.** *Every finite intersection* $\bigcap_{i=0,\dots,m} \operatorname{bst} w_i$ *is either empty or contractible.*

*Proof.* Let $\sigma = \{w_0, \dots, w_m\} \in K$. If the intersection is non-empty, then by the previous lemma every (geometric) simplex in $\bigcap_{i=0,\dots,m} \operatorname{bst} w_i$ has a coface in this set with $z(\sigma)$ as a vertex. Thus, $\bigcap_{i=0,\dots,m} \operatorname{bst} w_i$ is star-shaped with respect to $z(\sigma)$ and therefore contractible. □

**Proposition 4.2.10.** *The map* $\Psi$ *is carried by*

$$C\colon \{1, \dots, n\} \to \{\operatorname{bst} v_j \mid v_j \in N(\mathcal{F})^0\},\ i \mapsto \operatorname{bst} v_i,$$

*where* $v_i$ *is the vertex that corresponds to* $C_i$*.*

*Proof.* Recall the definition

$$\Psi\colon x \mapsto \sum_{i=1}^{n} \psi_i(x) \cdot e_{v_i}.$$

Note that if $x \in C_i$, then $\psi_i(x)$ is maximal among the $\psi_j(x)$ by construction. Hence, by Proposition 4.2.7 we know that $\Psi(x) \in \operatorname{bst}(v_i) = C(i)$ and the claim follows. □

We will now show that $\Gamma$ is a homotopy inverse to $\Psi$.

**Proposition 4.2.11.** *The map* $\Gamma$ *is carried by the map of covers*

$$D\colon \{\operatorname{bst}(v_j) \mid v_j \in N(\mathcal{F})^0\} \to \{1, \dots, n\},\ \operatorname{bst}(v_i) \mapsto i.$$

*Proof.* This follows directly from the convexity of the $C_j$. □

**Lemma 4.2.12.** *Let* $Y$ *be a topological space and let* $f\colon S^n \to Y$ *be a continuous map. Further assume that* $Y$ *is contractible. Then there exists a continuous map* $F\colon B^{n+1} \to Y$ *making the following diagram commute:*

$$\begin{array}{ccc} S^n & \xrightarrow{\ f\ } & Y \\ \big\downarrow & \overset{\exists F}{\nearrow} & \\ B^{n+1} & & \end{array} .$$

*Proof.* The contraction induces a homotopy $H\colon S^n \times I \to Y$ between a constant map and the map $f$. The map $F$ is now determined by requiring the commutativity of the diagram

$$\begin{array}{ccc} S^n \times I & \xrightarrow{H} & Y \\ {\scriptstyle (x,r)\mapsto r\cdot x}\big\downarrow & \nearrow_{\exists F} & \\ B^{n+1} & & \end{array}.$$

□

**Proposition 4.2.13.** *Let $K$ be a finite simplicial complex. Assume that $\varphi\colon |K| \to |K|$ is carried by the identity on the set of closed barycentric stars. Then there exists a homotopy*

$$\varphi \simeq \mathrm{id}_{|K|}\,.$$

*Proof.* We inductively construct homotopies

$$H^m\colon |\,\mathrm{sd}\,K^m| \times I \to |K|$$

between the inclusion of the $m$-skeleton, $i_m = |K^m| \hookrightarrow |K|$, and the restriction of $\varphi$ to $|K^m|$, such that $H^m$ is carried by

$$P\colon\ \mathrm{bst}(v_j) \times I \mapsto \mathrm{bst}(v_j).$$

If $m = n$ is the dimension of the simplicial complex, the map $H = H^n$ is the desired homotopy.

To establish the base case $m = 0$, let $p$ be a vertex of $\mathrm{sd}\,K$, and let $v_1, \ldots, v_k \in K^0$ be those vertices with $p \in \mathrm{bst}(v_i)$. By the assumption that $\varphi$ is carried by the identity, we know that both $p$ and $\varphi(p)$ are contained in $\bigcap_{i=1}^k \mathrm{bst}(v_i)$. We have seen in Corollary 4.2.9 that this set is contractible. Hence by Lemma 4.2.12 we can find the desired homotopy $H^0$. This map is carried by $P$.

For the induction step from $(m-1)$ to $m$, let $H^{m-1}$ be as above. Let $\sigma$ be an $m$-simplex in $\mathrm{sd}\,K^m$. Furthermore, let $v_1, \ldots, v_k \in K^0$ be those vertices with $\partial\sigma \subseteq \sigma \subseteq \mathrm{bst}(v_i)$. Then

$$H^{m-1}(\partial\sigma \times I) \subseteq J := \bigcap_{i=1}^{k} \mathrm{bst}(v_i)$$

by induction assumption. As $J$ is contractible, by Lemma 4.2.12 we can extend the homotopy $H^{m-1}|_{\partial\sigma\times I}$ to a homotopy $H^m|_{\sigma\times I}$ from $i_m|_\sigma$ to $\varphi|_\sigma$,

$$\begin{array}{ccc} (\partial\sigma \times I) \cup (\sigma \times \{0,1\}) \cong S^m & \xrightarrow{(H^{m-1},(i_m,\varphi))} & J \subseteq |K| \\ \big\downarrow & \nearrow_{H^m|_{\sigma\times I}} & \\ \sigma \times I \cong B^{m+1} & & \end{array},$$

and by gluing to $H^m\colon |K^m| \times I \to |K|$. This map is carried by $P$. □

*Proof of Theorem 4.2.2.* The map $\Gamma \circ \Psi$ is carried by the identity on $\{1, \ldots, n\}$ by Lemma 2.1.10 and thus it is homotopic to the identity $\mathrm{id}_X$ by the straight line homotopy, since the $C_i$ are convex.

The map $\Psi \circ \Gamma$ is carried by the identity on

$$\mathcal{S} = \{\mathrm{bst}(v_j) \mid v_j \in N(\mathcal{F})^0\}.$$

Hence, by Proposition 4.2.13 the map $\Psi \circ \Gamma$ is homotopic to $\mathrm{id}_{|N(\mathcal{F})|}$.

Therefore we have established a homotopy equivalence

$$|N(\mathcal{F})| \simeq X. \qquad \square$$

*Remark* 4.2.14. It is also possible to show that $\Psi \circ \Gamma$ is homotopic to $\mathrm{id}_{|N(\mathcal{F})|}$ by embedding $|N(\mathcal{F})|$ into a full simplex and showing that it is a neighborhood retract in which the straight line homotopy is fully contained. The composition of this homotopy with the retract then gives the desired homotopy.

*Remark* 4.2.15. One can also prove Theorem 4.2.2 by applying Nagórko's nerve theorem [Nag07]. But this requires some work.

### Functoriality

**Proposition 4.2.16.** *Let the assumptions be as in Theorem 4.2.2. Let $\mathcal{D}_\mathcal{F}$ denote the nerve diagram of $\mathcal{F}$. Then, the natural maps*

$$\begin{array}{ccccc} & & \operatorname{hocolim} \mathcal{D}_\mathcal{F} & & \\ & \swarrow^{p_f} & & \searrow^{p_b} & \\ X & & & & |\operatorname{sd} N(\mathcal{F})| \end{array}$$

*are homotopy equivalences.*

*Proof.* By Proposition 3.1.9 we know that $p_b$ is a homotopy equivalence. Consider the diagram

$$\begin{array}{ccccc} & & \operatorname{hocolim} \mathcal{D}_\mathcal{F} & & \\ & \swarrow^{p_f} & & \searrow^{p_b} & \\ X & & \xleftarrow[\Gamma]{} & & |\operatorname{sd} N(\mathcal{F})|. \end{array}$$

Every point $x \in \operatorname{hocolim} \mathcal{D}_\mathcal{F}$ can be described as a pair $x = (\beta, \alpha)$, where $\alpha$ is a point in some $|\sigma|$, $\sigma = \{J_0, \ldots, J_n\} \in \operatorname{sd} N(\mathcal{F})$, and $\beta \in C_{J_0}$. Then

$$\Gamma(p_b(x)) = \Gamma(\alpha) \in C_{J_n}$$

by construction and

$$p_f(x) = \beta \in C_{J_0} \subseteq C_{J_n}.$$

Therefore a straight line homotopy shows that the maps

$$\Gamma \circ p_b \simeq p_f$$

are homotopic. As $\Gamma$ and $p_b$ are homotopy equivalences the same is true for $p_f$. $\square$

## 4.3. A Nerve Theorem for Covers by Subcomplexes

**Proposition 4.3.1.** *Let $K$ be an abstract simplicial complex and let $X = |K|$ be its geometric realization. Let $\mathcal{F} = \{|A_i|\}_{i \in I}$ be a cover of $X$ by subcomplexes of $K$. Then, the natural map*

$$p_f \colon \operatorname{hocolim} \mathcal{D}_{\mathcal{F}} \to X$$

*is a homotopy equivalence.*

*Proof.* We use Proposition 3.1.15 and prove the statement for the space $\mathrm{B}(\mathcal{F})$. Let us construct a homotopy inverse

$$g \colon X \to \operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$$

to $p_f$. Define the function

$$\Gamma \colon \{\sigma \mid \sigma \in K\} \to \mathcal{P}(I),\ \sigma \mapsto \{i \in I \mid \sigma \in A_i\};$$

for $\tau \subseteq \sigma$ we have $\Gamma(\sigma) \subseteq \Gamma(\tau)$.

For every simplex $\mu \in K$, define

$$I_\mu = \operatorname{im}(|\mu| \times |\Gamma(\mu)| \to \mathrm{B}(\mathcal{F}))$$

and choose a point $p_\mu \in |\Gamma(\mu)|$. We introduce the carriers

$$C \colon S_1 := \{|\sigma| \mid \sigma \in K\} \to S_2 := \{I_\sigma \cup \bigcup_{\tau \in \partial\sigma} I_\tau \mid \sigma \in K\},\ |\sigma| \mapsto I_\sigma \cup \bigcup_{\tau \in \partial\sigma} I_\tau$$

and

$$D \colon S_2 \to S_1,\ I_\sigma \cup \bigcup_{\tau \in \partial\sigma} I_\tau \mapsto |\sigma|.$$

Note that $p_1$, as in Proposition 3.1.15, is carried by $D$.

Our next step is to construct, for every natural number $n \in \mathbb{N}$, a map

$$g^n \colon X^n := |K^n| \to \operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$$

such that

(i) $g^n|_{X^{n-1}} = g^{n-1}$

(ii) $g^n$ is carried by $C$.

Hence, $p_1 \circ g^n$ is carried by the identity on $S_1$.

For $n = 0$, define for any vertex $v$ in $K$

$$g^0(|v|) = (|v|, p_v) \in I_v.$$

Assume that $g^{n-1}$ has been constructed. Let $\sigma$ be an $n$-simplex with characteristic map $\chi_\sigma\colon \mathbb{B}^n \to X$[2]. For any boundary element $\eta \in \partial\sigma$, we have

$$p_2(g^{n-1}(|\eta|)) \subseteq |\Gamma(\eta) \cup \bigcup_{\tau\in\partial\eta} \Gamma(\tau)| =: L(\eta).$$

The full simplex $\Gamma(\sigma)$ is contained in the full simplices $\Gamma(\eta)$ and $\Gamma(\tau)$ for every $\tau \in \partial\eta$. Hence, $L(\eta)$ is contractible onto $p_\sigma$ by a straight line homotopy. This lifts to a straight line homotopy

$$h_\eta\colon |\eta| \times [0,1] \to \mathrm{B}(\mathcal{F})$$

with $h_\eta(\cdot, 0) = g^{n-1}|_\eta$ and such that for all $x \in |\eta|$ and $t \in [0,1]$ we have

$$p_1(h_\eta(x,t)) = p_1(h_\eta(x,0)),\ p_2(h_\eta(x,t)) \in L(\eta) \text{ and } p_2(h_\eta(x,1)) = p_\sigma.$$

As

$$\operatorname{im} h_\eta(\cdot,1) = \operatorname{im} p_1(g^{n-1}|_{|\eta|}) \times \{p_\sigma\} \subseteq |\eta| \times \{p_\sigma\},$$

we can append a straight line homotopy and assume without loss of generality that

$$h_\eta(x,1) = \mathrm{id}_{|\eta|}(x) \times \{p_\sigma\}$$

holds.

These glue together to a homotopy

$$h_{\partial\sigma}\colon |\partial\sigma| \times [0,1] \to \mathrm{B}(\mathcal{F}).$$

We define $g^n$ on $|\sigma|$ by using the characteristic map $\chi_\sigma$. Let $v$ be any unit vector:

- if $1/2 \le t \le 1$ we define

$$g^n(\chi_\sigma(t\cdot v)) = h_{\partial\sigma}(\chi_\sigma(v), 2 - 2\cdot t).$$

- if $0 \le t \le 1/2$ we define

$$g^n(\chi_\sigma(t\cdot v)) = \chi_\sigma(2t\cdot v) \times \{p_\sigma\}.$$

Then

$$g^n(|\sigma|) \subseteq I_\sigma \cup \bigcup_{\tau\in\partial\sigma} I_\tau = C(|\sigma|).$$

This completes the induction step. If $X$ is finite dimensional with dimension $n$, then we can take $g = g^n$. Otherwise, note that

$$\operatorname{colim}_n X^n \cong X$$

and we let $g$ be the unique morphism that is induced by the collection $\{g^n\}_{n\in\mathbb{N}_0}$.

[2] i.e. a homeomorphism such that $\operatorname{im}\chi_\sigma = |\sigma|$ and $\chi_\sigma(S^{n-1}) = |\partial\sigma|$

The map $p_1 \circ g$ is carried by the identity on $S_1$ and hence it is homotopic to the identity on $X$ via a straight line homotopy.

The map $g \circ p_1$ is carried by the identity on $S_2$. Given an element $(x,p) \in |\mu| \times |\sigma|$ with $\mu \in K$ and $\sigma \in \Gamma(\mu)$, then $g \circ p_1(x,p) = (x', p') \in |\mu'| \times |\sigma'|$ with $\mu' \in \partial\mu$ and $\sigma' \in \Gamma(\mu')$. Therefore, we can connect $(x,p)$ to $(x',p)$ by a straight line homotopy and then we connect, again by a straight line homotopy, the element $(x',p)$ to $(x',p')$. This is possible, because $p' = p_2((x',p')) = p_2 \circ g(p_1((x,p))) \in L(\mu)$ and $p = p_2((x,p)) \in |\Gamma(\mu)|$. In this way we can manufacture a homotopy between the identity on $\operatorname{hocolim} \mathcal{D}_{\mathcal{F}}$ and $g \circ p_1$. This completes the proof. □

We can easily prove the following nerve theorem.

**Theorem 4.3.2**

Let $K$ be an abstract simplicial complex and let $X = |K|$ be its geometric realization. Let $\mathcal{F} = \{|A_i|\}_{i \in I}$ be a cover of $X$ by subcomplexes of $K$. Assume that every finite and non-empty intersection of cover elements is contractible. Then, we have the following zigzag of homotopy equivalences.

$$X \xleftarrow[\simeq]{p_f} \operatorname{hocolim} \mathcal{D}_{\mathcal{F}} \xrightarrow[\simeq]{p_b} |\operatorname{sd} N(\mathcal{F})|.$$

*Proof.* By Proposition 4.3.1, the left map is a homotopy equivalence. It follows from the homotopy lemma that the right map is a homotopy equivalence, as well. □

*Remark* 4.3.3. By Whitehead's Theorem ([Hat02, Theorem 4.5]) any weak homotopy equivalence between CW-complexes is already a homotopy equivalence. Therefore, it is not necessary to state a weak nerve theorem, as in Theorem 4.1.3, for covers by subcomplexes.

*Remark* 4.3.4. Note that Theorem 4.2.2 is implied by Theorem 4.3.2 in the case of euclidean balls. This follows from the existence of *triangulations of semi-algebraic sets* [BCR98, Theorem 9.2.1], which states that in this case there exists a triangulation of the union of balls such that each finite and non-empty intersection is a subcomplex of the triangulation.

## 4.4. A Homological Nerve Theorem

Now we want to prove a purely chain complex version of the nerve theorem. The main task will be to prove that the fiber projection $p_f$ (Definition 3.1.24) is a weak equivalence under suitable assumptions. If not said otherwise, homology will always be considered over a commutative ring with unit $R$.

