Technische Universität München
Department of Electrical Engineering and Information Technology
Institute for Electronic Design Automation

Developing Optimization Methods for Design Decomposition of Inkjet-Printed Electronics

Master Thesis

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Abstract

The present master thesis is divided into two main portions.

The part on the subject “Mathematical Modeling for Printing-Based Microfabrication” portrays the algorithm to minimize the group number of the decomposed design and the corresponding drying-time simultaneously. Modeling the actions of the local solvent concentration with numerical methods plays an important role in solving the optimization problem.

The second on the subject “Newton’s Method” interprets the algorithm to find solutions of non-linear equations and systems. The conditions for the convergence and the estimations of its efficiency are presented.

For references, [1–7] are used for Part I Mathematical Modeling for Printing-Based Microfabrication and [8, 9] for Part II Newton’s Method. The Appendix is followed by [10].

Contents

Part I.

Mathematical Modeling for Printing-Based Microfabrication

Given that, the entire design is decomposed into small pieces, which will be mentioned in this context as *polygons* or *patterns*. The target is to assign these small pieces into different but finite groups for printing. Avoiding potential printing problems such as "Laplace pressure conflict" and "Proximity conflict" is required. We call these groups *printing-groups*. Only if the polygons in the previous printing-group dry down entirely, the patterns in the next printing-group can be printed.

Assuming the Gaussian function, denoted by f, models the drying-time-increase between two points and takes the distances as the argument. The local solvent concentration at a point will be modeled as the summation of the solvent concentration increase caused by all under-drying points. Therefore, the drying-time of a point in a printing-group is described as the sum of the definite integrals of the Gaussian function, defining on a closed area. Indeed, every closed area is the area of one polygon. The area of the i-th pattern P_i is denoted by the Cartesian product of two intervals, $\left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$ for $x_i^0, x_i^1, y_i^0, y_i^1 \in \mathbb{R}$. Then, in general, the drying-time of a given printing-group consisting of n polygons, will be described as

$$\max_{i=1,\dots,n} t_{P_i}$$

where

$$t_{P_i} = \max_{(x_i,y_i)\in P_i} \sum_{k=1}^{n} \int_{P_k} f\left(\operatorname{dist}\left((x,y),(x_i,y_i)\right)\right) \, \mathrm{d}^2(x,y)$$

is P_i's drying-time and for $i = 1, \dots, n$,

$$\operatorname{dist}\left((x,y),(x_i,y_i)\right) = \sqrt{(x-x_i)^2 + (y-y_i)^2},$$

is the distance between (x,y) and (x_i,y_i). Therefore, the drying-time of the printing-group, as shown in the figure below, needs to be calculated concerning the rules above.

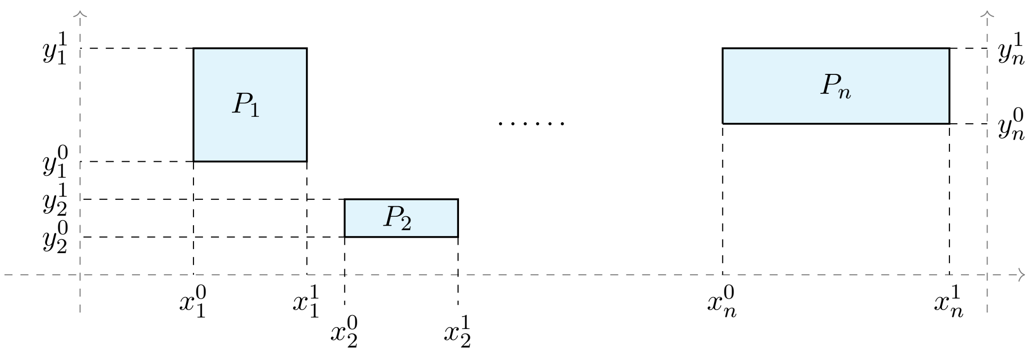

Figure 0.1.: An example of the printing-group consisting of n polygons.

Two models have been constructed for the computation of the drying-time — “Discrete Model” and “Continuous Model”. In the first model, polygons will be considered as collections of sampling points. The figure below shows how the points are distributed in the i-th pattern $P_i = \left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$.

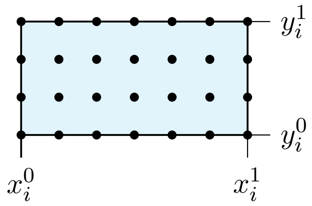

Figure 0.2.: The distribution of the points in the i-th pattern.

P_i's drying-time will be estimated the same as the certain point in the collection, which has the greatest drying-time-increase. The simple computational method guarantees the rapidity and high efficiency of the implementation with C++ program. However, error will be generated with fewer sampling points. In reality there exists infinite points, i.e., the maximum of the sampling points' drying-time is not guaranteed to be the global maximum in the entire printing-group. In order to find out the real drying-time for the multiple under-drying polygons, the second model is constructed in the following way.

According to Fermat's theorem, the local extrema of differentiable functions on open sets will be found by showing that every local extremum of the function is a stationary point.Newton's method is used as the root-finding algorithm to obtain all critical points. Finally, the maximum of their corresponding values is the drying-time without error of an arbitrary printing-group.

Newton's method is a powerful technique — in general the convergence is quadratic: as the method converges on the root, the distance between the root and the approximation is squared at each step. The implementation of Newton's method is realized by the GNU Scientific Library [7].

Consequently, with such approximations of drying-time of every printing-group, we use Gurobi [6] to find out the optimal solution of the following optimization problem,

Minimize #printing-groups $+ \sum$ drying-time of each printing-group,

Subject to: Avoiding of potential printing problems.

1. Basic Concept

Assuming electronics have been decomposed in some way, i.e., the number and the shape as well as the locations of the polygons are fixed and given.

Let N_P denote the number of polygons, N_G the number of printing-groups and P_i the i-th polygon for $i = 1, \ldots, N_P$. Let g_{P_i} refer to P_i's printing-group, then the boundary for g_{P_i} is described as,

$$1 \leq g_{P_i} \leq N_G \tag{1}$$

The patterns with four lines, i.e., rectangles, are concerned. The figure below shows the shape of the i-th polygon $P_i = \left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$.

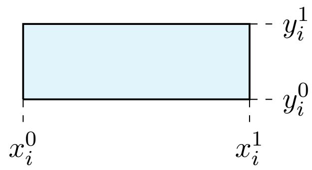

Figure 1.1.: The shape of the i-th polygon.

The goal is to find the optimal assignment of polygons such that both the total required drying-time and the number of printing-groups are minimized while avoiding conflicts.

2. Modeling of Laplace Pressure Conflict

As for each pair of polygons P_i and P_j that have the Laplace pressure conflict, the small object (P_i) needs to be printed earlier than the large object (P_j) to prevent possible ink absorption.

In order to present the relation between the patterns which possess of such characteristics, by [5, Section 3.3.1], we introduce the following constraint,

$$g_{P_i} + 1 \leq g_{P_j}. \tag{2}$$

3. Modeling of Proximity Conflict

This chapter is based on [5, Section 3.3.2].

Each pair of patterns P_i and P_j that have the proximity conflict, needs to be assigned into different groups. Therefore, the following constraint is obtained.

$$g_{P_i} \neq g_{P_j}.$$

Since this constraint is not a linear representation, we transform it into

$$\left(g_{P_i} + 1 \leq g_{P_j}\right) \vee \left(g_{P_j} + 1 \leq g_{P_i}\right),$$

i.e., either P_i is printed earlier than P_j or P_j is printed earlier than P_i. With the "Big M method", we can linearize the above constraint as,

$$\begin{aligned} & g_{P_i} + 1 \leq g_{P_j} + \mathcal{M} \cdot q_{ij}^{PC}, \\ & g_{P_j} + 1 \leq g_{P_i} + \mathcal{M} \cdot \left(1 - q_{ij}^{PC}\right), \end{aligned} \tag{3}$$

We set $\mathcal{M} = 1,000,000$ and q_{ij}^{PC} is a binary auxiliary variable, i.e.,

$$q_{ij}^{PC} = \begin{cases} 1, & g_{P_i} + 1 \leq g_{P_j} \\ 0, & g_{P_j} + 1 \leq g_{P_i} \end{cases}.$$

4. Modeling of Drying-Time

Assuming the Gaussian function models the drying-time-increase between two points and takes the distance as the argument. Within a printing-group, the quantities, relative-size and -placement of the patterns will have an influence on the corresponding drying-time-increase.

Consider the standard normal distribution $\mathcal{N}_{(0,1)}$ and use its probability density function as our Gaussian function,

$$f(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}.$$

4.1. Mathematical Model

In this section, I introduce the ideal mathematical model and without consideration of implementation.

4.1.1. Drying-Time of Two Points

Concentrate on two points A and B. The distribution of these two points is shown below.

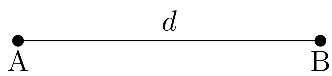

Figure 4.1.: The distribution of A and B.

Let d be the distance between A and B, with respect to the Euclidean norm.

(1) Now consider point A. The drying-time-increase caused by B is described as,

$$t_{B \to A} = f(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d^2}$$

and the drying-time of A itself is

$$t_{A\to A} = f(0) = \frac{1}{\sqrt{2\pi}}e^0 = \frac{1}{\sqrt{2\pi}}.$$

Therefore the drying-time of A including the increase caused by B is

$$t_A = t_{B\to A} + t_{A\to A} = f(d) + f(0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d^2} + \frac{1}{\sqrt{2\pi}}.$$

(2) Now consider point B. The drying-time-increase caused by A is

$$t_{A\to B} = f(d) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d^2}$$

and the drying-time of B itself is

$$t_{B\to B} = f(0) = \frac{1}{\sqrt{2\pi}}e^0 = \frac{1}{\sqrt{2\pi}}.$$

Thus the total drying-time of B is

$$t_B = t_{A\to B} + t_{B\to B} = f(d) + f(0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d^2} + \frac{1}{\sqrt{2\pi}}.$$

(3) Consider the printing-group containing two points.

Figure 4.2.: The printing-group composed of two points.

Then the drying-time of this printing-group is

$$t = \max\left\{t_A, t_B\right\} = f(d) + f(0) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d^2} + \frac{1}{\sqrt{2\pi}}.$$

4.1.2. Drying-Time of One Line and One Point

Consider the printing-group consisting of a line and a point. The distribution of the line and the point can been seen below,

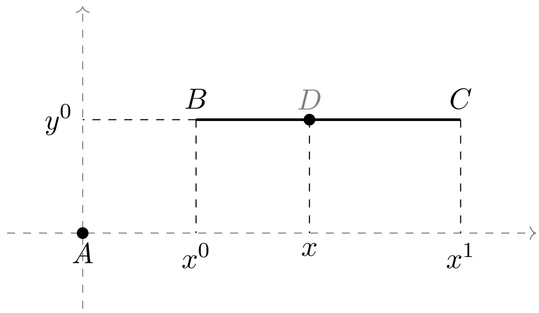

Figure 4.3.: The printing-group composed of a line and a point.

Let the coordinate of A be (x_A, y_A). For an arbitrary point $D = \left(x, y^0\right)$ in BC, the distance between A and D, with respect to the Euclidean norm, is

$$\text{dist}(A, D) = \sqrt{(x - x_A)^2 + (y^0 - y_A)^2}.$$

(1) Consider point A, the drying-time-increase caused by BC is

$$\begin{aligned} t_{BC \to A} &= \int_{[x^0, x^1]} f\left(\text{dist}\left(A, \left(x, y^0\right)\right)\right) \, \mathrm{d}(x) \\ &= \int_{x^0}^{x^1} f\left(\sqrt{(x - x_A)^2 + (y^0 - y_A)^2}\right) \, \mathrm{d}x = \int_{x^0}^{x^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x - x_A)^2 + \left(y^0 - y_A\right)^2\right]} \, \mathrm{d}x. \end{aligned}$$

Therefore the drying-time of point A is

$$\begin{aligned} t_A &= t_{BC \to A} + t_{A \to A} \\ &= \int_{x^0}^{x^1} f\left(\sqrt{(x - x_A)^2 + (y^0 - y_A)^2}\right) \, \mathrm{d}x + f(0) = \int_{x^0}^{x^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x - x_A)^2 + \left(y^0 - y_A\right)^2\right]} \, \mathrm{d}x + \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

(2) Consider the line BC.

- The drying-time-increase caused by A is

$$
\begin{aligned}
t_{A\to BC} &= \max_{D\in BC} f\left(\operatorname{dist}(A,D)\right)\\
&= \max_{x\in[x^0,x^1]} f\left(\sqrt{(x-x_A)^2+(y^0-y_A)^2}\right) = \max_{x\in[x^0,x^1]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_A)^2+(y^0-y_A)^2\right]},
\end{aligned}
$$

- The drying-time of BC itself is

$$
\begin{aligned}
t_{BC\to BC} &= \max_{x\in[x^0,x^1]} \int_{BC} f\left(\operatorname{dist}(\bar{x},x)\right)\ \mathrm{d}(\bar{x}) = \max_{x\in[x^0,x^1]} \int_{x^0}^{x^1} f\left(\sqrt{(\bar{x}-x)^2}\right)\ \mathrm{d}\bar{x}\\
&= \max_{x\in[x^0,x^1]} \int_{x^0}^{x^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\bar{x}-x)^2}\ \mathrm{d}\bar{x}
\end{aligned}
$$

Therefore the drying-time of BC including the drying-time-increase caused by A is

$$
\begin{aligned}
t_{BC} &= \max_{x\in[x^0,x^1]} \left\{ f\left(\operatorname{dist}\left(A,\left(x,y^0\right)\right)\right) + \int_{BC} f\left(\operatorname{dist}(\bar{x},x)\right)\ \mathrm{d}(\bar{x}) \right\}\\
&= \max_{x\in[x^0,x^1]} \left\{ f\left(\sqrt{(x-x_A)^2+(y^0-y_A)^2}\right) + \int_{x^0}^{x^1} f\left(\sqrt{(\bar{x}-x)^2}\ \right)\mathrm{d}\bar{x} \right\}\\
&= \frac{1}{\sqrt{2\pi}} \max_{x\in[x^0,x^1]} \left\{ e^{-\frac{1}{2}[(x-x_A)^2+(y^0-y_A)^2]} + \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2}\ \mathrm{d}\bar{x} \right\}
\end{aligned}
$$

(3) Finally, the drying-time of this printing-group is:

$$
\begin{aligned}
t &= \max\left\{t_A, t_{BC}\right\}\\
&= \frac{1}{\sqrt{2\pi}} \max\left\{ \int_{x^0}^{x^1} e^{-\frac{1}{2}[(x-x_A)^2+(y^0-y_A)^2]}\ \mathrm{d}x + 1,\ \max_{x\in[x^0,x^1]} \left\{ e^{-\frac{1}{2}[(x-x_A)^2+(y^0-y_A)^2]} + \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2}\ \mathrm{d}\bar{x} \right\}\right\}.
\end{aligned}
$$

4.1.3. Drying-Time of One Point and One Rectangle

Now consider a printing-group consisting of a point and a rectangle, which can be seen below.

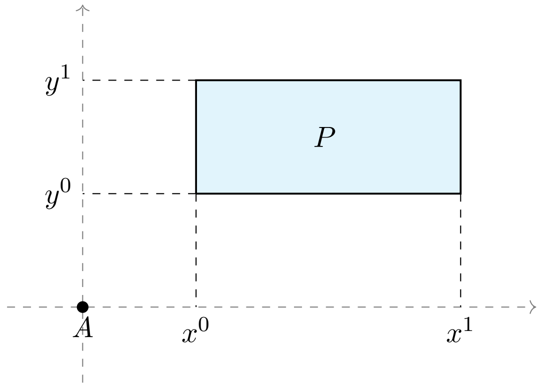

Figure 4.4.: An example of the printing-group consisting of a point and a rectangle.

As before, let the coordinate of A be (x_A, y_A). The area of the pattern P is denoted by the Cartesian product of two intervals $\left[x^0, x^1\right] \times \left[y^0, y^1\right]$.

(1) The drying-time-increase of point A caused by P is

$$t_{P \to A} = \int\limits_P f\left(\operatorname{dist}\left(A, (x,y)\right)\right)\ \mathrm{d}^2(x,y)$$

$$= \int\limits_{[x^0,x^1]\times[y^0,y^1]} f\left(\sqrt{(x-x_A)^2+(y-y_A)^2}\right)\ \mathrm{d}^2(x,y) = \int\limits_{x^0}^{x^1}\int\limits_{y^0}^{y^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_A)^2+(y-y_A)^2\right]}\ \mathrm{d}y\mathrm{d}x.$$

Thus the drying-time of A including the increase caused by P is

$$\begin{aligned} t_A &= t_{P\to A} + t_{A\to A} \\ &= \int\limits_{[x^0,x^1]\times[y^0,y^1]} f\left(\sqrt{(x-x_A)^2+(y-y_A)^2}\right)\ \mathrm{d}^2(x,y) + f(0) \\ &= \int\limits_{x^0}^{x^1}\int\limits_{y^0}^{y^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_A)^2+(y-y_A)^2\right]}\ \mathrm{d}y\mathrm{d}x + \frac{1}{\sqrt{2\pi}} \end{aligned}$$

(2) Now consider the pattern P.

- The drying-time-increase caused by A is

$$t_{A\to P} = \max_{p\in P} f\left(\operatorname{dist}(A,p)\right)$$

$$= \max_{(x,y)\in P} f\left(\sqrt{(x-x_A)^2+(y-y_A)^2}\right) = \max_{x\in[x^0,x^1],y\in[y^0,y^1]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_A)^2+(y-y_A)^2\right]}$$

- The drying-time of P itself is

$$t_{P\to P} = \max_{p\in P} \int_P f\left(\text{dist}\left(\bar{x},\bar{y}\right),p\right)\ \mathrm{d}^2\left(\bar{x},\bar{y}\right)$$
$$= \max_{(x,y)\in P} \int_P f\left(\sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2}\right)\ \mathrm{d}^2(\bar{x},\bar{y})$$
$$= \max_{x\in[x^0,x^1],y\in[y^0,y^1]} \int_{x^0}^{x^1}\int_{y^0}^{y^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]}\ \mathrm{d}\bar{y}\mathrm{d}\bar{x}.$$

Therefore the drying-time of P including the drying-time-increase caused by A is

$$t_P = \max_{p\in P}\left\{f\left(\text{dist}\left(A,p\right)\right) + \int_P f\left(\text{dist}\left(\bar{x},\bar{y}\right),p\right)\ \mathrm{d}^2\left(\bar{x},\bar{y}\right)\right\}$$
$$= \max_{(x,y)\in P}\left\{f\left(\sqrt{(x-x_A)^2+(y-y_A)^2}\right) + \int_P f\left(\sqrt{(\bar{x}-x)^2+(\bar{y}-y)^2}\right)\ \mathrm{d}^2(\bar{x},\bar{y})\right\}$$
$$= \frac{1}{\sqrt{2\pi}} \max_{x\in[x^0,x^1],y\in[y^0,y^1]}\left\{e^{-\frac{1}{2}\left[(x-x_A)^2+(y-y_A)^2\right]} + \int_{x^0}^{x^1}\int_{y^0}^{y^1} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]}\ \mathrm{d}\bar{y}\mathrm{d}\bar{x}\right\}$$

(3) Finally, the drying-time of the printing-group which consists of a point and a rectangle is

$$t = \max\left\{t_A, t_P\right\}$$
$$= \frac{1}{\sqrt{2\pi}}\max\left\{\int_{x^0}^{x^1}\int_{y^0}^{y^1} e^{-\frac{1}{2}\left[(x-x_A)^2+(y-y_A)^2\right]}\ \mathrm{d}y\mathrm{d}x + 1,\right.$$
$$\left.\max_{x\in[x^0,x^1],y\in[y^0,y^1]}\left\{e^{-\frac{1}{2}\left[(x-x_A)^2+(y-y_A)^2\right]} + \int_{x^0}^{x^1}\int_{y^0}^{y^1} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]}\ \mathrm{d}\bar{y}\mathrm{d}\bar{x}\right\}\right\}.$$

4.1.4. Drying-Time of Two Rectangles

Now, the printing-group consisting of two rectangles is concerned. This is a typical situation. The figure below shows an example of the distribution of two patterns.

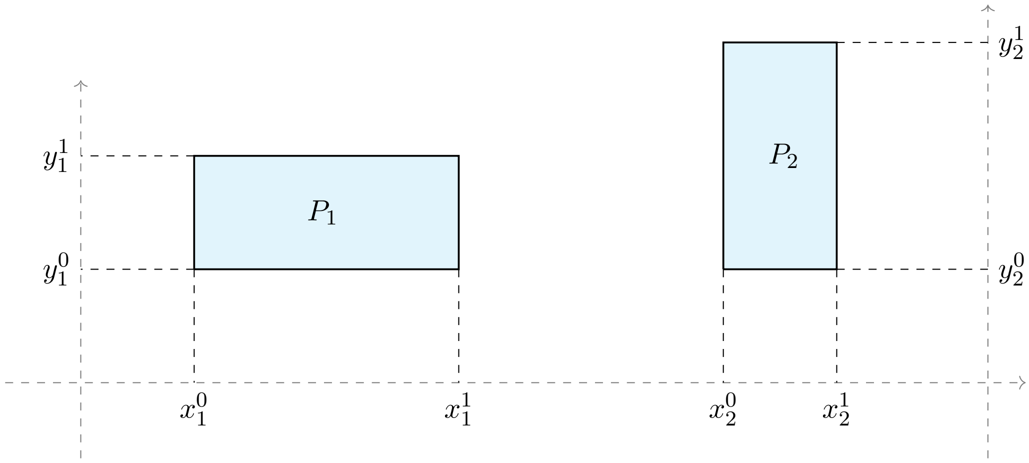

Figure 4.5.: The distribution of two rectangles in one printing-group.

(1) Consider the pattern P_1.

- The drying-time-increase caused by P_2 is

$$\begin{aligned} t_{P_2 \to P_1} &= \max_{p_1 \in P_1} \int_{P_2} f\left(\operatorname{dist}\left((x,y), p_1\right)\right) \, \mathrm{d}^2(x,y) \\ &= \max_{(x_1,y_1)\in P_1} \int_{P_2} f\left(\sqrt{(x-x_1)^2 + (y-y_1)^2}\right) \, \mathrm{d}^2(x,y) \\ &= \max_{x_1 \in [x_1^0, x_1^1], y_1 \in [y_1^0, y_1^1]} \int_{x_2^0}^{x_2^1} \int_{y_2^0}^{y_2^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_1)^2 + (y-y_1)^2\right]} \, \mathrm{d}y \mathrm{d}x \end{aligned}$$

- The drying-time of P_1 itself is

$$\begin{aligned} t_{P_1 \to P_1} &= \max_{p_1 \in P_1} \int_{P_1} f\left(\operatorname{dist}\left((x,y), p_1\right)\right) \, \mathrm{d}^2(x,y) \\ &= \max_{(x_1,y_1)\in P_1} \int_{P_1} f\left(\sqrt{(x-x_1)^2 + (y-y_1)^2}\right) \, \mathrm{d}^2(x,y) \\ &= \max_{x_1 \in [x_1^0, x_1^1], y_1 \in [y_1^0, y_1^1]} \int_{x_1^0}^{x_1^1} \int_{y_1^0}^{y_1^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_1)^2 + (y-y_1)^2\right]} \, \mathrm{d}y \mathrm{d}x \end{aligned}$$

Therefore the drying-time of P_1 including the drying-time-increase caused by P_2 is

$$
\begin{aligned}
t_{P_1} &= \max_{p_1 \in P_1} \left\{ \int_{P_1} f\left(\operatorname{dist}\left((x,y), p_1\right)\right) \, \mathrm{d}^2(x,y) + \int_{P_2} f\left(\operatorname{dist}\left((x,y), p_1\right)\right) \, \mathrm{d}^2(x,y) \right\} \\
&= \max_{(x_1,y_1) \in P_1} \left\{ \int_{P_1} f\left(\sqrt{(x-x_1)^2 + (y-y_1)^2}\right) \, \mathrm{d}^2(x,y) + \int_{P_2} f\left(\sqrt{(x-x_1)^2 + (y-y_1)^2}\right) \, \mathrm{d}^2(x,y) \right\} \\
&= \frac{1}{\sqrt{2\pi}} \max_{x_1 \in \left[x_1^0, x_1^1\right], y_1 \in \left[y_1^0, y_1^1\right]} \left\{ \int_{x_1^0}^{x_1^1} \int_{y_1^0}^{y_1^1} e^{-\frac{1}{2}\left[(x-x_1)^2 + (y-y_1)^2\right]} \, \mathrm{d}y \mathrm{d}x + \int_{x_2^0}^{x_2^1} \int_{y_2^0}^{y_2^1} e^{-\frac{1}{2}\left[(x-x_1)^2 + (y-y_1)^2\right]} \, \mathrm{d}y \mathrm{d}x \right\}
\end{aligned}
$$

(2) Analogous for P_2, we obtain

$$
\begin{aligned}
t_{P_2} &= \max_{p_2 \in P_2} \left\{ \int_{P_1} f\left(\operatorname{dist}\left((x,y), p_2\right)\right) \, \mathrm{d}^2(x,y) + \int_{P_2} f\left(\operatorname{dist}\left((x,y), p_2\right)\right) \, \mathrm{d}^2(x,y) \right\} \\
&= \max_{(x_2,y_2) \in P_2} \left\{ \int_{P_1} f\left(\sqrt{(x-x_2)^2 + (y-y_2)^2}\right) \, \mathrm{d}^2(x,y) + \int_{P_2} f\left(\sqrt{(x-x_2)^2 + (y-y_2)^2}\right) \, \mathrm{d}^2(x,y) \right\} \\
&= \frac{1}{\sqrt{2\pi}} \max_{x_2 \in \left[x_2^0, x_2^1\right], y_2 \in \left[y_2^0, y_2^1\right]} \left\{ \int_{x_1^0}^{x_1^1} \int_{y_1^0}^{y_1^1} e^{-\frac{1}{2}\left[(x-x_2)^2 + (y-y_2)^2\right]} \, \mathrm{d}y \mathrm{d}x + \int_{x_2^0}^{x_2^1} \int_{y_2^0}^{y_2^1} e^{-\frac{1}{2}\left[(x-x_2)^2 + (y-y_2)^2\right]} \, \mathrm{d}y \mathrm{d}x \right\}
\end{aligned}
$$

(3) Therefore, the drying-time of the printing-group with two rectangles is

$$
\begin{aligned}
t &= \max\left\{t_{P_1}, t_{P_2}\right\} \\
&= \frac{1}{\sqrt{2\pi}} \max_{\substack{x_1 \in \left[x_1^0, x_1^1\right], y_1 \in \left[y_1^0, y_1^1\right] \\ x_2 \in \left[x_2^0, x_2^1\right], y_2 \in \left[y_2^0, y_2^1\right]}} \left\{ \int_{x_1^0}^{x_1^1} \int_{y_1^0}^{y_1^1} e^{-\frac{1}{2}\left[(x-x_i)^2 + (y-y_i)^2\right]} \, \mathrm{d}y \mathrm{d}x + \int_{x_2^0}^{x_2^1} \int_{y_2^0}^{y_2^1} e^{-\frac{1}{2}\left[(x-x_i)^2 + (y-y_i)^2\right]} \, \mathrm{d}y \mathrm{d}x, i = 1, 2 \right\}
\end{aligned}
$$

4.1.5. Drying-Time of n Rectangles

Let $n \in \mathbb{N}$ be arbitrary and consider the printing-group with n patterns.

