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Using Affine GARCH Models in Portfolio Optimization

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Abstract

This dissertation studies different aspects of a dynamic portfolio allocation problem in a discrete-time environment where the underlying price process of the single risky asset follows the dynamics of an affine GARCH model. While prior to the work contained in this dissertation, a closed-form solution was known only in the case of a simple Gaussian GARCH model and an investor maximizing utility from terminal wealth via a constant relative risk aversion (CRRA) utility function, we extend this approach in three different directions. Firstly, we modify the setup to allow for consumption at intermediate time points before the end of the investment period. In a second step, we generalize the result for utility from terminal wealth to the class of hyperbolic absolute risk aversion (HARA) utility functions and relax an assumption on the investor's risk preferences. Finally, the model complexity is increased towards Lévy GARCH models, which, besides a Gaussian innovation, include an additional jump component. The optimal solution is available in approximate closed form in all these settings, enabling practitioners to implement the corresponding strategies efficiently. In the realm of discrete-time models, once renowned for the challenge of obtaining convenient solutions in the context of financial applications, this dissertation thus makes valuable contributions by addressing broadly formulated problems. We also investigate various facets of our topic numerically, discovering that accommodating heteroscedasticity in the underlying asset price model is highly beneficial for medium-term investors. In a study based on S&P 500 index data, a comparison of an investor implementing the strategy of a Gaussian GARCH model and another one following a homoscedastic variant shows a significant difference in the performance related to both consumption at intermediate time points and terminal wealth, in favor of the GARCH investor. Similarly, we use the generalizations with respect to HARA utilities and the investor's risk preferences to calculate the efficient frontier for an investor in the Gaussian GARCH model as well as in the homoscedastic variant and find that, given the same expected portfolio return, the heteroscedastic investor is exposed to significantly less portfolio variance. On the other hand, a comparison of a Gaussian and a non-Gaussian GARCH model within the framework of consumption and investment suggests that an extension to the latter model, which incorporates conditional skewness and excess kurtosis of log returns, has a relatively small impact. A different type of analysis in the context of Lévy GARCH models with jumps, based on the concept of wealth-equivalent loss (WEL), shows that correctly calibrated jump-free GARCH models can mimic very well the strategies suggested by models incorporating jumps.

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Contents

List of Abbreviations

ARCH	Autoregressive Conditional Heteroscedasticity
CPPI	Constant Proportion Portfolio Insurance
CRRA	Constant Relative Risk Aversion
DP	Dynamic Programming
EUT	Expected Utility Theory
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
GBM	Geometric Brownian Motion
HARA	Hyperbolic Absolute Risk Aversion
HN-GARCH	Affine Gaussian GARCH model introduced by Heston and Nandi (2000)
HS	Homoscedastic model with constant conditional variance
IG-GARCH	Inverse Gaussian GARCH model
MDP	Markovian Decision Process
MGF	Moment Generating Function
MJ	Merton Jump, referring to a possible choice for a Lévy jump innovation
MLE	Maximum Likelihood Estimate
MPR	Market Price of Risk
MV	Mean-Variance
NIG	Normal-Inverse Gaussian, referring to a possible choice for a Lévy jump innovation
PDF	Probability Density Function
SCM	Stochastic Control Model
SCM_{T-t}	$(T-t)$-stage optimization problem of maximizing the expected reward in an SCM
SFC	Self-Financing Condition
VG	Variance Gamma, referring to a possible choice for a Lévy jump innovation
VIX	Volatility index, based on S&P 500 index options, calculated and published by the Chicago Board Options Exchange
WEL	Wealth-Equivalent Loss

1 Introduction

In the realm of portfolio optimization and investment decision-making, the expected utility hypothesis is a fundamental assumption on the behavior of rational agents, stating that market participants maximize their expected subjective utility. In an uncertain environment involving, e.g., stochastic processes for the asset price processes, expected utility theory (EUT) provides a framework for developing portfolio strategies and evaluating their performance. Within this framework, the problem of finding the optimal portfolio selection has many facets, consisting of various ways to measure utility and different model choices concerning the underlying price processes.

In modern financial literature, two large model streams can be identified, interpreting the asset price process either as a continuous or a discrete function of time. Mossin (1968) and Samuelson (1969) made seminal contributions to EUT in a discrete-time setting, with their work also offering insights into the abovementioned question of how to measure an agent's utility. The former approach considers an investor that may re-allocate the investments at intermediate time points, but the strategy performance is evaluated solely at the end of the investment period, called the investor's (finite) time horizon. This formulation was extended by Samuelson (1969), who included the possibility of consuming parts of the allocated wealth at each intermediate time point. In continuous time, Merton (1969, 1971) pioneered the corresponding work in terms of optimizing consumption and portfolio allocation, delivering explicit formulas for both parts of the optimal solution. Merton's generalization from the earlier to the later paper shows yet another dimension of the problem formulation: To measure utility, he used the larger class of hyperbolic absolute risk aversion (HARA) utility functions instead of the nested constant relative risk aversion (CRRA) family. As a referral to this groundbreaking work, the problem of maximizing expected utility derived from terminal wealth and consumption is also known as *Merton's portfolio problem*.

Concerning the underlying asset price process model, the two streams identified above divide the research in the field into two parts. In his continuous-time approach, Merton (1969) relied on a geometric Brownian motion (GBM), assuming that asset prices are log-normally distributed. More advanced ways of modeling the underlying processes were brought up later, e.g., in the form of stochastic volatility models. One of the most famous approaches in this regard was published by Heston (1993), and the corresponding portfolio problem was solved a few years later by Kraft (2005). The major advantage of continuous-time models was rooted in their analytical tractability via Itô calculus and the abovementioned availability of closed-form solutions to pricing and portfolio allocation problems. On the other hand, practitioners always have to work with discrete financial data because of physical limitations, forming an inherent need for discrete-time models that can be estimated from historical data more easily than their continuous counterparts. On the side of discrete-time models, however, the famous modeling approaches, particularly the celebrated ARCH and GARCH family introduced by Engle (1982) and Bollerslev (1986), respectively, did not lead to convenient formulas for similar applications.

This challenge was overcome many years later when Heston and Nandi (2000) presented the first member of a class that would henceforth be called the affine GARCH models – the setup by Heston and Nandi (2000) is referred to as the HN-GARCH model. The crucial and characteristic property ensures that the joint moment generating function (MGF) of the log-asset return and its conditional variance is exponentially affine in these two variables. Initially invented for pricing purposes, models with this property bring about the advantage that closed-form solutions are available for values of various financial derivatives. Within the world of affine GARCH models, the research focused on extending the setup in the subsequent years. Christoffersen et al. (2006) introduced a new model and used inverse Gaussian innovations instead of the normal distribution in the HN-GARCH, and the corresponding model was named IG-GARCH after this choice. A few years later, Christoffersen et al. (2012) incorporated jump components in both the asset return and its volatility process. This feature, however, led to the loss of the crucial affine structure of the MGF and thus came at a great cost. Two years later, Ornthanalai (2014) found a formulation, based on the original HN-GARCH model, that allows for an additional (Lévy) jump component in the asset price process and still preserves the affine form of the MGF – the class of Lévy GARCH models was invented. Badescu et al. (2019) finally formulated a common framework that includes the Gaussian HN-GARCH as well as the IG-GARCH as special cases – the Lévy GARCH models, however, are not included. This setup has been generalized to a multi-factor model very recently (Augustyniak et al., 2023).

As mentioned previously, affine GARCH models were mainly used for pricing upon their invention. Heston and Nandi (2000) and Christoffersen et al. (2006) considered S&P 500 index options, and Badescu et al. (2019) found closed-form expressions for prices of variance swaps. Escobar-Anel et al. (2022a) were the first to exploit the availability of the MGF in a specific affine GARCH model in the context of portfolio optimization, solving an EUT problem with the investor deriving utility from terminal wealth via a CRRA function, with the underlying asset price following HN-GARCH dynamics. Figure 1.1 illustrates the state of the art in the

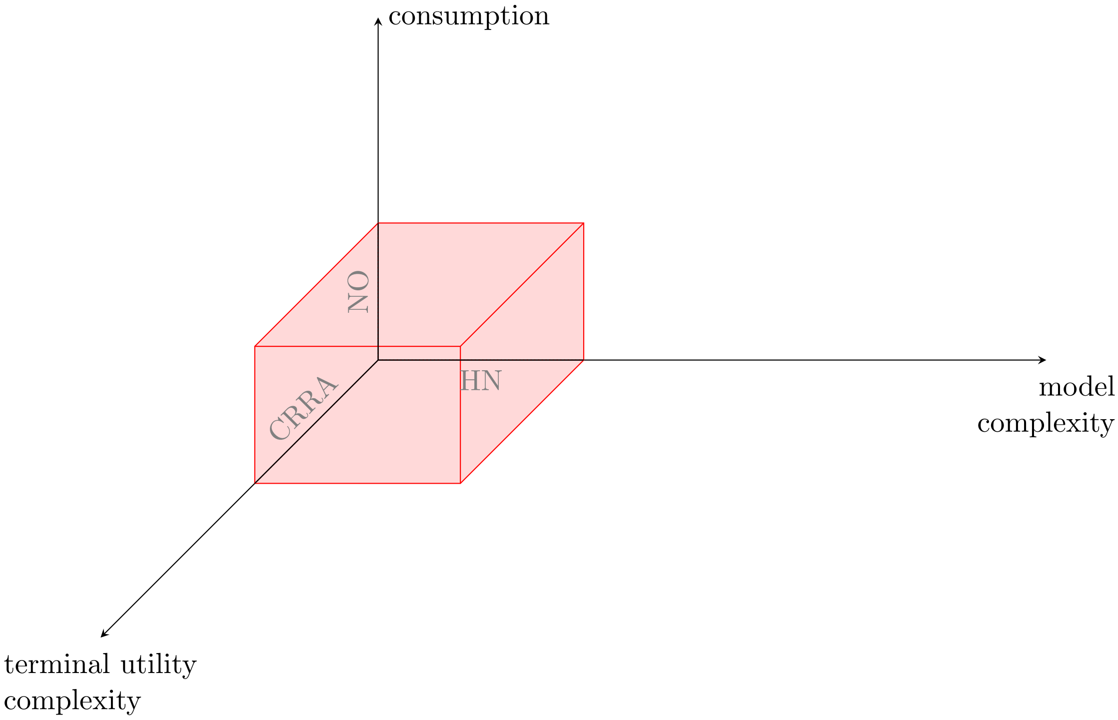


Figure 1.1: This Figure displays the state of the literature before the publications related to this dissertation, with only the solution to the EUT problem for an investor with CRRA utility in the HN-GARCH model available. This figure is extended as we proceed through the chapters of this dissertation, see Figures 3.1, 4.1, 5.1, and 6.1

area of optimal investment (and consumption) within affine GARCH models at that time, without the contributions to the field presented in this dissertation.[1] While at the intersection of the HN-GARCH model and CRRA utility from terminal wealth without any consumption at intermediate time points, the solution by Escobar-Anel et al. (2022a) existed, many connected research questions were still unanswered. This specifically involves the different problem dimensions described above and visualized in Figure 1.1:

- Is it possible to obtain closed-form solutions also allowing for consumption at intermediate time points before the end of the investment horizon?
- Can the assumption of CRRA utility from terminal wealth be relaxed and generalized to a larger class of utilities?
- Is it possible to solve the problem in a larger and more advanced class of models?

The present dissertation addresses all three of these questions, offering affirmative responses to each. Figure 1.1 will be extended successively as we move through the chapters and tackle the extensions, see Figures 3.1, 4.1, 5.1 and 6.1.

In the first step, we solved the EUT problem with a CRRA utility from terminal wealth for the class of general affine GARCH models in the formulation of Badescu et al. (2019). As specified above, this includes the non-Gaussian IG-GARCH model. Together with a numerical study about the impact of non-Gaussianity in conditional log-asset returns on optimal portfolio strategies, this work was published in Escobar-Anel et al. (2021) and accepted as the author's Master's thesis within the elite doctoral program TopMath. Since the theoretical results are fundamental for the subsequent chapters, we include the essential parts of this project in Chapter 3 in this dissertation. Chapter 3 does not contribute to the evaluation of this dissertation.

In Chapter 4, we solve a general problem including the possibility of consumption at intermediate time points while avoiding the undesirable scenario of consuming the entire wealth before the end of the time horizon. There are three types of numerical analysis connected to the theoretical contributions in Chapter 4: We first study the sensitivity of the optimal solution – referring to consumption as well as risky allocation – to important model parameters and focus on the impact of non-Gaussianity, leading to the eye-opening finding that conditional heteroscedasticity matters more to investors than non-Gaussianity of log-asset returns. In another numerical investigation, we use S&P 500 index data to show that investors following a homoscedastic instead of a Gaussian GARCH solution may face substantial losses. A complementary analysis of losses in the context of consumption on a theoretical level and results for different parameter sets can be found in Appendix C.2.

Chapter 5 targets another extension of the previous setup, focusing on utility from terminal wealth. While earlier results used CRRA utility functions to measure utility at the end of the time horizon, we prove an extension for the more general class of HARA utilities. Furthermore, we show that a restriction on the investor's risk aversion can be relaxed, making a connection to problem formulations in mean-variance (MV) theory and portfolio optimization based on higher return moments possible. Our corresponding numerical study focuses on the efficient frontier in the special case of MV optimization and shows, again based on S&P 500 index data, that an investor implementing an HN-GARCH strategy, given the same expected portfolio return, faces significantly less portfolio variance than another investor following a homoscedastic variant.

Chapter 6 extends the underlying asset price process type to Lévy GARCH models. We show that in the presence of an additional Lévy jump component, after imposing a condition on the model parameters, we can solve the corresponding EUT problem in closed form. This

[1] The increase in along the three axes is not meant to be continuous. We will have different sections in each dimension that correspond to model features, e.g., there are only going to be the two values *NO* and *YES* on the vertical axis, see Figure 4.1.

major theoretical contribution allows for a numerical investigation of the impact of jumps on the optimal solution and strategy performance. We simulate data from several Lévy GARCH models involving jumps and ask the question whether the jump-free Gaussian HN-GARCH model, with the parameters estimated based on the simulated dataset, is capable of producing comparable results. The awakening answer is that correctly calibrated jump-free models can produce outstanding results and suffer only minimal losses with respect to the optimal jump models.

By further developing the diagram in Figure 1.1 in the corresponding chapters, we visualize the main theoretical contributions of this dissertation. In summary, our contributions are the following:

- We present the first analytical solution to a consumption and investment problem in an affine GARCH environment in Chapter 4, based on the following article, which has been published as:

  M. Escobar-Anel et al. (2024b). "Optimal consumption and investment in general affine GARCH models." In: *OR Spectrum*. DOI: `10.1007/s00291-024-00749-z`

- In Chapter 5, we extend existing results for the EUT problem to the class of HARA functions to measure utility from terminal wealth and consider previously excluded ranges for the investor's risk aversion, establishing a connection to MV optimization. A short version of this chapter is published in:

  M. Escobar-Anel et al. (2024a). "Mean–variance optimization under affine GARCH: A utility-based solution." In: *Finance Research Letters* 59.104749. DOI: `10.1016/j.frl.2023.104749`

- We introduce a closed-form solution to the optimal investment problem in a Lévy GARCH environment in Chapter 6, allowing for jumps of different types in the asset price process. This chapter is based on the following article:

  M. Escobar-Anel et al. (2023). "Do Jumps Matter in Discrete-Time Portfolio Optimization?" Working paper submitted for publication

Chapter 7 concludes this dissertation, providing a concise summary of the main findings and contributions, and presenting prospects for potential future research.

2 Mathematical Preliminaries

2.1 Measure and Probability

The portfolio optimization problems in this dissertation are based on discrete-time financial market models and stochastic control. This section introduces basic notations for measurability and probability spaces to adequately describe the corresponding environments. All of the following results can be found in standard textbooks. Particularly concerning the part relevant in the context of stochastic control models, we refer to Hinderer et al. (2016, Appendix B).

Definition 2.1 (σ-algebra)**.** *A σ-algebra $\mathcal{F}$ on a non-empty set Ω is a system of subsets of Ω such that*

(i) $\Omega \in \mathcal{F}$,

(ii) $G \in \mathcal{F}$ *implies* $\Omega \setminus G \in \mathcal{F}$,

(iii) The countable union of sets in $\mathcal{F}$ still belongs to $\mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space, and the sets in $\mathcal{F}$ are said to be ($\mathcal{F}$-)measurable.

For a non-empty system $\mathcal{C}$ of subsets of Ω, there exists a smallest σ-algebra $\sigma(\mathcal{C})$ that contains $\mathcal{C}$. We say that $\sigma(\mathcal{C})$ is generated by $\mathcal{C}$ and call $\mathcal{C}$ the generator.

Definition 2.2 (Trace of a σ-algebra)**.** *For a measurable space $(\Omega, \mathcal{F})$ and a non-empty set $B \in \mathcal{F}$, we call*

$$B \cap \mathcal{F} := \{B \cap G \mid G \in \mathcal{F}\} = \{G \in \mathcal{F} \mid G \subseteq B\}$$

the trace σ-algebra of $\mathcal{F}$ on B.

In our applications, the underlying set usually is the set of real numbers $\mathbb{R}$. On this set, and in general, on all metric spaces, the most natural σ-algebra is the one generated by the open sets, i.e., by the generator $\{(-\infty, \alpha) \mid \alpha \in \mathbb{R}\}$.

Definition 2.3 (Product σ-algebra)**.** *For systems $\mathcal{C}_i$ of subsets of a set Ω_i, for $i = 1, \ldots, d$, we define the Cartesian product as $\times_1^d \mathcal{C}_i := \left\{\times_1^d B_i \mid B_i \in \mathcal{C}_i, 1 \le i \le d\right\}$. For measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i = 1, \ldots, d$, the product σ-algebra of the σ-algebras $\mathcal{F}_i$ is defined as $\otimes_1^d \mathcal{F}_i := \sigma\left(\times_1^d \mathcal{F}_i\right)$ on $\Omega := \times_1^d \Omega_i$.*

Definition 2.4 (Measurable mapping)**.** *Consider a mapping f from a non-empty set Ω into a measurable space $(\Omega', \mathcal{F}')$. The system of preimages $f^{-1}(\mathcal{F}') := \left\{f^{-1}(G') \mid G' \in \mathcal{F}'\right\}$ is a σ-algebra on Ω. For a σ-algebra $\mathcal{F}$ on Ω, the mapping f is called $\mathcal{F}$-$\mathcal{F}'$-measurable if $f^{-1}(\mathcal{F}') \subseteq \mathcal{F}$. An $\mathcal{F}$-$\mathcal{F}'$-measurable mapping is called a random variable on $(\Omega, \mathcal{F})$ with values in $(\Omega', \mathcal{F}')$. If the use of the corresponding σ-algebras is clear from the context, we will simply call f measurable.*

A collection of random variables $\{X_i\}_{i=1,\dots,d}$ generates a σ-algebra via $\sigma\left(\bigcup_{i=1}^{d}\sigma\left(X_i\right)\right)$, where $\sigma\left(X_i\right)$ is the σ-algebra generated by the preimages of X_i.

Definition 2.5 (Probability space)**.** *On a measurable space $(\Omega, \mathcal{F})$, we will call a mapping $\mathbb{P}: \mathcal{F} \to [0, \infty)$ a probability measure if it satisfies the following conditions:*

(i) $\mathbb{P}\left(\Omega\right) = 1$,

(ii) $\mathbb{P}$ *is countably additive, i.e.,* $\mathbb{P}\left(\emptyset\right) = 0$ *and for a collection of pairwise disjoint sets* $A_1, A_2, \ldots \in \mathcal{F}$, $\mathbb{P}$ *satisfies*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(A_i\right).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ forms a probability space.

In financial applications, market participants gain more and more information over time by observing stock and option prices. This kind of evolution is modeled via filtrations and filtered probability spaces.

Definition 2.6 (Filtered probability space)**.** *A collection $\{\mathcal{F}_i\}_{i=1,\dots,d}$ of σ-algebras that satisfy $\mathcal{F}_i \subseteq \mathcal{F}$ for all i and $\mathcal{F}_k \subseteq \mathcal{F}_l$ for $1 \leq k \leq l \leq d$ is called a filtration. The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}_i, \mathbb{P})$ is called a filtered probability space.*

In the sequel, we will consider real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., the random variables are mappings into the real numbers $\mathbb{R}$.

Definition 2.7 (Expectation)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a real-valued random variable. The expectation of X, denoted by $\mathbb{E}[X]$ is defined as*

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega),$$

where the integral is interpreted as the Lebesgue integral.

A random variable X is said to be (absolutely) integrable if $\mathbb{E}[|X|]$ is finite. The expectation is also referred to as the first moment of a random variable. Higher moments are defined below.

Definition 2.8 (Higher moments)**.** *Assuming that X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that the expressions below are well-defined, the variance, skewness, and kurtosis, respectively, of X, are given by:*

$$\begin{aligned} Var[X] &:= \sigma^2 := \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right], \\ \mathcal{S}[X] &:= \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sigma}\right)^3\right], \\ \mathcal{K}[X] &:= \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sigma}\right)^4\right]. \end{aligned}$$

Definition 2.9 (Conditional expectation)**.** *For an integrable random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the conditional expectation $\mathbb{E}[X \mid \mathcal{G}]$ of X given $\mathcal{G}$, where $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra, as the random variable Y such that*

(i) Y *is* $\mathcal{G}$*-measurable,*

(ii) $\mathbb{E}[|Y|] < \infty$,

(iii) $\mathbb{E}[\mathbb{1}_C Y] = \mathbb{E}[\mathbb{1}_C X]$ *for every* $C \in \mathcal{G}$,

with $\mathbb{1}_C(\omega)$ *the indicator function of* C *yielding* 1 *if and only if* $\omega \in C$ *and* 0 *otherwise.*

For random variables X and Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that $X = Y$ holds $\mathbb{P}$-almost surely if

$$\mathbb{P}(\{\omega \in \Omega \mid X(\omega) = Y(\omega)\}) = 1.$$

The following properties of conditional expectation will be helpful throughout this dissertation. the proofs are ommitted and can be found in many standard textbooks.

Proposition 2.10 (Properties of conditional expectation, cf. Bingham and Kiesel (2004), Proposition 2.5.1)**.** *Assume that X and Y are integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume that $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ are σ-algebras.*

(a) $\mathbb{E}[X \mid \{\emptyset, \Omega\}] = \mathbb{E}[X]$.

(b) If X *is* $\mathcal{G}$*-measurable, then* $\mathbb{E}[X \mid \mathcal{G}] = X$, $\mathbb{P}$*-almost surely.*

(c) Given constants $a_1, a_2 \in \mathbb{R}$, $\mathbb{E}[a_1 X + a_2 Y \mid \mathcal{G}] = a_1 \mathbb{E}[X \mid \mathcal{G}] + a_2 \mathbb{E}[Y \mid \mathcal{G}]$, $\mathbb{P}$*-almost surely.*

(d) If Y *is* $\mathcal{G}$*-measurable, then* $\mathbb{E}[YX \mid \mathcal{G}] = Y\mathbb{E}[X \mid \mathcal{G}]$, $\mathbb{P}$*-almost surely.*

(e) If $\mathcal{H} \subseteq \mathcal{G}$, *then* $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{H}] \mid \mathcal{G}] = \mathbb{E}[X \mid \mathcal{H}]$, $\mathbb{P}$*-almost surely.*

2.2 Stochastic Control Models

To solve this dissertation's discrete-time portfolio optimization problems, we first introduce the basic notation and some relevant results concerning dynamic programming and stochastic control models. This section is based on the comprehensive book by Hinderer et al. (2016) on various types of control models and decision processes.

2.2.1 Model Formulation

An investor in a typical setting considered in this dissertation starts at some time $t \geq 0$ with a certain amount of initial wealth. At this point, the investor has to decide how to allocate the available resources, choosing from the different assets available in the market, which in our problems consist of a riskless bank account and a risky asset following the dynamics of an affine GARCH model. According to this decision, and based on the random evolution of the asset price, the system then moves on to time $t+1$, characterized by a new value for the investor's wealth. This procedure continues to a predefined finite time horizon $T > t$.

For simplicity, we usually use $t = 0$ as the initial time point. In terms of a stochastic control model (SCM), we speak of an initial state $s_0 \in \mathbb{S}$, where $\mathbb{S}$ denotes the space of all possible states, and, together with the σ-algebra $\mathcal{S}$, forms a measurable space $(\mathbb{S}, \mathcal{S})$. The investor's allocation choice is taken from a space of actions $\mathbb{A}$, endowed with the σ-algebra $\mathcal{A}$. The so-called constraint set $\mathbb{D}$ contains all possible combinations of states and actions and is part of the product σ-algebra $\mathcal{S} \otimes \mathcal{A}$, equipped with the trace σ-algebra $\mathcal{D}$ of $\mathbb{D}$ in $\mathcal{S} \otimes \mathcal{A}$. For a specific $s \in \mathbb{S}$, we let $D(s) := \{a \in \mathbb{A} \mid (s, a) \in \mathbb{D}\}$.

While in a simpler form of a control model, the transition from one state to the next is fully deterministic and given via a mapping from $\mathbb{D}$ to $\mathbb{S}$, this does not apply to our investment problem. Instead, we let $(\mathbb{Y}, \mathcal{Y})$ be another measurable space, where $\mathbb{Y} \neq \emptyset$ is the space of so-called disturbances, in financial market models often named *innovations*. The disturbances are commonly assumed to be i.i.d., but in our applications, the distribution need not be the same at all time points – that is, we work with a slight relaxation of the mentioned assumption. The transition to the next state is now given by a measurable mapping $\mathbb{T} : \mathbb{D} \times \mathbb{Y} \to \mathbb{S}$, called the transition function. The combination of a disturbance distribution and a transition function can also be considered as a direct *transition probability* from $\mathbb{D}$ into $\mathbb{S}$. The central feature of this model is the fact that, even after the choice of an action at time point t, the state one time step ahead is still random and subject to the realization of the disturbance. The random state at time $t+1$ is therefore denoted by S_{t+1}, its realization is s_{t+1}, and similarly Y_t denotes a random variable with values in $(\mathbb{Y}, \mathcal{Y})$, while y_t is its realization. Furthermore, a *decision rule* π_t at time $t = 0, \ldots, T-1$ is a measurable mapping from $\mathbb{S}$ into $\mathbb{A}$ such that $(s_t, \pi_t(s_t)) \in \mathbb{D}$, i.e., a decision rule thus yields an action dependent on the realization of the state. A collection of decision rules $\pi = \{\pi_t\}_{t=0}^{T-1}$ is called *policy*, and we denote the set of all policies by $\mathbb{F}$. In the presence of a policy π, the random state S_t at some time point $t = 1, \ldots, T$ usually depends on π, but we omit this dependence for readability.

To properly define the transitions between the different states, we need the following formal definition of a transition probability:

Definition 2.11 (Transition probability, cf. Hinderer et al., 2016, Def. 16.1.1)**.** *A function* $\mathbb{P} : \mathbb{D} \times \mathcal{S} \to \mathbb{R}_+$ *is called a transition probability from* $\mathbb{D}$ *into* $\mathbb{S}$ *if*

- $(s, a) \mapsto \mathbb{P}(s, a, B)$ *is* $\mathcal{D}$*-measurable for all* $B \in \mathcal{S}$*,* $(s, a) \in \mathbb{D}$*.*
- $B \mapsto \mathbb{P}(s, a, B)$ *is a probability measure on* $\mathcal{S}$ *for all* $(s, a) \in \mathbb{D}$*.*

Given a decision rule π and using the fact that the composition of measurable mappings is again measurable, we can consider transition probabilities from $\mathbb{S}$ into $\mathbb{S}$ via

$$s \mapsto \mathbb{P}_\pi(s, B) := \mathbb{P}(s, \pi(s), B).$$

The performance of the investor's strategy is evaluated by a real-valued reward function ϕ_T at the end of the investment horizon, i.e., at time T, and in addition by a one-stage reward function[1] $r_w(s, a)$ from $\mathbb{D}$ into the real numbers $\mathbb{R}$. Both functions are assumed to be measurable. Finally, $\beta \in \mathbb{R}^+$ denotes a discount factor. We summarize the setup in the following definition:

Definition 2.12 (Stochastic control model, cf. Hinderer et al., 2016, Definitions 16.1.2 and 16.1.13)**.** *A stochastic control model (SCM) is a tuple*

$$(\mathbb{S}, \mathbb{A}, \mathbb{D}, \mathbb{Y}, \mathbb{T}, \mathbb{Q}, r_w, \phi_T, \beta)$$

with the following properties:

[1]The reward function is denoted by r_w with a w in the subscript to distinguish it from the riskless rate r.

- $(\mathbb{S}, \mathcal{S})$ *and* $(\mathbb{A}, \mathcal{A})$ *are measurable spaces.*
- $\mathbb{D} \in \mathcal{S} \otimes \mathcal{A}$ *contains the graph of a measurable mapping* $D : \mathbb{S} \to \mathbb{A}$*, such that* $D(s)$ *describes the actions that are admissible in state* $s \in \mathbb{S}$*.* $\mathbb{D}$ *is equipped with the trace* $\mathcal{D}$ *of* $\mathbb{D}$ *in* $\mathcal{S} \otimes \mathcal{A}$*.*
- $(\mathbb{Y}, \mathcal{Y})$ *is a measurable space, with* $\mathbb{Y} \neq \emptyset$ *the disturbance space.*
- $\mathbb{T} : \mathbb{D} \times \mathbb{Y} \to \mathbb{S}$ *is a measurable mapping.*
- $\mathbb{Q}$ *satisfies the conditions of a transition probability from* $\mathbb{D}$ *into* $\mathbb{Y}$ *according to Definition 2.11, i.e., the event* $\{Y \in C\}$ *for* $C \in \mathcal{Y}$ *occurs with probability* $\mathbb{Q}(s, a, C)$*. The resulting transition probability* $\mathbb{P}$ *from* $\mathbb{D}$ *into* $\mathbb{S}$ *is given by*

$$\mathbb{P}(s, a, B) = \mathbb{Q}(s, a, \{y \in \mathbb{Y} \mid \mathbb{T}(s, a, y) \in B\}), \qquad B \in \mathcal{S}, (s, a) \in \mathbb{D}. \tag{2.1}$$

- $r_w : \mathbb{D} \to \mathbb{R}$ *is measurable.*
- $\phi_T : \mathbb{S} \to \mathbb{R}$ *is measurable.*
- $\beta \in \mathbb{R}^+$ *is the discount factor.*

Instead of the combination of a disturbance and a transition function, one can define the model via the transition law $\mathbb{P}$ *in* (2.1) *directly. This variant is called the adjoint Markovian decision process (MDP) to the SCM.*

Remark 2.13 (Disturbance distributions)**.** *We note that there exist extensions towards time-dependent transition probabilities* $\{\mathbb{Q}_t\}_{t=0,\ldots,T-1}$ *and time-dependent one-step reward functions* $\{(r_w)_t\}_{t=0,\ldots,T-1}$ *(Hinderer et al., 2016, Def. 16.1.26).*

2.2.2 Value Iteration

In our application of an SCM, the investor is interested in maximizing the (expected) portfolio return via optimally choosing the investment strategy. For a policy π, define the $(T-t)$-stage random gain as

$$G_{t,\pi}(s_t) := r_w(s_t, \pi_t(s_t)) + \sum_{i=t+1}^{T-1} \beta^{i-t} \cdot r_w(S_i, \pi_i(S_i)) + \beta^{T-t} \phi_T(S_T). \tag{2.2}$$

Based on the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\{0,1,\ldots\}}, \mathbb{P}\right)$, the $(T-t)$-stage expected reward is given by

$$\phi_{t,\pi}(s_t) := r_w(s_t, \pi_t(s_t)) + \mathbb{E}\left[\sum_{i=t+1}^{T-1} \beta^{i-t} \cdot r_w(S_i, \pi_i(S_i)) + \beta^{T-t} \phi_T(S_T) \mid \mathcal{F}_t\right]. \tag{2.3}$$

Given our finance application, assuming that we are at some time point $t \geq 0$, we intend to solve the problem SCM_{T-t}, which, given $s_t \in \mathbb{S}$, is defined as $\max\{\phi_{t,\pi}(s_t) \mid \pi \in \mathbb{F}\}$. That is, the objective is to maximize the expected $(T-t)$-stage reward for policy π and a given initial state s_t. We define the concept of optimal policies and solutions for SCM_{T-t} as follows:

Definition 2.14 (Optimal policies and maximizers, Hinderer et al., 2016, Def. 11.3)**.**

(a) A $(T-t)$*-stage policy* π^* *is optimal for* SCM_{T-t} *if it maximizes* $\pi \mapsto \phi_{t,\pi}(s_t)$*.*

(b) *A solution of the* $(T-t)$*-stage problem consists of the* $(T-t)$*-stage optimal policy (if it exists) and the* $(T-t)$*-stage value function* $\phi_t : \mathbb{S} \to \mathbb{R}$, $\phi_t(s) := \sup\{\phi_{t,\pi}(s) \mid \pi \in \mathbb{F}\}$.

(c) *A decision rule* π_t *is called maximizer at time* t *if for all* $s \in \mathbb{S}$*, the action* $\pi_t(s)$ *is a maximum point of the mapping*

$$a \mapsto r_w(s,a) + \beta \cdot \mathbb{E}\left[\phi_{t+1}\left(\mathbb{T}\left(s,a,Y\right)\right)\right].$$

Note that, without any further assumptions, we are not sure whether SCM_{T-t} makes sense since, e.g., $\phi_t(s)$ as a supremum of an uncountable set of measurable functions is not necessarily measurable. To express the corresponding conditions adequately, certain operators turn out to be helpful. Assume that there exists a subset $\mathbb{M}$ of all integrable value functions $\phi : \mathbb{S} \to \mathbb{R}$ such that $\phi_T \in \mathbb{M}$. We define the following operators on $\mathbb{M}$:

$$\mathbb{L}\phi\left(s,a\right) := r_w\left(s,a\right) + \beta \cdot \mathbb{E}\left[\phi\left(\mathbb{T}\left(s,a,Y\right)\right)\right], \tag{2.4a}$$

$$\mathbb{U}\phi\left(s\right) := \sup_{a \in \mathbb{D}(s)} \mathbb{L}\phi\left(s,a\right). \tag{2.4b}$$

The dependencies on s and a may be suppressed in the sequel. We require the operators in (2.4) to be well-defined, and $\mathbb{U}$ to be an endomorphism on $\mathbb{M}$. Furthermore, for any $\phi \in \mathbb{M}$ and $t \in \{0,\ldots,T-1\}$, we assume the existence of some decision rule π_t^* maximizing $\pi_t \mapsto \mathbb{L}\phi\left(s,\pi_t\left(s\right)\right)$ on $\mathbb{A}$ for all $s \in \mathbb{S}$. Under these assumptions, we obtain the following important result:

Theorem 2.15 (Value iteration for SCM_{T-t}, Hinderer et al., 2016, Theorem 16.1.12)**.** *Let* $\mathbb{M}$ *be a set of integrable value functions* $\phi : \mathbb{S} \to \mathbb{R}$*, and let* $\phi_T \in \mathbb{M}$*. Assume that the operators in* (2.4) *exist and are well-defined at all time points* $0,\ldots,T-1$*, and furthermore require the following:*

(i) *There exists a decision rule* π_ϕ^* *maximizing* $\pi_\phi \mapsto \mathbb{L}\phi\left(s,\pi_\phi\left(s\right)\right)$ *on* $\mathbb{A}$ *for any* $\phi \in \mathbb{M}$, $s \in \mathbb{S}$.

(ii) $\mathbb{U}$ *is an endomorphism on* $\mathbb{M}$.

Then, for each $t = 0,\ldots,T-1$*, the decision rule* $\pi_t^* := \pi_{\phi_{t+1}}^*$ *is a maximizer at time* t*, and* $\pi^* := \{\pi_t^*\}_{t=0,\ldots,T-1}$ *is an optimal policy. Furthermore,* $\phi_t \in \mathbb{M}$ *for all* $t = 0,\ldots,T$ *and the so-called value iteration holds:*

$$\phi_t = \mathbb{U}\phi_{t+1}, \qquad t = 0,\ldots,T-1. \tag{2.5}$$

Proof. See Appendix A.1. □

2.2.3 Concave Value Functions

The results in this section, again primarily collected from Hinderer et al. (2016), will be helpful in our optimization techniques. The proofs of standard results are omitted and can be looked up in Roberts and Varberg (1973).

Definition 2.16 (Convex set)**.** *A set* $K \subseteq \mathbb{R}^k$ *is convex if, for any* $x, y \in K$ *and* $\alpha \in (0,1)$*,*

$$\alpha x + (1-\alpha)\, y \in K.$$

In particular, an interval $I \subset \mathbb{R}$ and the entire space $\mathbb{R}^k$ are convex.

Definition 2.17 (Convex and concave functions). *Let $K \subseteq \mathbb{R}^k$ be a convex set. The function $f : K \to \mathbb{R}$ is called convex if, for $x, y \in K$, where $x \neq y$, and $\alpha \in (0,1)$,*

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y). \tag{2.6}$$

If (2.6) *holds with strict inequality, f is said to be strictly convex. f is called (strictly) concave if $-f$ is (strictly) convex.*

Given the above relation between convexity and concavity, most results in the literature are stated for convex functions but can be translated straightforwardly.

Proposition 2.18 (Derivatives and convexity/concavity, cf. Roberts and Varberg (1973), Theorems I.12.B and I.12.C). *Let $K \subseteq \mathbb{R}$ be a one-dimensional convex set, and consider $f : K \to \mathbb{R}$. If the first derivative f' exists, then*

- *f is (strictly) convex if and only if f' is (strictly) increasing.*
- *f is (strictly) concave if and only if f' is (strictly) decreasing.*

If f'' exists, then

- *f is convex if and only if $f'' \geq 0$. If $f'' > 0$, then f is strictly convex.*
- *f is concave if and only if $f'' \leq 0$. If $f'' < 0$, then f is strictly concave.*

The following result is useful for combinations of multiple convex (or concave) functions and for functions of more than one variable. If not specified otherwise, we assume that the functions involved are real-valued.

Proposition 2.19 (Sum of convex/concave functions, Roberts and Varberg (1973), Theorem I.13.A). *If both f_1 and f_2 are (strictly) convex/concave on $K_1 \subseteq \mathbb{R}^k$ and on $K_2 \subseteq \mathbb{R}^k$, respectively, then $(x, y) \mapsto f_1(x) + f_2(y)$ is (strictly) convex/concave on $K_1 \times K_2$.*

In our optimization context, the next result for maximum points of concave functions, usually formulated for minimum points of convex functions, is particularly interesting.

Proposition 2.20 (Maximum points of concave functions, Roberts and Varberg (1973), Theorem V.51.A). *Assume that f is concave on a convex set $K \subseteq \mathbb{R}^k$. Then, the set of maximum points of f, denoted by K^*, is convex. Furthermore, if f is strictly concave, it has at most one maximum point.*

Lastly, we prove an essential proposition needed in the extended portfolio problem in Chapter 4, where parts of the wealth may be consumed at intermediate time points. The result is based on Hinderer et al. (2016, Prop. 7.1.7), where it was formulated for deterministic decision processes. The present extension targets an SCM as of Section 2.2.1. Note that the state space is two-dimensional here.

Proposition 2.21 (Argumentwise concavity, cf. Hinderer et al., 2016, Prop. 7.1.7). *Assume a stochastic control model with arbitrary disturbances from a set $\mathbb{Y}$, having state space $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$, action space $\mathbb{A}$, and constraint set $\mathbb{D} = \mathbb{S} \times \mathbb{A}$. With the transition function denoted by $\mathbb{T} : \mathbb{D} \times \mathbb{Y} \to \mathbb{S}$, the finite time horizon T, and ϕ_t the value function at time $t \in \{0, \ldots, T\}$, assume that the model satisfies the following properties:*

(i) $\mathbb{S}_1$, $\mathbb{S}_2$ and $\mathbb{A}$ are convex sets.

(ii) $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2)$, *where* $\mathbb{T}_2(s_1, s_2, a, y) = \mathbb{T}_2(s_2, y)$ *is independent of* (s_1, a), *and additionally,* $\mathbb{T}_1(s_1, s_2, a, y)$ *is increasing in* s_1 *and concave in* (s_1, a) *on* $\mathbb{S}_1 \times \mathbb{A}$ *for all* $s_2 \in \mathbb{S}_2$ *and* $y \in \mathbb{Y}$,

(iii) *The one-step return* $r_w(s_1, s_2, a)$ *is increasing in* s_1 *for all* $s_2 \in \mathbb{S}_2$ *and* $a \in \mathbb{A}$ *and concave in* (s_1, a) *for all* $s_2 \in \mathbb{S}_2$.

(iv) *The supremum*

$$\sup_{a \in \mathbb{D}(s)} \left\{ r_w(s_1, s_2, a) + \beta \cdot \mathbb{E}\left[\phi_{t+1}\left(\mathbb{T}\left(s_1, s_2, a, Y\right)\right)\right] \right\}$$

is finite for all s_1, s_2, *and for all* t.

(v) *The terminal value function* $\phi_T(s)$ *is increasing and concave in* s_1.

Then, all value functions $s \mapsto \phi_t(s)$, $t \in \{0, \ldots, T\}$, *are increasing and concave in* s_1.

Proof of Proposition 2.21. We prove by induction on $T - t$ that ϕ_t is increasing and concave in s_1. For $t = T$, this is trivially satisfied by (v). Now assume that the assertion is true for some ϕ_{t+1}. We will prove that it also holds true for ϕ_t. Fixing $\hat{s}_2 \in \mathbb{S}_2$ and $\hat{y} \in \mathbb{Y}$, we investigate $\phi_{t+1}\left(\mathbb{T}_1(s_1, \hat{s}_2, a, \hat{y}), \mathbb{T}_2(\hat{s}_2, \hat{y})\right)$. Using (ii) and the inductive hypothesis, we have that $\mathbb{T}_1$ is increasing in s_1 and concave in (s_1, a), and that ϕ_{t+1} is increasing and concave in s_1. Therefore, $\phi_{t+1}\left(\mathbb{T}_1(s_1, \hat{s}_2, a, \hat{y}), \mathbb{T}_2(\hat{s}_2, \hat{y})\right)$ is increasing in s_1 and concave in (s_1, a). Note that together with (iii), this implies in particular that the function

$$\mathbb{L}\phi_{t+1}(s_1, \hat{s}_2, a) = r_w(s_1, \hat{s}_2, a) + \bar{\beta}\mathbb{E}\left[\phi_{t+1}\left(\mathbb{T}\left(s_1, \hat{s}_2, a, Y\right)\right)\right]$$

is increasing in s_1 and concave in (s_1, a). Now define

$$s_1 \mapsto \mathbb{U}\phi_{t+1}(s_1) = \sup_{a \in \mathbb{A}} \mathbb{L}\phi_{t+1}(s_1, \hat{s}_2, a).$$

The concavity of $\mathbb{L}\phi_{t+1}$ in (s_1, a) combined with (i) implies that for $\alpha \in (0, 1)$, $s_1, s_1' \in \mathbb{S}_1$ and arbitrary $a, a' \in \mathbb{A}$,

$$\begin{aligned} \mathbb{U}\phi_{t+1}\left(\alpha s_1 + (1-\alpha)s_1'\right) &\geq \mathbb{L}\phi_{t+1}\left(\alpha s_1 + (1-\alpha)s_1', \hat{s}_2, \alpha a + (1-\alpha)a'\right) \\ &\geq \alpha \mathbb{L}\phi_{t+1}\left(s_1, \hat{s}_2, a\right) + (1-\alpha)\mathbb{L}\phi_{t+1}\left(s_1', \hat{s}_2, a'\right). \end{aligned}$$

Since α, a and a' were arbitrary, this yields

$$\mathbb{U}\phi_{t+1}\left(\alpha s_1 + (1-\alpha)s_1'\right) \geq \alpha \mathbb{U}\phi_{t+1}\left(s_1\right) + (1-\alpha)\mathbb{U}\phi_{t+1}\left(s_1'\right).$$

This reasoning is true for $\mathbb{P}$-almost all $\hat{s}_2 \in \mathbb{S}_2$. We can thus deduce that

$$\phi_t\left(s_1, s_2\right) = \mathbb{U}\phi_{t+1}\left(s_1, s_2\right)$$

again is increasing and concave in s_1, which concludes the proof. □

2.3 Maximum-Likelihood Estimation

The models considered in this dissertation are supposed to reflect the current market situation the investor faces in an actual application. To this end, the corresponding model parameters need to be estimated based on market data. One of the most common methods in this regard is maximum likelihood estimation. A short introduction of essential definitions and results, based on the extensive work by Bickel and Doksum (2006) and DeGroot and Schervish (2012), is given in this section.