**Proposition 4.4.1.** *Let $C$ be a chain complex of free $R$-modules with a fixed basis $B$. Further, let $\mathcal{U} = \{A_i\}_{i \in I}$ be a cover such that the basis elements that lie in $A_J$ ($J \subseteq I,\ |J| < \infty$) form a basis for it. Then, the fiber projection*

$$p_f \colon \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \to C$$

*is a weak equivalence, where* $\operatorname{hocolim} \mathcal{D}_{\mathcal{U}}$ *is as in Definition 3.1.19 and* $\mathcal{D}_{\mathcal{U}}$ *is the nerve diagram (Definition 3.1.23).*

*Remark* 4.4.2. If the reader is familiar with model categories, then he or she might notice that if $I$ is for example finite, then the existence of this specific basis $B$ is essentially saying that the nerve diagram $\mathcal{D}_{\mathcal{U}}$ is *Reedy cofibrant.* At this level of abstraction, the statement is then a formal consequence.

*Proof.* The proof follows closely the discussion on the Mayer-Vietoris spectral sequence in [Bro94, p. 166-168]. Recall that $\operatorname{hocolim} \mathcal{D}_{\mathcal{U}} = T(D_{\mathcal{D}_{\mathcal{U}}})$ and to ease notation we abbreviate $D_{\mathcal{D}_{\mathcal{U}}}$ by $D$.

Note that in this case, the set $N_m$ is just the set of $m$-simplices in $\operatorname{sd} N(\mathcal{U})$ together with an orientation. We would like to apply Corollary 2.3.21. Therefore, we have to show that the sequence of chain complexes

$$\cdots \xrightarrow{\delta} D_{\bullet,p} \xrightarrow{\delta} \cdots \xrightarrow{\delta} D_{\bullet,0} \xrightarrow{\epsilon} C_{\bullet} \to 0 \tag{4.4.1}$$

is exact, where $\epsilon$ is the map induced by the inclusions $\mathcal{D}_{\mathcal{U}}(J) = A_J \hookrightarrow C$.

To prove this, we rewrite the definition of

$$D_{p,q} = \bigoplus_{\gamma \in N_q} \mathcal{D}_{\mathcal{U}}(\gamma(0))_p.$$

Let $e \in B$ be a basis element of $C$. We define the subcomplex

$$K_e = \{\gamma \in N_m |\ m \in \mathbb{N}_0,\ e \in A_{\gamma(0)}\} \subseteq \operatorname{sd} N(\mathcal{U}).$$

Every basis element of $D_{p,q}$ can be described as a tuple $(\gamma, e)$ with $\gamma \in N_q$ and $e \in \mathcal{D}_{\mathcal{U}}(\gamma(0))_p$. By switching the tuple order, we get an isomorphism

$$D_{p,q} \cong \bigoplus_{e \in C_p \cap B} C_q(K_e).$$

We also have

$$C_p \cong \bigoplus_{e \in C_p \cap B} R.$$

Moreover, $\delta$ gets identified with

$$\bigoplus_{e \in C_p \cap B} \{\partial \colon C_q(K_e) \to C_{q-1}(K_e)\}$$

and $\epsilon$ with

$$\bigoplus_{e \in C_p \cap B} \{\epsilon' \colon C_0(K_e) \to R\},$$

where $\epsilon'$ is the augmentation map $(\sum \lambda_i v_i \mapsto \sum \lambda_i)$.

Thus, Eq. (4.4.1) is the sum of augmented chain complexes and it is exact if we can show that $K_e$ is acyclic for every basis element $e \in B$.

This holds true, because with $I_e = \{\alpha \in I \mid e \in A_\alpha\}$ we have

$$K_e = \mathrm{sd}\{\emptyset \neq J \subseteq I_e \mid |J| < \infty\} \subseteq \mathrm{sd}\, N(\mathcal{U}),$$

and thus every finite subcomplex of $K_e$ is contained in the barycentric subdivision of a full simplex. Therefore, every simplicial chain in $K_e$ is a boundary and $K_e$ is acyclic.

By exactness of Eq. (4.4.1), the vertical homology of $D$ is equal to $C$ concentrated on the $x$-axis. By Corollary 2.3.21, the fiber projection

$$p_f \colon \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} = T(D_{\mathcal{D}_{\mathcal{U}}}) = T(D) \to C$$

is a weak equivalence. □

Before stating the nerve theorem, let us recall that the barycentric subdivision of a simplicial complex does not affect its homology. Hence, in the next theorem one could replace $C^{simp}(\mathrm{sd}\, N(\mathcal{U}))$ by $C^{simp}(N(\mathcal{U}))$.

**Proposition 4.4.3.** *Let $K$ be a simplicial complex. Then, there exists a chain equivalence*

$$\lambda \colon C^{simp}(K) \xrightarrow{\simeq} C^{simp}(\mathrm{sd}\, K).$$

*The morphism $\lambda$ is called the* subdivision operator.

*Proof.* This statement is part of [Mun84, Theorem 17.2] if $R = \mathbb{Z}$. The general statement follows by taking tensor products with $R$. □

**Theorem 4.4.4**

Let $C$ be a chain complex of free $R$-modules with a fixed basis $B$. Further, let $\mathcal{U} = \{A_i\}_{i \in I}$ be a good cover such that the basis elements that lie in $A_J$ ($J \subseteq I$, $|J| < \infty$) form a basis for it. Moreover, assume that for the nerve diagram $\mathcal{D}_{\mathcal{U}}$, there exists a natural transformation

$$\eta \colon \mathcal{D}_{\mathcal{U}} \Rightarrow *$$

that is a pointwise weak equivalence to the constant diagram on the chain complex that has $R$ concentrated in degree zero. Then, the canonical morphisms

$$C \xleftarrow{p_f} \operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \xrightarrow{\eta_*} \operatorname{hocolim} * = C^{simp}(\mathrm{sd}\, N(\mathcal{U})),$$

where $\eta_*$ is induced by the functoriality of hocolim, are weak equivalences.
In particular, we get

$$H_*(C) \cong H_*^{simp}(\operatorname{sd} N(\mathcal{U})).$$

*Remark* 4.4.5. As all the involved chain complexes are non-negative and free, it follows from Remark 2.3.5 that the morphisms are actually chain equivalences.

*Remark* 4.4.6. Note that the assumptions in this theorem are just an abstraction of the case when one takes the simplicial chain complexes of a cover of a simplicial complex by subcomplexes (compare Corollary 4.4.10). In this setting, Theorem 4.4.4 follows from Theorem 4.3.2 together with Remark 3.1.25.

*Proof.* By Lemma 3.1.20, we have

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{U}} = T(D_{\mathcal{D}_{\mathcal{U}}}) \xrightarrow{\simeq} T(D_*) = C^{simp}(\operatorname{sd} N(\mathcal{U})).$$

The left hand side is the content of Proposition 4.4.1. □

*Remark* 4.4.7. The statement and proof of this theorem look quite similar to the ones for the other nerve theorems, like Theorem 4.1.1. In Remark 3.2.22 it was also mentioned that $\operatorname{hocolim} \mathcal{D}_{\mathcal{U}}$ is in fact a homotopy colimit in an abstract sense. This similarity will be elaborated on in a future publication.

### Functoriality

We didn't discuss the functoriality in Chapter 3 as the conditions we have to invoke in order to get it seem to be a bit artificial at first but are given if one starts with a topological setting (see Remark 4.4.12).

Let $C$ and $D$ be equipped with a cover $\mathcal{A} = \{A_i\}_{i\in I}$ and $\mathcal{B} = \{B_j\}_{j\in J}$, respectively. Assume that there exists a chain map $f\colon C \to D$ and a map $Q\colon I \to J$ such that

$$f(A_i) \subseteq B_{Q(i)}$$

holds for all $i \in I$.

Then, this induces a map of double complexes

$$D_{\mathcal{D}_{\mathcal{A}}} \to D_{\mathcal{D}_{\mathcal{B}}}$$

that is given for any $\gamma = J_0 \to \cdots \to J_k \in N_k^{\mathcal{A}}$, where $N_k^{\mathcal{A}}$ is the set of non-degenerate $k$-simplices in the nerve of the opposite face poset of $N(\mathcal{A})$, by

$$\mathcal{D}_{\mathcal{A}}(J_0)_n \to \mathcal{D}_{\mathcal{B}}(Q(J_0))_n = \begin{cases} f_{|\mathcal{D}_{\mathcal{A}}(J_0)_n} & \text{if } Q(J_0) \supsetneq \cdots \supsetneq Q(J_n) \\ 0 & \text{else.} \end{cases}$$

By passing to the total complex, we get a commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim} \mathcal{D}_{\mathcal{A}} & \longrightarrow & \operatorname{hocolim} \mathcal{D}_{\mathcal{B}} \\ \downarrow{\scriptstyle p_f} & & \downarrow{\scriptstyle p_f} \\ C & \xrightarrow{\quad f \quad} & D, \end{array}$$

where $p_f$ is the fiber projection as in Definition 3.1.24.

This shows half of the functoriality we want. It remains to show that the map

$$\operatorname{hocolim} \mathcal{D}_{\mathcal{U}} \to C^{simp}(\operatorname{sd} N(\mathcal{U}))$$

is functorial under suitable assumptions. Recall that we assume that there are natural transformations

$$\mathcal{D}_{\mathcal{A}} \overset{\Psi_{\mathcal{A}}}{\Rightarrow} *_{\mathcal{A}},\ \mathcal{D}_{\mathcal{B}} \overset{\Psi_{\mathcal{B}}}{\Rightarrow} *_{\mathcal{B}},$$

where $*_{\mathcal{A}}$ and $*_{\mathcal{B}}$ are the constant diagrams on the chain complex with $R$ concentrated in degree zero.

If for every $J \in N_0^{\mathcal{A}}$ we have

$$\Big((\Psi_{\mathcal{B}})_{Q(J)} \circ f_{|\mathcal{D}_{\mathcal{A}}(J)}\Big)_0 = \Big((\Psi_{\mathcal{A}})_J\Big)_0 \colon \mathcal{D}_{\mathcal{A}}(J)_0 \to R,$$

then the induced maps on the double complexes

$$\begin{array}{ccc} D_{\mathcal{D}_{\mathcal{A}}} & \longrightarrow & D_{\mathcal{D}_{\mathcal{B}}} \\ \downarrow & & \downarrow \\ D_{*_{\mathcal{A}}} = C^{simp}(\operatorname{sd} N(\mathcal{A})) & \xrightarrow{\tilde{Q}} & D_{*_{\mathcal{B}}} = C^{simp}(\operatorname{sd} N(\mathcal{B})). \end{array}$$

commute, where $\tilde{Q}$ is the map that is induced by $Q$ and the vertical morphisms are induced by the natural transformations. The desired result now follows by applying the total complex functor.

Thus, we have proven:

**Proposition 4.4.8.** *Let $C$ and $D$ be equipped with a cover $\mathcal{A} = \{A_i\}_{i \in I}$ and $\mathcal{B} = \{B_j\}_{j \in J}$, respectively. Assume that there exists a chain map $f \colon C \to D$ and a map $Q \colon I \to J$ such that*

$$f(A_i) \subseteq B_{Q(i)}$$

*holds for all $i \in I$. Moreover, assume that there are natural transformations $\mathcal{D}_{\mathcal{A}} \overset{\Psi_{\mathcal{A}}}{\Rightarrow} *_{\mathcal{A}}$ and $\mathcal{D}_{\mathcal{B}} \overset{\Psi_{\mathcal{B}}}{\Rightarrow} *_{\mathcal{B}}$ such that for every $J \in N_0^{\mathcal{A}}$ we have*

$$\Big((\Psi_{\mathcal{B}})_{Q(J)} \circ f_{|\mathcal{D}_{\mathcal{A}}(J)}\Big)_0 = \Big((\Psi_{\mathcal{A}})_J\Big)_0 \colon \mathcal{D}_{\mathcal{A}}(J)_0 \to R.$$

*Then, the following diagram commutes.*

$$\begin{array}{ccccc} C & \xleftarrow[p_f]{} & \operatorname{hocolim} \mathcal{D}_{\mathcal{A}} & \longrightarrow & C^{simp}(\operatorname{sd} N(\mathcal{A})) \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \tilde{Q}} \\ D & \xleftarrow[p_f]{} & \operatorname{hocolim} \mathcal{D}_{\mathcal{B}} & \longrightarrow & C^{simp}(\operatorname{sd} N(\mathcal{B})). \end{array}$$

*Remark* 4.4.9. The main difference to the topological setting is that we have to assume that there exist compatible natural transformations $\mathcal{D}_\mathcal{A} \overset{\Psi_\mathcal{A}}{\Rightarrow} *_\mathcal{A}$, $\mathcal{D}_\mathcal{B} \overset{\Psi_\mathcal{B}}{\Rightarrow} *_\mathcal{B}$. In Top these always exist and are compatible as the one point space is the terminal object, which is not true in $\mathrm{Ch}_{\geq 0}(\mathrm{Rmod})$ for the chain complex with $R$ concentrated in degree zero.

## Applications

We will now apply the homological nerve theorem to get two interesting statements.

**Corollary 4.4.10.** *Let $X$ be a CW-complex and let $\mathcal{U} = \{X_\alpha\}_\alpha$ be a cover by subcomplexes such that every non-empty and finite intersection of cover elements is acyclic. Then, there exists a chain equivalence*

$$C^{cell}(X) \overset{\simeq}{\to} C^{simp}(\mathrm{sd}\, N(\mathcal{U})).$$

*In particular,*

$$H_*(X) \cong H_*(\mathrm{sd}\, N(\mathcal{U})).$$

*Proof.* Apply Theorem 4.4.4 and Remark 4.4.5 to the cellular chain complex $C^{cell}(X)$ and the cover $\mathcal{U}' = \{C^{cell}(X_\alpha)\}_\alpha$. We get the diagram

$$C^{cell}(X) \overset{\simeq}{\leftarrow} \mathrm{hocolim}\, \mathcal{D}_{\mathcal{U}'} \overset{\simeq}{\to} C^{simp}(\mathrm{sd}\, N(\mathcal{U}')) = C^{simp}(\mathrm{sd}\, N(\mathcal{U})),$$

where all maps are chain equivalences. As chain equivalences have chain homotopy inverses, the claim follows. □

**Corollary 4.4.11.** *Let $X$ be a topological space and let $\mathcal{U} = \{X_i\}_{i\in I}$ be an open cover such that every non-empty and finite intersection of cover elements is acyclic. Then, there exists a chain equivalence*

$$S(X) \overset{\simeq}{\to} C^{simp}(\mathrm{sd}\, N(\mathcal{U})),$$

*where $S(X)$ is the singular chain complex of $X$. In particular,*

$$H_*(X) \cong H_*(\mathrm{sd}\, N(\mathcal{U})).$$

*Proof.* By [Mun84, Theorem 31.5] and tensoring this result with $R$ we get that the canonical inclusion

$$S^\mathcal{U}(X) \overset{\simeq}{\hookrightarrow} S(X)$$

is a chain equivalence, where $S^\mathcal{U}(X)$ is the $\mathcal{U}$-small singular chain complex (the image of a map $|\Delta^n| \to X$ is contained in a single cover element).

We apply Theorem 4.4.4 and Remark 4.4.5 to the cover $\mathcal{U}' = \{S(X_i)\}_{i\in I}$. Then we get

$$S^\mathcal{U}(X) \overset{\simeq}{\leftarrow} \mathrm{hocolim}\, \mathcal{D}_{\mathcal{U}'} \overset{\simeq}{\to} C^{simp}(\mathrm{sd}\, N(\mathcal{U}')) = C^{simp}(\mathrm{sd}\, N(\mathcal{U})),$$

where all maps are chain equivalences. As chain equivalences have chain homotopy inverses, the claim follows. □

*Remark* 4.4.12. Note that we can apply Proposition 4.4.8 in these settings. In both cases, there exists a unique natural transformation $\mathcal{D}_{\mathcal{U}} \Rightarrow *$ to the constant diagram on the one point space. By passing to the cellular/singular chain complex, we get the necessary natural transformation of diagrams of chain complexes. Moreover, a (cellular) map between covered spaces forces the natural transformations to be compatible in the sense of Proposition 4.4.8.