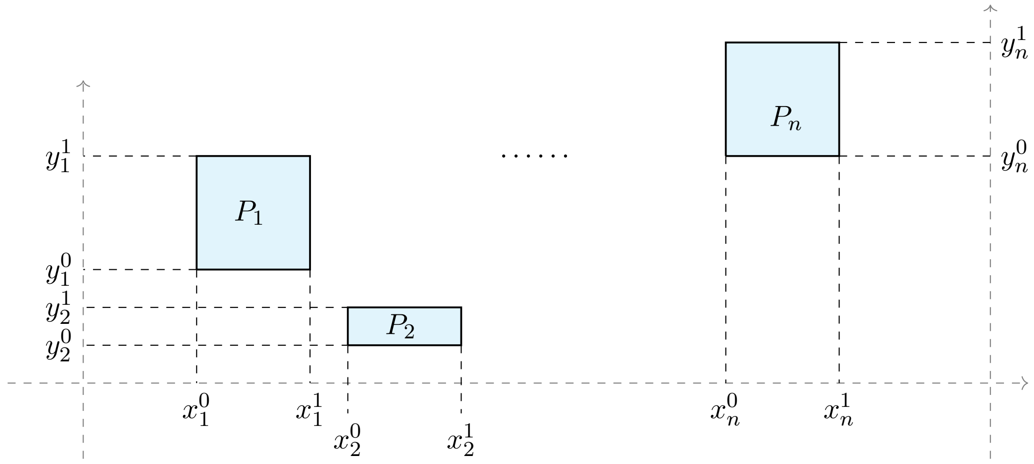

Figure 4.6.: The distribution of n patterns in one printing-group.

Consider the polygon P_i for $i = 1, \dots, n$ with $P_i = \left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$. Analogously,

(1) The drying-time-increase caused by P_i itself is

$$t_{P_i \to P_i} = \max_{x_i \in [x_i^0, x_i^1], y_i \in [y_i^0, y_i^1]} \int_{x_i^0}^{x_i^1} \int_{y_i^0}^{y_i^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_i)^2 + (y-y_i)^2\right]} \, \mathrm{d}y \mathrm{d}x$$

(2) The drying-time increase-caused by any other rectangle P_j with $i \neq j$ is

$$t_{P_j \to P_i} = \max_{x_i \in [x_i^0, x_i^1], y_i \in [y_i^0, y_i^1]} \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x-x_i)^2 + (y-y_i)^2\right]} \, \mathrm{d}y \mathrm{d}x$$

(3) Therefore, the drying-time of P_i including the drying-time-increase by other rectangles is

$$t_{P_i} = \frac{1}{\sqrt{2\pi}} \max_{x_i \in [x_i^0, x_i^1], y_i \in [y_i^0, y_i^1]} \sum_{j=1}^{n} \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} e^{-\frac{1}{2}\left[(x-x_i)^2 + (y-y_i)^2\right]} \, \mathrm{d}y \mathrm{d}x$$

This is analogous for each $i = 1, \ldots, n$. Finally, the drying-time of these n rectangles is

$$t = \max_{i=1,\ldots,n} t_{P_i} = \frac{1}{\sqrt{2\pi}} \max_{\substack{i=1,\ldots,n, \\ x_i \in [x_i^0, x_i^1], y_i \in [y_i^0, y_i^1]}} \sum_{j=1}^{n} \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} e^{-\frac{1}{2}[(x-x_i)^2+(y-y_i)^2]} \,\mathrm{d}y\mathrm{d}x$$

4.2. Finding the Maximum-Point with Mathematical Method

The description of the drying-time of a given printing-group have been already obtained with "max". In this section, I introduce the approach to find the solution of our expression by using mathematical method.

4.2.1. Mathematical Background

Definition 4.1 (Hessian-matrix). *Let $U \subseteq \mathbb{R}^d$ be open and $h : U \to \mathbb{R}$; if all second partial derivatives of h exist and are continuous over U, then the **Hessian-matrix** $H_h(x)$ of h is a square $d \times d$ matrix, usually defined and arranged as follows:*

$$H_h(x) := \left(D^2 h\right)(x) = \left[\partial_j \partial_k h(x)\right]_{j,k} = \begin{pmatrix} \partial_1^2 h & \partial_1\partial_2 h & \cdots & \partial_1\partial_d h \\ \partial_2\partial_1 h & \partial_2^2 h & \cdots & \partial_2\partial_d h \\ \vdots & & \ddots & \vdots \\ \partial_d\partial_1 h & \partial_d\partial_2 h & \cdots & \partial_d^2 h \end{pmatrix}$$

Notation.

$$\partial_j \partial_k h(x) = \partial_j \left(\partial_k h(x)\right) = \frac{\partial\left(\frac{\partial h(x)}{\partial x_k}\right)}{\partial x_j} = \frac{\partial^2 h(x)}{\partial x_j \partial x_k} \qquad \diamond$$

Remark 4.2. In our case, consider solely $d = 2$, i.e., $h : \mathbb{R}^2 \to \mathbb{R}$, hence the Hessian-matrix has the form,

$$H_h(\bar{x}) = \begin{pmatrix} \partial_1^2 h(\bar{x}) & \partial_1\partial_2 h(\bar{x}) \\ \partial_2\partial_1 h(\bar{x}) & \partial_2^2 h(\bar{x}) \end{pmatrix} = \begin{pmatrix} \partial_x^2 h(x,y) & \partial_x\partial_y h(x,y) \\ \partial_y\partial_x h(x,y) & \partial_y^2 h(x,y) \end{pmatrix} \qquad \text{for } \bar{x} = (x,y). \qquad \diamond$$

`Theorem 4.3 (H.A.Schwarz).` By [3, Chapter 8, Section 8.1, Theorem 8.8], let $U \subseteq \mathbb{R}^d$ be open, $j, k \in \{1, \dots, d\}$ and $h : U \to \mathbb{R}$ be partially differentiable; furthermore let $\partial_k \partial_j h$ exist in U and be continuous. Then $\partial_j \partial_k h$ exists in U and $\partial_j \partial_k h = \partial_k \partial_j h$. △

Definition 4.4 ((strict) local minimum/maximum). *Let $U \subseteq \mathbb{R}^d$, $h : U \to \mathbb{R}$. h is said to have*

*(1) **local minimum** at point $x_0 \in U$, if there exists a neighborhood U_{x_0} of x_0 and for all $y \in U_{x_0} \cap U$, $h(x_0) \leq h(y)$.*

*(2) **local maximum** at point $x_0 \in U$, if for all $y \in U_{x_0} \cap U$, $h(x_0) \geq h(y)$.*

*(3) **strict local maximum** at point $x_0 \in U$, if for all $y \in (U_{x_0} \cap U) \setminus \{x_0\}$, $h(x_0) > h(y)$.*

*(4) and **strict local minimum** at point $x_0 \in U$, if for all $y \in (U_{x_0} \cap U) \setminus \{x_0\}$, $h(x_0) < h(y)$.*

*We said h has a **local extremum** if h has a local minimum or a local maximum.*

Definition 4.5 (Gradient). *By [3, Chapter 8, Section 8.1, Definition 8.4], let $U \subseteq \mathbb{R}^d$ be open and $h : U \to \mathbb{R}$ be partially differentiable. The **gradient** of h is denoted by ∇h and $\operatorname{grad} h := \nabla h : U \to \mathbb{R}^d$ with $x \mapsto ((\partial_1 h)(x), \dots, (\partial_d h)(x))$. In particular, $\nabla := (\partial_1, \dots, \partial_d)$ is **Nabla-operator**.*

`Theorem 4.6 (Necessary Condition for the local Extrema).` By [3, Chapter 8, Section 8.4, Theorem 8.34], let $U \subseteq \mathbb{R}^d$ be open and $h : U \to \mathbb{R}$ be partially differentiable. If h have a local extremum at point $x \in U$, then $(\nabla h)(x) = 0$. △

Let $\operatorname{Mat}(d \times d, \mathbb{R})$ denote the set of $d \times d$ real matrices.

Definition 4.7. *By [3, Chapter 8, Section 8.4, Definition 8.35], let $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a inner product, $M \in \operatorname{Mat}(d \times d, \mathbb{R})$ be symmetric. Then*

*(1) M is **(strict) positive definite** if and only if $\langle x, Mx \rangle > 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$;*

*(2) M is **positive semidefinite** if and only if $\langle x, Mx \rangle \geq 0$ for all $x \in \mathbb{R}^d$;*

*(3) M is **(strict) negative definite (resp. semidefinite)** if and only if $-M$ (strict) positive definite (resp. semidefinite);*

*(4) M is **indefinite** if and only if there exists $x, y \in \mathbb{R}^d$ with $\langle x, Mx \rangle > 0$ and $\langle y, My \rangle < 0$.*

`Theorem 4.8 (Sufficient Condition for Extrema).` By [3, Chapter 8, Section 8.4, Theorem 8.38], let $U \subseteq \mathbb{R}^d$ open, $x \in U$ and $h : U \to \mathbb{R}$ twice continuously differentiable. Then

(1) If $(\nabla h)(x) = 0$ and $\left(D^2 h\right)(x)$ strict positive, then x is the strict local minimum of h.

(2) If $(\nabla h)(x) = 0$ and $\left(D^2 h\right)(x)$ strict negative, then x is the strict local maximum of h.

Warning: If $\left(D^2 h\right)(x)$ is positive or negative semidefinite, then there is no statement. △

Definition 4.9. *By [2, Page 1] and [4, Page 2], let $M \in \mathrm{Mat}(d \times d, \mathbb{R})$. A $k \times k$ submatrix of M formed by deleting $n-k$ rows of M, and the same $n-k$ columns of M, is called* ***principal submatrix*** *of M. The determinant of a principal submatrix of M is called a* ***principal minor*** *of M and denoted by $\triangle_k$.*

The k-th oder principal submatrix of M obtained by deleting the last $d-k$ rows and columns of M is called the k-th oder ***leading principal submatrix*** *of M and denoted by $M^{(k)}$.*

$$M^{(k)} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1k} \\ m_{21} & m_{22} & \cdots & m_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k1} & m_{k2} & \cdots & m_{kk} \end{pmatrix}.$$

In particular, $M^{(1)} = (m_{11})$ and $M^{(d)} = M$. The determinant of $M^{(k)}$ is called the k-th order ***leading principal minor*** *of M and denoted by D_k.*

`Theorem 4.10 (Sylvester's Criterion).` By [4, Page 5], let M be a symmetric $d \times d$ matrix. Then,

(1) M is positive definite if and only if $D_k > 0$ for all the leading principal minors.

(2) M is negative definite if and only if $(-1)^k D_k > 0$ for all the leading principal minors.

(3) M is positive semidefinite if and only if $\triangle_k \geq 0$ for all the principal minors.

(4) M is negative semidefinite if and only if $(-1)^k \triangle_k \geq 0$ for all the principal minors.

(5) M is indefinite, if one of its k-th order leading principal minors is negative for an even k or if there are two odd leading principal minors that have different signs. △

Consider our Hessian-matrix, $H_h(x) = \begin{pmatrix} \partial_x^2 h(x,y) & \partial_x \partial_y h(x,y) \\ \partial_y \partial_x h(x,y) & \partial_y^2 h(x,y) \end{pmatrix}$. By `Theorem 4.3` we have $\partial_x \partial_y h(x,y) = \partial_y \partial_x h(x,y)$. Thus,

(1) $H_h(x,y) > 0$ if and only if $\partial_x^2 h(x,y) > 0$ and $\det(H_h(x,y)) = \partial_x^2 h(x,y) \cdot \partial_y^2 h(x,y) - (\partial_x \partial_y h(x,y))^2 > 0$.

(2) $H_h(x,y) < 0$ if and only if $\partial_x^2 h(x,y) < 0$ and $\det(H_h(x,y)) = \partial_x^2 h(x,y) \cdot \partial_y^2 h(x,y) - (\partial_x \partial_y h(x,y))^2 > 0$.

4.2.2. Conclusions

`Claim 4.11.` Consider an arbitrary printing-group containing only one pattern $P = \left[x^0, x^1\right] \times \left[y^0, y^1\right]$. Define the *central-point* of P equals to $\left(\frac{x^0+x^1}{2}, \frac{y^0+y^1}{2}\right)$. Then the drying-time of this printing-group is the drying-time of the central-point of P. ◊

There are two methods to proof the claim. The first is theoretical and the second is intuitive.

Proof (Method 1). By 4.1 Mathematical Model, the drying-time of P is

$$\max_{(x,y)\in P} \frac{1}{\sqrt{2\pi}} \int_{x^0}^{x^1} \int_{y^0}^{y^1} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \,\mathrm{d}\bar{y}\mathrm{d}\bar{x}.$$

We define

$$g(x,y) = \frac{1}{\sqrt{2\pi}} \int_{x^0}^{x^1} \int_{y^0}^{y^1} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \,\mathrm{d}\bar{y}\mathrm{d}\bar{x} = \frac{1}{\sqrt{2\pi}} \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2} \,\mathrm{d}\bar{x} \int_{y^0}^{y^1} e^{-\frac{1}{2}(\bar{y}-y)^2} \,\mathrm{d}\bar{y}$$

then the partial derivatives of g are

- $\partial_x g(x,y) = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}\left(x^0-x\right)^2} - e^{-\frac{1}{2}\left(x^1-x\right)^2}\right) \int_{y^0}^{y^1} e^{-\frac{1}{2}(\bar{y}-y)^2} \,\mathrm{d}\bar{y}$;

- $\partial_y g(x,y) = \frac{1}{\sqrt{2\pi}} \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2} \,\mathrm{d}\bar{x} \left(e^{-\frac{1}{2}\left(y^0-y\right)^2} - e^{-\frac{1}{2}\left(y^1-y\right)^2}\right)$;

- $\partial_x^2 g(x,y) = \frac{1}{\sqrt{2\pi}} \left(\left(x^0-x\right) e^{-\frac{1}{2}\left(x^0-x\right)^2} - \left(x^1-x\right) e^{-\frac{1}{2}\left(x^1-x\right)^2}\right) \int_{y^0}^{y^1} e^{-\frac{1}{2}(\bar{y}-y)^2} \,\mathrm{d}\bar{y}$;

- $\partial_y^2 g(x,y) = \frac{1}{\sqrt{2\pi}} \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2} \,\mathrm{d}\bar{x} \left(\left(y^0-y\right) e^{-\frac{1}{2}\left(y^0-y\right)^2} - \left(y^1-y\right) e^{-\frac{1}{2}\left(y^1-y\right)^2}\right)$;

- $\partial_x \partial_y g(x,y) = \partial_y \partial_x g(x,y) = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}\left(x^0-x\right)^2} - e^{-\frac{1}{2}\left(x^1-x\right)^2}\right) \left(e^{-\frac{1}{2}\left(y^0-y\right)^2} - e^{-\frac{1}{2}\left(y^1-y\right)^2}\right)$.

Now we calculate the critical point by setting $\nabla g(x,y) = (\partial_x g(x,y), \partial_y g(x,y)) = 0$.
$\partial_x g(x,y) = 0$ implies

$$\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}(x^0-x)^2} - e^{-\frac{1}{2}(x^1-x)^2}\right)\underbrace{\int_{y^0}^{y^1} e^{-\frac{1}{2}(\bar{y}-y)^2}\, \mathrm{d}\bar{y}}_{>0} = 0,$$

i.e., $\underbrace{e^{-\frac{1}{2}(x^0-x)^2}}_{>0} - \underbrace{e^{-\frac{1}{2}(x^1-x)^2}}_{>0} = 0$ and $\partial_y g(x,y) = 0$ implies

$$\frac{1}{\sqrt{2\pi}}\underbrace{\int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2}\, \mathrm{d}\bar{x}}_{>0}\left(e^{-\frac{1}{2}(y^0-y)^2} - e^{-\frac{1}{2}(y^1-y)^2}\right) = 0,$$

i.e., $\underbrace{e^{-\frac{1}{2}(y^0-y)^2}}_{>0} - \underbrace{e^{-\frac{1}{2}(y^1-y)^2}}_{>0} = 0$. Therefore, there exists an unique critical point $(\hat{x},\hat{y}) = \left(\frac{x^0+x^1}{2}, \frac{y^0+y^1}{2}\right)$.
`Theorem 4.8` and Theorem 4.10 (2) can be used to confirm this critical point is the strict local maximum of g, i.e.,

$$\begin{aligned}\partial_x^2 g(\hat{x},\hat{y}) &= \frac{1}{\sqrt{2\pi}}\left(\left(\frac{x^0-x^1}{2}e^{-\frac{1}{4}(x^0-x^1)^2} - \frac{x^1-x^0}{2}e^{-\frac{1}{4}(x^1-x^0)^2}\right)\int_{\frac{y^0-y^1}{2}}^{\frac{y^1-y^0}{2}} e^{-\frac{1}{2}(\bar{y}-y)^2}\, \mathrm{d}\bar{y}\right)\\ &= \frac{1}{\sqrt{2\pi}}\underbrace{\left(x^0-x^1\right)}_{<0}\overbrace{e^{-\frac{1}{4}(x^0-x^1)^2}}^{>0}\underbrace{\int_{\frac{y^0-y^1}{2}}^{\frac{y^1-y^0}{2}} e^{-\frac{1}{2}(\bar{y}-y)^2}\, \mathrm{d}\bar{y}}_{>0} < 0\end{aligned}$$

and

$$\begin{aligned}\det\left(H_g(\hat{x},\hat{y})\right) &= \partial_x^2 g(\hat{x},\hat{y})\partial_y^2 g(\hat{x},\hat{y}) - \left(\partial_x\partial_y g(\hat{x},\hat{y})\right)^2\\ &= \frac{1}{2\pi}\underbrace{\left(x^0-x^1\right)e^{-\frac{1}{4}(x^0-x^1)^2}\int_{\frac{y^0-y^1}{2}}^{\frac{y^1-y^0}{2}} e^{-\frac{1}{2}(\bar{y}-y)^2}\, \mathrm{d}\bar{y}}_{<0}\overbrace{\left(y^0-y^1\right)e^{-\frac{1}{4}(y^0-y^1)^2}\int_{\frac{x^0-x^1}{2}}^{\frac{x^1-x^0}{2}} e^{-\frac{1}{2}(\bar{x}-x)^2}\, \mathrm{d}\bar{x}}^{<0} - 0 > 0.\end{aligned}$$

Hence, the claim follows. ■

Proof (Method 2). Consider the function from *Proof (Method 1)*,

$$g(x,y) = \frac{1}{\sqrt{2\pi}} \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2}\, \mathrm{d}\bar{x} \int_{y^0}^{y^1} e^{-\frac{1}{2}(\bar{y}-y)^2}\, \mathrm{d}\bar{y} =: \frac{1}{\sqrt{2\pi}} g_1(x) g_2(y)$$

where g_1 only depends on x and g_2 only depends on y. The maximum of g will be achieved if and only if g_1 and g_2 achieve their maximum respectively. Consider g_1 (analogous for g_2),

$$g_1(x) = \int_{x^0}^{x^1} e^{-\frac{1}{2}(\bar{x}-x)^2}\, \mathrm{d}\bar{x} = \int_{x^0-x}^{x^1-x} e^{-\frac{1}{2}\bar{x}^2}\, \mathrm{d}\bar{x}.$$

Since $x \in [x^0, x^1]$, i.e., $x^0 - x \leq 0$ and $x^1 - x \geq 0$. The figure of the value of g_1 at an arbitrary point x is shown below.

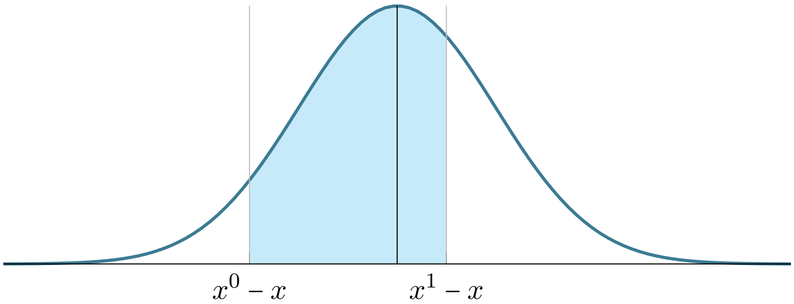

Figure 4.7.: The area of the shadow is the corresponding value of g_1.

Therefore, the area of shadow will be maximal if and only if $|x^0 - x| = |x^1 - x|$, i.e., $x - x^0 = x^1 - x$, then $x = \frac{x^0+x^1}{2}$. The figure of g_1's value at the point $x = \frac{x^0+x^1}{2}$ is shown below.

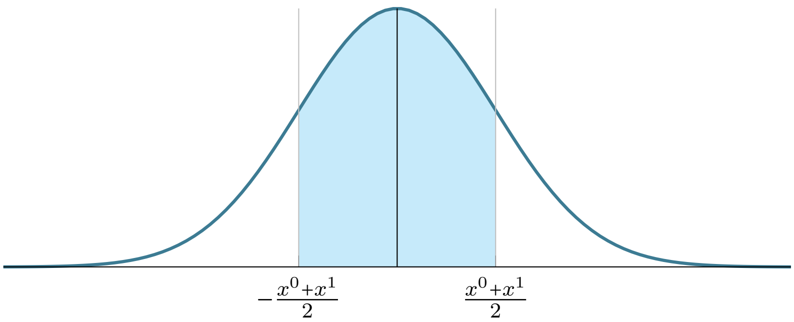

Figure 4.8.: The value of $g_1\left(\frac{x^0+x^1}{2}\right)$.

Analogous for g_2, it will achieve its maximum when $y = \frac{y^0+y^1}{2}$. In other words, g achieves its maximum at the central-point. ■

`Conclusion 4.12.` Consider the pattern $P = \left[x^0, x^1\right] \times \left[y^0, y^1\right]$ with the central-point $p = \left(\frac{x^0+x^1}{2}, \frac{y^0+y^1}{2}\right)$. Concentrate on some special points in P, whose locations are shown in the following figure.

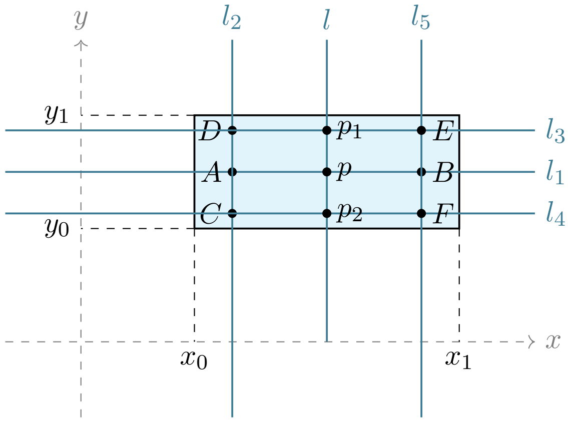

Figure 4.9.: Some special points in P.

We assume that

$$\begin{aligned} \operatorname{dist}(A,p) = \operatorname{dist}(B,p) = \operatorname{dist}(D,p_1) = \operatorname{dist}(E,p_1) = \operatorname{dist}(C,p_2) = \operatorname{dist}(F,p_2) =: \varepsilon_x; \\ \operatorname{dist}(C,A) = \operatorname{dist}(D,A) = \operatorname{dist}(E,B) = \operatorname{dist}(F,B). \end{aligned}$$

(1) The drying-time of A is the same as of B and of C is the same as of D. Indeed, the drying-time of C, D, E and F are the same.

Proof. By *Proof (Method 2)*, all the points in l_1 have the same value of g_2, since they have the same y-coordinate. Therefore, the difference between the drying-time of A and of B depends on their values of g_1. The figures below show the value of g_1 at point A and at point B.

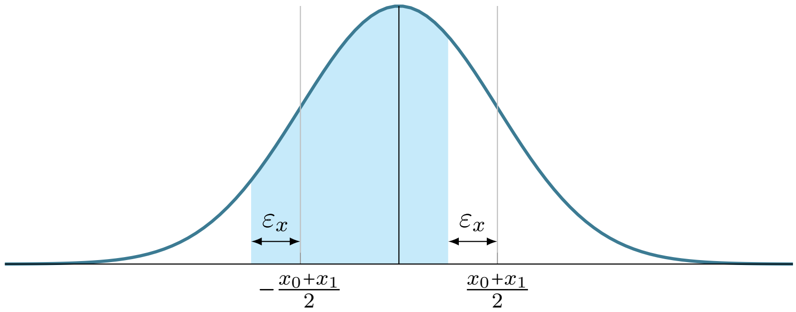

Figure 4.10.: The value of g_1 at point A is the area of the shadow.

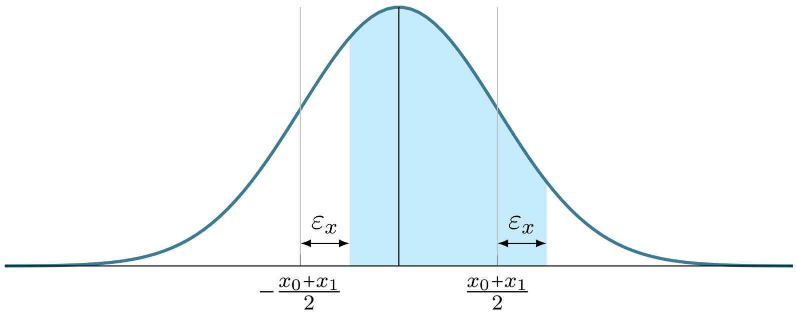

Figure 4.11.: The value of g_1 at point B is the area of the shadow.

According to the characteristic of the standard normal distribution and the figures, the area of the shadow in Figure 4.10 is the same as in Figure 4.11. The claim follows. Analogously, we obtain the rest statements. ■

(2) Those polygons, whose central-points have the same x-coordinate $\hat{x}$ (resp. y-coordinate $\hat{y}$), the maximum point will be on the line $x = \hat{x}$ (resp. $y = \hat{y}$).

Proof. Let $P_1, \ldots, P_n$ be n arbitrary patterns with $P_i = \left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$ for $i = 1, \ldots, n$. Without loss of generality, let the central-points of them have the same y-coordinate $\hat{y}$, i.e., for all $i, j = 1, \ldots, n$, $\hat{y} = \frac{y_i^0 + y_i^1}{2} = \frac{y_j^0 + y_j^1}{2}$. The figure below shows an example of the locations of the polygons in such situation above.

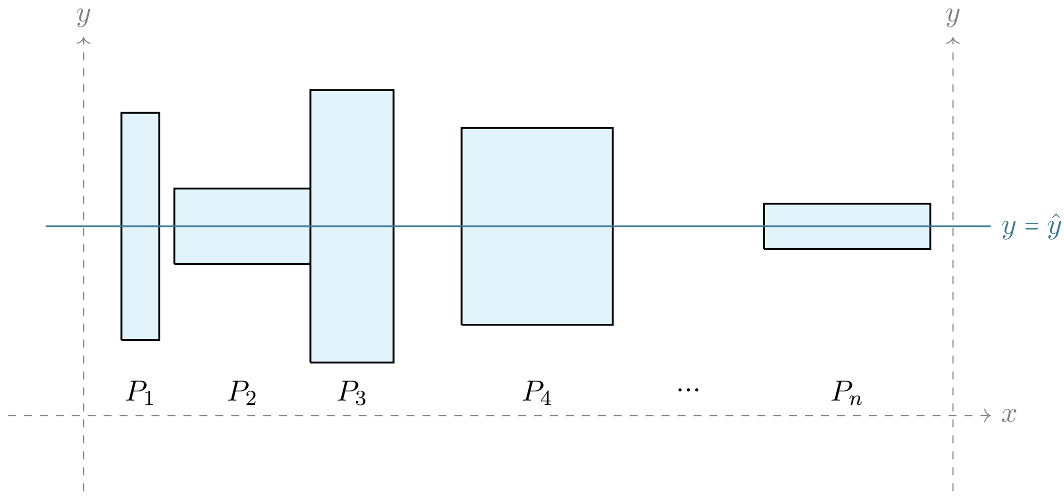

Figure 4.12.: An example of the distributions of the patterns, whose central-points have the same y-coordinate.