The general idea of maximum likelihood estimation is to choose the set of values for the model parameters that is the "most likely", based on the given dataset. The method avoids assumptions on prior distributions and the use of loss functions. We will consider a set of random variables $X_1, \ldots, X_n$, $n \in \mathbb{N}$, forming a random sample from a distribution with parameter (vector) θ in a parameter space Θ. The central instrument for the maximum likelihood estimation is the likelihood function:

Definition 2.22 (Likelihood function)**.** *For an observed vector $x = (x_1, \ldots, x_n)$ of values, the likelihood function is the joint probability density function (PDF) $f_n(x \mid \theta)$ of the observations in a random sample, seen as a function of θ.*

Based on this notion of a likelihood function, we introduce the terminology of a maximum likelihood estimator:

Definition 2.23 (Maximum likelihood estimator)**.** *For each possible observed vector x, let $\theta(x) \in \Theta$ denote a value of $\theta \in \Theta$ for which the likelihood function $f_n(x \mid \theta)$ is a maximum. The estimator $\hat{\theta} = \theta(X)$ defined in this way is called a maximum likelihood estimator of θ. After $X = x$ is observed, the value $\delta(x)$ is called a maximum likelihood estimate (MLE) of θ.*

Note that the parameter value maximizing $f_n(x \mid \theta)$ is the same as the one that maximizes $\log f_n(x \mid \theta)$, which often turns out to be more convenient to handle in our applications. The abbreviation MLE is often used inconsistently for both the estimator and the estimate.

Definition 2.24 (Fisher information matrix, DeGroot and Schervish, 2012, Def. 8.8.4)**.** *Suppose that $X = (X_1, \ldots, X_n)$ form a random sample from a distribution with joint PDF $f_n(x \mid \theta)$, where the value of θ lies in an open subset $\Theta \subseteq \mathbb{R}^k$. Let $l_n(x \mid \theta) := \log f_n(x \mid \theta)$. Assume that the mapping $\theta \mapsto \{x \in \mathbb{R}^n \mid f_n(x \mid \theta) > 0\}$ is constant for all θ and that $l_n(x \mid \theta)$ is twice differentiable with respect to θ. The Fisher information matrix $I(\theta)$ in the random sample X is defined as the $k \times k$ matrix*

$$I(\theta) = Cov_\theta \left[\nabla_\theta \, l_n(X \mid \theta)\right], \tag{2.7}$$

where Cov_θ indicates the covariance given the paramter vector θ, and ∇_θ is the gradient with respect to θ.

Assuming sufficient regularity of the underlying densities so that we can change the order of integration and differentiation, the Fisher information matrix also admits the representation (cf. DeGroot and Schervish, 2012, Theorem 8.8.1)

$$I(\theta) = -\mathbb{E}_\theta \left[\nabla_\theta^2 \, l_n(X \mid \theta)\right] \tag{2.8}$$

Under certain regularity conditions, implying that the MLE $\hat{\theta}$ is consistent and asymptotically normal, as well as the assumptions of Definition 2.24, one can derive that (cf. Bickel and Doksum, 2006, Theorem 6.2.2)

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \overset{n\to\infty}{\longrightarrow} Y \sim \mathcal{N}\left(0, I(\theta)^{-1}\right), \tag{2.9}$$

where n is the sample size and the convergence is meant in distribution. In particular, the standard errors of the estimates can be derived via the Fisher information matrix.

3 Expected Utility Theory on General Affine GARCH Models

3.1 Introduction

This chapter presents a first result concerning expected utility theory (EUT) in the environment of general affine GARCH models. As visualized in Figure 3.1, the main contribution consists of tackling the entire class of general affine GARCH models instead of the Gaussian model version only. An extended version of this chapter was published in Escobar-Anel et al. (2021) and accepted as the author's M.Sc. thesis in the TopMath program at the Technical University of Munich. Therefore, this part does not contribute to the evaluation of the present doctoral dissertation.

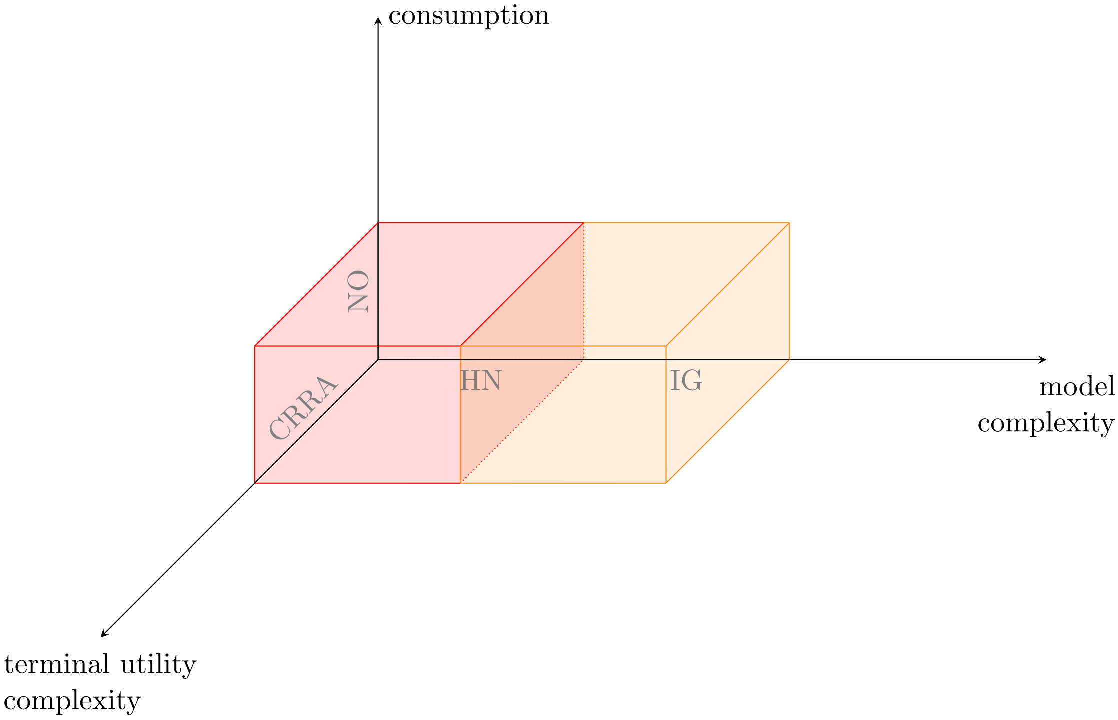


Figure 3.1: Overview of the contributions of this dissertation with regard to portfolio optimization under affine GARCH models, increasing the model complexity as well as the complexity of terminal and intermediate utility, see also Figures 1.1, 4.1, 5.1 and 6.1. The orange box coresponds to Chapter 3.

As described in the context of the general motivation for this dissertation in Chapter 1, the idea of approaching optimal resource allocation problems in multiple discrete time steps measuring expected utility derived from terminal wealth originated from Mossin (1968) and was extended by Samuelson (1969). In a continuous-time setting, Merton (1969) presented closed-form solutions for constant relative risk aversion (CRRA) utility functions, considering a geometric

Brownian motion (GBM) as the underlying asset dynamics. In the subsequent decades, more advanced asset price models were introduced, for example, the popular stochastic volatility model by Heston (1993), which Kraft (2005) solved in the context of EUT. However, just as in the abovementioned example, most considered models were continuous-time, still bearing difficulties regarding implementation and testing. These issues can be addressed using discrete-time GARCH models, first introduced by Engle (1982) and Bollerslev (1986). Unfortunately, these usually do not yield closed-form solutions in the context of derivative pricing and portfolio optimization.

This lack of closed-form expressions for derivative pricing was overcome in Heston and Nandi (2000), who introduced the so-called HN-GARCH model. The key feature, the availability of the moment generating function (MGF) in exponentially affine form, allowed for closed-form solutions to asset pricing problems. The same model property can also be exploited in the context of portfolio optimization, as demonstrated by Escobar-Anel et al. (2022a), who solved the EUT problem for an investor maximizing a CRRA utility function from terminal wealth, where the underlying risky asset follows the HN-GARCH dynamics, obtaining an approximate closed-form optimal solution.

While the innovations in the HN-GARCH model are i.i.d. normally distributed, Christoffersen et al. (2006) introduced an extension with time-dependent non-Gaussian innovations a few years later. Badescu et al. (2019) recently formulated general conditions for the affine structure of GARCH models, explicitly requiring the characteristic exponentially affine form of the MGF. The extended paper version (Escobar-Anel et al., 2021) of this chapter solves the EUT problem for general affine GARCH dynamics as of Badescu et al. (2019) and focuses on the implications of the non-Gaussian setting as compared to the Gaussian model or a homoscedasctic variant.

By considering the CRRA class of utility functions and allowing for utility derived from terminal wealth only, the problem setup considered here forms a simple base case of the several extensions considered in Chapters 4, 5 and 6. The theoretical contributions presented in this chapter are twofold:

- Assuming a CRRA utility function for the decision maker and the implied EUT setting of Campbell and Viceira (1999), we demonstrate that the optimal portfolio strategy in a general affine GARCH setting is available in recursive form, and the optimal wealth process is again an affine GARCH process.

- We demonstrate that a slight relaxation of the conditions on general affine GARCH models suffices to include the IG-GARCH model as a special case.

This chapter is organized as follows: In Section 3.2, we define the portfolio optimization problem and give an outline of the approach. Section 3.3 presents the main result for a general affine GARCH model, and Section 3.4 specifically targets the necessary extensions to include the non-Gaussian IG-GARCH model in the setup.

3.2 Outline of the Approach

This section presents the general optimization framework for our investment problem and embeds general affine GARCH models in this context. We assume that the log-price process $X_t = \log P_t^{(1)}$, the price at time t being $P_t^{(1)}$, follows an affine GARCH model (as defined by Badescu et al., 2019), where for simplicity of exposition we assume $\Delta = 1$ for the length of the time step. Let θ be a vector of parameters and $\{\epsilon_t\}_{t=1,2,\ldots}$ a sequence of $\mathcal{F}_t$-measurable and

$\mathcal{F}_{t-1}$-conditional i.i.d. random variables with zero mean and a finite moment generating function. Based on the associated filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,1,\dots}, \mathbb{P}\right)$ with physical measure $\mathbb{P}$, the model dynamics is:

$$Y_t := X_t - X_{t-1} = f_1(h_t, \theta) + \sqrt{h_t}\,\epsilon_t, \tag{3.1a}$$

$$h_{t+1} = f_2(h_t, \theta) + f_3(h_t, \epsilon_t, \theta), \tag{3.1b}$$

where f_1, f_2 are affine in h_t, while f_3 is such that the following representation for the conditional MGF holds:

$$\begin{aligned}\Psi_{(X_t, h_{t+1})}(u, v \,|\, \mathcal{F}_{t-1}) &:= \mathbb{E}\left[\exp\{u \cdot X_t + v \cdot h_{t+1}\} \,|\, \mathcal{F}_{t-1}\right] \\ &= \exp\{uX_{t-1} + A(u, v; t-1, t) + B(u, v; t-1, t) \cdot h_t\}.\end{aligned} \tag{3.2}$$

The conditional expectation w.r.t. $\mathcal{F}_t$ will also be denoted via the subindex t, e.g., with $\mathbb{E}_t$, in the sequel. Note that the moment generating function in (3.2) can also be formulated for the log return instead of the log price as the first element, making the term uX_{t-1} in the exponential function vanish. The choice of f_1, f_2, and f_3 for a particular application shapes the coefficients A and B in Equation (3.2) according to this requirement. The form of said coefficients will also play a significant role in our framework. Since it is clear from the context that we use the one-step conditional MGF, the last two time arguments of A and B are usually obsolete, and we will omit them where possible.

We consider a portfolio optimization problem with a finite time horizon T in a setting with one risky asset and the cash account $P_t^{(0)}$. For $t \in \{0, \dots T-1\}$, let π_t denote the fraction of wealth invested in the risky asset, let V_t be the corresponding wealth and $w_0 = \log v_0$ the log of the initial wealth. Following the reasoning in Escobar-Anel et al. (2022a), we approximate returns in the self-financing condition (SFC) by log prices. That is, we start with the true equation

$$\frac{V_t}{V_{t-1}} = \pi_{t-1}\frac{P_t^{(1)}}{P_{t-1}^{(1)}} + (1-\pi_{t-1})\frac{P_t^{(0)}}{P_{t-1}^{(0)}} = \pi_{t-1}\frac{P_t^{(1)}}{P_{t-1}^{(1)}} + (1-\pi_{t-1})\,e^r, \tag{3.3}$$

where r is the continuously compounded riskless rate, and use the following Taylor approach: For the log return of the risky asset, we write

$$\begin{aligned}X_t - X_{t-1} = \log\left(\frac{P_t^{(1)}}{P_{t-1}^{(1)}}\right) &\approx \frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} - \frac{1}{2}\left(\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}}\right)^2 \\ &\approx \frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} - \frac{1}{2}\mathit{Var}\left[\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} \mid \mathcal{F}_{t-1}\right],\end{aligned} \tag{3.4}$$

where we approximated the squared return by its conditional variance in the last step. Observing from (3.4) that $\mathit{Var}\left[\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} \mid \mathcal{F}_{t-1}\right] \approx \mathit{Var}\left[X_t - X_{t-1} \mid \mathcal{F}_{t-1}\right] = h_t$, we receive

$$\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} \approx X_t - X_{t-1} + \frac{1}{2}h_t. \tag{3.5}$$

For the cash account, the linear approximation $\log x \approx x - 1$ for x around 1 yields

$$\frac{P_t^{(0)} - P_{t-1}^{(0)}}{P_{t-1}^{(0)}} \approx \log\left(\frac{P_t^{(0)}}{P_{t-1}^{(0)}}\right) = r. \tag{3.6}$$

Now, looking at the return on wealth and performing the same steps as for the return of the risky asset above (with the variance of the return on wealth being $\pi_{t-1}^2 h_t$ according to (3.3)), we arrive at

$$\begin{aligned}\log\left(\frac{V_t}{V_{t-1}}\right) &\approx \left(\frac{V_t}{V_{t-1}}-1\right)-\frac{1}{2}\left(\frac{V_t}{V_{t-1}}-1\right)^2\\ &\approx \pi_{t-1}\frac{P_t^{(1)}-P_{t-1}^{(1)}}{P_{t-1}^{(1)}}+(1-\pi_{t-1})\frac{P_t^{(0)}-P_{t-1}^{(0)}}{P_{t-1}^{(0)}}-\frac{1}{2}\pi_{t-1}^2 h_t\\ &\approx \pi_{t-1}\left(X_t-X_{t-1}+\frac{1}{2}h_t\right)+(1-\pi_{t-1})\,r-\frac{1}{2}\pi_{t-1}^2 h_t\end{aligned}$$

All in all, for $t=1,\dots,T$, this leads to an approximation W_t of the log wealth $\log V_t$ that reads

$$W_t = W_{t-1}+\pi_{t-1}(X_t-X_{t-1})+\frac{1}{2}\left(\pi_{t-1}-\pi_{t-1}^2\right)h_t+(1-\pi_{t-1})r, \tag{3.7}$$

with $W_0=w_0$. This second-order approximation is well-studied – in particular, it corresponds to Equation (16) in Campbell and Viceira (1999). Equation (3.7) exactly equals (3.3) in the limit for $\Delta\to 0$, i.e., as the length of the trading interval approaches zero. We refer to the full-length paper Escobar-Anel et al. (2021) for an investigation of the quality of this approximation in this framework.

The goal is to maximize the expected utility from terminal wealth over the set of admissible, self-financing, relative portfolio strategies $\{\pi_t\}_{t=0}^{T-1}$. To this end, assume wealth is assessed by the decision maker according to a CRRA utility function of the form $U(v)=\frac{1}{\gamma}v^\gamma$ for some parameter $\gamma<1$, implying a relative risk aversion of $1-\gamma$. Using the approximation of the log wealth via W_T as described above, the stochastic control problem can be respresented as follows:

$$\sup_{\{\pi_t\}_{t=0}^{T-1}}\mathbb{E}\left[\frac{1}{\gamma}(V_T)^\gamma \mid \mathcal{F}_0\right]\approx \max_{\{\pi_t\}_{t=0}^{T-1}}\mathbb{E}\left[\frac{\exp\{\gamma W_T\}}{\gamma}\,\middle|\,\mathcal{F}_0\right]=:\phi_0(w_0,h_1), \tag{3.8}$$

where h denotes the conditional variance process of the log return of the underlying asset.

In the spirit of Section 2.2, let $\mathbb{A}$ be the set of admissible portfolios, $\mathbb{H}=(0,\infty)$ and $\mathbb{Y}=(0,\infty)$ be the domains of the conditional variance and the log return of the stock prices, respectively. Let $\mathbb{W}\subset\mathbb{R}$ be the set of possible values for the log wealth, and set $\mathbb{S}=\mathbb{W}\times\mathbb{H}$. Then, the transition function is given by $\mathbb{T}:\mathbb{S}\times\mathbb{A}\times\mathbb{Y}\to\mathbb{S}$,

$$\begin{aligned}\mathbb{T}(s,a,Y)=\mathbb{T}(W,h,a,Y) := \Bigg(&W+aY+\frac{1}{2}\left(a-a^2\right)h+(1-a)r,\\ &f_2(h,\theta)+f_3\left(h,\frac{Y-f_1(h,\theta)}{\sqrt{h}},\theta\right)\Bigg).\end{aligned} \tag{3.9}$$

Now assume that there exists a subset $\mathbb{M}$ of all integrable value functions $\phi:\mathbb{S}\to\mathbb{R}$ such that $\phi_T(W_T)=U(\exp\{W_T\})\in\mathbb{M}$. With the operators $\mathbb{L}$ and $\mathbb{U}$,

$$\begin{aligned}\mathbb{L}\,\phi(W,h,a) &:= \mathbb{E}\left[\phi\left(\mathbb{T}(W,h,a,Y)\right)\right],\\ \mathbb{U}\,\phi(W,h) &:= \sup_{a\in\mathbb{A}}\mathbb{L}\,\phi(W,h,a),\end{aligned}$$

assumed to be well-defined for all $\phi\in\mathbb{M}$, we moreover require for any $\phi\in\mathbb{M}$ the existence of some $a:\mathbb{W}\times\mathbb{H}\to\mathbb{A}$ with $a(W,h)$ maximizing $a\mapsto\mathbb{L}\,\phi(W,h,a)$ on $\mathbb{A}$ for all $W\in\mathbb{W}$ and $h\in\mathbb{H}$, and $\mathbb{U}:\mathbb{M}\to\mathbb{M}$.

With the above requirements, we ensure that the conditions of Theorem 2.15 are satisfied and that we can use the value iteration. We thus maximize recursively step by step in order to obtain the optimal solution $\{\pi_t^*\}_{t=0}^{T-1}$, defined as $\pi_t^*(W_t, h_{t+1}) := \arg\max_{a\in\mathbb{A}} \mathbb{L}\phi_{t+1}(W_t, h_{t+1}, a)$, and the value iteration

$$\begin{aligned}\phi_t(W_t, h_{t+1}) &= \mathbb{U}\,\phi_{t+1}(W_t, h_{t+1}) \\ &= \max_{a\in\mathbb{A}} \mathbb{E}_t\left[\phi_{t+1}\left(\mathbb{T}\left(W_t, h_{t+1}, a, Y_t\right)\right)\right], \quad t \in \{0, \dots, T-1\},\end{aligned} \tag{3.10}$$

with the obvious terminal condition $\phi_T(W_T) = U(\exp\{W_T\})$.

3.3 Solution to the Portfolio Optimization Problem

This section presents the solution to the portfolio optimization problem outlined in Section 3.2. The optimal solution to (3.8) is provided in the next theorem, which is followed by a corollary on the affine GARCH nature of the optimal wealth process.

Theorem 3.1 (Maximum expected utility representation)**.** *Assume $\gamma < 0$, and let the log price of the risky asset follow an affine GARCH model, where for all $t \in \{0, \dots, T-1\}$, and all admissible π_t and v, the coefficients in* (3.2) *satisfy*

$$\frac{\partial^2}{\partial u^2} A(\gamma\pi_t, v) \geq 0 \quad \textit{and} \quad \frac{\partial^2}{\partial u^2} B(\gamma\pi_t, v) > 0. \tag{3.11}$$

Then at time t, the maximum expected utility from terminal wealth can be written as

$$\phi_t(W_t, h_{t+1}) = \frac{1}{\gamma} \exp\{\gamma W_t + D_{t,T}(\pi_t^*) + E_{t,T}(\pi_t^*) \cdot h_{t+1}\}, \tag{3.12}$$

with $D_{t,T}$ and $E_{t,T}$ given by the recursive representations

$$D_{t,T}(\pi_t^*) = D_{t+1,T}(\pi_{t+1}^*) + (1 - \pi_t^*)\gamma r + A\left(\gamma\pi_t^*, E_{t+1,T}(\pi_{t+1}^*)\right), \tag{3.13a}$$

$$E_{t,T}(\pi_t^*) = B\left(\gamma\pi_t^*, E_{t+1,T}(\pi_{t+1}^*)\right) + \frac{\gamma}{2}\left(\pi_t^* - (\pi_t^*)^2\right). \tag{3.13b}$$

with $E_{T,T} = D_{T,T} = 0$. The optimal fraction of wealth invested in the risky asset at time t is given as a solution π_t^* to the equations

$$\frac{\partial}{\partial u} A\left(\gamma\pi_t, E_{t+1,T}(\pi_{t+1}^*)\right) = r, \tag{3.14a}$$

$$\frac{\partial}{\partial u} B\left(\gamma\pi_t, E_{t+1,T}(\pi_{t+1}^*)\right) = \pi_t - \frac{1}{2}. \tag{3.14b}$$

Moreover, a solution π_t^* satisfying (3.11), (3.14a), *and* (3.14b) *is a maximum.*

Note that an optimal solution is required to satisfy both equations in (3.14), which in general might not be possible. However, in the special cases of HN-GARCH (c.f. Heston and Nandi, 2000) and IG-GARCH models (c.f. Christoffersen et al., 2006), (3.14a) is trivially satisfied and there remains only (3.14b) to solve.

Proof of Theorem 3.1. As outlined above, we use Bellman's value iteration in order to maximize via backwards induction and the optimal solution at time $t+1$ for our approach at time t. In general, this results in the calculations below, where for the terminal step at time $t = T - 1$ we just work with

$$\phi_T(W_T, h_{T+1}) \coloneqq \phi_T(W_T) = U(\exp\{W_T\}) = \frac{1}{\gamma}\exp\{\gamma W_T\},$$

resulting in $D_{T,T} = E_{T,T} = 0$. We obtain:

$$\begin{aligned}
\phi_t(W_t, h_{t+1}) &= \max_{\pi_t} \mathbb{E}_t\left[\phi_{t+1}(W_{t+1}, h_{t+2})\right] \\
&= \max_{\pi_t} \mathbb{E}_t\left[\frac{1}{\gamma}\exp\left\{D_{t+1,T}(\pi^*_{t+1}) + \gamma W_{t+1} + E_{t+1,T}(\pi^*_{t+1}) \cdot h_{t+2}\right\}\right] \qquad (3.15)\\
&= \max_{\pi_t} \frac{1}{\gamma}\exp\left\{D_{t+1,T}(\pi^*_{t+1})\right\} \\
&\quad \times \mathbb{E}_t\Bigg[\exp\Bigg\{\gamma \cdot \Bigg[W_t + \pi_t(X_{t+1} - X_t) + \frac{1}{2}\left(\pi_t - \pi_t^2\right)h_{t+1} \\
&\qquad + (1 - \pi_t)\,r\Bigg] + E_{t+1,T}(\pi^*_{t+1}) \cdot h_{t+2}\Bigg\}\Bigg] \\
&= \max_{\pi_t} \frac{1}{\gamma}\exp\left\{D_{t+1,T}(\pi^*_{t+1}) + (1 - \pi_t)\gamma r + \gamma W_t + \frac{\gamma}{2}\left(\pi_t - \pi_t^2\right)h_{t+1}\right\} \\
&\quad \times \mathbb{E}_t\left[\exp\left\{\gamma\pi_t(X_{t+1} - X_t) + E_{t+1,T}(\pi^*_{t+1}) \cdot h_{t+2}\right\}\right] \\
&= \max_{\pi_t} \frac{1}{\gamma}\exp\left\{D_{t+1,T}(\pi^*_{t+1}) + (1 - \pi_t)\gamma r + \gamma W_t + \frac{\gamma}{2}\left(\pi_t - \pi_t^2\right)h_{t+1}\right\} \\
&\quad \times \Psi_{(X_{t+1} - X_t,\, h_{t+2})}\left(\gamma\pi_t, E_{t+1,T}(\pi^*_{t+1}) \,\middle|\, \mathcal{F}_t\right) \\
&= \max_{\pi_t} \frac{1}{\gamma}\exp\left\{D_{t,T}(\pi_t) + \gamma W_t + E_{t,T}(\pi_t) \cdot h_{t+1}\right\}, \qquad (3.16)
\end{aligned}$$

where we used the availability of the MGF in the affine GARCH setting according to (3.2) and define D and E recursively according to (3.13). Note that $\gamma < 0$ together with the second-order conditions in (3.11) are sufficient for the function in (3.16) to be concave in π_t. The first-order conditions in (3.14) thus characterize the optimal solution to our portfolio problem. In particular, a solution satisfying (3.11), (3.14a), and (3.14b) is a maximum. □

Corollary 3.2 (Affine GARCH process for optimal log wealth)**.** *The optimal log-wealth process $\{W_t\}_t$ follows an affine GARCH process. Furthermore, the optimal solution π_t^* does not depend on the conditional variance or the wealth.*

Proof. We start with the self-financing condition in Equation (3.7). The equation shows that the conditional variance of W_t is given by $\pi_{t-1}^2 h_t$. The affine GARCH representation for the optimal log-wealth process can now be deduced from the proof of Theorem 3.1. That is, starting in line (3.15), disregarding the maximum operator and unnecessary factors, we can proceed to line (3.16) to solve the expectation below for the conditional bivariate moment generating function of W_t and h_{t+1}. Adopting also the definitions in (3.13) results in:

$$\begin{aligned}
\Psi_{(W_t, h_{t+1})}(u, v \,|\, \mathcal{F}_{t-1}) &= \mathbb{E}_{t-1}\left[\exp\left\{uW_t + vh_{t+1}\right\}\right] \\
&= \exp\Big\{uW_{t-1} + (1 - \pi_{t-1})\,ur + A(u\pi_{t-1}, v) \\
&\qquad + \left(B(u\pi_{t-1}, v) + \frac{u}{2}\left(\pi_{t-1} - (\pi_{t-1})^2\right)\right) \cdot h_t\Big\}.
\end{aligned}$$

This satisfies the requirements for affine GARCH processes outlined in Equations (3.1) and (3.2). An investigation of the equations defining the optimal solution yields the second statement of the corollary. □

It is not difficult to see that the HN-GARCH model (Heston and Nandi, 2000), which was already tackled in this context by Escobar-Anel et al. (2022a), is a particular case of Theorem 3.1. In particular, we introduce the set of parameters $\theta = (\lambda, \omega, \beta, \alpha, \rho)$ and an i.i.d. sequence $\{\epsilon_t\}_t$ of standard normal innovations. Then, choosing $f_1(h_t, \theta) = r + \lambda h_t$, $f_2(h_t, \theta) = \omega + \beta h_t$ and $f_3(h_t, \epsilon_t, \theta) = \alpha\left(\epsilon_t - \rho\sqrt{h_t}\right)^2$, we obtain the following dynamics for the log return and its conditional variance process in the HN-GARCH model:

$$X_t - X_{t-1} = r + \lambda h_t + \sqrt{h_t}\epsilon_t, \tag{3.17a}$$

$$h_{t+1} = \omega + \beta h_t + \alpha\left(\epsilon_t - \rho\sqrt{h_t}\right)^2. \tag{3.17b}$$

The proof by Heston and Nandi (2000) for the exponentially affine structure of the MGF and the derivation for the coefficients A and B can be found in Appendix B.1.

3.4 Application to IG-GARCH Model

In this section, we present the IG-GARCH model (c.f. Christoffersen et al., 2006) as a special case of our main result. Assume the following dynamics for the log return $X_t - X_{t-1} = \log P_t^{(1)} - \log P_{t-1}^{(1)}$ and its conditional variance process $h = \{h_t\}_{t=1,\dots,T}$:

$$X_t - X_{t-1} = r + \nu h_t + \eta y_t, \tag{3.18a}$$

$$h_{t+1} = w + b h_t + c y_t + a\frac{h_t^2}{y_t}, \tag{3.18b}$$

with $\eta < 0$ and $\{y_t\}_t$ a sequence of random variables with inverse Gaussian distribution and single parameter $\delta_t = \delta(t) = {}^{h_t}/_{\eta^2}$. In particular, translating this single-parameter form to the known representation of an inverse Gaussian distribution[1] , we have $y_t \sim \text{IG}\left(\delta_t, \delta_t^2\right)$ and $\mathbb{E}_{t-1}[y_t] = Var_{t-1}[y_t] = \delta_t$.

Comparing Equation (3.18a) to (3.1), we first note that scaling of the inverse Gaussian random variable with some parameter $\alpha > 0$ yields $\alpha y_t \sim \text{IG}\left(\alpha\delta_t, \alpha\delta_t^2\right)$. Together with the relation

$$\eta y_t = \sqrt{h_t}\cdot\frac{\eta}{\sqrt{h_t}}y_t = \sqrt{h_t}\cdot\left(-\frac{1}{\sqrt{\delta_t}}y_t\right) =: \sqrt{h_t}\cdot\xi_t, \tag{3.19}$$

this implies that, in order to obtain the form of (3.1), we need to set $-\xi_t \sim \text{IG}\left(\sqrt{\delta_t}, \sqrt{\delta_t}^3\right) = \text{IG}\left({}^{\sqrt{h_t}}/_{|\eta|}, {}^{\sqrt{h_t}^3}/_{|\eta|^3}\right)$. From the variance of a two-parameter inverse Gaussian variable, we obtain $Var_{t-1}[\xi_t] = 1$. The minus compensates for $\eta < 0$. Standardizing this random variable further w.r.t. the mean by subtracting $-{}^{\sqrt{h_t}}/_{\eta}$ finally allows us to present the embedding of the IG-GARCH model into the general setting by Badescu et al. (2019). With a set of parameters denoted by $\theta = (\nu, \eta, w, b, c, a)$, we have

$$f_1(h_t, \theta) = r + \nu\cdot h_t, \quad f_2(h_t, \theta) = w + b\cdot h_t, \quad f_3(h_t, y_t, \theta) = c y_t + a\frac{h_t^2}{y_t},$$

[1] An inverse Gaussian random variable $\xi \sim \text{IG}(\mu, \lambda)$ with $\mu, \lambda > 0$ (see, e.g., Seshadri, 1999) has support $(0, \infty)$. Furthermore, $\mathbb{E}[\xi] = \mu$ and $Var[\xi] = \frac{\mu^3}{\lambda}$.

where $\{\epsilon_t\}_t$, $\epsilon_t = \xi_t + \sqrt{h_t}/\eta$, are standardized inverse Gaussian random variables. However, we usually work with $\tilde{f}_1 = r + \left(\nu + \frac{1}{\eta}\right) h_t$ and use $\{\xi_t\}_t$ as innovations in (3.1). From this mean-standardized formulation, we derive the market price of risk as $\lambda = \nu + \frac{1}{\eta}$.[2] The derivation of the exponentially affine structure of the MGF in the IG-GARCH model can be found in Appendix B.2.

Remark 3.3 (Inverse Gaussian innovations)**.**

(1) Note that the shape of y_t still depends on h_t. Thus, the IG-GARCH model only fulfills a relaxation of the condition on the sequence of shocks in the general affine setting. In particular, instead of explicitly relying on the innovations $\{\epsilon_t\}_{t=1,\dots T}$ (or $\{\xi_t\}_{t=1,\dots T}$) in (3.1) *being i.i.d., we require that the choice of distribution allows for the exponentially affine representation of the conditional bivariate generating function in* (3.2)*. This slight adjustment accommodates the IG-GARCH model while preserving our framework's necessary and important properties.*

(2) Via sending $\eta \to 0$, the conditional skewness $\mathcal{S}$ and excess kurtosis $\mathcal{K}$ on the one-step distribution, which are determined by (c.f. Christoffersen et al., 2006)

$$\mathcal{S}_t\left[X_{t+1} - X_t\right] = 3\eta\left(h_{t+1}\right)^{-1/2}, \quad \mathcal{K}_t\left[X_{t+1} - X_t\right] = 15\eta^2\left(h_{t+1}\right)^{-1}, \tag{3.20}$$

vanish and the standardized shock becomes standard normal in the limit. This can be used to derive the continuous-time Heston model (Heston, 1993) in the limit (cf. Christoffersen et al., 2006, pp. 258).[3]

The first two conditional moments of the log return are provided next, where we use the length of a time step being exactly one day, i.e., $\Delta = 1$.

$$\mathbb{E}_t\left[X_{t+1} - X_t\right] = r + \left(\nu + \frac{1}{\eta}\right) \cdot h_{t+1}, \tag{3.21a}$$

$$Var_t\left[X_{t+1}\right] = h_{t+1}. \tag{3.21b}$$

The following corollary applies our main results to the IG-GARCH setting.

Corollary 3.4 (IG-GARCH model)**.** *Assuming existence, the optimal solution to the stochastic control problem in* (3.8)*, with the log-price dynamics represented in the form of* (3.18)*, is given as a real solution to the equation*

$$\nu + \frac{\sqrt{\left(1 - 2E^*_{t+1,T}a\eta^4\right)}}{\eta\sqrt{\left(1 - 2\eta\gamma\pi_t - 2E^*_{t+1,T}c\right)}} = \pi_t - \frac{1}{2}, \tag{3.22}$$

where $E^*_{t+1,T} := E_{t+1,T}(\pi^*_{t+1})$, if the following two conditions are satisfied for all admissible values of π_t and $t \in \{0, \dots, T-1\}$:

$$1 - 2E^*_{t+1,T}a\eta^4 > 0 \quad \textit{and} \quad 1 - 2\eta\gamma\pi_t - 2E^*_{t+1,T}c > 0. \tag{3.23}$$

[2] It is possible to derive a set of HN-GARCH parameters matching the first two moments of a given IG-GARCH parameter set (Christoffersen et al., 2006). For this pair of models, the market price of risk in the IG-GARCH coincides with the parameter λ of the HN-GARCH model in (3.17a).

[3] It is known (Escobar-Anel et al., 2022a) that the optimal strategy in the HN-GARCH model converges to the continuous-time Heston solution as of Kraft (2005) for $\Delta \to 0$ under certain conditions. Thus, imposing these requirements and taking the limit $\eta \to 0$, our optimal solution approaches the continuous-time Heston strategy.

The solutions of Equation (3.22) *can be found via the roots of a cubic polynomial with real coefficients, and there is exactly one real solution if the discriminant of this polynomial is negative. A solution* π_t^* *satisfying* (3.22) *and* (3.23) *is a maximum.*

Proof. As of Christoffersen et al., 2006, and as stated in Appendix B.2, the coefficients A and B of the conditional bivariate moment generating function in (3.2) in the IG-GARCH model are given by

$$\begin{aligned}
A\left(\gamma\pi_t, E^*_{t+1,T}\right) &= r\gamma\pi_t + E^*_{t+1,T}\,w - \frac{1}{2}\log\left(1 - 2E^*_{t+1,T}a\eta^4\right), \\
B\left(\gamma\pi_t, E^*_{t+1,T}\right) &= E^*_{t+1,T}b + \nu\gamma\pi_t + \eta^{-2} \\
&\quad - \eta^{-2}\sqrt{\left(1 - 2E^*_{t+1,T}a\eta^4\right)\left(1 - 2\eta\gamma\pi_t - 2E^*_{t+1,T}c\right)}.
\end{aligned}$$

Combining these formulas with Theorem 3.1, i.e., plugging the above expressions for A and B into (3.13), we directly obtain the explicit representations for $D_{t,T}$ and $E_{t,T}$ in terms of the parameters of the IG-GARCH model:

$$D_{t,T} = D_{t+1,T} + \gamma r + E^*_{t+1,T}w - \frac{1}{2}\log\left(1 - 2E^*_{t+1,T}a\eta^4\right), \tag{3.24a}$$

$$\begin{aligned}
E_{t,T}(\pi_t) &= \frac{\gamma}{2}\left(\pi_t - \pi_t^2\right) + E^*_{t+1,T}b + \nu\gamma\pi_t \\
&\quad + \eta^{-2}\left(1 - \sqrt{\left(1 - 2E^*_{t+1,T}a\eta^4\right)\left(1 - 2\eta\gamma\pi_t - 2E^*_{t+1,T}c\right)}\right).
\end{aligned} \tag{3.24b}$$

Thus, the two Equations (3.14a) and (3.14b) reduce to (3.22), which needs to be solved for π_t in order to obtain the optimal fraction invested in the risky asset. With the second-order conditions (3.23) fulfilled, we deduce that a solution must be a maximum.

Working towards solving Equation (3.22), we first isolate the radicals, square the equation and rearrange terms in order to obtain the following cubic equation:

$$\begin{aligned}
& \frac{1 - 2E^*_{t+1,T}a\eta^4}{\eta^2\cdot\left(1 - 2\eta\gamma\pi_t - 2E^*_{t+1,T}c\right)} = \pi_t^2 - (1+2\nu)\cdot\pi_t + \left(\nu + \frac{1}{2}\right)^2 \\
\Leftrightarrow\quad & \frac{1}{\eta^2}\cdot\left(1 - 2E^*_{t+1,T}a\eta^4\right) = -2\eta\gamma\pi_t^3 + \left(1 - 2E^*_{t+1,T}c\right)\pi_t^2 \\
&\qquad + 2\eta\gamma(1+2\nu)\cdot\pi_t^2 - (1+2\nu)\left(1 - 2E^*_{t+1,T}c\right)\pi_t \\
&\qquad - 2\eta\gamma\left(\nu + \frac{1}{2}\right)^2\pi_t + \left(1 - 2E^*_{t+1,T}c\right)\left(\nu + \frac{1}{2}\right)^2 \\
\Leftrightarrow\quad & p_3\cdot\pi_t^3 + p_2\cdot\pi_t^2 + p_1\cdot\pi_t + p_0 = 0,
\end{aligned} \tag{3.25}$$

where the coefficients of the polynomial in (3.25) are given by

$$p_3 = 2\eta\gamma, \tag{3.26a}$$

$$p_2 = -2\eta\gamma(2\nu+1) - \left(1 - 2E^*_{t+1,T}c\right), \tag{3.26b}$$

$$p_1 = 2\eta\gamma\left(\nu + \frac{1}{2}\right)^2 + (2\nu+1)\cdot\left(1 - 2E^*_{t+1,T}c\right), \tag{3.26c}$$

$$p_0 = \frac{1}{\eta^2}\left(1 - 2E^*_{t+1,T}a\eta^4\right) - \left(\nu + \frac{1}{2}\right)^2\cdot\left(1 - 2E^*_{t+1,T}c\right). \tag{3.26d}$$

The polynomial in (3.25) has three roots, and the number of real solutions among these can be deduced from the sign of the discriminant of the polynomial, which in the cubic case is known to be (Karpfinger and Meyberg, 2013, p. 366)

$$\mathcal{D} = 18\, p_3\, p_2\, p_1\, p_0 + p_2^2\, p_1^2 - 4\, p_3\, p_1^3 - 4\, p_2^3\, p_0 - 27\, p_3^2\, p_0^2,$$

where we have a single real solution (and two non-real complex conjugate roots) if $\mathcal{D} < 0$. □

The fact that the number of real solutions of a cubic polynomial can be deduced from the sign of its discriminant makes it worth pointing out that in Equations (3.26), the arrangement ensures that the signs of three coefficients can easily be derived. Here, we take into account Conditions (3.23) and use that both η and γ are assumed to be strictly negative, while, according to the estimates by Christoffersen et al. (2006, Table 2) and, more recently, by Babaoğlu et al. (2018, Table 2), ν is positive with order of magnitude 3. In particular, we have $p_3 > 0$ and $p_1 > 0$, while $p_2 < 0$ and no immediate decision can be made for p_0. However, from this position, the sign of $\mathcal{D}$ cannot be told immediately. We thus refer to our numerical analyses in Escobar-Anel et al. (2021), which show that for relevant values of γ, we have exactly one real solution to the optimality equation.

Note that we can make sure that Condition (3.11) is satisfied by assuming that the arguments of both square roots in (3.22) are positive, see (3.23). This implicitly imposes conditions on π_t^*, which need to be checked in practice. However, due to the order of magnitude of the parameter η being 10^{-4} (considering the estimates from Christoffersen et al. (2006) and Babaoğlu et al. (2018) again), the conditions can be expected to be satisfied at least by the optimal terminal solution.

3.5 Conclusion

This chapter presents an approximate closed-form solution to a portfolio optimization problem with one risky asset whose log return follows a general affine GARCH process, allowing for non-Gaussian innovations. The investor maximizes a CRRA utility from terminal wealth. Using a second-order approximation for the self-financing condition supported in the literature, we apply the value iteration (Bellman's principle) iteratively to obtain the optimal strategy in recursive form. The optimal wealth process is shown to follow an affine GARCH process as well. Furthermore, the IG-GARCH model is developed as a particular case of the main result, which constitutes the first EUT closed-form solution available in the literature for a non-Gaussian GARCH model. The analysis shows that we have a unique real solution to the optimality equation for popular sets of parameter estimates. One outstanding advantage of the new model is that asset returns follow a leptokurtic, negatively skewed distribution, allowing for an investigation of the impact of these non-Gaussian features on portfolio allocation.

4 Optimal Consumption and Investment

4.1 Introduction

After the introduction to the expected utility theory (EUT) problem in an affine GARCH framework in Chapter 3, this chapter presents the solution to an extended problem, where the investor may also derive utility from consumption at intermediate time points. Figure 4.1 illustrates that we target the vertical dimension in the diagram. Hence, the present chapter corresponds to the green box. This chapter is pubslished as a full-length paper, entitled "Optimal Consumption and Investment in General Affine GARCH Models" (Escobar-Anel et al., 2024b).

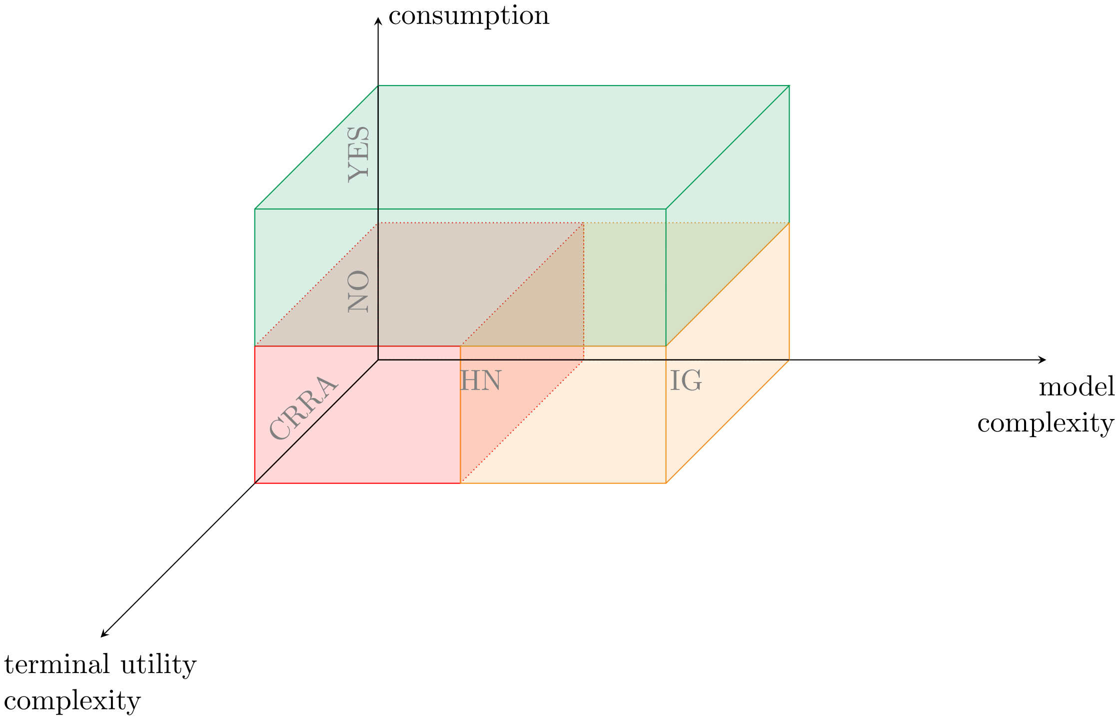


Figure 4.1: Overview of the contributions of this dissertation with regard to portfolio optimization under affine GARCH models, increasing the model complexity as well as the complexity of terminal and intermediate utility, see also Figures 1.1, 3.1, 5.1 and 6.1. The green box to Chapter 4.