# 5. Persistent Nerve Theorems

It is a general scheme in topological data analysis to adapt classical notions of (algebraic) topology to an “approximate” setting. Hence, it is tempting to relax the notion of a good cover to an *$\epsilon$-good cover*, which will be explained in the first section of this chapter. In recent years this idea has been picked up to establish *persistent nerve theorems* [CS18; GS18].

Throughout this chapter we fix a field $k$. Homology, if not said otherwise, is always considered over $k$. By $P$ we will always mean either $\mathbb{Z}$ or $\mathbb{R}$.

Recall the following concepts as can be found for $\mathcal{C} = \mathrm{Vec}_k$ in [BL15].

**Definition 5.0.1.** Let $\mathcal{C}$ be a category. For any $\delta \in P$, the *shift functor* $(\cdot)(\delta)\colon \mathcal{C}^P \to \mathcal{C}^P$ is the endofunctor on the category of functors $F\colon P \to \mathcal{C}$ defined by $F(\delta)_t = F_{t+\delta}$, $F(\delta)(s \leq t) = F(s+\delta \leq t+\delta)$. For any morphism $f$ in $\mathcal{C}^P$ we define $f(\delta)_t = f_{t+\delta}$.

**Definition 5.0.2.** Let $\mathcal{C}$ be a category. Two functors

$$F, G\colon P \to \mathcal{C}$$

are called *$\delta$-interleaved* if there exist natural transformations $\varphi\colon F \to G(\delta)$ and $\psi\colon G \to F(\delta)$ such that the equations

$$\psi_{t+\delta} \circ \varphi_t = F(t \leq t+2\delta),\ \ \varphi_{t+\delta} \circ \psi_t = G(t \leq t+2\delta)$$

hold for every $t \in P$. Then, we write $F \overset{\delta}{\sim} G$. The *interleaving or bottleneck distance* between $F$ and $G$ is defined to be

$$d_B(F, G) = \inf\{\delta \mid \exists F \overset{\delta}{\sim} G\}.$$

We state two elementary observations about interleavings that follow directly from the definition and that come in handy shortly.

**Lemma 5.0.3.** *Let $F, G\colon P \to \mathcal{C}$ be two $\delta$-interleaved functors. If $H\colon \mathcal{C} \to \mathcal{D}$ is any functor, then $H \circ F$ and $H \circ G$ are also $\delta$-interleaved.*

**Lemma 5.0.4.** *Let $F, G, H\colon P \to \mathcal{C}$ be functors. If $F$ and $G$ are $\delta$-interleaved and $G$ and $H$ are $\epsilon$-interleaved, then $F$ and $H$ are $(\delta+\epsilon)$-interleaved.*

Let us introduce more terminology.

**Definition 5.0.5.** A *persistently covered space* $F$ is a functor

$$F = (X_r, \mathcal{U}_r = \{U_i^r\}_{i \in I_r})_r \colon P \to \mathrm{Cov},$$

where Cov is the category of covered spaces (Definition 2.1.8). For any two numbers $t, s \in P$ with $t \leq s$ we write

$$F(t \leq s) = (F(t \leq s)_f, F(t \leq s)_C) \colon (X_t, \mathcal{U}_t) \to (X_s, \mathcal{U}_s).$$

**Proposition 5.0.6.** *Let* $F = (X_r, \mathcal{U}_r = \{U_i^r\}_{i \in I_r})_r \colon P \to \mathrm{Cov}$ *be a persistently covered space and assume that* $F$ *is* $\delta$*-interleaved with a functor* $G = (Y_r, \mathcal{V}_r = \{V_j^r\}_{j \in J_r})_r \colon P \to \mathrm{Cov}$*. Then, the functor*

$$\Phi_S \circ F \colon P \to \mathrm{Top},\ r \mapsto X_r$$

*is* $\delta$*-interleaved with*

$$\Phi_S \circ G \colon P \to \mathrm{Top},\ r \mapsto Y_r.$$

*Moreover,*

$$N \circ F \colon P \to \mathrm{Simp},\ r \mapsto N(\mathcal{U}_r)$$

*is* $\delta$*-interleaved with*

$$N \circ G \colon P \to \mathrm{Simp},\ r \mapsto N(\mathcal{V}_r).$$

*Proof.* This follows directly from Lemma 5.0.3. □

A naive formulation of a persistent nerve theorem is the following.

**Theorem 5.0.7**

If in addition to the assumptions of Proposition 5.0.6 all the covered spaces $G(t)$, $t \in P$, satisfy either Theorem 4.1.1, Proposition 4.2.16 or Theorem 4.3.2, then

$$\pi \circ \Phi_S \circ F \colon P \to \mathrm{hTop},\ r \mapsto X_r$$

is $2\delta$-interleaved with

$$\pi \circ |\cdot| \circ N \circ F \colon P \to \mathrm{hTop},\ r \mapsto |N(\mathcal{U}_r)|,$$

where $\pi \colon \mathrm{Top} \to \mathrm{hTop}$ is the canonical quotient functor as in Definition 3.2.1.

*Proof.* By the mentioned theorems, we know that $Y_t$ and $|N(\mathcal{V}_t)|$ are *functorially isomorphic* in hTop, which means that $\pi \circ (Y_r)_r \cong \pi \circ (|N(\mathcal{V}_r)|)_r$ are isomorphic functors. By concatenation,

$$\pi \circ (X_r)_r \overset{\delta}{\sim} \pi \circ (Y_r)_r \cong \pi \circ (|N(\mathcal{V}_r)|)_r \overset{\delta}{\sim} \pi \circ (|N(\mathcal{U}_r)|)_r$$

we get that $\pi \circ (X_r)_r$ is $2\delta$-interleaved with $(\pi \circ |N(\mathcal{U}_r)|)_r$. □

*Remark* 5.0.8. It is an interesting question, whether the upper bound of $2\delta$ in Theorem 5.0.7 on the bottleneck distance is sharp. This is also a topic of further research.

**Corollary 5.0.9.** *Let the setting be as in Theorem 5.0.7. Then,* $(H_*(X_r))_r$ *is* $2\delta$*-interleaved with* $(H_*(N(\mathcal{U}_r)))_r$*.*

*Proof.* This is a direct consequence of the fact that homotopic maps induce the same map in homology. ☐

Let us give a small example how the theorem can be applied.

**Corollary 5.0.10.** *Let* $F\colon \mathbb{R} \to \mathbf{Cov}$ *be a functor and consider*

$$F^\delta\colon \mathbb{R} \to \mathbf{Cov},\ F^\delta(r) = F(r^\delta := \max\{m \in \delta\mathbb{Z} \mid m \leq r\}),\ F^\delta(r \leq r') = F(r^\delta \leq r'^\delta),$$

*i.e. we discretize the functor. Then,* $N \circ F^\delta$ *converges to* $N \circ F$ *in terms of the bottleneck distance*

$$d_B(N \circ F^\delta, N \circ F) \leq \delta \overset{\delta \searrow 0}{\to} 0.$$

*In particular, as homology is a functor, we have the same convergence behavior on the level of homology by Lemma 5.0.3.*

*Proof.* By Theorem 5.0.7 it suffices to define a $\delta$-interleaving by $(F^\delta_r \to F_{r+\delta}) = F(r^\delta \leq r + \delta)$ and $(F_r \to F^\delta_{r+\delta}) = F(r \leq (r+\delta)^\delta)$ for any $r \in \mathbb{R}$. ☐

Even though this persistent nerve theorem gives an interleaving between the spaces and the nerves, it has two drawbacks. On the one hand it has quite serious assumptions as the conclusion is more or less a formal consequence. On the other hand, one gets an interleaving in hTop. When working with homology this is not an issue, but if one wants to work on the level of homotopy theory, say in the form of the *homotopy interleaving distance* as introduced in [BL17], then this is a major problem.

In this chapter we will discuss a way to weaken the assumptions of the theorem, while accepting a worse bound on the interleaving distance. Before doing so, let us quote the two main versions of a persistent nerve theorem that can be found in the literature. The reader who is not familiar with the notions in the following statements can be reassured that all necessary definitions will be given, in our terminology, later.

**Theorem 5.0.11 ([GS18])**

Given a space $X$ endowed with a function $f$ and a (filtered) cover $\mathcal{U}$, if every non-empty finite intersection of cover elements is $\epsilon$-acyclic, then there exists a function on the nerve $g\colon N(\mathcal{U}) \to \mathbb{R}$, such that the bottleneck distance $d_B(\cdot)$ is bounded by

$$d_B(\mathrm{Dgm}(X, f), \mathrm{Dgm}(N(\mathcal{U}), g)) \leq 2(Q + 1)\epsilon,$$

where

$$Q = \min\{\dim(X), \dim(N(\mathcal{U}))\}.$$

The authors proved this version by closely analyzing the *Mayer-Vietoris spectral sequences*, which we used implicitly in the proof of Theorem 4.4.4, associated to the covers.
In this thesis we will be more interested in the proof strategy of the following version.

**Theorem 5.0.12 ([CS18])**

Given a finite collection of finite simplicial filtrations $\mathcal{U} = \{U_0, \ldots, U_n\}$, where $U_i := (U_i^\alpha)_{\alpha \geq 0}$ and all $U_i^\alpha$ are subcomplexes of a sufficiently large simplicial complex, if $\mathcal{U}$ is an $\epsilon$-good cover filtration of $\mathcal{W} := (\bigcup_{i=0}^n U_i^\alpha)_{\alpha \geq 0}$, then

$$d_B(\mathrm{Dgm}_K(\mathcal{W}), \mathrm{Dgm}_K(N(\mathcal{U}))) \leq (K+1)\epsilon,$$

where $K \in \mathbb{N}$ is a natural number.

*Remark* 5.0.13. It would be an interesting task to compare the proofs of these two persistent nerve theorems to see if there are any similarities. Moreover, it is still a mystery to the author why in Theorem 5.0.11 the dimension of $X$ is appearing in the bound on the bottleneck distance. This is not at all apparent with the proof strategy of Theorem 5.0.12.

## 5.1. $\epsilon$-Good Covers

In this section we will introduce *$\epsilon$-goodness*. We will also fix the proof of a key fact in [CS18], which is interesting in its own right.

**Definition 5.1.1.** Let $F = (X_r, \mathcal{U}_r = \{U_i^r\}_{i \in I_r})\colon P \to \mathrm{Cov}$, be a persistently covered space and let $\epsilon \in P$ be a number. We call $F$ *$\epsilon$-good* if for every number $n \in P$ and every simplex $J \in N(\mathcal{U}_n)$ we have

$$\tilde{H}_*(F(n \leq n+\epsilon)_f\colon U_J^n \to U^{n+\epsilon}_{F(n\leq n+\epsilon)_C(J)}) = 0,$$

where $\tilde{H}_*$ is the reduced singular homology with coefficients in $k$.

The key fact we are interested in relates this condition to a more homotopy theoretic property.

**Theorem 5.1.2**

Let $f\colon C_\bullet \to D_\bullet$ be a chain map between chain complexes of vector spaces over $k$. Assume that $f_*\colon H_*(C_\bullet) \to H_*(D_\bullet)$ is the zero map. Then, $f$ is chain homotopic to the zero morphism.

*Remark* 5.1.3. From a very abstract point of view this theorem is a direct consequence of the fact that the category $\mathrm{Rep}_k A_2$ of *representations of $A_2$-quivers* is *hereditary*, which means that the functor $\operatorname{Ext}^2$ vanishes identically. Chain complexes in such categories are always related to their homology, viewed as a chain complex with the boundary morphisms being identically zero, by a zigzag of weak equivalences. See for example [Sey]. But we will not pursue this idea as it will be of no help for what follows.

*Proof.* For every natural number $i \in \mathbb{N}_0$ we decompose $C_i$ in the following way

$$C_i \cong \ker \partial_i \oplus V_i \cong (H_i \oplus \operatorname{im} \partial_{i+1}) \oplus V_i,$$

where $V_i$ and $H_i$ are chosen complements in $C_i$ and $\ker \partial_i$, respectively. Note that the boundary map is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \partial_{i|V_i} \\ 0 & 0 & 0 \end{pmatrix}$$

and that $\partial_{i|V_i}$ is an isomorphism. In this form, we can directly give a chain homotopy. Choose a basis for $H_i$. By assumption we have $f_* = 0$ and therefore we can choose for any basis element $x \in H_i$ an element $z \in D_{i+1}$ such that

$$f_i(x) = \overline{\partial}_{i+1} z$$

holds. Then, we define $h_i(x) = z$. Further, we define

$$h_{i|\operatorname{im} \partial_{i+1}} = f_{i+1} \circ \partial_{i+1}{}^{-1}_{|V_{i+1}}$$

and $h_{i|V_i} = 0$. One easily checks that the equation

$$f_i = \overline{\partial}_{i+1} \circ h_i + h_{i-1} \circ \partial_i$$

holds. □

*Remark* 5.1.4. Note that in the very first step in the proof we use the fact that complements always exist. This is, of course, not true when working with abelian groups.

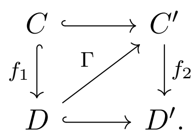
**Corollary 5.1.5.** *Let $f, g \colon C_\bullet \to D_\bullet$ be chain maps between chain complexes of vector spaces over $k$. Assume that both induce the same map in homology $f_* = g_*$. Then, $f$ and $g$ are chain homotopic.*

*Proof.* Consider $f - g$ and apply Theorem 5.1.2. □

We now enhance the proof of Theorem 5.1.2 to deduce the following theorem.

**Theorem 5.1.6**

Consider the following commutative diagram of chain complexes over the field $k$:



Assume that there exists a null-homotopy $h = \{h_i\}_{i\in\mathbb{N}_0}$ for $f_1$ and that $f_2$ induces the zero morphism in homology. Then, there exists a null-homotopy $h' = \{h'_i\}_{i\in\mathbb{N}_0}$ for $f_2$ such that
$$h'_{i|C_i} = h_i.$$

*Remark* 5.1.7. Theorem 5.1.2 follows easily from Theorem 5.1.6 by setting $C = D = 0$. Nevertheless the proof of Theorem 5.1.2 is so short that it is included as a warm up to grasp the idea of the proof of Theorem 5.1.6.

*Proof.* Let $i \in \mathbb{N}_0$ be any natural number. We consider the following decomposition as in the proof of Theorem 5.1.2
$$C_i \cong (H_i \oplus \operatorname{im}\partial^C_{i+1}) \oplus V_i$$
with boundary map
$$\partial^C_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \partial^C_{i\,|V_i} \\ 0 & 0 & 0 \end{pmatrix}.$$
Note that $\partial_{|V_i}$ is an isomorphism.

Further, we choose for $C'_i$ a decomposition
$$C'_i \cong (H'_i \oplus \operatorname{im}\partial^{C'}_{i+1}) \oplus V'_i$$
such that the inclusion $C \hookrightarrow C'$, using that it is null-homologous and hence cycles of $C$ get mapped injectively into the image of $\partial^{C'}_{i+1}$, takes the form
$$\begin{pmatrix} 0 & 0 & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{pmatrix}.$$
Hence, by choosing a complement $W_i$ in $\operatorname{im}\partial^{C'}_{i+1}$ we get the decomposition
$$C'_i \cong H'_i \oplus (H_i \oplus \operatorname{im}\partial^C_{i+1} \oplus W_i) \oplus V'_i$$
such that the inclusion $C \hookrightarrow C'$ gets identified with the obvious inclusion
$$(H_i \oplus \operatorname{im}\partial^C_{i+1}) \oplus V_i \hookrightarrow H'_i \oplus (H_i \oplus \operatorname{im}\partial^C_{i+1} \oplus W_i) \oplus V'_i,\ (x, y, z) \mapsto (0, x, y, 0, z).$$
As
$$\partial^{C'}_{i+1|V'_{i+1}} : V'_{i+1} \to H_i \oplus \operatorname{im}\partial^C_{i+1} \oplus W_i$$
is an isomorphism, we can pull back the decomposition to get the following direct sum decomposition
$$V'_{i+1} \cong T^1_{i+1} \oplus V_{i+1} \oplus T^2_{i+1}$$

such that the boundary map is given by

$$\partial^{C'}_{i+1|T^1_{i+1}\oplus V_{i+1}\oplus T^2_{i+1}} = \begin{pmatrix} \partial^{C'}_{|T^1_{i+1}} & 0 & 0 \\ 0 & \partial^{C'}_{|V_{i+1}} & 0 \\ 0 & 0 & \partial^{C'}_{|T^2_{i+1}} \end{pmatrix} : T^1_{i+1} \oplus V_{i+1} \oplus T^2_{i+1} \to H_i \oplus \operatorname{im} \partial^C_{i+1} \oplus W_i$$

and all appearing maps are isomorphisms. The inclusion $C \hookrightarrow C'$ now takes the form

$$(H_i \oplus \operatorname{im} \partial^C_{i+1}) \oplus V_i \hookrightarrow H'_i \oplus (H_i \oplus \operatorname{im} \partial^C_{i+1} \oplus W_i) \oplus (T^1_i \oplus V_i \oplus T^2_i),\ (x, y, z) \mapsto (0, x, y, 0, 0, z, 0).$$

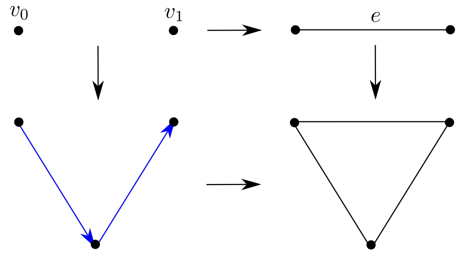
We define the homotopy $h' = \{h'_i \colon C'_i \to D'_{i+1}\}_{i\in\mathbb{N}_0}$ in the following order by considering different cases.