Define for $i = 1, \dots, n$,

$$g_{P_i}(x,y) = \frac{1}{\sqrt{2\pi}} \int_{x_i^0}^{x_i^1} \int_{y_i^0}^{y_i^1} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \, \mathrm{d}\bar{y}\mathrm{d}\bar{x} = \frac{1}{\sqrt{2\pi}} \int_{x_i^0}^{x_i^1} e^{-\frac{1}{2}(\bar{x}-x)^2} \, \mathrm{d}\bar{x} \int_{y_i^0}^{y_i^1} e^{-\frac{1}{2}(\bar{y}-y)^2} \, \mathrm{d}\bar{y}$$

and

$$g_{P_i,1}(x) = \int_{x_i^0}^{x_i^1} e^{-\frac{1}{2}(\bar{x}-x)^2} \, \mathrm{d}\bar{x}, \qquad g_{P_i,2}(y) = \int_{y_i^0}^{y_i^1} e^{-\frac{1}{2}(\bar{y}-y)^2} \, \mathrm{d}\bar{y}.$$

By 4.1 Mathematical Model, the drying-time for this printing-group can be described as

$$\max_{i=1,\dots,n} t_{P_i} = \max_{(x,y)\in \bigsqcup_{i=1}^{n} P_i} \sum_{i=1}^{n} g_{P_i}(x,y) = \frac{1}{\sqrt{2\pi}} \max_{(x,y)\in \bigsqcup_{i=1}^{n} P_i} \sum_{i=1}^{n} g_{P_i,1}(x) g_{P_i,2}(y).$$

`Claim 4.11` shows that for $i = 1, \dots, n$, $g_{P_i,2}(y)$ achieves its maximum value at $\hat{y} = \frac{y_i^0+y_i^1}{2}$. Then the prove is finished. In particular, all the patterns have the same y-coordinate of their central points. The value of $g_{P_i,2}$ at $\hat{y}$ is denoted as a constant $c_{i,y} \in \mathbb{R}$. Therefore,

$$\max_{i=1,\dots,n} t_{P_i} = \frac{1}{\sqrt{2\pi}} \max_{x\in \bigsqcup_{i=1}^{n} [x_i^0, x_i^1]} \sum_{i=1}^{n} c_{i,y} g_{P_i,1}(x). \qquad \blacksquare$$

4.2.3. Introduction to Newton's Method

Newton's method is an iterative method designed to provide a sequence $(x_n)_{n\in\mathbb{N}_0}$ that converges to a zero of a given function h. With the concepts in 4.1 Mathematical Model and 4.2.1 Mathematical Background, Newton's method is used to find all the critical points in order to obtain the global maximum, i.e., the drying-time. The Newton's iteration is based on [1, Chapter 6, Section 6.3, Page 127-128].

If $U \subseteq \mathbb{R}$ and $h : U \to \mathbb{R}$ is differentiable, then Newton's method is defined by the recursion

$$x_0 \in U, \qquad x_{n+1} := x_n - \frac{h(x_n)}{h'(x_n)}, \qquad \text{for each } n \in \mathbb{N}_0.$$

Analogously, Newton's method can also be defined for differentiable $h : U \to \mathbb{R}^d$ with $U \subseteq \mathbb{R}^d$,

$$x_0 \in U, \qquad x_{n+1} := x_n - (Dh(x_n))^{-1} h(x_n), \qquad \text{for each } n \in \mathbb{N}_0.$$

In practice, in each step of Newton's method, one will determine x_{n+1} as the solution to the linear system

$$Dh(x_n)\, x_{n+1} = Dh(x_n)\, x_n - h(x_n).$$

Notation. $Dh(x)$ is the *Jacobian-matrix*

$$Dh(x) := [\partial_j h_i(x)]_{ij} = \begin{pmatrix} \partial_1 h_1(x) & \partial_2 h_1(x) & \cdots & \partial_n h_1(x) \\ \partial_1 h_2(x) & \partial_2 h_2(x) & \cdots & \partial_n h_2(x) \\ \vdots & & \ddots & \vdots \\ \partial_1 h_m(x) & \partial_2 h_m(x) & \cdots & \partial_n h_m(x) \end{pmatrix}. \qquad \diamond$$

Here for $d = 2$ and $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ with $h(x,y) = \begin{pmatrix} h_1(x,y) \\ h_2(x,y) \end{pmatrix}$, the Jacobian-matrix is $Dh(x,y) = \begin{pmatrix} \partial_x h_1(x,y) & \partial_y h_1(x,y) \\ \partial_x h_2(x,y) & \partial_y h_2(x,y) \end{pmatrix}$. Therefore, Newton's method for two-dimensions is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \partial_x h_1(x_n, y_n) & \partial_y h_1(x_n, y_n) \\ \partial_x h_2(x_n, y_n) & \partial_y h_2(x_n, y_n) \end{pmatrix}^{-1} \begin{pmatrix} h_1(x_n, y_n) \\ h_2(x_n, y_n) \end{pmatrix}.$$

We determine x_{n+1}, y_{n+1} as the solution to the linear system

$$\begin{pmatrix} \partial_x h_1(x_n, y_n) & \partial_y h_1(x_n, y_n) \\ \partial_x h_2(x_n, y_n) & \partial_y h_2(x_n, y_n) \end{pmatrix} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \partial_x h_1(x_n, y_n) & \partial_y h_1(x_n, y_n) \\ \partial_x h_2(x_n, y_n) & \partial_y h_2(x_n, y_n) \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} h_1(x_n, y_n) \\ h_2(x_n, y_n) \end{pmatrix},$$

i.e., the linear system is

$$\begin{cases} \partial_x h_1(x_n, y_n) x_{n+1} + \partial_y h_1(x_n, y_n) y_{n+1} = \partial_x h_1(x_n, y_n) x_n + \partial_y h_1(x_n, y_n) y_n - h_1(x_n, y_n) \\ \partial_x h_2(x_n, y_n) x_{n+1} + \partial_y h_2(x_n, y_n) y_{n+1} = \partial_x h_2(x_n, y_n) x_n + \partial_y h_2(x_n, y_n) y_n - h_2(x_n, y_n) \end{cases}.$$

With Newton's method we can determine the drying-time of the printing-group consisting of n patterns. In 4.1.5 Drying-Time of n Rectangles, we have concluded that the drying-time of P_i including the drying-time increase by other rectangles is

$$t_{P_i} = \frac{1}{\sqrt{2\pi}} \max_{x_i \in [x_i^0, x_i^1], y_i \in [y_i^0, y_i^1]} \sum_{j=1}^{n} \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} e^{-\frac{1}{2}[(x-x_i)^2 + (y-y_i)^2]} \, \mathrm{d}y \mathrm{d}x$$

for $i = 1, \ldots, n$. In oder to obtain the maximum of the sum for $x_i \in [x_i^0, x_i^1], y_i \in [y_i^0, y_i^1]$ we define

$$g^i(x_i, y_i) := \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} e^{-\frac{1}{2}[(x-x_i)^2 + (y-y_i)^2]} \, \mathrm{d}y \mathrm{d}x.$$

Then the partial derivatives of g are

$$\partial_{x_i} g^i(x_i, y_i) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} \left(e^{-\frac{1}{2}\left(x_j^0 - x_i\right)^2} - e^{-\frac{1}{2}\left(x_j^1 - x_i\right)^2} \right) \int_{y_j^0}^{y_j^1} e^{-\frac{1}{2}(y-y_i)^2} \, \mathrm{d}y$$

and

$$\partial_{y_i} g^i(x_i, y_i) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} \int_{x_j^0}^{x_j^1} e^{-\frac{1}{2}(x-x_i)^2} \, \mathrm{d}x \left(e^{-\frac{1}{2}\left(y_j^0 - y_i\right)^2} - e^{-\frac{1}{2}\left(y_j^1 - y_i\right)^2} \right).$$

As for each pattern, we apply Newton's method on the objective function $h^i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, \ldots, n$, $h^i(x_i, y_i) = \begin{pmatrix} h_1^i(x_i, y_i) \\ h_2^i(x_i, y_i) \end{pmatrix} = \begin{pmatrix} \partial_{x_i} g^i(x_i, y_i) \\ \partial_{y_i} g^i(x_i, y_i) \end{pmatrix}$ and let the corresponding central-point $\left(\frac{x_i^0 + x_i^1}{2}, \frac{y_i^0 + y_i^1}{2} \right)$ be the initial guess. Finally, there are n critical points, $p_1 = (\hat{x}_1, \hat{y}_1), \ldots, p_n =$

$(\hat{x}_n, \hat{y}_n)$. Using `Theorem 4.8`, the critical points can be evaluated. However the evaluation is redundant in our case. Define for $i = 1, \ldots, n$, $\hat{t}_i := g^i(\hat{x}_i, \hat{y}_i)$. By doing the comparison of all $\hat{t}_i$ for $i = 1, \ldots, n$ the global maximum will be obtained directly.

4.2.4. The Selection of the Initial-Guess

By Claim 4.11, when there is only one piece of polygon, the maximum will be at the central-point. If there are other patterns, the maximum point of each pattern will shift from the central-point. Let x_0 be the initial-guess and h be the objective function. The method will usually converge, provided this initial-guess is close enough to the unknown zero, and the fact $h'(x_0) \neq 0$. If the method diverges, which means that the shift is too far from the corresponding central-point. This indicates the existence of other patterns, whose drying-time is much greater than the drying-time of this pattern. If there exists a critical point, which is not located in any patterns, it is a meaningless point, although its drying-time may be greater than the drying-time of any other points.

According to the information above, the central-points of the underlying patterns in each printing-group will be chosen as initial-guesses for Newton's method.

The approach to process the valid results of Newton's method with the central-points as initial-guesses will be expressed later in 5.2.1 Computing the Drying-Time with Newton's Method. Moreover, the conditions for the convergence of Newton's method will be introduced in Part II. Newton's Method.

5. Implementation

Mixed Integer Linear Programming (MILP) is used to solve the optimization-problem. C++ and Gurobi[6] are required as the solving tools. In 5.1 Discrete Model, finite sampling points will be distributed uniformly in each pattern and the drying-time of them will be computed. On the other hand, infinite sampling points will be considered in 5.2 Continuous Model. Meanwhile, some typical examples for the calculation of drying-time with Newton's method will be introduced.

5.1. Discrete Model

In order to obtain the drying-time, finite sampling points are distributed uniformly in each pattern, which can be seen in the following figure.

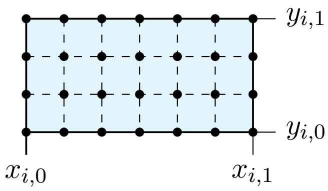

Figure 5.1.: The distribution of points in the i-th pattern.

The entire design is decomposed into $\#\text{tile}_x \times \#\text{tile}_y$ rectangular tiles. For example, in Figure 5.1, $\#\text{tile}_x = 6$, $\#\text{tile}_y = 3$. Define $n_x := \#\text{tile}_x$, $n_y := \#\text{tile}_y$. Then, the coordinate of first point is $(x_{i,0}, y_{i,0})$ and then $(x_{i,0} + n_x, y_{i,0})$, $(x_{i,0} + 2n_x, y_{i,0}), \ldots$ Finally, the coordinate of the last point is $(x_{i,1}, y_{i,1})$. Now assume $n_x = n_y = 10$ for $i = 1, \ldots, N_P$. Let n_i be the number of points in each pattern, i.e., $n_i = 121$ for all $i \in \{1, \ldots, N_P\}$. Let x_{i_r} and y_{i_r}, $r = 1, \ldots, n_i$ indicate the position of these points in P_i.

Define binary variable $q_{i,m} = 1$ when P_i is assigned to the m-th printing-group. Then for $i = 1, \ldots, N_P$ and $m = 1, \ldots, N_G$,

$$\sum_{m=1}^{N_G} q_{i,m} = 1, \tag{4}$$

i.e., each pattern is exactly assigned to one printing-group.

Define binary variable $q_{(i,j),m} = 1$ when both P_i, P_j are assigned to the m-th printing-group. We obtain for $i, j = 1, \dots, N_P$ and $m = 1, \dots, N_G$,

$$q_{(i,j),m} = q_{i,m} \cdot q_{j,m}. \tag{5.1.1}$$

To linearize the constraint above, it can be transformed into the following in-equation,

$$q_{i,m} + q_{j,m} \leq q_{(i,j),m} + 1 \tag{5}$$

By the minimization of the sum of drying-time and the number of printing-group, (5.1.1) can be realized.

5.1.1. Drying-time of the Points in a Printing-Group

Consider one fixed pattern P_i, i.e., i fixed with $i \in \{1, \dots, N_P\}$ and fixed m, i.e., fixed printing-group with $m \in \{1, \dots, N_G\}$. Let $t_{i,m,k},\ k = 1, \dots, n_i$ denote the drying-time of the point $p_{i_r} = (x_{i_r}, y_{i_r})$ in the m-th printing-group. Let $[x_j^0, x_j^1] \times [y_j^0, y_j^1]$ denote the area of the j-th pattern. Then, for $i = 1, \dots, N_P$ and $m = 1, \dots, N_G$, $k = 1, \dots, n_i$,

$$t_{i,m,k} = \sum_{j=1}^{N_P} \left(\int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} f\left(\sqrt{(x - x_{i_k})^2 + (y - y_{i_k})^2}\right) \mathrm{d}y \mathrm{d}x \cdot q_{(i,j),m} \right).$$

By the definition of our Gaussian function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$,

$$t_{i,m,k} = \sum_{j=1}^{N_p} \left(\int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x - x_{i_k})^2 + (y - y_{i_k})^2\right]} \mathrm{d}y \mathrm{d}x \cdot q_{(i,j),m} \right).$$

Define the coefficient for $i, j = 1, \dots, N_P$ and $k = 1, \dots, n_i$, $c_k^{ij} = \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[(x - x_{i_k})^2 + (y - y_{i_k})^2\right]} \mathrm{d}y \mathrm{d}x$, which will be computed with the GNU Scientific Library [7]. Thus, for $k = 1, \dots, n_i$,

$$t_{i,m,k} = \sum_{j=1}^{N_P} c_k^{ij} \cdot q_{(i,j),m} \tag{6}$$

In particular, c_k^{ij} is not the same as c_k^{ji}. c_k^{ij} is about the integral over the points in the i-th pattern and c_k^{ji} in the j-th pattern.

5.1.2. Drying-Time of the Patterns in a Printing-Group

Let $t_{i,m,\max P_i}$ denote the drying-time of P_i in the m-th printing-group, then for $i = 1, \ldots, N_P$, $m = 1, \ldots, N_G$ and $k = 1, \ldots, n_i$,

$$t_{i,m,\max P_i} \geq t_{i,m,k} \tag{7}$$

5.1.3. Drying-Time of a Printing-Group

Let $t_{m,\max m}$ denote the drying-time of the m-th printing-group. Then, for $i = 1, \ldots, N_P$ and $m = 1, \ldots, N_G$,

$$t_{m,\max m} \geq t_{i,m,\max P_i} \tag{8}$$

Finally, the complete optimization problem can be modelled as,

$$\text{Minimize} \sum_{m=1}^{N_G} t_{m,\max m} + N_G \qquad \text{Subject to } \textit{(1)-(8)}$$

5.2. Continuous Model

In this section, we introduce the model using Newton's method to compute the drying-time.

5.2.1. Computing the Drying-Time with Newton's Method

Although with Newton's method the unknown zero of the function and then the maximal value can be computed, it requires the composition of the printing-group. I.e., we must know which patterns are contained in each printing-group. For this reason we introduce a new concept—*combination*.

If there are N_P patterns, $P_1, \ldots, P_{N_P}$, then there exists 2^{N_P} combinations. Let the l-th combination C_l contain n_l patterns for $l = 1, \ldots, 2^{N_P}$, then C_l has the following form,

$$C_l = \left\{P_1^l, \ldots, P_{n_l}^l\right\},$$

where $P_1^l, \ldots, P_{n_l}^l \in \{P_1, \ldots, P_{N_P}\}$. Let N_C be the number of combinations. For $i = 1, \ldots, n_l$, the area of the i-th pattern is $P_i^l = \left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$. As in 4.2.3 Introduction to Newton's

Method we define for the point $(x_i, y_i) \in P_i^l$,

$$g_i^l(x_i, y_i) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n_l} \int_{x_j^0}^{x_j^1} \int_{y_j^0}^{y_j^1} e^{-\frac{1}{2}\left[(x-x_i)^2+(y-y_i)^2\right]} \, \mathrm{d}y\mathrm{d}x.$$

`Algorithm 5.1.` For each pattern we apply Newton's method. Suppose that the root we have found for the i-th pattern in one printing-group is the point $p_i = (\hat{x}_i, \hat{y}_i)$.

(1) If $(\hat{x}_i, \hat{y}_i) \in \left[x_i^0, x_i^1\right] \times \left[y_i^0, y_i^1\right]$, then we take the value of $g_i^l(\hat{x}_i, \hat{y}_i)$ as the drying-time of P_i^l.

(2) If $\hat{x}_i < x_i^0$ and $\hat{y}_i \in \left[y_i^0, y_i^1\right]$, then we take the value of $g_i^l\left(x_i^0, \hat{y}_i\right)$ as the drying-time of P_i^l.

(3) If $\hat{x}_i > x_i^1$ and $\hat{y}_i \in \left[y_i^0, y_i^1\right]$, then we take the value of $g_i^l\left(x_i^1, \hat{y}_i\right)$ as the drying-time of P_i^l.

(4) If $\hat{x}_i \in \left[x_i^0, x_i^1\right]$ and $\hat{y}_i < y_i^0$, then we take the value of $g_i^l\left(\hat{x}_i, y_i^0\right)$ as the drying-time of P_i^l.

(5) If $\hat{x}_i \in \left[x_i^0, x_i^1\right]$ and $\hat{y}_i > y_i^1$, then we take the value of $g_i^l\left(\hat{x}_i, y_i^1\right)$ as the drying-time of P_i^l.

(6) If $\hat{x}_i < x_i^0$ and $\hat{y}_i < y_i^0$, then we take the value of $g_i^l\left(x_i^0, y_i^0\right)$ as the drying-time of P_i^l.

(7) If $\hat{x}_i < x_i^0$ and $\hat{y}_i > y_i^1$, then we take the value of $g_i^l\left(x_i^0, y_i^1\right)$ as the drying-time of P_i^l.

(8) If $\hat{x}_i > x_i^1$ and $\hat{y}_i < y_i^0$, then we take the value of $g_i^l\left(x_i^1, y_i^0\right)$ as the drying-time of P_i^l.

(9) If $\hat{x}_i > x_i^1$ and $\hat{y}_i > y_i^1$, then we take the value of $g_i^l\left(x_i^1, y_i^1\right)$ as the drying-time of P_i^l. ◇

This drying-time may not be the real drying-time since we do not use `Theorem 4.8` to verify whether the critical point is a local extremum. However it will not influence the result for this particular combination.

5.2.2. Examples for the Special Printing-Group

`Conclusion 5.2.` If there are n patterns, $P_1, \ldots, P_n$.

(1) Consider the following two extreme situations:

(i) Let the distance between each pair of $P_1, \ldots, P_n$ be great enough such that there exists no influence between each pair of patterns. I.e., the situation can be treated as: $P_1, \ldots, P_n$ are separately assigned to n printing-group. Then by `Claim 4.11` there exist n critical points, which are the central-points of $P_1, \ldots, P_n$. In particular, they are local maxima.

(ii) Let patterns be located directly to each other, which can be seen in the following figure.

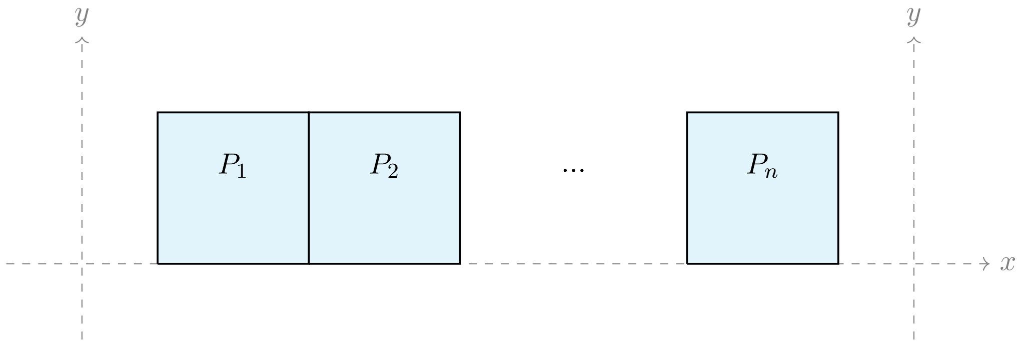

Figure 5.2.: An example of the printing-group containing n patterns, which are located directly to each other.

These n patterns can be incorporated to one pattern. By `Claim 4.11`, the drying-time of this printing-group is the drying-time of the central-point of the merged pattern.

We conclude from the extreme situations that if there are n patterns then there exists maximal n local maxima.

(2) If the figure of the printing-group is completely symmetric and the distance between patterns is suitable, i.e., they locate not too faraway from each other, then the geometric center of the figure is the unique local maximum, i.e., the global maximum. $\diamond$

We close this subsection with a discussion about some special locations of patterns in one printing-group. With the examples below, `Conclusion 5.2` will be confirmed. Before beginning, define the area of each pattern as before, $P = \left[x^0, x^1\right] \times \left[y^0, y^1\right]$,

$$g_P\left(x, y\right) = \int_{x^0}^{x^1} \int_{y^0}^{y^1} e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \,\mathrm{d}\bar{y}\mathrm{d}\bar{x}, \qquad \text{for all } \left(x, y\right) \in \left[x^0, x^1\right] \times \left[y^0, y^1\right].$$

Example 5.3. *Consider two patterns P_1^1 and P_2^1 with $P_1^1 = [0,1]\times[0,1]$, $P_2^1 = [2,3]\times[0,1]$. The locations of P_1^1 and P_2^1 are described in the following figure, let G_1 denote the printing-group, which consists of these two patterns.*

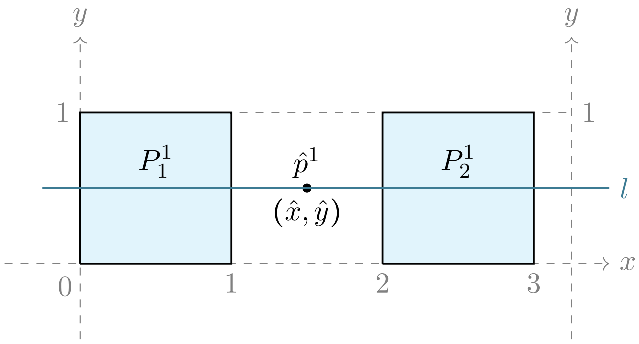

Figure 5.3.: The printing-group $G_1 = \{P_1^1, P_2^1\}$. l is the line $y = \frac{0+1}{2} = .5$.

By 4.1.4 Drying-Time of Two Rectangles, the drying-time of G_1 can be described as

$$\begin{aligned} t_{G_1} &= \max_{(x,y)\in P_1^1 \sqcup P_2^1} g_{P_1^1}(x,y) + g_{P_2^1}(x,y) \\ &= \frac{1}{\sqrt{2\pi}} \max_{x\in[0,1]\sqcup[2,3], y\in[0,1]} \left\{ \int_0^1 \int_0^1 e^{-\frac{1}{2}[(\bar{x}-x)^2+(\bar{y}-y)^2]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \int_0^1 \int_2^3 e^{-\frac{1}{2}[(\bar{x}-x)^2+(\bar{y}-y)^2]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \right\}. \end{aligned}$$

Take the central-points $(.5, .5)$ and $(2.5, .5)$ as the initial guesses for Newton's method and the results are the same. I.e., there exists only one critical point $\hat{p}^1 = (\hat{x}, \hat{y}) = (1.5, .5)$. Thus, this point is the unique local maximum point and then the global maximum point. However, this point belongs to none of these two patterns. ***Conclusion 4.12 (2)*** *shows that the maximum point will be on the line l. Therefore, the corresponding values of the points on the left and right side of $\hat{p}$ are smaller, i.e., for all $x < \hat{x}$ or $x > \hat{x}$, we have*

$$g_{P_1^1}(x, \hat{y}) + g_{P_2^1}(x, \hat{y}) < g_{P_1^1}(\hat{x}, \hat{y}) + g_{P_2^1}(\hat{x}, \hat{y}).$$

In particular, the closer to the point $\hat{p}$ the greater the value. With such consideration, we use ***Algorithm 5.1*** *to find the drying-time of P_1^1 and P_2^1.*

(P_1^1) *The original result of Newton's method conforms to* ***Algorithm 5.1 (3)***. *Thus, the maximum point of P_1^1 is $\hat{p}_1^1 = (1, .5)$ and the drying-time of P_1^1 is $g_{P_1^1}(\hat{p}_1^1) + g_{P_2^1}(\hat{p}_1^1) =$* .4580884948.

(P_2^1) *The original result of Newton's method conforms to* ***Algorithm 5.1 (2)***. *Thus, the maximum point of P_2^1 is $\hat{p}_2^1 = (2, .5)$ and the drying-time of P_2^1 is $g_{P_1^1}(\hat{p}_2^1) + g_{P_2^1}(\hat{p}_2^1) =$* .4580884948.

It can be seen that the drying-time of these two patterns are the same, since their shapes are exactly the same and they are located symmetrically to each other. ***Conclusion 5.2 (2)*** *is also* *be verified through this example. Finally, the drying-time of this printing-group is*

$$\max\{.4580884948, .4580884948\} = .4580884948.$$

In the next example, consider two identical patterns. However, the distance between them is a little greater than Example 5.3. We will see the influence on the critical point shift, which caused by the change of the distance between patterns.

Example 5.4. Consider two patterns P_1^2 and P_2^2 with $P_1^2 = [0,1] \times [0,1]$, $P_2^2 = [1,2] \times [2,3]$. Let G_2 denote the printing-group containing these two patterns and the distribution of them is shown in the figure below. In particular, line l is generated from the central-points of P_1^2 and P_2^2. We also draw the results of Newton's method, denoted by $\hat{p}_1^2$ and $\hat{p}_2^2$ respectively.

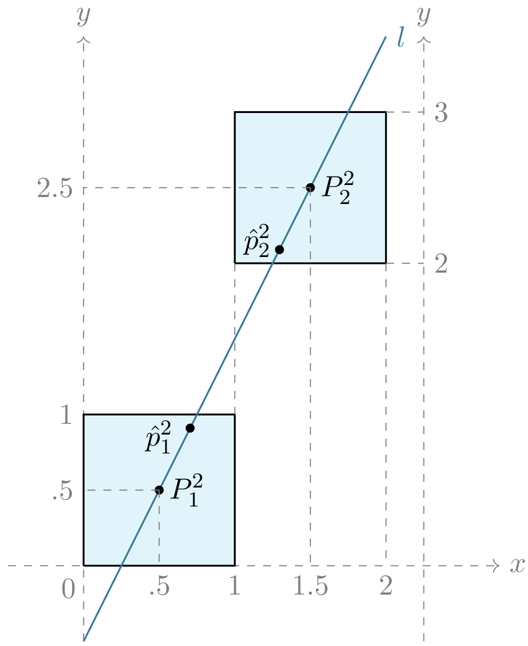

Figure 5.4.: The printing-group $G_2 = \{P_1^2, P_2^2\}$.

By 4.1.4 Drying-Time of Two Rectangles, the drying-time of G_2 can be described as

$$\begin{aligned} t_{G_2} &= \max_{(x,y)\in P_1^2 \sqcup P_2^2} g_{P_1^2}(x,y) + g_{P_2^2}(x,y) \\ &= \frac{1}{\sqrt{2\pi}} \max_{x\in[0,2], y\in[0,1]\sqcup[2,3]} \left\{ \int_0^1 \int_0^1 e^{-\frac{1}{2}[(\bar{x}-x)^2+(\bar{y}-y)^2]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \int_2^3 \int_1^2 e^{-\frac{1}{2}[(\bar{x}-x)^2+(\bar{y}-y)^2]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \right\}. \end{aligned}$$

Take the central-points $(.5, .5)$ and $(1.5, 2.5)$ as the initial guesses for Newton's method and *the results are different. The result of* P_1^2 *is* $\hat{p}_1^2 = (.7045386024, .9098098415)$ *and the result of* P_2^2 *is* $\hat{p}_2^2 = (1.2954613976, 2.0901901585)$. *Thus, these points are the local maxima and there must be a local minimum between these two maxima. In particular, it can be verified by* `Theorem 4.8`*. According to* `Algorithm 5.1 (1)`*, the drying-time of* P_1^2 *is* $g_{P_1^2}(\hat{p}_1^2) + g_{P_2^2}(\hat{p}_1^2) = .4195087470$ *and of* P_2^2 *is* $g_{P_1^2}\left(\hat{p}_2^2\right) + g_{P_2^2}\left(\hat{p}_2^2\right) = .4195087470$. *Finally, the drying-time of* G_2 *is* $\max\left\{.4195087470, .4195087470\right\} = .4195087470$. *Furthermore, the local maxima located on the line* l.