The optimization of consumption and investment has been of great importance to insurance companies, pension funds, and individual investors for decades. We have already mentioned the seminal contributions made by Merton (1969, 1971), based on a geometric Brownian motion (GBM) model for the underlying risky asset with utility functions of the constant relative risk aversion (CRRA) and hyperbolic absolute risk aversion (HARA) type, respectively. Since then, the problem has been tackled for a variety of different models, including stochastic factors like volatility, interest rate, correlations, and different environments like complete and incomplete markets settings. We have pointed out in Chapter 1 that the vast majority of models considered in the literature have been continuous-time in nature, the reason being the availability of closed-form, analytical solutions rooted in the convenience of Itô's processes and partial differential equations. For instance, Kim and Omberg (1996) explored the behavior of investors maximizing HARA utility functions, with the risk premium of the single risky asset following an Ornstein-Uhlenbeck process. The original solution approach was complemented by a duality-based proof of the same problem provided by Battauz et al. (2015). As mentioned earlier, Kraft (2005) solved the portfolio allocation problem for the famous Heston (1993) model, while Liu and Pan (2003) found closed-form solutions in the presence of stochastic volatility and price jumps. Various consumption and investment problems have been tackled, e.g., by Chacko and Viceira (2005) (with Duffie and Epstein (1992a,b) utility) and in a multivariate setting by Liu (2007) (for CRRA utility).

In the area of discrete-time models, specifically for the most important class in finance and economics, i.e., GARCH processes, very little has been done regarding analytical portfolio optimization solutions. This is not surprising due to the lack of analytical representations for basic functions of the process, e.g., the conditional probability density. The difficulties in approaching GARCH models in this context can also be observed in Chen (2005), who could solve their rich optimization problem only via scenario generation, i.e., via finite discrete approximations of the return distribution. As mentioned earlier in this dissertation, the picture finally changed with the crucial paper by Heston and Nandi (2000), who introduced the first affine GARCH model (HN-GARCH) for pricing purposes. New affine GARCH models were added to the family in the following years, more notably, the IG-GARCH of Christoffersen et al. (2006), who extended the innovations from Gaussian to inverse Gaussian, hence allowing for negatively skewed and leptokurtic one-period stock returns, while incorporating the HN-GARCH as a limiting case. More recently, Badescu et al. (2019) provided a general definition for the class of affine GARCH models, requiring the conditional bivariate moment generating function (MGF) of the log-asset return and its conditional variance to be exponentially affine in the latter.

The benefit of this property, in the context of an investor maximizing a CRRA power utility function on terminal wealth, was first exploited by Escobar-Anel et al. (2022a) for the HN-GARCH model, and later generalized to the class of general affine GARCHs (see Chapter 3 and Escobar-Anel et al., 2021). In both cases, the authors use a well-established approximation of the self-financing condition (SFC) to produce explicit expressions for the optimal risky allocation, optimal wealth process, and value function.

These developments have excluded the critical aspect of investors' consumption preferences. Not only is consumption a crucial decision-making component for individuals, insurers and pension funds, but its presence can also drastically disrupt optimal risky allocations and wealth behavior. This chapter tackles this problem by considering a large family of affine GARCH models while allowing for consumption and terminal wealth preferences by the investor. We assume a special HARA utility for consumption with a cap on wealth, ensuring that consumption cannot become greater than wealth, which is a drawback of existing solutions in the literature. After the standard approximation of the SFC, we obtain closed-form solutions for optimal consumption, allocation strategy, wealth process, and value function. In particular, the work in this chapter differs from Chen (2005) in that the affine nature of our model allows for an entirely different solution approach without scenario generation.

The main contributions of this chapter are the following:

- We present the first analytical solutions for optimal consumption and allocation in an expected utility setting for affine GARCH models.

- Optimal wealth and consumption processes are shown to also follow affine GARCH models.[1]
- A numerical application to Gaussian and non-Gaussian GARCH, as well as to the homoscedastic model, reveals a significant impact of the presence/absence of consumption on the optimal allocation level. It also shows significant differences in risky allocations compared to homoscedastic cases, with the greater impact coming from heteroscedasticity rather than non-Gaussianity.
- In a study based on real-world market data, over a five-year horizon, the total consumption of investors following a GARCH strategy can be up to 10% higher than with the optimal homoscedastic solution at the same time also achieving 8% more terminal wealth.

The remainder of this chapter is structured as follows: In Section 4.2, we describe the consumption and investment problem and derive a formulation in terms of stochastic control models, starting with the standard approximation of the SFC in 4.2.1, formally introducing the class of affine GARCH models in 4.2.2 and presenting the final optimization problem in 4.2.3. Section 4.3 contains our main results, for a general affine GARCH model in Subsection 4.3.1 and for the two special cases, HN-GARCH and IG-GARCH, in 4.3.2. The numerical analysis in Section 4.4 is divided into a part about a study of the optimal solution to the problem in an IG-GARCH environment (Section 4.4.1) and an assessment of the strategy performance of a Gaussian GARCH model based on real-world data from the S&P 500 stock index in 4.4.2. We conclude in Section 4.5.

4.2 Problem Description

This section derives a formulation of our investment problem in the context of affine GARCH models, adapting the approach by Campbell and Viceira (1999) in the presence of consumption.

4.2.1 Self-Financing Condition

Let V_t denote the wealth at time t, and let $P_t^{(0)}$ and $P_t^{(1)}$ be the value of the bank account and the risky asset at time t, respectively. With π_t the fraction of wealth invested in the risky asset and describing the consumption at time t by C_t, we start with the exact relation

$$\frac{V_{t+1}}{V_t - C_t} = \pi_t \frac{P_{t+1}^{(1)}}{P_t^{(1)}} + (1 - \pi_t)\, \frac{P_{t+1}^{(0)}}{P_t^{(0)}}. \tag{4.1}$$

Two Taylor expansions of order 2 around 1, using the notation $\bar{C}_t = {}^{C_t}/_{V_t}$ for the relative consumption, yield

$$\log\left(\frac{P_{t+1}^{(1)}}{P_t^{(1)}}\right) \approx \frac{P_{t+1}^{(1)}}{P_t^{(1)}} - 1 - \frac{1}{2}\left(\frac{P_{t+1}^{(1)}}{P_t^{(1)}} - 1\right)^2,$$

$$\log\left(\frac{V_{t+1}}{V_t\left(1 - \bar{C}_t\right)}\right) \approx \frac{V_{t+1}}{V_t\left(1 - \bar{C}_t\right)} - 1 - \frac{1}{2}\left(\frac{V_{t+1}}{V_t\left(1 - \bar{C}_t\right)} - 1\right)^2.$$

[1] This is beneficial in terms of closed-form risk analysis and pricing of derivatives on the optimal portfolio.

We use the variance of the return instead of its square, denoted by $h_{t+1} = Var\left(\log\left({P_{t+1}^{(1)}}/{P_t^{(1)}}\right)\right)$ and obtain

$$\log\left(\frac{P_{t+1}^{(1)}}{P_t^{(1)}}\right) \approx \frac{P_{t+1}^{(1)}}{P_t^{(1)}} - 1 - \frac{1}{2} Var\left(\frac{P_{t+1}^{(1)}}{P_t^{(1)}} - 1\right) \approx \frac{P_{t+1}^{(1)}}{P_t^{(1)}} - 1 - \frac{1}{2} h_{t+1},$$

$$\log\left(\frac{V_{t+1}}{V_t\left(1-\bar{C}_t\right)}\right) \approx \frac{V_{t+1}}{V_t\left(1-\bar{C}_t\right)} - 1 - \frac{1}{2} Var\left(\frac{V_{t+1}}{V_t\left(1-\bar{C}_t\right)} - 1\right)$$
$$\approx \frac{V_{t+1}}{V_t\left(1-\bar{C}_t\right)} - 1 - \frac{1}{2}\pi_t^2 h_{t+1}.$$

Denoting the continuously compounded interest rate by $r > 0$, a linear approximation of the logarithm around 1 furthermore leads to

$$r = \log\left(\frac{P_{t+1}^{(0)}}{P_t^{(0)}}\right) \approx \frac{P_{t+1}^{(0)}}{P_t^{(0)}} - 1.$$

All in all, this leads to the following approximation of the SFC:

$$\log\left(\frac{V_{t+1}}{V_t\left(1-\bar{C}_t\right)}\right) + 1 + \frac{1}{2}\pi_t^2 h_{t+1} = \pi_t\left(\log\left(\frac{P_{t+1}^{(1)}}{P_t^{(1)}}\right) + 1 + \frac{1}{2} h_{t+1}\right) + \left(1-\pi_t\right)\left(1+r\right). \quad (4.2)$$

Solving for $W_t := \log V_t$, and defining $\hat{C}_t := \log\left(\frac{V_t}{V_t - C_t}\right)$ as well as $X_t := \log P_t^{(1)}$, we arrive at

$$W_{t+1} = W_t - \hat{C}_t + \pi_t\left(X_{t+1} - X_t\right) + \left(1-\pi_t\right) r + \frac{1}{2}\left(\pi_t - \pi_t^2\right) h_{t+1}. \quad (4.3)$$

Remark 4.1 (Relation to SFC approximation without consumption)**.** *Equation* (4.2) *is identical to the approach by Campbell and Viceira (1999) for a setting without consumption, used only for the reinvested part of wealth, and thus well-supported in the literature. In a homoscedastic setting, the authors combined this with a log-linearization around the mean consumption-wealth ratio, which is not necessary in the above approach. Note that there is no approximation concerning consumption in the procedure leading to* (4.3)*. Thus, examining the quality of this approach and the error of the approximation essentially boils down to the analysis performed in Escobar-Anel et al. (2021) for the case without consumption.*

4.2.2 Affine GARCH Models and Transition Function

Now assume a general affine GARCH setting as already presented in (3.1), with the log-asset return and its conditional variance being defined as

$$Y_t := X_t - X_{t-1} = f_1\left(h_t, \theta\right) + \sqrt{h_t}\,\epsilon_t, \quad (4.4a)$$
$$h_{t+1} = f_2\left(h_t, \theta\right) + f_3\left(h_t, \epsilon_t, \theta\right), \quad (4.4b)$$

according to Badescu et al. (2019), with the innovation denoted by ϵ_t and having mean zero, and θ being a vector of model parameters.[2] In particular, assume that the conditional bivariate

[2] Note that we do not require the innovations to be i.i.d., but rely on the structure of the MGF in (4.5) directly to also accommodate the IG-GARCH model. This framework forms a special case of the recent affine multifactor model introduced in Augustyniak et al. (2023).

MGF of the log-asset return and its conditional variance admits the representation

$$\begin{aligned}\Psi_{(Y_t,h_{t+1})}(u,v \mid \mathcal{F}_{t-1}) &:= \mathbb{E}\left[\exp\{uY_t + vh_{t+1}\} \mid \mathcal{F}_{t-1}\right] \\ &= \exp\{A(u,v) + B(u,v)\cdot h_t\}\end{aligned} \tag{4.5}$$

for some coefficients A and B dependent on the model parameters. On this basis, let $\mathbb{W} \subset \mathbb{R}$ be the set of possible values for the log wealth, and let $\mathbb{A}_\pi$ and $\mathbb{A}_{\hat{C}}$ be the sets of admissible values for the portfolio strategy and consumption control $\hat{C}$, respectively. We set $\mathbb{Y}$ the set of possible log return values and define $\mathbb{H} = (0,\infty)$. The transition function according to (4.3) thus can be written as $\mathbb{T} : \mathbb{W} \times \mathbb{H} \times \mathbb{A}_\pi \times \mathbb{A}_{\hat{C}} \times \mathbb{Y} \to \mathbb{W} \times \mathbb{H}$,

$$\begin{aligned}\mathbb{T}\left(W,h,\pi,\hat{C},Y\right) &= \left(\mathbb{T}_1\left(W,h,\pi,\hat{C},Y\right), \mathbb{T}_2(h,Y)\right) \\ &= \Bigg(W - \hat{C} + \pi Y + (1-\pi)r + \frac{1}{2}\left(\pi - \pi^2\right)h, \\ &\qquad f_2(h,\theta) + f_3\left(h, \frac{Y - f_1(h,\theta)}{\sqrt{h}}, \theta\right)\Bigg).\end{aligned} \tag{4.6}$$

Note that $\mathbb{W}, \mathbb{H}, \mathbb{A}_\pi$ and $\mathbb{A}_{\hat{C}}$ are intervals in $\mathbb{R}$ and thus convex sets.

4.2.3 Optimization Problem

We suppose that the investor derives utility from consumption at each time point $t = 0,\dots,T-1$, and utility from terminal wealth at time T. In particular, for some constant parameter $\bar{\gamma} > 1$, we consider utility functions for consumption of the form $u_t(C_t) = -a_t^{\bar{\gamma}}\left[(V_t - C_t)^+\right]^{\bar{\gamma}}$, with some time-dependent $a_t > 0$ and $t \in \{0,\dots,T-1\}$. This utility is concave, increasing in C_t on the interval $(0, V_t)$, and flat on $[V_t, \infty)$, implying that the current wealth will never be exceeded by consumption. Note that an increase in consumption at the same time negatively affects potential future utilities, see (4.3). This choice of utility is popularly known as HARA, see Merton (1971), and given in general form by

$$U_H(C) = \frac{1-\gamma_H}{\gamma_H}\left(\frac{\beta_H \cdot C}{1-\gamma_H} + \eta_H\right)^{\gamma_H}.$$

We can match Merton's and our notation as follows:

$$\gamma_H = \bar{\gamma}, \qquad \eta_H = \beta_H \cdot \frac{V_t}{\gamma_H - 1}, \qquad \beta_H = \frac{a}{b},$$

with

$$a = a_t(\gamma_H - 1), \qquad b = \left(\frac{\gamma_H - 1}{\gamma_H}\right)^{\frac{1}{\gamma_H}}.$$

In the sequel, we will work with the parameter $a_t := \bar{a}/V_t$, for some constant $\bar{a} > 0$, and define

$$U_t(c) = \begin{cases} u_t(c), & c \in [0,\infty), \\ -\infty, & \text{otherwise.}\end{cases}$$

In terms of the Arrow-Pratt measure for absolute risk aversion, we thus obtain for $C_t \in (0, V_t)$:

$$\mathcal{A}_t(C_t) = -\frac{U_t''(C_t)}{U_t'(C_t)} = \frac{\bar{\gamma} - 1}{V_t - C_t},$$

supporting the well-known feature of absolute risk aversion decreasing in wealth. Note that $\mathcal{A}_t(C_t)$ is increasing in $\bar{\gamma}$. In terms of the relative risk aversion, we obtain

$$\mathcal{R}_t(C_t) = -C_t \frac{U_t''(C_t)}{U_t'(C_t)} = \frac{C_t}{V_t - C_t} \cdot (\bar{\gamma} - 1) = \frac{\bar{C}_t}{1 - \bar{C}_t} \cdot (\bar{\gamma} - 1).$$

Furthermore, on $[0, V_t]$, we can transform $U_t(C_t)$ to be a function of our control on consumption $\hat{C}_t$ via

$$\begin{aligned} U_t(C_t) &= -a_t^{\bar{\gamma}} \left[(V_t - C_t)^+ \right]^{\bar{\gamma}} \\ &= -\bar{a}^{\bar{\gamma}} \left(\frac{V_t - C_t}{V_t} \right)^{\bar{\gamma}} &\quad (4.7) \\ &= -\bar{a}^{\bar{\gamma}} \exp\left\{ -\bar{\gamma} \log\left(\frac{V_t}{V_t - C_t} \right) \right\} \\ &= -\bar{a}^{\bar{\gamma}} \exp\left\{ -\bar{\gamma} \hat{C}_t \right\}, &\quad (4.8) \end{aligned}$$

which means that we are maximizing an exponential utility function w.r.t. $\hat{C}_t \in [0, \infty)$. For convenience, we denote U_t as a function of $\hat{C}_t$ as in (4.8) in the sequel. At time T, utility only depends on the final wealth V_T, following a CRRA power utility function $U_T(v) = \frac{1}{\gamma} v^\gamma$ for some $\gamma < 1$. This implies that the investor's absolute risk aversion $\mathcal{A}_T(v) = -\frac{U_T''(v)}{U_T'(v)} = \frac{1-\gamma}{v}$ is decreasing in γ. The Arrow-Pratt measure for relative risk aversion yields $\mathcal{R}_T(v) \equiv \mathcal{R}_T = 1 - \gamma$.

Following the strict separation between evaluating the utility of consumption at all intermediate time points and measuring the utility from terminal wealth via the power utility, we intend to solve:

$$\max_{\left\{ (\pi_s, \hat{C}_s) \right\}_t^{T-1}} \mathbb{E}\left[\sum_{s=t}^{T-1} \bar{\beta}^{s-t} \cdot U_s(C_s) + \bar{\beta}^{T-t} \cdot U_T(V_T) \;\middle|\; \mathcal{F}_t \right],$$

where $\bar{\beta} \in [0, 1]$ is the corresponding factor for intertemporal substitution. We work with the log wealth and, as outlined in Chapter 3, apply the value iteration in Theorem 2.15, using the SFC approximation (4.3). This leads to the following value function:

$$\begin{cases} \phi_T(W_T, h_{T+1}) \coloneqq \phi_T(W_T) = U_T\left(e^{W_T}\right) = \frac{1}{\gamma} \exp\{\gamma W_T\} \\ \phi_t(W_t, h_{t+1}) = \max\limits_{\pi_t, \hat{C}_t} \left\{ -\bar{a}^{\bar{\gamma}} \exp\left\{ -\bar{\gamma} \hat{C}_t \right\} + \bar{\beta} \cdot \mathbb{E}\left[\phi_{t+1}(\mathbb{T}(\Xi_t)) \;\middle|\; \mathcal{F}_t \right] \right\}, \\ t = 0, \ldots, T-1. \end{cases} \quad (4.9)$$

Here, $\Xi_t = \left(W_t, h_{t+1}, \pi_t, \hat{C}_t, Y_{t+1} \right)$.

Remark 4.2 (On the choice of utility functions)**.** *Concerning the optimization setting introduced above, note the following:*

(1) For $\bar{C}_t = C_t/V_t$ close to zero, the first-order Taylor approximation

$$\hat{C}_t = -\log\left(1 - \bar{C}_t\right) \approx \bar{C}_t$$

works well. Plugging this into the utility for consumption leads to

$$U_t(C_t) = -\bar{a}^{\bar{\gamma}} \exp\left\{ -\bar{\gamma} \hat{C}_t \right\} \approx -\bar{a}^{\bar{\gamma}} \exp\left\{ -\bar{\gamma} \bar{C}_t \right\} = -\bar{a}^{\bar{\gamma}} \exp\left\{ -\frac{\bar{\gamma}}{V_t} C_t \right\}.$$

Calculating the Arrow-Pratt measure of absolute risk aversion for the expression on the right-hand side yields $\mathcal{A}_t(C_t) = \bar{\gamma}/V_t$*, again supporting the idea of risk aversion decreasing with the level of wealth. Having the same relative risk aversion as for the CRRA function measuring utility from terminal wealth requires* $\mathcal{R}_t(C_t) = C_t \cdot \mathcal{A}_t(C_t) \approx \bar{C}_t\bar{\gamma} \stackrel{!}{=} 1-\gamma$*.*[3]

(2) *Dropping the convention that* $C_T = 0$ *leads to an optimization problem with one additional variable at time* T *– the proportion of wealth to consume at maturity. The question of how much to consume (value* C_T*, measured via utility for consumption) and how much to leave untouched (*$V_T - C_T$*, measured via power utility), will essentially be determined by the exact shape of the two utility functions.*

(3) *Using power utility also for consumption at time* $t \in \{0, \dots, T-1\}$ *leads to absolute risk aversion being inversely proportional to* C_t *and disconnected from wealth. Furthermore, there is no inherent bound preventing consumption from exceeding the current wealth. The approach above solves this puzzle by connecting consumption directly to wealth.*

4.3 Solution to the Portfolio Optimization Problem

4.3.1 Main Results

This section presents our main theoretical results, deriving an optimal strategy for the dynamic optimization problem in Section 4.2.3. For the sake of readability, the proofs are to be found in Appendix C.1.

In order to prove our main result in a general context, a structural result about the concavity of value functions in stochastic control models is needed, see Proposition 2.21 in the Mathematical Preliminaries. In particular, we refer to our model as an stochastic control model (SCM) with arbitrary disturbances, based on Defintion 2.12.

Remark 4.3. *We can relax the requirement of i.i.d. disturbances to accommodate the IG-GARCH model, asking that the distribution of innovations allow for the exponentially affine representation of the conditional bivariate generating function of the log-asset return and its conditional variance instead, since all necessary properties of the model are still given in each individual step.*

Theorem 4.4. *In the environment described above, assume that the log asset price of the risky asset follows an affine GARCH model, and that* $\gamma < 0$ *and* $\bar{\gamma} > 1$*. The value function can be written as*

$$\begin{aligned}\phi_t(W_t, h_{t+1}) = &- \bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t^*\right\} - \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t}\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_s^*\right\}\right] \\ &+ \frac{\bar{\beta}^{T-t}}{\gamma} \exp\left\{D_{t,T}(\pi_t^*) + E_{t,T}^{\hat{C}}\hat{C}_t^* + E_{t,T}^W W_t + E_{t,T}^h(\pi_t^*)h_{t+1}\right\},\end{aligned} \tag{4.10}$$

[3]The occurrence of relative consumption in this equation can be solved via considering the initial time point $t = 0$ and replacing the level of consumption implied by a homoscedastic Gaussian model in our setting with the solution by Merton (1969). Denoting Merton's solution at time $t = 0$, discretized for step length Δt equal to one day, by $C^{(M)}$, we set $\bar{C}_0 = \frac{C^{(M)}}{v_0}$, where v_0 denotes the investor's initial wealth. For more details, see Section 4.4.

with the optimal solution $\left\{\left(\pi_t^*, \hat{C}_t^*\right)\right\}_t$ *and coefficients* D, E^W, $E^{\hat{C}}$, E^h *as defined in Equation* (C.2) *in the proof. The optimal relative consumption* $\hat{C}_t^*$ *is affine in the log wealth and the conditional variance, and admits the representation*

$$\hat{C}_t^* = P_{t,T}\left(\pi_t^*\right) + Q_{t,T} \cdot W_t + R_{t,T}\left(\pi_t^*\right) \cdot h_{t+1}, \tag{4.11}$$

for coefficients P, Q, R *as defined in* (C.6)*. By denoting the coefficients of the bivariate MGF of the log-asset return and its conditional variance* A *and* B*, the optimal risky allocation* π_t^* *satisfies*

$$\frac{\partial}{\partial u} A\left(E_{t,T}^W \pi_t^*,\, E_{t+1,T}^h\left(\pi_{t+1}^*\right) + E_{t+1,T}^{\hat{C}} R_{t+1,T}\left(\pi_{t+1}^*\right)\right) = r, \tag{4.12a}$$

$$\frac{\partial}{\partial u} B\left(E_{t,T}^W \pi_t^*,\, E_{t+1,T}^h\left(\pi_{t+1}^*\right) + E_{t+1,T}^{\hat{C}} R_{t+1,T}\left(\pi_{t+1}^*\right)\right) = \pi_t^* - \frac{1}{2}. \tag{4.12b}$$

Remark 4.5 (On Theorem 4.4)**.**

(1) The conditional expectation in the value function (4.10) *can be solved recursively using* (4.11)*, the SFC approximation* (4.3) *and the conditional bivariate MGF of the log-asset return and its conditional variance in the underlying affine GARCH model.*

(2) Note that the (first-order) optimality equations for π_t^* *are the same as in the CRRA case without consumption, although the arguments of the coefficients* A *and* B *are different, of course.*

Using the approximation of the SFC (4.3) and the conditional bivariate generating function of the log-asset return and its conditional variance in the underlying affine GARCH model analogously to the proof of Theorem 4.4, we can establish that the optimal log-wealth process actually is an affine GARCH process itself.

Corollary 4.6 (Affine GARCH process for optimal log wealth)**.** *The optimal log-wealth process* $\{W_t\}_t$ *follows an affine GARCH process. Specifically,*

$$\begin{aligned}
&\Psi_{\left(W_t,(\pi_t^*)^2 h_{t+1}\right)}\left(u, v \mid \mathcal{F}_{t-1}\right) \\
&= \exp\Bigg\{ A\left(u\pi_{t-1}^*, v\left(\pi_t^*\right)^2\right) - uP_{t-1}\left(\pi_{t-1}^*\right) + \left(1-\pi_{t-1}^*\right) ur + u\left(1-Q_{t-1}\right) W_{t-1} \\
&\qquad + \frac{1}{\left(\pi_{t-1}^*\right)^2}\left[B\left(u\pi_{t-1}^*, v\left(\pi_t^*\right)^2\right) - uR_{t-1}\left(\pi_{t-1}^*\right) + \frac{u}{2}\left(\pi_{t-1}^* - \left(\pi_{t-1}^*\right)^2\right)\right] \cdot \left(\pi_{t-1}^*\right)^2 h_t \Bigg\}.
\end{aligned} \tag{4.13}$$

Furthermore, the optimal investment strategy π_t^* *does not depend on the conditional variance or the wealth.*

Remark 4.7. *Since the optimal control on consumption,* $\hat{C}_t^*$*, is affine in* W_t *and* h_{t+1}*, the MGF of* $\hat{C}_t^*$ *is also exponentially affine in the log wealth and the conditional variance of the log return.*

Using Corollary 4.6, and the MGF (4.13) in particular, we can now establish a shorter form of the value function (4.10).

Corollary 4.8. *For $t \in \{0, \dots, T-1\}$, the value function* (4.10) *can be written as*

$$\phi_t(W_t, h_{t+1}) = -\exp\left\{\log(T-t+1) + E^W_{t,T}(1-Q_{t,T})W_t + \left(E^h_{t,T}(\pi^*_t) - E^W_{t,T}R_{t,T}(\pi^*_t)\right)h_{t+1}\right\} - \sum_{\tau=t}^{T} \exp\left\{\tilde{D}_{t,\tau,T}\left(\{\pi_k\}^{\tau}_t\right)\right\}, \tag{4.14}$$

for deterministic coefficients $\tilde{D}_{t,\tau,T}$ with $\tau \in \{t, \dots, T\}$.

We omit the dependence of $\tilde{D}_{t,\tau,T}$ on the optimal strategy for readability.

Remark 4.9 (Wealth-equivalent loss). *Given the results of Corollary 4.8, we can introduce a concept for measuring the wealth-equivalent loss (WEL) arising from following a suboptimal strategy instead of the optimum according to Theorem 4.4. We assume the same transition law* (4.6) *for both the optimal and the suboptimal solution, dependent on $\left\{\left(\pi^*_t, \hat{C}^*_t\right)\right\}_t$ and $\left\{\left(\pi^s_t, \hat{C}^s_t\right)\right\}_t$, respectively. In particular, the value function for the suboptimal variant is given by*

$$\phi^s_t(W_t, h_{t+1}) = -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}^s_t\right\} + \bar{\beta} \cdot \mathbb{E}_t\left[\phi^s_{t+1}\left(\mathbb{T}\left(W_t, h_{t+1}, \pi^s_t, \hat{C}^s_t, Y_{t+1}\right)\right)\right].$$

We now define the wealth-equivalent loss from following the suboptimal strategy instead of the optimal as the solution L^s_t to the equation

$$\phi_t\left(\log\left(V_t\left(1-L^s_t\right)\right), h_{t+1}\right) = \phi^s_t\left(\log\left(V_t\right), h_{t+1}\right). \tag{4.15}$$

If the suboptimal allocation is deterministic and the suboptimal control on consumption is affine in the log wealth and the conditional variance of the log-asset return, by Corollary 4.8, (4.15) *is equivalent to*

$$\begin{aligned}
&\exp\left\{E^W_{t,T}(1-Q_{t,T})\log(V_t) + \left(E^h_{t,T}(\pi^s_t) - E^W_{t,T}R_{t,T}(\pi^s_t)\right)h_{t+1}\right\} \\
&- \exp\left\{E^W_{t,T}(1-Q_{t,T})\log\left(V_t\left(1-L^s_t\right)\right) + \left(E^h_{t,T}(\pi^*_t) - E^W_{t,T}R_{t,T}(\pi^*_t)\right)h_{t+1}\right\} \\
=&\sum_{\tau=t}^{T}\left(\exp\left\{\tilde{D}^*_{t,\tau,T} - \log(T-t+1)\right\} - \exp\left\{\tilde{D}^s_{t,\tau,T} - \log(T-t+1)\right\}\right).
\end{aligned} \tag{4.16}$$

Solving for L^s_t leads to a closed-form expression for the wealth-equivalent loss. The above requirement in particular accommodates all affine GARCH models as potential candidates for suboptimal choices.

Lastly, we note that the previously tackled investment problem for an investor deriving utility from terminal wealth only is nested in the more general environment described here.

Corollary 4.10 (Special case without consumption). *The setting in this document nests the setup without consumption, i.e., with an investor deriving utility from terminal wealth only, as presented in Chapter 3 (Escobar-Anel et al., 2022a, 2021), as a limiting case for the parameter choice $\bar{a} = 1$ and letting $\bar{\gamma} \to \infty$.*

4.3.2 Examples of Applications to Affine GARCH Models

HN-GARCH Model

Assume that the log asset price and its conditional variance follow an affine GARCH model as of Heston and Nandi (2000), and as already specified in (3.17), such that the set of parameters is given by $\theta = (\lambda, \omega, \beta, \alpha, \rho)$, $\{\epsilon_t\}_t$ is a sequence of standard normal innovations, and (4.4) is of the form

$$Y_t = X_t - X_{t-1} = r + \lambda h_t + \sqrt{h_t}\epsilon_t, \tag{4.17a}$$

$$h_{t+1} = \omega + \beta h_t + \alpha \left(\epsilon_t - \rho\sqrt{h_t}\right)^2. \tag{4.17b}$$

As presented in Appendix B.1, this implies in particular that for $t = 0, \dots, T-1$,

$$\begin{aligned} \Psi_{(Y_{t+1}, h_{t+2})}(u, v \mid \mathcal{F}_t) &:= \mathbb{E}\left[\exp\{u \cdot Y_{t+1} + v \cdot h_{t+2}\} \mid \mathcal{F}_t\right] \\ &= \exp\left\{A(u,v) + B(u,v) \cdot h_{t+1}\right\} \end{aligned} \tag{4.18}$$

with the two coefficients A and B given by

$$A(u,v) = ur + v\omega - \frac{1}{2}\log(1 - 2v\alpha), \tag{4.19a}$$

$$B(u,v) = u\lambda + v\left(\alpha\rho^2 + \beta\right) + \frac{(u - 2v\alpha\rho)^2}{2(1 - 2v\alpha)}. \tag{4.19b}$$

Therefore, the HN-GARCH model is an affine GARCH model and yields a special case of our main result. Using that the equations in (4.12) have the same structure as in the case without consumption, we obtain the following corollary:

Corollary 4.11 (HN-GARCH). *In the situation of Theorem 4.4, if the underlying log asset price process follows an HN-GARCH model and is given by* (4.17)*, the value function and the optimal control on consumption are given by* (4.10) *and* (4.11)*, respectively, using the explicit expressions* (4.19) *for the coefficients A and B. Concerning the optimal allocation,* (4.19) *reduces* (4.12) *to*

$$\pi_t^* = \frac{(\lambda + 1/2)(1 - 2v\alpha) - 2v\alpha\rho}{1 - E_{t,T}^W - 2v\alpha}, \tag{4.20}$$

using the abbreviation

$$v := E_{t+1,T}^h\left(\pi_{t+1}^*\right) + E_{t+1,T}^{\hat{C}} R_{t+1,T}\left(\pi_{t+1}^*\right). \tag{4.21}$$

Proof. See Appendix C.1. □

Remark 4.12 (On Corollary 4.11).

(1) A common way to look at an optimal allocation like (4.20) *is to split the solution into two terms. A myopic term, independent of the horizon and related to the celebrated solution by Merton (1973), and a time-dependent term coming from hedging the extra sources of randomness, see Kraft (2005) for a popular continuous-time example with stochastic volatility without consumption. The same exercise for the HN-GARCH without consumption, which is a special case of our equation* (4.20)*, can be found in Escobar-Anel et al. (2022a).*

(2) *In the presence of consumption, our Equation* (4.20) *is different from Merton's solution even in the absence of GARCH (i.e., in the homoscedastic case, where* $\alpha = \beta = 0$*). More importantly, in such a scenario, our optimal allocation is time-dependent and non-myopic (i.e.,* $\pi_t^* = \frac{\lambda+1/2}{1-\gamma\bar{\gamma}(\bar{\gamma}-(T-t-1)\gamma)^{-1}}$ *as opposed to* $\pi_t^M = \frac{\lambda+1/2}{1-\gamma}$ *for Merton), therefore the split into a myopic and a non-myopic term is no longer feasible.*

IG-GARCH Model

Assume that the log asset price and its conditional variance follow the IG-GARCH model (cf. Christoffersen et al., 2006), where the set of parameters is given by $\theta = (\nu, \eta, w, a, b, c)$, having $\eta < 0$. The log return and its conditional variance, as presented in (3.18), are given by

$$Y_t = X_t - X_{t-1} = r + \nu h_t + \eta y_t, \tag{4.22a}$$

$$h_{t+1} = w + b h_t + c y_t + a\frac{h_t^2}{y_t}, \tag{4.22b}$$

where $\{y_t\}_t$ is a sequence of random variables with inverse Gaussian distribution and single parameter $\delta(t) = \delta_t = h_t/\eta^2$. As stated by Christoffersen et al. (2006), the MGF is again given by (4.18) with the coefficients A and B defined as (see Appendix B.2)

$$A(u,v) = ur + vw - \frac{1}{2}\log\left(1 - 2va\eta^4\right), \tag{4.23a}$$

$$B(u,v) = vb + u\nu + \frac{1}{\eta^2}\left(1 - \sqrt{(1-2va\eta^4)(1-2u\eta-2vc)}\right). \tag{4.23b}$$

This matches the characterization of general affine GARCH models as of Section 4.2.2 and, together with the fact that the structure of the optimality equations (4.12) for the optimal risky allocation is the same as in the case without consumption, yields the following corollary:

Corollary 4.13 (IG-GARCH)**.** *In the situation of Theorem 4.4, if the underlying log asset price process follows an HN-GARCH model and is given by* (4.22)*, the value function and the optimal control on consumption are given by* (4.10) *and* (4.11)*, respectively, using the explicit expressions* (4.23) *for the coefficients* A *and* B*. Concerning the optimal allocation, plugging in* (4.23) *reduces* (4.12) *to*

$$\nu + \frac{\sqrt{1-2va\eta^4}}{\eta\sqrt{1-2\eta E_{t,T}^W \pi_t^* - 2vc}} = \pi_t^* - \frac{1}{2}, \tag{4.24}$$

where v *is again defined as in* (4.21)*.*

Proof. See Appendix C.1. □

Remark 4.14 (On Corollary 4.13)**.**

(1) *Note that Equation* (4.24) *is not linear in* π_t^**. Instead, separating radicals and squaring the equation yields a cubic equation. The optimal portfolio strategy* π_t^* *thus is a root of a polynomial of degree* 3*, the discriminant of which indicates the number of real solutions, see Section 3.4. In the case of a CRRA utility without consumption, we could show numerically that there is exactly one real solution (cf. Escobar-Anel et al., 2021).*

(2) *Given the connection between IG- and HN-GARCH models, we can show numerically, that the real solution to equation* (4.24) *converges to the solution of the HN-GARCH model in equation* (4.20). *To this end, we have to let* $\eta \to 0$ *according to the parametrization in Equation (6) in Christoffersen et al. (2006), see Figure 4.4 and the corresponding comments.*

4.4 Numerical Analysis

The purpose of this section is to investigate the behavior of the solution to our optimization problem numerically, based on the special cases of the IG-GARCH and the HN-GARCH model from Section 4.3.2, and on the homoscedastic case. Two different types of analysis are provided: The first focuses on the sensitivity of the optimal allocation and consumption strategy w.r.t. the model and investment parameters, and studies in particular the impact of conditional skewness and kurtosis on consumption and risky allocation. The second uses real-world data from the S&P 500 stock index to compare the strategy performance over two separate periods of five years each.

4.4.1 Optimal Solution Analysis

This subsection shows how the optimal solution to our optimization problem as presented in Section 4.3, implemented for the special cases of an HN-GARCH model, an IG-GARCH model, and a homoscedastic model (HS, constant variance), depends on important model and investment parameters. In particular, provided that the inverse Gaussian distribution embeds the HN and HS models while allowing for negatively skewed and leptokurtic innovations for the log-asset return, we also explore the impact of skewness and kurtosis on our solution. We will focus on the sensitivity of the optimal risky allocation and of the optimal consumption process, measured in relative consumption $\bar{C}_t^* = {}^{C_t^*}/_{V_t} = 1 - \exp\left\{-\hat{C}_t^*\right\}$. The analysis is conducted using the maximum likelihood estimates (MLEs) from Babaoğlu et al. (2018) as parameter values (see Table 4.1), an additional analysis with the earlier estimates from Christoffersen et al. (2006) can be found in the appendix. If not stated otherwise, the time horizon is five years with 252 trading days each, the interest rate is $r = {}^{0.01}/_{252}$, the parameter for intertemporal substitution is $\bar{\beta} = 1$ and the initial wealth is $v_0 = 1$. In view of Corollary 4.10, we work with $\bar{a} = 1$. Concerning the investor's risk aversion, the default value for utility from terminal wealth is $\gamma = -4$, corresponding to $\mathcal{R}_T = 5$ as the level of relative risk aversion. For the parameter $\bar{\gamma}$, we refer to Remark 4.2. The proxy for relative consumption needed to obtain the same level of risk aversion as at time T is derived via matching the consumption of a homoscedastic Gaussian model in our setting with the optimal solution proposed by Merton (1969) at the initial time point $t = 0$. Plugging in the corresponding value for relative consumption amounts to $\bar{\gamma} = 7.559 \times 10^3$.

Figure 4.2 shows how the optimal initial solution depends on the investor's time horizon and how solutions to different special cases of our general model compare. The plots show the optimal relative consumption and the optimal risky allocation, respectively, for the IG-GARCH model, for the HN-GARCH model, and for a homoscedastic Gaussian model. The MLEs for the parameter values are taken from Babaoğlu et al. (2018), and all models are calibrated to the same data set. For the optimal risky allocation, we also plot the solution without consumption, i.e., for an investor deriving utility from terminal wealth only. Given that this case is nested in our model as the limit for $\bar{\gamma} \to \infty$ (see Corollary 4.10), we calculate these solutions via setting this parameter to a very large value and leaving the rest unchanged. Here, we use $\bar{\gamma} = 1 \times 10^8$.

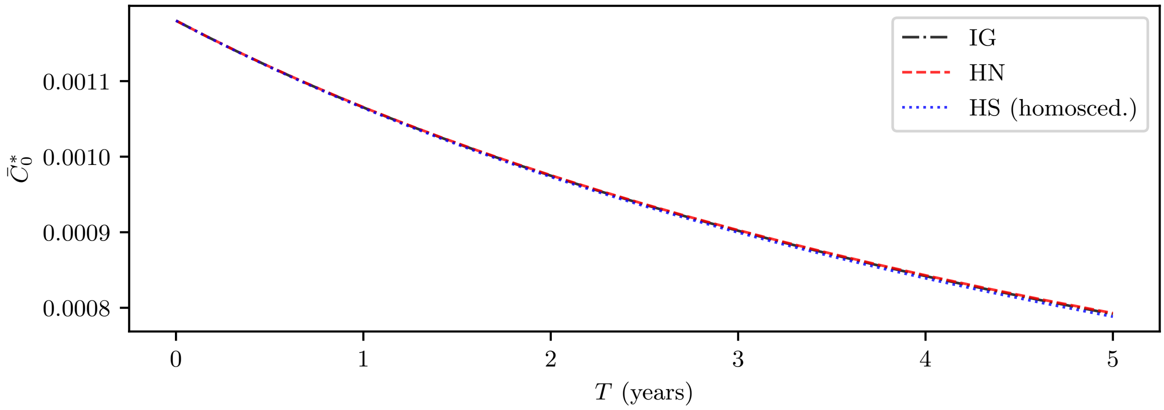
Table 4.1: Values for the parameters used for the IG-GARCH, the HN-GARCH model and the homoscedastic Gaussian – consisting of the MLEs as of Babaoğlu et al. (2018) for all parameters. For the IG-GARCH model, the value of ν implies that the market price of risk is $\nu + \eta^{-1} \approx 1.16$.

Panel A: IG-GARCH		**Panel B: HN-GARCH**		**Panel C: Hom. Gauss.**	
Param.	Values	Param.	Values	Param.	Values
ν	1.747×10^{3}	λ	1.100	λ	0.78
η	-5.729×10^{-4}	ω	-1.396×10^{-6}	ω	1.373×10^{-4}
w	-1.469×10^{-6}	β	9.000×10^{-1}		
a	3.190×10^{7}	α	3.761×10^{-6}		
b	-2.182×10^{1}	ρ	1.457×10^{2}		
c	4.047×10^{-6}				

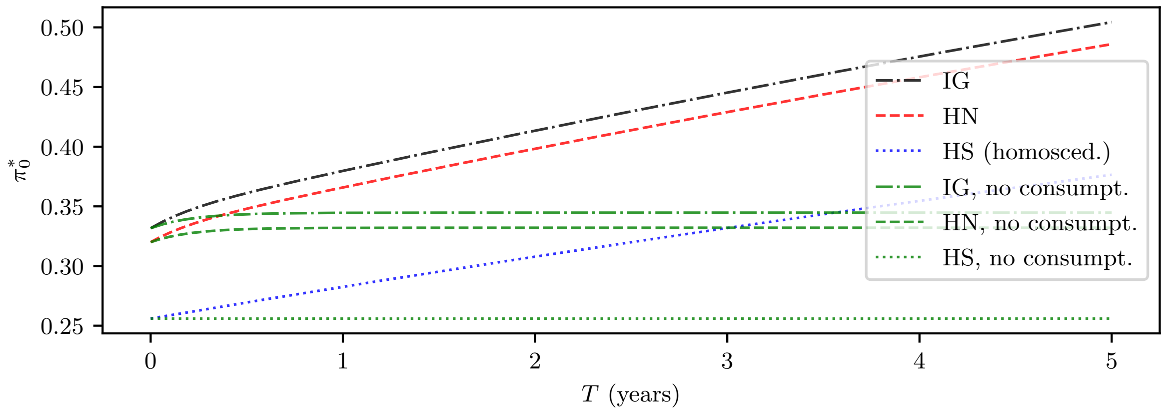
Figure 4.2a shows that the specific model for the log-asset return within the class of affine GARCH models is almost irrelevant for the optimal relative consumption – the difference is of order 1×10^{-6}. In this context, we note that, from the proof of Theorem 4.4, the coefficient $Q_{t,T}$ for the affine relation of $\hat{C}_t^*$ and W_t is independent of the specific model parameters. This picture is changed when we look at the optimal π_0^*: In Figure 4.2b, we observe a clear difference for the optimal risky allocation, with the highest fraction invested in the risky asset when the IG-GARCH model is used, directly followed by the Gaussian HN-GARCH model. With a larger difference to the heteroscedastic models, a homoscedastic investor invests the least fraction of wealth in the risky asset. Note that a similar pattern has been observed without consumption by Escobar-Anel et al. (2021). In general, the green lines reproducing the solutions without consumption show that the relative risky allocation is increased significantly over time as consumption is added to the model.