- Choose a basis for $H'_i$. For any basis element $x \in H'_i$ we have $\partial^{C'}_i x = 0$ and so we can choose an element $z \in D'_{i+1}$ such that $\partial^{D'}_{i+1} z = f_2(x)$. Then, we define $h'_i(x) = z$.
- Let $x \in H_i \oplus \operatorname{im} \partial^C_{i+1} \oplus V_i$. We define $h'_i(x) = h_i(x)$. This shows that $h'$ extends $h$.
- Let $x \in W_i$. We define $h'_i(x) = f_2((\partial^{C'}_{|T^2_{i+1}})^{-1}(x))$.
- Let $x \in T^2_i$. We define $h'_i(x) = 0$.
- Choose a basis for $T^1_i$. For every basis element $x \in T^1_i$ we need to find an element $z \in D'_{i+1}$ such that
  $$f_2(x) = h'_{i-1}(\partial^{C'}_i(x)) + \partial^{D'}_{i+1} z,$$
  because then we can define $h'_i(x) := z$.
  First, recall that we have defined for $\partial^{C'}_i(x) \in H_{i-1}$
  $$h'_{i-1}(\partial^{C'}_i(x)) = h_{i-1}(\partial^{C'}_i(x)) = f_2 \circ \Gamma_i(h_{i-1}(\partial^{C'}_i(x))).$$
  Hence, it suffices to show that
  $$\partial^{C'}_i[x - \Gamma_i h_{i-1}(\partial^{C'}_i(x))] = 0$$
  holds, because then the element $z$ exists by the assumption $H_i(f_2) = 0$.
  We compute like mad
  $$\begin{aligned} &\partial^{C'}_i(x) - \partial^{C'}_i \Gamma_i h_{i-1}(\partial^{C'}_i(x)) = \partial^{C'}_i(x) - \Gamma_{i-1}\partial^D_i h_{i-1}(\partial^{C'}_i(x)) \\ =&\partial^{C'}_i(x) - \Gamma_{i-1}[-h_{i-2}\partial^C_{i-1}(\partial^{C'}_i(x)) + f_1(\partial^{C'}_i(x))] \\ =&\partial^{C'}_i(x) - \Gamma_{i-1} \circ f_1(\partial^{C'}_i(x)) \\ =&\partial^{C'}_i(x) - \partial^{C'}_i(x) = 0. \end{aligned}$$

This datum defines the desired null-homotopy. □

*Remark* 5.1.8. Note that it is actually not necessary to assume that the map $D \to D'$ is an injection. This was only done to ease notation.

*Remark* 5.1.9. The statement does not hold true, in general, without the existence of $\Gamma$. Consider the augmented simplicial chain complexes of the following diagram of simplicial complexes




with $h_0(1) = v_0, h_1(v_0) = 0$ and $h_1(v_1)$ is the blue chain as indicated in the picture. Then, $e - h_1(\partial e)$ is the boundary of a two simplex but this is not bounded in the lower right complex. Hence, there does not exist an extension $h'$ of $h$.

*Remark* 5.1.10. The previous counterexample illustrates the only obstruction to the existence of the extension. Simply said, the only chains that are problematic are those in $C'$ but not in $C$ whose boundary is completely contained in $C$.

**Corollary 5.1.11.** *Let $\{K_i\}_{i\in\mathbb{N}_0}$ be simplicial complexes such that for every natural number $i \in \mathbb{N}_0$ we have $K_i \subseteq K_{i+1}$. Assume in addition, that all inclusions induce the zero morphism in reduced simplicial homology. Choose any vertex $v \in K_0$. Then, there exist homotopies $\{h^i : C_\bullet(K_i) \to C_{\bullet+1}(K_{i+1})\}$ between the inclusion and the maps*

$$v^i : C_\bullet(K_i) \to C_\bullet(K_{i+1}),\ \sigma \mapsto \begin{cases} v & \textit{if } \dim\sigma = 0 \\ 0 & \textit{else.} \end{cases}$$

*such that*

$$h^{i+1}_{|C_\bullet(K_i)} = h^i.$$

*Proof.* Recall that the reduced simplicial homology is the homology of the augmented chain complex

$$k \xleftarrow{\epsilon} C_0(K_i) \xleftarrow{\partial} \cdots ,$$

where $\epsilon$ sends every vertex to 1. Consider the canonical embedding

$$\begin{array}{ccccccc} k & \xleftarrow[\epsilon]{} & \langle v\rangle & \xleftarrow[\partial]{} & 0 & \xleftarrow[\partial]{} & \cdots \\ \downarrow{\scriptstyle\text{id}} & & \downarrow & & \downarrow & & \\ k & \xleftarrow[\epsilon]{} & C_0(K_0) & \xleftarrow[\partial]{} & C_1(K_0) & \xleftarrow[\partial]{} & \cdots . \end{array}$$

The identity on the upper chain complex is null-homotopic with chain homotopy $\tilde{h}_0(1) = v$ and $\tilde{h}_j = 0$ for $j > 0$. By Theorem 5.1.6, there exists a null-homotopy $\tilde{h}^0 : \tilde{C}_\bullet(K_0) \to$

$\tilde{C}_{\bullet+1}(K_1)$ that extends $\tilde{h}$, where $\tilde{C}$ is the augmented chain complex. Now note that in degree 1 we have for a basis element $w \in \tilde{C}_1(K_0) = C_0(K_0)$

$$w = \tilde{h}_0^0(\epsilon(w)) + \partial\tilde{h}_1^0(w) = \tilde{h}_0(1) + \partial\tilde{h}_1^0(w) = v + \partial\tilde{h}_1^0(w).$$

Thus, by defining $\{h_j^0 = \tilde{h}_{j+1}^0\}_{j\in\mathbb{N}_0}$ we get the desired homotopy.

We proceed by induction. Assume the $\tilde{h}^n$ and $h^n$ have been constructed. Then, again by Theorem 5.1.6, there exists a null-homotopy $\tilde{h}^{n+1}\colon \tilde{C}_\bullet(K_{n+1}) \to \tilde{C}_{\bullet+1}(K_{n+2})$ that extends $\tilde{h}^n$. A similar computation as before shows that $\{h_j^{n+1} = \tilde{h}_{j+1}^{n+1}\}_{j\in\mathbb{N}_0}$ is the desired homotopy. □

### A Counterexample

We now give an example of the fact that the previous results are not true when working with abelian groups. Consider the following simplicial decomposition of the *real projective plane* $\mathbb{R}P^2$:

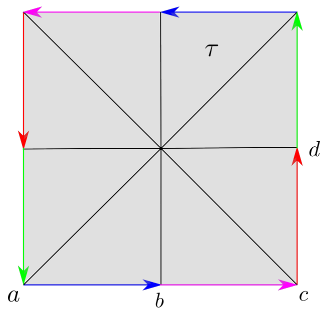



Figure 5.1.: A triangulation of $\mathbb{R}P^2$.

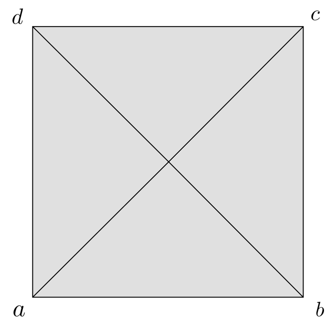



Figure 5.2.: A triangulation of the square.

From this we construct a second space $T$ by attaching a square as indicated above. Then, the homologies with coefficients in $\mathbb{Z}$ are given by:

| | $\tilde{H}_0$ | $\tilde{H}_1$ | $\tilde{H}_2$ |
|---|---|---|---|
| $\mathbb{R}P^2$ | 0 | $\mathbb{Z}/2\mathbb{Z}$ | 0 |
| $T$ | 0 | 0 | $\mathbb{Z}$. |

Hence, the canonical embedding $\iota\colon \mathbb{R}P^2 \hookrightarrow T$ is null-homologous.

We will argue that the induced map on the simplicial chain complexes

$$\iota_*\colon C_*^{simp}(\mathbb{R}P^2) \to C_*^{simp}(T)$$

is not chain homotopic to the chain map

$$x_a\colon C_*^{simp}(\mathbb{R}P^2) \to C_*^{simp}(T),\ \sigma \mapsto \begin{cases} a & \dim(\sigma) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assume this was the case and let $h = \{h_k\}_{k\in\mathbb{N}_0}$ be the chain homotopy. Then, by the definition of chain homotopy and not writing the zero maps, the following equation holds

$$\iota_2 = h_1 \circ \partial.$$

Let $e$ be the canonical generator of $H_1(\mathbb{R}P^2)$, i.e. it consists of every colored one simplex in Fig. 5.1 with the indicated orientation, and let $c$ be the chain in $C_2^{simp}(\mathbb{R}P^2)$ that bounds $2 \cdot e$, i.e. it consists of every two simplex with the orientations that are consistent with the indicated ones in Fig. 5.1.

Then,

$$c' = \iota_2(c) = h_1 \circ \partial(c) = h_1(2 \cdot e) = 2 \cdot h_1(e),$$

where $c'$ is the chain in $C_*^{simp}(T)$ that corresponds to $c$. But this is a contradiction, because by considering the coefficients of the simplex $\tau$ as in Fig. 5.1 one gets the equation

$$1 = c'_{\iota_2(\tau)} = 2 \cdot h_1(e)_{\iota_2(\tau)}$$

that has no solution in $\mathbb{Z}$. This contradicts the assumption.

## 5.2. The Proof Idea of a Persistent Nerve Theorem

Before getting into the details, let us explain the guiding topological picture of the proof of a persistent nerve theorem we will present. Things will get more difficult though, because we will work on the level of chain complexes.

Let $F = (X_r, \mathcal{U}_r = \{U_i^r\}_{i\in I_r})\colon \mathbb{N}_0 \to \mathrm{Cov}$ be an $\epsilon$-good persistently covered space. The first fundamental idea is to work with the homotopy colimit of the nerve diagram (Definition 3.1.4) and the following diagram, that we know from Proposition 3.1.10.

$$\begin{array}{ccc} X_r & \longrightarrow & X_{r+t} \\ \uparrow & & \uparrow \\ \operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r} & \longrightarrow & \operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+t}} \\ \downarrow & & \downarrow \\ |\operatorname{sd} N(\mathcal{U}_r)| & \longrightarrow & |\operatorname{sd} N(\mathcal{U}_{r+t})|. \end{array}$$

Later on, we will only consider covers of simplicial complexes by subcomplexes and hence, the upwards pointing arrows will be homotopy equivalences by Proposition 4.3.1. Therefore, it will suffice to show the interleaving for the lower half of the diagram as the result we chase after is about the homology of these spaces.

For illustrative purposes, assume the stronger condition that for every natural number $r \in \mathbb{N}_0$ and every simplex $J \in N(\mathcal{U}_r)$ the map

$$F(r \le r+\epsilon)_f \colon U_J^r \to U^{r+\epsilon}_{F(r\le r+\epsilon)_C(J)}$$

is homotopic to the constant map onto a point $p_J^r \in U^{r+\epsilon}_{F(r\le r+\epsilon)_C(J)}$.

Similar to Proposition 4.2.1, we can now try to construct a map

$$\Gamma_r\colon |\operatorname{sd} N(\mathcal{U}_r)| \to X_{r+t}$$

for some suitable $t \in \mathbb{N}_0$. One can do so by inductively constructing maps

$$\Gamma_r^d\colon |\operatorname{sd} N(\mathcal{U}_r)^d| \to X_{r+d\epsilon}$$

such that for any $d$-simplex $\sigma = \{J_0, \dots, J_d\} \in \operatorname{sd} N(\mathcal{U}_r)$ we have

$$\Gamma_r^d(|\sigma|) \subseteq U^{r+d\epsilon}_{F(r\leq r+d\epsilon)_C(J_d)}.$$

Let us sketch this construction. For $d = 0$ choose for any vertex $J \in \operatorname{sd} N(\mathcal{U}_r)^0$ a point $q_J^r \in U_J^r$ and define

$$\Gamma_r^0(J) = q_J^r.$$

Assume the map has been constructed for $d-1$. Take a $d$-simplex $\sigma = \{J_0, \dots, J_d\} \in \operatorname{sd} N(\mathcal{U}_r)$, then by induction assumption we have

$$\Gamma_r^{d-1}(|\partial\sigma|) \subseteq U^{r+(d-1)\epsilon}_{F(r\leq r+(d-1)\epsilon)_C(J_d)}$$

and we can extend $\Gamma_r^{d-1}$ to $\Gamma_r^d$ by gluing in the homotopy onto the constant map on $p^{r+(d-1)\epsilon}_{F(r\leq r+(d-1)\epsilon)_C(J_d)}$.

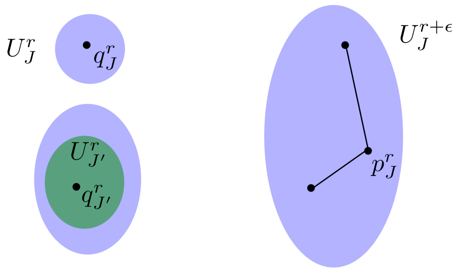



Figure 5.3.: Illustration of the construction with $J, J' \in N(\mathcal{U}_r), J \subseteq J'$.

This finishes the construction and one can now lift this map to get a diagram of the form

$$\begin{array}{ccc} X_r & \longrightarrow & X_{r+t} \\ \uparrow & & \uparrow \\ \operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r} & \longrightarrow & \operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+t}} \\ \downarrow & \nearrow & \downarrow \\ |\operatorname{sd} N(\mathcal{U}_r)| & \longrightarrow & |\operatorname{sd} N(\mathcal{U}_{r+t})| \end{array},$$

from which one can try to deduce an interleaving (up to homotopy).

This proof strategy has two main difficulties. First, we want to work on the level of chain complexes. Hence, desirable properties that might hold in the topological setting for free need to be verified by hands on computations. Second, one needs to be extremely careful as the constructions must fit together nicely over the parameter $r$ .

## 5.3. The Proof of a Persistent Nerve Theorem

Before moving on, let us make some simplifying assumptions such that the setting resembles the one that can be found in [CS18], which is the main source for this section. Let $F = (X_r, \mathcal{U}_r = \{U_r^i\}_{i\in I_r})_r \colon \mathbb{N}_0 \to \mathrm{Cov}$ be a persistently covered space[1]. We will assume that every space $X_r$ is a finite simplicial complex[2] and the cover $\mathcal{U}_r$ is a finite cover by subcomplexes. Moreover, we assume that for every pair of natural numbers $m, n \in \mathbb{N}_0$ with $m \le n$ the morphism $F(m \le n)_f$ is an embedding of simplicial complexes and that $F(m \le n)_C$ is injective. Hence, the complexes $\{X_r\}_{r\in\mathbb{N}_0}$ and $\{N(\mathcal{U}_r)\}_{r\in\mathbb{N}_0}$ can be identified with subcomplexes of a sufficiently large simplicial complex $X$ and $N$, respectively.

We will show the following.