Compare Example 5.3 *and* Example 5.4. *The patterns in the both printing-group are exactly the same. However, the distance between* P_1^2 *and* P_2^2 *in* G_2 *is greater than* P_1^1 *and* P_2^1 *in* G_1. *The result refers to* `Conclusion 5.2 (1)`.

`Example 5.5.` *Consider four patterns* P_1^3, P_2^3, P_3^3 *and* P_4^3 *with* $P_1^3 = [0,1] \times [0,1]$, $P_2^3 = [2,3] \times [0,1]$, $P_3^3 = [0,1] \times [2,3]$, $P_4^3 = [2,3] \times [2,3]$. *Let them be totally symmetrically distributed, i.e., the figure of this printing-group is completely symmetric, which can be seen below. Such a printing-group is denoted by* G_3.

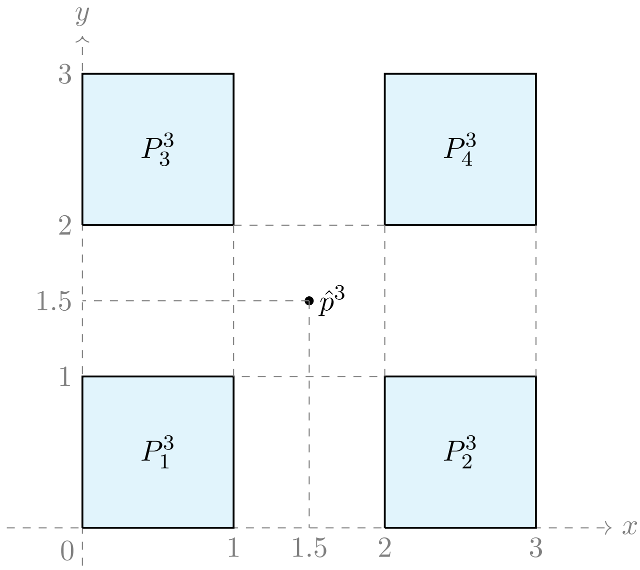

Figure 5.5.: The printing-group $G_3 = \left\{P_1^3, P_2^3, P_3^3, P_4^3\right\}$.

By 4.1.5 Drying-Time of n Rectangles, the drying-time of G_3 can be describe as

$$t_{G_3} = \max_{(x,y)\in P_1^3\sqcup P_2^3\sqcup P_3^3\sqcup P_4^3} g_{P_1^3}(x,y) + g_{P_2^3}(x,y) + g_{P_3^3}(x,y) + g_{P_4^3}(x,y)$$

$$= \frac{1}{\sqrt{2\pi}} \max_{(x,y)\in[0,1]\times[0,1]\sqcup[2,3]\times[0,1]\sqcup[0,1]\times[2,3]\sqcup[2,3]\times[2,3]} \left\{ \int_0^1 \int_0^1 e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \right.$$

$$\left. + \int_0^1 \int_2^3 e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \int_2^3 \int_0^1 e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \int_2^3 \int_2^3 e^{-\frac{1}{2}\left[(\bar{x}-x)^2+(\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \right\}.$$

Take the central-points $(.5,.5)$, $(2.5,.5)$,$(.5,2.5)$ and $(2.5,2.5)$ as the initial guesses for Newton's method and the results are the same. I.e., there exists only one critical point $\hat{p}^3 = (\hat{x},\hat{y}) = (1.5,1.5)$. Therefore, this point is the unique local maximum point and then the global maximum point. However, this point belongs to none of these three patterns. Analogously, use the same method as in Example 5.3.

(P_1^3) *The original result of Newton's method conforms to* `Algorithm 5.1 (9)`. *Thus, the maximum point of P_1^3 is $\hat{p}_1^3 = (1,1)$ and the drying-time of P_1^3 is $g_{P_1^3}\left(\hat{p}_1^3\right) + g_{P_2^3}\left(\hat{p}_1^3\right) + g_{P_3^3}\left(\hat{p}_1^3\right) + g_{P_4^3}\left(\hat{p}_1^3\right) = .5709282965$.*

(P_2^3) *The original result of Newton's method conforms to* `Algorithm 5.1 (7)`. *Thus, the maximum point of P_2^3 is $\hat{p}_2^3 = (2,1)$ and the drying-time of P_2^3 is $g_{P_1^3}\left(\hat{p}_2^3\right) + g_{P_2^3}\left(\hat{p}_2^3\right) + g_{P_3^3}\left(\hat{p}_2^3\right) + g_{P_4^3}\left(\hat{p}_2^3\right) = .5709282965$.*

(P_3^3) *The original result of Newton's method conforms to* `Algorithm 5.1 (8)`. *Thus, the maximum point of P_3^3 is $\hat{p}_3^3 = (1,2)$ and the drying-time of P_3^3 is $g_{P_1^3}\left(\hat{p}_3^3\right) + g_{P_2^3}\left(\hat{p}_3^3\right) + g_{P_3^3}\left(\hat{p}_3^3\right) + g_{P_4^3}\left(\hat{p}_3^3\right) = .5709282965$.*

(P_4^3) *The original result of Newton's method conforms to* `Algorithm 5.1 (6)`. *Thus, the maximum point of P_4^3 is $\hat{p}_4^3 = (2,2)$ and the drying-time of P_4^3 is $g_{P_1^3}\left(\hat{p}_4^3\right) + g_{P_2^3}\left(\hat{p}_4^3\right) + g_{P_3^3}\left(\hat{p}_4^3\right) + g_{P_4^3}\left(\hat{p}_4^3\right) = .5709282965$. Finally, the drying-time of G_3 is*

$$\max\left\{.5709282965, .5709282965, .5709282965, .5709282965\right\} = .5709282965.$$

The example shows `Conclusion 5.2 (2)`. *If let the distance between $P_1^3, P_2^3, P_3^3, P_4^3$ be a little greater than before but not too far away from each other, i.e., suitable. Then, the result will be like in Example 5.4. There will exists four critical-points with totally symmetric distribution. Furthermore, their drying-time will be exactly the same, i.e., it is the drying-time of such printing-group.*

Let a special printing-group from our test-case be the following example. The complete pattern is decomposed into three rectangles, i.e., it is not regular.

`Example 5.6.` *Consider three patterns* P_1^4, P_2^4 *and* P_3^4 *with* $P_1^4 = [10, 12.891] \times [848.473, 851.363]$, $P_2^4 = [12.891, 15.781] \times [849.238, 850.598]$, $P_3^4 = [15.781, 17.164] \times [847.945, 851.891]$. *The locations of them are drawn in the following figure and let* G_4 *denote this printing-group. In particular, line* l *is generated from the central-points of them, i.e., the equation of* l *is*

$$y = \frac{848.473 + 851.363}{2} = \frac{849.238 + 850.598}{2} = \frac{847.945 + 851.891}{2} = 849.918.$$

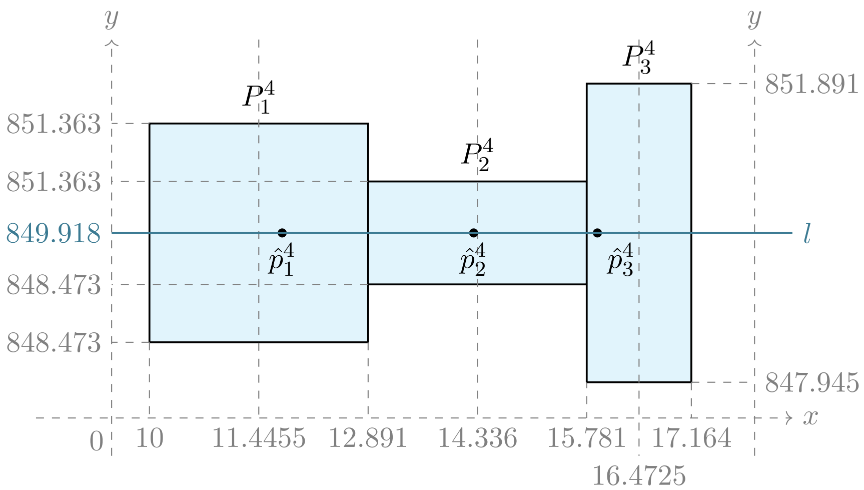

Figure 5.6.: The printing-group $G_4 = \{P_1^4, P_2^4, P_3^4\}$.

By 4.1.5 Drying-Time of n Rectangles*, the drying-time of* G_4 *can be described as*

$$\begin{aligned} t_{G_4} &= \max_{(x,y) \in P_1^4 \sqcup P_2^4 \sqcup P_3^4} g_{P_1^4}(x,y) + g_{P_2^4}(x,y) + g_{P_3^4}(x,y) \\ &= \frac{1}{\sqrt{2\pi}} \max_{\substack{x \in [10,12.891] \sqcup [12.891,15.781] \sqcup [15.781,17.164] \\ y \in [848.473,851.363] \sqcup \times [849.238,850.598] \sqcup [847.945,851.891]}} \left\{ \int_{848.473}^{851.363} \int_{10}^{12.891} e^{-\frac{1}{2}\left[(\bar{x}-x)^2 + (\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \right. \\ &+ \int_{849.238}^{850.598} \int_{12.891}^{15.781} e^{-\frac{1}{2}\left[(\bar{x}-x)^2 + (\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} + \left. \int_{847.945}^{851.891} \int_{15.781}^{17.164} e^{-\frac{1}{2}\left[(\bar{x}-x)^2 + (\bar{y}-y)^2\right]} \, \mathrm{d}\bar{x}\mathrm{d}\bar{y} \right\}. \end{aligned}$$

Take the central-points $(11.4455, 849.918)$, $(14.336, 849.918)$ *and* $(16.4725, 849.918)$ *as the initial guesses for Newton's method and the results are different. The result of* P_1^4 *is* $\hat{p}_1^4 = (11.7552380787, 849.918)$, *the result of* P_2^4 *is* $\hat{p}_2^4 = (14.2861194800, 849.918)$ *and the result of* P_3^4 *is* $\hat{p}_3^4 = (15.9221029508, 849.918)$. *In particular, the results also verify* `Conclusion 4.12 (2)`. $\hat{p}_1^4$ *and* $\hat{p}_3^4$ *are the local maxima,* $\hat{p}_2^4$

is the local minimum. This can be confirmed by `Theorem 4.8`*. The corresponding values are*

$$g_{P_1^4}(\hat{p}_1^4) + g_{P_2^4}(\hat{p}_1^4) + g_{P_3^4}(\hat{p}_1^4) = 1.9382906117,$$
$$g_{P_1^4}(\hat{p}_2^4) + g_{P_2^4}(\hat{p}_2^4) + g_{P_3^4}(\hat{p}_2^4) = 1.4041474740,$$
$$g_{P_1^4}(\hat{p}_3^4) + g_{P_2^4}(\hat{p}_3^4) + g_{P_3^4}(\hat{p}_3^4) = 1.6321114754.$$

Finally, the drying-time of this printing-group is

$$\max\left\{1.6321114754, 1.4041474740, 1.9382906117\right\} = 1.9382906117.$$

Compare `Example 5.5` *and* `Example 5.6`*. The distance between patterns in G_3 is greater than in G_4. However there is only one critical point in the printing-group G_3 while three in G_4. Therefore, the relative-size and -placement of the patterns will have an influence on the critical point shift. In particular,* `Conclusion 5.2 (1)` *has also confirmed.*

5.2.3. Implementation of the Continuous Model

Construct the combination with the consideration of the Laplace pressure conflict and the proximity conflict in the following way.

(1) If P_i and P_j have the Laplace pressure conflict then consider only the combinations without $\{P_i, P_j\}$.

(2) Pairs that have proximity conflict cannot be assigned to the same combination.

Use Newton's method to compute the drying-time for the l-th combination and denote the result by d_l. Let $q_{l,m}$ be binary variable. If the l-th combination is the m-th printing-group, then let $q_{l,m} = 1$. Let C_l denote the l-th combination. For example, the l-th combination consists of P_i and P_j, then $C_l = \{P_i, P_j\}$.

Define a matrix $A_l \in \{0,1\}^{N_p}$ for each combination to indicate, whether a pattern is assigned to the l-th combination.

$$A_l = \left(a_1^l, \ldots, a_{N_P}^l\right) \in \{0,1\}^{N_P}$$

is defined in the following way. For each $i = 1, \ldots, N_p$, if $P_i \in C_l$ then $a_i^l = 1$ and if $P_i \notin C_l$, then $a_i^l = 0$. Furthermore, we need to guarantee that each pattern is assigned to exactly one printing-group, i.e., for $i = 1, \dots, N_P$,

$$\sum_{l=1}^{N_C} \sum_{m=1}^{N_G} a_i^l \cdot q_{l,m} = 1. \tag{9}$$

With the same notation as before,

$$g_{P_i} = \sum_{l=1}^{N_C} \sum_{m=1}^{N_G} m \cdot a_i^l \cdot q_{l,m}. \tag{10}$$

Each combination appears at most once, then for $l = 1, \dots, N_C$,

$$\sum_{m=1}^{N_G} q_{l,m} \leq 1. \tag{11}$$

Finally, the complete optimization problem can be modelled as

$$\text{Minimize} \sum_{l=1}^{N_C} \sum_{m=1}^{N_G} d_l \cdot q_{l,m} + N_G \qquad \text{Subject to } (1),(2),(9),(10) \text{ and } (11).$$

5.3. Relation Between Two Models

N_P patterns have at most 2^{N_P} combinations. So, in practice, with the increasing quantity of patterns usually follows an enlargement of the scale of calculation. Concerning the Laplace pressure conflict and proximity conflict results in reduction of number of combinations. Still, the number of combinations is large, furthermore the existence of unnecessary combinations cannot be avoided. This is an obvious obstacle to the implementation with C++. In order to solve this problem, utilizing the previous two models one after another provides a way of further optimization.

Through the first model, we obtain the maximal number of patterns in one printing-group and denote it by n_P. Subsequently using the result from the discrete model and consider those combinations that contain less equal than n_P patterns. At the same time, let $\varepsilon_n \in \mathbb{N}_+$ be the number that not too large, such that the combinations with $n_P + \varepsilon_n$ patterns can also be concerned but without huge impact during the implementation with C++. Insert those combinations that meet this criteria. Finally, comparing the results implemented by the two models.

Part II.

Newton's Method

It can be seen that even for a polynomial of one complex variable we cannot decide if Newton's method will converge to a root of the polynomial on a given initial guess. We introduce quantities α, β and γ which play an important role in analyzing the complexity of algorithms that approximate the solutions of systems of equations in both chapters.

Our main results in the first chapter, `Theorem 6.3` and `Theorem 6.12`, give the speed of convergence to a root in terms of these quantities, while other results such as `Proposition 6.16` estimate them. In particular, `Theorem 6.12` gives us a criterion, computable at a point x, to confirm that x is "close" to an actual zero ζ of a system of equations. Furthermore, the proof of `Theorem 6.12` with the constant $\alpha_0 = .130707$ is given in 7 n-Dimensional Generalization.

In the second chapter, we deduce consequences from data at a single point. This point of view has valuable features for computation and its theory. The idea is simply to apply the theorems to a finite sequence of equations of the form $f(z) - t_i f(z_0) = 0,\ 0 \le t_i \le 1$, to solve $f(z) = 0$.

For purposes of exposition the one-variable case is treated first. Then it its noted how the results extend to systems of equations $f : \mathbb{C}^n \to \mathbb{C}^n$ and even maps of Banach spaces $f : \mathbb{E} \to \mathbb{F}$.

6. Newton's Method in One-Variable Case

This chapter, without indication, we follow [8, Chapter 8, Section 1-2].

6.1. Approximate Zeros

We begin this section by solving linear equations. Given a linear equation in one varaible $f(x) = ax + b$ with $a \neq 0$, we solve the equation $f(x) = 0$ by $x = -a^{-1}b$. For quadratic equations $f(z) = az^2 + bz + c$, $a \neq 0$, we solve for the two roots, $f(z) = 0$, by the quadratic formula $\zeta_+ = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $\zeta_- = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

Newton's method is an iterative method designed to approximate the roots of nonlinear equations. Given an initial approximation a to a root of the equation $f = 0$, Newton's method replaces a by the exact solution a' of the best linear approximation to f which is given by the tangent to the graph of f at the point $(a, f(a))$. The process of convergence of Newton's method will be shown in the following figure.

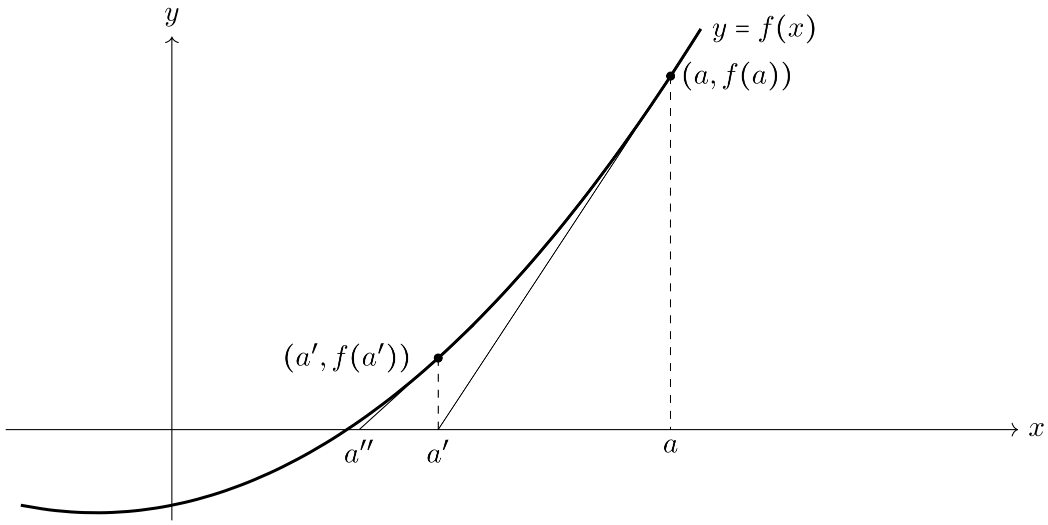

Figure 6.1.: Starting from a, two steps of Newton's method give a'', a close approximation to a zero of f.

Suppose that $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots = \sum_{i=0}^{\infty} a_i z^i$ is an analytic function of one complex (or real) variable defined on all of $\mathbb{C}$ (or $\mathbb{R}$). Thus, for example, f may be a polynomial, the sine or cosine functions, the exponential function, or sums, products, and composition of these, and so on. Our main application for the theory developed in this chapter is to polynomials.

Newton's method is an iteration based on the map from $\mathbb{C}$ to itself,

$$N_f(z) = z - \left(f'(z)\right)^{-1} f(z),$$

where $f'(z)$ is the derivative of f at z. This formula is defined as long as $\left(f'(z)\right)^{-1}$ exists. The formula for N_f is also written $N_f(z) = z - \left(\frac{f(z)}{f'(z)}\right)$. We say $\left(f'(z)\right)^{-1}$ exists in place of $f'(z) \neq 0$ since the theory we are presenting is valid in the much more general context of maps between n-dimensional or even Banach spaces. In this context the derivative $f'(z)$ is a continuous linear map that we assume has an inverse. We also write $N_f'(z)$ as we do since the formula is valid in n-dimensional or Banach spaces where linear maps do not necessarily commute.

We recall that if $f(\zeta) = 0$ and $f'(\zeta)^{-1}$ exists, then $N_f(\zeta) = \zeta - \left(f'(\zeta)\right)^{-1} f(\zeta) = \zeta$ and in that case $N_f'(\zeta) = f'(\zeta)^{-1} f''(\zeta) f'(\zeta)^{-1} f(\zeta) = 0$. The Taylor series of N_f near ζ is then

$$T_{N_f}(z,\zeta) = \sum_{k=0}^{\infty} \frac{N_f^{(k)}(\zeta)}{k!} (z-\zeta)^k = N_f(\zeta) + (z-\zeta)\, N_f'(\zeta) + \frac{1}{2}(z-\zeta)^2 N_f''(\zeta) + \cdots = \zeta + c_2 (z-\zeta)^2 + \cdots,$$

where $c_2 = \frac{1}{2} N_f''(\zeta)$. I.e.,

$$N_f(z) - \zeta = c_2 (z-\zeta)^2 + \text{higher order terms}.$$

Thus the distance from $N_f(z)$ to ζ is decreasing quadratically. We now proceed to make this more precise.

Definition 6.1. *Say that z is an **approximate zero** of f if the sequence given by $z_0 = z$ and $z_{i+1} = N_f(z_i)$ is defined for all $i \in \mathbb{N}_0^+$, and there is a ζ such that $f(\zeta) = 0$ with*

$$|z_i - \zeta| \leq \left(\frac{1}{2}\right)^{2^i - 1} |z - \zeta|.$$

*Call ζ the **associated zero**.*

Definition 6.2. *First we need to define an auxiliary quantity. Let*

$$\gamma(f,z) = \sup_{k \geq 2} \left| \frac{f'(z)^{-1} f^{(k)}(z)}{k!} \right|^{\frac{1}{k-1}},$$

where we use $f^{(k)}$ to denote the k-th derivative of f. This definition applies to analytic functions

f. If f is analytic and $f'(z)^{-1}$ exists, then this sup *exists as well since $\frac{f^{(k)}}{k!} = a_k$ has a geometric growth rate.*

We discuss more details about $\frac{f^{(k)}}{k!}$ in A Analytic Function.

`Theorem 6.3.` Suppose that $f(\zeta) = 0$ and that $f'(\zeta)^{-1}$ exists. If

$$|z - \zeta| \leq \frac{3 - \sqrt{7}}{2\gamma(f, \zeta)},$$

then z is an approximate zero of f with associated zero ζ. △

In order to prove this theorem we first prove two lemmas and a proposition.

`Lemma 6.4.` We have for $0 \leq r < 1$,

(a) $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$.

(b) $\sum_{i=1}^{\infty} i r^{i-1} = \frac{1}{(1-r)^2}$. △

Proof. In (a) we have summed the geometric series which gives an analytic function of r. We define $S_n := 1 + r + r^2 + \cdots + r^{n-1}$ then $S_n r = r + r^2 + \cdots + r^n$. Therefore, $(1-r) S_n = 1 - r^n$, i.e., $S_n = \frac{1-r^n}{1-r}$. Thus, for $0 \leq r < 1$, $S_n = \frac{1-r^n}{1-r} \xrightarrow{n\to\infty} \frac{1}{1-r}$.

In (b) we have differentiated both sides of (a), term by term on the left.

$$\frac{\partial}{\partial r}\left(\sum_{i=0}^{\infty} r^i\right) = \sum_{i=0}^{\infty} i r^{i-1} = \sum_{i=1}^{\infty} i r^{i-1} \qquad \text{and} \qquad \frac{\partial}{\partial r}\left(\frac{1}{1-r}\right) = \frac{(1-r)\cdot 0 - (-1)\cdot 1}{(1-r)^2} = \frac{1}{(1-r)^2}.$$

The claim follows. ∎

The following simple quadratic polynomial plays an important role in the estimates in this section.

$$\psi(u) = 1 - 4u + 2u^2. \tag{6.1.1}$$

`Lemma 6.5.` If $u := |z_1 - z| \gamma(f, z) < 1 - \frac{\sqrt{2}}{2}$, then

(a) $f'(z)^{-1} f'(z_1) = 1 + B$, where $|B| \leq \frac{1}{(1-u)^2} - 1 < 1$;

(b) $\left|f'(z_1)^{-1}f'(z)\right| \le \frac{(1-u)^2}{\psi(u)}$. △

Proof. (a) Using the Taylor expansion of f' at z,

$$\begin{aligned} T_f(z_1,z) &= f'(z) + f''(z)\,(z_1-z) + \frac{1}{2}f'''(z)\,(z_1-z)^2 + \cdots \\ &= \sum_{k=0}^{\infty}\frac{f^{(k+1)}(z)\,(z_1-z)^k}{k!} = \sum_{k=1}^{\infty}\frac{f^{(k)}(z)\,(z_1-z)^{k-1}}{(k-1)!} = f'(z) + \sum_{k=2}^{\infty}\frac{f^{(k)}(z)\,(z_1-z)^{k-1}}{(k-1)!}. \end{aligned}$$

Thus

$$f'(z)^{-1}f'(z_1) = f'(z)^{-1}\left(f'(z) + \sum_{k=2}^{\infty}\frac{f^{(k)}(z)\,(z_1-z)^{k-1}}{(k-1)!}\right) = 1 + B$$

where

$$B = \sum_{k=2}^{\infty} k\frac{f'(z)^{-1}f^{(k)}(z)\,(z_1-z)^{k-1}}{k!}.$$

By the definition of $\gamma(f,z)$ in Definition 6.2,

$$|B| \le \sum_{k=2}^{\infty} k(\underbrace{\gamma(f,z)\,|z_1-z|}_{u})^{k-1} = \left(\frac{1}{(1-u)^2}\right) - 1\cdot u^{1-1} - 0 = \left(\frac{1}{(1-u)^2}\right) - 1$$

which is less than 1 since $u < 1 - \frac{\sqrt{2}}{2}$.

(b) By Lemma 6.4 (a) we conclude that $\sum_{k=0}^{\infty}|B|^k = \frac{1}{1-|B|}$, then

$$|1+B| = |1-(-B)| \ge \big|1-|B|\big| \overset{|B|<1}{=\!=\!=} 1-|B|,$$

which implies $\frac{1}{|1+B|} \le \frac{1}{1-|B|} = \sum_{k=0}^{\infty}|B|^k$. Therefore,

$$\begin{aligned} \left|f'(z_1)^{-1}f'(z)\right| &= \left|\left(f'(z)^{-1}f'(z_1)\right)^{-1}\right| = \left|(1+B)^{-1}\right| \le \sum_{k=0}^{\infty}|B|^k \\ &\le \frac{1}{1-\left(\frac{1}{(1-u)^2}-1\right)} = \frac{(1-u)^2}{2(1-u)^2-1} = \frac{(1-u)^2}{1-4u+2u^2} = \frac{(1-u)^2}{\psi(u)}. \end{aligned}$$ ■

Proposition 6.6. Let $f(\zeta) = 0$ and let $u = |z-\zeta|\gamma(f,\zeta)$. Suppose $u < \frac{5-\sqrt{7}}{4}$. Then

(a) $\left|N_f(z) - \zeta\right| < \frac{\gamma(f,\zeta)|z-\zeta|^2}{\psi(u)} = \frac{u|z-\zeta|}{\psi(u)}$.