Since the IG-GARCH contains the HN-GARCH as a special case, we continue the analysis with the IG-GARCH model. Figure 4.3a shows the impact of the risk aversion parameters – with γ corresponding to the utility derived from terminal wealth (left plot) and $\bar{\gamma}$ corresponding to the utility derived from consumption (right plot) – on the initial value of relative consumption. We use $\gamma \in [-6, -0.1]$ and $\bar{\gamma} \in \left[5 \times 10^2, 1 \times 10^4\right]$ to cover reasonable risk aversion levels for terminal wealth and consumption, respectively. Our default parameter choice $\gamma = -4$ corresponds to a risk aversion level of $\mathcal{R}_T = 5$, and the default value $\bar{\gamma} = 7.559 \times 10^3$ is chosen according to Remark 4.2 based on this value. In general, risk aversion for intermediate consumption influences both relative consumption and the relative risky investment (right plots in Figure 4.3a and 4.3b), but the risk aversion for utility from terminal wealth only has a minor effect on consumption (left plot in Figure 4.3a); the decrease in $\bar{C}_0^*$ is much larger as $\mathcal{R}_0\left(\bar{C}_0^*\right)$ increases. The left plot in Figure 4.3b shows that as the investor's risk aversion concerning utility derived from terminal wealth decreases ($\mathcal{R}_T \to 0$), the optimal risky allocation increases. The impact is similar to the one caused by a decreasing risk aversion for consumption in the right plot in Figure 4.3b. Again, the optimal risky allocation increases as the investor's risk aversion decreases.

Given the presence of non-Gaussianity and heteroscedasticity in the model, we study the impact of these two features on the solution to our optimization problem next. Taking advantage of the



(a) Optimal initial relative consumption, dependent on the time horizon.



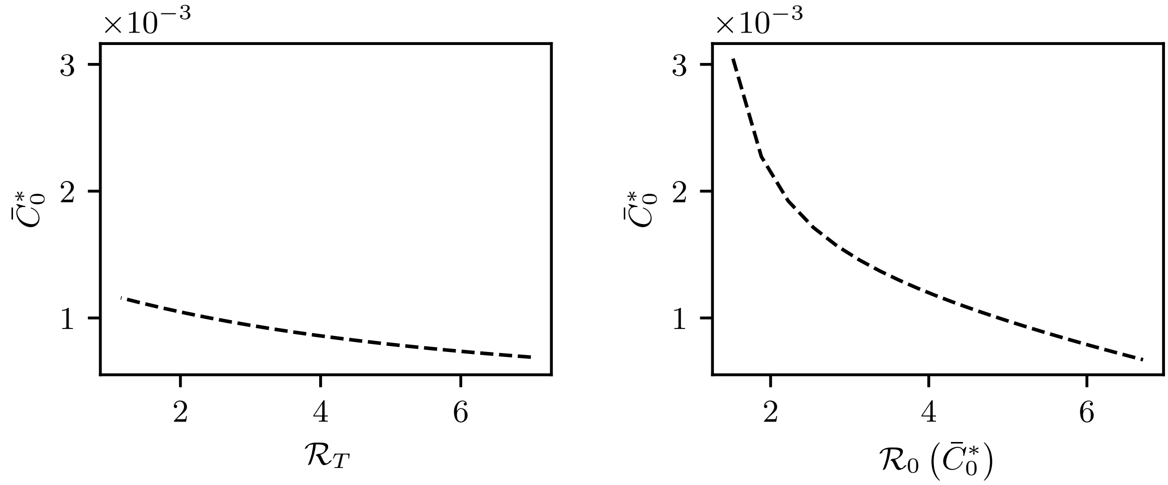
(b) Optimal initial relative risky allocation, dependent on the time horizon.

Figure 4.2: Values of the optimal initial strategy at time $t = 0$, dependent on the time horizon. The plots include an IG-GARCH, an HN-GARCH and a homoscedastic Gaussian, all fitted to the same data set. The solutions without consumption are obtained via setting $\bar{\gamma} = 1 \times 10^8$ to approximate the case without consumption. One year is assumed to have 252 trading days. We used the parameter MLEs of Babaoğlu et al. (2018).

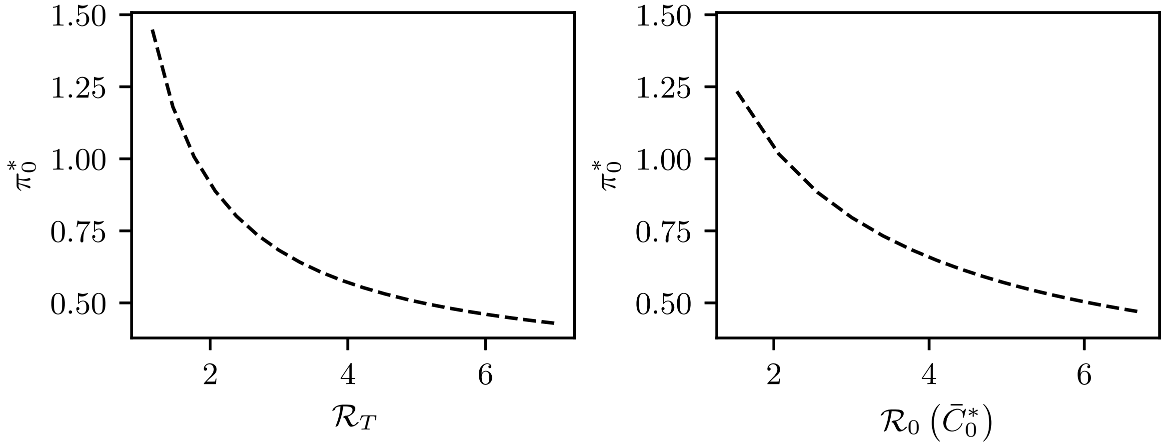
more general nature of the IG-GARCH among the three models, we transform this analysis into measuring the impact of skewness and kurtosis on the optimal solution while assuming the first two unconditional moments remain the same. As presented by Christoffersen et al. (2006), the IG-GARCH model allows for negatively skewed and leptokurtic log-asset returns. The one-step skewness $\mathcal{S}$ and excess kurtosis $\mathcal{K}$ are calculated dependent on the value of the parameter η via the formulas

$$\mathcal{S}_t\left[Y_{t+1}\right] = 3\eta\left(h_{t+1}\right)^{-1/2}, \qquad \mathcal{K}_t\left[Y_{t+1}\right] = 15\eta^2\left(h_{t+1}\right)^{-1}. \tag{4.25}$$

Since changes in η also influence the expected log-asset return and its variance, we apply the approach of Escobar-Anel et al. (2021) and adjust the remaining parameters in the IG-GARCH model with changes in η such that the first two moments are kept constant. This isolates the impact of negatively skewed and leptokurtic log-asset returns on the optimal strategy. The right plot in Figure 4.4 indicates that relative consumption decreases as negative skewness and excess kurtosis increase (η decreases), but the scale also shows that the impact is very little. The left plot shows a similar picture for the optimal risky allocation; we refer to Escobar-Anel et al. (2021) for more detailed analysis in the context of a pure investment problem and note



(a) Optimal initial relative consumption, dependent on the investor's risk aversion concerning utility from terminal wealth (left plot) and from consumption (right plot).



(b) Optimal initial relative risky allocation, dependent on the investor's risk aversion concerning utility from terminal wealth (left plot) and from consumption (right plot).

Figure 4.3: Values of the optimal initial relative consumption and optimal initial risky allocation, dependent on relative risk aversion. Relative risk aversion for terminal wealth in the left plots is governed by the parameter γ, and relative risk aversion for consumption in the right plots is modified via $\bar{\gamma}$, see Remark 4.2. The time horizon is five years.

that the direction of the impact of η on π_0^* is analogous if the model is extended by allowing for consumption. For both figures, we also insert the limiting HN-GARCH solution and the embedded HS solution, all with the same first two unconditional moments.

This figure conveys one of the main conclusions of this chapter, which is that the most consequential feature between heteroscedasticity and non-Gaussianity is the former. In light of this finding, we pick the HN-GARCH and the homoscedastic model for an application on real data in the next section.

Further parameters of the IG-GARCH model have a small impact of the order of 1×10^{-5} (0.001%) on the optimal relative consumption, see Appendix C.2.1. However, it is essential to note that changes in the parameters w and r do change the solution, as opposed to the special case without consumption.

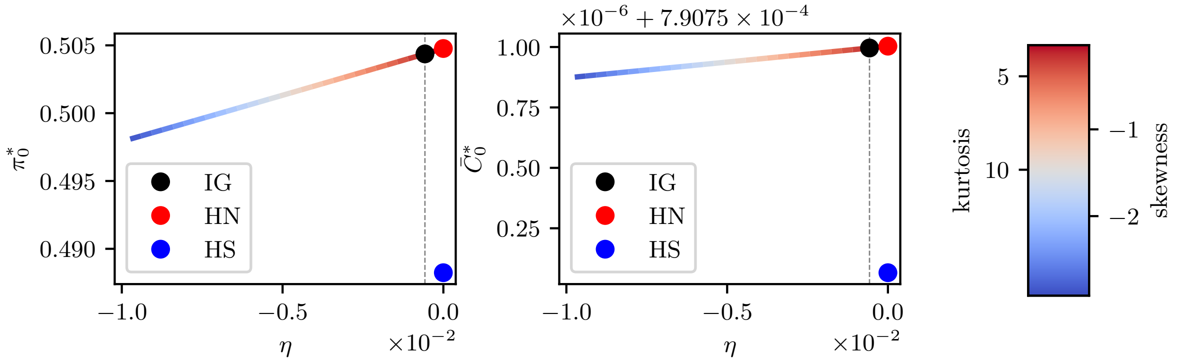


Figure 4.4: Values of the optimal initial risky allocation and relative consumption dependent on the parameter η. The time horizon is five years. The color bar on the right-hand side indicates the corresponding values for the skewness and excess kurtosis of log-asset returns. The vertical dashed line marks the original MLE for η in the considered parameter set. The first two moments are kept constant under changes in η. The blue and red dots mark the solutions for HN-GARCH and homoscedastic models with the same first two unconditional moments.

4.4.2 Data-based Performance Analysis

This section presents a data-based analysis of the strategy performance, using parameters estimated from S&P 500 return values in two periods from January 3, 2013, through December 30, 2017, and from January 3, 2018, through December 30, 2022. In our study, we focus on the HN-GARCH model in comparison to the embedded homoscedastic variant. As explained before in the context of Figure 4.4, skewness and kurtosis have very little effect on risky allocation and consumption. At the same time, the impact of heteroscedasticity is more significant, particularly on the optimal allocation. Therefore, we select the HN-GARCH and the homoscedastic as the models for the comparison.

For each of the two five-year periods, we estimate the HN-GARCH and the homoscedastic parameters using the returns-only estimation of Escobar-Anel et al. (2022b) for the path of S&P 500 log returns $\{x_t\}_{t=1,\ldots,T}$. Given the vector of parameters $\theta = (\lambda, \omega, \alpha, \beta, \rho)$, with $\alpha = \beta = 0$ in the homoscedastic case, and the log return x_{t-1}, the law for X_t is

$$X_t \mid (x_{t-1}, \theta) \sim \mathcal{N}\left(x_{t-1} + r + \lambda h_t, h_t\right). \tag{4.26}$$

The joint probability density function (PDF) $f_{X_1,\ldots,X_T}$ of the log returns can be expressed via conditioning on the previous values:

$$f_{X_1,\ldots,X_T}\left(x_1, \ldots, x_T\right) = f_{X_1}\left(x_1\right) \cdot f_{X_2|X_1}\left(x_2\right) \cdot \ldots \cdot f_{X_T|X_{T-1}}\left(x_T\right). \tag{4.27}$$

Combining (4.26) and (4.27) with the PDF of a normally distributed random variable and subsequently taking the logarithm leads to the log-likelihood function

$$l_T\left(x_1, \ldots, x_T \mid \theta\right) = -\frac{1}{2} \sum_{t=1}^{T} \left(\log\left(2\pi\right) + \log\left(h_t\right) + \epsilon_t^2\right), \tag{4.28a}$$

where

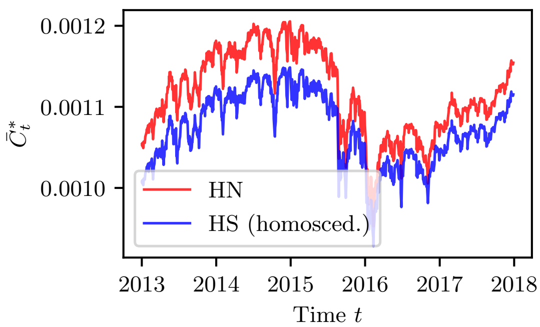
$$\epsilon_t = \frac{x_t - x_{t-1} - r - \lambda h_t}{\sqrt{h_t}}. \tag{4.28b}$$

The estimates $\hat{\theta}$ for θ are given by $\hat{\theta} = \arg\max_\theta l_T(x_1, \ldots, x_T \mid \theta)$. The standard errors of the estimates are calculated via the Fisher information matrix (see Equation 2.9). The combined results with standard errors are provided in Table 4.2.

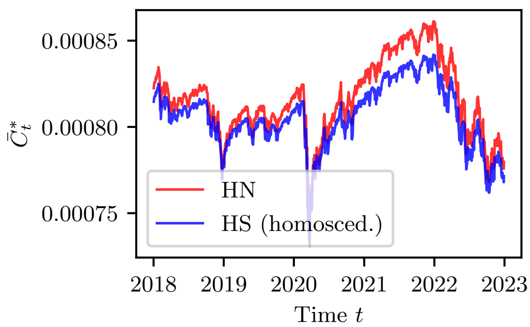
Table 4.2: MLEs for the HN-GARCH model and the homoscedastic model, based on the S&P 500 levels from January 3, 2013, through December 30, 2017, and from January 3, 2018, through December 30, 2022, with standard errors in parentheses.

Param.	**HN-GARCH**		**Homoscedastic**	
	2013-17	**2018-22**	**2013-17**	**2018-22**
λ	8.97	1.79	8.59	1.47
	(3.90)	(2.13)	(3.81)	(2.08)
ω	2.07×10^{-6}	4.61×10^{-8}	5.58×10^{-5}	1.91×10^{-4}
	(3.75×10^{-7})	(5.52×10^{-7})	(1.45×10^{-6})	(2.86×10^{-6})
α	3.05×10^{-6}	1.08×10^{-5}		
	(4.91×10^{-7})	(9.65×10^{-7})		
β	5.86×10^{-1}	7.16×10^{-1}		
	(4.26×10^{-2})	(2.92×10^{-2})		
ρ	3.22×10^{2}	1.41×10^{2}		
	(4.26×10^{1})	(1.30×10^{1})		

Based on these parameters, we calculate the optimal HN-GARCH and homoscedastic strategies for the default investment parameters described in Section 4.4.1 and evaluate the performance in-sample, with the S&P 500 in the role of the single risky asset in the market. The plots for



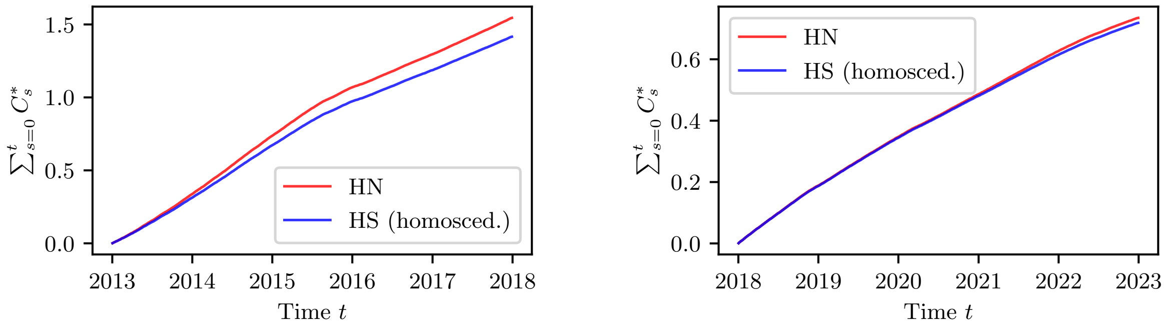
(a) Relative consumption from 2013-2017.



(b) Relative consumption from 2018-2022.

Figure 4.5: Relative consumption $\bar{C}_t^*$ for both periods of five years, plotting the HN-GARCH solution and the homoscedastic variant and using the parameters from Table 4.2.

relative consumption in Figure 4.5 show that, in general, the HN-GARCH and the homoscedastic solution follow the same major trends. However, in both Figure 4.5a and 4.5b, the blue line for the heteroscedastic model clearly exceeds the homoscedastic line for most of the time. The reverse, i.e., a period where the homoscedastic investor consumes significantly more than the one following the HN-GARCH, cannot be found.

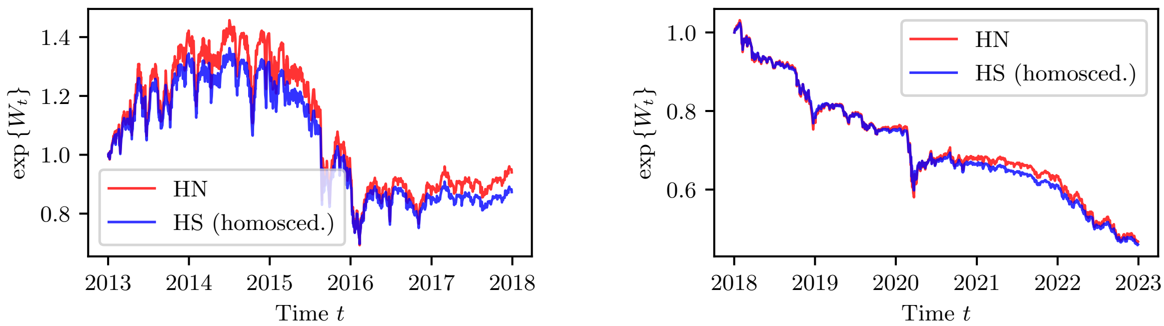


(a) Cumulative consumption from 2013-2017.

(b) Cumulative consumption from 2018-2022.

Figure 4.6: Cumulative consumption $\sum_{s=0}^{t} C_s^*$ for both periods of five years, plotting the HN-GARCH solution and the homoscedastic variant and using the parameters from Table 4.2.

This observation is also reflected in Figure 4.6, which plots the sum of all consumption values at intermediate time points up to time t for both periods. The HN-GARCH investor's total consumption exceeds the homoscedastic investor's in both cases, although the difference is larger in Figure 4.6a for the period 2013-2017, amounting to almost 10% more total consumption by an HN-GARCH investor.



(a) Wealth process from 2013-2017.

(b) Wealth process from 2018-2022.

Figure 4.7: The approximate wealth process $\exp\{W_t\}$ for both periods of five years, plotting the HN-GARCH solution and the homoscedastic variant and using the parameters from Table 4.2.

The wealth processes of the HN-GARCH and the homoscedastic model again develop simultaneously in both periods of five years, see Figure 4.7. A significant decrease in the level of wealth can be observed in 2015 and in the whole period from 2018 to the end of 2022. We note, however, that in both models, the initial level is clearly exceeded between 2013 and 2015. While there is a relatively rapid decrease in wealth in 2015, the total amount of money invested in the risky asset and the bank account falls gradually (except for the beginning of the pandemic in early 2020) during 2018-2022. For the period in Figure 4.7a, the non-consumed wealth of an HN-GARCH investor exceeds the wealth of a homoscedastic investor most of the time, and the former is 8% larger at December 30, 2018.

4.5 Conclusion

In this chapter, we derive the first closed-form solution to a consumption and investment problem for a model where the underlying log-asset price follows a general affine GARCH process. In particular, this general solution includes the HN-GARCH, the IG-GARCH, and a homoscedastic model as special cases. We show that the optimal log-wealth process and the optimal control on consumption also follow affine GARCH processes. Our setup includes the existing solution to the problem without consumption, i.e., for an investor maximizing only utility from terminal wealth, as a limiting case. Our numerical findings highlight a greater impact of heteroscedasticity on the solution than non-Gaussianity, where the latter is measured via skewness and excess kurtosis. Based on S&P 500 index data, in two periods of five years each, we analyze the strategy performance of a Gaussian GARCH model against a homoscedastic variant. Our real-world study shows that the heteroscedastic strategy can lead to up to 10% more total consumption while also ending up with 8% more wealth at the end of the investment horizon. These numbers suggest that the model choice is highly important for the strategy performance if consumption is included.

5 HARA Utilities and Mean-Variance Optimization

5.1 Introduction

This chapter presents an extension of the expected utility theory (EUT) problem for the more general class of hyperbolic absolute risk aversion (HARA) utility functions and for risk aversion levels previously excluded from consideration. Referring to Figure 5.1, where the contributions of Chapters 3 and 4 are already included, we now focus on generalizing the setup for utility derived from terminal wealth by adding the purple box. The essence of this paper was published in *Finance Research Letters* (Escobar-Anel et al., 2024a).

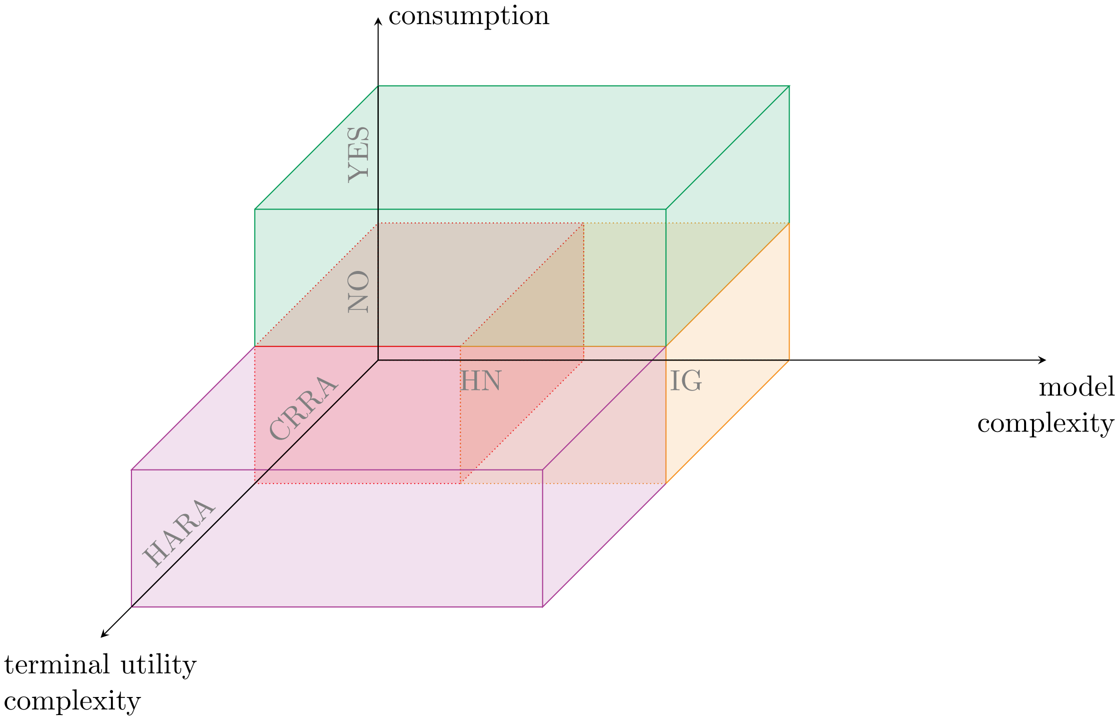


Figure 5.1: Overview of the contributions of this dissertation with regard to portfolio optimization under affine GARCH models, increasing the model complexity as well as the complexity of terminal and intermediate utility, see also Figures 1.1, 3.1, 4.1 and 6.1. The orange box coresponds to Chapter 3, the green box to Chapter 4, and the purple box to Chapter 5.

In the fifties, Harry M. Markowitz laid the foundation of modern portfolio theory, introducing the classical mean-variance (MV) problem in groundbreaking publications (Markowitz, 1952, 1959), earning him the Nobel Prize in 1990. The 1997 Nobel laureate, Robert C. Merton, further advanced portfolio analyses by finding, in the early seventies, closed-form solutions in EUT for a large family of utilities, some of which interpretable as mean-variance objective functions, (Merton, 1969, 1971). These timeless works remain cornerstones and sources of new applications in finance. In the eighties, Robert F. Engle developed the ARCH model (Engle, 1982), fundamentally changing the way market time series are analyzed and clearing the path to his Nobel Prize in 2003. Bollerslev (1986) extended Engle's work with the GARCH model before Heston and Nandi (2000) introduced the first affine GARCH, drastically improving usability with regard to pricing problems through the existence of closed-form solutions. As mentioned in the preceding chapters of this dissertation, other affine GARCH models were introduced by, e.g., Christoffersen et al. (2006, Inverse-Gaussian GARCH) and Ornthanalai (2014, Lévy GARCH). Finally, Badescu et al. (2019) defined a general framework for the class of affine GARCH models. More recent activities in the field of affine GARCH models include a multivariate setting introduced by Escobar-Anel et al. (2020), the derivation of semi-closed-form solutions for VIX and target volatility options (Cao et al., 2020), and closed-form expressions for risk-maximizing hedging strategies (Augustyniak and Badescu, 2021). A generalization of the framework by Badescu et al. (2019) was recently presented by Augustyniak et al. (2023).

Financial institutions still heavily rely on the closed-form solutions of Markowitz and Merton for portfolio decisions. In more sophisticated and realistic stochastic settings, such as those involving stochastic volatility or jumps, achieving reliable estimates of the underlying Itô processes is undeniably a challenging task. Furthermore, the lack of explicit formulas often leads to the use of models that fail to account for important stylized facts of asset returns, such as conditional heteroscedasticity. Our solution bridges this gap for practitioners seeking to move beyond simplistic models while benefiting from analytical solutions.

This work connects the three branches mentioned above, extending the recent developments in EUT for GARCH models of Escobar-Anel et al. (2021), presented in Chapter 3 in this dissertation, in two directions. The first concerns the class of utility functions: We go beyond the previously tackled subset of constant relative risk aversion (CRRA) and fully solve the whole constant relative risk aversion (CRRA) as well as the HARA class of utilities in Merton (1971). The second direction consists of solving the mean-variance problem after establishing a link via the work of Zhou and Li (2000) and Zhu and Escobar-Anel (2022).

Numerous results have been published in the fields of EUT and MV for rich continuous-time models – see, for instance, Liu (2007) for closed-form solutions to EUT problems, and Basak and Chabakauri (2010) for time consistent and inconsistent solutions within MV. To the best of our knowledge, no work concerning HARA utilities and mean-variance theory exists in the important world of affine GARCH models.

The contributions of this chapter are the following:

- We develop an innovated approach to approximate the self-financing condition (SFC) to accommodate HARA utility functions.
- We derive a closed-form optimal portfolio strategy for an investor maximizing a HARA utility function with general exponent $\gamma \in \mathbb{R} \setminus \{0, 1\}$, where the underlying stock exhibits general affine GARCH dynamics.
- We use the HARA approach to reveal solutions to two interesting problems for practitioners, the constant proportion portfolio insurance (CPPI) and the MV problem.
- We provide numerical insight to MV optimization with affine GARCH models via an analysis based on 20 years of S&P 500 data, showing that a heteroscedastic GARCH investor could outperform a homoscedastic strategy by three percentage points in annualized standard deviation over a five-year horizon.

The remainder of this chapter is organized as follows: In Section 5.2, we state the general optimization problem and present two approaches to deriving a suitable approximation for the self-financing condition (SFC). Section 5.3 contains the main theorem concerning the optimal solution. Section 5.4 connects the result to mean-variance and higher-order portfolio theory, and Section 5.5 presents the numerical analysis. We conclude in Section 5.6.

5.2 Formulation of the EUT Problem

Assume the investor maximizes a HARA utility function of terminal wealth given by

$$U(v) = \frac{1-b_H}{b_H}\left(\frac{a_H}{1-b_H}\cdot v + d_H\right)^{b_H} - e_H, \tag{5.1}$$

with $b_H < 1$, $b_H \neq 0$, $a_H > 0$ and $\frac{a_H}{1-b_H}\cdot v + d_H > 0$. Dropping the intercept e_H and setting $\gamma := b_H$, $K := (1-\gamma)^{1-\gamma}\cdot a_H^{\gamma}\cdot(\text{sgn}\,(v-L))^{\gamma}$, and $L := -\frac{d_H(1-\gamma)}{a_H}$, we can transform this function to a shorter representation:

$$U(v) = \frac{1}{\gamma}\cdot K\cdot|v-L|^{\gamma}, \tag{5.2}$$

Note that (5.2) emerges as a direct transformation from (5.1) via the definitions of K and L. In case $\gamma < 1$, the original constraint $\frac{a_H}{1-b_H}v + d_H > 0$ for (5.1) translates into $v > L$, and L can be interpreted as a lower bound on wealth that is exceeded with certainty. If $\gamma > 1$, the same constraint imposes an upper bound on wealth – in particular, the domain of v is bounded by $v - L < 0$. All in all, we thus require

$$\text{sgn}\,(v-L) = \text{sgn}\,(1-\gamma) \tag{5.3}$$

for (5.2), and consequently note that the expression for K simplifies to $K = \text{sgn}\,(1-\gamma)\cdot a_H^{\gamma}\cdot|1-\gamma|^{1-\gamma}$. In the sequel of this chapter, we will mainly use the representation in (5.2) and refer to the parameters γ, K, and L, with v bounded via (5.3).

Let π_t denote the share of wealth invested in the risky asset in the time interval $(t, t+1]$, and let $X_t = \log P_t^{(1)}$ be the log asset price. Assume that the log asset price and its conditional variance h_t follow an affine GARCH model that can be written in the general form as presented earlier in (3.1) and (4.4) (Badescu et al., 2019):

$$X_t - X_{t-1} = f_1(h_t, \theta) + \sqrt{h_t}\,\epsilon_t \tag{5.4a}$$

$$h_{t+1} = f_2(h_t, \theta) + f_3(h_t, \epsilon_t, \theta), \tag{5.4b}$$

for a vector of parameters θ and a sequence of innovations $\{\epsilon_t\}_t$. In particular, f_1 and f_2 are affine in h_t and f_3 is such that the conditional joint bivariate moment generating function (MGF) of the log-asset return and its conditional variance has the representation

$$\mathbb{E}\left[\exp\{u\cdot(X_t - X_{t-1}) + v\cdot h_{t+1}\} \mid \mathcal{F}_{t-1}\right] = \exp\Big\{A(u,v) + B(u,v)\cdot h_t\Big\}, \tag{5.5}$$

for some coefficients A and B dependent on the model parameters.

Maximizing utility from terminal wealth and taking into account the representation of our utility function in (5.2), we define $\hat{V}_t := |V_t - L_t|$, where V_t is the investor's wealth at time t and $L_t := L\cdot e^{-r\cdot(T-t)}$ with the continuously compounded interest rate r. The following two subsections present two different ways of obtaining the same approximation of $\log \hat{V}_t$, which will play a crucial role in the derivation of the optimal solution to the EUT problem in Section 5.3.

5.2.1 Taylor Approach

The budget constraint for the portfolio reads:

$$\frac{V_t - V_{t-1}}{V_{t-1}} = \pi_{t-1} \cdot \frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} + (1 - \pi_{t-1}) \cdot \frac{P_t^{(0)} - P_{t-1}^{(0)}}{P_{t-1}^{(0)}}, \tag{5.6}$$

with $\{V_t\}_t$, $\left\{P_t^{(1)}\right\}_t$ and $\left\{P_t^{(0)}\right\}_t$ being the wealth process, the price process of the risky asset and the price process of the riskless bank account, respectively.

Along the lines of Campbell and Viceira (1999) and Escobar-Anel et al. (2022a, 2021)[1], we transform returns into log returns by applying a second-order Taylor approximation of the logarithm around 1 to the processes $\log\left(\frac{V_t - L_t}{V_{t-1} - L_{t-1}}\right)$, $\log\left(\frac{P_t^{(1)}}{P_{t-1}^{(1)}}\right)$, $\log\left(\frac{P_t^{(0)}}{P_{t-1}^{(0)}}\right)$, $\log\left(\frac{L_t}{L_{t-1}}\right)$, as well as an approximation of the squared return, both consistent with the popular continuous-time literature. The approximation was demonstrated to be accurate in the context of affine GARCH models by Escobar-Anel et al. (2022a) and Escobar-Anel et al. (2021). Note that the cited works seek an approximation for $\log\left(\frac{V_t}{V_{t-1}}\right)$ and do not include the additional terms L_t and L_{t-1}. To obtain an extension based on the true SFC (5.6), we follow the steps below:

(a) Use a second-order Taylor approximation of the logarithm around 1 for $\log\left(\frac{V_t - L_t}{V_{t-1} - L_{t-1}}\right)$.

(b) Use the same second-order Taylor approximation of the logarithm of the return of the risky asset around 1 to obtain:

$$X_t - X_{t-1} = \log\left(\frac{P_t^{(1)}}{P_{t-1}^{(1)}}\right) \approx \frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} - \frac{1}{2}\left(\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}}\right)^2 \tag{5.7}$$

$$\Leftrightarrow \qquad \frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} \approx X_t - X_{t-1} + \frac{1}{2}\left(\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}}\right)^2. \tag{5.8}$$

Moreover, with a small size of the time step, we can approximate the squared return by its conditional variance in (5.8), analogously to a continuous-time setting. Subsequently, we observe from (5.8) that $Var\left[\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} \mid \mathcal{F}_{t-1}\right] \approx Var\left[X_t - X_{t-1} \mid \mathcal{F}_{t-1}\right] = h_t$. Together, this leads to:

$$\frac{P_t^{(1)} - P_{t-1}^{(1)}}{P_{t-1}^{(1)}} \approx X_t - X_{t-1} + \frac{1}{2}h_t. \tag{5.9}$$

(c) Use the linear approximation $\log x \approx x - 1$ with x close to 1 for the riskless asset and the discounted floor to find

$$\frac{P_t^{(0)} - P_{t-1}^{(0)}}{P_{t-1}^{(0)}} \approx \log\left(\frac{P_t^{(0)}}{P_{t-1}^{(0)}}\right) = r, \qquad L_t - L_{t-1} = \left(e^r - 1\right) L_{t-1} \approx r L_{t-1}. \tag{5.10}$$

[1] See also Sections 3.2 and 4.2.1.

(d) Use the same approach as in (b), now for the return on wealth, noting that, according to (5.9), $Var\left[\frac{V_t - V_{t-1}}{V_{t-1}} \mid \mathcal{F}_{t-1}\right] \approx \pi_{t-1}^2 \cdot h_t$. This leads to the approximation

$$\left(\frac{V_t - V_{t-1}}{V_{t-1}} - \frac{L_t - L_{t-1}}{V_{t-1}}\right)^2 \approx \pi_{t-1}^2 \cdot h_t. \tag{5.11}$$

All in all, with $\hat{\pi}_{t-1} = \pi_{t-1} \cdot \frac{V_{t-1}}{V_{t-1} - L_{t-1}}$, we obtain

$$\begin{aligned}
\log\left(\frac{V_t - L_t}{V_{t-1} - L_{t-1}}\right) &\overset{(a)}{\approx} \left(\frac{V_t - L_t}{V_{t-1} - L_{t-1}} - 1\right) - \frac{1}{2}\left(\frac{V_t - L_t}{V_{t-1} - L_{t-1}} - 1\right)^2 \\
&= \left(\frac{V_{t-1}}{V_{t-1} - L_{t-1}}\right)\left(\frac{V_t - L_t - V_{t-1} + L_{t-1}}{V_{t-1}}\right) \\
&\quad - \frac{1}{2}\left(\frac{V_{t-1}}{V_{t-1} - L_{t-1}}\right)^2 \left(\frac{V_t - V_{t-1}}{V_{t-1}} - \frac{L_t - L_{t-1}}{V_{t-1}}\right)^2 \\
&\overset{(c),(d)}{\approx} \left(\frac{V_{t-1}}{V_{t-1} - L_{t-1}}\right)\left(\frac{V_t - V_{t-1}}{V_{t-1}} - \frac{L_{t-1}\left(e^r - 1\right)}{V_{t-1}}\right) \\
&\quad - \frac{1}{2}\left(\frac{V_{t-1}}{V_{t-1} - L_{t-1}}\right)^2 \pi_{t-1}^2 h_t \\
&\overset{(b),(c)}{\approx} \left(\frac{V_{t-1}}{V_{t-1} - L_{t-1}}\right)\left(\pi_{t-1}\left(X_t - X_{t-1} + \frac{1}{2}h_t\right) + \left(1 - \pi_{t-1}\right) r - \frac{L_{t-1}}{V_{t-1}} r\right) \\
&\quad - \frac{1}{2}\left(\frac{V_{t-1}}{V_{t-1} - L_{t-1}}\right)^2 \pi_{t-1}^2 h_t \\
&= \hat{\pi}_{t-1}\left(X_t - X_{t-1} + \frac{1}{2}h_t\right) + \left(1 - \hat{\pi}_{t-1}\right) r - \frac{1}{2}\hat{\pi}_{t-1}^2 h_t.
\end{aligned} \tag{5.12}$$

This approach is independent of $V_t - L_t$ being positive or negative. If $V_t - L_t$ and $V_{t-1} - L_{t-1}$ are both negative, we arrive at the same equation (5.12), but with π_{t-1} and $\hat{\pi}_{t-1}$ having different signs. Reordering terms in (5.12), we obtain the following expression for $\hat{W}_t$ as an approximation of $\log \hat{V}_t$:

$$\hat{W}_t = \hat{W}_{t-1} + \hat{\pi}_{t-1} \cdot \left(X_t - X_{t-1}\right) + \left(1 - \hat{\pi}_{t-1}\right) r + \frac{1}{2}\left(\hat{\pi}_{t-1} - \hat{\pi}_{t-1}^2\right) h_t. \tag{5.13}$$

We note that this expression is identical to the approximation of the log wealth in (3.7) for the CRRA case (cf. Escobar-Anel et al., 2021) if $\gamma < 0$ and $L = 0$, which holds exactly in continuous time when Itô's Lemma can be applied to the budget constraint (5.6).

5.2.2 CPPI Approach

Alternatively to the approach presented in Section 5.2.1, we may treat the modified CRRA utility in (5.2) as a CPPI setting.[2] Assume that $L \leq v_0 \cdot e^{rT}$, with v_0 the initial wealth. Note that this yields an implicit condition on the parameter d_H of the original HARA utility function, since for $\gamma < 1$

$$L \leq v_0 \cdot e^{rT} \quad \Leftrightarrow \quad d_H \geq -\frac{a_H \cdot v_0 \cdot e^{rT}}{1 - \gamma}. \tag{5.14}$$

[2] We restrict this approach to $\gamma < 1$, where (5.3) yields the relation $V_T \geq L$.

Assuming d_H as fixed, this can also be seen as a restriction on the investor's initial wealth. Now define the current (time-dependent) level of the *floor* L_t as above via

$$L_t := L \cdot e^{-r\cdot(T-t)}.$$

L_t can be interpreted as the amount of money that needs to be invested in the riskless asset at time t to reach the terminal threshold L for sure. Furthermore, define $\hat{V}_t := V_t - L_t$ as the *cushion*, i.e., the money beyond the threshold L_t that is used for return maximization. For this part of the wealth, $\hat{V}_t$, we can choose our allocation between the riskless and the risky asset freely. This leads to the existing approach using CRRA utility as of Chapter 3, replacing V_t by $\hat{V}_t$ and writing $\hat{\pi}_t$ for the share of $\hat{V}_t$ that is invested in the risky asset in the time interval $(t, t+1]$. In particular, we approximate $\hat{V}_t$ via $e^{\hat{W}_t}$, where

$$\hat{W}_t = \hat{W}_{t-1} + \hat{\pi}_{t-1} \cdot (X_t - X_{t-1}) + (1 - \hat{\pi}_{t-1}) \cdot r + \frac{1}{2}\left(\hat{\pi}_{t-1} - \hat{\pi}_{t-1}^2\right) \cdot h_t, \tag{5.15}$$

according to (3.7). Note that (5.15) is identical to (5.13). Furthermore, the above reasoning implies that the wealth dropping below the floor in this environment is equivalent to the wealth dropping below zero in the CRRA environment, which is excluded by construction.[3]

5.3 Optimal Investment Maximizing HARA Utility

Now, with $T > 0$ some finite time horizon, and the log asset price following an affine GARCH model, consider the expected utility problem from terminal wealth for an investor maximizing a HARA utility function U in the form of (5.2). Our goal is to solve the following optimization problem:

$$\max_{\{\pi_s\}_t^{T-1}} \frac{K}{\gamma} \mathbb{E}_t \left[\exp\left\{\gamma \hat{W}_T\right\}\right] =: \phi_t\left(\hat{W}_t, h_{t+1}\right), \tag{5.16}$$

with $\mathbb{E}_t$ again denoting the conditional expectation under $\mathcal{F}_t$ at time t. On the optimal solution to this problem, we establish the following result:

Theorem 5.1. *Assume that the log asset price follows an affine GARCH model, and assume that the investor maximizes a HARA utility function of terminal wealth in the form of* (5.2) *with a general exponent* $\gamma \in \mathbb{R} \setminus \{0, 1\}$. *Furthermore, assume that the coefficient* A *of the conditional joint bivariate MGF* (5.5) *satisfies*

$$\frac{\partial}{\partial u} A(u, v) = r \qquad \forall u, v \in \mathbb{R}. \tag{5.17}$$

Then, the solution to the problem (5.16) *is given by*

$$\phi_t\left(\hat{W}_t, h_{t+1}\right) = \frac{K}{\gamma} \exp\left\{\gamma \hat{W}_t + D_{t,T}\left(\hat{\pi}_t^*\right) + E_{t,T}\left(\hat{\pi}_t^*\right) \cdot h_{t+1}\right\}, \tag{5.18}$$

[3] Since we are working with an approximation of the log wealth, the wealth process in the model is always going to stay positive. Plugging the model parameters, e.g., of the HN-GARCH model, into the true budget constraint (5.6) and then solving for the innovation of the log-asset return under the condition that $V_t \leq 0$ yields an interval for the innovation where the actual wealth process would be pulled below zero. Working with parameter estimates for the HN-GARCH model (Christoffersen et al., 2006) and numerically evaluating the probability of the innovation ending up in this extreme interval (e.g., via the software R) yields zero, i.e., the probability is so small that it is not numerically measurable.

with some recursive coefficients D and E dependent on the (partial) optimal solution $\hat{\pi}_t^*$, which solves the optimality equation

$$\frac{\partial}{\partial u} B\left(\gamma\hat{\pi}_t^*, E_{t,T}\left(\hat{\pi}_t^*\right)\right) = \hat{\pi}_t^* - \frac{1}{2} \tag{5.19}$$

and the second-order condition

$$\frac{\partial^2}{\partial u^2} B\left(\gamma\hat{\pi}_t^*, E_{t,T}\left(\hat{\pi}_t^*\right)\right) \begin{cases} > \frac{1}{\gamma}, & \gamma \in \mathbb{R}\setminus[0,1], \\ < \frac{1}{\gamma}, & \gamma \in (0,1). \end{cases} \tag{5.20}$$

The optimal strategy π_t^* is obtained via

$$\pi_t^* = \hat{\pi}_t^* \cdot \frac{V_t - L_t}{V_t}. \tag{5.21}$$

Remark 5.2 (On Theorem 5.1)**.**

(1) Theorem 5.1 generalizes the known solution presented in Chapter 3 (Escobar-Anel et al., 2021) in several directions. Besides the use of the more general HARA class of utility functions, the range of exponents γ is extended to investors with very low risk aversion levels ($0 < \gamma < 1$*) and to exponents $\gamma > 1$, which establishes a connection to mean-variance theory and higher-moment portfolio selection, as we will see in the following sections.*

(2) The additional requirement $\frac{\partial}{\partial u} A\left(u,v\right) = r$ is satisfied for both the HN-GARCH (Heston and Nandi, 2000) and the IG-GARCH (Christoffersen et al., 2006) model, which form the two well-known special cases of the general formulation of an affine GARCH model as of (5.4)*. The condition can be relaxed to $\frac{\partial^2}{\partial^2 u} A\left(u,v\right) \geq 0$, and the theorem still stands, although with more tedious expressions, especially for the second-order conditions.*

(3) The optimality equation (5.19) *is the same for all values of γ. This includes, in particular, the known case of $\gamma < 0$ with the second-order condition* (5.20)*.*

Proof of Theorem 5.1. Focusing on the partial strategy $\hat{\pi}_t$ only, we assume the investor's initial wealth to be equal to $\hat{v}_0 = v_0 - L_0$, we set $\hat{W}_0 = \log\hat{v}_0$, and approximate the log wealth via (5.13). This sub-problem is identical to the case of an investor maximizing a CRRA utility and can be solved recursively for the deterministic optimal strategy $\{\hat{\pi}_t^*\}_{t=0}^{T-1}$ via guessing the correct form for the value function, using the MGF (5.5), and then applying Bellman's value iteration to obtain (5.18), see Theorem 3.1. $D_{t,T}$ and $E_{t,T}$ are defined via $D_{T,T} = E_{T,T} = 0$ and via the coefficients A and B in (5.5):

$$D_{t,T}(\hat{\pi}_t) = D_{t+1,T}(\hat{\pi}_{t+1}^*) + (1-\hat{\pi}_t)\,\gamma r + A\left(\gamma\hat{\pi}_t, E_{t+1,T}(\hat{\pi}_{t+1}^*)\right), \tag{5.22a}$$

$$E_{t,T}(\hat{\pi}_t) = B\left(\gamma\hat{\pi}_t, E_{t+1,T}(\hat{\pi}_{t+1}^*)\right) + \frac{\gamma}{2}\left(\hat{\pi}_t - \hat{\pi}_t^2\right). \tag{5.22b}$$

Taking derivatives of the objective function w.r.t. $\hat{\pi}_t$, using the definitions of $D_{t,T}$ and $E_{t,T}$, yields the equations (5.17) and (5.19). These first-order conditions are identical to the CRRA case, and (5.17) is satisfied by assumption. Concerning the second-order condition, assume that

$\hat{\pi}_t^*$ satisfies (5.19) and define

$$\begin{aligned} f(\hat{\pi}_t) &= \frac{K}{\gamma} \exp\left\{\gamma \hat{W}_t + D_{t,T}(\hat{\pi}_t) + E_{t,T}(\hat{\pi}_t) \cdot h_{t+1}\right\}, \\ g(\hat{\pi}_t) &= -\gamma r + \gamma \frac{\partial}{\partial u} A\left(\gamma \hat{\pi}_t, E_{t+1,T}\left(\hat{\pi}_{t+1}^*\right)\right) \\ &\quad + \left(\frac{\gamma}{2} - \gamma \hat{\pi}_t + \gamma \frac{\partial}{\partial u} B\left(\gamma \hat{\pi}_t, E_{t+1,T}\left(\hat{\pi}_{t+1}^*\right)\right)\right) \cdot h_{t+1} \\ &= \left(\frac{\gamma}{2} - \gamma \hat{\pi}_t + \gamma \frac{\partial}{\partial u} B\left(\gamma \hat{\pi}_t, E_{t+1,T}\left(\hat{\pi}_{t+1}^*\right)\right)\right) \cdot h_{t+1}, \end{aligned}$$

using again (5.17) in the last equality. Note that $g(\hat{\pi}_t^*) = 0$ due to (5.19), and the second derivative of the objective function f can be expressed as

$$\frac{\partial^2}{\partial \hat{\pi}_t^2} f(\hat{\pi}_t) = f(\hat{\pi}_t)\left[g^2(\hat{\pi}_t) + \left(\gamma^2 \frac{\partial^2}{\partial u^2} B\left(\gamma \hat{\pi}_t, E_{t+1,T}\left(\hat{\pi}_{t+1}^*\right)\right) - \gamma\right) h_{t+1}\right]. \tag{5.23}$$

We distinguish between two cases to justify (5.20):

- $\gamma \notin [0,1]$: If $\gamma < 0$ or $\gamma > 1$, we have $\frac{K}{\gamma} < 0$ and negativity of (5.23) at the extreme point can be ensured via making the coefficient in front of the conditional variance h_{t+1} positive.
- $\gamma \in (0,1)$: In this case, $\frac{K}{\gamma} > 0$, implying that the coefficient in front of h_{t+1} in (5.23) needs to be negative.