**Theorem 5.3.1**

Let $D \in \mathbb{N}_0$ be a natural number. If in addition to the assumptions above $F$ is $\epsilon$-good, then the inequality

$$d_B(H_D(\Phi_S \circ F), H_D(N \circ F)) \le (D+1)\epsilon$$

holds, where $H_D$ is the $D$-dimensional simplicial homology over a field $k$ of characteristic 2.

*Remark* 5.3.2. If the cover is a good cover in the sense that $F$ is 0-good, then it follows from the fact that the interleaving distance is a natural number that $H_D(\Phi_S \circ F)$ is isomorphic to $H_D(N \circ F)$. This relates Theorem 5.3.1 to Theorem 4.4.4.

To prove this theorem, let us first introduce all necessary chain maps. For every simplex $J \in N$ choose a vertex

$$x_J \in U_J^\alpha,$$

where $\alpha = \min\{r \mid U_J^r \neq \emptyset\}$ is the first time when $J$ appears. Now, define for every natural number $r \in \mathbb{N}_0$ the map

$$x_J^r \colon C_\bullet(U_J^r) \to C_\bullet(U_J^{r+\epsilon}),\ x_J^r(\sigma) = \begin{cases} x_J & \text{if } \dim\sigma = 0 \\ 0 & \text{else.} \end{cases}$$

By Corollary 5.1.11 there exists for every $s \in \{0, \dots, \epsilon - 1\}$ a family of consistent chain homotopies

$$\{c_J^{\alpha+s+j\epsilon} \colon C_\bullet(U_J^{\alpha+s+j\epsilon}) \to C_{\bullet+1}(U_J^{\alpha+s+(j+1)\epsilon})\}_{j\in\mathbb{N}_0}$$

between the inclusions $\{F(\alpha + sj\epsilon \le \alpha + s + (j+1)\epsilon)_f \colon U_J^{\alpha+s+j\epsilon} \to U_J^{\alpha+s+(j+1)\epsilon}\}_{j\in\mathbb{N}_0}$ and $\{x_J^{\alpha+s+j\epsilon}\}_{j\in\mathbb{N}_0}$.

[1] We view $F$ as a functor $\mathbb{Z} \to \mathrm{Cov}$ with $F(r) = \emptyset$ for $r < 0$.

[2] At this point, we abuse the notation and identify an abstract simplicial complex with its realization.

These will now be used to define the maps $\Gamma_r$, as advertised in the previous section. First, define for any natural number $n \in \mathbb{N}_0$ a map

$$c^{r,n}\colon \bigoplus_{\sigma=\{J_0,\dots,J_n\}\in \operatorname{sd} N(\mathcal{U}_r)} \bigoplus_{j=n}^{D} C_{j-n}(J_0)\otimes\sigma \to C_{\leq D+1}(X_{r+(n+1)\epsilon}),$$

where $C_{\leq D+1}$ denotes the $(D+1)$-truncated chain complex (Definition 2.3.1), on the cellular basis element $\tau\otimes\sigma$[3] $\in C_{j-n}(J_0)\otimes\sigma$, by

$$c^{r,n}(\tau\otimes\sigma) = (c^{r+n\epsilon}_{J_n}\circ\cdots\circ c^{r}_{J_0})(\tau).$$

With these we can define

$$c^r\colon C_{\leq D}(\operatorname{hocolim}\mathcal{D}_{\mathcal{U}_r}) \to C_{\leq D+1}(X_{r+(D+1)\epsilon}),\ \tau\otimes\sigma \mapsto i^{r+(\dim\sigma+1)\epsilon,r+(D+1)\epsilon}\Big(c^{r,\dim\sigma}(\tau\otimes\sigma)\Big),$$

where

$$i^{r+(\dim\sigma+1),r+(D+1)\epsilon} = F(r+(\dim\sigma+1)\leq r+(D+1)\epsilon)_f\colon X_{r+(\dim\sigma+1)} \to X_{r+(D+1)\epsilon}$$

is the inclusion.

Finally, we are able to define for any natural number $n\in\{0,\dots,D+1\}$

$$\Gamma^n_r\colon C_n(\operatorname{sd} N(\mathcal{U}_r)) \to C_n(X_{r+n\epsilon})$$
$$\sigma=\{J_0,\dots,J_n\}\mapsto \begin{cases} c^{r,n-1}(x_{J_0}\otimes\{J_1,\dots,J_n\}) & \text{if } n\geq 1\\ x_{J_0} & \text{if } \dim\sigma=0.\end{cases}$$

and moreover

$$\Gamma_r\colon C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)) \to C_{\leq D+1}(X_{r+(D+1)\epsilon}),\ \sigma\mapsto i^{r+\dim\sigma\epsilon,r+(D+1)\epsilon}\Big(\Gamma^{\dim\sigma}_r(\sigma)\Big).$$

It is not at all obvious whether this map is a chain map or not. Unfortunately, the following is the case.

**Proposition 5.3.3.** *Let $n\in\{1,\dots,D+1\}$ be a natural number and let $\sigma\in C_n(\operatorname{sd} N(\mathcal{U}_r))$ be a simplex. Then, we have*

$$\partial\Gamma^n_r(\sigma) = i^{r+(n-1)\epsilon,r+n\epsilon}\Big((-1)^n\Gamma^{n-1}_r(\partial\sigma)\Big).$$

*Moreover, if $n=0$ we trivially have $\partial\Gamma^0_r(\sigma)=0$.*

*Proof.* We prove the statement inductively. Let $\sigma=\{J_0,\dots,J_n\}\in C_n(\operatorname{sd} N(\mathcal{U}_r))$ be a simplex. For $\dim\sigma=1$ we compute

$$\begin{aligned}\partial\Gamma^1_r(\sigma) &= \partial c^{r,0}(x_{J_0}\otimes\{J_1\}) \overset{\text{Def.}}{=} \partial c^r_{J_1}(x_{J_0}) \overset{\text{ch.hom.}}{=} -c^r_{J_1}(\partial x_{J_0}) + i^{r,r+\epsilon}(x_{J_0}) - x^r_{J_1}(x_{J_0})\\ &= i^{r,r+\epsilon}(x_{J_0}) - i^{r,r+\epsilon}(x_{J_1}) = i^{r,r+\epsilon}\Big(-\Gamma^0_r(\partial\sigma)\Big).\end{aligned}$$

[3] Compare the beginning of Section 3.1.2

Now, let $2 \leq n = \dim \sigma \leq D+1$ and assume that the statement holds for all dimensions smaller than $n$. Then, we can compute

$$\begin{aligned}
&\partial\Gamma_r^n(\sigma) = \partial c^{r,n-1}(x_{J_0} \otimes \{J_1, \dots, J_n\}) = \partial c_{J_n}^{r+(n-1)\epsilon} c^{r,n-2}(x_{J_0} \otimes \{J_1, \dots, J_{n-1}\}) \\
&= -c_{J_n}^{r+(n-1)\epsilon}(\partial c^{r,n-2}(x_{J_0} \otimes \{J_1, \dots, J_{n-1}\})) + i^{r+(n-1)\epsilon, r+n\epsilon}\Big(c^{r,n-2}(x_{J_0} \otimes \{J_1, \dots, J_{n-1}\})\Big) \\
&- x_{J_n}^{r+(n-1)\epsilon}(\underbrace{c^r(x_{J_0} \otimes \{J_1, \dots, J_{n-1}\})}_{\dim(\cdot)\geq 1}) \\
&= -c_{J_n}^{r+(n-1)\epsilon}(\partial\Gamma_r^{n-1}(\{J_0, \dots, J_{n-1}\})) + i^{r+(n-1)\epsilon, r+n\epsilon}\Big(c^{r,n-2}(x_{J_0} \otimes \{J_1, \dots, J_{n-1}\})\Big) - 0 \\
&\overset{\text{Ind.Ass.}}{=} -c_{J_n}^{r+(n-1)\epsilon}(i^{r+(n-2)\epsilon, r+(n-1)\epsilon}\Big((-1)^{n-1}\Gamma_r^{n-2}(\partial\{J_0, \dots, J_{n-1}\})\Big) \\
&+ i^{r+(n-1)\epsilon, r+n\epsilon}\Big(\Gamma_r^{n-1}(\{J_0, J_1, \dots, J_{n-1}\})\Big) \\
&\overset{\text{consistency}}{=} (-1)^n i^{r+(n-1)\epsilon, r+n\epsilon} c_{J_n}^{r+(n-2)\epsilon}(\Gamma_r^{n-2}(\partial\{J_0, \dots, J_{n-1}\})) \\
&+ i^{r+(n-1)\epsilon, r+n\epsilon}\Big(\Gamma_r^{n-1}(\{J_0, J_1, \dots, J_{n-1}\})\Big) \\
&= i^{r+(n-1)\epsilon, r+n\epsilon}\Big((-1)^n\Big[\sum_{i=0}^{n-1}(-1)^i\Gamma_r^{n-1}(\{J_0, \dots, \hat{J}_i, \dots, J_n\}) + (-1)^n\Gamma_r^{n-1}(\{J_0, J_1, \dots, J_{n-1}\})\Big]\Big) \\
&= i^{r+(n-1)\epsilon, r+n\epsilon}\Big((-1)^n\Gamma_r^{n-1}(\partial\sigma)\Big).
\end{aligned}$$

This completes the proof. □

**Corollary 5.3.4.** *Let $\sigma \in C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r))$ be a simplex. The map $\Gamma_r$ satisfies*

$$\partial\Gamma_r(\sigma) = (-1)^{\dim\sigma}\Gamma_r(\partial\sigma).$$

*Proof.* By the previous proposition

$$\begin{aligned}
&\partial\Gamma_r(\sigma) = \partial i^{r+\dim\sigma\epsilon, r+(D+1)\epsilon}\Big(\Gamma_r^{\dim\sigma}(\sigma)\Big) \\
&= i^{r+\dim\sigma\epsilon, r+(D+1)\epsilon}\Big(\partial\Gamma_r^{\dim\sigma}\Big) \\
&= i^{r+\dim\sigma\epsilon, r+(D+1)\epsilon} \circ i^{r+(\dim\sigma-1)\epsilon, r+\dim\sigma\epsilon}\Big((-1)^{\dim\sigma}\Gamma_r^{\dim\sigma-1}(\partial\sigma)\Big) \\
&= (-1)^{\dim\sigma} i^{r+(\dim\sigma-1)\epsilon, r+(D+1)\epsilon}\Big(\Gamma_r^{\dim\sigma-1}(\partial\sigma)\Big) = (-1)^{\dim\sigma}\Gamma_r(\partial\sigma).
\end{aligned}$$

□

*Remark* 5.3.5. This non-commutativity with the boundary map is the technical reason why we stick to characteristic 2, because then $\Gamma_r$ is a chain map. Nevertheless, there is no obvious reason why the theorem should not hold true for arbitrary fields. It is a topic of further research to find a way to drop this assumption.

So far, we have achieved the following situation.

$$
\begin{array}{ccc}
C_{\leq D+1}(X_r) & \xrightarrow{\quad i_1 \quad} & C_{\leq D+1}(X_{r+(D+1)\epsilon}) \\
\uparrow{\scriptstyle (p_f)_*} & & \uparrow{\scriptstyle (p_f)_*} \\
C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}) & \xrightarrow{\quad i_3 \quad} & C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}}) \\
\downarrow{\scriptstyle (p_b)_*} & \nearrow{\scriptstyle \Gamma_r} & \downarrow{\scriptstyle (p_b)_*} \\
C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)) & \xrightarrow{\quad i_2 \quad} & C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_{r+(D+1)\epsilon})),
\end{array}
$$

where $i_1, i_2$ and $i_3$ are the ones that are given by $F$ and all other maps are induced by the continuous maps as in Proposition 3.1.10. More concretely, we have

$$(p_f)_*\colon C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}) \to C_{\leq D+1}(X_r),\ \tau \otimes \sigma \mapsto \begin{cases} \tau & \text{if } \dim \sigma = 0 \\ 0 & \text{else} \end{cases}$$

and similarly

$$(p_b)_*\colon C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}) \to C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)),\ \tau \otimes \sigma \mapsto \begin{cases} \sigma & \text{if } \dim \tau = 0 \\ 0 & \text{else.} \end{cases}$$

**From now on we assume** $\operatorname{char} k = 2$**.**

Before describing how to lift $\Gamma_r$ to take values in $C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}})$, we will need one more fact later on.

**Proposition 5.3.6.** *For any basis element* $\tau \otimes \sigma \in C_{\leq D}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r})$ *with* $\sigma = \{J_0, \ldots, J_n\}$ *we have for* $n = 0$

$$\begin{aligned} &\partial c^{r,0}(\tau \otimes \sigma) + c^{r,0}((\partial \tau) \otimes \sigma) \\ &= i^{r,r+\epsilon}[(p_f)_*(\tau \otimes \sigma)] - i^{r,r+\epsilon}[\Gamma_r^0 \circ (p_b)_*(\tau \otimes \sigma)]. \end{aligned}$$

*and for* $n > 0$

$$\begin{aligned} &\partial c^{r,n}(\tau \otimes \sigma) + c^{r,n}((\partial \tau) \otimes \sigma) + i^{r+n\epsilon,r+(n+1)\epsilon} c^{r,n-1}(\tau \otimes (\partial \sigma)) \\ &= i^{r,r+(n+1)\epsilon}[(p_f)_*(\tau \otimes \sigma)] - i^{r+n\epsilon,r+(n+1)\epsilon}[\Gamma_r^n \circ (p_b)_*(\tau \otimes \sigma)]. \end{aligned}$$

*Proof.* To show this statement, we will consider the following cases

1. a) $\dim \tau = \dim \sigma = 0$
   b) $\dim \tau = 0,\ \dim \sigma = 1$
   c) $\dim \tau = 0,\ 1 < \dim \sigma \leq D$
2. a) $\dim \tau > 0,\ \dim \sigma = 0$
   b) $\dim \tau > 0,\ \dim \sigma = 1$

c) $\dim \tau > 0,\ 1 < \dim \sigma \leq D$

Let us start with 1.a):

$$\begin{aligned}
&\partial c^{r,0}(\tau \otimes \sigma) = \partial c^r_{J_0}(\tau) = -c^r_{J_0}(\partial\tau) + i^{r,r+\epsilon}(\tau) - i^{r,r+\epsilon}(x_{J_0}) \\
&= 0 + i^{r,r+\epsilon}((p_f)_*(\tau \otimes \sigma)) - i^{r,r+\epsilon}(\Gamma^0_r(\sigma)) \\
&= -c^{r,0}(\underbrace{(\partial\tau)\otimes\sigma}_{=0}) + i^{r,r+\epsilon} \circ (p_f)_*(\tau\otimes\sigma) - i^{r,r+\epsilon}(\Gamma^0_r \circ (p_b)_*(\tau\otimes\sigma)).
\end{aligned}$$

1.b):

$$\begin{aligned}
&\partial c^{r,1}(\tau\otimes\sigma) = \partial c^{r+\epsilon}_{J_1} c^r_{J_0}(\tau) = -c^{r+\epsilon}_{J_1}(\partial c^r_{J_0}(\tau)) + i^{r+\epsilon,r+2\epsilon}(c^r_{J_0}(\tau)) \\
&- x^r_{J_1}(\underbrace{c^r_{J_0}(\tau)}_{\dim\geq 1}) = -c^{r+\epsilon}_{J_1}[-c^r_{J_0}(\partial\tau) + i^{r,r+\epsilon}(\tau) - \underbrace{x^r_{J_0}(\tau)}_{=i^{r,r+\epsilon}(x_{J_0})}] + i^{r+\epsilon,r+2\epsilon}(c^r_{J_0}(\tau)) \\
&= c^{r,1}((\partial\tau)\otimes\sigma) - c^{r+\epsilon}_{J_1}(i^{r,r+\epsilon}(\tau)) + i^{r+\epsilon,r+2\epsilon}(c^r_{J_0}(\tau)) + c^{r+\epsilon}_{J_1}(i^{r,r+\epsilon}(x_{J_0})) \\
&\overset{\text{consistency}}{=} c^{r,1}((\partial\tau)\otimes\sigma) + [-i^{r+\epsilon,r+2\epsilon}(c^r_{J_1}(\tau)) + i^{r+\epsilon,r+2\epsilon}(c^r_{J_0}(\tau))] + i^{r+\epsilon,r+2\epsilon}(c^r_{J_1}(x_{J_0})) \\
&= c^{r,1}((\partial\tau)\otimes\sigma) - i^{r+\epsilon,r+2\epsilon}[c^{r,0}(\tau\otimes\{J_1\}) - c^{r,0}(\tau\otimes\{J_0\})] + i^{r+\epsilon,r+2\epsilon}(\underbrace{c^{r,0}(\{x_{J_0}\}\otimes\{J_1\})}_{=\Gamma^1_r(\sigma)}) \\
&\overset{\text{char}\,k=2}{=} -c^{r,1}(\partial(\tau)\otimes\sigma) - i^{r+\epsilon,r+2\epsilon}(c^{r,0}(\tau\otimes(\partial\sigma))) - i^{r+\epsilon,r+2\epsilon}(\Gamma^1_r((p_b)_*(\tau\otimes\sigma))) \\
&+ i^{r,r+2\epsilon}(\underbrace{(p_f)_*(\tau\otimes\sigma)}_{=0}).
\end{aligned}$$

2.b): The proof is analogous to the previous case, except that the term $x^r_{J_0}(\tau)$ vanishes. This is not an issue as $(p_b)_*(\tau\otimes\sigma) = 0$ holds.