(b) $\left|N_f^k(z) - \zeta\right| \le \left(\frac{u}{\psi(u)}\right)^{2^k-1} |z-\zeta|$ for all $k \ge 0$. △

Proof. (a) We consider the Taylor expansion of f and f' at ζ

$$f(z) = f(\zeta) + \sum_{k=1}^{\infty} \frac{f^{(k)}(\zeta)\,(z-\zeta)^k}{k!} = \sum_{k=1}^{\infty} \frac{f^{(k)}(\zeta)}{k!}\,(z-\zeta)^k$$

and

$$f'(z) = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(\zeta)\,(z-\zeta)^k}{k!} = \sum_{k=1}^{\infty} \frac{f^{(k)}(\zeta)}{(k-1)!}(z-\zeta)^{k-1}$$

so

$$\begin{aligned} f'(z)(z-\zeta) - f(z) &= \sum_{k=1}^{\infty} \frac{f^{(k)}(\zeta)}{(k-1)!}(z-\zeta)^{k-1} \cdot (z-\zeta) - \sum_{k=1}^{\infty} \frac{f^{(k)}(\zeta)}{k!}(z-\zeta)^k \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right) f^{(k)}(\zeta)(z-\zeta)^k = \sum_{k=1}^{\infty} (k-1)\frac{f^{(k)}(\zeta)}{k!}(z-\zeta)^k. \end{aligned}$$

Then

$$\begin{aligned} \left|N_f(z) - \zeta\right| &= \left|(z-\zeta) - f'(z)^{-1}\left(f(z)\right)\right| = f'(z)^{-1}\left(f'(z)(z-\zeta) - f(z)\right) \\ &= \left| f'(z)^{-1} f'(\zeta) \sum_{k=1}^{\infty} (k-1) \frac{f'(\zeta)^{-1} f^{(k)}(\zeta)}{k!} (z-\zeta)^k \right| \\ &\le \left|f'(z)^{-1} f'(\zeta)\right| |z-\zeta| \sum_{k=1}^{\infty} (k-1)\left(\gamma(f,z)\,|z-\zeta|\right)^{k-1} \end{aligned}$$

In particular,

$$\sum_{k=1}^{\infty}(k-1)\left(\gamma(f,z)\,|z-\zeta|\right)^{k-1} = \sum_{k=1}^{\infty}(k-1)u^{k-1} = \sum_{k=1}^{\infty} k u^{k-1} - \sum_{k=1}^{\infty} u^{k-1} = \frac{1}{(1-u)^2} - \frac{1}{1-u}.$$

Then by `Lemma 6.5 (b)`

$$\left|N_f(z)-\zeta\right| \le \frac{(1-u)^2}{\psi(u)}|z-\zeta| \underbrace{\left(\frac{1}{(1-u)^2}-\frac{1}{(1-u)}\right)}_{\frac{u}{(1-u)^2}} \le \frac{u|z-\zeta|}{\psi(u)}.$$

Note that if $u=\psi(u)$ we have $2u^2-5u+1=0$, which implies $u=\frac{5\pm\sqrt{17}}{4}$. I.e., $\frac{u}{\psi(u)}<1$ for $0\le u<\frac{5-\sqrt{17}}{4}$.

(b) We prove it by mathematical induction. Base Case ($k=0$): $\left|N_f^0(z)-\zeta\right| = |z-\zeta| \le |z-\zeta|$.

Inductive Hypothesis: For $k\ge 1$ assume by induction that

$$\left|N_f^{k-1}(z)-\zeta\right| < \left(\frac{u}{\psi(u)}\right)^{2^{k-1}-1}|z-\zeta|.$$

Then apply `(a)` to get

$$\begin{aligned}\left|N_f(z)-\zeta\right| = \left|N_f\left(N_f^{k-1}(z)\right)-\zeta\right| &< \frac{\gamma(f,z)\left|N_f^{k-1}(z)-\zeta\right|^2}{\psi(u)}\\ &= \frac{\gamma(f,z)}{\psi(u)}\left(\left(\frac{u}{\psi(u)}\right)^{2^{k-1}-1}\right)^2|z-\zeta|^2 = \frac{u}{\psi(u)}\left(\frac{u}{\psi(u)}\right)^{2^k-2}|z-\zeta| = \left(\frac{u}{\psi(u)}\right)^{2^k-1}|z-\zeta|\end{aligned}$$

and we are done. ■

We can now give the proof of `Theorem 6.3`.

Proof (Theorem 6.3.). We consider the equation $\frac{u}{\psi(u)}=\frac{u}{1-4u+2u^2}=\frac{1}{2}$, i.e., $u=\frac{3\pm\sqrt{7}}{2}$. Thus $\frac{3-\sqrt{7}}{2}$ is the first positive solution of $\frac{u}{\psi(u)}=\frac{1}{2}$. In other words, if $u<\frac{3-\sqrt{7}}{2}$, then $\frac{u}{\psi(u)}<\frac{1}{2}$ and `Proposition 6.6 (b)` finishes the proof. ■

Remark 6.7. `Proposition 6.6` implies that Newton's method converges if $\frac{u}{\psi(u)}<1$; that is $|z-\zeta|\gamma(f,\zeta)<\frac{5-\sqrt{17}}{4}$. The constant is better than in `Theorem 6.3`, but z is not guaranteed to be an approximate zero. ◇

The following result follows immediately from the preceding remark.

`Corollary 6.8.` If ζ, ζ' are zeros of f, then they are separated by a distance that can be estimated from below by

$$\left|\zeta' - \zeta\right| \geq \frac{5 - \sqrt{17}}{4\gamma(f, \zeta)}. \qquad \triangle$$

Example 6.9. As an example of an application of `Theorem 6.3` *let us consider the problem of computing the d-th roots of the unity; that is, we want to compute the roots of the polynomial*

$$f(x) = x^d - 1.$$

Let $\zeta \in \mathbb{C}$ be such that $f(\zeta) = 0$. The k-th derivative of f at ζ is $f^{(k)}(\zeta) = d(d-1)\cdot\dots\cdot(d-k+1)\zeta^{d-k}$, in particular, $f'(\zeta) = d\zeta^{d-1}$. Thus,

$$\gamma(f, \zeta) = \sup_{k\geq 2}\left|\frac{f'(\zeta)^{-1}f^{(k)}(\zeta)}{k!}\right|^{\frac{1}{k-1}} = \sup_{k\geq 2}\left|\frac{d(d-1)\cdot\dots\cdot(d-k+1)\zeta^{d-k}}{d\zeta^{d-1}\cdot k!}\right|^{\frac{1}{k-1}} = \sup_{k\geq 2}\left(\frac{(d-1)\cdot\dots\cdot(d-k+1)}{k!}\right)^{\frac{1}{k-1}}$$

We consider only the fraction in the bracket,

$$\frac{(d-1)\cdot\dots\cdot(d-k+1)}{k!} \leq \frac{(d-1)\cdot\dots\cdot(d-k+1)}{2^{k-1}} = \frac{d-1}{2}\cdot\dots\cdot\frac{d-k+1}{2} \leq \left(\frac{d}{2}\right)^{k-1}.$$

Therefore, $\gamma(f, z) \leq \frac{d}{2}$. According to `Theorem 6.3` *all points z such that $|z - \zeta| < \frac{3-\sqrt{7}}{d}$ are approximate zeros of f with associated zero ζ.*

Remark 6.10. The invariant $\gamma(f, \zeta)$, `Theorem 6.3`, its proof, and its corollaries extend immediately to systems $f : \mathbb{C}^n \to \mathbb{C}^n$ and even to maps of Banach spaces. See `Theorem 7.7` in 7 n-Dimensional Generalization ◇

6.2. Point Estimates for Approximate Zeros

`Theorem 6.3` is useful if we have information about one or more of the roots of f, but we would like a criterion computable at the point z itself that guarantees that z is an approximate zero of f. To this end we define two more auxiliary quantities.

Definition 6.11 (The Length of the Newton Step).

$$\beta(f, z) = \left|z - N_f(z)\right| = \left|f'(z)^{-1}f(z)\right|$$

and

$$\alpha(f,z) = \beta(f,z)\gamma(f,z).$$

In `Theorem 6.12` we show that if $\alpha(f,z) < \alpha_0$ for some universal constant α_0, then z is an approximate zero of f. `Proposition 6.14` estimates the reduction in the absolute value of f after one iterate of Newton's method. As a consequence of `Theorem 6.25` we obtain the following result.

`Theorem 6.12`. There is a universal constant α_0 with the following property. If $\alpha(f,z) < \alpha_0$, then z is an approximate zero of f in the sense of Definition 6.2. Moreover, the distance from z to the associated zero ζ is at most $2\beta(f,z)$. △

Remark 6.13. The invariant $\alpha(f,z)$ depends only on derivatives of f at the point z, which can be computed if f is a polynomial map. Thus `Theorem 6.12` gives a criterion that can be used in principle and in practice to give certainty that z is indeed an approximation to a solution.◇

`Proposition 6.14`. Let $z' = N_f(z)$. If $\alpha(f,z) < 1$, then

$$\frac{|f(z')|}{|f(z)|} \le \frac{\alpha(f,z)}{1-\alpha(f,z)}.$$ △

Remark 6.15. This is the only result in this part that does not generalize to n-dimensional or Banach spaces. In the proof we use the fact that $f'(z)$ and $f^{(k)}(z)$ commute. ◇

Proof. Since $z' = N_f(z)$ one has $z' - z = -\left(\frac{f(z)}{f'(z)}\right)$, by the Taylor expansion of f at the point z

$$\begin{aligned} f(z') &= f(z) - f'(z)(z'-z) + \sum_{k=2}^{\infty} \frac{f^{(k)}(z)}{k!}\left(z'-z\right)^k = f(z) + f'(z)\left(-\frac{f(z)}{f'(z)}\right) + \sum_{k=2}^{\infty}\frac{f^{(k)}(z)}{k!}\left(-\frac{f(z)}{f'(z)}\right)^k \\ &= f(z) - f'(z)\left(\frac{f(z)}{f'(z)}\right) + \sum_{k=2}^{\infty}(-1)^k\frac{f^{(k)}(z)}{k!}\left(\frac{f(z)}{f'(z)}\right)^k = \sum_{k=2}^{\infty}(-1)^k\frac{f^{(k)}(z)}{k!}\left(\frac{f(z)}{f'(z)}\right)^k \end{aligned}$$

so

$$\begin{aligned} \left|f(z')\right| &\le |f(z)|\sum_{k=2}^{\infty}\left|\frac{f^{(k)}(z)}{k!f'(z)}\right|\left|f'(z)^{-1}f(z)\right|^{k-1} \le |f(z)|\sum_{k=2}^{\infty}\gamma(f,z)^{k-1}\beta(f,z)^{k-1} \\ &= |f(z)|\sum_{k=2}^{\infty}\alpha(f,z)^{k-1} = |f(z)|\left(\frac{1}{1-\alpha(f,z)} - 1\right) = |f(z)|\frac{\alpha(f,z)}{1-\alpha(f,z)} \end{aligned}$$

as long as $\alpha(f,z) < 1$. ■

The next proposition estimates α, β and γ at a point z_1 near z in terms of the values of these quantities at z.

`Proposition 6.16.` If $u < 1 - \frac{\sqrt{2}}{2}$ and $|z_1 - z|\gamma(f,z) = u$, then

(a) $\beta(f, z_1) \le \frac{(1-u)}{\psi(u)}\left((1-u)\beta(f,z) + |z_1 - z|\right)$;

(b) $\gamma(f, z_1) \le \frac{\gamma(f,z)}{\psi(u)(1-u)}$;

(c) $\alpha(f, z_1) \le \frac{(1-u)\alpha(f,z)+u}{\psi(u)^2}$. △

We use the following two lemmas to prove the proposition.

`Lemma 6.17.` Let $0 \le r < 1$ and k be a positive integer; then

$$\sum_{l=0}^{\infty} \frac{(k+l)!}{k!l!} r^l = \frac{1}{(1-r)^{k+1}}.$$ △

Proof. By mathematical induction, we first prove

$$\left(\sum_{i=0}^{\infty} r^i\right)^{(k)} = \sum_{l=0}^{\infty} \frac{(k+l)!r^l}{l!} \qquad \text{and} \qquad \left(\frac{1}{1-r}\right)^{(k)} = \frac{k!}{(1-r)^{k+1}}$$

Base Case ($k = 0, k = 1$): For $k = 0$ this is trivial. Let $k = 1$, then

$$\frac{\partial}{\partial r}\left(\sum_{i=0}^{\infty} r^i\right) = \sum_{i=1}^{\infty} i r^{i-1} = \sum_{l=0}^{\infty}(1+l)r^l = \sum_{l=0}^{\infty}\frac{(1+l)!}{l!}r^l \quad \text{and} \quad \frac{\partial}{\partial r}\left(\frac{1}{1-r}\right) = \frac{1}{(1-r)^2} = \frac{1!}{(1-r)^{1+1}}.$$

Inductive Hypothesis: For $k \ge 1$ assume by induction that

$$\left(\sum_{i=0}^{\infty} r^i\right)^{(k)} = \sum_{l=0}^{\infty} \frac{(k+l)!r^l}{l!} \qquad \text{and} \qquad \left(\frac{1}{1-r}\right)^{(k)} = \frac{k!}{(1-r)^{k+1}}$$

Then for $k+1$ we conclude that

$$\begin{aligned}\left(\sum_{i=0}^{\infty} r^i\right)^{(k+1)} &= \frac{\partial}{\partial r}\left(\sum_{l=0}^{\infty}\frac{(k+l)!r^l}{l!}\right) = \sum_{l=0}^{\infty}\frac{(k+l)!lr^{l-1}}{l!} \\ &= \sum_{l=1}^{\infty}\frac{(k+l)!lr^{l-1}}{l!} = \sum_{l=0}^{\infty}\frac{(k+1+l)!(l+1)r^l}{(l+1)!} = \sum_{l=0}^{\infty}\frac{(k+1+l)!r^l}{l!}\end{aligned}$$

and

$$\left(\frac{1}{1-r}\right)^{(k+1)} = \frac{\partial}{\partial r}\left(\frac{k!}{(1-r)^{k+1}}\right) = k!\frac{(k+1)(1-r)^k}{(1-r)^{2k+2}} = \frac{(k+1)!}{(1-r)^{(k+1)+1}}$$

As in `Lemma 6.4`, $\left(\sum_{i=0}^{\infty} r^i\right)^{(k)} = \left(\frac{1}{(1-r)}\right)^{(k)}$. Then we finish the proof. ∎

`Lemma 6.18.` If $u < 1 - \frac{\sqrt{2}}{2}$ and $u := |z_1 - z|\,\gamma(f,z)$, then

(a) $\left|\frac{f'(z_1)^{-1}f^{(k)}(z_1)}{k!}\right| \le \frac{1}{\psi(u)}\left(\frac{\gamma(f,z)}{1-u}\right)^{k-1}$ for $k \ge 2$;

(b) $\left|f'(z)^{-1}f(z_1)\right| \le \beta(f,z) + \frac{|z_1-z|}{1-u}$. △

Proof. (a) Write γ for $\gamma(f,z)$. Using the Taylor expansion of $f^{(k)}$ at z and `Lemma 6.5 (b)`,

$$\begin{aligned}\left|\frac{f'(z_1)^{-1}f^{(k)}(z_1)}{k!}\right| &\le \left|f'(z_1)^{-1}f'(z)\right|\left|\frac{f'(z)^{-1}}{k!}\sum_{l=0}^{\infty}\frac{f^{(k+l)}(z)\,(z_1-z)^l}{l!}\right| \\ &\le \frac{(1-u)^2}{\psi(u)}\left|\sum_{l=0}^{\infty}\frac{(k+l)!}{k!l!}\frac{f'(z)^{-1}f^{(k+l)}(z)\,(z_1-z)^l}{(k+l)!}\right| \le \frac{(1-u)^2}{\psi(u)}\sum_{l=0}^{\infty}\frac{(k+l)!}{k!l!}\gamma^{k+l-1}\,|z_1-z|^l.\end{aligned}$$

According to `Lemma 6.17` we conclude that

$$\sum_{l=0}^{\infty}\frac{(k+l)!}{k!l!}\underbrace{\gamma^l\,|z_1-z|^l}_{u^l} = \frac{1}{(1-u)^{k+1}}.$$

Thus

$$\left|\frac{f'(z_1)^{-1}f^{(k)}(z_1)}{k!}\right| \le \frac{(1-u)^2}{\psi(u)}\gamma^{k-1}\frac{1}{(1-u)^{k+1}} = \frac{1}{\psi(u)}\left(\frac{\gamma}{1-u}\right)^{k-1};$$

(b) Using the Taylor expansion of f at z,

$$\begin{aligned}\left|f'(z)^{-1}f(z_1)\right| &= \left|f'(z)^{-1}f(z) + f'(z)^{-1}f'(z)(z_1-z) + \sum_{k=2}^{\infty}\frac{f'(z)^{-1}f^{(k)}(z)}{k!}(z_1-z)^k\right| \\ &\le \left|f'(z)^{-1}f(z)\right| + |z_1-z|\left|1 + \sum_{k=2}^{\infty}\gamma^{k-1}\,|z_1-z|^{k-1}\right| \\ &= \beta(f,z) + |z_1-z|\left|1 + \left(\frac{1}{1-u} - 1\right)\right| = \beta(f,z) + \frac{|z_1-z|}{|1-u|} = \beta(f,z) + \frac{|z_1-z|}{1-u}.\end{aligned}$$

The last step follows from the bound $u < 1$. ■

Proof (Proposition 6.16). (a) By `Lemma 6.5 (b)` and `Lemma 6.18 (b)`

$$\begin{aligned}\beta(f, z_1) &= \left|f'(z_1)^{-1} f(z_1)\right| = \left|f'(z_1)^{-1} f'(z)\right| \left|f'(z)^{-1} f(z_1)\right| \\ &\le \frac{(1-u)^2}{\psi(u)} \left(\beta(f,z) + \frac{|z_1 - z|}{1-u}\right) = \frac{(1-u)}{\psi(u)} \left((1-u)\,\beta(f,z) + |z_1 - z|\right).\end{aligned}$$

(b) By definition $\gamma(f, z_1) = \sup_{k\ge 2} \left|\frac{f'(z_1) f^{(k)}(z_1)}{k!}\right|^{\frac{1}{k-1}}$ and by `Lemma 6.18 (a)`,

$$\gamma(f, z_1) \le \sup_{k\ge 2} \left(\frac{1}{\psi(u)}\right)^{\frac{1}{k-1}} \frac{\gamma(f,z)}{1-u}.$$

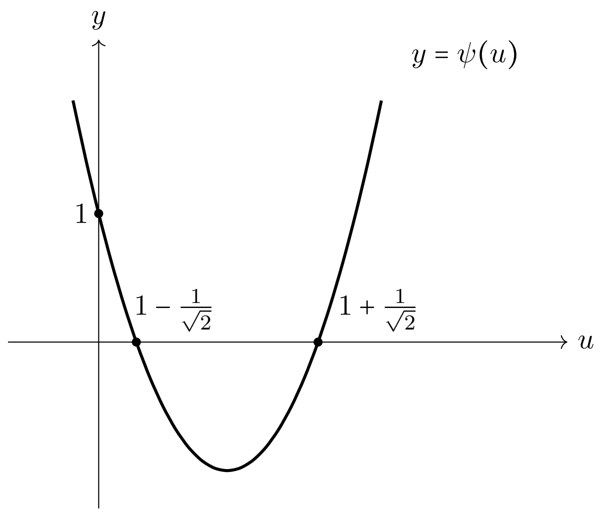

Figure 6.2.: The Plot of ψ.

Since $\psi(u) < 1$ for $0 \le u < 1 - \frac{\sqrt{2}}{2}$, the supremum is achieved at $k = 2$,

$$\gamma(f, z_1) \le \left[\left(\frac{1}{\psi(u)}\right)^{\frac{1}{k-1}} \frac{\gamma(f,z)}{1-u}\right]_{k=2} = \frac{\gamma(f,z)}{\psi(u)(1-u)}$$

and we are done.

(c) Multiplying the inequalities in `(a)` and `(b)` proves `(c)`. ■

Next we bound the derivative of Newton's map in terms of α.

Proposition 6.19. For all analytic f and all z, $\left|N_f'(z)\right| \le 2\alpha(f,z)$. △

Proof.

$$\left|N_f'(z)\right| = \left|\frac{\partial}{\partial z}\left(z - f'(z)^{-1}f(z)\right)\right| = \left|1 - \frac{f'(z)^2 - f(z)f''(z)}{[f'(z)]^2}\right| = \left|1 - 1 + f'(z)^{-2}f''(z)f(z)\right|$$
$$= \left|f'(z)^{-1}f''(z)f'(z)^{-1}f(z)\right| = 2\left|\frac{f'(z)^{-1}f''(z)}{2}\right|\left|f'(z)^{-1}f(z)\right| \le 2\gamma(f,z)\beta(f,z) = 2\alpha(f,z).$$

Then we finish the proof. ■

The next proposition states a fact about contraction maps of complete metric spaces X. For most of our applications X is a closed ball and $d(x,y) = |x-y|$. We use $B(r,z)$ to denote the closed ball of radius r around z defined by $B(r,z) = \{z' : \ d(z,z') \le r\}$.

Definition 6.20. *Suppose that X is a complete metric space. A map $f : X \to X$ satisfying that, for all x, y in X and $c < 1$,*

$$d\left(f(x), f(y)\right) \le c d(x,y)$$

is called a ***contraction map*** *with* ***contraction constant*** *c.*

Proposition 6.21. Let $f : X \to X$ be a contraction map with contraction constant c. Then there is a unique fixed point $p \in X$, $f(p) = p$ and $f^n(x)$ converges to p as $n \to \infty$ for all x in X. Moreover, for any $x \in X$,

$$\frac{d(x, f(x))}{1+c} \le d(x,p) \le \frac{d(x, f(x))}{1-c}.$$ △

Proof. By mathematical induction, we first prove for $n \ge 1$,

$$d\left(f^n(x), f^{n+1}(x)\right) \le c^n d(x, f(x)).$$

Base Case ($n = 1$): Since f is a contraction map, there exists a $c < 1$ such that

$$d\left(f(x), f^2(x)\right) = d\left(f(x), f(f(x))\right) \le c d\left(x, f(x)\right)$$

Inductive Hypothesis: For $n \ge 1$ assume by induction that

$$d\left(f^n(x), f^{n+1}(x)\right) \le c^n d(x, f(x)).$$

Then for $n+1$ we conclude that,

$$d\left(f^{n+1}(x), f^{n+2}(x)\right) = d\left(f\left(f^{n}(x)\right), f\left(f^{n+1}(x)\right)\right) \leq cd\left(f^{n}(x), f^{n+1}(x)\right) \leq c^{n} \cdot cd\left(x, f(x)\right) = c^{n+1} d\left(x, f(x)\right).$$

In particular, for $k \geq 1$,

$$\begin{aligned} d\left(f^{n}(x), f^{n+1}(x)\right) &\leq d\left(f^{n}(x), f^{n+1}(x)\right) + d\left(f^{n+1}(x), f^{n+2}(x)\right) + \cdots + d\left(f^{n+k-1}(x), f^{n+k}(x)\right) \\ &\leq \left(c^{n} + c^{n+1} + \cdots + c^{n+k-1}\right) d\left(x, f(x)\right) = \sum_{i=n}^{n+k-1} c^{n} d\left(x, f(x)\right) \\ &= \frac{c^{n}\left(1-c^{k}\right)}{1-c} d\left(x, f(x)\right) < \frac{c^{n}}{1-c} d\left(x, f(x)\right). \end{aligned}$$

The last step follows from $c < 1$. I.e., we have proved that, for each $n \geq 1$ and for all $m \geq n$,

$$d\left(f^{n}(x), f^{m}(x)\right) \leq \frac{c^{n}}{1-c} d\left(x, f(x)\right).$$

Since c^n tends to zero $\left(f^{n}(x)\right)_{n\geq 1}$ is a Cauchy sequence. By the completeness of X we conclude that $\left(f^{n}(x)\right)_{n\geq 1}$ converges to a point p in X. The sequence $\left(f^{n+1}(x)\right)_{n\geq 1}$ also converges to p so by continuity of f,

$$f^{n+1}(p) = f\left(f^{n}(p)\right) \xrightarrow{n\to\infty} f(p),$$

i.e., $f(p) = p$. Let p, q be different fix points, i.e., $f(p) = p$ and $f(q) = q$. Then

$$d(p, q) = d\left(f(p), f(q)\right) \leq cd(p, q),$$

it follows that p is the unique fixed point of f and that every orbit $f^n(x)$ converges to p as $n \to \infty$. Since

$$d(x,p) \leq d\left(x, f(x)\right) + d\left(f(x), f^{2}(x)\right) + \cdots \leq (1 + c^{1} + c^{2} + \cdots) d\left(x, f(x)\right) = \sum_{n=0}^{\infty} c^{n} d\left(x, f(x)\right) = \frac{1}{1+c} d\left(x, f(x)\right),$$

which means

$$d(x,p) \leq \frac{1}{1-c} d\left(x, f(x)\right).$$

Finally, by the triangle inequality,

$$d(x, f(x)) \leq d(x,p) + d\left(p, f(x)\right) = d(x,p) + d\left(f(p), f(x)\right) \leq (1+c) d(x,p).$$

The last step follows from the fact that f is a contraction map. ∎

Theorem 6.22. If $r < \frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,z)}$, then

(a) for all z_1 with $|z_1 - z| < r$, $u = r\gamma(f,z)$ and $\psi(u) = 1 - 4u + 2u^2$,

$$\left|N_f'(z_1)\right| \leq \frac{2(\alpha(f,z)+u)}{\psi(u)^2}.$$

(b) Define $r' = \frac{2(\alpha(f,z)+u)}{\psi(u)^2} r$, then

$$N_f\left(B\left(r,z\right)\right) \subseteq B\left(r', N_f(z)\right). \quad \triangle$$

Proof. Part (a) follows immediately from Proposition 6.19 and Proposition 6.16 (c),

$$\alpha(f,z_1) \leq \frac{(1-u)\alpha(f,z)+u}{\psi(u)^2} < \frac{\alpha(f,z)+u}{\psi(u)^2}.$$

For Part (b) we need the following lemma. ■

Lemma 6.23. Suppose $g : B(r,z) \to B(r,z)$ is continuously differentiable with $|g'(z_1)| \leq c$ for all $z_1 \in B(r,z)$. Then, $|g(z_1) - g(z_2)| \leq c|z_1 - z_2|$ for all $z_1, z_2 \in B(r,z)$. △

Proof. Let L be the straight-line segment connecting z_1 and z_2. So the length of L is $|z_1 - z_2|$ and $L \subseteq B(r,z)$.

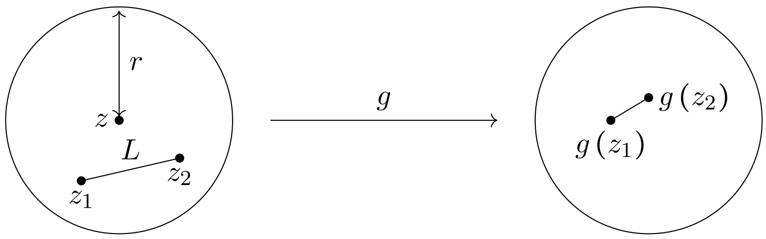

Figure 6.3.: The figure of $B(r,z)$ and the map g.

The distance $|g(z_1) - g(z_2)|$ equals the length of the straight-line segment connection $g(z_1)$ and $g(z_2)$, which is the shortest differentiable curve joining them. In particular, by the mean value theorem, there exists a ξ in (z_1, z_2) such that $g'(\xi) = \frac{g(z_2)-g(z_1)}{z_2-z_1}$, which implies,

$$|g(z_2) - g(z_1)| \leq |z_2 - z_1| \cdot \max_{z' \in L} \left|g'(z')\right|.$$

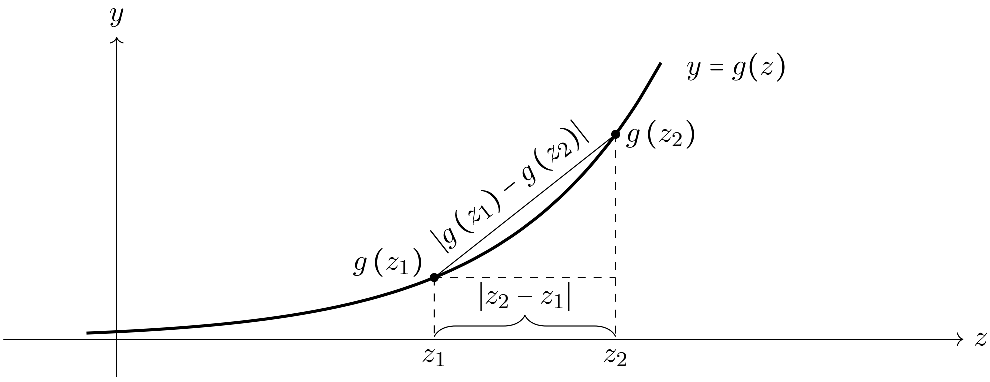

Figure 6.4.: The relation between $|z_2 - z_1|$ and $|g(z_2) - g(z_1)|$.