The actual strategy π_t^* can be derived via scaling the partial strategy $\hat{\pi}_t^*$, including also the additional investment L_t in the riskless asset. Using the true process V_t, this leads to a share of $\hat{\pi}_t^* \cdot \frac{V_t - L_t}{V_t}$ of the total wealth invested in the risky asset, and to a proportion of $1 - \hat{\pi}_t{}^* \cdot \frac{V_t - L_t}{V_t} = (1 - \hat{\pi}_t^*) \cdot \frac{V_t - L_t}{V_t} + \frac{L_t}{V_t}$ for the riskless asset. □

5.4 Connection to Mean-Variance Theory

In a dynamic mean-variance portfolio problem, the objective is to maximize expected terminal wealth, given an upper bound for the variance of terminal wealth. The Pareto-optimal solutions to this trade-off can be found by solving the Markowitz-λ problem

$$\max_{\{\pi_s\}_t^{T-1}} \left\{\mathbb{E}\left[V_T \mid \mathcal{F}_t\right] - \lambda \cdot Var\left[V_T \mid \mathcal{F}_t\right]\right\} \tag{5.24}$$

for different values of $\lambda > 0$. The optimization problem in (5.24) is not solvable via dynamic programming due to time inconsistency, but the following result, in its original version presented by Zhou and Li (2000, Thm. 3.1)[4] provides a workaround.

Proposition 5.3 (cf. Zhou and Li, 2000, Theorem 3.1)**.** *Any pre-commitment solution to* (5.24) *will also solve*

$$\min_{\{\pi_s\}_t^{T-1}} \left\{\mathbb{E}\left[(V_T - \mu)^2 \mid \mathcal{F}_t\right]\right\}, \tag{5.25}$$

where $\mu := \frac{1}{2\lambda} + \mathbb{E}\left[V_T \mid \mathcal{F}_t\right]$.

[4]See also Zhu and Escobar-Anel (2022, Thm. 2).

Proof of Proposition 5.3. Assume that $\pi = \{\pi_t\}_{t=0}^{T-1}$ is optimal for (5.24), and denote the corresponding wealth process by V. Furthermore, assume that π is not optimal for (5.25), i.e., assume the existence of a pair π^* and V^* such that

$$\begin{aligned} & \mathbb{E}_t\left[(V_T-\mu)^2\right] > \mathbb{E}_t\left[(V_T^*-\mu)^2\right] \\ \Leftrightarrow\quad & \mathbb{E}_t\left[V_T^2\right] - 2\mu\mathbb{E}_t\left[V_T\right] + \mu^2 > \mathbb{E}_t\left[(V_T^*)^2\right] - 2\mu\mathbb{E}_t\left[V_T^*\right] + \mu^2 \\ \Leftrightarrow\quad & \mathbb{E}_t\left[V_T^2\right] - 2\mu\mathbb{E}_t\left[V_T\right] > \mathbb{E}_t\left[(V_T^*)^2\right] - 2\mu\mathbb{E}_t\left[V_T^*\right]. \end{aligned} \tag{5.26}$$

As in Zhou and Li (2000), we define $f(x,y) := \lambda x - \lambda y^2 - y$, which is a concave function in (x,y) having first partial derivatives $\frac{\partial}{\partial x} f(x,y) = \lambda$ and $\frac{\partial}{\partial y} f(x,y) = -(1+2\lambda y)$. It is easy to see that $f\left(\mathbb{E}_t\left[V_T^2\right], \mathbb{E}_t\left[V_T\right]\right)$ yields the negative of the optimal value of the objective function in (5.24). But by the concavity of f, using its partial derivatives mentioned above and $2\lambda\mu = 1 + \mathbb{E}_t\left[V_T\right]$ according to (5.25), we obtain

$$\begin{aligned} f\left(\mathbb{E}_t\left[(V_T^*)^2\right], \mathbb{E}_t\left[V_T^*\right]\right) \le\ & f\left(\mathbb{E}_t\left[V_T^2\right], \mathbb{E}_t\left[V_T\right]\right) \\ & + \lambda\left(\mathbb{E}_t\left[(V_T^*)^2\right] - \mathbb{E}_t\left[V_T^2\right]\right) - \left(1 + 2\lambda\mathbb{E}_t\left[V_T\right]\right)\left(\mathbb{E}_t\left[V_T^*\right] - \mathbb{E}_t\left[V_T\right]\right) \\ =\ & f\left(\mathbb{E}_t\left[V_T^2\right], \mathbb{E}_t\left[V_T\right]\right) \\ & + \lambda\left(\mathbb{E}_t\left[(V_T^*)^2\right] - \mathbb{E}_t\left[V_T^2\right] - 2\mu\mathbb{E}_t\left[V_T^*\right] + 2\mu\mathbb{E}_t\left[V_T\right]\right) \\ <\ & f\left(\mathbb{E}_t\left[V_T^2\right], \mathbb{E}_t\left[V_T\right]\right), \end{aligned}$$

where we used (5.26) and $\lambda > 0$ for the last inequality. This is equivalent to

$$\mathbb{E}\left[V_T \mid \mathcal{F}_t\right] - \lambda \cdot \mathit{Var}\left[V_T \mid \mathcal{F}_t\right] < \mathbb{E}\left[V_T^* \mid \mathcal{F}_t\right] - \lambda \cdot \mathit{Var}\left[V_T^* \mid \mathcal{F}_t\right],$$

which is a contradiction to the assumption that π with wealth process V is optimal for (5.24). The pair of π and V thus is also optimal for (5.25). □

With Proposition 5.3, assuming existence of the optimal solution to (5.24) and uniqueness of the optimal solution to (5.25), the mean-variance problem can be solved via the auxiliary problem (5.25). On the other hand, (5.25) is a special case of (5.2) if we set $\gamma = 2$, $K = -1$, and $L = \mu$. Fixing $\bar{\lambda} > 0$ as the parameter for (5.24) and choosing $L = \bar{\mu} > \frac{1}{2\bar{\lambda}}$ fixes the level of expected terminal wealth to $\bar{\mu} - \frac{1}{2\bar{\lambda}}$. Theorem 5.1 then yields the optimal solution to

$$\begin{aligned} \max_{\{\pi_s\}_t^{T-1}} \left\{ -\frac{1}{2}\mathbb{E}\left[(V_T-\bar{\mu})^2 \mid \mathcal{F}_t\right]\right\} &= \max_{\{\pi_s\}_t^{T-1}} \left\{ -\frac{1}{2}\mathbb{E}\left[V_T^2 - 2V_T\bar{\mu} + \bar{\mu}^2 \mid \mathcal{F}_t\right]\right\} \\ &= \max_{\{\pi_s\}_t^{T-1}} \left\{ -\frac{1}{2}\mathbb{E}\left[V_T^2 \mid \mathcal{F}_t\right] + \bar{\mu}\left(\bar{\mu} - \frac{1}{2\bar{\lambda}}\right) - \frac{1}{2}\bar{\mu}^2\right\} \\ &= \max_{\{\pi_s\}_t^{T-1}} \left\{ -\frac{1}{2}\mathbb{E}\left[V_T^2 \mid \mathcal{F}_t\right] + \frac{1}{2}\left(\bar{\mu}^2 - \frac{\bar{\mu}}{\bar{\lambda}}\right)\right\}. \end{aligned} \tag{5.27}$$

The variance of the optimal portfolio can be recovered using this optimal value of the objective function (5.27) and the fact that $\mathit{Var}\left[V_T \mid \mathcal{F}_t\right] = \mathbb{E}\left[V_T^2 \mid \mathcal{F}_t\right] - \left(\mathbb{E}\left[V_T \mid \mathcal{F}_t\right]\right)^2$, yielding the corresponding value on the efficient frontier.

Remark 5.4 (Higher-order moments in portfolio selection)**.** *Numerous authors have investigated the impact of higher moments on the portfolio return (see, e.g., Guidolin and Timmermann, 2008; Harvey et al., 2010; Lai, 1991). In the present version, the general formulation of*

Theorem 5.1 allows for a connection to a problem including skewness or skewness and kurtosis, choosing exponents of 3 *or* 4 *in the utility function. A numerical analysis studying the impact of higher-order moments on the portfolio choice could be subject to further research.*

5.5 Numerical Analysis

For our numerical analysis, we focus on the well-studied HN-GARCH model (see Section 4.3.2), forming a special case of our general setup (5.4) with $\theta = (\lambda, \omega, \alpha, \beta, \rho)$, $\{\epsilon_t\}_t$ a sequence of i.i.d. standard normal random variables and

$$f_1(h_t, \theta) = r + \lambda h_t, \qquad f_2(h_t, \theta) = \omega + \beta h_t, \qquad f_3(h_t, \epsilon_t, \theta) = \alpha \left(\epsilon_t - \rho\sqrt{h_t}\right)^2, \tag{5.28}$$

leading the following representations of the coefficients A and B in (5.5), see Appendix B.1:

$$A(u, v) = ur + v\omega - \frac{1}{2}\log(1 - 2v\alpha), \tag{5.29a}$$

$$B(u, v) = u\lambda + v\left(\alpha\rho^2 + \beta\right) + \frac{(u - 2v\alpha\rho)^2}{2(1 - 2v\alpha)}. \tag{5.29b}$$

Our parameters are estimated based on the S&P 500 levels from January 2, 2003, through December 30, 2022. While working with larger datasets could affect parameter stability, our time series includes two major crises and can thus be considered a representative sample. The 20-year time span falls within the range of various other studies that estimated affine GARCH models, such as Ornthanalai (2014) and Babaoğlu et al. (2018), with time frames of 15 and 23 years, respectively. We once more follow the lines of the returns-only estimation described in Section 4.4.2 (cf. Escobar-Anel et al., 2022b), exploiting the normal distribution of the HN-GARCH innovations to obtain the log-likelihood function

$$l_T(x_1, \ldots, x_T \mid \theta) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log(2\pi) + \log(h_t) + \left(\frac{x_t - x_{t-1} - r - \lambda h_t}{\sqrt{h_t}}\right)^2\right), \tag{5.30}$$

for the path of log returns $\{x_t\}_{t=1,\ldots,T}$. The estimates $\hat{\theta}$ for the vector of parameters θ are given by $\hat{\theta} = \arg\max_\theta l(x_1, \ldots, x_T \mid \theta)$. The standard errors of the estimates are again calculated via the Fisher information matrix. The same dataset of S&P 500 returns is used to obtain the parameter values for the corresponding homoscedastic Gaussian model (HS), i.e., an HN-GARCH model with constant conditional variance ($\alpha = \beta = 0$). The combined results with standard errors are provided in Table 5.1.[5]

We start our numerical study with the time horizon T set to five years with 252 trading days, aiming to study medium-term horizons. The interest rate is set to $r = 2\%$, slightly higher than the average Federal Funds Rate of 1.31% over the time of the underlying dataset. We choose $a_H = 1$ with the initial wealth $V_0 = 1$.

Focussing on the exponent $\gamma = 2$ and the connection to mean-variance theory established in Section 5.4, we target a special case accessible only through the generalizations in Theorem 5.1. Figure 5.2 displays the efficient frontiers[6] both for the HN-GARCH and the homoscedastic model with the parameters from Table 5.1. The HN-GARCH dominates the homoscedastic

[5]Since $\omega < 0$ in the context of the HN-GARCH model is known to render the model ill-defined and given the large standard error, we set $\omega = 0$ when working with the HN-GARCH parameter set from Table 5.1.

[6]In the presence of a riskless asset in the market, the efficient frontier (or *capital market line*, *CML*), is a straight line with the leftmost point marking the return of this riskless asset.

Table 5.1: Maximum likelihood estimates (MLEs) for the HN-GARCH model and the homoscedastic model, based on the S&P 500 levels from January 2, 2003, through December 30, 2022, with standard errors in parentheses.

Param.	**HN-GARCH**		**Homoscedastic**	
λ	2.46	(1.19)	1.94	(1.17)
ω	-1.31×10^{-7}	(1.53×10^{-7})	1.48×10^{-4}	(1.09×10^{-6})
α	5.85×10^{-6}	(2.85×10^{-7})		
β	7.69×10^{-1}	(1.16×10^{-2})		
ρ	1.76×10^{2}	(8.40)		

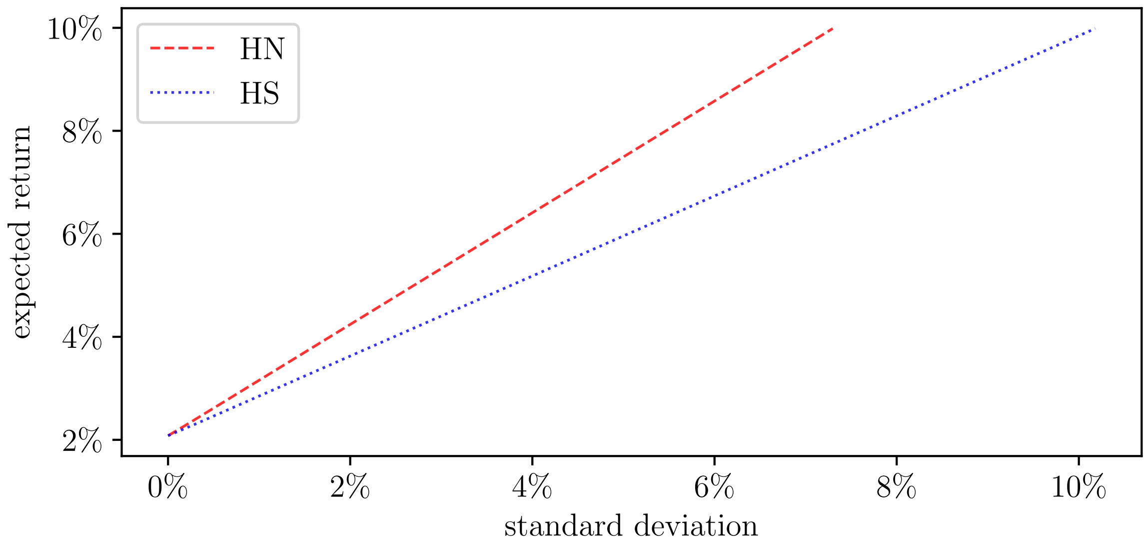


Figure 5.2: Efficient frontiers for the HN-GARCH and the homoscedastic model with the parameter values as of Table 5.1. The time horizon is five years with 252 trading days; the results are annualized.

solution in this context, yielding lower values in terms of the standard deviation for the same expected return. The annualized difference between the two models increases as the expected return is increased and amounts to about three percentage points in standard deviation for an expected return of 10%. This emphasizes the importance of accounting for the time-changing nature of return volatility in portfolio selection and thus concerns a widely debated feature in the area.

Figure 5.3 shows the optimal initial strategy π_0^* for different lengths of the time horizon T. While the optimality equation (5.19) delivers the partial strategy $\hat{\pi}_0^*$, the optimal overall strategy π_0^* is obtained via scaling the partial strategy by the factor $\frac{V_0 - L_0}{V_0}$. For the plot in Figure 5.3, we used $\frac{V_0 - L}{V_0}$ with $V_0 = 1$ and L as described in Section 5.4 as a proxy for this factor. Note that (5.3) implies that $\frac{V_0 - L}{V_0} < 0$. The plot shows that, while the homoscedastic strategy is independent of

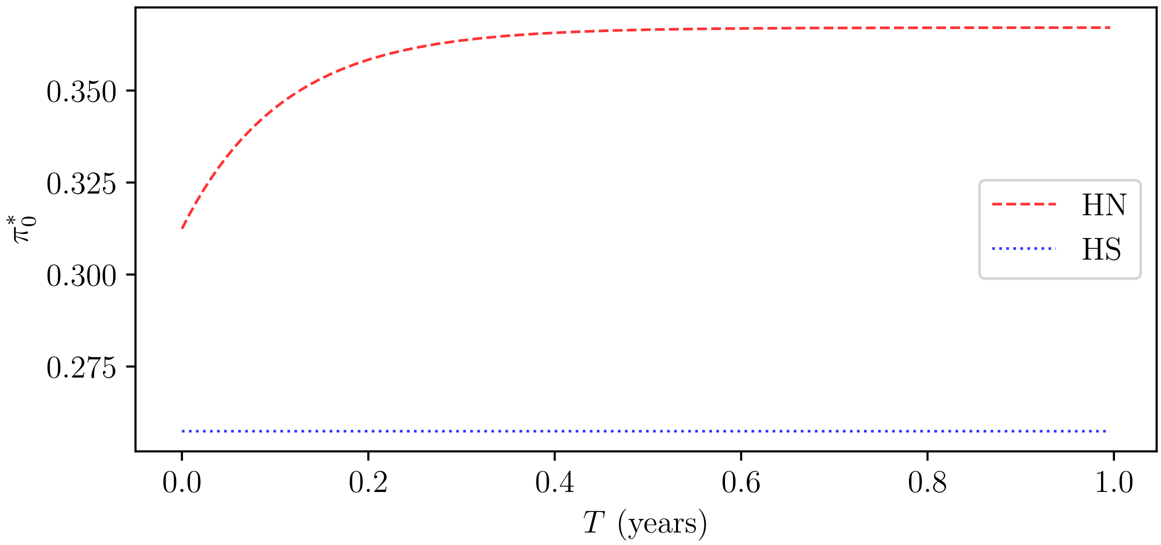
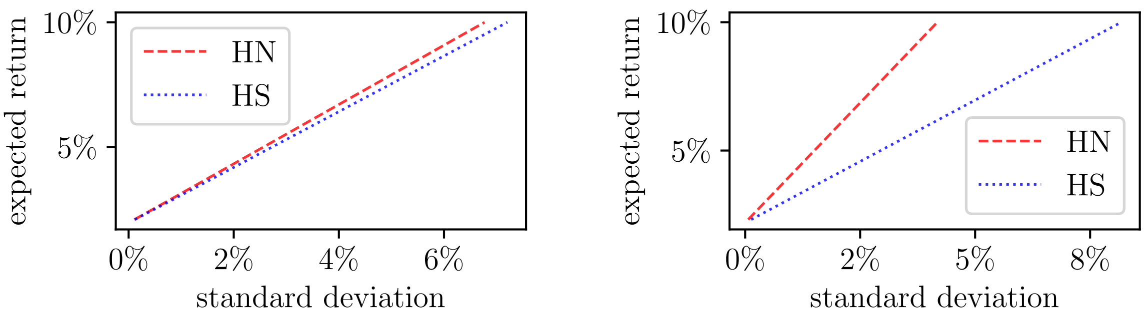


Figure 5.3: Optimal initial risky allocations for the HN-GARCH and the homoscedastic model with the parameter values as of Table 5.1 and $\gamma = 2$, dependent on the time horizon T.

T, the initial HN-GARCH strategy decreases significantly for short horizons below half a year. Since there are no significant changes for longer time horizons, we limit Figure 5.3 to $T \leq 1$.



(a) Efficient frontiers for a one-year horizon.

(b) Efficient frontiers for a ten-year horizon.

Figure 5.4: Efficient frontiers for the HN-GARCH and the homoscedastic model with the parameter values as of Table 5.1. One year is assumed to have 252 trading days, the results are annualized.

Figure 5.4 targets the changes in the difference between the two models as the time horizon varies. Both plots show that the HN-GARCH model outperforms the homoscedastic variant. However, the difference increases significantly as the time horizon is extended to ten years, while it is relatively small for one year with 252 trading days. Overall, the results emphasize the key finding that incorporating the heteroscedasticity of log returns in the underlying model is of significant importance for investors.

5.6 Conclusion

In this chapter, we present a dynamic portfolio optimization problem within the class of affine GARCH models for an investor maximizing a HARA utility function under significantly generalized conditions. Theorem 5.1 not only extends the family of utilities in use but also solves the problem for previously excluded ranges of risk aversion. Choosing the exponent of the utility function to be 2, we establish a connection to mean-variance theory, which we analyze numerically using real-world market data. Based on 20 years of S&P 500 log returns, we estimate the parameters both for an HN-GARCH model and a homoscedastic variant, and show that the heteroscedastic strategy clearly dominates by calculating the efficient frontier. The numerical results suggest that the advantage of a heteroscedastic investor is larger if longer time horizons are considered.

6 Expected Utility Theory within Lévy GARCH Models

6.1 Introduction

In this chapter, we extend the expected utility theory (EUT) approach presented in Chapter 3 to a different model class, allowing for jumps in the asset price process. While we considered only one – potentially non-Gaussian – innovation per time step in the setting of the previous chapters, the model for the underlying price process used here allows for a Gaussian and an additional jump component at each time step. As visualized in Figure 6.1, we thus add more complexity concerning the model for the underlying asset. This chapter was submitted for publication in a similar form (Escobar-Anel et al., 2023).

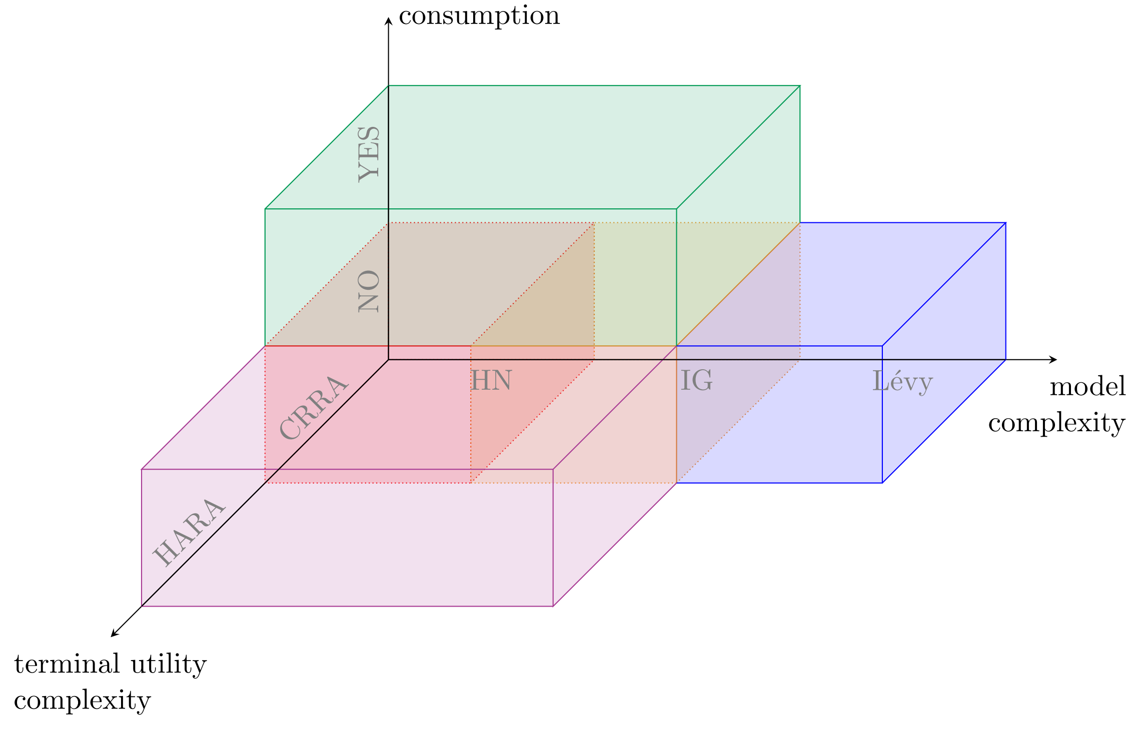


Figure 6.1: Overview of the contributions of this dissertation with regard to portfolio optimization under affine GARCH models, increasing the model complexity as well as the complexity of terminal and intermediate utility, see also Figures 1.1, 3.1, 4.1 and 5.1. The orange box coresponds to Chapter 3, the green box to Chapter 4, the purple box to Chapter 5, and the blue box to Chapter 6.

Jumps have been a topic of immense interest in portfolio optimization for decades, particularly in a continuous-time setting thanks to analytical solutions – see, for instance, Merton (1971). Since empirical evidence suggests that financial markets exhibit occasional large price movements, considering jump components allows investors to model the underlying asset price movements more adequately with the corresponding improvement in financial decisions, from derivative pricing to risk analysis and portfolio management. On the other hand, models are desired to remain mathematically tractable and, in the best case, deliver closed-form expressions for pricing and portfolio optimization problems, a property that is easily imperiled.

The first affine GARCH models by Heston and Nandi (2000) and Christoffersen et al. (2006) overcame the lack of analyitical solutions, but did not account for potential jumps in the underyling processes. The most common approach to extending these models and incorporate jumps consists of adding a (dynamic) Compound Poisson component in addition to the normal noise. In this manner, Christoffersen et al. (2012) introduced a model of GARCH type with stochastic volatility and separate dynamics for jumps, related to the non-affine work by Maheu and McCurdy (2004) and Duan et al. (2006a,b). In their version, however, Christoffersen et al. (2012) allow for jumps not only in the dynamics of the log-price process but also in the variance of the normal innovation and in the jump intensity, jeopardizing the exponentially affine form of the moment generating function (MGF). Ornthanalai (2014) formulated the corresponding model with jumps incorporated in the return dynamics only, preserving the affine nature of the model. The model includes two contemporaneously independent innovations: A Gaussian innovation and a Lévy jump component. The general framework for the latter allows for finite and infinite-activity jumps and extends the range of jump innovations beyond the Compound Poisson approach. This framework has been generalized recently and forms a special case of the affine multi-factor model introduced by Augustyniak et al. (2023).

Both the HN-GARCH and the Lévy GARCH framework were originally introduced for option pricing; Escobar-Anel et al. (2022a, 2021) first used the form of the generating function to derive the solution to a discrete-time portfolio optimization problem based on the jump-free model formulation by Badescu et al. (2019), as presented in Chapter 3. This chapter extends this approach to the Lévy GARCH framework and thus provides the opportunity to study the relevance of modeling jumps for portfolio optimization problems in the affine GARCH environment.

The impact of capturing extreme events on investment strategies has been studied extensively, although mostly in continuous time.[1] The analysis by Liu et al. (2003) suggests that price jumps have a larger effect than jumps in volatility. Ascheberg et al. (2016) claim that jumps matter more if the expected jump size or the jump intensity is large. These lines serve as a guideline for creating extreme jump settings to challenge the capability of jump-free models to deliver comparable investment strategies. Our paper differs from the existing literature on one key aspect. We take advantage of the capacity of the HN-GARCH model to deliver accurate estimates of the parameters (cf. Escobar-Anel et al., 2022b) in order to assess if the portfolio of an investor with a simple model could perform as well as that of an investor with the most advanced Lévy GARCH model. In other words, given that the true model of reality is unknown, we act as practical investors, working with a simple model to make decisions. This allows us to check in a control experiment if a simple, albeit wrong, model could match the performance of a correct advanced model. Our conclusions are eye-opening concerning the power of simple models in a discrete-time portfolio optimization setting.

Many authors have investigated the role of non-Gaussianity in asset returns in the context of portfolio selection. One prominent example is the work by Bekaert et al. (1998) suggesting that it is rather immaterial for an investor considering emerging market allocations to accommodate skewness and excess kurtosis. The crucial point that a wide range of expected utility problems can be well approximated by a mean-variance approach was summarized by Markowitz (2012).

[1] We are unaware of any work investigating the impact of jumps in discrete-time portfolio optimization.

Our results are not contradictory with the continuous-time literature for three main reasons. Firstly, a continuous-time investor, although hypothetical, could, in principle, realize small benefits between a true and a wrong model more frequently. Such dynamics could be very profitable in the long term, and a detailed analysis in this regard could certainly be subject to further research. Secondly, most studies of complex models rely on sensitivity analyses for the parameters. However, the same model with different parameters does not describe the same reality, so results could be misleading. Lastly, even when two models are estimated on the same data, small sample sizes may incorporate estimation errors into the analysis, possibly creating a distorted conclusion since the estimation error could exacerbate a small difference in performance. In conclusion, our analyses provide a fresh perspective on the importance of jumps for portfolio decisions.

Our main contributions are:

- We fully develop a discrete-time dynamic portfolio optimization problem in an affine GARCH framework with jumps and provide a closed-form solution based on an established and reliable approximation of the self-financing condition.
- We apply the general result to three different types of jump structures: the Merton jump, the Normal Inverse Gaussian, and the Variance Gamma model. Furthermore, we derive the well-known HN-GARCH model as a jump-free special case of our setting.
- We study the impact of jumps on investment strategies via a comparison to the HN-GARCH model and a homoscedastic variant. We find that correctly calibrated jump-free models can imitate all three investigated jump models very well. This result is reflected in a low wealth-equivalent loss (WEL) for investors following suboptimal jump-free strategies, and it is robust with respect to more extreme jump environments.

The remainder of this chapter is organized as follows: In Section 6.2, we introduce the framework for the portfolio optimization problem, which is solved in Section 6.3, providing optimality conditions in general form and explicitly for three different types of jump increments. The numerical analysis follows in Section 6.4, targeting specifically the impact of jumps and a comparison to jump-free HN-GARCH and homoscedastic models. We conclude in Section 6.5.

6.2 The Lévy GARCH Model

Adopting the setting proposed by Ornthanalai (2014), we use two contemporaneously independent random variables z_t and y_t to model the log return $X_t - X_{t-1} = \log\left(P_t^{(1)}/P_{t-1}^{(1)}\right)$ of the risky asset. While z_t follows a normal $\mathcal{N}\left(0, h_{z,t}\right)$ distribution, y_t is a pure Lévy jump innovation with generalized conditional Fourier transform

$$F_y\left(u; t-1, t\right) = \mathbb{E}_{t-1}\left[e^{u y_t}\right] = \exp\left\{\psi_y\left(u; t-1, t\right)\right\},$$

where ψ denotes the cumulant exponent (or cumulant generating function) of y_t. The Lévy jump innovation y_t is assumed to be *time-homogeneous* in one of the parameters of its distribution, which we will denote by h_y, meaning that (Ornthanalai, 2014, Def. 1) the cumulant exponent is multiplicatively separable with respect to h_y and thus admits the representation

$$\psi_y\left(u; t-1, t\right) = h_{y,t} \cdot \xi_y(u), \tag{6.1}$$

for some $\xi_y(u)$ independent of $h_{y,t}$. Note that the explicit distribution of y_t remains unspecified at this point. Potential choices include the Merton jump (MJ), a Normal-inverse Gaussian

(NIG) or a Variance Gamma (VG) innovation. Note that the Gaussian innovation z_t is time-homogeneous in its variance. In particular, we have

$$\psi_z\left(u; t-1, t\right) = h_{z,t} \cdot \frac{u^2}{2}, \quad \text{i.e.,} \quad \xi_z(u) = \frac{u^2}{2}.$$

Let the log-return process follow the dynamics given in the equation

$$R_t \coloneqq X_t - X_{t-1} = r_t + (\lambda_z - \xi_z(1)) \cdot h_{z,t} + (\lambda_y - \xi_y(1)) \cdot h_{y,t} + z_t + y_t, \tag{6.2}$$

where ξ_z and ξ_y are involved as convexity adjustments making $z_t + y_t$ a martingale, r_t is time-deterministic, and $\lambda_z h_{z,t} + \lambda_y h_{y,t}$ is referred to as the conditional equity premium (rate of return in excess of the risk-free rate). Note that by the properties of the cumulant generating function, Equation (6.2), in connection with the setting introduced above, implies that

$$Var_t\left[X_t - X_{t-1}\right] \equiv \sigma_t^2 = h_{z,t} + h_{y,t} \cdot \xi_y''(0). \tag{6.3}$$

Concerning the evolution of the parameters $h_{z,t}$ and $h_{y,t}$, we follow Ornthanalai (2014) and Heston and Nandi (2000), respectively, and assume the following dynamics:

$$h_{z,t+1} = \omega_z + b_z \cdot h_{z,t} + \frac{a_z}{h_{z,t}}\left(z_t - c_z h_{z,t}\right)^2, \tag{6.4a}$$

$$h_{y,t+1} = \omega_y + b_y \cdot h_{y,t} + \frac{a_y}{h_{z,t}}\left(z_t - c_y h_{z,t}\right)^2. \tag{6.4b}$$

This is equivalent to the evolution of the conditional variance in the HN-GARCH model by Heston and Nandi (2000), adapted for non-standardized innovations, but nonetheless converging to the Cox-Ingersoll-Ross process if the length of the time step is shrunk to zero (cf. Badescu et al., 2019).

Following the approach by Campbell and Viceira (1999) and Escobar-Anel et al. (2021) presented in Chapter 3, we use a second-order approach in order to approximate the self-financing condition. With (6.3) as the variance of the log return, we obtain

$$\begin{aligned} W_t &= W_{t-1} + \pi_{t-1} R_t + \frac{1}{2}\left(\pi_{t-1} - \pi_{t-1}^2\right)\sigma_t^2 + \left(1 - \pi_{t-1}\right) r_t \\ &= W_{t-1} + \pi_{t-1} R_t + \frac{1}{2}\left(\pi_{t-1} - \pi_{t-1}^2\right)\left(h_{z,t} + h_{y,t}\xi_y''(0)\right) + \left(1 - \pi_{t-1}\right) r_t. \end{aligned} \tag{6.5}$$

In order to derive a representation of the maximum expected utility later, we investigate the generating function of the log-price process under the physical measure. For enhanced readability, proofs are postponed to Appendix D.1. We establish the following result:

Proposition 6.1. *Let T be some finite time horizon and let $t < T-1$. In the setting of a Lévy GARCH model as introduced above, the one-step multivariate generating function of the log return R_{t+1} and the homogeneous parameters $h_{y,t+2}$ and $h_{z,t+2}$, under the physical measure, conditioned on the information available at time t, is given by*

$$\begin{aligned} \Psi_{(R_{t+1}, h_{z,t+2}, h_{y,t+2})}\left(u, v_z, v_y \,|\, \mathcal{F}_t\right) &\coloneqq \mathbb{E}_t\left[\exp\left\{u R_{t+1} + v_z h_{z,t+2} + v_y h_{y,t+2}\right\}\right] \\ &= \exp\Big\{A\left(u, v_z, v_y; t, t+1\right) \\ &\qquad + B\left(u, v_z, v_y; t, t+1\right) \cdot h_{z,t+1} \\ &\qquad + C\left(u, v_y; t, t+1\right) \cdot h_{y,t+1}\Big\}, \end{aligned} \tag{6.6}$$

where we define

$$A(u, v_z, v_y; t, t+1) = ur_{t+1} + v_z\omega_z + v_y\omega_y - \frac{1}{2}\log\left(1 - 2\left(v_z a_z + v_y a_y\right)\right), \tag{6.7a}$$

$$\begin{aligned} B(u, v_z, v_y; t, t+1) &= u\left(\lambda_z - \xi_z(1)\right) + v_z\left(b_z + a_z c_z^2\right) \\ &\quad + v_y a_y c_y^2 + \frac{\left(u - 2v_z a_z c_z - 2v_y a_y c_y\right)^2}{2\left(1 - 2\left(v_z a_z + v_y a_y\right)\right)}, \end{aligned} \tag{6.7b}$$

$$C(u, v_y; t, t+1) = u\left(\lambda_y - \xi_y(1)\right) + v_y b_y + \xi_y(u). \tag{6.7c}$$

In the sequel, we will omit the arguments t and $t+1$ from A, B, and C if the use of the one-step generating function is clear.

6.3 Optimal Investment Maximizing CRRA Utility

6.3.1 Maximum Utility Representation

The main result concerning dynamic portfolio optimization in the setting introduced in Section 6.2 is presented in Theorem 6.2. Again, for the sake of readability, the proofs in this section are postponed to Appendix D.1.

Theorem 6.2 (Maximum utility representation)**.** *Assume that the decision maker maximizes a power utility function with corresponding parameter $\gamma < 0$. Furthermore, let the log price of the risky asset be described by a Lévy GARCH model as outlined above, satisfying the additional condition that $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$ holds for some $\Lambda \in \mathbb{R}$, for all $z_t \in \mathbb{R}$ and $t \in \{0, \dots, T-1\}$. Then, the optimal value function at time t can be written as*

$$\begin{aligned} \phi_t\left(W_t, h_{z,t+1}, h_{y,t+1}\right) &= \frac{1}{\gamma}\exp\left\{D_{t,T}(\pi_t^*) + \gamma W_t + E_{t,T}^z(\pi_t^*)\cdot h_{z,t+1} + E_{t,T}^y(\pi_t^*)\cdot h_{y,t+1}\right\}, \\ &= \frac{1}{\gamma}\exp\left\{D_{t,T}(\pi_t^*) + \gamma W_t + \left[E_{t,T}^z(\pi_t^*) + \Lambda E_{t,T}^y(\pi_t^*)\right]\cdot h_{z,t+1}\right\}, \end{aligned} \tag{6.8}$$

with $D_{t,T}$, $E_{t,T}^z$ and $E_{t,T}^y$ given via $D_{T,T} = E_{T,T}^z = E_{T,T}^y = 0$ and the recursive equations

$$D_{t,T}(\pi_t^*) = D_{t+1,T}(\pi_{t+1}^*) + (1 - \pi_t^*)\,\gamma r_{t+1} + A\left(\gamma\pi_t^*,\, E_{t+1,T}^z(\pi_{t+1}^*),\, E_{t+1,T}^y(\pi_{t+1}^*)\right), \tag{6.9a}$$

$$E_{t,T}^z(\pi_t^*) = \frac{\gamma}{2}\left(\pi_t^* - (\pi_t^*)^2\right) + B\left(\gamma\pi_t^*,\, E_{t+1,T}^z(\pi_{t+1}^*),\, E_{t+1,T}^y(\pi_{t+1}^*)\right), \tag{6.9b}$$

$$E_{t,T}^y(\pi_t^*) = \frac{\gamma}{2}\left(\pi_t^* - (\pi_t^*)^2\right)\cdot\xi_y''(0) + C\left(\gamma\pi_t^*,\, E_{t+1,T}^y(\pi_{t+1}^*)\right). \tag{6.9c}$$

The optimal solution π_t^* satisfies

$$\begin{aligned} &\left[\lambda_z - \pi_t^* + \frac{\gamma\pi_t^* - 2E_{t+1,T}^z\left(\pi_{t+1}^*\right)a_z c_z - 2E_{t+1,T}^y\left(\pi_{t+1}^*\right)a_y c_y}{1 - 2E_{t+1,T}^z\left(\pi_{t+1}^*\right)a_z - 2E_{t+1,T}^y\left(\pi_{t+1}^*\right)a_y}\right] \\ &\qquad + \Lambda\cdot\left[\xi_y''(0)\left(\frac{1}{2} - \pi_t^*\right) + \left(\lambda_y - \xi_y(1)\right) + \xi_y'\left(\gamma\pi_t^*\right)\right] = 0, \end{aligned} \tag{6.10}$$

together with the second-order condition

$$\Lambda\cdot\xi_y''\left(\gamma\pi_t^*\right) > \frac{1 + \Lambda\xi_y''(0)}{\gamma} - \frac{1}{1 - 2E_{t+1,T}^z\left(\pi_{t+1}^*\right)a_z - 2E_{t+1,T}^y\left(\pi_{t+1}^*\right)a_y}. \tag{6.11}$$

In the case of $h_{y,t+1} \neq \Lambda \cdot h_{z,t+1}$, the optimal strategy needs to satisfy two equations to meet the corresponding optimality condition (explicitly given as (D.5) in the proof) for any values of $h_{z,t+1}$ and $h_{y,t+1}$. In particular, both coefficients in square brackets in (6.10) need to be set to zero. Since the coefficient corresponding to $h_{z,t+1}$ (first square bracket) is linear in π_t, we obtain an explicit expression for the optimal strategy dependent on the other parameters. Note that this solution candidate – name it $\pi_t^{(z)}$ – is time-dependent because of $E_{t+1,T}^{z,*}$ and $E_{t+1,T}^{y,*}$.[2] Concerning the coefficient corresponding to $h_{y,t+1}$ (second square bracket), this yields a condition for ξ_y, stating that

$$\xi_y''(0)\left(\frac{1}{2}-\pi_t^{(z)}\right)+(\lambda_y-\xi_y(1))+\xi_y'\left(\gamma\pi_t^{(z)}\right)=0, \tag{6.12}$$

for all $t \in \{0,\ldots,T-1\}$. These T different conditions, however, need to be satisfied for all investors, i.e., for all possible choices for γ and the corresponding values of $\pi_t^{(z)}$. Thus, we need to look at (6.12) as a differential equation for ξ_y, which after the substitution $u = \gamma\pi_t^{(z)}$ has the form

$$\xi_y'(u)=\frac{k_2}{\gamma}u+k_1-\frac{k_2}{2}-\lambda_y, \tag{6.13}$$

subject to the two conditions $\xi_y(1) = k_1$ and $\xi_y''(0) = k_2$. This still – counter-intuitively – imposes a condition on the statistical properties of log-asset returns related to the investor's risk aversion parameter γ. Furthermore, integrating yields

$$\xi_y(u)=\frac{k_2}{2\gamma}u^2+\left(k_1-\frac{k_2}{2}-\lambda_y\right)u+C \tag{6.14}$$

for some constant $C \in \mathbb{R}$, with

$$\xi_y(1)=\frac{k_2}{2\gamma}+k_1-\frac{k_2}{2}-\lambda_y+C=k_1, \tag{6.15a}$$

$$\xi_y''(0)=\frac{k_2}{\gamma}=k_2. \tag{6.15b}$$

It is easy to see that (6.15b) requires $\gamma = 1$ for the risk aversion parameter, which implies $C = \lambda_y$ from (6.15a). Hence, this restricts the analysis to risk-neutral investors and contradicts our assumption that $\gamma < 0$.