1.c): We prove this case by induction over $\dim\sigma$. The base case has been established in 1.b). Let $\sigma$ have dimension $1 < n \leq D$ and assume the statement holds for $n-1$. Then, we can compute with $\sigma' = \{J_0, \dots, J_{n-1}\}$

$$\begin{aligned}
&\partial c^{r,n}(\tau\otimes\sigma) = \partial c^{r+n\epsilon}_{J_n} c^{r,n-1}(\tau\otimes\sigma') = -c^{r+n\epsilon}_{J_n}(\partial c^{r,n-1}(\tau\otimes\sigma')) + i^{r+n\epsilon,r+(n+1)\epsilon}(c^{r,n-1}(\tau\otimes\sigma')) \\
&+ x^{r+n\epsilon}_{J_n}(\underbrace{c^{r,n-1}(\tau\otimes\sigma')}_{\dim\geq 1}) \\
&\overset{\text{Ind.Ass}}{=} -c^{r+n\epsilon}_{J_n}\Big( - c^{r,n-1}((\partial\tau)\otimes\sigma') - i^{r+(n-1)\epsilon,r+n\epsilon}c^{r,n-2}(\tau\otimes(\partial\sigma')) + i^{r,r+n\epsilon}[\underbrace{(p_f)_*(\tau\otimes\sigma')}_{=0}] \\
&- i^{r+(n-1)\epsilon,r+n\epsilon}[\Gamma^{n-1}_r \circ (p_b)_*(\tau\otimes\sigma')]\Big) + i^{r+n\epsilon,r+(n+1)\epsilon}(c^{r,n-1}(\tau\otimes\sigma'))
\end{aligned}$$

$$\overset{\text{consistency}}{=} c^{r,n}((\partial\tau)\otimes\sigma) + i^{r+n\epsilon,r+(n+1)\epsilon}c_{J_n}^{r+(n-1)\epsilon}c^{r,n-2}(\tau\otimes(\partial\sigma'))$$
$$+ i^{r+n\epsilon,r+(n+1)\epsilon}(c_{J_n}^{r+(n-1)\epsilon}[\Gamma_r^{n-1}\circ(p_b)_*(\tau\otimes\sigma')]) + i^{r+n\epsilon,r+(n+1)\epsilon}(c^{r,n-1}(\tau\otimes\sigma'))$$
$$= c^{r,n}((\partial\tau)\otimes\sigma) + i^{r+n\epsilon,r+(n+1)\epsilon}[c_{J_n}^{r+(n-1)\epsilon}c^{r,n-2}(\tau\otimes(\partial\sigma')) + c^{r,n-1}(\tau\otimes\sigma')]$$
$$+ i^{r+n\epsilon,r+(n+1)\epsilon}(\Gamma_r^n\circ(p_b)_*(\tau\otimes\sigma))$$
$$= c^{r,n}((\partial\tau)\otimes\sigma) + i^{r+n\epsilon,r+(n+1)\epsilon}[c^{r,n-1}(\tau\otimes(\partial\sigma))] + i^{r+n\epsilon,r+(n+1)\epsilon}(\Gamma_r^n\circ(p_b)_*(\tau\otimes\sigma))$$
$$+ i^{r,r+(n+1)\epsilon}[\underbrace{(p_f)_*(\tau\otimes\sigma)}_{=0}]$$

2.a):

$$\partial c^{r,0}(\tau\otimes\sigma) = \partial c_{J_0}^r(\tau) = -c_{J_0}^r(\partial\tau) + i^{r,r+\epsilon}(\tau) - \underbrace{x_{J_0}^r(\tau)}_{=0} = -c^{r,0}((\partial\tau)\otimes J_0)$$
$$+ i^{r,r+\epsilon}[(p_f)_*(\tau\otimes\sigma)] - i^{r,r+\epsilon}(\Gamma_r^0\circ\underbrace{(p_b)_*(\tau\otimes\sigma)}_{=0})$$

2.c): One proves this case by induction over $\dim\sigma$. The base case has been established in 2.b). The rest of the proof is analogous to the one for 1.c). ☐

**Corollary 5.3.7.** *The map $c^r$ is a D-chain homotopy between*

$$i_1\circ(p_f)_* \text{ and } \Gamma_r\circ(p_b)_*.$$

*Proof.* Consider any basis element $\tau\otimes\sigma\in C_{\leq D}(\operatorname{hocolim}\mathcal{D}_{\mathcal{U}_r})$ with $\sigma=\{J_0,\dots,J_n\}$. If $n=0$, then

$$\partial c^r(\tau\otimes\sigma) = i^{r+\epsilon,r+(D+1)\epsilon}(\partial c^{r,0}(\tau\otimes\sigma))$$
$$= i^{r+\epsilon,r+(D+1)\epsilon}\Big(c^{r,0}((\partial\tau)\otimes\sigma) + i^{r,r+\epsilon}[(p_f)_*(\tau\otimes\sigma)]$$
$$+ i^{r,r+\epsilon}[\Gamma_r^0\circ(p_b)_*(\tau\otimes\sigma)\Big)$$
$$= c^r((\partial\tau)\otimes\sigma) + i_1[(p_f)_*(\tau\otimes\sigma)] + \Gamma_r\circ(p_b)_*(\tau\otimes\sigma)$$
$$= c^r(\partial(\tau\otimes\sigma)) + i_1[(p_f)_*(\tau\otimes\sigma)] + \Gamma_r\circ(p_b)_*(\tau\otimes\sigma)$$

If $n>0$, then

$$\partial c^r(\tau\otimes\sigma) = i^{r+(n+1)\epsilon,r+(D+1)\epsilon}(\partial c^{r,n}(\tau\otimes\sigma))$$
$$= i^{r+(n+1)\epsilon,r+(D+1)\epsilon}\Big(c^{r,n}((\partial\tau)\otimes\sigma) + i^{r+n\epsilon,r+(n+1)\epsilon}c^{r,n-1}(\tau\otimes(\partial\sigma))$$
$$+ i^{r,r+(n+1)\epsilon}[(p_f)_*(\tau\otimes\sigma)] + i^{r+n\epsilon,r+(n+1)\epsilon}[\Gamma_r^n\circ(p_b)_*(\tau\otimes\sigma)\Big)$$
$$= c^r((\partial\tau)\otimes\sigma) + c^r(\tau\otimes(\partial\sigma)) + i_1[(p_f)_*(\tau\otimes\sigma)] + \Gamma_r\circ(p_b)_*(\tau\otimes\sigma)$$
$$= c^r(\partial(\tau\otimes\sigma)) + i_1[(p_f)_*(\tau\otimes\sigma)] + \Gamma_r\circ(p_b)_*(\tau\otimes\sigma)$$

This proves the claim. ☐

To go on and define the liftings, we need to introduce the *Alexander-Whitney diagonal approximation*.

**Definition 5.3.8.** Let $K$ be an ordered simplicial complex. We define the chain map

$$AW\colon C_\bullet(K) \to C_\bullet(K) \otimes C_\bullet(K),\ \sigma \mapsto \sum_{i=0}^{n} \sigma_i \otimes \overline{\sigma}_i,$$

where $\sigma = v_0 \to \cdots \to v_n \in K$ is a simplex and we write $\sigma_i$ for $v_0 \to \cdots \to v_i$ and $\overline{\sigma}_i$ for $v_i \to \cdots \to v_n$.

*Remark* 5.3.9. According to [DK01, p. 66] the geometric root of the *Alexander-Whitney diagonal approximation* lies in the need of finding a cellular approximation to the diagonal map $|K| \to |K| \times |K|,\ x \mapsto (x, x)$.

**Definition 5.3.10.** Let $X$ and $Z$ be simplicial complexes and let $Y$ be an ordered simplicial complex. Let $S \subseteq C_\bullet(X) \otimes C_\bullet(Y)$ be a subchain complex that is spanned by elements of the form $\tau \otimes \sigma$ with $\tau \in X$ and $\sigma \in Y$ such that whenever $\sigma' \subseteq \sigma$ is a face, then $\tau \otimes \sigma'$ is in $S$ as well. Assume we are given a chain map $f\colon S \to C_\bullet(Z)$. Then the chain map

$$\hat{f}\colon S \to C_\bullet(Z) \otimes C_\bullet(Y),\ \hat{f} = (f \otimes \mathrm{id}_{C_\bullet(Y)}) \circ (\mathrm{id}_{C_\bullet(X)} \otimes AW)$$

is called the *lift of* $f$. Explicitly, we have for $\tau \otimes \sigma \in S$

$$\hat{f}(\tau \otimes \sigma) = \sum_{i=0}^{n} f(\tau \otimes \sigma_i) \otimes \overline{\sigma}_i.$$

*Example* 5.3.11. The lift of $\Gamma_r$, where $X = *, Y = \operatorname{sd} N(\mathcal{U}_r), Z = X_{r+(D+1)\epsilon}$ and $S = C_{\leq D+1}(Y)$, is given by

$$\hat{\Gamma}_r\colon C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)) \to C_{\leq D+1}(X_{r+(D+1)\epsilon}) \otimes C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)),\ \sigma \mapsto \sum_{i=0}^{n} \Gamma_r(\sigma_i) \otimes \overline{\sigma}_i.$$

We have recalled in Section 3.1.2 that the cellular chain complex $C_\bullet(\operatorname{hocolim} \mathcal{D}_{\mathcal{F}})$ of the homotopy colimit of a nerve diagram is a subchain complex of the tensor product $C_\bullet(X) \otimes C_\bullet(\operatorname{sd} N(\mathcal{F}))$.

**Lemma 5.3.12.** *The lift of* $\Gamma_r$*, viewed as a chain map*

$$\hat{\Gamma}_r\colon C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)) \to C_\bullet(X_{r+(D+1)\epsilon}) \otimes C_\bullet(\operatorname{sd} N(\mathcal{U}_r)) \subseteq C_\bullet(X_{r+(D+1)\epsilon}) \otimes C_\bullet(\operatorname{sd} N(\mathcal{U}_{r+(D+1)\epsilon})),$$

*takes values in* $C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}})$.

*Proof.* Let $n \in \mathbb{N}_0$ be a natural number with $n \leq D+1$ and let $\sigma = \{J_0, \ldots, J_n\} \in \operatorname{sd} N(\mathcal{U}_r)$ be a simplex. Then

$$\hat{\Gamma}_r(\sigma) = \sum_{i=0}^{n} \Gamma_r(\sigma_i) \otimes \overline{\sigma}_i = x_{J_0} \otimes \sigma + \sum_{i=1}^{n} c^r(x_{J_0} \otimes \{J_1, \ldots, J_i\}) \otimes \overline{\sigma}_i.$$

We have

$$x_{J_0} \otimes \sigma \in C_n(U_{J_0}^{r+(D+1)\epsilon} \times |\sigma|) \subseteq C_n(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}})$$

and

$$c^r(x_{J_0} \otimes \{J_1, \ldots, J_i\}) \otimes \overline{\sigma}_i \subseteq C_n(U_{J_i}^{r+(D+1)\epsilon} \times |\overline{\sigma}_i|) \subseteq C_n(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}}).$$

Hence, $\hat{\Gamma}_r(\sigma)$ is contained in $C_n(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}})$. □

In sum, we were able to define a diagram of the following shape.

$$\begin{array}{ccc}
C_{\leq D+1}(X_r) & \xrightarrow{\;i_1\;} & C_{\leq D+1}(X_{r+(D+1)\epsilon}) \\
\uparrow{\scriptstyle (p_f)_*} & & \uparrow{\scriptstyle (p_f)_*} \\
C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}) & \xrightarrow{\;i_3\;} & C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}}) \\
\downarrow{\scriptstyle (p_b)_*} & \nearrow{\scriptstyle \hat{\Gamma}_r} & \downarrow{\scriptstyle (p_b)_*} \\
C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)) & \xrightarrow{\;i_2\;} & C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_{r+(D+1)\epsilon})),
\end{array}$$

Before proving Theorem 5.3.1, we collect a few more facts.

**Lemma 5.3.13.** *The relations* $\widehat{\Gamma_r \circ (p_b)_*} = \hat{\Gamma}_r \circ (p_b)_*$, $(p_b)_* \circ \hat{\Gamma}_r = i_2$ *and* $\widehat{i_1 \circ (p_f)_*} = i_3$ *hold.*

*Proof.* All three relations are shown in the same fashion. For illustrative purposes we will show the first one. Let $\tau \otimes \sigma \in C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r})$ be a basis element. Assume first that $\dim \tau > 0$. Then,

$$\widehat{\Gamma_r \circ (p_b)_*}(\tau \otimes \sigma) = \sum_{i=0}^{n} \Gamma_r \circ (p_b)_*(\tau \otimes \sigma_i) \otimes \overline{\sigma_i} = 0 = \hat{\Gamma}_r \circ (p_b)_*(\tau \otimes \sigma).$$

If $\dim \tau = 0$, then

$$\widehat{\Gamma_r \circ (p_b)_*}(\tau \otimes \sigma) = \sum_{i=0}^{n} \Gamma_r \circ (p_b)_*(\tau \otimes \sigma_i) \otimes \overline{\sigma_i} = \sum_{i=0}^{n} \Gamma_r(\sigma_i) \otimes \overline{\sigma_i} = \hat{\Gamma}_r \circ (p_b)_*(\tau \otimes \sigma).$$

□

**Lemma 5.3.14.** *The inclusion* $i_3$ *is* $D$*-chain homotopic to* $\hat{\Gamma}_r \circ (p_b)_*$.

*Proof.* By Corollary 5.3.7, the map $c^r$ is a $D$-chain homotopy between $i_1 \circ (p_f)_*$ and $\Gamma_r \circ (p_b)_*$. One can easily check (compare [CS18, Lemma 11]) that the map

$$\begin{aligned}
\hat{c}^r &: C_{\leq D}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}) \to C_\bullet(X_{r+(D+1)\epsilon}) \otimes C_\bullet(\operatorname{sd} N(\mathcal{U}_{r+(D+1)\epsilon})) \\
\hat{c}^r &= (c^r \otimes \mathrm{id}) \circ (\mathrm{id} \otimes AW) \\
\tau \otimes \sigma &\mapsto \sum_{i=0}^{n} c^r(\tau \otimes \sigma_i) \otimes \overline{\sigma}_i
\end{aligned}$$

is, as the composition of the chain homotopy $c^r \otimes \mathrm{id}$ with the chain map $\mathrm{id} \otimes AW$, a $D$-chain homotopy between $\widehat{i_1 \circ (p_f)_*} \overset{Lemma\ 5.3.13}{=} i_3$ and $\widehat{\Gamma_r \circ (p_b)_*} \overset{Lemma\ 5.3.13}{=} \hat{\Gamma}_r \circ (p_b)_*$.

It remains to show that $\hat{c}^r$ takes values in $C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}})$. But this holds true as for any natural number $0 \leq n \leq D$ and $\tau \otimes \sigma \in C_n(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r})$ we have for any $0 \leq i \leq \dim \sigma$

$$c^r(\tau \otimes \{J_0, \ldots, J_i\}) \otimes \overline{\sigma}_i \subseteq C_{n+1}(U_{J_i}^{r+(D+1)\epsilon} \times |\overline{\sigma}_i|) \subseteq C_{n+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}})$$

by construction.