Finally,

$$|g(z_1) - g(z_2)| \leq |z_2 - z_1| \cdot \max_{z' \in L} \left| g'(z') \right| \leq c |z_1 - z_2| . \qquad \blacksquare$$

Proof (Theorem 6.22 (b)). By `Theorem 6.22 (a)` and `Lemma 6.23`, for all z_1 in $B(r, z)$,

$$\left| N_f(z_1) - N_f(z) \right| \leq \frac{2(\alpha(f,z) + u)}{\psi(u)^2} |z_1 - z| \leq \frac{2(\alpha(f,z) + u)}{\psi(u)^2} r. \qquad \blacksquare$$

`Corollary 6.24.` If $u := r\gamma(f,z) < 1 - \frac{\sqrt{2}}{2}$, $c := \frac{2(\alpha(f,z)+u)}{\psi(u)^2} < 1$ and $\alpha(f,z) + cu \leq u$, then N_f is a contraction map of the ball $B\left(\frac{u}{\gamma(f,z)}, z\right)$ into itself with contraction constant c. Hence there is a unique root $\zeta \in B\left(\frac{u}{\gamma(f,z)}, z\right)$ of f and all $z' \in B\left(\frac{u}{\gamma(f,z)}, z\right)$ tend to ζ under iteration of N_f. △

Proof. By `Theorem 6.22 (a)`, c is a contraction constant on $B\left(\frac{u}{\gamma(f,z)}, z\right)$. By `Theorem 6.22 (b)` and the triangle inequality, if $\beta(f,z) + \frac{cu}{\gamma(f,z)} < \frac{u}{\gamma(f,z)}$, then for all z_1 in $B\left(\frac{u}{\gamma(f,z)}, z\right)$,

$$\left| N_f(z_1) - z \right| = \left| N_f(z_1) - N_f(z) + N_f(z) - z \right| \leq \left| N_f(z_1) - N_f(z) \right| + \left| N_f(z) - z \right| \leq \frac{cu}{\gamma(f,z)} + \beta(f,z) < \frac{u}{\gamma(f,z)},$$

i.e.,

$$N_f\left(B\left(\frac{u}{\gamma(f,z)}, z\right)\right) \subseteq B\left(\frac{u}{\gamma(f,z)}, z\right)$$

In particular, $\beta(f,z) + \frac{cu}{\gamma(f,z)} < \frac{u}{\gamma(f,z)}$ follows from $\alpha(f,z) + cu \leq u$ by dividing by $\gamma(f,z)$.

Now the rest of the proof follows from `Proposition 6.21`. For $N_f : B\left(\frac{u}{\gamma(f,z)}, z\right) \to B\left(\frac{u}{\gamma(f,z)}, z\right)$, there exists a unique fixed point $\zeta \in B\left(\frac{u}{\gamma(f,z)}, z\right)$ such that $N_f(\zeta) = \zeta$ and $N_f^n(z') \xrightarrow{n\to\infty} \zeta$, for all z' in $B\left(\frac{u}{\gamma(f,z)}, z\right)$. In other word, ζ is the unique root of f in $B\left(\frac{u}{\gamma(f,z)}, z\right)$. ■

`Corollary 6.24` gives us a good criterion in terms of α and γ for convergence of the iterates of Newton's map by a contraction map in a neighborhood of a point z. The next theorem gives a simpler criterion in terms of α and u. The three inequalities in `Corollary 6.24` hold if α and u are small enough. Further restrictions on α and u guarantee that $B\left(\frac{u}{\gamma(f,z)}, z\right)$ consists of approximate zeros.

`Theorem 6.25 (Robust α Theorem).` There are positive real numbers α_0 and u_0 such that: if $\alpha(f,z) < \alpha_0$, then there is a root ζ of f such that

$$B\left(\frac{u_0}{\gamma(f,z)}, z\right) \subseteq B\left(\frac{3-\sqrt{7}}{2\gamma(f,\zeta)}, \zeta\right)$$

and

$$N_f : B\left(\frac{u_0}{\gamma(f,z)}, z\right) \to B\left(\frac{u_0}{\gamma(f,\zeta)}, \zeta\right)$$

with contraction constant less than or equal to $\frac{1}{2}$. △

Remark 6.26. It follows from `Theorem 6.3` that $B\left(\frac{u_0}{\gamma(f,\zeta)}, \zeta\right)$ consists of approximate zeros with associated zero ζ. ◇

Proof (Theorem 6.25). Choose $\alpha_0 > 0, u_0 > 0, l_0 > 2$ to satisfy,

(6.2.1) $c_0 = \frac{2(\alpha_0+u_0)}{\psi(u_0)^2} < \frac{1}{l_0} < \frac{1}{2} < 1;$

(6.2.2) $\alpha_0 + c_0 u_0 < u_0;$

(6.2.3) $\left(\frac{\alpha_0}{1+c_0} + u_0\right)\left(\frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)}\right) < \frac{3-\sqrt{7}}{2};$

(6.2.4) $\frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)} \leq \frac{l_0}{2}.$

Let ζ be the root of f given by `(6.2.1)`,`(6.2.2)` and `Corollary 6.24`. Then by `Proposition 6.21` we obtain the following in-equation.

(6.2.5) $|z-\zeta| \le \frac{|z-N_f(z)|}{1-c_0} = \frac{\beta(f,z)}{1-c_0}$.

By the triangle inequality, if $z' \in B\left(\frac{u_0}{\gamma(f,z)}, z\right)$, then

$$\left|z'-\zeta\right| \le \left|z'-z\right| + |z-\zeta| \le \frac{\beta(f,z)}{1-c_0} + \frac{u_0}{\gamma(f,z)}.$$

Multiplying by $\gamma(f,z)$ gives $|z'-\zeta|\,\gamma(f,z) \le \frac{\alpha(f,z)}{1-c_0} + u_0$ and then multiplying by $\frac{\gamma(f,\zeta)}{\gamma(f,z)}$,

$$\left|z'-\zeta\right|\gamma(f,\zeta) \le \left(\frac{\alpha(f,z)}{1-c_0} + u_0\right)\frac{\gamma(f,\zeta)}{\gamma(f,z)}.$$

By `Proposition 6.16 (b)` and `(6.2.5)` multiplied by $\gamma(f,z)$,

$$\gamma(f,\zeta) \le \frac{\gamma(f,z)}{\underbrace{\psi\underbrace{(|z-\zeta|\,\gamma(f,z))}_{\le\frac{\alpha_0}{1-c_0}}}_{\ge\psi\left(\frac{\alpha_0}{1-c_0}\right)}\underbrace{(1-|z-\zeta|\,\gamma(f,z))}_{\ge 1-\frac{\alpha_0}{1-c_0}}} \le \frac{\gamma(f,z)}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)},$$

i.e.,

(6.2.6) $\frac{\gamma(f,\zeta)}{\gamma(f,z)} \le \frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)}$.

Thus,

$$\left|z'-\zeta\right|\gamma(f,\zeta) < \left(\frac{\alpha(f,z)}{1-c_0} + u_0\right)\frac{\gamma(f,\zeta)}{\gamma(f,z)}\frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)} < \frac{3-\sqrt{7}}{2}$$

and

$$B\left(\frac{u_0}{\gamma(f,z)}, z\right) \subseteq B\left(\frac{3-\sqrt{7}}{2\gamma(f,\zeta)}, \zeta\right).$$

Moreover, by `Corollary 6.24`, `(6.2.1)` and `(6.2.2)`, $\zeta \in B\left(\frac{u_0}{\gamma(f,z)}, z\right)$ and N_f has contraction constant less than $\frac{1}{l_0}$ on $B\left(\frac{u_0}{\gamma(f,z)}, z\right)$. Hence if z_1 belongs to the ball $B\left(\frac{u_0}{\gamma(f,z)}, z\right)$, then

$$|z_1-\zeta| = |z_1-z+z-\zeta| \le |z_1-z| + |z-\zeta| \le \frac{2u_0}{\gamma(f,z)}$$

and by (6.2.4) and (6.2.5),

$$\begin{aligned}\left|N_f(z_1) - \zeta\right| \gamma(f,\zeta) &= \left|N_f(z_1) - N_f(\zeta)\right| \gamma(f,\zeta) \\ &< \frac{1}{l_0} |z_1 - \zeta| \gamma(f,\zeta) = \frac{2}{l_0} \frac{u_0}{\gamma(f,z)} \gamma(f,\zeta) \le \frac{2}{l_0} u_0 \frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1 - \frac{\alpha_0}{1-c_0}\right)} \le u_0,\end{aligned}$$

so we are done. ■

Remark 6.27. We may take $l_0 = 3$ and $\alpha_0 = .03, u_0 = .05$. This may be checked by substitution.

- $\frac{\alpha_0}{1-c_0} = .0398$;
- $\psi(u_0) = 1 - 4u_0 + 2u_0^2 = .805$;
- $\psi\left(\frac{\alpha_0}{1-c_0}\right) = 1 - 4\frac{\alpha_0}{1-c_0} + 2\left(\frac{\alpha_0}{1-c_0}\right)^2 = .8438$;
- $c_0 = \frac{2(\alpha_0+u_0)}{\psi(u_0)^2} = \frac{2\times(.03+.05)}{.805^2} = .2469 < \frac{1}{3} = \frac{1}{l_0} < \frac{1}{2} < 1$;
- $\alpha_0 + c_0 u_0 = .03 + .2469 \times .05 = .0423 < .05 = u_0$;
- $\left(\frac{\alpha_0}{1+c_0} + u_0\right)\left(\frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)}\right) = (.0398 + .05)\left(\frac{1}{.8438\times.9601}\right) = .1109 < .1771 = \frac{3-\sqrt{7}}{2}$;
- $\frac{1}{\psi\left(\frac{\alpha_0}{1-c_0}\right)\left(1-\frac{\alpha_0}{1-c_0}\right)} = 1.234 < 1.5 = \frac{l_0}{2}$. ◊

Theorem 6.12 will be proved with the constant $\alpha_0 = .130707$ in the 7 n-Dimensional Generalization. The distance from z to the associated zero ζ, by (6.2.5), is

$$|z - \zeta| \le \frac{\beta(f,z)}{1 - c_0} < \frac{\beta(f,z)}{\frac{1}{2}} = 2\beta(f,z),$$

i.e., at most $2\beta(f,z)$.

We close this chapter with a discussion about the level of generality of the results we have just proved. We began this chapter assuming that f was an analytic function of one complex or real variable defined on all of $\mathbb{C}$ or $\mathbb{R}$. In fact, we have been careful to present our definitions, theorems, and proofs to be valid in a broader context. Now we explain the context.

We suppose that $\mathcal{E}$ and $\mathcal{F}$ are complete normed vector spaces, that is, Banach spaces, over the real or complex numbers. So $\mathcal{E}$ and $\mathcal{F}$ might be $\mathbb{R}^n$ or $\mathbb{C}^m$ or subspaces of them, or they might even be infinite-dimensional spaces such as $C^0([0,1],\mathbb{R})$, the space of continuous functions ϕ

with domain the closed unit interval $[0,1]$ and taking real values. When dealing with elements of $\mathcal{E}$ or $\mathcal{F}$ where we have used absolute value it should be replaced by the norm so, for example, in $C^0([0,1],\mathbb{R})$ a standard norm which makes it a complete normed vector space is

$$|\psi| = \sup_{x\in[0,1]} |\psi(x)|.$$

Next f is presumed to be defined and analytic on some open set $D \subseteq \mathcal{E}$ with values in $\mathcal{F}$. Where we have written f' it should be considered as a continuous linear operator $f' : \mathcal{E} \to \mathcal{F}$, which is the derivative of f. Then $f^{(k)}$ is the k-th derivative of f and is a symmetric multilinear operator, operation on k-tuples of elements in $\mathcal{E}$ with values in $\mathcal{F}$. When the k-tuple has a vector x repeated l times, $f^{(k)}x^l$ denotes the operator on $k-l$-tuples obtained by substituting x in l places.

In the definition of $\gamma(f,z)$, $f'(z)^{-1}f^{(k)}(z)$ is a composition so that it operates on k-tuples of elements of $\mathcal{E}$ and takes values in $\mathcal{E}$. Absolute values of operators are understood to be operator norms; that is, for an operator A, its operator norm is

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

That $f'(z)^{-1}$ exists means that $f'(z)$ has a continuous linear operator inverse. So that now

$$N_f'(z) = f'(z)^{-1}f''(z)f'(z)^{-1}f(z)$$

makes sense as a linear operator from $\mathcal{E}$ to itself and is indeed the derivative of Newton's map. That $f'(z) = 0$ means it is identically zero as linear operator. Several places where we have written 1, such as in `Lemma 6.5`, should be read as the identity linear map.

The entire section now makes sense for analytic $f : \mathcal{E} \to \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are Banach spaces over the real or complex numbers. Our definitions, theorems, corollaries, lemmas, and propositions remain the same with the exception of `Proposition 6.14` which is restricted to one dimension.

When our map f is defined on an open set $D \subseteq \mathcal{E}$ and not on all of $\mathcal{E}$, $f : D \to \mathcal{F}$, our theorems, corollaries, lemmas, and propositions remain valid with the additional hypothesis that $B\left(\frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,z)}, z\right) \subseteq D$. In fact it is natural to have the open ball of radius $\frac{1}{\gamma(f,z)}$ contained in D as the next proposition shows.

`Proposition 6.28.` Let f be analytic at z and r be the radius of convergence of the Taylor series of f at z. Then $r \geq \frac{1}{\gamma(f,z)}$. △

Proof. The Taylor expansion of f at z is $T_f(x,z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!}(x-z)^n$ then by `(A.2.1)` in A.2.1

Basics,

$$r = \frac{1}{\limsup\limits_{k\to\infty} \left|\frac{f^{(k)}(z)}{k!}\right|^{\frac{1}{k}}}.$$

and

$$\limsup_{k\to\infty}\left|\frac{f^{(k)}(z)}{k!}\right|^{\frac{1}{k}} \le \limsup_{k\to\infty}\left|f'(z)\right|^{\frac{1}{k}}\left|\frac{f'(z)^{-1}f^{(k)}(z)}{k!}\right|^{\frac{1}{k}} \le \limsup_{k\to\infty}\left|\frac{f'(z)^{-1}f^{(k)}(z)}{k!}\right|^{\frac{1}{k}}$$
$$\le \limsup_{k\to\infty}\left|\frac{f'(z)^{-1}f^{(k)}(z)}{k!}\right|^{\frac{1}{k-1}} \le \sup_{k\to\infty}\left|\frac{f'(z)^{-1}f^{(k)}(z)}{k!}\right|^{\frac{1}{k-1}} \le \gamma(f,z). \qquad \blacksquare$$

We end this section with a version of the inverse function theorem that is valid in this context and which gives an estimate of the size of the ball on which the inverse is defined in terms of $\gamma(f,z)$.

If $f'(z)^{-1}$ exists, the inverse function theorem asserts that there is an inverse function f_z^{-1} defined on a ball B around $f(z)$, with the property that, for all $w \in B$, $f_z^{-1}(f(z)) = z$, $f\left(f_z^{-1}(w)\right) = w$ and f_z^{-1} is differentiable. We use `Theorem 6.25` to estimate the size of this ball.

`Proposition 6.29 (Inverse Function Theorem).` Let $f : B(r, z_0) \to \mathcal{F}$ be analytic. Then

$$B\left(\frac{\alpha_0}{|f'(z_0)^{-1}|\gamma(f,z_0)}, f(z_0)\right) \subseteq f\left(B\left(\frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,z_0)}, z_0\right)\right)$$

and $f_{z_0}^{-1}$ exists and is differentiable on this ball. △

Proof. Let $c \in \mathcal{F}$ with $|c| \le \frac{\alpha_0}{|f'(z_0)^{-1}|\gamma(f,z_0)}$ and we define $f_c(z) := f(z) - c - f(z_0)$. Then, $f_c'(z) = f'(z)$, which means $f_c^{(k)}(z) = f^{(k)}(z)$. In particular, $\gamma(f_c, z_0) = \gamma(f, z_0)$ and

$$\beta(f_c,z_0) = \left|f_c'(z_0)^{-1}c\right| = \left|f'(z_0)^{-1}c\right| \le \left|f'(z_0)^{-1}\right||c| \le \left|f'(z_0)^{-1}\right|\frac{\alpha_0}{|f'(z_0)^{-1}|\gamma(f,z_0)} = \frac{\alpha_0}{\gamma(f,z_0)} = \frac{\alpha_0}{\gamma(f_c,z_0)}.$$

Thus $\alpha(f_c, z_0) < \alpha_0$ and by `Theorem 6.25` $N_{f_c}^k(z_0)$ converges to the unique root ζ_c of f_c in the open ball $B\left(\frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,z_0)}, z_0\right)$. Moreover, by $f_c(\zeta_c) = 0$, we obtain $f(\zeta_c) = c + f(z_0)$ and $f'(\zeta_c)^{-1}$ exists by `Lemma 6.5`. Thus

$$|f(\zeta_c) - f(z_0)| = |c + f(z_0) - f(z_0)| = |c| \le \frac{\alpha_0}{\left|f'(z_0)^{-1}\right|\gamma(f,z_0)},$$

which means for $\zeta_c \in B\left(\frac{1-\frac{\sqrt{2}}{2}}{\gamma(f,z_0)}, z_0\right)$, $f(\zeta_c) \in B\left(\frac{\alpha_0}{|f'(z_0)^{-1}|\gamma(f,z_0)}, f(z_0)\right)$. The proposition follows.■

More details about the n-dimensional generalization will be discussed in the next chapter.

7. n-Dimensional Generalization

This chapter, without indication, we follow [9].

A standing hypothesis in this section is that $f : \mathcal{E} \to \mathcal{F}$ is an analytic map from one Banach space to another, both $\mathcal{E}$ and $\mathcal{F}$ are real or both are complex. Main examples are the finite dimensional cases $\mathcal{E} = \mathbb{C}^n, \mathcal{F} = \mathbb{C}^n$, where $n \in \mathbb{N}$. The map f could be given by a system of polynomials.

7.1. Approximate Zeros

The derivative of $f : \mathcal{E} \to \mathcal{F}$ at $z \in \mathcal{E}$ is a linear map $Df(z) : \mathcal{E} \to \mathcal{F}$. If $Df(z)$ is invertible, Newton's method provides a new vector z' from z by $z' = z - Df(z)^{-1}f(z) = N_f(z)$.

Let $\beta(f,z)$ denote the norm of this Newton step $z' - z$, i.e., $\beta(f,z) = \left\|Df(z)^{-1}f(z)\right\|$. In case $Df(z)$ is not invertible, let $\beta(z,f) = \infty$. For a point $z_0 \in \mathcal{E}$, define inductively the sequence $z_n = z_{n-1} - Df\left(z_{n-1}\right)^{-1} f\left(z_{n-1}\right)$, if possible.

Definition 7.1. *Say that z_0 is an **approximate zero** of f if z_n is defined for all n and satisfies, for all $n \in \mathbb{N}$.*

$$\left\|z_n - z_{n-1}\right\| \le \left(\frac{1}{2}\right)^{2^{n-1}-1} \left\|z_1 - z_0\right\|.$$

Clearly, this implies that z_n is a Cauchy sequence with a limit, say $\zeta \in \mathcal{E}$. That $f(\zeta) = 0$ can be seen as follows. Since $z_{n+1} - z_n = -Df\left(z_n\right)^{-1} f\left(z_n\right)$, then

$$\left\|f\left(z_n\right)\right\| = \left\|Df\left(z_n\right)\left(z_{n+1} - z_n\right)\right\| \le \left\|Df\left(z_n\right)\right\| \left\|z_{n+1} - z_n\right\|.$$

Take the limit as $n \to \infty$ and f is continuous differentiable, so

$$\|f(\zeta)\| \le \|Df(\zeta)\| \lim_{n\to\infty} \left\|z_{n+1} - z_n\right\| = 0.$$

`Proposition 7.2.` If z_0 is an approximate zero and $z_n \to \zeta$ as $n \to \infty$, then

$$\|z_n - \zeta\| \le \left(\frac{1}{2}\right)^{2^{n-1}} \|z_1 - z_0\| \, K,$$

where $K = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2^k} \le \frac{7}{4}$. △

Proof. For the proof, sum both sides in the definition of approximate zero.

$$\|z_N - z_n\| = \|z_N - z_{N-1} + z_{N-1} - \cdots - z_{n+1} + z_{n+1} - z_n\| \le \sum_{k=n+1}^{N} \|z_k - z_{k-1}\| \le \|z_1 - z_0\| \sum_{k=n+1}^{N} \left(\frac{1}{2}\right)^{2^{k-1}-1}.$$

Let $N \to \infty$,

$$\|z_n - \zeta\| = \lim_{N\to\infty} \|z_N - z_n\| \le \|z_1 - z_0\| \sum_{k=n+1}^{\infty} \left(\frac{1}{2}\right)^{2^{k-1}-1} = \|z_1 - z_0\| \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2^{k+n}-1} \le \|z_1 - z_0\| \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2^k} \left(\frac{1}{2}\right)^{2^n-1}.$$

Then we are done. ■

Toward giving criteria for z to be an approximate zero define

$$\gamma(f,z) = \sup_{k>1} \left\| Df(z)^{-1} \frac{D^k f(z)}{k!} \right\|^{\frac{1}{k-1}}$$

or, if $Df(z)^{-1}$ or the supremum does not exist, let $\gamma(f,z) = \infty$. Here $D^k f(z)$ is the k-th derivative of f at z as a k-linear map. Define $\alpha(f,z) = \beta(f,z)\gamma(f,z)$.

`Theorem 7.3.` There is a naturally defined number α_0 approximately equal to .130707 such that if $\alpha(f,z) < \alpha_0$, then z is an approximate zero of f. △

Suppose now $f : \mathcal{E} \to \mathcal{F}$ is a map which is expressed as $f(z) = \sum_{k=0}^{d} a_k z^k$, for all $z \in \mathcal{E}, 0 < d \le \infty$. Here $\mathcal{E}$ and $\mathcal{F}$ are Banach spaces and a_k is a bounded symmetric k-linear map from $\mathcal{E} \times \cdots \times \mathcal{E}$ (k times) to $\mathcal{F}$. Thus $a_k z^k$ is a homogeneous polynomial of degree k. For $\mathcal{E} = \mathbb{C}^n$, this is the case in the usual sense, and in one variable a_k is the k-th coefficient (real or complex) of f. Then if d is finite, f is a polynomial. Define $\|f\| = \sup_{k\ge 0} \|a_k\|$, where $\|a_k\|$ is the norm of a_k as a bounded map. Define

$$\phi_d(r) = \sum_{i=0}^{d} r^i, \qquad \phi_d'(r) = \frac{\partial}{\partial r}\phi_d(r) = \sum_{i=1}^{d} i r^i$$

and $\phi(r) = \phi_\infty(r)$.

`Theorem 7.4.`

$$\gamma(f,z) < \left\|Df(z)^{-1}\right\| \|f\| \frac{\phi_d'(\|z\|)^2}{\phi_d \|z\|}.$$

Here, if $Df(z)$ is not invertible interpret $\left\|Df(z)^{-1}\right\| = \infty$ as usual. △

Thus combining `Theorem 7.3` and `Theorem 7.4`, we have a first derivative criterion (at z) for z to be an approximate zero.

`Corollary 7.5.` If

$$\left\|Df(z)^{-1}\right\| \|f\| \frac{\phi_d'(\|z\|)^2}{\phi_d(\|z\|)} \beta(f,z) < \alpha_0$$

then z is an approximate zero of f. △

Proof. By the definition of $\alpha(f,z)$, we finish the proof. ■

There is a reason to consider an alternate definition of approximate zero. This second notion is in terms of an actual zero ζ of $f : \mathcal{E} \to \mathcal{F}$.

Definition 7.6. *We say that z_0 is an **approximate zero of the second kind** of $f : \mathcal{E} \to \mathcal{F}$ provided there is some $\zeta \in \mathcal{E}$ with $f(\zeta) = 0$ and for $n \geq 1$,*

$$\|z_n - \zeta\| \leq \left(\frac{1}{2}\right)^{2^{n-1}} \|z_0 - \zeta\|,$$

where $z_n = z_{n-1} - Df\left(z_{n-1}\right)^{-1} f\left(z_{n-1}\right)$.

While the first definition of approximate zero deals with information at hand, and computable quantities, an approximate zero of the second kind can often be studied statistically or theoretically more handily.

`Theorem 7.7.` Suppose that $f : \mathcal{E} \to \mathcal{F}$ is analytic, $\zeta \in \mathcal{E}$, $f(\zeta) = 0$ and $z \in \mathcal{E}$ satisfies

$$\|z - \zeta\| < \left(\frac{3-\sqrt{7}}{2}\right) \frac{1}{\gamma(f,\zeta)}.$$

Then z is an approximate zero of the second kind. △

This result gives more evidence for the importance of the invariant $\gamma(f,\zeta)$.

7.2. Proofs of Preparations for the Main Theorems

Here we prove some lemmas and propositions from which the main results will follow easily. Suppose $\mathcal{E}$ and $\mathcal{F}$ are Banach spaces, both real or both complex.

Lemma 7.8. Let $A, B : \mathcal{E} \to \mathcal{F}$ be bounded linear maps with A invertible such that $\left\|A^{-1}B - I\right\| < c < 1$. Then B is invertible and $\left\|B^{-1}A\right\| < \frac{1}{1-c}$. △

Proof. Let $v = I - A^{-1}B$. Since $\|v\| = \left\|I - A^{-1}B\right\| < c < 1$, $\sum_{i=0}^{\infty} v^i$ exists with norm

$$\left\|\sum_{i=0}^{\infty} v^i\right\| = \left\|\frac{1}{I-v}\right\| = \frac{1}{\|I-v\|} < \frac{1}{1-c}.$$

The last step follows from $\|I - v\| \le |\|I\| - \|v\|| = 1 - \|v\| > 1 - c$. Note

$$(I-v)\sum_{i=0}^{n} v^i = (I-v)\frac{I-v^{n+1}}{I-v} = I - v^{n+1}.$$

By taking limits and $A^{-1}B = I - v$, we obtain $A^{-1}B \sum_{i=0}^{\infty} v^i = I$, i.e., $A^{-1}B$ is seen to be invertible with inverse $\sum_{i=0}^{\infty} v^i$. By the last equation, B can be written as the composition of invertible maps, $\sum_{i=0}^{\infty} v^i = B^{-1}A$. Therefore, $\left\|B^{-1}A\right\| = \left\|\sum_{i=0}^{\infty} v^i\right\| < \frac{1}{1-c}$. Then the proof is finished. ■

Lemma 7.9. Suppose $f : \mathcal{E} \to \mathcal{F}$ is analytic, $z', z \in \mathcal{E}$ such that $\|z' - z\|\gamma(f,z) < 1 - \frac{\sqrt{2}}{2}$. Then

(a) $Df(z')$ is invertible.