An analog to our condition $h_{y,t} = \Lambda h_{z,t}$ can be found, e.g., in the setting by Liu and Pan (2003), where the stochastic jump intensity is connected to the instantaneous variance process. In practice, we ensure the linear dependence of time-homogeneous factors by connecting the parameters in (6.4b) to the ones in (6.4a), i.e., by assuming the initial condition $h_{y,1} = \Lambda h_{z,1}$ together with

$$c_y = c_z, \quad b_y = b_z, \quad \omega_y = \Lambda\omega_z, \quad \text{and} \quad a_y = \Lambda a_z, \tag{6.16}$$

implying a slightly simplified expression for (6.10). We emphasize that a simple reduction of our model delivers a well-known special case without jumps:

[2] $E_{t+1,T}^{x,*}$ is short for $E_{t+1,T}^{x}(\pi_{t+1}^*)$, with $x \in \{y,z\}$

Corollary 6.3 (HN-GARCH). *Disregarding the jump innovation, i.e., setting $\xi_y(u) = 0$, $\Lambda = 0$ and $\lambda_y = 0$, yields the HN-GARCH model.*[3] *Note that due to the convexity adjustment present in* (6.2)*, we need to choose $\lambda_z = \lambda_{\mathrm{HN}} + \frac{1}{2}$, where λ_{HN} is the maximum likelihood estimate (MLE) for the risk premium in the HN-GARCH model. In particular,* (6.10) *leads to*

$$\pi_t^* = \frac{\lambda_z \left(1 - 2E_{t+1,T}^{z,*} a_z\right) - 2E_{t+1,T}^{z,*} a_z c_z}{1 - 2E_{t+1,T}^{z,*} a_z - \gamma}, \tag{6.17}$$

and $1 - 2E_{t+1,T}^{z,*} a_z > 0$ ensures (6.11).

Note that the HN-GARCH model satisfies the second-order condition

$$\Lambda \xi_y'' \left(\gamma \pi_t\right) > \frac{1 + \Lambda \xi_y''(0)}{\gamma} - \frac{1}{1 - 2E_{t+1,T}^{z}\left(\pi_{t+1}^*\right) a_z - 2E_{t+1,T}^{y}\left(\pi_{t+1}^*\right) a_y} \tag{6.18}$$

on the entire real line for π_t, i.e., we know that the objective function is concave in this case, and the solution to the first-order condition is a global maximum.

6.3.2 Examples of Jump Innovations

This section considers several choices for the Lévy jump innovation and its distribution. First, the above optimality problem is investigated under the well-known MJ model. Subsequently, we also tackle NIG and VG innovations.

Merton Jump Model

In the Merton jump (MJ) model (cf. Ornthanalai, 2014), each jump size is drawn independently from $\mathcal{N}\left(\theta, \delta^2\right)$. The number of jumps η_{t+1} occurring in the interval $(t, t+1]$ is a Poisson process with predictable intensity $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$, i.e., this intensity serves as the time-homogeneous parameter in our Lévy GARCH model. Thus, let $y_{t+1} = \sum_{k=1}^{\eta_{t+1}} Y_k$, with $\{Y_k\}_k$ i.i.d. according to $\mathcal{N}\left(\theta, \delta^2\right)$. Note that

$$\mathbb{E}_t \left[\exp\left\{u \cdot y_{t+1}\right\} \middle| \eta_{t+1} = n \right] = \exp\left\{ \left(u\theta + \frac{1}{2}\delta^2 u^2\right) \cdot n \right\},$$

which allows us to deduce from the tower property that

$$\mathbb{E}_t \left[\exp\left\{u \cdot y_{t+1}\right\}\right] = \exp\left\{ \left(\exp\left\{ u\theta + \frac{1}{2}\delta^2 u^2 \right\} - 1 \right) \cdot h_{y,t+1} \right\}.$$

Having performed the above analysis conditioned on the information at time t, we obtain the cumulant exponent from the last equation and state that in the Merton jump model,

$$\xi_y(u) = \exp\left\{ u\theta + \frac{1}{2}\delta^2 u^2 \right\} - 1. \tag{6.19}$$

[3] The HN-GARCH model was originally introduced by Heston and Nandi (2000). The notation λ_{HN} thus refers to this original formulation without convexity adjustment, see, e.g., Chapter 3. Contrary to the previous chapters, this chapter will use the HN-GARCH model in the Lévy GARCH parametrization for clarity.

Transferring this representation back to our optimality equation (6.10), we now need to solve

$$\left[\lambda_z - \pi_t + \frac{\gamma\pi_t - 2\left(E^{z,*}_{t+1,T} - \Lambda E^{y,*}_{t+1,T}\right)a_z c_z}{1 - 2a_z\left(E^{z,*}_{t+1,T} + \Lambda E^{y,*}_{t+1,T}\right)}\right]$$
$$+\Lambda \cdot \left[\left(\theta^2 + \delta^2\right)\left(\frac{1}{2} - \pi_t\right) + \left(\lambda_y - \exp\left\{\left(\theta + \frac{1}{2}\delta^2\right)\right\} + 1\right)\right.$$
$$\left. + \left(\theta + \delta^2\gamma\pi_t\right)\cdot \exp\left\{\theta\gamma\pi_t + \frac{1}{2}\delta^2\gamma^2\pi_t^2\right\}\right] = 0. \tag{6.20}$$

The optimality equation above yields the following statement on existence and uniqueness of an optimal solution in the prevailing model:

Corollary 6.4. *Suppose* (6.18) *holds with strict inequality for all* $t \in \{0, \dots, T-1\}$ *in the Merton jump model with normally distributed jump size. Then, a unique solution exists for the optimal relative portfolio strategy* π_t^*.

However, the strategy π_t is involved both in linear and in exponential terms, making (6.20) difficult to solve for π_t explicitly. One straightforward way to an approximate solution uses that the values for the parameters θ and δ are very small (referring to the MLEs reported by Ornthanalai, 2014), which implies that the approximation $\exp\{x\} \approx 1 + x$ is sufficiently good in this case.

Normal Inverse Gaussian Model

A Normal-inverse Gaussian (NIG) process (Barndorff-Nielsen, 1997; Ornthanalai, 2014) can be constructed via a Brownian motion, which is time-changed using an inverse Gaussian process. In particular, we introduce $\alpha > 0$ and β with $|\beta| < \alpha$. In order to obtain an innovation that is time-homogeneous in one of its parameters later, we refer to Barndorff-Nielsen (1997) and set the starting point for the bivariate Brownian motion, which is used to construct the NIG process, to zero (corresponding to $\mu = 0$ in the original parametrization). This then leads to:

$$\psi_y(u; t, t+1) = -\left(\sqrt{\alpha^2 - (\beta + u)^2} - \sqrt{\alpha^2 - \beta^2}\right)\cdot h_{y,t+1},$$

with the part excluding $h_{y,t+1}$ (note the minus in front!) as $\xi_y(u)$. In this setting too, we rely on the condition $h_{y,t+1} = \Lambda \cdot h_{z,t+1}$, which then yields the following optimality equation:

$$\left[\lambda_z - \pi_t + \frac{\gamma\pi_t - 2\left(E^{z,*}_{t+1,T} - \Lambda E^{y,*}_{t+1,T}\right)a_z c_z}{1 - 2a_z\left(E^{z,*}_{t+1,T} + \Lambda E^{y,*}_{t+1,T}\right)}\right]$$
$$+\Lambda \cdot \left[\frac{\alpha^2 - 2\beta^2}{(\alpha^2 - \beta^2)^{\frac{3}{2}}}\left(\frac{1}{2} - \pi_t\right) + \lambda_y + \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2}\right.$$
$$\left. + \frac{\beta + \gamma\pi_t}{\sqrt{\alpha^2 - \beta^2 - 2\beta\gamma\pi_t - \gamma^2\pi_t^2}}\right] = 0. \tag{6.21}$$

Variance Gamma Model

Another potential choice concerning the model for the jump innovation is the increment of a Variance Gamma (VG) process (Madan and Seneta, 1990). Following Ornthanalai (2014), we consider a normal density $\mathcal{N}(\theta x, x)$, conditional on $x \sim \text{Gamma}(\alpha, \beta)$, for $\alpha, \beta > 0$. Again referring to Ornthanalai (2014), we obtain via conditioning on x:

$$\begin{aligned}\mathbb{E}_t\left[\exp\left\{u \cdot y_{t+1}\right\}\right] &= \left(\frac{\beta}{\beta - \frac{1}{2}u^2 - \theta u}\right)^{\alpha} \\ &= \exp\left\{-\alpha \cdot \log\left(1 - \frac{\theta u + \frac{1}{2}u^2}{\beta}\right)\right\}.\end{aligned}$$

Thus, choosing $h_{y,t+1}$ to take the role of α, we arrive at

$$\xi_y(u) = \log\left(\frac{\beta}{\beta - \frac{1}{2}u^2 - \theta u}\right). \tag{6.22}$$

Using this in (6.10) yields the optimality equation

$$\begin{aligned}&\left[\lambda_z - \pi_t + \frac{\gamma\pi_t - 2\left(E^{z,*}_{t+1,T} - \Lambda E^{y,*}_{t+1,T}\right)a_z c_z}{1 - 2a_z\left(E^{z,*}_{t+1,T} + \Lambda E^{y,*}_{t+1,T}\right)}\right] \\ +\Lambda \cdot &\left[\frac{\beta + \theta^2}{\beta^2}\left(\frac{1}{2} - \pi_t\right) + \lambda_y - \log\left(\frac{\beta}{\beta - \theta - \frac{1}{2}}\right) + \frac{\theta + \gamma\pi_t}{\beta - \theta\gamma\pi_t - \frac{1}{2}\gamma^2\pi_t^2}\right] = 0.\end{aligned} \tag{6.23}$$

Multiplying both sides with the denominator of the last term leads to a third-order polynomial.

Remark 6.5. *Although Equations* (6.20), (6.21) *and* (6.23) *seem to be tedious to solve for* π_t, *we can check numerically that the second-order condition is satisfied for the entire real line at all time points (for reasonable parameter values), i.e., that the objective function is concave in* π_t *and that a solution will be a unique maximum.*

6.4 Numerical Analysis

This section is devoted to the numerical study of the prevailing Lévy GARCH model. In particular, the sensitivity of the optimal portfolio allocation to the central parameters of the model is investigated, as well as the WEL occurring when the investor follows suboptimal strategies. We are interested specifically in the impact of jumps on the optimal strategy and on potential losses and want to compare the Lévy GARCH models to the nested HN-GARCH (which does not incorporate jumps) and a homoscedastic variant. All analyses are going to be presented for the MJ model as well as for NIG and VG innovations.

If not specified otherwise, the choices for the model's investment parameters are given in Table 6.1.

Table 6.1: Standard values for the investment parameters used for the plots in Section 6.4.

Parameter	T	r	γ
Value	252	$0.01/252$	-1

6.4.1 Parameter Estimates

Our analysis is based on the MLEs provided by Ornthanalai (2014). Since in the estimation process for these MLEs the linear dependence between the two time-homogeneous parameters, $h_{y,t} = \Lambda h_{z,t}$ for $t \in \{1, \dots, T\}$, was not assumed, we need to modify the original parameter set. The central question in this regard is how to derive the value for the parameter Λ, since this, by construction, will govern the behavior of the process $\{h_{y,t}\}_t$. The approach presented here uses the unconditional mean of the process $\{h_{y,t}\}_t$ and derives a value for Λ such that the unconditional means of $\Lambda h_{z,t}$ and $h_{y,t}$ are matched. Assuming that both time-homogeneous parameters h_z and h_y satisfy the stationarity conditions $b_z + a_z c_z^2 < 1$ and $b_y < 1$, the unconditional first moments of h_z and h_y, respectively, are given by (Ornthanalai, 2014, Equation (9))

$$\bar{h}_z^{(1)} = \frac{\omega_z + a_z}{1 - b_z - a_z c_z^2}, \qquad \bar{h}_y^{(1)} = \frac{\omega_y + a_y + a_y c_y^2 \bar{h}_z^{(1)}}{1 - b_y}. \tag{6.24}$$

We replace the parameters a_y, b_y, c_y and ω_y in (6.24) according to the relations in (6.16) to obtain the Λ-dependent version $\hat{h}_y^{(1)}$. Subsequently, we choose the value for Λ such that $\hat{h}_y^{(1)}$ and $\bar{h}_y^{(1)}$ are equal, leading to the equation

$$\frac{\Lambda\omega_z + \Lambda a_z + \Lambda a_z c_z^2 \bar{h}_z^{(1)}}{1 - b_z} = \frac{\omega_y + a_y + a_y c_y^2 \bar{h}_z^{(1)}}{1 - b_y}, \tag{6.25}$$

which is equivalent to

$$\Lambda = \frac{(1 - b_z)\left(\omega_y + a_y + a_y c_y^2 \bar{h}_z^{(1)}\right)}{(1 - b_y)\left(\omega_z + a_z + a_z c_z^2 \bar{h}_z^{(1)}\right)}. \tag{6.26}$$

Note that one could also include higher unconditional moments in the above approach and then derive the best value for Λ, e.g., based on the method of least squares. This, however, would sacrifice an exact match of the long-term mean of h_y, forming the reason why we remain with the exact choice in (6.26).

6.4.2 Methodology for Comparisons and WEL

In order to correctly investigate the impact of jumps on the solution to our portfolio optimization problem, we want to compare the prevailing jump model to the closest HN-GARCH model, which is nested in our setup as described in Corollary 6.3 and does not incorporate jumps. In order to avoid estimation errors from small samples, for every jump model, we simulate 100 paths with a long sample of $N = 2 \times 10^4$ steps of log returns for the prevailing jump model and then derive MLEs for the HN-GARCH parameters based on this dataset. To this end, we follow the lines of the returns-only estimation of Escobar-Anel et al. (2022b) that we described in Section 4.4.2, exploiting the normal distribution of the HN-GARCH innovations to obtain the log-likelihood function l. For one path of log returns $\{x_n\}_{n=1,\dots,N}$, the estimates $\hat{\theta}$ for $\theta = (\lambda_z, \omega_z, a_z, b_z, c_z)$ are given by

$$\hat{\theta} = \arg\max_{\theta} l\left(x_1, \dots, x_N \mid \theta\right). \tag{6.27}$$

After finding estimates $\hat{\theta}_1, \ldots, \hat{\theta}_{100}$ for all paths, we take the median as the best proxy for the HN parametrization[4], which can be understood as the best HN-GARCH approximating a given Lévy GARCH. The same dataset is used to obtain the corresponding homoscedastic Gaussian model (HS), i.e., an HN-GARCH model with $a_z = b_z = c_z = 0$. In the following, the jump models – we use the MJ, the NIG and the VG model as examples – are always compared to their re-calibrated versions of the HN-GARCH and the homoscedastic variant.

We use the exponentially affine value function from Theorem 6.2 to measure investors' losses when following suboptimal strategies. In particular, we compare the optimal value function at time t, ϕ_t, which is obtained via using the optimal strategy $\{\pi_t^*\}_t$ in the recursive parameters, to the expected utility corresponding to the suboptimal strategy $\{\pi_t^s\}_t$, denoted by ϕ_t^s. The WEL L_t^s is the percentage of wealth at time t that an investor following the optimal strategy can forgo in order to obtain the same expected utility as another investor following the suboptimal strategy $\{\pi_t^s\}_t$. Thus, L_t^s satisfies the equation

$$\phi_t^s \left(\log V_t, h_{z,t+1}, h_{y,t+1}\right) = \phi_t \left(\log \left(V_t \left(1 - L_t^s\right)\right), h_{z,t+1}, h_{y,t+1}\right). \tag{6.28}$$

Using (6.8) and exploiting $h_{y,t+1} = \Lambda h_{z,t+1}$ yields the following closed-form representation of the WEL:

$$\begin{aligned} L_t^s = 1 - \exp \Bigg\{ \frac{1}{\gamma} \cdot \Big[& D_{t,T}^s \left(\pi_t^s\right) - D_{t,T} \left(\pi_t^*\right) \\ & + \Big(E_{t,T}^{z,s} \left(\pi_t^s\right) - E_{t,T}^{z} \left(\pi_t^*\right) + \Lambda E_{t,T}^{y,s} \left(\pi_t^s\right) - \Lambda E_{t,T}^{y} \left(\pi_t^*\right) \Big) h_{z,t+1} \Big] \Bigg\}. \end{aligned} \tag{6.29}$$

6.4.3 Results for the Merton Jump Model

This subsection presents an analysis of the sensitivity of the optimal solution to the most relevant investment parameters and aims at comparing the Merton jump model to the closest HN-GARCH and homoscedastic solution. To this end, we derived MLEs for the latter two jump-free models according to the methodology presented in Section 6.4.2, based on the MJ parameters from Ornthanalai (2014), adapted to our setting as described in Section 6.4.1. We add that, compared to the sample statistics for the original S&P 500 dataset used to estimate the MJ parameters in Table 6.2, namely a sample skewness of −0.194 and a kurtosis of 11.15 (Ornthanalai, 2014), the first 15 years of observations in our artificial dataset created with these MJ parameters yield a sample skewness of 0.010 and a kurtosis of 3.47.[5] For all three investigated models, similar sample statistics are obtained when simulating with or without modifying the parameter set according to Section 6.4.1 to achieve the dependence in (6.16).

Comparing the solutions to the portfolio optimization problem for MJ to its no-jump analogues as of Table 6.2, Figure 6.2 shows that the re-calibrated HN-GARCH model without jumps replicates the MJ very well, with the difference in the optimal initial allocation decreasing as shorter time horizons are considered. In contrast to this, the homoscedastic solution is not sensitive to the length of the time horizon and stays constant, also causing an increasing difference to the MJ as the time horizon increases.

Figure 6.3a addresses the sensitivity of the optimal solution with respect to the investor's level of relative risk aversion, which in the prevailing setting of a power utility function with parameter

[4]We use the median rather than the average due to outliers in the estimates (i.e., potential local minima on some samples). We also report the standard error from the 100 samples to confirm the estimate's accuracy.

[5]While these numbers for the artificial set of MJ returns are less extreme than the actual sample statistics of the original dataset, this picture changes when looking at different Lévy GARCH models, see Section 6.4.4 for the NIG model.

Table 6.2: MLEs for the MJ model and the recalibrated HN-GARCH and homoscedastic model, respectively. Standard errors for the recalibrated models are in parentheses. The MJ parameters are based on Ornthanalai (2014) and the methodology presented in Section 6.4.1, the two jump-free models have been estimated according to Section 6.4.2.

Par.	**MJ**	**HN**		**HS**	
λ_z	1.47	2.67	(2.70×10^{-1})	2.66	(1.85×10^{-1})
ω_z	-2.39×10^{-6}	-1.23×10^{-6}	(5.67×10^{-7})	6.88×10^{-4}	(3.50×10^{-5})
a_z	2.98×10^{-6}	4.45×10^{-6}	(5.80×10^{-7})		
b_z	9.41×10^{-1}	9.59×10^{-1}	(5.59×10^{-3})		
c_z	1.35×10^{2}	9.04×10^{1}	(1.19×10^{1})		
λ_y	6.18×10^{-2}				
Λ	2.31×10^{2}				
θ	-2.39×10^{-3}				
δ	2.85×10^{-2}				

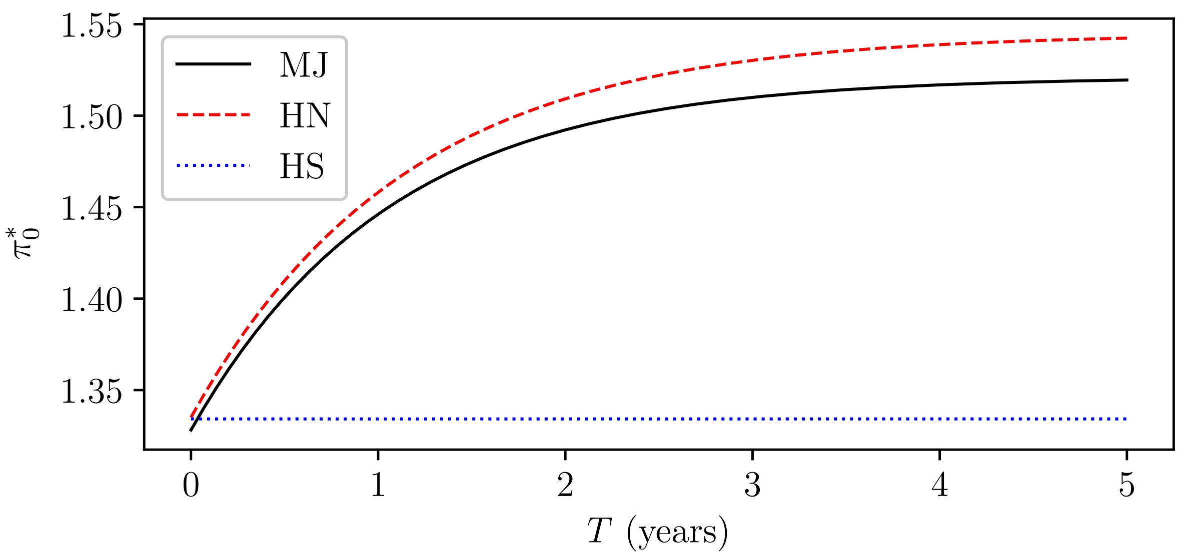
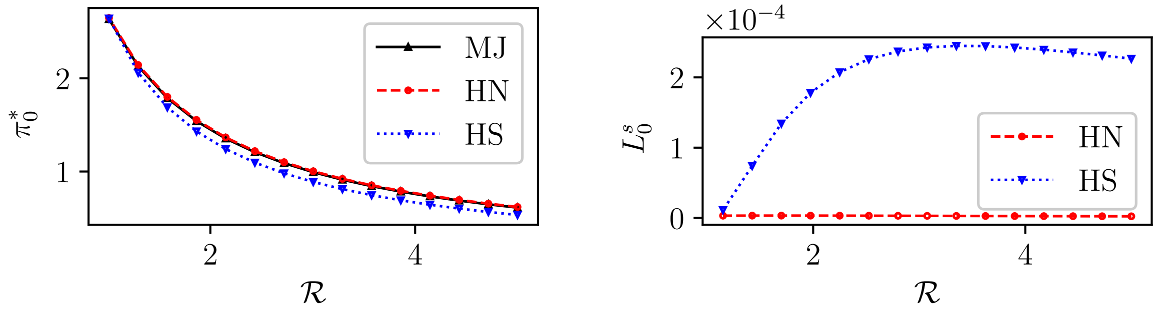


Figure 6.2: Values of the optimal strategy π_0^* in the MJ model and the corresponding re-calibrated HN-GARCH and homoscedastic model, dependent on the length of the time horizon, for T up to five years. The parameters are given in Table 6.1 (investment parameters) and in Table 6.2 (MJ, HN, HS).

$\gamma < 1$ is given via $\mathcal{R} = 1-\gamma$. Clearly, the fraction of wealth invested in the risky asset increases in all three models as the level of relative risk aversion decreases. This can lead to major differences between investors using the same model, but having different risk preferences. Following the MJ, for instance, with a relative risk aversion of $\mathcal{R} \approx 1.3$ yields an initial risky allocation of around 200%, whereas $\mathcal{R} \approx 4$ leads to an allocation below 100%. For the entire plotted range,

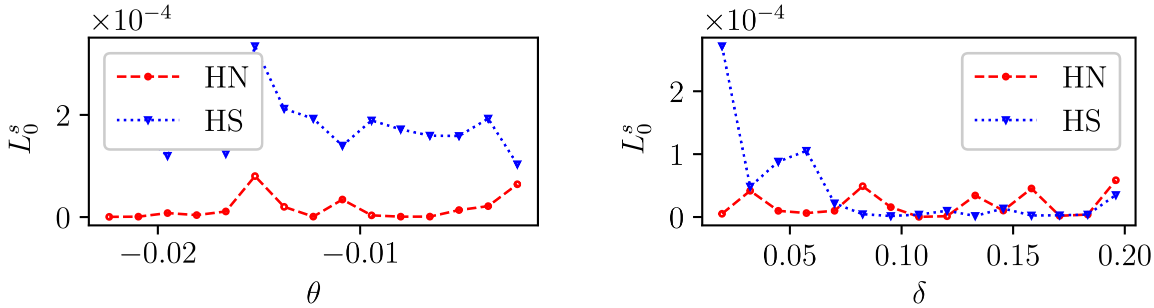


(a) Values of π_0^* dependent on $\mathcal{R}$. The lines for the MJ and the HN are overlapping.

(b) WEL dependent on $\mathcal{R}$.

Figure 6.3: Values of the optimal initial allocation π_0^* and of wealth-equivalent losses, dependent on the investor's level of relative risk aversion $\mathcal{R} = 1 - \gamma \in [1.1, 5]$. The parameters are given in Table 6.1 (investment parameters) and in Table 6.2 (MJ, HN, HS).

the HN-GARCH strategy remains very close to the optimal strategy in the MJ, while investors following the homoscedastic variant have a slightly lower risky allocation. Figure 6.3b uses the same parameter range and considers the WEL that investors incur when following the HN-GARCH or the homoscedastic strategy, while log returns actually evolve according to an MJ. In general, the losses remain at a very low level for a one-year time horizon, yielding values below 5 basis points for the homoscedastic model and even less for the HN-GARCH. This confirms the impression from Figures 6.2 and 6.3a that the jump-free models lead to very similar investment strategies.



(a) WEL dependent on θ, up to an absolute increase by a factor of 10.

(b) WEL dependent on δ, up to an increase by a factor of 10.

Figure 6.4: Values of the wealth-equivalent losses, dependent on the jump parameters in the MJ model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 6.4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

In Figure 6.4, we consider more extreme jump settings, which are obtained via increasing the absolute mean of the normally distributed jump size in the MJ up to a factor of 10 and via increasing the standard deviation of the jump size by the same factor. For each grid point between the original parameter estimate as of Table 6.2 and the most extreme setting, we re-calibrate the jump-free models to a set of simulated log return according to Section 6.4.2.[6] Both

[6] The recalibration exercise leads to natural inaccuracies in the estimates, which explains the minor variation in the calculation of WEL among grid points.

plots suggest that the capability of the HN-GARCH and the homoscedastic model to imitate the MJ is not significantly weakened, i.e., losses remain negligible for all considered parameter settings.

Besides modifications concerning the jump size in the MJ, the jump intensity can of course play a huge role too, bearing in mind, e.g., the analysis by Ascheberg et al. (2016), suggesting that jumps matter more if the intensity is large. In the Merton jump model, the intensity can easily be modified via the parameter Λ, governing the jump intensity given by $\Lambda h_{z,t}$ in the interval $(t, t+1]$. One has to take into account, however, that the long-run equity premium, obtained via

$$\bar{\lambda} := \lambda_z \bar{h}_z^{(1)} + \lambda_y \bar{h}_y^{(1)} = \lambda_z \bar{h}_z^{(1)} + \lambda_y \Lambda \bar{h}_z^{(1)}, \tag{6.30}$$

changes accordingly, which will have a major impact on the optimal solution. Since we wish to isolate the effect of modifications to the jump size, we compensate for changes is the long-run equity premium by adapting λ_z such that $\bar{\lambda}$ remains constant. Figure 6.5 uses this approach

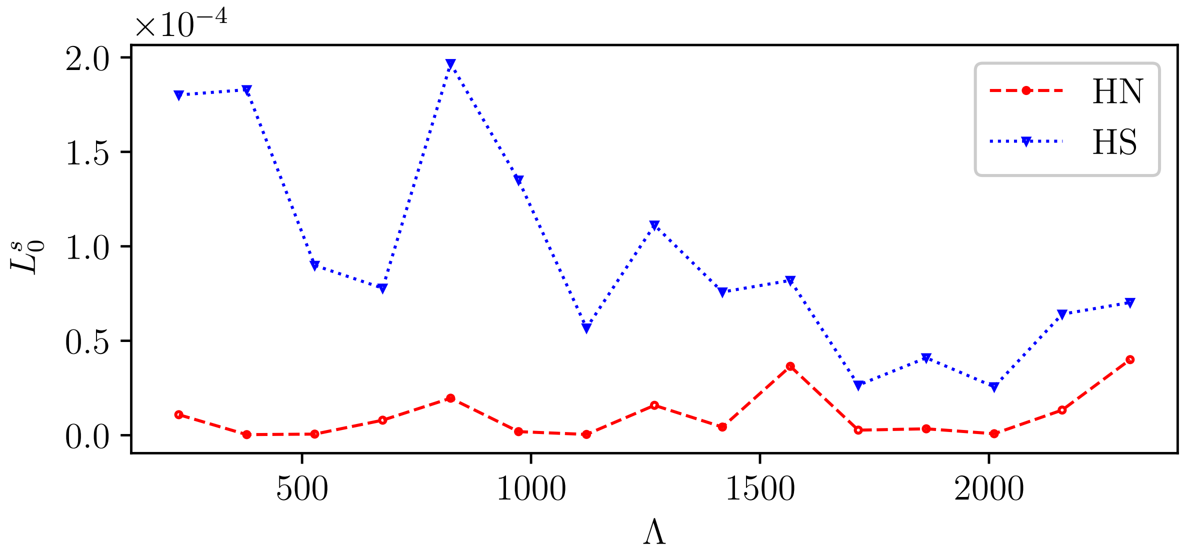
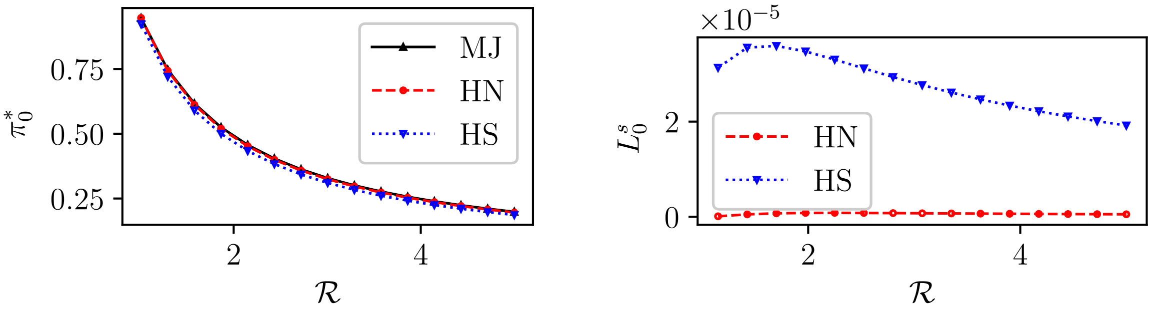


Figure 6.5: Values of the wealth-equivalent losses for the HN-GARCH and the homoscedastic model, dependent on the jump intensity in the MJ model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 6.4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

and increases Λ up to one order of magnitude from the original value in the MJ model (see Table 6.2). Again, both jump-free models are re-calibrated for every grid point. As seen for the parameters affecting the jump size, the parameter Λ does not have a significant impact on the WEL occurring for the closest HN-GARCH and homoscedastic models, as all values stay below or close to 2 basis points, with the HN-GARCH outperforming the constant solution and yielding even lower values.

To conclude our numerical investigation of the Merton jump model, we modify both the mean jump size and the intensity at the same time. In particular, we increase the parameters θ and Λ by a factor of 10. Concerning changes in the jump intensity, implied by an increase in Λ, we again use the approach described above to avoid changes in the long-run equity premium. Figure 6.6a shows the values of the initial optimal risky allocation for investors with different risk aversion levels, for the MJ as well as for the two jump-free models. The plot once more suggests that the strategies obtained from the HN-GARCH and the homoscedastic model are similar to the one in the MJ. Concerning the WEL investors incur when following the former



(a) Values of π_0^* dependent on $\mathcal{R}$.

(b) WEL dependent on $\mathcal{R}$.

Figure 6.6: Values of the optimal initial allocation π_0^* and of wealth-equivalent losses, dependent on the investor's level of relative risk aversion $\mathcal{R} = 1 - \gamma \in [1.1, 5]$. These plots are based on a modified, extreme parameter setting for the MJ with θ and Λ one order of magnitude larger than the original estimates in Table 6.2. The HN-GARCH and the homoscedastic model have been re-calibrated according to Section 6.4.2. The time horizon is one year with 252 trading days.

two suboptimal strategies, Figure 6.6b suggests a very good performance of the re-calibrated models, yielding values below one basis point for the entire range of relative risk aversion.

6.4.4 Results for the Normal Inverse Gaussian Model

In this subsection, we address the Normal Inverse Gaussian model in terms of the sensitivity of the optimal solution to the main model parameters and compare the model to the Gaussian HN-GARCH model without jumps and to a homoscedastic variant. The parameters for these three models are displayed in Table 6.3. The estimates for the NIG model are based on Ornthanalai (2014), adapted to our setting according to Section 6.4.1. As described in 6.4.2, the HN-GARCH and the homoscedastic model parameters are re-estimated based on a set of simulated log returns from the NIG model. The sample skewness for the first 15 years of the simulated log returns is -2.99, the kurtosis amounts to 73.90.[7]

Concerning the evolution over time, Figure 6.7 shows that an investor with a time horizon of more than 1.5 years will invest 2.5 percentage points more than an investor with a very short time horizon. While the HN-GARCH solution only shows slight differences in the initial allocation for very small values of T, the homoscedastic solution does not change for different time horizons.

Plotting the values of the initial optimal portfolio allocation for a one-year time horizon for investors with different levels of relative risk aversion, Figure 6.8a shows that the small differences observed in Figure 6.7 for the special case of $\mathcal{R} = 2$ are confirmed for other reasonable values, too. This is also reflected in Figure 6.8b, suggesting extremely low values for the WEL occurring when following the HN-GARCH or the homoscedastic solution, while returns actually evolve according to the NIG model.

The interpretation of the NIG innovation as a normal variance-mean mixture $y_t \sim \mathcal{N}(\beta x, x)$, with $x \sim \mathrm{IG}\left(h_{y,t}, \sqrt{\alpha^2 - \beta^2}\right)$, helps to get an intuition about the role of the different parameters

[7] These values are much more extreme than the numbers observed in the original dataset from the S&P 500 (see Section 6.4.3), similar to the picture for implied average conditional skewness and kurtosis observed by Ornthanalai (2014).

Table 6.3: MLEs for the NIG model and the recalibrated HN-GARCH and homoscedastic model, respectively. Standard errors for the recalibrated models are in parentheses. The NIG parameters are based on Ornthanalai (2014) and the methodology presented in Section 6.4.1, the two jump-free models have been estimated according to Section 6.4.2.

Par.	**NIG**	**HN**		**HS**	
λ_z	8.00×10^{-1}	1.83	(4.30×10^{-1})	1.83	(3.97×10^{-1})
ω_z	-1.65×10^{-6}	3.40×10^{-6}	(8.81×10^{-8})	3.21×10^{-4}	(2.96×10^{-5})
a_z	2.41×10^{-6}	4.21×10^{-7}	(4.62×10^{-8})		
b_z	9.40×10^{-1}	7.04×10^{-1}	(2.25×10^{-2})		
c_z	1.43×10^{2}	6.70×10^{2}	(6.14×10^{1})		
λ_y	6.88×10^{-1}				
Λ	2.62				
α	1.16×10^{1}				
β	-6.86				

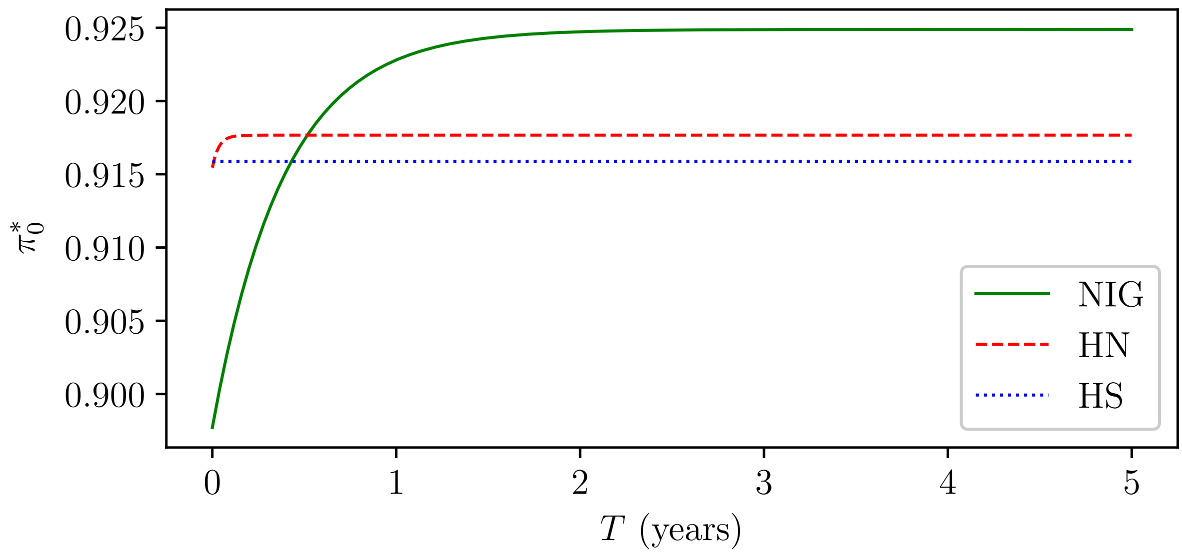
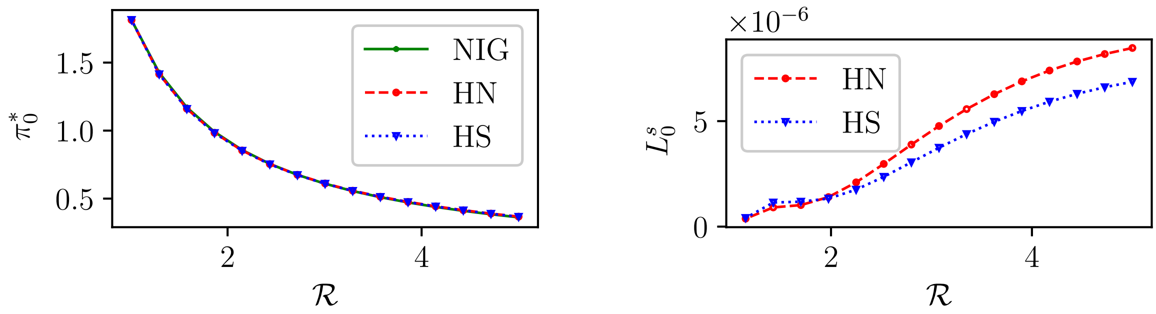


Figure 6.7: Values of the optimal strategy π_0^* in the NIG model and the corresponding re-calibrated HN-GARCH and homoscedastic model, dependent on the length of the time horizon, for T up to five years. The parameters are given in Table 6.3.

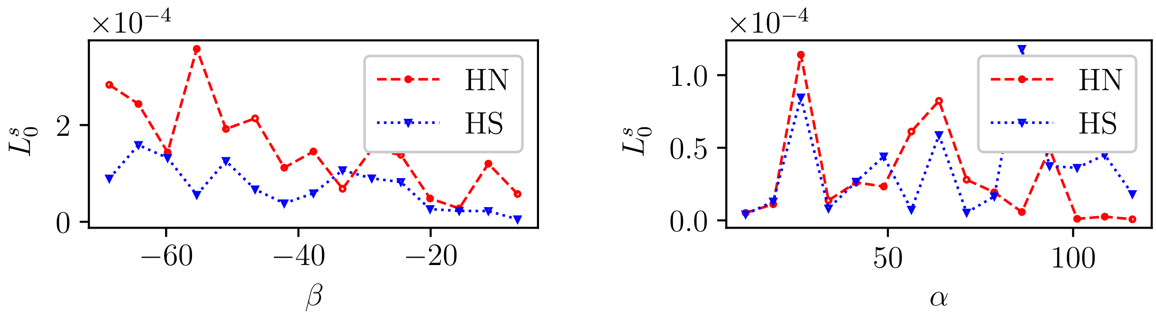
in this setting. As for the MJ model in Section 6.4.3, we intend to increase the average jump size, which can be achieved via modifications to β. As we change β, we correct the inverse Gaussian distribution to remain the same by adapting α accordingly. Increasing the absolute value of β up to one order of magnitude in this manner, and then re-calibrating the HN-GARCH and the homoscedastic model leads to the WEL plotted in Figure 6.9a. The loss increases towards higher absolute jump sizes for both suboptimal strategies, but the numbers remain on a very



(a) Values of π_0^* dependent on $\mathcal{R}$.

(b) WEL dependent on $\mathcal{R}$.

Figure 6.8: Values of the optimal initial allocation π_0^* and of wealth-equivalent losses in the NIG model, dependent on the investor's level of relative risk aversion $\mathcal{R} = 1 - \gamma \in [1.1, 5]$. The time horizon is one year with 252 trading days, the parameters are given in Table 6.3.



(a) WEL dependent on β, up to an absolute increase by a factor of 10.

(b) WEL dependent on α, up to an increase by a factor of 10.

Figure 6.9: Values of the wealth-equivalent losses, dependent on the jump parameters in the NIG model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 6.4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

low level in general. It is remarkable that in this case, the homoscedastic model could produce a lower WEL than the heteroscedastic HN-GARCH for some very extreme settings.

We add that the third parameter that can influence the jumps, Λ, plays a different role in the NIG as compared to the MJ model. While we were looking at an increase of said parameter in the MJ to raise the jump intensity, we may now increase Λ to obtain a larger mean of the inverse Gaussian random variable in the normal variance-mean mixture. Looking at an increase of up to one order of magnitude and re-calibrating the HN-GARCH and the homoscedastic model at each of the grid points also shows insignificant values for the WEL. The WEL evaluated for an investor with $\mathcal{R} = 2$ does not exceed one basis point.

6.4.5 Results for the Variance Gamma Model

This subsection is devoted to the numerical results for the special case of the Variance Gamma model, see Section 6.3.2. We follow the lines of Sections 6.4.3 and 6.4.4 and investigate the impact of more extreme jump settings in the VG model on the capability of the HN-GARCH

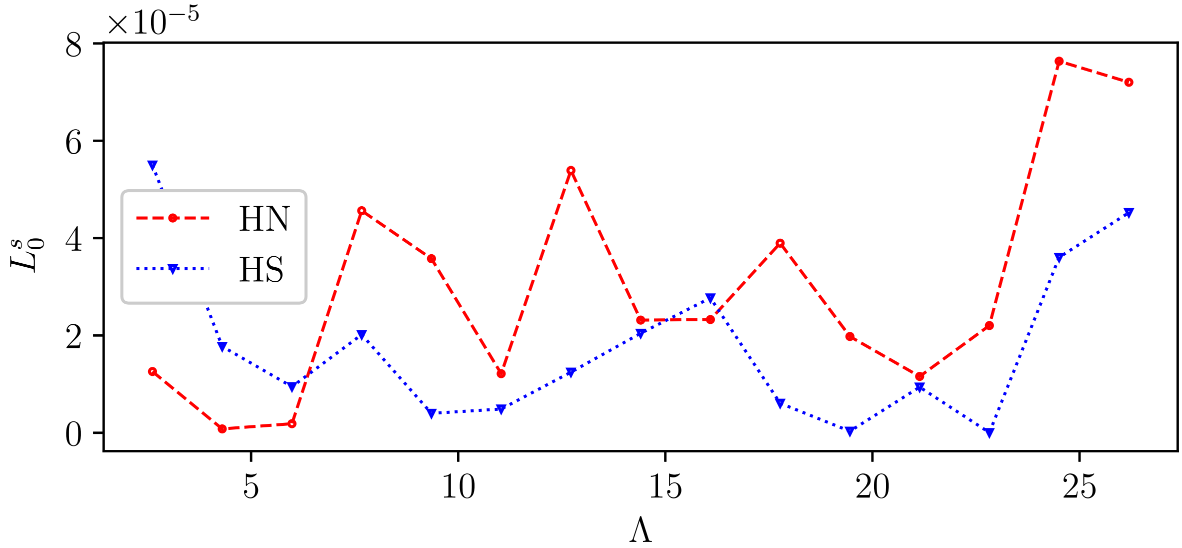


Figure 6.10: Values of wealth-equivalent losses for HN-GARCH and homoscedastic model, compared to the NIG model and dependent on the parameter Λ. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 6.4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

Table 6.4: MLEs for the VG model and the recalibrated HN-GARCH and homoscedastic model, respectively. Standard errors for the recalibrated models are in parentheses. The VG parameters are based on Ornthanalai (2014) and the methodology presented in Section 6.4.1, the two jump-free models have been estimated according to Section 6.4.2.