□

With these tools we can now easily prove the persistent nerve theorem.

*Proof of Theorem 5.3.1.* As mentioned in Section 5.2, it is enough to consider the following diagram

$$\begin{array}{ccccc}
C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}) & \xrightarrow{i_3} & C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+(D+1)\epsilon}}) & \xrightarrow{i_3'} & C_{\leq D+1}(\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_{r+2(D+1)\epsilon}}) \\
\downarrow (p_b)_* & \nearrow \hat{\Gamma}_r & \downarrow (p_b)_* & \nearrow \hat{\Gamma}_{r+(D+1)\epsilon} & \downarrow (p_b)_* \\
C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_r)) & \xrightarrow[i_2]{} & C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_{r+(D+1)\epsilon})) & \xrightarrow[i_2']{} & C_{\leq D+1}(\operatorname{sd} N(\mathcal{U}_{r+2(D+1)\epsilon})).
\end{array}$$

By Lemma 5.3.14 we know that $i_3$ is $D$-chain homotopic to $\hat{\Gamma}_r \circ (p_b)_*$. Of course, the same holds for $i_3'$ and $\hat{\Gamma}_{r+(n+1)\epsilon} \circ (p_b)_*$. Let us check the interleaving relations, where the candidates for the interleaving are given on objects for any $r$ by $\hat{\Gamma}_r$ and $i_2 \circ (p_b)_*$:

1. $i_3' \circ \hat{\Gamma}_r \simeq_{\leq D} \hat{\Gamma}_{r+(n+1)\epsilon} \circ (p_b)_* \circ \hat{\Gamma}_r \overset{\text{Lemma 5.3.13}}{=} \hat{\Gamma}_{r+(n+1)\epsilon} \circ i_2$
2. $i_2' \circ (i_2 \circ (p_b)_*) = i_2' \circ ((p_b)_* \circ i_3) = (i_2' \circ (p_b)_*) \circ i_3$
3. $(i_2' \circ (p_b)_*) \circ \hat{\Gamma}_r = i_2' \circ ((p_b)_* \circ \hat{\Gamma}_r) \overset{\text{Lemma 5.3.13}}{=} i_2' \circ i_2$
4. $\hat{\Gamma}_{r+(n+1)\epsilon} \circ (i_2 \circ (p_b)_*) = \hat{\Gamma}_{r+(n+1)\epsilon} \circ ((p_b)_* \circ i_3) = (\hat{\Gamma}_{r+(n+1)\epsilon} \circ (p_b)_*) \circ i_3 \simeq_{\leq D} i_3' \circ i_3$.

By employing Lemma 2.3.3, we can conclude that these relations hold on the nose on the level of homology up to dimension $D$. Unfortunately, this does not quite show the interleaving as we do not know if the collection $\{H_*(\hat{\Gamma}_r)\}_{r \in \mathbb{N}_0}$ forms a natural transformation. Nevertheless, it follows from the relations three and four that the natural transformation induced by $i_2 \circ (p_b)_*$ has $2(D+1)\epsilon$-trivial kernel and cokernel on the level of homology up to dimension $D$ and hence the claim is a consequence of Proposition A.2.5. □

**Corollary 5.3.15.** *Let the assumptions be as in Theorem 5.3.1 but replace $\mathbb{N}_0$ by $\mathbb{R}$. Let $D \in \mathbb{N}_0$ be a natural number. Assume that $F$ is $\epsilon$-good and that there exists a number $r_0 \in \mathbb{R}$ such that for all $r \leq r_0$ we have*

$$F(r) = F(r_0), \ F(r \to r_0) = \mathrm{id}_{F(r_0)}.$$

*Then the inequality*

$$d_B(H_D(\Phi_S \circ F), H_D(N \circ F)) \leq (D+1)\epsilon$$

*holds, where $H_D$ is the $D$-dimensional simplicial homology over a field $k$ of characteristic 2.*

*Proof.* We want to apply our persistent nerve theorem (Theorem 5.3.1). To do so, we discretize $F$ as in Corollary 5.0.10. Let $F^\delta \colon \mathbb{R} \to \mathrm{Cov}$ be this discretization and note that $F^\delta$ factors through

$$G \colon \delta\mathbb{Z} \hookrightarrow \mathbb{R} \overset{F^\delta}{\to} \mathrm{Cov}$$

by construction. Now consider

$$L \colon \mathbb{N}_0 \to \delta\mathbb{Z},\ n \mapsto r_0^\delta - (D+1)\tilde{\epsilon} + \delta n,$$

where $\tilde{\epsilon}$ is $\min\{m \in \delta\mathbb{N}_0 \mid m \geq \epsilon\}$. Then, the functor $F' = G \circ L$ is $\tilde{\epsilon}/\delta$-good and hence we get the bound

$$d_B(H_D(\Phi_S \circ F'), H_D(N \circ F')) \leq (D+1)\tilde{\epsilon}/\delta$$

by Theorem 5.3.1.

As this interleaving distance only takes discrete values, it is always attained and therefore there exists a $(D+1)\tilde{\epsilon}/\delta$ interleaving between $H_D(\Phi_S \circ F')$ and $H_D(N \circ F')$. This extends to a $(D+1)\tilde{\epsilon}$ interleaving between $H_D(\Phi_S \circ G)$ and $H_D(N \circ G)$. By a rescaled version of Lemma A.2.3 the functors $H_D(\Phi_S \circ F^\delta)$ and $H_D(N \circ F^\delta)$ are also $(D+1)\tilde{\epsilon}$ interleaved. By the triangle inequality we finally get

$$\begin{aligned} d_B(H_D(\Phi_S \circ F), H_D(N \circ F)) &\leq d_B(H_D(\Phi_S \circ F), H_D(\Phi_S \circ F^\delta)) + d_B(H_D(\Phi_S \circ F^\delta), H_D(N \circ F^\delta)) \\ &\quad + d_B(H_D(N \circ F^\delta), H_D(N \circ F)) \\ &\leq \delta + (D+1)\tilde{\epsilon} + \delta \overset{\delta \searrow 0}{\to} (D+1)\epsilon. \end{aligned}$$

This proves the claim. ∎

# 6. Outlook

The notion of *coherence* is fundamental in modern homotopy theory and is a foundational idea of higher category theory (compare [Lur09; Rie14]). At this point, it is enough to know that a *coherent diagram* $\mathcal{C} \to \mathrm{Top}$ is one that commutes up to homotopy, those homotopies are linked by "higher homotopies" and so on. For a precise definition see [Rie18, p.7 and p.16] or [CP86, p.67 ff.].

One interesting property is the following.

**Theorem 6.0.1 ([CP86, Theorem 1.1.])**

Let $G, H \colon \mathcal{D} \to \mathrm{Top}$ be diagrams and let there be a natural transformation $G \Rightarrow H$ that is a pointwise homotopy equivalence. Then, there exists a *coherent transformation* $H \Rightarrow G$ such that the compositions of these two transformations are *homotopic* to the appropriate identities.

In this general setting it is possible to study *homotopy coherent $\delta$-interleavings* as in [BL17]. There, the *$\delta$-interleaving category* $I^\delta$ is considered that comes equipped with two functors $E^0, E^1 \colon \mathbb{R} \to I^\delta$. These should be seen as being like the inclusions of the endpoints of the interval $[0,1]$. An ordinary $\delta$-interleaving between two functors $F, G \colon \mathbb{R} \to \mathrm{Top}$ is a functor $H \colon I^\delta \to \mathrm{Top}$ such that $H \circ E^0 \cong F$ and $H \circ E^1 \cong G$.

A *homotopy coherent $\delta$-interleaving* between $F$ and $G$ is then a coherent diagram $H \colon I^\delta \to \mathrm{Top}$ such that there are *homotopy equivalences* $H \circ E^0 \simeq F$ and $H \circ E^1 \simeq G$.

In this chapter we will make an effort to advertise how these ideas might help to understand persistent nerve theorems.

Let $F = (X_r, \mathcal{U}_r = \{U_r^i\}_{i \in I_r})_r \colon \mathbb{R} \to \mathrm{Cov}$ be a persistently covered space. We will assume that every space $X_r$ is a simplicial complex and the cover $\mathcal{U}_r$ is a cover by subcomplexes. Moreover, we assume that for any two numbers $m, n \in \mathbb{R}$ with $m \leq n$ the morphism $F(m \leq n)_f$ is an embedding of simplicial complexes and that $F(m \leq n)_C$ is injective. Hence, the complexes $\{X_r\}_r$ and $\{N(\mathcal{U}_r)\}_r$ can be identified with subcomplexes of a sufficiently large simplicial complex $X$ and $N$, respectively.

For each natural number $r \in \mathbb{R}$ consider the opposite face poset $P_r$ of $N(\mathcal{U}_r)$ as a subposet of the opposite face poset $P$ of $N$. We view the nerve diagrams as functors

$$T_r \colon P \to \mathrm{Top},$$

where for every $p \notin P_r$ we have $T_r(p) = \emptyset$. Moreover, consider the functor

$$S_r \colon P \to \mathrm{Top}$$

that is obtained from $F_r$ by replacing every non-empty space by the one point space. Now, we can interpret $T$ and $S$ as functors

$$T, S\colon \mathbb{R} \to \mathrm{Fun}(P, \mathrm{Top}).$$

According to [Rie18, p.9] there exists a *quasi-category* $\mathrm{Coh}(P, \mathrm{Top})$ whose $n$-simplices are given by coherent diagrams in Top of shape $P \times [n]$. A *coherent diagram* $\mathcal{C} \to \mathrm{Fun}(P, \mathrm{Top})$ is then a map of simplicial sets

$$N(\mathcal{C}) \to \mathrm{Coh}(P, \mathrm{Top}).$$

Coherent $\delta$-interleavings are defined as before.

**Conjecture 6.0.2**

Assume that $T$ and $S$ are *coherently $\delta$-interleaved.* Then, the functors

$$\Phi_S \circ F,\ |\cdot| \circ N \circ F\colon \mathbb{R} \to \mathrm{Top},$$

where $\Phi_S \circ F$ is the space filtration and $N \circ F$ the nerve filtration, are coherently $\delta$-interleaved.

More generally, from an abstract point of view it might not be hard to drop the assumption on the injectivity of the structure morphisms of $F$. To this end, it is certainly useful to consider the *category of small diagrams.*

**Definition 6.0.3.** Let $\mathcal{C}$ be a category. The *category of small diagrams* $\mathrm{Diag}\,\mathcal{C}$ has as objects functors $G\colon I \to \mathcal{C}$, where $I$ is a small category. A morphism from $G\colon I \to \mathcal{C}$ to $H\colon J \to \mathcal{C}$ is given by a functor $\gamma\colon I \to J$ together with a natural transformation

$$G \Rightarrow H \circ \gamma.$$

With this notion, we can view $T$ and $S$ as functors

$$T, S\colon \mathbb{R} \to \mathrm{Diag}\,\mathrm{Top},$$

where the maps on the indexing categories are induced by the simplicial maps on the nerves.

It is still subject to further research to figure out what we mean by a coherent diagram in Diag Top. But certainly, we want the following statements to hold true.

**Conjecture 6.0.4**

Let $F$ be as above; in particular the structure morphisms need not be monomorphisms. Assume that $T$ and $S$ are *coherently $\delta$-interleaved.* Then, the functors

$$\Phi_S \circ F,\ |\cdot| \circ N \circ F\colon \mathbb{R} \to \mathrm{Top},$$

where $\Phi_S \circ F$ is the space filtration and $N \circ F$ the nerve filtration, are coherently

$\delta$-interleaved.

**Conjecture 6.0.5**

The homotopy colimit sends a coherent diagram in Diag Top to a coherent diagram in Top.

*Remark* 6.0.6. If set up correctly, this should follow from general principles.

Assuming the second conjecture, the proof of Conjecture 6.0.4 is straightforward.

*Proof of Conjecture 6.0.4 assuming Conjecture 6.0.5.* Apply hocolim to the coherent $\delta$-interleaving of $T$ and $S$. This yields a coherent $\delta$-interleaving of $\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r}$ and $|N(\mathcal{U}_r)|$. The claim now follows from the fact that the natural transformation $\operatorname{hocolim} \mathcal{D}_{\mathcal{U}_r} \to X_r$, which is a pointwise homotopy equivalence, can be inverted (up to homotopy) by Theorem 6.0.1. □

For simplicity, assume again that the structure morphisms of $F$ are injective. Instead of assuming that the functors $T$ and $S$ are coherently $\delta$-interleaved, we can also only assume that for each simplex $J \in N$ the non-empty finite intersections $U_J^r$ are coherently $\delta$-interleaved with the diagram that we get by replacing $U_J^r$ by a single point. This is the analogue of a $\delta$-trivial cover.

**Conjecture 6.0.7**

Under the assumptions above, the functor $T$ is coherently $(D+1)\delta$-interleaved with $S$, where $D = \sup_r \dim N(\mathcal{U}_r)$. Hence, $\Phi_S \circ F$ and $|\cdot| \circ N \circ F$ are coherently $(D+1)\delta$-interleaved.

This is the analogue to the persistent nerve theorem (Theorem 5.3.1).

Recall that in the proof of Theorem 5.3.1 we had two main difficulties:

- there were very few conceptual arguments and a lot of computations.
- we had to force the homotopies $c^r$ to fit together over the parameter $r$, which only worked partially. As a result, it is not clear whether the constructions yield a true interleaving.

We hope to achieve the following through a homotopy theoretic ansatz:

- to give a conceptual proof of a generalized persistent nerve theorem that subsumes Theorem 5.3.1. In particular one that does not use the assumption $\operatorname{char} k = 2$.
- to gain the freedom, by replacing strict equalities by “higher homotopies”, to deduce a true coherent interleaving.

If this approach succeeds, then this would strengthen the role of abstract homotopy theory in the field of applied topology.

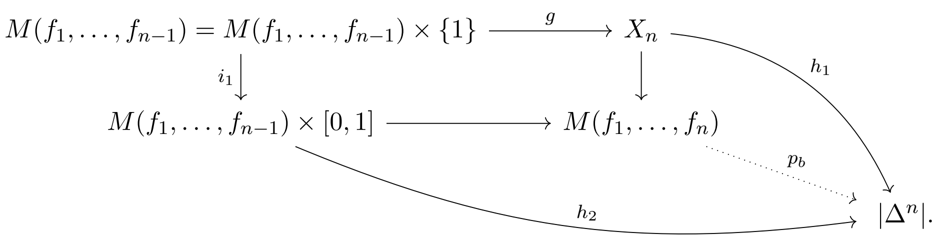
# A. Appendix

## A.1. The Homotopy Colimit according to Hatcher

To see that the iterated mapping cylinders we constructed in Section 3.1.1 fit together, we define for any simplex $\sigma = v_0 \to \cdots \to v_n$ the *natural base projection*

$$p_b \colon M(f_1, \ldots, f_n) \to |\Delta^n|.$$

For $n = 0$, this is the unique map to the one point space and for $n = 1$, this is just the projection of $M(f_1) = (X_0 \times [0,1]) \cup X_1/\sim$ onto the second coordinate. For $n > 0$, we define $p_b$ by the following pushout diagram




(A.1.1)

The map $h_1$ is the constant map onto the last vertex of $|\Delta^n|$ and $h_2$ is the composite

$$M(f_1, \ldots, f_{n-1}) \times [0,1] \overset{p_b \times id}{\to} |\Delta^{n-1}| \times [0,1] \to |\Delta^n|,$$

where the last map identifies the points $|\Delta^{n-1}| \times \{1\}$ with the last vertex of $|\Delta^n|$.

Similarly, the *natural fiber projection*

$$p_f \colon M(f_1, \ldots, f_n) \to \operatorname{colim}(X_0 \overset{f_1}{\to} X_1 \overset{f_2}{\to} \cdots \overset{f_n}{\to} X_n)$$

is induced by the pushout diagram with

$$M(f_1, \ldots, f_{n-1}) \times [0,1] \overset{p_1}{\to} M(f_1, \ldots, f_{n-1}) \to X_{n-1} \to \operatorname{colim}(X_0 \overset{f_1}{\to} X_1 \overset{f_2}{\to} \cdots \overset{f_n}{\to} X_n)$$

and

$$X_n \to \operatorname{colim}(X_0 \overset{f_1}{\to} X_1 \overset{f_2}{\to} \cdots \overset{f_n}{\to} X_n),$$

where the last maps are the canonical maps into the colimit, respectively.