(b) $\left\|Df(z')^{-1}Df(z)\right\| < \frac{1}{2-\phi'(\|z'-z\|\gamma(f,z))}$

(c) $\gamma(f,z') \le \gamma(f,z)\frac{1}{2-\phi'(\|z'-z\|\gamma(f,z))}\left(\frac{1}{1-\|z'-z\|\gamma(f,z)}\right)^3$

Here $\phi'(r) = \frac{1}{(1-r)^2}$ could be replaced by ϕ'_d. △

Proof. Take a Taylor series expansion of Df about z, i.e., the map $\mathcal{E} \to L(\mathcal{E},\mathcal{F})$, as follows

$$Df(z') = \sum_{k=0}^{\infty} \frac{D^{k+1}f(z)}{k!}\left(z'-z\right)^k.$$

From this,

$$\begin{aligned} Df(z)^{-1}Df(z') &= \sum_{k=0}^{\infty} Df(z)^{-1}\frac{D^{k+1}f(z)}{k!}\left(z'-z\right)^k \\ &= \underbrace{Df(z)^{-1}\frac{Df(z)}{1}(z'-z)^0}_{I} + \sum_{k=1}^{\infty}(k+1)\,\frac{Df(z)^{-1}D^{k+1}f(z)}{(k+1)!}\left(z'-z\right)^k \end{aligned}$$

and

$$\begin{aligned} \left\|Df(z)^{-1}Df(z')-I\right\| &\le \sum_{k=1}^{\infty}(k+1)\underbrace{\left\|\frac{Df(z)^{-1}D^{k+1}f(z)}{(k+1)!}\right\|}_{\le\gamma(f,z)^k}\left\|z'-z\right\|^k \\ &\le \sum_{k=1}^{\infty}(k+1)\left(\gamma(f,z)\left\|z'-z\right\|\right)^k = \sum_{k=2}^{\infty}k\left(\gamma(f,z)\left\|z'-z\right\|\right)^{k-1} \\ &\overset{\texttt{Lemma 6.4 (b)}}{=\!=\!=} \sum_{k=1}^{\infty}k\left(\gamma(f,z)\left\|z'-z\right\|\right)^{k-1}-1 \\ &= \frac{1}{\left(1-\gamma(f,z)\left\|z'-z\right\|\right)^2}-1 = \phi'\left(\gamma(f,z)\left\|z'-z\right\|\right)-1. \end{aligned}$$

Observe, that since $\gamma(f,z)\left\|z'-z\right\| < 1-\frac{\sqrt{2}}{2} < 1$, all the series converge. Moreover, note that $\phi'(r) = \frac{1}{(1-r)^2}$, so that if $r < 1-\frac{\sqrt{2}}{2}$, then $\phi'(r)-1 < 1$. Thus `Lemma 7.8` applies to yield parts `(a)` and `(b)` of `Lemma 7.9` with $A := Df(z),\ B := Df(z'),\ c := \phi'\left(\gamma(f,z)\left\|z'-z\right\|\right)-1$. ■

The following simple formulae come up frequently.

$$\frac{\phi^{(l)}}{l!}(r) = \sum_{k=0}^{\infty}\binom{l+k}{k}r^k = \sum_{l=0}^{\infty}\frac{(k+l)!}{k!l!}r^l \overset{\texttt{Lemma 6.17}}{=\!=\!=} \frac{1}{(1-r)^{l+1}}.$$

Both quantities on the right are seen to be equal to the l-th derivative.

$$\frac{1}{2-\phi'(r)}\cdot\frac{1}{(1-r)^2} = \frac{2(1-r)^2-1}{(1-r)^2}\frac{(1-r)^2}{2(1-r)^2-1} = \frac{1}{2(1-r)^2-1} = \frac{1}{2r^2-2r+1} = \frac{1}{\psi(r)},$$

i.e.,

$$\frac{1}{2-\phi'(r)}\frac{1}{(1-r)^2} = \frac{1}{\psi(r)}. \tag{7.2.1}$$

where $\psi(r) = 2r^2-4r+1$.

Proof (Lemma 7.9 (c)). Let $\gamma_k(f,z) = \left\| Df(z)^{-1}\frac{D^k f(z)}{k!}\right\|^{\frac{1}{k-1}}$ and $\gamma(f,z) = \sup_{k>1} \gamma_k(f,z)$. Then by Taylor's theorem

$$\begin{aligned}
\gamma_k\left(f,z'\right)^{k-1} &= \left\| Df(z')^{-1}Df(z)\sum_{l=0}^{\infty}\frac{Df(z)^{-1}D^{k+l}f(z)(z'-z)^l}{l!k!}\right\| \\
&= \left\|Df(z')^{-1}Df(z)\right\|\sum_{l=0}^{\infty}\binom{k+l}{l}\left\|\frac{Df(z)^{-1}D^{k+l}f(z)(z'-z)^l}{(k+l)!}\right\| \\
&\le \left\|Df(z')^{-1}Df(z)\right\|\gamma(f,z)^{k+l-1}\sum_{l=0}^{\infty}\binom{k+l}{l}\left\|z'-z\right\|^l \\
&= \left\|Df(z')^{-1}Df(z)\right\|\gamma(f,z)^{k-1}\sum_{l=0}^{\infty}\binom{k+l}{l}\left(\left\|z'-z\right\|\gamma(f,z)\right)^l \\
&\le \left\|Df(z')^{-1}Df(z)\right\|\gamma(f,z)^{k-1}\left(\frac{1}{1-\left\|z'-z\right\|\gamma(f,z)}\right)^{k+1}
\end{aligned}$$

Now use `Lemma 7.9 (b)` and take the $(k-1)$ root to obtain

$$\gamma_k(f,z') \le \frac{\gamma(f,z)}{\left(2-\phi'\left(\gamma(f,z)\left\|z'-z\right\|\right)\right)^{\frac{1}{k-1}}}\left(\frac{1}{1-\left\|z'-z\right\|\gamma(f,z)}\right)^{\frac{k+1}{k-1}}.$$

The supremum is achieved at $k=2$, yielding the statement of `Lemma 7.9 (c)`. ■

`Lemma 7.10.` (a) Let $\alpha(f,z)<1$ and $z' = z - Df(z)^{-1}f(z)$. Then

$$\left\|Df(z)^{-1}f(z')\right\| \le \beta(f,z)\left(\frac{\alpha(f,z)}{1-\alpha(f,z)}\right).$$

(b) Let $z, z' \in \mathcal{E}$ with $f(z)=0$ and $\left\|z'-z\right\|\gamma(f,z)<1$. Then

$$\left\|Df(z)^{-1}f(z')\right\| \le \frac{\left\|z'-z\right\|}{1-\left\|z'-z\right\|\gamma(f,z)}$$ △

Proof. We first prove part `(a)`. The Taylor series yields

$$Df(z)^{-1}f(z') = \sum_{k=0}^{\infty} Df(z)^{-1}\frac{D^k f(z)(z'-z)^k}{k!}.$$

The first two terms drop out. Since $\beta(f,z) = \|z'-z\|$,

$$\begin{aligned}\left\|Df(z)^{-1}f(z')\right\| &\le \beta(f,z)\sum_{k=2}^{\infty}(\gamma(f,z)\beta(f,z))^{k-1} = \beta(f,z)\sum_{k=2}^{\infty}\alpha(f,z)^{k-1}\\ &= \beta(f,z)\sum_{k=1}^{\infty}\alpha(f,z)^{k} = \beta(f,z)\left(\frac{1}{1-\alpha(f,z)}-1\right) = \beta(f,z)\frac{\alpha(f,z)}{1-\alpha(f,z)}\end{aligned}$$

For part `(b)`, we start as above and now the first term is zero since $f(z)=0$. We have

$$\begin{aligned}Df(z)^{-1}f(z') &= \underbrace{\frac{Df(z)^{-1}f(z)(z'-z)}{0!}}_{0} + \underbrace{\frac{Df(z)^{-1}Df(z)(z'-z)}{1!}}_{z'-z} + \sum_{k=2}^{\infty}Df(z)^{-1}\frac{D^kf(z)(z'-z)^k}{k!}\\ &= (z'-z)\left(1+\sum_{k=2}^{\infty}Df(z)^{-1}\frac{D^kf(z)(z'-z)^{k-1}}{k!}\right) \le (z'-z)\left(1+\sum_{k=2}^{\infty}\gamma_k(f,z)^{k-1}\left(z'-z\right)^{k-1}\right)\\ &= (z'-z)\sum_{k=0}^{\infty}\gamma_k(f,z)^k\left(z'-z\right)^k \le (z'-z)\sum_{k=0}^{\infty}\gamma(f,z)^k\left(z'-z\right)^k = (z'-z)\frac{1}{1-\gamma(f,z)(z'-z)},\end{aligned}$$

i.e.,

$$\left\|Df(z)^{-1}f(z')\right\| \le \left\|z'-z\right\|\frac{1}{1-\gamma(f,z)\left\|z'-z\right\|}.$$

This finishes the proof of `Lemma 7.10`. ■

`Proposition 7.11.` Let f be an analytic map from the Banach space $\mathcal{E}$ to $\mathcal{F}$ as usual.

(a) if $\alpha(f,z) < 1-\frac{\sqrt{2}}{2}$, then

$$\beta(f,z') \le \beta(f,z)\left(\frac{\alpha(f,z)}{1-\alpha(f,z)}\right)\left(\frac{1}{2-\phi'(\alpha(f,z))}\right).$$

(b) if $f(z)=0$ and $\gamma(f,z)\left\|z'-z\right\| < 1-\frac{\sqrt{2}}{2}$ then

$$\beta(f,z') \le \left\|z'-z\right\|\left(\frac{1}{2-\phi'\left(\gamma(f,z)\left\|z'-z\right\|\right)}\right)\left(\frac{1}{1-\gamma(f,z)\left\|z'-z\right\|}\right).$$

△

Proof. Write

$$\begin{aligned}\beta(f,z') &= \left\|Df(z')^{-1}f(z')\right\| = \left\|Df(z')^{-1}Df(z)Df(z)^{-1}f(z')\right\| \\ &\le \left\|Df(z')^{-1}Df(z)\right\| \left\|Df(z)^{-1}f(z')\right\| \le \left(\frac{1}{2-\phi'(\alpha(f,z))}\right)\beta(f,z)\left(\frac{\alpha(f,z)}{1-\alpha(f,z)}\right).\end{aligned}$$

The last step uses `Lemma 7.9 (b)` and `Lemma 7.10 (a)`.

Similarly for the second part of the proposition. If $f(z)=0$, use `Lemma 7.9 (b)` and `Lemma 7.10 (b)` as follows,

$$\beta(f,z') \le \left\|Df(z')^{-1}Df(z)\right\| \left\|Df(z)^{-1}f(z')\right\| \le \left(\frac{1}{2-\phi'\left(\gamma(f,z)\left\|z'-z\right\|\right)}\right)\left(\frac{\left\|z'-z\right\|}{1-\gamma(f,z)\left\|z'-z\right\|}\right).$$

This proves `Proposition 7.11`. ∎

`Proposition 7.12.` Recalling $\psi(r) = 2r^2-4r+1$ and using the notation of `Proposition 7.11`,

(a) if $\alpha < 1-\frac{\sqrt{2}}{2}$,

$$\alpha(f,z') \le \left(\frac{\alpha(f,z)}{\psi(\alpha(f,z))}\right)^2.$$

(b) if $f(\zeta)=0$ and $\gamma(f,\zeta)\left\|z'-\zeta\right\| < 1-\frac{\sqrt{2}}{2}$,

$$\alpha(f,z') \le \frac{\gamma(f,\zeta)\left\|z'-\zeta\right\|}{\psi\left(\gamma(f,\zeta)\left\|z'-\zeta\right\|\right)^2}.$$ △

Proof. For the proof of `(a)`, not that $\alpha(f,z') = \beta(f,z')\gamma(f,z')$. Use `Lemma 7.9 (c)` and `Proposition 7.11 (a)` to obtain

$$\begin{aligned}\alpha(f,z') &\le \beta(f,z)\left(\frac{\alpha(f,z)}{1-\alpha(f,z)}\right)\left(\frac{1}{2-\phi'(\alpha(f,z))}\right)\gamma(f,z)\frac{1}{2-\phi'\left(\left\|z'-z\right\|\gamma(f,z)\right)}\left(\frac{1}{1-\left\|z'-z\right\|\gamma(f,z)}\right)^3 \\ &= \beta(f,z)\gamma(f,z)\alpha(f,z)\left(\left(\frac{1}{1-\alpha(f,z)}\right)^2\left(\frac{1}{2-\phi'(\alpha(f,z))}\right)\right)^2,\end{aligned}$$

by (7.2.1),

$$\alpha(f,z') \le (\beta(f,z)\gamma(f,z))\alpha(f,z)\left(\frac{1}{\psi(\alpha(f,z))}\right)^2 = \left(\frac{\alpha(f,z)}{\psi(\alpha(f,z))}\right)^2.$$

The proof of Proposition 7.12 (b) is similar by using Lemma 7.9 (c) and Proposition 7.11 (b).

$$\begin{aligned}\alpha(f,z') &= \beta(f,z')\gamma(f,z')\\ &\le \frac{\|z'-\zeta\|}{2-\phi'(\gamma(f,\zeta)\|z'-\zeta\|)}\left(\frac{1}{1-\gamma(f,\zeta)\|z'-\zeta\|}\right)\frac{\gamma(f,\zeta)}{2-\phi'(\gamma(f,\zeta)\|z'-\zeta\|)}\left(\frac{1}{1-\gamma(f,\zeta)\|z'-\zeta\|}\right)^3\\ &= \|z'-\zeta\|\gamma(f,\zeta)\left(\left(\frac{1}{1-\gamma(f,\zeta)\|z'-\zeta\|}\right)^2\left(\frac{1}{2-\phi'(\gamma(f,\zeta)\|z'-\zeta\|)}\right)\right)^2 = \frac{\|z'-\zeta\|\gamma(f,\zeta)}{\psi(\gamma(f,\zeta)\|z'-\zeta\|)^2}.\end{aligned}$$

Then the claim follows. ■

Proposition 7.13. Suppose that $A > 0$, $a_i > 0$, $i \in \mathbb{N}_0$ satisfy, for all $i \in \mathbb{N}_0$, $a_{i+1} \le Aa_i^2$. Then, for all $k \in \mathbb{N}_0$,

$$a_k \le (Aa_0)^{2^k-1} a_0. \qquad \triangle$$

Proof. By mathematical induction, we first prove for Base Case ($k = 0$): $a_0 \le (Aa_0)^{2^0-1} a_0 = a_0$.

Inductive Hypothesis: for all $k \ge 0$,

$$a_k \le (Aa_0)^{2^k-1} a_0.$$

Then for $k + 1$ we conclude that,

$$a_{k+1} \le Aa_k^2 \le A\left((Aa_0)^{2^k-1} a_0\right)^2 = A\left((Aa_0)^{2^{k+1}-2} a_0^2\right) = A^{2^{k+1}-1} a_0^{2^{k+1}-1} a_0 = (Aa_0)^{2^{k+1}-1} a_0 \quad ■$$

We end this section with a short discussion of sharpness. Lemma 7.9 (b) can be seen to be sharp by taking $z = 0$ and for $0 < a < 1 - \frac{\sqrt{2}}{2}$,

$$f(z) = 2z - \phi(z) + 1 - a.$$

Then, for $z' = a$,

- $f(z) = 2z - \frac{1}{1-z} + 1 - a$, $f(0) = 0 - 1 + 1 - a = -a$;

- $Df(z) = 2 - \frac{1}{(1-z)^2}$, $Df(0) = 2 - 1 = 1$.

It is easy to see that for $k \geq 2$,

$$D^k f(z) = \frac{k!}{(1-z)^{k+1}}.$$

Therefore,

$$\gamma(f, 0) = \sup_{k>1} \left\| Df(z)^{-1} \frac{D^k f(z)}{k!} \right\|^{\frac{1}{k-1}} \Bigg|_{z=0} = \sup_{k>1} \left\| Df(z)^{-1} \frac{1}{(1-z)^{k+1}} \right\|^{\frac{1}{k-1}} \Bigg|_{z=0} = 1$$

and $\|z' - z\| \gamma(f, z) = a$ at $z = 0$. Thus,

$$\left\| Df(z')^{-1} Df(z) \right\| \Big|_{z=0, z'=a} = \frac{1}{2 - \frac{1}{(1-a)^2}} = \frac{1}{2 - \phi'(a)}.$$

The same example may be used to see that `Lemma 7.9 (c)` is sharp. One only needs to make the easy computation that

$$\begin{aligned}
\gamma(f, a) &= \sup_{k>1} \left\| Df(z)^{-1} \frac{D^k f(z)}{k!} \right\|^{\frac{1}{k-1}} \Bigg|_{z=a} \\
&= \sup_{k>1} \left\| \frac{1}{\left(2 - \frac{1}{(1-z)^2}\right)(1-z)^{k+1}} \right\|^{\frac{1}{k-1}} \Bigg|_{z=a} = \sup_{k>1} \left\| \left(\frac{1}{2 - \phi'(z)} \right)^{\frac{1}{k-1}} \left(\frac{1}{1-z} \right)^{\frac{k+1}{k-1}} \right\| \Bigg|_{z=a}.
\end{aligned}$$

The supremum is achieved at $k = 2$, so

$$\gamma(f, a) = \left(\frac{1}{2 - \phi'(z)} \right) \left(\frac{1}{1-z} \right)^3 \Bigg|_{z=a} = \left(\frac{1}{1-a} \right)^3 \left(\frac{1}{2 - \phi'(a)} \right).$$

Again, the same example shows that `Lemma 7.10 (a)` is sharp, just observing that

- $\alpha(f, 0) = \beta(f, 0)\gamma(f, 0) = \|z' - z\| \gamma(f, 0) = a$;
- $f(a) = 2a - \frac{1}{1-a} + 1 - a = 1 - \frac{1}{1-a} + a = \frac{1-a-1+a-a^2}{1-a} = \frac{a^2}{1-a}$.

Thus,

- $Df(z)^{-1} f(z') \big|_{z=0, z'=a} = f(a) = \frac{a^2}{1-a}$;

- $\beta(f,z)\left(\frac{\alpha(f,z)}{1-\alpha(f,z)}\right)\Big|_{z=0,z'=a} = a \cdot \frac{a}{1-a} = \frac{a^2}{1-a}$.

One can see that `Lemma 7.10 (b)` is sharp with the example. Let $z = 0$ and $0 < z' < 1$,

$$g(z) = \phi(z) - 1.$$

Similarly,

- $D^k g(z) = \frac{k!}{(1-z)^{k+1}}$, $Dg(0) = \frac{1}{(1-z)^2}\Big|_{z=0} = 1$;
- $\gamma(g,0) = \left(\frac{1}{2-\phi'(z)}\right)\left(\frac{1}{1-z}\right)^3\Big|_{z=0} = \frac{1}{2-\phi'(0)}\left(\frac{1}{1-0}\right)^3 = 1$.

Then

- $\left\|Dg(z)^{-1}g(z')\right\|\Big|_{z=0,0<z'<1} = \|g(z')\| = \left\|\frac{1}{1-z'} - 1\right\| = \left\|\frac{z'}{1-z'}\right\| = \frac{z'}{1-z'}$;
- $\frac{\|z'-z\|}{1-\|z'-z\|\gamma(g,z)}\Big|_{z=0,0<z'<1} = \frac{\|z'\|}{1-\|z'\|} = \frac{z'}{1-z'}$.

`Proposition 7.11 (a)` is sharp with the example of `Lemma 7.9`. The same applies to `Proposition 7.12 (a)`. We have for `Proposition 7.11 (a)`,

- $\beta(f,z')|_{z'=a} = \left\|Df(z')^{-1}f(z')\right\|\Big|_{z'=a} = \left(\frac{1}{2-\phi'(a)}\right)\left(\frac{a^2}{1-a}\right)$;
- $\beta(f,z)\left(\frac{\alpha(f,z)}{1-\alpha(f,z)}\right)\left(\frac{1}{2-\phi'(\alpha(f,z))}\right)\Big|_{z=0,z'=a} = a \cdot \frac{a}{1-a} \cdot \frac{1}{2-\phi'(a)} = \left(\frac{1}{2-\phi'(a)}\right)\left(\frac{a^2}{1-a}\right)$

and for `Proposition 7.12 (a)`

$$\begin{aligned}\alpha(f,z')\big|_{z'=a} = \alpha(f,a) = \beta(f,a)\gamma(f,a) &\le \left(\frac{1}{2-\phi'(a)}\right)\left(\frac{a^2}{1-a}\right)\left(\frac{1}{1-a}\right)^3\left(\frac{1}{2-\phi'(a)}\right)\\ &= \left(\frac{a}{(1-a)^2}\left(\frac{1}{2-\phi'(a)}\right)\right)^2 = \left(\frac{a}{\psi(a)}\right)^2 = \left(\frac{\alpha(f,z')}{\psi(\alpha(f,z'))}\right)^2\Bigg|_{z'=a}\end{aligned}$$

7.3. The Proofs of Main Results

In this final section we finish the proofs of our main results. Toward the proof of `Theorem 7.3`, consider our polynomial $\psi(r) = 2r^2 - 4r + 1$ and the function $\left(\frac{\alpha(f,z)}{\psi(\alpha(f,z))}\right)^2$ of `Proposition 7.11 (a)`.

In the range of concern to us, $0 \le r \le 1 - \frac{\sqrt{2}}{2}$, $\psi(r)$ is a parabola decreasing from 1 to 0 as r

goes from 0 to $1-\frac{\sqrt{2}}{2}$, which can be seen in `Figure 6.2`. Therefore, $\frac{r}{\psi(r)^2}$ increases form 0 to ∞ as r goes from 0 to $1-\frac{\sqrt{2}}{2}$.

Let α_0 be the unique r such that $\frac{r}{\psi(r)^2}=\frac{1}{2}$. Thus α_0 is a zero of the real quadric polynomial $\psi(r)^2-2r$. Using Newton's method one calculates approximately, $\alpha_0 = .130707$.

With this discussion, `Theorem 7.3` is a consequence of the following proposition where $a=\frac{1}{2}$.

`Proposition 7.14`. Let $f:\mathcal{E}\to\mathcal{F}$ be analytic, $z=z_0\in\mathcal{E}$ and $\frac{\alpha(f,z)}{(\psi(\alpha(f,z)))^2}=a<1$. Let $z_k = z_{k-1}-Df\left(z_{k-1}\right)^{-1}f\left(z_{k-1}\right)$ for $k=1,2,\dots$, then

(a) z_k is defined for all k.

Let $\alpha_k=\alpha\left(f,z_k\right),\ \psi_k=\psi\left(\alpha(f,z_k)\right)$, $k=1,2,\dots$,

(b) $\alpha_k\le a^{2^k-1}\alpha\left(f,z_0\right)$, $k=1,2,\dots$,

(c) $\|z_k-z_{k-1}\|\le a^{2^{k-1}-1}\|z_1-z_0\|$, for all k. △

Proof. Note that `(a)` follows from `(b)` and that `(b)` is a consequence of `Proposition 7.12 (a)` and `Proposition 7.13`,

$$\alpha_{k+1}\le\left(\frac{\alpha_k}{\psi\left(\alpha_k\right)}\right)^2=\frac{\alpha_k}{\psi\left(\alpha_k\right)^2}\cdot\alpha_k<a\alpha_k,$$

i.e., $\alpha_k\xrightarrow{k\to\infty}0$. Therefore, the condition in `Proposition 7.13` is satisfied. Thus,

$$\alpha_k\le\left(a\alpha\left(f,z_0\right)\right)^{2^k-1}\alpha\left(f,z_0\right)=a^{2^k-1}\cdot\alpha\left(f,z_0\right)^{2^k}<a^{2^k-1}\cdot\alpha\left(f,z_0\right).$$

It remains to check `(c)`. The case $k=1$ is trivial so assume $k>1$. We may write using `Proposition 7.11 (a)` and our relation between ϕ' and ψ,

$$\begin{aligned}\|z_k-z_{k-1}\|&\le\|z_{k-1}-z_{k-2}\|\left(\frac{\alpha_{k-2}}{1-\alpha_{k-2}}\right)\left(\frac{1}{2-\phi'\left(\alpha_{k-2}\right)}\right)\\&=\|z_{k-1}-z_{k-2}\|\,\alpha_{k-2}\left(1-\alpha_{k-2}\right)\underbrace{\left(\frac{1}{\left(1-\alpha_{k-2}\right)^2}\right)\left(\frac{1}{2-\phi'\left(\alpha_{k-2}\right)}\right)}_{\frac{1}{\psi\left(\alpha_{k-2}\right)}}=\|z_{k-1}-z_{k-2}\|\,\frac{\alpha_{k-2}\left(1-\alpha_{k-2}\right)}{\psi\left(\alpha_{k-2}\right)}.\end{aligned}$$

Now use part (b) and induction on this inequality to obtain

$$\|z_k - z_{k-1}\| \le \underbrace{a^{2^{k-2}-1}}_{<1} \|z_1 - z_0\| \, a^{2^{k-1}-1} \alpha(f, z_0) \left(\frac{1-\alpha_{k-2}}{\psi_{k-2}}\right) \le a^{2^{k-1}-1} \|z_1 - z_0\| \frac{\alpha(f, z_0)}{\psi_{k-2}}.$$

However,

$$\frac{\alpha(f, z_0)}{\psi_{k-2}} \le \frac{\alpha(f, z_0)}{\psi(\alpha(f, z_0))} = \frac{\alpha(f, z_0)}{\psi(\alpha(f, z_0))^2} \cdot \psi(\alpha(f, z_0)) = a\psi(\alpha(f, z_0)) < 1,$$

thus

$$\|z_k - z_{k-1}\| \le a^{2^{k-1}-1} \|z_1 - z_0\| \frac{\alpha(f, z_0)}{\psi_{k-2}} \le a^{2^{k-1}-1} \|z_1 - z_0\|.$$

■

Remark 7.15. Theorem 7.3 and most of the lemmas and propositions of 7.2 Proofs of Preparations for the Main Theorems can be slightly sharpened in case that f is a polynomial map $\mathcal{E} \to \mathcal{F}$ of Banach spaces of degree $d < \infty$. Replace $\phi(r)$ by $\phi_d(r) = \sum_{i=0}^{d} r^i$ everywhere in the proofs and conclusions. ◇

For example, going through proofs this way yields the following generalization of Proposition 7.12 (a).

Proposition 7.16. If $f : \mathcal{E} \to \mathcal{F}$ has degree d, then if $\phi_d'(\alpha(f, z)) < 2$,

$$\alpha(f, z') < \alpha(f, z)^2 \frac{\phi_{d-2}(\alpha(f, z))}{\phi_d(\alpha(f, z))} \left(\frac{\phi_d'(\alpha(f, z))}{2 - \phi_d'(\alpha(f, z))}\right)^2$$

If $d = \infty$, this reverts to Proposition 7.12 (a). △

Proof. First of all, we prove Lemma 7.10 (a) and Lemma 7.9 (b) for ϕ_d.

$$\begin{aligned}\left\|Df(z)^{-1} f(z')\right\| &\le \beta(f, z) \sum_{k=2}^{d} \left\|\frac{Df(z)^{-1} D^k f(z)}{k!}\right\| \beta(f, z)^{k-1} = \beta(f, z) \sum_{k=2}^{d} (\gamma(f, z)\beta(f, z))^{k-1} \\ &= \beta(f, z) \sum_{k=1}^{d-1} \alpha(f, z)^k = \beta(f, z) \frac{\alpha(f, z)\left(1 - \alpha(f, z)^{d-1}\right)}{1 - \alpha} = \beta(f, z)\alpha(f, z)\phi_{d-2}(\alpha(f, z)).\end{aligned}$$

For `Lemma 7.9 (b)`,

$$Df(z)^{-1}Df(z') = \sum_{k=0}^{\infty} \frac{Df(z)^{-1}D^{k+1}f(z)}{k!}\left(z'-z\right)^k$$
$$= I + \sum_{k=1}^{\infty}(k+1)\frac{Df(z)^{-1}D^{k+1}f(z)}{(k+1)!}\left(z'-z\right)^k = I + \sum_{k=1}^{d-1}(k+1)\frac{Df(z)^{-1}D^{k+1}f(z)}{(k+1)!}\left(z'-z\right)^k$$

Therefore,

$$\left\|Df(z)^{-1}Df(z') - I\right\| = \sum_{k=1}^{d-1}(k+1)\underbrace{\left\|\frac{Df(z)^{-1}D^{k+1}f(z)}{(k+1)!}\right\|}_{\leq\gamma(f,z)^k}\beta(f,z)^k \leq \sum_{k=1}^{d-1}(k+1)\alpha(f,z)^k = \phi_d'(\alpha(f,z)) - 1$$

We also need `Lemma 7.8` for ϕ_d, as before we define $v = I - A^{-1}B$. Then

$$(1-v)\sum_{i=0}^{d} v^i = (1-v)\frac{1-v^{d+1}}{1-v} = 1 - v^{d+1},$$

which implies $(1-v)\sum_{i=0}^{d+1} v^i = 1$, i.e., $A^{-1}B\sum_{i=0}^{d+1} v^i = 1$ and $B^{-1}A = \sum_{i=0}^{d+1} v^i = \frac{1-v^{d+2}}{1-v}$. Thus

$$\left\|B^{-1}A\right\| = \left\|\frac{1-v^{d+2}}{1-v}\right\| = \frac{\left\|1-v^{d+2}\right\|}{\left\|1-v\right\|} < \frac{1+c^{d+2}}{1-c} < \frac{1+c}{1-c}.$$

We let $c = \phi_d'(\alpha(f,z)) - 1$, thus

$$\left\|Df(z')^{-1}Df(z)\right\| < \frac{1+\phi_d'(\alpha(f,z)) - 1}{1-\left(\phi_d'(\alpha(f,z)) - 1\right)} = \frac{\phi_d'(\alpha(f,z))}{2-\phi_d'(\alpha(f,z))}. \tag{7.3.1}$$

Finally, we obtain

$$\beta(f,z') \leq \left\|Df(z')^{-1}Df(z)\right\|\left\|Df(z)^{-1}f(z')\right\| \leq \beta(f,z)\alpha(f,z)\phi_{d-2}(\alpha(f,z))\frac{\phi_d'(\alpha(f,z))}{2-\phi_d'(\alpha(f,z))}.$$

Analogously,

$$\gamma(f,z') \leq \gamma(f,z)\frac{1}{\phi_d(\alpha(f,z))}\frac{\phi_d'(\alpha(f,z))}{2-\phi_d'(\alpha(f,z))}. \tag{7.3.2}$$

In order to prove this claim, we first prove the following two statements.