Par.	**VG**	**HN**		**HS**	
λ_z	6.82×10^{-1}	2.33	(3.34×10^{-1})	2.35	(2.76×10^{-1})
ω_z	-1.75×10^{-6}	1.25×10^{-7}	(3.18×10^{-7})	4.49×10^{-4}	(2.13×10^{-5})
a_z	2.57×10^{-6}	2.85×10^{-6}	(3.23×10^{-7})		
b_z	9.38×10^{-1}	9.44×10^{-1}	(6.50×10^{-3})		
c_z	1.47×10^{2}	1.31×10^{2}	(1.50×10^{1})		
λ_y	1.68×10^{-2}				
Λ	1.17×10^{2}				
θ	-8.19				
β	9.49×10^{2}				

and the homoscedastic model when it comes to dynamic investment strategies. Table 6.4 shows the estimates for the VG model, derived from Ornthanalai (2014) and adapted according to Section 6.4.1. The HN-GARCH and the homoscedastic parameter estimates are obtained according to the methodology described in Section 6.4.2. In this case, the sample skewness for the first 15 years of simulated log returns is −0.13, the sample kurtosis is 4.29.

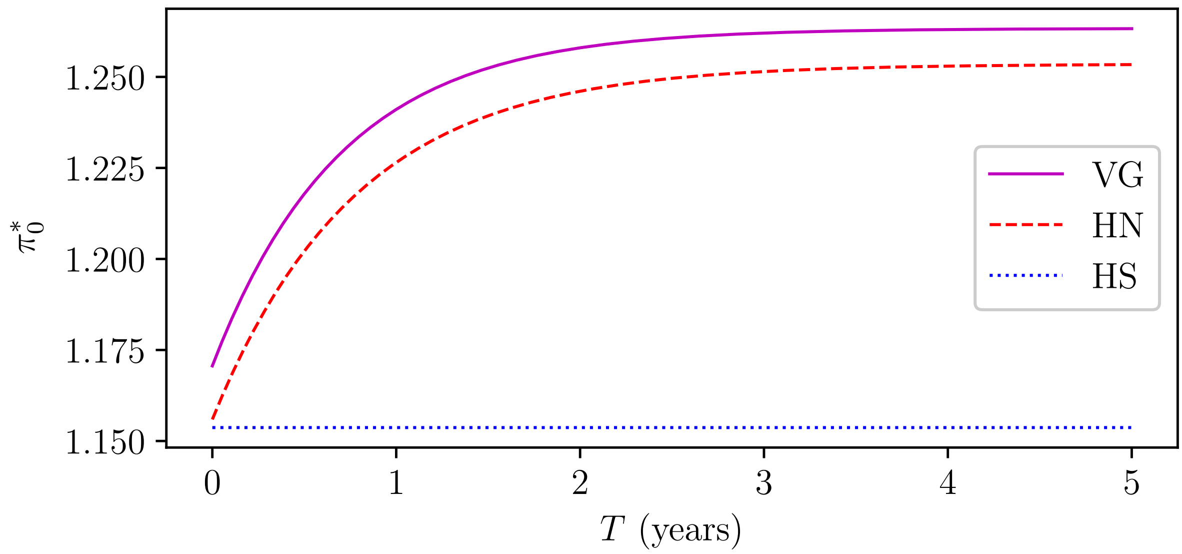
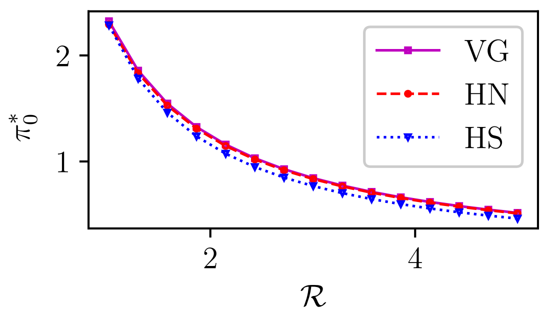
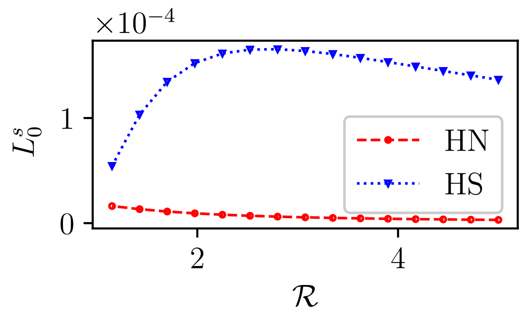


Figure 6.11: Values of the optimal strategy π_0^* in the VG model and the corresponding re-calibrated HN-GARCH and homoscedastic model, dependent on the length of the time horizon, for T up to five years. The parameters are given in Table 6.4.

Figure 6.11 shows that the re-calibrated HN-GARCH model, based on a simulated dataset of VG returns, produces an initial value for the risky allocation which is quite close to the one obtained directly from the VG model for all time horizons up to five years. The initial allocation resulting from the homoscedastic model is constant over different time horizons and almost identical to the HN-GARCH and the VG strategy for very small T.



(a) Values of π_0^* dependent on $\mathcal{R}$.

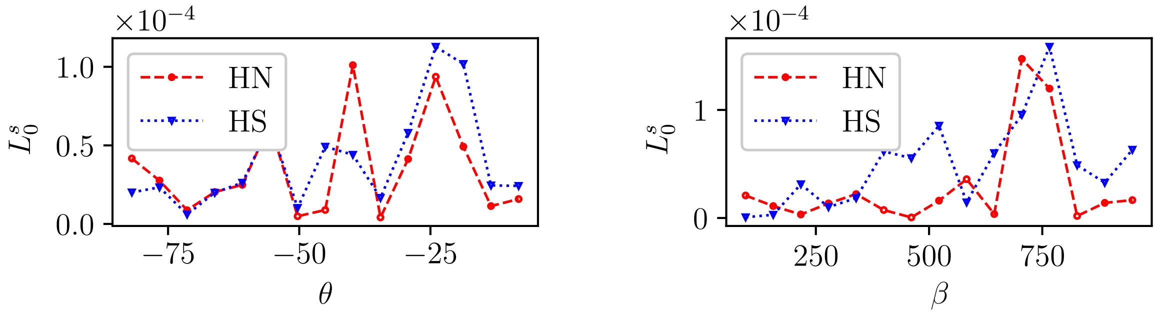


(b) WEL dependent on $\mathcal{R}$.

Figure 6.12: Values of the optimal initial allocation π_0^* and of wealth-equivalent losses in the VG model, dependent on the investor's level of relative risk aversion $\mathcal{R} = 1 - \gamma \in [1.1, 5]$. The time horizon is one year with 252 trading days, the parameters are given in Table 6.4.

This is also reflected in the plots in Figure 6.12, showing the difference of the initial risky allocations for a one-year time horizon across different levels of relative risk aversion. The HN-GARCH yields a value almost identical to the one from the VG model, the homoscedastic model seems to be quite close, too. Concerning the WEL, the picture observed for the MJ and the NIG models in Sections 6.4.3 and 6.4.4 is confirmed once more, since the values for the losses when following the HN-GARCH are very low, exceeded by the losses produced with the homoscedastic solution, which, however, also remain small – around one basis point for $\mathcal{R}$ between 2 and 4.

Concerning the jump parameters in the VG model, we again look at the normal variance-mean mixture interpretation of the underlying distribution. In particular, considering $y_t \sim \mathcal{N}(\theta x, x)$, based on $x \sim \text{Gamma}(h_{y,t}, \beta)$, we quickly identify θ as the key to modifications in the jump size. Since the mean of x is given by $h_{y,t}/\beta$, and this again is reflected in both the mean and the variance of y_t, we can modify these two measures via decreasing β and leaving the behavior of the process $h_{y,t}$ unchanged. Repeating the same procedure as for the other WEL evaluations



(a) WEL dependent on θ, up to an absolute increase by a factor of 10.

(b) WEL dependent on β, up to a decrease by a factor of 10.

Figure 6.13: Values of the wealth-equivalent losses, dependent on the jump parameters in the VG model. For each grid point, both the HN-GARCH and the homoscedastic model are re-calibrated using the methodology in Section 6.4.2. The time horizon is one year with 252 trading days, and we consider an investor with $\gamma = -1$.

here yields a very familiar picture, as shown in Figure 6.13: Re-calibrated using simulations from the jump model for different parameter values, the losses for the HN-GARCH and the homoscedastic model remain very low, barely exceeding one basis point for a one-year time horizon in the VG setting.

The evidence in this section indicates that Gaussian models excel at replicating the simulated data of jump processes across different jump models and parameter scenarios. In this context, we note that our Gaussian models are fitted based on a relatively large data set, see Section 6.4.2. Other results, e.g., in the context of regime switches (Ang and Bekaert, 2002; Guidolin and Timmermann, 2007; Campani et al., 2021, among others), are based on comparatively short time series. This suggests that the large amount of data points we use helps to eliminate the estimation error and significantly improves the performance of the Gaussian models.

6.5 Conclusion

This chapter establishes an approximate closed-form solution for a discrete-time portfolio optimization problem in a Lévy GARCH framework and therefore extends the results presented in Chapter 3 in a different direction than Chapters 4 and 5 do. We provide optimality criteria for different special cases of the framework, including both finite-activity and infinite-activity jump components. Our numerical study focuses on the impact of jumps on the optimal strategy. It compares three different jump models to the jump-free Gaussian HN-GARCH model and a homoscedastic Gaussian model. Across all considered jump models, the results demonstrate that investors following the closest possible jump-free variants obtain similar strategies and incur only insignificant wealth-equivalent losses. Our finding remains true for modified parameter settings indicating extreme jump behavior in the Lévy GARCH models. This surprising performance of simple models might be explained by two factors. One is the connection between expected utility maximization and mean-variance (MV) theory, stating that expected utility maximization can be well approximated using MV, i.e., by the first two moments (Markowitz, 2012). This highlights the importance of matching the first two moments of the wealth distribution. The quality of this approximation in other settings, e.g., with a disappointment-averse investor (Dahlquist et al., 2017) could be subject to further research. Secondly, the frequency of re-balancing in a discrete-time setting leaves little room to take advantage of sudden jumps. Overall, our analysis shows that correctly calibrated jump-free models in the discrete-time GARCH framework could diminish the necessity of modeling jumps for investors.

7 Conclusion

After the first closed-form solution to the EUT problem in a general affine GARCH environment presented in Chapter 3, we tackled the extension towards consumption at intermediate time points in Chapter 4. Our construction with a HARA type utility function at all time points $t = 0, \ldots, T-1$ and a change of control for the investor's consumption avoid the consumption of the entire available wealth before the end of the investment horizon. We furthermore proved that the optimal log-wealth process again follows an affine GARCH process and that the simpler case without consumption at intermediate time points is still nested in this setup. In Chapter 5, we established two generalizations concerning the utility derived from terminal wealth. The first allows for the more general class of HARA utilities instead of the embedded CRRA family. For a risk-averse investor, we learned that the HARA function of terminal wealth can be interpreted as a CRRA utility based on a surplus beyond some fixed threshold that is achieved with certainty, which provides us with an intuitive interpretation as a CPPI approach. The second generalization relaxes an assumption on the investor's risk aversion, allowing in particular also for a quadratic utility and a connection to the prominent field of MV optimization. In this generalized framework, one may even include higher moments of the portfolio return and consider investors' preferences on skewness and excess kurtosis. While Chapters 3, 4 and 5 extended the problem formulation in several directions, the theory was based on the log asset price evolving according to a general affine GARCH model. Chapter 6 finally broke through these barriers and considered the Lévy GARCH setting, where an additional jump component besides a Gaussian one drives asset price evolution. Potential choices for the distribution of the jump innovation include the Merton jump (MJ) with finite activity and more involved constructions such as Normal-inverse Gaussian (NIG) and Variance Gamma (VG) processes. By disregarding the jump innovation, the setup still nests the Gaussian HN-GARCH model.

We have seen various numeric examples throughout this dissertation. In the model with consumption in Chapter 4, we studied the impact of including consumption at intermediate time points on optimal portfolio strategies and noted that this additional possibility of deriving utility disrupts optimal allocation levels. Since the model setup allows for Gaussian and non-Gaussian innovations, we also investigated the effect of non-Gaussianity, i.e., of conditional skewness and excess kurtosis in asset returns, on the optimal strategy. This led to the eye-opening conclusion that, among conditional non-Gaussianity and heteroscedasticity, the latter is the more important feature in asset return models. In a study based on S&P 500 index returns, we estimated the Gaussian HN-GARCH model and a homoscedastic variant. The five-year period from 2013 through 2017 revealed that following a homoscedastic strategy can lead to 10% less total consumption and, at the same time, 8% less terminal wealth as compared to the Gaussian GARCH model. Focusing on utility derived from terminal wealth again in Chapter 5, we chose the newly established connection to MV theory and studied the efficient frontiers for a Gaussian GARCH and a homoscedastic model. The results indicated that, given the same expected portfolio return, a GARCH investor faces significantly less variance, with the difference to a homoscedastic investor increasing with the length of the investment horizon. This confirmed the general numerical finding of Chapter 4 and can be summarized in the brief statement that heteroscedasticity matters. The numerical study in Chapter 6 took a different point of view and considered the question of whether jumps in the asset price process make an impact. Instead of calibrating all involved models based on the same dataset, which in real-world applications is usually limited to small time series, we wanted to know if the best possible jump-free Gaussian GARCH model is capable of imitating a Lévy GARCH model with jumps. To this end, we sampled a large number of returns with jumps and calibrated the HN-GARCH model as well as a homoscedastic variant based on this artificial dataset. Although the results are hypothetical because of the limited availability of market data in real applications, the results indicate that correctly calibrated GARCH models produce only negligible losses in various extreme jump environments. This finding is extremely relevant for practitioners and connects very well to the numerical studies from Chapters 4 and 5, as the most significant feature to consider in the underlying asset return model among heteroscedasticity, non-Gaussianity, and jumps, seems to be the first one.

The results presented in this dissertation leave room for open questions. The most natural ones target the unfilled space in Figure 6.1, i.e., the connection between HARA utilities for terminal wealth, consumption at intermediate time points, and more general asset price models. Since all the extensions presented in this dissertation preserve the exponentially affine structure of the MGF, a bold conjecture could be that the combination works in a similar fashion. In the projects related to this dissertation, we chose to deal with these issues separately for brevity and lucidity. In addition to these opportunities, a new class of affine multi-factor models was introduced very recently by Augustyniak et al. (2023), nesting all models of the present dissertation. In particular, while the IG-GARCH model is not included in the Lévy GARCH setup, both are contained in the new family, paving the way to further generalizations of our existing results. Another natural extension of our environment concerns the presence of more than one risky asset, i.e., the transition to a multivariate setting, which in the world of affine GARCH models was introduced by Escobar-Anel et al. (2020). Note, however, that the general multi-factor model of Augustyniak et al. (2023) is formulated for the univariate case. Last but not least, it would be interesting to study many of the numerical results in this dissertation with respect to their dependence on the length of the time step. For simplicity, we assume this length to be exactly one day throughout this dissertation, and so it remains an open question whether more intermediate time points change the effect that heteroscedasticity, non-Gaussianity, and jumps have on optimal portfolio strategies and strategy performance. For clarity and in the spirit of addressing challenges methodically, we chose not to delve into the abovementioned extensions in this dissertation.

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A Appendix for Chapter 2

A.1 Proof of Theorem 2.15

The following proof is based on Hinderer et al. (2016, Theorem 16.1.12), relying on further results that can be found in earlier chapters of the same book.

Proof of Theorem 2.15. We verify the result by induction on $n \geq 1$, using the assertion (I_n) that the three following properties hold:

- $\phi_{T-n+1} \in \mathbb{M}$,
- π^*_{T-n} is a maximizer at time $T-n$,
- $\phi_{T-n} = \mathbb{U}\phi_{T-n+1}$.

Note that (I_1) holds since $\phi_T \in \mathbb{M}$ by assumption and π^*_{T-1} is a maximizer by Condition (i). Furthermore, by (2.4b) and Definition 2.14, we deduce $\phi_{T-1} = \mathbb{U}\phi_T$. Assuming that for some $n \geq 1$, the hypothesis (I_ν) is true for all $\nu = 1, \dots, n$, Condition (ii) immediately yields $\phi_{T-n} \in \mathbb{M}$. We know by Condition (i) that π^*_{T-n-1} is a maximizer at time $T-n-1$, since $\phi_{T-n} \in \mathbb{M}$. The set of true hypotheses $(I_n), \dots, (I_1)$ shows that $\{\pi^*_t\}_{t=T-n}^{T-1}$ is an optimal policy for the problem SCM_n. Thus, there remains to show that $\phi_{T-n-1} = \mathbb{U}\phi_{T-n}$. Note that for $s \in \mathbb{S}$, denoting $\pi' := \{\pi_t\}_{T-n}^{T-1}$,

$$\begin{aligned}
\phi_{T-n-1}(s) &= \sup_{(\pi_{T-n-1},\pi')} \phi_{T-n-1,(\pi_{T-n-1},\pi')}(s) \\
&= \sup_{\pi_{T-n-1}} \sup_{\pi'} \Big\{ r_w\left(s, \pi_{T-n-1}(s)\right) + \beta \cdot \mathbb{E}\left[\phi_{T-n,\pi'}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right] \Big\} \\
&= \sup_{\pi_{T-n-1}} \left\{ r_w\left(s, \pi_{T-n-1}(s)\right) + \beta \sup_{\pi'} \mathbb{E}\left[\phi_{T-n,\pi'}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right] \right\}. \qquad \text{(A.1)}
\end{aligned}$$

If we can change the order of taking the supremum and the expectation in the last line, then we are done. For all $s \in \mathbb{S}$, by Definition 2.14, we have that $\phi_{T-n,\pi'}(s) \leq \phi_{T-n}(s)$. Since this inequality also holds after taking the conditional expectation and the supremum over π', we arrive at

$$\sup_{\pi'} \mathbb{E}\left[\phi_{T-n,\pi'}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right] \leq \mathbb{E}\left[\phi_{T-n}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right]. \qquad \text{(A.2)}$$

On the other hand, since there exists an optimal policy $\pi^* = \{\pi^*_t\}_{t=T-n}^{T-1}$, we have

$$\begin{aligned}
\sup_{\pi'} \mathbb{E}\left[\phi_{T-n,\pi'}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right] &\geq \mathbb{E}\left[\phi_{T-n,\pi^*}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right] \\
&= \mathbb{E}\left[\phi_{T-n}\left(\mathbb{T}\left(s, \pi_{T-n-1}(s), Y\right)\right)\right].
\end{aligned} \qquad \text{(A.3)}$$

This shows that we can change the order of expectation and supremum in (A.1) and with the maximizer π^*_{T-n-1} at time $T-n-1$ we obtain

$$\phi_{T-n-1} = \mathbb{U}\phi_{T-n}, \tag{A.4}$$

also implying that $\{\pi^*_t\}_{t=T-n-1}^{T-1}$ is an optimal policy for SCM_{n+1}. □

B Appendix for Affine GARCH Models

B.1 MGF for the HN-GARCH Model

This section contains the proof for the exponentially affine nature of the moment generating function (MGF) of the log-asset return in an HN-GARCH model as presented and used throughout the main body of this dissertation. In the setting of Heston and Nandi (2000), in general denoting the length of the time step by Δ, we let $\theta = (\lambda, \omega, \beta, \alpha, \rho)$ be a vector of parameters, and r be the continuously compounded riskless rate for the bank account $P^{(0)}$. Assume the following dynamics for the log-asset return and its conditional variance, respectively:

$$X_{t+\Delta} - X_t \coloneqq \log\left(\frac{P^{(1)}_{t+\Delta}}{P^{(1)}_t}\right) = r + \lambda h_{t+\Delta} + \sqrt{h_{t+\Delta}}\epsilon_{t+\Delta}, \tag{B.1a}$$

$$h_{t+\Delta} = \omega + \beta h_t + \alpha\left(\epsilon_t - \rho\sqrt{h_t}\right)^2, \tag{B.1b}$$

with $\{\epsilon_t\}_t$ a sequence of i.i.d. standard normally distributed random variables. Note that we consider only the single-lag version here, where $P^{(1)}$ and h are allowed to be autoregressive and dependent only on the previous realization, and that in the main body, we work with the length of the time step set to one day, i.e., $\Delta = 1$. Along the lines of Heston and Nandi (2000, Appendix A), we prove the following result:

Proposition B.1 (Heston and Nandi, 2000)**.** *Assuming the dynamics described in* (B.1)*, the generating function takes the form*

$$\Psi_{X_T}(u; t, T) \coloneqq \mathbb{E}_t\left[\left(P^{(1)}_T\right)^u\right] = \left(P^{(1)}_t\right)^u \exp\left\{A(u; t, T) + B(u; t, T)\, h_{t+\Delta}\right\}, \tag{B.2}$$

with the coefficients satisfying

$$A(u; t, T) = A(u; t+\Delta, T) + ur + \omega B(u; t+\Delta, T) - \frac{1}{2}\log\left(1 - 2\alpha B(u; t+\Delta; T)\right), \tag{B.3a}$$

$$B(u; t, T) = u(\lambda + \rho) - \frac{1}{2}\rho^2 + \beta B(u, t+\Delta, T) + \frac{\frac{1}{2}(u-\rho)^2}{1 - 2\alpha B(u; t+\Delta, T)}, \tag{B.3b}$$

and the terminal conditions $A(u; T, T) = B(u; T, T) = 0$.

Proof. We prove Proposition B.1 via induction on $n \coloneqq \frac{T-t}{\Delta}$. That is, we start at the end of the time horizon and move backwards step by step. The terminal conditions follow directly from evaluating Equation (B.2) at $t = T$, corresponding to $n = 0$. Now, assuming that the induction hypothesis is true up to some $n - 1 \geq 0$ corresponding to $t + \Delta$, we start by using the tower property and apply the induction hypothesis in the second equation. This yields:

$$\begin{aligned}\Psi_{X_T}(u; t, T) &= \mathbb{E}_t\left[\Psi_{X_T}(u; t+\Delta, T)\right] \\ &= \mathbb{E}_t\left[\exp\left\{uX_{t+\Delta} + A(u; t+\Delta, T) + B(u; t+\Delta, T)\cdot h_{t+2\Delta}\right\}\right]\end{aligned} \tag{B.4}$$

$$
\begin{aligned}
&= \mathbb{E}_t \Bigg[\exp \Bigg\{ uX_t + ur + u\lambda h_{t+\Delta} + u\sqrt{h_{t+\Delta}}\epsilon_{t+\Delta} + A(u; t+\Delta, T) \\
&\qquad + B(t+\Delta, T, u) \left(\omega + \beta h_{t+\Delta} + \alpha \left(\epsilon_{t+\Delta} - \rho\sqrt{h_{t+\Delta}} \right)^2 \right) \Bigg\} \Bigg] \\
&= \mathbb{E}_t \Bigg[\exp \Bigg\{ u\left(X_t + r\right) + A(u; t+\Delta, T) + B(u; t+\Delta, T)\omega \\
&\qquad + B(u; t+\Delta, T)\alpha \left(\epsilon_{t+\Delta}^2 - 2\epsilon_{t+\Delta}\rho\sqrt{h_{t+\Delta}} + \rho^2 h_{t+\Delta} \right) \\
&\qquad + u\epsilon_{t+\Delta}\sqrt{h_{t+\Delta}} + \left(u\lambda + B(u; t+\Delta, T)\beta\right) h_{t+\Delta} \Bigg\} \Bigg] .
\end{aligned} \tag{B.5}
$$

Via completing the square and using that everything except for $\epsilon_{t+\Delta}$ is known at time t, we obtain, continuing from (B.5):

$$
\begin{aligned}
\Psi_{X_T}(u; t, T) &= \mathbb{E}_t \Bigg[\exp \Bigg\{ u\left(X_t + r\right) + A(u; t+\Delta, T) + B(u; t+\Delta, T)\omega \\
&\qquad + B(u; t+\Delta, T)\alpha \left(\epsilon_{t+\Delta} - \left(\rho - \frac{u}{2\alpha B(u; t+\Delta, T)} \right) \sqrt{h_{t+\Delta}} \right)^2 \\
&\qquad + \left(u\lambda + B(u; t+\Delta, T)\beta + \rho u - \frac{u^2}{4\alpha B(u; t+\Delta, T)} \right) h_{t+\Delta} \Bigg\} \Bigg] \\
&= \exp \Bigg\{ u\left(X_t + r\right) + A(u; t+\Delta, T) + B(u; t+\Delta, T)\omega \\
&\qquad + \left(u\lambda + B(u; t+\Delta, T)\beta + \rho u - \frac{u^2}{4\alpha B(u; t+\Delta, T)} \right) h_{t+\Delta} \Bigg\} \\
&\quad + \mathbb{E}_t \left[\exp \left(a\left(\epsilon_{t+\Delta} + b\right)^2 \right) \right] ,
\end{aligned} \tag{B.6}
$$

setting $a = \alpha \cdot B(u; t+\Delta, T)$ and $b = -\left(\rho - \frac{u}{2\alpha B(u; t+\Delta, T)} \right) \sqrt{h_{t+\Delta}}$ in (B.6). For a standard normally distributed variable z, an integration by substitution and completing the square shows that (Heston and Nandi, 2000, p. 619; Badescu et al., 2019, p. 33)

$$
\mathbb{E}\left[\exp\left(a(z+b)^2 \right) \right] = \exp\left(-\frac{1}{2}\log(1-2a) + \frac{ab^2}{1-2a} \right) . \tag{B.7}
$$

We use (B.7) in (B.6) in order to derive:

$$
\begin{aligned}
\Psi_{X_T}(u; t, T) &= \exp \Bigg\{ u\left(X_t + r\right) + A(u; t+\Delta, T) + B(u; t+\Delta, T)\omega \\
&\qquad - \frac{1}{2}\log\left(1 - 2\alpha B(u; t+\Delta, T)\right) + \left(u\lambda + B(u; t+\Delta, T)\beta + \rho u\right) h_{t+\Delta} \\
&\qquad + \Bigg[\frac{1}{1 - 2\alpha B(u; t+\Delta, T)} \Bigg(\alpha B(u; t+\Delta, T)\rho^2 - \rho u \\
&\qquad\quad + \frac{u^2}{4\alpha B(u; t+\Delta, T)} \Bigg) - \frac{u^2}{4\alpha B(u; t+\Delta, T)} \Bigg] h_{t+\Delta} \Bigg\} \\
&= \exp \Bigg\{ u\left(X_t + r\right) + A(u; t+\Delta, T) + B(u; t+\Delta, T)\omega \\
&\qquad - \frac{1}{2}\log\left(1 - 2\alpha B(u; t+\Delta, T)\right) \\
&\qquad + \left(u(\lambda + \rho) - \frac{1}{2}\rho^2 + B(u; t+\Delta, T)\beta + \frac{\frac{1}{2}(u-\rho)^2}{1 - 2\alpha B(u; t+\Delta, T)} \right) h_{t+\Delta} \Bigg\} ,
\end{aligned}
$$

which gives exactly the coefficients in (B.3) and thus proves the hypothesis for also for n and t, respectively. It is easy to check that the following representation for B is equivalent to (B.3b):

$$B(u;t,T) = u\lambda + B(u,t+\Delta,T)\left(\alpha\rho^2+\beta\right) + \frac{\left(u-2\alpha\rho B(u,t+\Delta,T)\right)^2}{2\left(1-2\alpha B(u;t+\Delta,T)\right)}. \tag{B.8}$$

□

Remark B.2. *Instead of the general form in* (B.2) *with the coefficients* (B.3)*, we will use the conditional bivariate one-step MGF of the log return* $Y_{t+\Delta} = X_{t+\Delta} - X_t$ *and the conditional variance* $h_{t+2\Delta}$*, which is defined as*

$$\Psi_{(Y_{t+\Delta},h_{t+2\Delta})}\left(u,v \mid \mathcal{F}_t\right) := \mathbb{E}_t\left[\exp\left\{u\cdot Y_{t+\Delta} + v\cdot h_{t+2\Delta}\right\}\right], \tag{B.9}$$

for $t \leq T-2\Delta$*. Analogously to the calculations in the proof of Proposition B.1, we can show that:*

$$\Psi_{(Y_{t+\Delta},h_{t+2\Delta})}\left(u,v \mid \mathcal{F}_t\right) = \exp\left\{A(u,v;t,t+\Delta) + B(u,v;t,t+\Delta)\cdot h_{t+\Delta}\right\}, \tag{B.10}$$

with the coefficients A *and* B *given by*

$$A(u,v;t,t+\Delta) = ur + \omega v - \frac{1}{2}\log\left(1-2\alpha v\right), \tag{B.11a}$$

$$B(u,v;t,t+\Delta) = u\left(\lambda+\rho\right) - \frac{1}{2}\rho^2 + \beta v + \frac{\frac{1}{2}\left(u-\rho\right)^2}{1-2\alpha v}. \tag{B.11b}$$

In this dissertation, we omit the last two time arguments of A *and* B *in the one-step MGF where they are obsolete. Furthermore, note that the representations in* (B.11) *assume that* $1-2\alpha v > 0$*.*

B.2 MGF for the IG-GARCH Model

In the setting of Christoffersen et al. (2006), let the vector of parameters be $\theta = (\nu, \eta, w, a, b, c)$. We assume the following dynamics for the log asset price and its conditional variance, respectively:

$$\log\left(\frac{P^{(1)}_{t+\Delta}}{P^{(1)}_t}\right) = r\Delta + \nu h_{t+\Delta} + \eta y_{t+\Delta}, \tag{B.12a}$$

$$h_{t+\Delta} = w + bh_t + cy_t + a\frac{h_t^2}{y_t}, \tag{B.12b}$$

where $\{y_t\}_t$ is a sequence of random variables with inverse Gaussian distribution and single parameter $\delta(t) = \delta_t = {}^{h_t}/_{\eta^2}$. Christoffersen et al. (2006, Appendix A) proved the following result:

Proposition B.3 (Christoffersen et al., 2006)**.** *Assuming the dynamics described in Equation B.12, the generating function takes the form*

$$\Psi_{X_T}(u;t,T) = \mathbb{E}_t\left[\left(P^{(1)}_T\right)^u\right] = \left(P^{(1)}_t\right)^u \exp\left\{A(u;t,T) + B(u;t,T)\,h_{t+\Delta}\right\}, \tag{B.13}$$

with the coefficients satisfying

$$\begin{aligned} A(u;t,T) &= A\left(u;t+\Delta,T\right) + ur\Delta + wB\left(u;t+\Delta,T\right) \\ &\quad - \frac{1}{2}\log\left(1-2\eta^4 aB\left(u;t+\Delta,T\right)\right), \end{aligned} \tag{B.14a}$$

$$\begin{aligned} B(u;t,T) &= bB\left(u;t+\Delta,T\right) + u\nu + \frac{1}{\eta^2} \\ &\quad - \frac{1}{\eta^2}\sqrt{\left(1-2a\eta^4 B\left(u;t+\Delta,T\right)\right)\left(1-2u\eta-2B\left(u;t+\Delta,T\right)c\right)}, \end{aligned} \tag{B.14b}$$

and the terminal conditions $A(u;T,T) = B(u;T,T) = 0$.

Proof. We again use induction on $n := \frac{T-t}{\Delta}$. The terminal conditions follow directly from evaluating $\Psi_{X_T}(u;T,T)$, proving the hypothesis for $n=0$. Assuming that this hypothesis holds true up to some $n-1 \geq 0$, corresponding to $t+\Delta$, we apply the tower property to find:

$$\begin{aligned} \Psi_{X_T}(u;t,T) &= \mathbb{E}_t\left[\Psi_{X_T}(u;t+\Delta,T)\right] \\ &= \mathbb{E}_t\left[\left(P_t^{(1)}\right)^u \exp\left\{u\log\left(\frac{P_{t+\Delta}^{(1)}}{P_t^{(1)}}\right) + A\left(u;t+\Delta,T\right) + B\left(u;t+\Delta,T\right)h_{t+2\Delta}\right\}\right] \\ &= \left(P_t^{(1)}\right)^u \cdot \mathbb{E}_t\Bigg[\exp\Bigg\{u\left(r\Delta+\nu h_{t+\Delta}+\eta y_{t+\Delta}\right) + A\left(u;t+\Delta,T\right) \\ &\qquad\qquad + B\left(u;t+\Delta,T\right)\left(w + bh_{t+\Delta} + cy_{t+\Delta} + a\frac{h_{t+\Delta}^2}{y_{t+\Delta}}\right)\Bigg\}\Bigg] \\ &= \left(P_t^{(1)}\right)^u \cdot \mathbb{E}_t\Bigg[\exp\Bigg\{ur\Delta + A\left(u;t+\Delta,T\right) + B\left(u;t+\Delta,T\right)w \\ &\qquad\qquad + \left(u\nu + B\left(u;t+\Delta,T\right)b\right)\cdot h_{t+\Delta} \\ &\qquad\qquad + \left(u\eta + B\left(u;t+\Delta,T\right)c\right)\cdot y_{t+\Delta} \\ &\qquad\qquad + B\left(u;t+\Delta,T\right)a\cdot\frac{h_{t+\Delta}^2}{y_{t+\Delta}}\Bigg\}\Bigg]. \end{aligned} \tag{B.15}$$

Note that the only parameter that is not known by time t is $y_{t+\Delta}$, so the first to lines in (B.15) can just be taken out of the expectation. For what is left in the expectation we use the generalization of the MGF of the inverse Gaussian distribution, Formula (2) in Christoffersen et al. (2006). If y inversely Gaussian distributed with single parameter δ, and $\alpha,\beta \in \mathbb{R}$, then (Christoffersen et al., 2006, Equation (2))

$$\mathbb{E}\left[\exp\left(\alpha y + \frac{\beta}{y}\right)\right] = \frac{\delta}{\sqrt{\delta^2-2\beta}}\exp\left\{\delta - \sqrt{\left(\delta^2-2\beta\right)\left(1-2\alpha\right)}\right\}. \tag{B.16}$$

Continuing from (B.15), we set $\alpha = u\eta + B\left(u;t+\Delta,T\right)c$ and $\beta = B\left(u;t+\Delta,T\right)\cdot a\cdot h_{t+\Delta}^2$, and $\delta = h_{t+\Delta}/\eta^2$. Writing $A_{t+\Delta}$ and $B_{t+\Delta}$ short for $A\left(u;t+\Delta,T\right)$ and $B\left(u;t+\Delta,T\right)$, respectively, we obtain:

$$\begin{aligned} \Psi_{X_T}(u;t,T) &= \exp\left\{ur\Delta + A_{t+\Delta} + B_{t+\Delta}w + \left(u\nu + B_{t+\Delta}b\right)\cdot h_{t+\Delta}\right\} \\ &\quad \times \left(1-2\eta^4 aB_{t+\Delta}\right)^{-\frac{1}{2}} \\ &\quad \times \exp\left\{\left(\frac{1}{\eta^2} - \sqrt{\left(\frac{1}{\eta^4} - 2aB_{t+\Delta}\right)\left(1-2u\eta-2B_{t+\Delta}c\right)}\right)h_{t+\Delta}\right\}. \end{aligned} \tag{B.17}$$

Comparing coefficients gives the recursions in (B.14), proving the hypothesis also for n. □

Remark B.4. *In the spirit of Remark B.2, we can derive the conditional bivariate one-step MGF of the log return and the conditional variance as*

$$\begin{aligned}\Psi_{(Y_{t+\Delta},h_{t+2\Delta})}\left(u,v \mid \mathcal{F}_t\right) &= \mathbb{E}_t\left[\exp\left\{u\cdot Y_{t+\Delta} + v\cdot h_{t+2\Delta}\right\}\right] \\ &= \exp\left\{A(u,v;t,t+\Delta) + B(u,v;t,t+\Delta)\cdot h_{t+\Delta}\right\},\end{aligned} \tag{B.18}$$

with the coefficients

$$A\left(u,v;t,t+\Delta\right) = ur + vw - \frac{1}{2}\log\left(1-2va\eta^4\right), \tag{B.19a}$$

$$B\left(u,v;t,t+\Delta\right) = vb + u\nu + \frac{1}{\eta^2}\left(1-\sqrt{\left(1-2va\eta^4\right)\left(1-2u\eta-2vc\right)}\right). \tag{B.19b}$$

Note that the above calculations assume that $1-2va\eta^4 > 0$ *and* $1-2u\eta-2vc > 0$.

C Appendix for Chapter 4

C.1 Proofs from Section 4.3

Proof of Theorem 4.4. We start by guessing the correct form of the value function and using it as an ansatz. For lack of better terms, the factors in (4.11) are denoted by P, Q, and R. The recursive parameters P, R, D, and E^h depend on the optimal solution, but we omit the dependencies in the following lines for readability. Furthermore, referring to the coefficients A and B of the moment generating function of the affine GARCH model in (4.5), let us introduce the following short notation:

$$\bar{A} = A\left(\left(E^W_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right)\pi_t, E^h_{t+1,T} + E^{\hat{C}}_{t+1,T} R_{t+1,T}\right), \tag{C.1a}$$

$$\bar{B} = B\left(\left(E^W_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right)\pi_t, E^h_{t+1,T} + E^{\hat{C}}_{t+1,T} R_{t+1,T}\right). \tag{C.1b}$$

Using the above-mentioned ansatz and the affine form of the optimal control on consumption, we arrive at:

$$\begin{aligned}
&\phi_t\left(W_t, h_{t+1}\right) \\
&= \max_{\pi_t, \hat{C}_t} \left\{ -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} + \bar{\beta} \cdot \mathbb{E}_t\left[\phi_{t+1}\left(\mathbb{T}\left(W_t, h_{t+1}\right)\right)\right]\right\} \\
&= \max_{\pi_t, \hat{C}_t} \Bigg\{ -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} - \bar{\beta} \cdot \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t-1} \bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}^*_s\right\}\right] \\
&\qquad + \frac{\bar{\beta}}{\gamma} \cdot \mathbb{E}_t\Bigg[\bar{\beta}^{T-t-1} \exp\Bigg\{D_{t+1,T} + E^{\hat{C}}_{t+1,T} \cdot \hat{C}^*_{t+1} \\
&\qquad\qquad + E^W_{t+1,T} \cdot W_{t+1} + E^h_{t+1,T} \cdot h_{t+2}\Bigg\}\Bigg]\Bigg\} \\
&= \max_{\pi_t, \hat{C}_t} \Bigg\{ -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} - \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t} \bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}^*_s\right\}\right] \\
&\qquad + \frac{\bar{\beta}^{T-t}}{\gamma} \cdot \mathbb{E}_t\Bigg[\exp\Bigg\{D_{t+1,T} + E^{\hat{C}}_{t+1,T} \cdot P_{t+1,T} \\
&\qquad\qquad + \left(E^W_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right) \cdot W_{t+1} \\
&\qquad\qquad + \left(E^h_{t+1,T} + E^{\hat{C}}_{t+1,T} R_{t+1,T}\right) \cdot h_{t+2}\Bigg\}\Bigg]\Bigg\}.
\end{aligned}$$

Together with the approximation of the self-financing condition (4.3) and the bivariate generating function of the log return and the conditional variance in the underlying affine GARCH model (4.5), this leads to:

$$\phi_t\left(W_t, h_{t+1}\right)$$

$$
\begin{aligned}
=\max_{\pi_t,\hat{C}_t} \Bigg\{ & -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} - \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t}\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_s^*\right\}\right] \\
& + \frac{\bar{\beta}^{T-t}}{\gamma} \cdot \mathbb{E}_t \Bigg[\exp\Bigg\{D_{t+1,T} + E_{t+1,T}^{\hat{C}} \cdot P_{t+1,T} \\
& \qquad + \left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \\
& \qquad \times \left(W_t - \hat{C}_t + (1-\pi_t)\, r + \pi_t \left(X_{t+1} - X_t\right) + \frac{1}{2}\left(\pi_t - \pi_t^2\right) h_{t+1}\right) \\
& \qquad + \left(E_{t+1,T}^{h} + E_{t+1,T}^{\hat{C}} R_{t+1,T}\right) \cdot h_{t+2}\Bigg\}\Bigg] \Bigg\} \\
=\max_{\pi_t,\hat{C}_t} \Bigg\{ & -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} - \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t}\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_s^*\right\}\right] \\
& + \frac{\bar{\beta}^{T-t}}{\gamma} \exp\Bigg\{D_{t+1,T} + E_{t+1,T}^{\hat{C}} P_{t+1,T} \\
& \qquad + \left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) (1-\pi_t)\, r \\
& \qquad + \left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \left(W_t - \hat{C}_t\right) \\
& \qquad + \frac{1}{2}\left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \left(\pi_t - \pi_t^2\right) h_{t+1}\Bigg\} \\
& \times \mathbb{E}_t \Bigg[\exp\Bigg\{\left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \pi_t \left(X_{t+1} - X_t\right) \\
& \qquad + \left(E_{t+1,T}^{h} + E_{t+1,T}^{\hat{C}} R_{t+1,T}\right) h_{t+2}\Bigg\}\Bigg] \Bigg\} \\
=\max_{\pi_t,\hat{C}_t} \Bigg\{ & -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} - \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t}\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_s^*\right\}\right] \\
& + \frac{\bar{\beta}^{T-t}}{\gamma} \exp\Bigg\{D_{t+1,T} + E_{t+1,T}^{\hat{C}} \cdot P_{t+1,T} + \bar{A} \\
& \qquad + \left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \cdot (1-\pi_t)\, r \\
& \qquad + \left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \cdot \left(W_t - \hat{C}_t\right) \\
& \qquad + \left[\frac{1}{2}\left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \left(\pi_t - \pi_t^2\right) + \bar{B}\right] \cdot h_{t+1}\Bigg\} \Bigg\},
\end{aligned}
$$

using the coefficients A and B from the generating function of the affine GARCH model. This justifies (4.10) with the recursive definitions[1]

$$
\begin{aligned}
D_{t,T}\left(\pi_t\right) = D_{t+1,T}\left(\pi_{t+1}^*\right) &+ E_{t+1,T}^{\hat{C}} \cdot P_{t+1,T}\left(\pi_{t+1}^*\right) + \bar{A} \\
&+ \left(E_{t+1,T}^{W} + E_{t+1,T}^{\hat{C}} Q_{t+1,T}\right) \cdot (1-\pi_t)\, r,
\end{aligned}
\tag{C.2a}
$$

[1] For completeness, dependencies on the optimal solution are displayed in (C.2), but omitted again afterwards.

$$E^{\hat{C}}_{t,T} = -\left(E^{W}_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right), \tag{C.2b}$$

$$E^{W}_{t,T} = \left(E^{W}_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right), \tag{C.2c}$$

$$E^{h}_{t,T}(\pi_t) = \frac{1}{2}\left(E^{W}_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right) \cdot \left(\pi_t - \pi_t^2\right) + \bar{B}, \tag{C.2d}$$

for all $t < T$, where in the terminal case $D_{T,T} = E^{\hat{C}}_{T,T} = E^{h}_{T,T} = 0$ and $E^{W}_{T,T} = \gamma$. Note that in particular, for all $t < T-1$,

$$E^{W}_{t,T} = \left(E^{W}_{t+1,T} + E^{\hat{C}}_{t+1,T} Q_{t+1,T}\right) = E^{W}_{t+1,T}\left(1 - Q_{t+1,T}\right) = -E^{\hat{C}}_{t+1,T}\left(1 - Q_{t+1,T}\right).$$

Concerning the solution to the optimization problem, we apply the first-order conditions w.r.t. π_t to find the optimality equations from

$$\frac{\partial}{\partial \pi_t} D_{t,T}(\pi_t) = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial u}\bar{A} = r, \tag{C.3a}$$

and

$$\frac{\partial}{\partial \pi_t} E^{h}_{t,T}(\pi_t) = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial u}\bar{B} = \pi_t - \frac{1}{2}. \tag{C.3b}$$