Further, let us construct maps $h_\sigma \colon X_0 \times |\Delta^n| \to M(f_1, \ldots, f_n)$ such that the composite

$$X_0 \times |\Delta^n| \overset{h_\sigma}{\to} M(f_1, \ldots, f_n) \to X_n$$

is the map

$$X_0 \times |\Delta^n| \xrightarrow{p_1} X_0 \to X_n$$

and such that

$$X_0 \times |\Delta^n| \xrightarrow{h_\sigma} M(f_1, \dots, f_n) \xrightarrow{p_b} |\Delta^n|$$

is

$$X_0 \times |\Delta^n| \xrightarrow{p_2} |\Delta^n|.$$

For $n = 0$, this is the identity $\mathrm{id}_{X_0}$ and for $n = 1$, this is the canonical map into $M(f_1) = (X_0 \times [0,1]) \cup X_1 / \sim$. We construct $h_\sigma$ inductively. Let $n \geq 2$ and consider

$$X_0 \times |\Delta^{n-1}| \times [0,1] \xrightarrow{h_{d_n\sigma} \times \mathrm{id}} M(f_1, \dots, f_{n-1}) \times [0,1] \to M(f_1, \dots, f_n),$$

which factors through

$$X_0 \times |\Delta^{n-1}| \times [0,1] \to X_0 \times |\Delta^n|,$$

by the universal property of quotient spaces, because for any $x \in X_0$ the space $\{x\} \times |\Delta^{n-1}| \times \{1\}$ gets mapped to a point in $M(f_1, \dots, f_n)$. This gives the map $h_\sigma$.

**Proposition A.1.1.** *We consider the iterated mapping cylinders* $M_0 = M(f_2, \dots, f_n)$, $M_n = M(f_1, \dots, f_{n-1})$[1] *and for* $1 \leq i < n$, $M_i = M(f_1, \dots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \dots, f_n)$. *Then, we have the following commutative diagram and both squares are pullback diagrams*

$$\begin{array}{ccc} X_0 \times |\Delta^{n-1}| & \xhookrightarrow{\mathrm{id} \times d^i} & X_0 \times |\Delta^n| \\ \downarrow s & & \downarrow h \\ M_i & \hookrightarrow & M(f_1, \dots, f_n) \\ \downarrow p_b & & \downarrow p_b \\ |\Delta^{n-1}| & \xhookrightarrow{d^i} & |\Delta^n| , \end{array} \qquad p_2 : X_0 \times |\Delta^{n-1}| \to |\Delta^{n-1}|, \quad p_2 : X_0 \times |\Delta^n| \to |\Delta^n| \tag{A.1.2}$$

*where* $s$ *is* $h_{d_i\sigma}$, *if* $i \neq 0$ *and* $s$ *is the composition*

$$X_0 \times |\Delta^{n-1}| \xrightarrow{f_1 \times \mathrm{id}} X_1 \times |\Delta^{n-1}| \xrightarrow{h_{d_0\sigma}} M(f_2, \dots, f_n),$$

*if* $i = 0$.

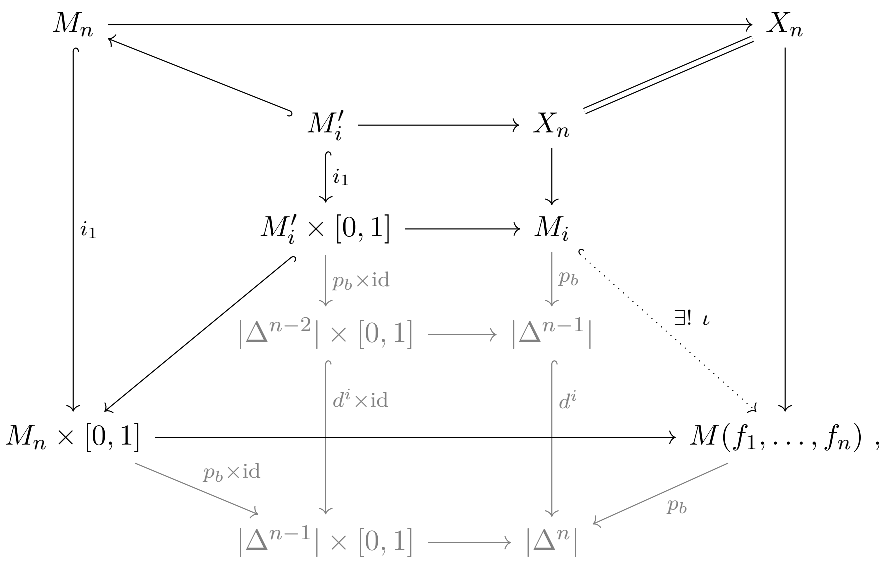
*Proof.* We show the statement by induction over $n$.

The base case $n = 1$ is easily verified, as $M(f_1)$ is simply the mapping cylinder of $f_1$.

Let us prove the induction step $n - 1 \to n$. For $i = n$, Eq. (A.1.2) can be shown by identifying $M_n$ with $M(f_1, \dots, f_{n-1}) \times \{0\}$ in Eq. (A.1.1).

[1] If $n = 1$, then $M_0 = X_1$ and $M_1 = X_0$.

We show the statement for $0 \leq i < n$. Let $M'_i$ be the iterated mapping cylinders as in the proposition for $M(f_1, \ldots, f_{n-1})$. The following commutative diagram shows how the embedding is manufactured




(A.1.3)

where the black diagram illustrates the morphism of pushout diagrams that induces $\iota$ and the colored part is for the pullback diagram. We have $p_b \circ \iota = d^i \circ p_b$, because both morphisms coincide with the morphism that is given by the universal property of pushouts.

It can be checked directly that $\iota$, as being induced by a pushout morphism of embeddings, is indeed an embedding.

A diagram chase shows that $p_b^{-1}(\operatorname{im} d^i) = \operatorname{im} \iota$ and hence the lower rectangle in Eq. (A.1.2) is a pullback diagram.

The commutativity of the remaining diagram in Eq. (A.1.2) follows directly from the definitions. Note, that the composed rectangle is a pullback diagram and because pullbacks satisfy a 2 out of 3 property ([Mac98, p.72]) it follows that the upper rectangle is a pullback diagram as well. ☐

**Definition A.1.2.** The homotopy colimit, as in [Hat02, Section 4.G. Gluing Constructions], of a diagram of spaces $\mathcal{D}$ over $K$ is defined to be

$$\operatorname{Hocolim} \mathcal{D} = \operatorname{colim} M(f_1, \ldots, f_n),$$

where the colimit diagram is indexed over all simplices in $K$ and the morphisms are given by the embeddings described in Proposition A.1.1.

**Definition A.1.3.** If the diagram of spaces $\mathcal{D}$ is the nerve diagram of a cover $\mathcal{F}$, then the natural base projections of the iterated mapping cylinders respect all identifications and glue together to the *base projection*

$$p_b \colon \operatorname{Hocolim} \mathcal{D} \to |\operatorname{sd} N(\mathcal{F})|.$$

Moreover, the natural fiber projections glue together to the the *fiber projection*

$$p_f\colon \operatorname{Hocolim}\mathcal{D} \to \operatorname{colim}\mathcal{D}.$$

**Lemma A.1.4.** *For $n \geq 1$, let $\mathcal{D}$ be a diagram of spaces over the standard $n$-simplex $\sigma = e_0 \to \cdots \to e_n$. We write $M_{i,j}$ for $(M_i)_j$ as in Proposition A.1.1. Then, we have a pushout diagram*

$$\begin{array}{ccc}
\mathcal{D}(e_0)\times|\partial\sigma| & \hookrightarrow & \mathcal{D}(e_0)\times|\sigma| \\
\Big\downarrow{\scriptstyle \overline{h}} & \ulcorner & \Big\downarrow{\scriptstyle h_\sigma} \\
\operatorname{colim}\{M_{i,j}, M_i\} & \hookrightarrow & M(f_1,\dots,f_n),
\end{array}$$

*where the colimit runs over all possible $i,j$ and the morphisms are given by the embeddings*

$$M_i \longleftarrow M_{i,j} \longrightarrow M_j\ .$$

*The map $\overline{h}$ is given on $D(e_0)\times|d_i\sigma|$ as follows. For $i \neq 0$ it is given by $h_{d_i\sigma}\colon \mathcal{D}(e_0)\times|d_i\sigma| \to M_i$ and for $i = 0$ by the morphism $\mathcal{D}(e_0)\times|d_0\sigma| \overset{\mathcal{D}(e_0\to e_1)\times\mathrm{id}}{\to} \mathcal{D}(e_1)\times|d_0\sigma| \overset{h_{d_0\sigma}}{\to} M_0$.*

*Proof.* We show the statement by induction over $n$. For $n = 1$, this is easily verified, because $M(f_1)$ is simply the mapping cylinder of $f_1$.

Let us prove the induction step $n-1 \to n$. By the induction assumption, we have the pushout diagram

$$\begin{array}{ccc}
\mathcal{D}(e_0)\times|\partial\Delta^{n-1}| & \hookrightarrow & \mathcal{D}(e_0)\times|\Delta^{n-1}| \\
\Big\downarrow{\scriptstyle \tilde{h}} & \ulcorner & \Big\downarrow{\scriptstyle h_{d_n\sigma}} \\
\operatorname{colim}_{i,j}\{(M_{n,i})_j, M_{n,i}\} & \hookrightarrow & M(f_1,\dots,f_{n-1}).
\end{array}$$

The pushout diagram in Eq. (A.1.1) and Proposition A.1.1 show that

$$\begin{array}{ccc}
\mathcal{D}(e_0)\times|\Lambda^n_n| & \hookrightarrow & \mathcal{D}(e_0)\times|\Delta^n| \\
\Big\downarrow{\scriptstyle \overline{h}} & \ulcorner & \Big\downarrow{\scriptstyle h_\sigma} \\
\operatorname{colim}_{i,j,\ i\neq n}\{M_{i,j}, M_i\} & \hookrightarrow & M(f_1,\dots,f_n)
\end{array}$$

is a pushout diagram, where $\Lambda^n_k$ is the boundary of the standard $n$-simplex minus the $k$-th face. The induction assumption implies the lemma. □

**Proposition A.1.5.** *Let $\mathcal{D}$ be a diagram of spaces over $K$. Then there exists a homeomorphism*

$$\operatorname{hocolim}\mathcal{D} \cong \operatorname{Hocolim}\mathcal{D}.$$

*Moreover, this homeomorphism identifies the fiber and base projections.*

*Proof.* We prove this fact by induction over the skeleton of $K$. Denote the homotopy colimits over $K^n$ by $\mathrm{hocolim}^n$ and $\mathrm{Hocolim}^n$, respectively.

For $n = 0$, we have

$$\mathrm{hocolim}^0 \cong \coprod_{v \in K^0} \mathcal{D}(v) \cong \mathrm{Hocolim}^n .$$

Denote this homeomorphism by $\varphi_0$. It identifies the fiber and base projection.

Let $n > 0$ be arbitrary. The span

$$\begin{array}{ccc} \coprod_{\sigma = v_0 \to \cdots \to v_n \in K} \mathcal{D}(v_0) \times |\partial \Delta^n| & \hookrightarrow & \coprod_{\sigma = v_0 \to \cdots \to v_n \in K} \mathcal{D}(v_0) \times |\Delta^n| \\ \big\downarrow \bar{h} & & \\ \mathrm{Hocolim}^{n-1} & & \end{array}$$

is, by induction assumption, isomorphic to

$$\begin{array}{ccc} \coprod_{\sigma = v_0 \to \cdots \to v_n \in K} \mathcal{D}(v_0) \times |\partial \sigma| & \hookrightarrow & \coprod_{\sigma = v_0 \to \cdots \to v_n \in K} \mathcal{D}(v_0) \times |\sigma| \\ \big\downarrow p & & \\ \mathrm{hocolim}^{n-1} & & \end{array} ,$$

where $p$ is the canonical projection into the quotient. Note, that the colimit of the second diagram is $\mathrm{hocolim}^n$ and by Lemma A.1.4, the colimit of the first diagram is $\mathrm{Hocolim}^n$. It follows that

$$\mathrm{hocolim}^n \cong \mathrm{Hocolim}^n,$$

by the defining universal property of the colimit. This homeomorphism identifies the fiber and base projection.

If $K$ is finite dimensional, this proves the claim. Otherwise, we conclude by passing to the colimit

$$\mathrm{hocolim}\,\mathcal{D} \cong \mathrm{colim}_n \,\mathrm{hocolim}^n \cong \mathrm{colim}_n \,\mathrm{Hocolim}^n \cong \mathrm{Hocolim}\,\mathcal{D}. \qquad \square$$

## A.2. Single Morphism Characterization of Interleavings

**Definition A.2.1.** Let $P$ be either $\mathbb{Z}$ or $\mathbb{R}$ and let $\delta \in P$ be an element. We call a functor $F\colon P \to \mathrm{Vec}_k$ *$\delta$-trivial* if for every $t \in P$ the equation

$$F(t \leq t + \delta) = 0$$

holds.

**Proposition A.2.2** ([BL15, Corollary 6.6.]). *Two pointwise finite dimensional persistence modules $M, N\colon \mathbb{R} \to \mathrm{Vec}_k$ are $\delta$-interleaved if and only if there exists a morphism $f\colon M \to N(\delta)$ with $\ker f$ and $\operatorname{coker} f$ both $2\delta$-trivial.*

There is an obvious way to apply this result to $\mathbb{Z}$ indexed diagrams. Let $F\colon \mathbb{Z} \to \mathrm{Vec}_k$ be a functor. We define

$$\iota(F)\colon \mathbb{R} \to \mathrm{Vec}_k,\ t \mapsto F_{\lfloor t \rfloor},\ \iota(F)(t \leq t') = F(\lfloor t \rfloor \leq \lfloor t' \rfloor),$$

where $\lfloor \cdot \rfloor$ is the floor function. This defines a natural embedding $\iota\colon\ \mathrm{Vec}_k^{\mathbb{Z}} \to \mathrm{Vec}_k^{\mathbb{R}}$.

**Lemma A.2.3.** *Let $F, G\colon \mathbb{Z} \to \mathrm{Vec}_k$ be two functors. If $F$ and $G$ are $\delta$-interleaved, then so are $\iota(F)$ and $\iota(G)$. Further, the embedding $\iota$ preserves the bottleneck distance.*

*Proof.* It is clear that the first statement holds and thus we have

$$d_B(\iota(F), \iota(G)) \leq d_B(F, G).$$

For the converse inequality note that if $\phi\colon \iota(F) \to \iota(G)(\delta)$ and $\psi\colon \iota(G) \to \iota(F)(\delta)$ form a $\delta$-interleaving, then

$$\tilde{\phi}_t\colon F_t \overset{\phi_t}{\to} G_{t+\delta} = G_{t+\lfloor \delta \rfloor} \text{ and } \tilde{\psi}_t\colon G_t \overset{\psi_t}{\to} F_{t+\delta} = F_{t+\lfloor \delta \rfloor}$$

form a $\lfloor \delta \rfloor$-interleaving between $F$ and $G$. This shows

$$d_B(\iota(F), \iota(G)) \geq d_B(F, G).$$

□

*Remark* A.2.4. In particular, the bottleneck distance between $\iota(F)$ and $\iota(G)$ is always attained and a natural number.

**Proposition A.2.5.** *Two pointwise finite dimensional persistence modules $M, N\colon \mathbb{Z} \to \mathrm{Vec}_k$ are $\delta$-interleaved if and only if there exists a morphism $f\colon M \to N(\delta)$ with* $\ker f$ *and* $\operatorname{coker} f$ *both $2\delta$-trivial.*

*Proof.* If $M, N$ are $\delta$-interleaved, then it follows readily from the definition of interleaving that both interleaving morphisms $M \to N(\delta)$ and $N \to M(\delta)$ have $2\delta$-trivial kernels and cokernels (compare [BL15, Lemma 6.3.]).

The converse follows from the previous statements and the fact that $\delta$-triviality of the kernel and cokernel is preserved by $\iota$. □

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