(7.3.3) $\phi_d^{(k)}(r) = \left(1-\alpha^{d+1}\right)\phi^{(k)}(r) - \sum_{i=1}^{k}\binom{k}{i}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(k-i)}(r)$;

(7.3.4) $\frac{\phi_d^{(k)}(r)}{k!} \le \frac{1}{\phi_d(r)}$.

By $\phi_d(r) = \sum_{i=0}^{d} r^i = \frac{1-r^{d-1}}{1-r} = \left(1-r^{d+1}\right)\phi(r)$ and mathematical induction, we obtain (7.3.3).
Base Case ($k=1$): For $k=0$ this is trivial. Let $k=1$, then

$$\phi_d'(r) = \left(1-r^{d+1}\right)\phi'(r) - (d+1)r^d\phi(r) = \left(1-r^{d+1}\right)\phi'(r) - \sum_{i=1}^{1}\binom{1}{i}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(1-i)}(r).$$

Inductive Hypothesis: For $k>1$ assume by induction that

$$\phi_d^{(k)}(r) = \left(1-\alpha^{d+1}\right)\phi^{(k)}(r) - \sum_{i=1}^{k}\binom{k}{i}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(k-i)}(r).$$

Therefore, $\phi_d^{(k+1)}(r)$ can be described as,

$$\begin{aligned}
&\left(1-r^{d+1}\right)\phi^{(k+1)}(r) - (d+1)r^d\phi^{(k)}(r) - \sum_{i=1}^{k}\binom{k}{i}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(k-i+1)}(r)\\
&-\sum_{i=1}^{k}\binom{k}{i}(d+1)d\cdots(d-i+2)(d-i+1)r^{d-i}\phi^{k-i}(r)\\
&= \left(1-r^{d+1}\right)\phi^{(k+1)}(r) - (d+1)r^d\phi^{(k)}(r) - \binom{k}{1}(d+1)r^d\phi^{(k)}(r) - \sum_{i=2}^{k}\binom{k}{i}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(k+1-i)}(r)\\
&-\sum_{i=2}^{k}\binom{k}{i-1}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(k+1-i)}(r) - \binom{k}{k}(d+1)d\cdots(d-k+1)r^{d-k}\phi(r)\\
&= \left(1-r^{d+1}\right)\phi^{(k+1)}(r) - \sum_{i=1}^{k+1}\binom{k+1}{i}(d+1)d\cdots(d-i+2)r^{d-i+1}\phi^{(k+1-i)}(r),
\end{aligned}$$

the last step follows from $\binom{k+1}{i} = \binom{k}{i-1} + \binom{k}{i}$.

Thus, $\phi_d^{(k)}(r) \le \left(1-r^{d+1}\right)\phi^k(r) = \left(1-r^{d+1}\right)\frac{k!}{(1-r)^{k+1}}$ for $r<1$ and

$$\frac{\phi_d^{(k)}(r)}{k!} \le \frac{(1-r)^{k+1}}{(1-r^{d+1})} \le \frac{1-r}{(1-r^{d+1})} = \frac{1}{\phi_d(r)}.$$

Using the same notation in `Lemma 7.9 (c)`,

$$\begin{aligned}
\gamma_k\left(f,z'\right)^{k-1} &= \left\| Df(z')^{-1}Df(z)\sum_{l=0}^{\infty}\frac{Df(z)^{-1}D^{k+l}f(z)(z'-z)^l}{l!k!}\right\| \\
&= \left\| Df(z')^{-1}Df(z)\right\| \sum_{l=0}^{d-k}\binom{k+l}{l}\left\|\frac{Df(z)^{-1}D^{k+l}f(z)(z'-z)^l}{(k+l)!}\right\| \\
&\le \left\| Df(z')^{-1}Df(z)\right\| \gamma(f,z)^{k+l-1}\sum_{l=0}^{d-k}\binom{k+l}{l}\left\|z'-z\right\|^l \\
&= \left\| Df(z')^{-1}Df(z)\right\| \gamma(f,z)^{k-1}\sum_{l=0}^{d-k}\binom{k+l}{l}\left(\left\|z'-z\right\|\gamma(f,z)\right)^l \\
&= \left\| Df(z')^{-1}Df(z)\right\| \gamma(f,z)^{k-1}\left(\frac{\phi_d^{(k)}\left(\alpha(f,z)\right)}{k!}\right).
\end{aligned}$$

Now use `(7.3.1)`, `(7.3.3)` and `(7.3.4)` and take the $(k-1)$ root to obtain

$$\gamma_k\left(f,z'\right) \le \left(\frac{\phi_d'\left(\alpha(f,z)\right)}{2-\phi_d'\left(\alpha(f,z)\right)}\right)^{\frac{1}{k-1}}\gamma(f,z)\left(\frac{1}{\phi_d(\alpha(f,z))}\right)^{\frac{1}{k-1}}.$$

The supremum is achieved at $k=2$, yielding `(7.3.2)`. Then the claim follows. ■

Example 7.17. The following shows that α_0 must be less than or equal $3-2\sqrt{2}$ in Theorem 7.3. Let $f_a:\mathbb{C}\to\mathbb{C}$ be

$$f_a(z) = 2z - \frac{z}{1-z} - a, \qquad a>0.$$

Then

- $D^k f_a(z) = -\frac{k!}{(1-z)^{k+1}}$;
- $\gamma\left(f_a,0\right) = \sup_{k\ge 2}\left\|Df_a(z)^{-1}\frac{D^k f_a(z)}{k!}\right\|^{\frac{1}{k-1}}\Big|_{z=0} = \sup_{k\ge 2}\left\|1\cdot\left(-\frac{k!}{k!}\right)\right\|^{\frac{1}{k-1}} = 1$;
- $\beta\left(f_a,0\right) = \left\|Df_a(z)^{-1}f_a(z)\right\|\Big|_{z=0} = \|1\cdot(-a)\| = a$;
- $\alpha\left(f_a,0\right) = \beta\left(0,f_a\right)\gamma\left(f_a,0\right) = a$

and $f_a(\zeta)=0$ where

$$\zeta = \frac{(1+a)\pm\sqrt{(1+a)^2-8a}}{4}.$$

If $\alpha = a > 3 - 2\sqrt{2}$, i.e., $(1+a)^2 - 8a < 0$, these roots are not real, so that Newton's method for solving $f_a(\zeta) = 0$, starting at $z_0 = 0$ will never converge.

Toward the proof of `Theorem 7.7`, we use the following proposition.

`Proposition 7.18.` Let $f : \mathcal{E} \to \mathcal{F}, \zeta, z \in \mathcal{E}$ satisfy $f(\zeta) = 0$ and $\gamma(f,\zeta)\|z-\zeta\| < 1 - \frac{\sqrt{2}}{2}$. Then

$$\left\|Df(z)^{-1}f(z) - (z-\zeta)\right\| \le \frac{\gamma(f,\zeta)\|z-\zeta\|^2}{\psi(\gamma(f,\zeta)\|z-\zeta\|)}. \qquad \triangle$$

Note that this proposition gives an estimate on how well the Newton vector $-Df(z)^{-1}f(z)$ approximates $\zeta - z$, the exact vector from z to ζ.

Proof. For the proof we consider the two Taylor series, $f(z) = \sum_{k=0}^{\infty} \frac{D^k f(\zeta)}{k!}(z-\zeta)^k = \sum_{k=1}^{\infty} \frac{D^k f(\zeta)}{k!}(z-\zeta)^k$ and $Df(z) = \sum_{k=0}^{\infty} \frac{D^{k+1} f(\zeta)}{k!}(z-\zeta)^k$. Now apply the second to $(z-\zeta)$ and subtract it from the first to obtain,

$$\begin{aligned} f(z) - Df(z)(z-\zeta) &= \sum_{k=1}^{\infty} \frac{D^k f(\zeta)}{k!}(z-\zeta)^k - \sum_{k=0}^{\infty} \frac{D^{k+1} f(\zeta)}{k!}(z-\zeta)^{k+1} \\ &= \sum_{k=1}^{\infty} \frac{D^k f(\zeta)}{k!}(z-\zeta)^k - \sum_{k=1}^{\infty} \frac{D^k f(\zeta)}{(k-1)!}(z-\zeta)^k \\ &= \sum_{k=1}^{\infty} \underbrace{\left(\frac{1}{k!} - \frac{1}{(k-1)!}\right)}_{(1-k)\frac{1}{k!}} D^k f(z)(z-\zeta)^k = -\sum_{k=1}^{\infty}(k-1)\frac{D^k f(\zeta)}{k!}(z-\zeta)^k. \end{aligned}$$

Then multiple both sides with $Df(z)^{-1}$,

$$Df(z)^{-1}f(z) - (z-\zeta) = -Df(z)^{-1}Df(\zeta)\sum_{k=1}^{\infty}(k-1)\frac{Df(\zeta)^{-1}D^k f(\zeta)(z-\zeta)^k}{k!}.$$

Take norms and apply `Lemma 7.9 (b)` to obtain,

$$\begin{aligned}
\left\|Df(z)^{-1}f(z)-(z-\zeta)\right\| &= \left\|Df(z)^{-1}Df(\zeta)\right\| \sum_{k=1}^{\infty}(k-1)\left\|\frac{Df(\zeta)^{-1}D^k f(\zeta)(z-\zeta)^k}{k!}\right\| \\
&\le \left(\frac{1}{2-\phi'(\gamma(f,\zeta)\left\|z-\zeta\right\|)}\right)\sum_{k=2}^{\infty}(k-1)\left(\gamma(f,\zeta)\left\|z-\zeta\right\|\right)^{k-1}\left\|z-\zeta\right\| \\
&= \left(\frac{1}{2-\phi'(\gamma(f,\zeta)\left\|z-\zeta\right\|)}\right)\underbrace{\sum_{k=1}^{\infty}k\left(\gamma(f,\zeta)\left\|z-\zeta\right\|\right)^{k}}_{\gamma(f,\zeta)\|z-\zeta\|\sum_{k=1}^{\infty}k(\gamma(f,\zeta)\|z-\zeta\|)^{k-1}}\left\|z-\zeta\right\| \\
&= \left(\frac{1}{2-\phi'(\gamma(f,\zeta)\left\|z-\zeta\right\|)}\right)\gamma(f,\zeta)\left\|z-\zeta\right\|\left(\frac{1}{\gamma(f,\zeta)\left\|z-\zeta\right\|}\right)^2\left\|z-\zeta\right\| \\
&= \frac{\gamma(f,\zeta)\left\|z-\zeta\right\|^2}{\psi\left(\gamma(f,\zeta)\left\|z-\zeta\right\|\right)},
\end{aligned}$$

which proving the proposition. ■

`Corollary 7.19.` Suppose f,ζ,z are as in the proposition. Let $A=\frac{\gamma(f,\zeta)\|z-\zeta\|}{\psi(\gamma(f,\zeta)\|z-\zeta\|)}<1$ or equivalently

$$\left\|z-\zeta\right\| < \frac{5-\sqrt{17}}{4}\left(\frac{1}{\gamma(f,\zeta)}\right).$$

Then

$$\left\|z_n-\zeta\right\| \le A^{2^n-1}\left\|z-\zeta\right\|,$$

where $z=z_0$, $z_n=z_{n-1}-Df\left(z_{n-1}\right)^{-1}f\left(z_{n-1}\right)$. △

Proof. This follows from `Proposition 7.18` using `Proposition 7.13`. Using notations of propositions, we obtain,

$$\left\|Df(z)^{-1}f(z)-(z-\zeta)\right\| \le A\left\|z-\zeta\right\|$$

and

$$\left\|Df(z)^{-1}f(z)-(z-\zeta)\right\| = \left\|-N_f(z)+\zeta\right\|,$$

i.e., for all $k\ge 0$,

$$\left\|z_{k+1}-\zeta\right\| \le A\left\|z_k-\zeta\right\|,$$

i.e., $\|z_k - \zeta\| \xrightarrow{k\to\infty} 0$. By `Proposition 7.13`,

$$\|z_k - \zeta\| \le (A\|z-\zeta\|)^{2^k-1}\|z-\zeta\| = A^{2^k-1}\underbrace{\|z-\zeta\|^{2^{k-1}}}_{<1}\|z-\zeta\| \le A^{2^k-1}\|z-\zeta\|. \qquad \blacksquare$$

Now `Theorem 7.7` follows by choosing $A = \frac{1}{2}$ in the corollary.

For the sharpness of the corollary, consider

$$f(z) = \frac{z}{1-z}$$

with $\zeta = 0$. Then

- $D^k f(z) = \frac{k!}{(1-z)^{k+1}}$
- $\gamma(f,\zeta) = \sup_{k\ge 2}\left\|Df(z)^{-1}\frac{D^k f_a(z)}{k!}\right\|^{\frac{1}{k-1}}\Bigg|_{z=0} = \sup_{k\ge 2}\left\|1\cdot\left(\frac{k!}{k!}\right)\right\|^{\frac{1}{k-1}} = 1;$

In particular,

$$z_n = z_{n-1} - Df(z_{n-1})^{-1} f(z_{n-1}) = z_{n-1} - (1-z_{n-1})^2\frac{z_{n-1}}{1-z_{n-1}} = z_{n-1} - \left(z_{n-1} - z_{n-1}^2\right) = z_{n-1}^2,$$

i.e.,

- $z_n = z_{n-1}^2$. Therefore,

  $$A = \frac{\gamma(f,\zeta)\|z-\zeta\|}{\psi(\gamma(f,\zeta)\|z-\zeta\|)} = \frac{z}{\psi(z)}$$

  and the root for $\psi(z) - z = 0$ is $\frac{5-\sqrt{17}}{4}$.

A. Analytic Function

A.1. Line Integrals

This section is followed by [10, Section 0.5.1, 0.5.2, Page 18-20].

A.1.1. Paths in $\mathbb{C}$

We consider continuous functions $g : [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$ and $a < b$. Two continuous functions $g_1 : [a, b] \to \mathbb{C}, g_2 : [c, d] \to \mathbb{C}$ are called *equivalent* if there is a continuous monotone increasing function $\varphi : [a, b] \to [c, d]$ such that $g_1 = g_2 \circ \varphi$. The equivalence classes of this relation are called *path* (in $\mathbb{C}$), and a function $g : [a, b] \to \mathbb{C}$ representing a path is called a *parametrization* of the path.

A *(continuously) differentiable path* is a path represented by a (continuously) differentiable function $g : [a, b] \to \mathbb{C}$.

Let γ be a path. Choose a parametrization $g : [a, b] \to \mathbb{C}$ of γ. We call $g(a)$ the *start point* and $g(b)$ the *end point* of γ. Further, $g([a, b])$ is called the *support* of γ. By saying that a function is continuous on γ, or that γ is contained in a particular set, etc., we mean the support of γ.

The path γ is said to be *closed* if its end point is equal to its start point, i.e., if $g(a) = g(b)$. The path γ is called a *contour* if it is closed, has no self-intersections, and is traversed counterclockwise.

Let γ_1, γ_2 be paths, such that the end point of γ_1 is equal to the start point of γ_2. We define $\gamma_1 + \gamma_2$ to be the path obtained by first traversing γ_1 and then γ_2. For instance, if $g_1 : [a, b] \to \mathbb{C}$ is a parametrization of γ_1 then we may choose a parametrization $g_2 : [b, c] \to \mathbb{C}$ of γ_2; then $g : [a, c] \to \mathbb{C}$ defined by

$$g(t) := \begin{cases} g_1(t), & \text{if } a \le t \le b, \\ g_2(t), & \text{if } b \le t \le c \end{cases}$$

is a parametrization of $\gamma_1 + \gamma_2$.

Given a path γ, we define $-\gamma$ to be the path traversed in the opposite direction, i.e., the start point of $-\gamma$ is the end point of γ and conversely.

Let γ be a path and $F : \gamma \to \mathbb{C}$ a continuous function on (the support of) γ. Then $F(\gamma)$ is the path such that if $g : [a,b] \to \mathbb{C}$ is a parametrization of γ then $F \circ g : [a,b] \to \mathbb{C}$ is a parametrization of $F(\gamma)$.

Definition A.1 (Homotopy). *Let $U \subseteq \mathbb{C}$ and γ_1, γ_2 two paths in U with start point z_0 and end point z_1. Then γ_1, γ_2 are homotopic in U if one can be continuously deformed into the other within U. More precisely this means the following. There are parametrizations $g_1 : [0,1] \to \mathbb{C}$ of γ_1, $g_2 : [0,1] \to \mathbb{C}$ of γ_2 and a continuous map $H : [0,1] \times [0,1] \to U$ with the following properties,*

$$H(0,t) = g_1(t), \qquad H(1,t) = g(t), \qquad \text{for } 0 \le t \le 1;$$
$$H(s,0) = z_0, \qquad H(s,1) = z_1, \qquad \text{for } 0 \le s \le 1.$$

A.1.2. Definition of Line Integrals

All paths occurring in our context will be built up from circle segments and line segments. So for our purposes, it suffices to define integrals of continuous functions along *piecewise continuously differentiable paths*, these are paths of the shape $\gamma_1 + \gamma_2 + \cdots + \gamma_r$, where $\gamma_1, \gamma_2, \ldots, \gamma_r$ are continuously differentiable paths, and for $i = 1, \ldots, r-1$, the end point of γ_i coincides with the start point of γ_{i+1}.

Let γ be a continuously differentiable path, and $f : \gamma \to \mathbb{C}$ a continuous function. Choose a continuously differentiable parametrization $g : [a,b] \to \mathbb{C}$ of γ. Then we define

$$\int_\gamma f(z)\, \mathrm{d}z := \int_a^b f(g(t))\, g'(t)\, \mathrm{d}t.$$

Further, we define the *length* of γ by

$$L(\gamma) := \int_a^b |g'(t)|\, \mathrm{d}t.$$

If $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_r$ is a piecewise continuously differentiable path with continuously differentiable pieces $\gamma_1, \gamma_2, \ldots, \gamma_r$ and $f : \gamma \to \mathbb{C}$ is continuous, we define

$$\int_\gamma f(z)\, \mathrm{d}z := \sum_{i=1}^{r} \int_{\gamma_i} f(z)\, \mathrm{d}z$$

and

$$L(\gamma) := \sum_{i=1}^{r} L(\gamma_i).$$

In case that γ is closed, we write

$$\oint_\gamma f(z)\, \mathrm{d}z.$$

A.2. Complex Analysis

This section is based on [10, Section 0.7.1-0.7.3, Page 25-34].

A.2.1. Basics

Let U be a non-empty open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. We say that f is *holomorphic* or *analytic in* $z_0 \in U$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

In that case, the limit is denoted by $f'(z_0)$. We say that f is *analytic on* U if f is analytic in every $z \in U$; in that case, the derivative $f'(z)$ is defined for every $z \in U$. We say that f is *analytic around* z_0 if it is analytic on some open disk $D(z_0, \delta) = \{z \in \mathbb{C} : |z - z_0| < \delta\}$ for some $\delta > 0$. Finally, given a not necessarily open subset A of $\mathbb{C}$ and a function $f : A \to \mathbb{C}$, we say that f is *analytic on* A if there is an open set $A \subseteq U$ such that f is defined on U and analytic on U. An everywhere analytic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.

Recall that a power series around $z_0 \in \mathbb{C}$ is an infinite sum

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}_0^+$. The radius of convergence of this series is given by

$$R = R_f = \left(\limsup_{n \to \infty} \sqrt[n]{|a_n|}\right)^{-1}. \tag{A.2.1}$$

We state without proof the following fact.

`Theorem A.2.` By [10, Theorem 0.19], let $z_0 \in \mathbb{C}$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ a power series around $z_0 \in \mathbb{C}$ with radius of convergence $R > 0$. Then f defines a function on $D(z_0, R)$, which is analytic infinitely often. For $k \geq 0$ the k-th derivative $f^{(k)}$ of f has a power series expansion with radius of convergence R given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (z - z_0)^{n-k}. \qquad \triangle$$

A.2.2. Cauchy's Theorem and Some Applications

`Theorem A.3 (Cauchy's Theorem).` Let $U \subseteq \mathbb{C}$ be a non-empty open set and $f : U \to \mathbb{C}$ an analytic function. Further, let γ_1, γ_2 be two paths in U with the same start point and end point that are homotopic in U. Then

$$\int_{\gamma_1} f(z)\, \mathrm{d}z = \int_{\gamma_2} f(z)\, \mathrm{d}z. \qquad \triangle$$

`Corollary A.4.` By [10, Corollary 0.21], let γ_1, γ_2 be two contours, such that γ_2 is contained in the interior of γ_1. Let $U \subseteq \mathbb{C}$ be an open set which contains γ_1, γ_2 and the region between γ_1 and γ_2. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then

$$\oint_{\gamma_1} f(z)\, \mathrm{d}z = \oint_{\gamma_2} f(z)\, \mathrm{d}z. \qquad \triangle$$

Proof. Let z_0, z_1 be points on γ_1, γ_2 respectively and let α be a path from z_0 to z_1 lying inside the region between γ_1 and γ_2 without self-intersections. Then γ_1 is homotopic in U to the path $\alpha + \gamma_2 - \alpha$, which consists of first traversing α, then γ_2 and then α in the opposite direction. Hence, by `Theorem A.3`,

$$\oint_{\gamma_1} f(z)\, \mathrm{d}z = \left(\int_{\alpha} + \oint_{\gamma_2} - \int_{\alpha} \right) f(z)\, \mathrm{d}z = \oint_{\gamma_2} f(z)\, \mathrm{d}z. \qquad \blacksquare$$

`Corollary A.5 (Cauchy's Integral Formula).` By [10, Corollary 0.22], let γ be a contour in $\mathbb{C}$, $U \subseteq \mathbb{C}$ an open set containing γ and its interior, z_0 a point in the interior of γ and $f : U \to \mathbb{C}$

an analytic function. Then

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{z-z_0}\,\mathrm{d}z = f(z_0)\,. \qquad \triangle$$

Proof. Let $\gamma_{z_0,\delta}$ be the circle with center z_0 and radius δ, traversed counterclockwise. Then by `Corollary A.4` we have for any sufficiently small $\delta > 0$,

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{z-z_0}\,\mathrm{d}z = \frac{1}{2\pi i}\oint_{\gamma_{z_0,\delta}}\frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

Furthermore, f is continuous, hence uniformly continuous on any sufficiently small compact set containing z_0,

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)}{z-z_0}\,\mathrm{d}z - f(z_0)\right| &= \left|\frac{1}{2\pi i}\oint_{\gamma_{z_0,\delta}}\frac{f(z)}{z-z_0}\,\mathrm{d}z - f(z_0)\right| = \left|\int_0^1 \frac{f\left(z_0+\delta e^{2\pi i t}\right)}{\delta e^{2\pi i t}}\,\delta e^{2\pi i t}\,\mathrm{d}t - f(z_0)\right| \\ &= \left|\int_0^1 f\left(z_0+\delta e^{2\pi i t}\right) - f(z_0)\,\mathrm{d}t\right| \le \sup_{0\le t\le 1}\left|f\left(z_0+\delta e^{2\pi i t}\right) - f(z_0)\right| \xrightarrow{\delta\searrow 0} 0.\end{aligned}$$

This completes our proof. ■

A.2.3. Taylor Series

`Theorem A.6.` By [10, Theorem 0.25], let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Further, let $z_0 \in U$ and $R > 0$ be such that $D(z_0, R) \subseteq U$. Then f has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converging for $z \in D(z_0, R)$. Further, we have for $n \in \mathbb{Z}_0^+$,

$$a_n = \frac{1}{2\pi i}\oint_{z_0,r}\frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z \qquad \text{(A.2.2)}$$

for any r with $0 < r < R$. △

Proof. We fix $z \in D(z_0, R)$ and use w to indicate a complex variable. Choose r with $|z - z_0| < r < R$. By `Corollary A.5`,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

We rewrite the integrand.

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w - z_0) - (z - z_0)} = \frac{f(w)}{w - z_0} \left(1 - \frac{z - z_0}{w - z_0}\right)^{-1} = \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{f(w)}{w - z_0^{n+1}} (z - z_0)^n.$$

The lattr series converges uniformly on $\gamma_{z_0,r}$. Let $M := \sup_{w \in \gamma_{z_0,r}} |f(w)|$. Then

$$\sup_{w \in \gamma_{z_0,r}} \left| \frac{f(w)}{w - z_0^{n+1}} (z - z_0)^n \right| \leq \frac{M}{r} \left(\frac{|z - z_0|}{r} \right)^n =: M_n$$

and $\sum_{n=0}^{\infty} M_n$ converges since $|z - z_0| < r$. Consequently,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z} \, \mathrm{d}w = \frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \sum_{n=0}^{\infty} \frac{f(w)}{w - z_0^{n+1}} (z - z_0)^n \, \mathrm{d}w = \sum_{n=0}^{\infty} (z - z_0)^n \left(\frac{1}{2\pi i} \oint_{\gamma_{z_0,r}} \frac{f(w)}{w - z_0^{n+1}} \, \mathrm{d}w \right).$$

Now `Theorem A.6` follows since by `Corollary A.4` the integral in (A.2.2) is independent of r. ■

`Corollary A.7.` By [10, Corollary 0.26], let $U \subseteq \mathbb{C}$ be a non-empty, open set and $f : U \to \mathbb{C}$ an analytic function. Then f is analytic on U infinitely often, i.e., for every $k \geq 0$ the k-the derivative $f^{(k)}$ exists and is analytic on U. △

Proof. Let z arbitrary in U. Choose $\delta > 0$ such that $D(z, \delta) \subseteq U$. Then for $w \in D(z, \delta)$ we have for $0 < r < \delta$,

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n, \qquad a_n = \frac{1}{2\pi i} \oint_{\gamma_{z,r}} \frac{f(w)}{(w - z)^{n+1}} \, \mathrm{d}w.$$

Now for every $k \geq 0$, the k-th derivative $f^{(k)}(z)$ exists and is equal to $k! a_k$. Since, by `Theorem A.2`,

$$\begin{aligned} f^{(k)}(w)\Big|_{w=z} &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (w - z)^{n-k} \Big|_{w=z} \\ &= n(n-1)\cdots(n-k+1)\, a_n (w - z)^{n-k} \Big|_{w=z, n=k} = k! a_k. \end{aligned}$$

■

`Corollary A.8.` By [10, Corollary 0.27], let γ be a contour in $\mathbb{C}$ and U an open subset of $\mathbb{C}$ containing γ and its interior. Further, let $f : U \to \mathbb{C}$ be an analytic function. Then for every z in the interior of γ and every $k \geq 0$ we have

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z)^{k+1}} \, \mathrm{d}w.$$ △

Proof. Choose $\delta > 0$ such that $\gamma_{z,\delta}$ lies in the interior of γ. By `Corollary A.4`,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z)^{k+1}} \, \mathrm{d}w = \frac{1}{2\pi i} \oint_{\gamma_{z,\delta}} \frac{f(w)}{(w - z)^{k+1}} \, \mathrm{d}w.$$

By the argument in `Corollary A.7`, this is equal to $\frac{f^{(k)}(z)}{k!}$. ■

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Declaration of Authenticity

The work contained in this thesis is original and has not been previously submitted for examination which has led to the award of a degree.

To the best of my knowledge and belief, this thesis contains no material previously published or written by another person except where due reference is made.

Place, Date

Signature