Taking derivatives w.r.t. $\hat{C}_t$ yields the following first-order condition:

$$\bar{\gamma}\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} = -\frac{\bar{\beta}^{T-t} E^{\hat{C}}_{t,T}}{\gamma} \exp\left\{D_{t,T} + E^{\hat{C}}_{t,T}\hat{C}_t + E^{W}_{t,T} W_t + E^{h}_{t,T} h_{t+1}\right\}, \tag{C.4}$$

which is equivalent to

$$\hat{C}^*_t = -\frac{1}{\bar{\gamma} + E^{\hat{C}}_{t,T}}\left(\log\left(-\frac{\bar{\beta}^{T-t} E^{\hat{C}}_{t,T}}{\gamma\bar{\gamma}\bar{a}^{\bar{\gamma}}}\right) + D_{t,T} + E^{W}_{t,T} W_t + E^{h}_{t,T} h_{t+1}\right). \tag{C.5}$$

Requiring that $E^{\hat{C}}_{t,T} > 0$ for all $t < T$, this justifies (4.11) with[2]

$$P_{t,T}(\pi^*_t) = -\frac{1}{\bar{\gamma} + E^{\hat{C}}_{t,T}}\left(\log\left(-\frac{\bar{\beta}^{T-t} E^{\hat{C}}_{t,T}}{\gamma\bar{\gamma}\bar{a}^{\bar{\gamma}}}\right) + D_{t,T}(\pi^*_t)\right), \tag{C.6a}$$

$$Q_{t,T} = -\frac{E^{W}_{t,T}}{\bar{\gamma} + E^{\hat{C}}_{t,T}}, \tag{C.6b}$$

$$R_{t,T}(\pi^*_t) = -\frac{E^{h}_{t,T}(\pi^*_t)}{\bar{\gamma} + E^{\hat{C}}_{t,T}}. \tag{C.6c}$$

Since $C_T = 0$, we start the recursion with $P_{T,T} = Q_{T,T} = R_{T,T} = 0$. Plugging (C.6b) into (C.2b) and using that $\bar{\gamma} > 0$ yields for $t < T-1$:

$$E^{\hat{C}}_{t,T} = E^{\hat{C}}_{t+1,T} - E^{\hat{C}}_{t+1,T} Q_{t+1,T} = E^{\hat{C}}_{t+1,T} + \frac{E^{\hat{C}}_{t+1,T} E^{W}_{t+1,T}}{\bar{\gamma} + E^{\hat{C}}_{t+1,T}}$$

[2] Again, we display dependencies on the optimal solution in (C.6) for completeness.

$$= E^{\hat{C}}_{t+1,T} - \frac{\left(E^{\hat{C}}_{t+1,T}\right)^2}{\bar{\gamma} + E^{\hat{C}}_{t+1,T}} > E^{\hat{C}}_{t+1,T} - E^{\hat{C}}_{t+1,T} = 0,$$

Now since $E^{\hat{C}}_{t,T} > 0$ and $E^{W}_{t,T} = -E^{\hat{C}}_{t,T}$ for all $t < T$, we also find that $E^{W}_{t,T} < 0$ for all $t \in \{0, \dots, T\}$. With all these ingredients, we can use Proposition 2.21 to show that the objective function is concave on $\mathbb{A}_\pi \times \mathbb{A}_{\hat{C}}$. To see this, note that in the notation and terminology of stochastic control models, our state space is $\mathbb{W} \times \mathbb{H}$, containing elements (W, h). The action space is $\mathbb{A}_\pi \times \mathbb{A}_{\hat{C}} = \mathbb{R} \times [0, \infty)$. We emphasize that the set of admissible actions stays the same for all states, implying in particular that the constraint set of states and actions is given by $\mathbb{W} \times \mathbb{H} \times \mathbb{A}_\pi \times \mathbb{A}_{\hat{C}}$. The space of disturbances is $\mathbb{Y}$, containing all possible log returns $Y_t = X_t - X_{t-1}$, $t \in \{1, \dots, T\}$. The transition function is given by (4.6). Note that the first component of the transition function, denoted by $\mathbb{T}_1$, is increasing in W and concave in $(W, \pi, \hat{C})$ on $\mathbb{W} \times \mathbb{A}_\pi \times \mathbb{A}_{\hat{C}}$ for all $h \in \mathbb{H}$ and $Y \in \mathbb{Y}$, since for $\lambda \in (0, 1)$ and $W_1, W_2 \in \mathbb{W}$, $\pi_1, \pi_2 \in \mathbb{A}_\pi$ and $\hat{C}_1, \hat{C}_2 \in \mathbb{A}_{\hat{C}}$, we find, using that $x \mapsto -x^2$ is concave,

$$\begin{aligned}
&\mathbb{T}_1\left(\lambda W_1 + (1-\lambda)W_2, h, \lambda\pi_1 + (1-\lambda)\pi_2, \lambda\hat{C}_1 + (1-\lambda)\hat{C}_2, Y\right) \\
&= \lambda W_1 + (1-\lambda)W_2 - \lambda\hat{C}_1 - (1-\lambda)\hat{C}_2 + (\lambda\pi_1 + (1-\lambda)\pi_2)\, Y \\
&\quad + (1 - \lambda\pi_1 - (1-\lambda)\pi_2))\, r \\
&\quad + \left(\lambda\pi_1 + (1-\lambda)\pi_2 - (\lambda\pi_1 + (1-\lambda)\pi_2)^2\right) h \\
&\geq \lambda W_1 + (1-\lambda)W_2 - \lambda\hat{C}_1 - (1-\lambda)\hat{C}_2 + (\lambda\pi_1 + (1-\lambda)\pi_2)\, Y \\
&\quad + (1 - \lambda\pi_1 - (1-\lambda)\pi_2))\, r \\
&\quad + \left(\lambda\pi_1 + (1-\lambda)\pi_2 - \lambda\pi_1^2 - (1-\lambda)\pi_2^2\right) h \\
&= \lambda \cdot \left[W_1 - \hat{C}_1 + \pi_1 Y + (1-\pi_1)\, r + \left(\pi_1 - \pi_1^2\right) h\right] \\
&\quad + (1-\lambda) \cdot \left[W_2 - \hat{C}_2 + \pi_2 Y + (1-\pi_2)\, r + \left(\pi_2 - \pi_2^2\right) h\right] \\
&= \lambda\mathbb{T}_1\left(W_1, h, \pi_1, \hat{C}_1, Y\right) + (1-\lambda)\, \mathbb{T}_1\left(W_2, h, \pi_2, \hat{C}_2, Y\right).
\end{aligned}$$

The second component $\mathbb{T}_2$ of the transition function is independent of $(W, \pi, \hat{C})$. The "one-step" return of the present stochastic control model is given via the utility derived from immediate consumption in the current time step. For $t \in \{0, \dots, T-1\}$, we defined this utility to be

$$U_t\left(\hat{C}_t\right) = -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\},$$

which is an increasing and concave function in $\hat{C}_t$ on $\mathbb{A}_{\hat{C}}$ if $\bar{\gamma} > 0$. Finally, note that for $\gamma < 0$, the value function is bounded from above by zero, and the terminal value function, defined as

$$\phi_T\left(W_T\right) = \frac{1}{\gamma} \exp\left\{\gamma W_T\right\},$$

is increasing and concave in W_T. All in all, this leads to the requirements (i)-(v) in Proposition 2.21 being satisfied, implying in particular that the objective function at time t, denoted by

$$\begin{aligned} f_t(W_t, h_{t+1}, \pi_t, \hat{C}_t) = & -\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_t\right\} - \mathbb{E}_t\left[\sum_{s=t+1}^{T-1} \bar{\beta}^{s-t}\bar{a}^{\bar{\gamma}} \exp\left\{-\bar{\gamma}\hat{C}_s^*\right\}\right] \\ & + \frac{\bar{\beta}^{T-t}}{\gamma} \exp\left\{D_{t,T} + E_{t,T}^{\hat{C}}\hat{C}_t + E_{t,T}^{W}W_t + E_{t,T}^{h}h_{t+1}\right\} \end{aligned}$$

for $t \in \{0, \dots, T-1\}$, is concave on $\mathbb{W} \times \mathbb{A}_\pi \times \mathbb{A}_{\hat{C}}$ for all $t \in \{0, \dots, T-1\}$, and thus that a solution $\left\{\pi_t^*, \hat{C}_t^*\right\}$ to (C.3) and (C.5) is actually a maximum. □

Proof of Corollary 4.6. From (4.3), we deduce that

$$Var_{t-1}\left[W_t\right] = \left(\pi_{t-1}^*\right)^2 \cdot h_t.$$

Concerning the required conditional bivariate moment generating function of W_t and its conditional variance, we note that the optimal strategy $\{\pi_t^*\}_t$ is non-random. For the sake of readability, we drop the asterisk for the optimal risky allocation and calculate:

$$\begin{aligned} & \Psi_{\left(W_t, \pi_t^2 h_{t+1}\right)}\left(u, v \mid \mathcal{F}_{t-1}\right) \\ = & \mathbb{E}_{t-1}\left[\exp\left\{u \cdot W_t + v \cdot \pi_t^2 h_{t+1}\right\}\right] \\ = & \mathbb{E}_{t-1}\left[\exp\left\{u \cdot \left(W_{t-1} - \hat{C}_{t-1} + \pi_{t-1}Y_t + \left(1 - \pi_{t-1}\right)r + \frac{1}{2}\left(\pi_{t-1} - \pi_{t-1}^2\right)h_t\right) + v \cdot \pi_t^2 h_{t+1}\right\}\right] \\ = & \exp\left\{u\left(W_{t-1} - \hat{C}_{t-1}\right) + \left(1 - \pi_{t-1}\right)ur + \frac{u}{2}\left(\pi_{t-1} - \pi_{t-1}^2\right)h_t\right\} \\ & \times \mathbb{E}_{t-1}\left[\exp\left\{u\pi_{t-1}Y_t + v\pi_t^2 h_{t+1}\right\}\right] \\ = & \exp\left\{ A\left(u\pi_{t-1}, v\pi_t^2\right) + \left(1 - \pi_{t-1}\right)ur + u\left(W_{t-1} - \hat{C}_{t-1}\right)\right. \\ & \left. + \frac{1}{\pi_{t-1}^2}\left[B\left(u\pi_{t-1}, v\pi_t^2\right) + \frac{u}{2}\left(\pi_{t-1} - \pi_{t-1}^2\right)\right] \cdot \pi_{t-1}^2 h_t\right\} \\ = & \exp\left\{ A\left(u\pi_{t-1}, v\pi_t^2\right) - uP_{t-1} + \left(1 - \pi_{t-1}\right)ur + u\left(1 - Q_{t-1}\right)W_{t-1}\right. \\ & \left. + \frac{1}{\pi_{t-1}^2}\left[B\left(u\pi_{t-1}, v\pi_t^2\right) - uR_{t-1} + \frac{u}{2}\left(\pi_{t-1} - \pi_{t-1}^2\right)\right] \cdot \pi_{t-1}^2 h_t\right\}. \end{aligned}$$

This shows that the conditional bivariate moment generating function is affine in the conditional variance, implying that $\{W_t\}_t$ follows an affine GARCH process (cf. Badescu et al., 2019).

Checking the optimality equations for the optimal risky allocation, we note that there is no dependence on the conditional variance or on the level of wealth. □

Proof of Corollary 4.8. We start by deriving the closed-form expression

$$\phi_t\left(W_t, h_{t+1}\right) = -\sum_{\tau=t}^{T} \exp\left\{\tilde{D}_{t,\tau,T}\left(\{\pi_k^*\}_t^\tau\right) + \tilde{E}_{t,\tau,T}^{W} \cdot W_t + \tilde{E}_{t,\tau,T}^{h}\left(\{\pi_k^*\}_t^\tau\right) \cdot h_{t+1}\right\}. \tag{C.7}$$

Again, for the sake of readability, we omit the dependencies of recursive parameters on the optimal solution except for definitions. Comparing (C.7) and (4.10), it is clear that

$$\tilde{D}_{t,t,T}\left(\pi_t^*\right) = \log\left(\bar{a}^{\bar{\gamma}}\right) - \bar{\gamma} P_{t,T}\left(\pi_t^*\right), \tag{C.8a}$$

$$\tilde{E}^W_{t,t,T} = -\bar{\gamma} Q_{t,T}, \tag{C.8b}$$

$$\tilde{E}^h_{t,t,T}\left(\pi_t^*\right) = -\bar{\gamma} R_{t,T}\left(\pi_t^*\right), \tag{C.8c}$$

for the first term corresponding to the utility derived from consumption at time t, and furthermore that

$$\tilde{D}_{t,T,T}\left(\pi_t^*\right) = \log\left(-\frac{\bar{\beta}^{T-t}}{\gamma}\right) + D_{t,T}\left(\pi_t^*\right) - E^W_{t,T} P_{t,T}\left(\pi_t^*\right), \tag{C.9a}$$

$$\tilde{E}^W_{t,T,T} = E^W_{t,T} \cdot \left(1 - Q_{t,T}\right), \tag{C.9b}$$

$$\tilde{E}^h_{t,T,T}\left(\pi_t^*\right) = E^h_{t,T}\left(\pi_t^*\right) - E^W_{t,T} R_{t,T}\left(\pi_t^*\right). \tag{C.9c}$$

For brevity, we now define, based on the coefficients A and B from (4.5):

$$\tilde{A} = A\left(-\bar{\gamma} Q_{\tau,T} \cdot \pi^*_{\tau-1}, -\bar{\gamma} R_{\tau,T}; \tau-1, \tau\right),$$
$$\tilde{B} = B\left(-\bar{\gamma} Q_{\tau,T} \cdot \pi^*_{\tau-1}, -\bar{\gamma} R_{\tau,T}; \tau-1, \tau\right).$$

With this notation, we now use the conditional bivariate MGF of $(W_\tau, h_{\tau+1})$ from Corollary 4.6 to obtain the recursive parameters for general $\tau \in \{t+1, \ldots, T-1\}$:

$$\begin{aligned}
\mathbb{E}_{\tau-1}\left[\exp\left\{-\bar{\gamma}\hat{C}^*_\tau\right\}\right] &= \mathbb{E}_{\tau-1}\left[\exp\left\{-\bar{\gamma} P_{\tau,T} - \bar{\gamma} Q_{\tau,T} W_\tau - \bar{\gamma} R_{\tau,T} h_{\tau+1}\right\}\right] \\
&= \exp\left\{-\bar{\gamma} P_{\tau,T}\right\} \cdot \Psi_{(W_\tau, h_{\tau+1})}\left(-\bar{\gamma} Q_{\tau,T}, -\bar{\gamma} R_{\tau,T} \mid \mathcal{F}_{\tau-1}\right) \\
&= \exp\Bigg\{ -\bar{\gamma} P_{\tau,T} + \tilde{A} - \left(1-\pi^*_{\tau-1}\right)\bar{\gamma} Q_{\tau,T} r - \bar{\gamma} Q_{\tau,T}\left(W_{\tau-1} - \hat{C}^*_{\tau-1}\right) \\
&\qquad + \left[\tilde{B} - \frac{1}{2}\bar{\gamma} Q_{\tau,T}\left(\pi^*_{\tau-1} - \left(\pi^*_{\tau-1}\right)^2\right)\right] \cdot h_\tau \Bigg\} \\
&= \exp\Bigg\{ -\bar{\gamma} P_{\tau,T} + \tilde{A} - \left(1-\pi^*_{\tau-1}\right)\bar{\gamma} Q_{\tau,T} r + \bar{\gamma} Q_{\tau,T} P_{\tau-1,T} \\
&\qquad - \bar{\gamma} Q_{\tau,T}\left(1 - Q_{\tau-1,T}\right) W_{\tau-1} \\
&\qquad + \left[\tilde{B} - \frac{1}{2}\bar{\gamma} Q_{\tau,T}\left(\pi^*_{\tau-1} - \left(\pi^*_{\tau-1}\right)^2\right) + \bar{\gamma} Q_{\tau,T} R_{\tau-1,T}\right] \cdot h_\tau \Bigg\},
\end{aligned}$$

leading to

$$\begin{aligned}
\tilde{D}_{\tau-1,\tau,T}\left(\{\pi^*_k\}^\tau_{\tau-1}\right) &= \log\left(\bar{\beta}^{\tau-(\tau-1)}\bar{a}^{\bar{\gamma}}\right) - \bar{\gamma} P_{\tau,T}\left(\pi^*_\tau\right) + \tilde{A} \\
&\quad - \left(1-\pi^*_{\tau-1}\right)\bar{\gamma} Q_{\tau,T} r + \bar{\gamma} Q_{\tau,T} P_{\tau-1,T}\left(\pi^*_{\tau-1}\right), \\
\tilde{E}^W_{\tau-1,\tau,T} &= -\bar{\gamma} Q_{\tau,T}\left(1 - Q_{\tau-1,T}\right), \\
\tilde{E}^h_{\tau-1,\tau,T}\left(\{\pi^*_k\}^\tau_{\tau-1}\right) &= \tilde{B} - \frac{1}{2}\bar{\gamma} Q_{\tau,T}\left(\pi^*_{\tau-1} - \left(\pi^*_{\tau-1}\right)^2\right) + \bar{\gamma} Q_{\tau,T} R_{\tau-1,T}\left(\pi^*_{\tau-1}\right).
\end{aligned}$$

In general, if $t < \tau$, we can use the conditional bivariate moment generating function of the log wealth and the conditional variance of the log-asset return from Corollary 4.6 again to recursively calculate

$$\begin{aligned}&\bar{\beta}\cdot\mathbb{E}_t\left[\exp\left\{\tilde{D}_{t+1,\tau,T}\left(\{\pi_k^*\}_{t+1}^{\tau}\right)+\tilde{E}_{t+1,\tau,T}^{W}W_{t+1}+\tilde{E}_{t+1,\tau,T}^{h}\left(\{\pi_k^*\}_{t+1}^{\tau}\right)h_{t+2}\right\}\right]\\ =&\exp\left\{\tilde{D}_{t,\tau,T}\left(\{\pi_k^*\}_t^{\tau}\right)+\tilde{E}_{t,\tau,T}^{W}\cdot W_t+\tilde{E}_{t,\tau,T}^{h}\left(\{\pi_k^*\}_t^{\tau}\right)\cdot h_{t+1}\right\},\end{aligned}$$

with

$$\begin{aligned}\tilde{D}_{t,\tau,T}\left(\{\pi_k^*\}_t^{\tau}\right)&=\log\left(\bar{\beta}\right)+\tilde{D}_{t+1,\tau,T}\left(\{\pi_k^*\}_{t+1}^{\tau}\right)+A\left(\tilde{E}_{t+1,\tau,T}^{W}\cdot\pi_t^*,\tilde{E}_{t+1,\tau,T}^{h}\left(\{\pi_k^*\}_t^{\tau}\right)\right)\\&\quad+\left(1-\pi_t^*\right)\tilde{E}_{t+1,\tau,T}^{W}r-\tilde{E}_{t+1,\tau,T}^{W}P_{t,T}\left(\pi_t^*\right),\end{aligned}\tag{C.10a}$$

$$\tilde{E}_{t,\tau,T}^{W}=\tilde{E}_{t+1,\tau,T}^{W}\left(1-Q_{t,T}\right),\tag{C.10b}$$

$$\begin{aligned}\tilde{E}_{t,\tau,T}^{h}\left(\{\pi_k^*\}_t^{\tau}\right)&=B\left(\tilde{E}_{t+1,\tau,T}^{W}\cdot\pi_t^*,\tilde{E}_{t+1,\tau,T}^{h}\left(\{\pi_k^*\}_{t+1}^{\tau}\right)\right)\\&\quad+\frac{1}{2}\tilde{E}_{t+1,\tau,T}^{W}\left(\pi_t^*-\left(\pi_t^*\right)^2\right)-\tilde{E}_{t+1,\tau,T}^{W}R_{t,T}\left(\pi_t^*\right).\end{aligned}\tag{C.10c}$$

We can continue repeating this last step until we have reached time point t to obtain an expression which is exponentially affine in W_t and h_{t+1}. Concerning the factors $\tilde{E}_{t,\tau,T}^{W}$ for $\tau\in\{t,\dots,T\}$, given by (C.8b), (C.9b) and (C.10b), we note that

$$\begin{aligned}\tilde{E}_{t,t,T}^{W}&\overset{\text{(C.8b)}}{=}-\bar{\gamma}Q_{t,T}\overset{\text{(C.6b)}}{=}\bar{\gamma}\frac{E_{t,T}^{W}}{\bar{\gamma}+E_{t,T}^{\hat{C}}}=E_{t,T}^{W}\left(\frac{E_{t,T}^{\hat{C}}+\bar{\gamma}-E_{t,T}^{\hat{C}}}{\bar{\gamma}+E_{t,T}^{\hat{C}}}\right)\\&=\;E_{t,T}^{W}\left(1-\frac{E_{t,T}^{\hat{C}}}{\bar{\gamma}+E_{t,T}^{\hat{C}}}\right)\overset{\text{(C.6b)}}{=}E_{t,T}^{W}\left(1-Q_{t,T}\right)\overset{\text{(C.9b)}}{=}\tilde{E}_{t,T,T}^{W}.\end{aligned}\tag{C.11}$$

This implies in particular that at time $t=T-1$, both coefficients corresponding to the log wealth are equal. For $t<T-1$ and each $\tau\in\{t+1,\dots,T-1\}$, we know that $\tilde{E}_{t,\tau,T}^{W}$ is determined recursively, starting with $\tilde{E}_{\tau,\tau,T}^{W}$ and then (repeatedly) using (C.10b). Thus, combining these steps leads to

$$\begin{aligned}\tilde{E}_{t,\tau,T}^{W}&\overset{\text{(C.10b)}}{=}\tilde{E}_{t+1,\tau,T}^{W}\left(1-Q_{t,T}\right)\\&\overset{\text{(C.10b)}}{=}\tilde{E}_{\tau,\tau,T}^{W}\left(1-Q_{\tau-1,T}\right)\cdot\ldots\cdot\left(1-Q_{t+1,T}\right)\left(1-Q_{t,T}\right)\\&\overset{\text{(C.11)}}{=}\tilde{E}_{\tau,T,T}^{W}\cdot\left(1-Q_{\tau-1,T}\right)\cdot\ldots\cdot\left(1-Q_{t+1,T}\right)\left(1-Q_{t,T}\right)\\&\overset{\text{(C.9b)}}{=}E_{\tau,T}^{W}\left(1-Q_{\tau,T}\right)\cdot\left(1-Q_{\tau-1,T}\right)\cdot\ldots\cdot\left(1-Q_{t+1,T}\right)\left(1-Q_{t,T}\right)\\&\overset{\text{(C.2c)}}{=}E_{t,T}^{W}\left(1-Q_{t,T}\right)\\&\overset{\text{(C.9b)}}{=}\tilde{E}_{t,T,T}^{W},\end{aligned}\tag{C.12}$$

showing that all coefficients corresponding to the log wealth are equal. The same reasoning for $\tilde{E}_{t,\tau,T}^{h}$ with $\tau\in\{t,\dots,T\}$, given by (C.8c), (C.9c) and (C.10c) yields:

$$\begin{aligned}\tilde{E}_{t,t,T}^{h}\left(\pi_t^*\right)&\overset{\text{(C.8c)}}{=}-\bar{\gamma}R_{t,T}\left(\pi_t^*\right)\overset{\text{(C.6c)}}{=}\bar{\gamma}\frac{E_{t,T}^{h}\left(\pi_t^*\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}}=E_{t,T}^{h}\left(\pi_t^*\right)\left(\frac{E_{t,T}^{\hat{C}}+\bar{\gamma}-E_{t,T}^{\hat{C}}}{\bar{\gamma}+E_{t,T}^{\hat{C}}}\right)\\&=\;E_{t,T}^{h}\left(\pi_t^*\right)\left(1-\frac{E_{t,T}^{\hat{C}}}{\bar{\gamma}+E_{t,T}^{\hat{C}}}\right)\overset{\text{(C.6c)}}{=}E_{t,T}^{h}\left(\pi_t^*\right)-E_{t,T}^{W}R_{t,T}\left(\pi_t^*\right)\end{aligned}$$

$$\overset{(C.9c)}{=} \tilde{E}^h_{t,T,T}\left(\pi^*_t\right). \tag{C.13}$$

Furthermore, using the next to last expression in (C.11) and (C.13), respectively, in $(*)$, we obtain

$$\begin{aligned}
\tilde{E}^h_{\tau-1,\tau,T}\left(\{\pi^*_k\}^\tau_{\tau-1}\right) &\overset{(C.10c)}{=} B\left(\tilde{E}^W_{\tau,\tau,T}\cdot\pi^*_{\tau-1}, \tilde{E}^h_{\tau,\tau,T}\left(\pi^*_\tau\right); \tau-1,\tau\right) \\
&\quad + \frac{1}{2}\tilde{E}^W_{\tau,\tau,T}\left(\pi^*_{\tau-1} - \left(\pi^*_{\tau-1}\right)^2\right) - \tilde{E}^W_{\tau,\tau,T}R_{\tau-1,T}\left(\pi^*_{\tau-1}\right) \\
&\overset{(*)}{=} B\left(E^W_{\tau,T}\left(1-Q_{\tau,T}\right)\cdot\pi^*_{\tau-1}, E^h_{\tau,T}\left(\pi^*_\tau\right) - E^W_{\tau,T}R_{\tau,T}\left(\pi^*_\tau\right); \tau-1,\tau\right) \\
&\quad + \frac{1}{2}E^W_{\tau,T}\left(1-Q_{\tau,T}\right)\left(\pi^*_{\tau-1} - \left(\pi^*_{\tau-1}\right)^2\right) \\
&\quad - \tilde{E}^W_{\tau,\tau,T}R_{\tau-1,T}\left(\pi^*_{\tau-1}\right) \\
&\overset{(C.2d)}{=} E^h_{\tau-1,T}\left(\pi^*_{\tau-1}\right) - \tilde{E}^W_{\tau,\tau,T}R_{\tau-1,T}\left(\pi^*_{\tau-1}\right) \\
&\overset{(C.11)}{=} E^h_{\tau-1,T}\left(\pi^*_{\tau-1}\right) - E^W_{\tau,T}\left(1-Q_{\tau,T}\right)R_{\tau-1,T}\left(\pi^*_{\tau-1}\right) \\
&\overset{(C.2c)}{=} E^h_{\tau-1,T}\left(\pi^*_{\tau-1}\right) - E^W_{\tau-1,T}R_{\tau-1,T}\left(\pi^*_{\tau-1}\right) \\
&\overset{(C.13)}{=} \tilde{E}^h_{\tau-1,\tau-1,T}\left(\pi^*_{\tau-1}\right).
\end{aligned} \tag{C.14}$$

Using this step (C.14), we can show by backward induction on t, for $t \in \{1,\ldots,\tau\}$, that $\tilde{E}^h_{t,\tau,T}\left(\{\pi^*_k\}^\tau_t\right) = \tilde{E}^h_{t,t,T}\left(\pi^*_t\right)$. In particular, using the induction hypothesis $\tilde{E}^h_{t+1,\tau,T}\left(\{\pi^*_k\}^\tau_{t+1}\right) = \tilde{E}^h_{t+1,t+1,T}\left(\pi^*_{t+1}\right)$ in $(*')$ and proceeding as in (C.14), we arrive at

$$\begin{aligned}
\tilde{E}^h_{t,\tau,T}\left(\{\pi^*_k\}^\tau_t\right) &\overset{(C.10c)}{=} B\left(\tilde{E}^W_{t+1,\tau,T}\cdot\pi^*_t, \tilde{E}^h_{t+1,\tau,T}\left(\{\pi^*_k\}^\tau_{t+1}\right)\right) \\
&\quad + \frac{1}{2}\tilde{E}^W_{t+1,\tau,T}\left(\pi^*_t - \left(\pi^*_t\right)^2\right) - \tilde{E}^W_{t+1,\tau,T}R_{t,T}\left(\pi^*_t\right) \\
&\overset{(*')}{=} B\left(\tilde{E}^W_{t+1,t+1,T}\cdot\pi^*_t, \tilde{E}^h_{t+1,t+1,T}\left(\pi^*_{t+1}\right)\right) \\
&\quad + \frac{1}{2}\tilde{E}^W_{t+1,t+1,T}\left(\pi^*_t - \left(\pi^*_t\right)^2\right) - \tilde{E}^W_{t+1,t+1,T}R_{t,T}\left(\pi^*_t\right) \\
&\overset{(*)}{=} B\left(E^W_{t+1,T}\left(1-Q_{t+1,T}\right)\cdot\pi^*_t, E^h_{t+1,T}\left(\pi^*_{t+1}\right) - E^W_{t+1,T}R_{t+1,T}\left(\pi^*_{t+1}\right)\right) \\
&\quad + \frac{1}{2}E^W_{t+1,T}\left(1-Q_{t+1,T}\right)\left(\pi^*_t - \left(\pi^*_t\right)^2\right) - \tilde{E}^W_{t+1,t+1,T}R_{t,T}\left(\pi^*_t\right) \\
&\overset{(C.2d)}{=} E^h_{t,T}\left(\pi^*_t\right) - \tilde{E}^W_{t+1,t+1,T}R_{t,T}\left(\pi^*_t\right) \\
&\overset{(C.11)}{=} E^h_{t,T}\left(\pi^*_t\right) - E^W_{t+1,T}\left(1-Q_{t+1,T}\right)R_{t,T}\left(\pi^*_t\right) \\
&\overset{(C.2c)}{=} E^h_{t,T}\left(\pi^*_t\right) - E^W_{t,T}R_{t,T}\left(\pi^*_t\right) \\
&\overset{(C.13)}{=} \tilde{E}^h_{t,t,T}\left(\pi^*_t\right),
\end{aligned} \tag{C.15}$$

implying that all coefficients corresponding to the conditional variance of the log-asset return are equal. Using (C.11), (C.12), (C.13) and (C.15) in (C.7) and re-ordering terms now yields (4.14). □

Proof of Corollary 4.10. We show that in the limit for $\bar{\gamma} \to \infty$, all three coefficients $P_{t,T}$, $Q_{t,T}$ and $R_{t,T}$ in (4.11) become zero for all $t \in \{0,\ldots,T-1\}$ if $\bar{a} = 1$, resulting in $\hat{C}_t = 0$, which is equivalent to $V_t = V_t - C_t$ and thus $C_t = 0$. In particular, take into account that $P_{T,T} =$

$Q_{T,T} = R_{T,T} = 0$. Now, fixing $t \in \{0, \dots, T-1\}$ and assuming that $P_{s,T} = Q_{s,T} = R_{s,T} = 0$ for all $s \in \{t+1, \dots, T\}$ in (C.2), we find that

$$D_{t,T}\left(\pi_t^*\right) = D_{t+1,T}\left(\pi_{t+1}^*\right) + E_{t+1,T}^W\left(1-\pi_t^*\right)r + A\left(E_{t+1,T}^W \pi_t, E_{t+1,T}^h\left(\pi_{t+1}^*\right)\right), \tag{C.16a}$$

$$E_{t,T}^{\hat{C}} = -E_{t+1,T}^W \cdot \mathbb{1}_{\{t<T\}}, \tag{C.16b}$$

$$E_{t,T}^W = E_{t+1,T}^W = \gamma, \tag{C.16c}$$

$$E_{t,T}^h\left(\pi_t^*\right) = \frac{1}{2}E_{t+1,T}^W\left(\pi_t - \pi_t^2\right) + B\left(E_{t+1,T}^W \pi_t^*, E_{t+1,T}^h\left(\pi_{t+1}^*\right)\right), \tag{C.16d}$$

are independent of $\bar{\gamma}$. Checking (C.6) now reveals that $Q_{t,T}$ and $R_{t,T}$ approach zero as $\bar{\gamma} \to \infty$. Furthermore, we calculate for $\gamma < 0$, using the above independences and de l'Hôpital's rule in $(*)$:

$$\begin{aligned}
\lim_{\bar{\gamma}\to\infty} P_{t,T}\left(\pi_t^*\right) &= \lim_{\bar{\gamma}\to\infty}\left\{-\frac{1}{\bar{\gamma}+E_{t,T}^{\hat{C}}}\left(\log\left(-\frac{\bar{\beta}^{T-t}E_{t,T}^{\hat{C}}}{\gamma\bar{\gamma}\bar{a}^{\bar{\gamma}}}\right) + D_{t,T}\left(\pi_t^*\right)\right)\right\} \\
&= \lim_{\bar{\gamma}\to\infty}\left\{-\frac{\log\left(\bar{\beta}^{T-t}E_{t,T}^{\hat{C}}\right) - \log\left(-\gamma\bar{\gamma}\bar{a}^{\bar{\gamma}}\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}} - \frac{D_{t,T}\left(\pi_t^*\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}}\right\} \\
&= \lim_{\bar{\gamma}\to\infty}\left\{-\underbrace{\frac{\log\left(\bar{\beta}^{T-t}E_{t,T}^{\hat{C}}\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}}}_{\to 0} + \frac{\log\left(-\gamma\bar{\gamma}\bar{a}^{\bar{\gamma}}\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}} - \underbrace{\frac{D_{t,T}\left(\pi_t^*\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}}}_{\to 0}\right\} \\
&= 0 + \lim_{\bar{\gamma}\to\infty}\frac{\log\left(-\gamma\bar{\gamma}\bar{a}^{\bar{\gamma}}\right)}{\bar{\gamma}+E_{t,T}^{\hat{C}}} + 0 \\
&\overset{(*)}{=} \lim_{\bar{\gamma}\to\infty}\frac{\log\left(\bar{a}\right) + 1/\bar{\gamma}}{1} \\
&= \log\left(\bar{a}\right).
\end{aligned}$$

This shows that the coefficient $P_{t,T}$ approaches zero in the limit for $\bar{\gamma} \to \infty$ if and only if $\bar{a} = 1$ holds. Together with $E_{T,T}^W = \gamma$, having $P_{t,T} = Q_{t,T} = R_{t,T} = 0$ for all $t \in \{0, \dots, T\}$ reproduces the formulas from Escobar-Anel et al. (2021) in (C.16) (i.e., in (C.2)) and (C.3). □

Proof of Corollary 4.11. The formulas for the value function and optimal control on consumption follow directly from the fact that the HN-GARCH model is a special case of an affine GARCH model and that Theorem 4.4 can be applied. With A and B defined according to (4.19), we can easily check that (4.12a) is always satisfied. Using the abbreviation in (4.21), Equation (4.12b) requires

$$\begin{aligned}
& \lambda + \frac{2E_{t,T}^W \pi_t^* - 4v\alpha\rho}{2\left(1-2v\alpha\right)} = \pi_t^* - \frac{1}{2} \\
\Leftrightarrow\quad & \frac{E_{t,T}^W \pi_t^*}{1-2v\alpha} - \pi_t^* = \frac{2v\alpha\rho}{1-2v\alpha} - \lambda - \frac{1}{2} \\
\Leftrightarrow\quad & \frac{E_{t,T}^W - 1 + 2v\alpha}{1-2v\alpha}\cdot\pi_t^* = \frac{2v\alpha\rho}{1-2v\alpha} - \lambda - \frac{1}{2} \\
\Leftrightarrow\quad & \pi_t^* = \frac{1-2v\alpha}{E_{t,T}^W - 1 + 2v\alpha}\left(\frac{2v\alpha\rho}{1-2v\alpha} - \left(\lambda + \frac{1}{2}\right)\right)
\end{aligned}$$

$$\Leftrightarrow \qquad \pi_t^* = \frac{2v\alpha\rho}{E_{t,T}^W - 1 + 2v\alpha} - \frac{\left(\lambda + \frac{1}{2}\right)(1 - 2v\alpha)}{E_{t,T}^W - 1 + 2v\alpha}$$

$$\Leftrightarrow \qquad \pi_t^* = \frac{\left(\lambda + \frac{1}{2}\right)(1 - 2v\alpha) - 2v\alpha\rho}{1 - E_{t,T}^W - 2v\alpha}.$$

Note that we already implicitly assumed that $1 - 2v\alpha > 0$ for the representations of A and B in (4.19), see Appendix B.1. Furthermore, we showed in the proof of Theorem 4.4 that $E_{t,T}^W < 0$ for all $t = 0, \ldots, T$. Together, this implies that the above representation of the optimal allocation is well-defined. □

Proof of Corollary 4.13. Analogous to Corollary 4.11, we plug the expressions from (4.23) into (4.12). While (4.12a) is again trivially satisfied, (4.12b) with the abbreviation in (4.21) yields

$$\nu - \frac{1}{\eta^2}\left(\frac{\left(1 - 2va\eta^4\right)2\eta}{2\sqrt{\left(1 - 2va\eta^4\right)\left(1 - 2E_{t,T}^W \pi_t^* \eta - 2vc\right)}}\right) = \pi_t^* - \frac{1}{2}$$

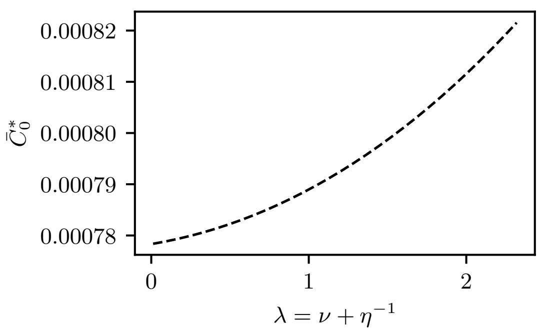
$$\Leftrightarrow \qquad \nu - \frac{\sqrt{1 - 2va\eta^4}}{\eta\sqrt{1 - 2E_{t,T}^W \pi_t^* \eta - 2vc}} = \pi_t^* - \frac{1}{2},$$

where the conditions for the arguments of the squareroots have already been assumed to be positive for the representations of A and B in (4.23), see Appendix B.2. □

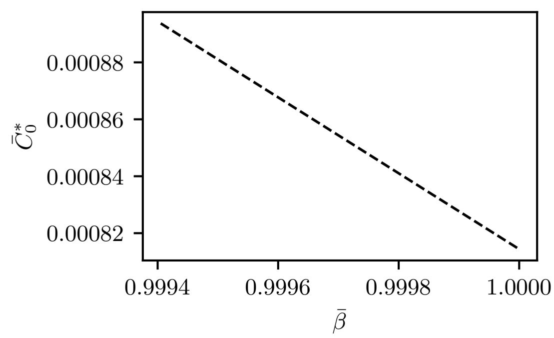
C.2 Complementary Material

C.2.1 Additional Plots for Section 4.4.1: Sensitivity Analysis

In addition to the analysis in Section 4.4.1 regarding the sensitivity of the optimal solution to changes in parameter values, this section presents some additional insights, again using the set of maximum likelihood estimates (MLEs) from Table 4.1.



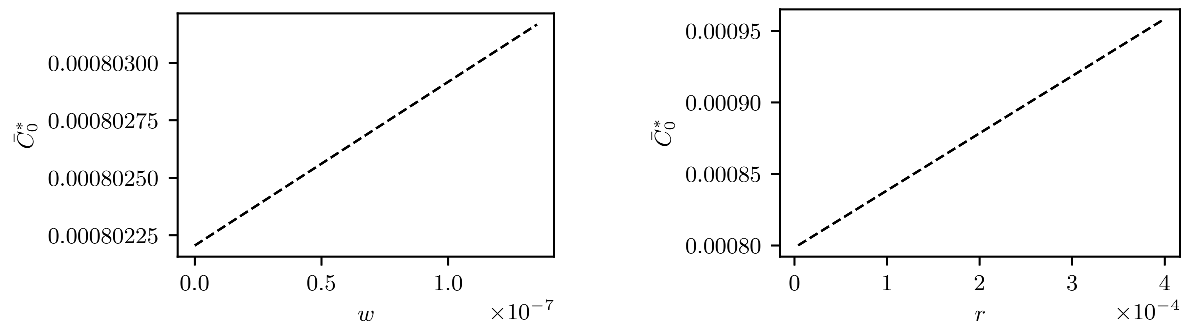
(a) Values of the optimal initial relative consumption dependent on the market price of risk.



(b) Values of the optimal initial relative consumption dependent on the parameter $\bar{\beta}$ for intertemporal substitution.

Figure C.1: Values of the optimal initial relative consumption, dependent on the market price of risk (modified via ν) and on the parameter $\bar{\beta}$ for intertemporal substitution. The time horizon is five years with 252 trading days each.

The first of these additional investigations targets changes in the market price of risk (MPR), which in the IG-GARCH model is given via $\lambda = \nu + \eta^{-1}$. Keeping the parameter η constant, we investigate sensitivity w.r.t. the market price of risk via changes in ν. Figure C.1a shows that the initial relative consumption increases with the market price of risk, although the changes are relatively small with an order of 1×10^{-5} for the entire displayed range $\lambda \in [0, 2.5]$ (with the original MPR for the set in Table 4.1 being 1.16). Another interesting analysis considers the factor $\bar{\beta}$ for intertemporal substitution. As Figure C.1b shows, the initial consumption-wealth ratio increases as the value $\bar{\beta}$ is decreased, i.e., as utilities at future time points are valued less.



(a) Values of the optimal initial relative consumption, dependent on w.

(b) Values of the optimal initial relative consumption, dependent on r.

Figure C.2: Values of the optimal initial relative consumption, dependent on the parameters w and r. The time horizon is five years with 252 trading days each.

Contrary to the optimal relative investment strategy $\{\pi_t^*\}_t$, the optimal consumption process does depend on the parameters w and r. In both cases, the level of relative consumption increases as the parameter increases. For w, which has an impact mainly on the conditional variance of the log-asset return and the long-term mean of this variance, we consider the positive part of the confidence interval for the MLE, since negative values of w may allow for negative values for the variance. Concerning the interest rate r, we consider the range $[0.1\%, 10\%]$ of annualized interest rates.

We note that the initial level of relative consumption slightly increases as the parameters a, b and c of the underlying IG-GARCH model are increased within the corresponding confidence intervals reported with the MLEs presented in Table 4.1.

C.2.2 Alternative Parametric Choice

Complementary to the analysis in Section 4.4, we provide the central figures also for the MLEs by Christoffersen et al. (2006). As in Section 4.4, the analysis is based on the MLEs for the three different models – IG-GARCH, HN-GARCH and the homoscedastic variant – using the same underlying data set for all models. The parameter values for all three models can be found in Table C.1. Again, if not stated otherwise, we use the default investment parameters as described in Section 4.4.1.

Figure C.3a shows that the behavior of the optimal initial relative consumption $\bar{C}_0^*$, as the time horizon changes, is quite similar to what is observed for the first parameter set in Figure 4.2a. In particular, the value for initial relative consumption decreases as the length of the time horizon increases. Comparing Figure C.3b to 4.2b, however, shows a different picture in terms of the optimal risky allocation. While the special cases without consumption still lead to the smallest fractions of wealth invested in the risky asset, the IG-GARCH now yields the lowest

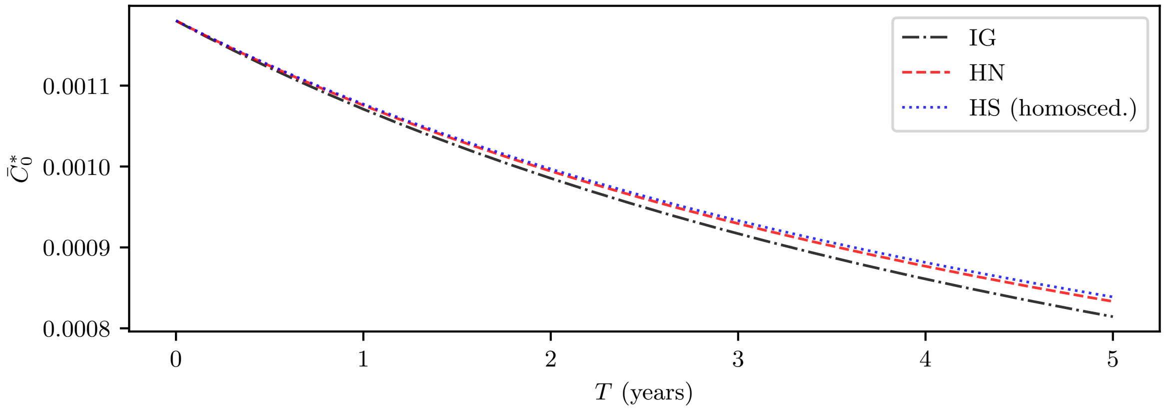
Table C.1: Values for the parameters used for the IG-GARCH, the HN-GARCH and the homoscedastic Gaussian model – consisting of the MLEs as of Christoffersen et al. (2006) for all parameters and models. The value of ν for the IG-GARCH model implies that the market price of risk is $\nu + \eta^{-1} \approx 2.150$.

Panel A: IG-GARCH		**Panel B: HN-GARCH**		**Panel C: Hom. Gauss.**	
Param.	Values	Param.	Values	Param.	Values
ν	1.625×10^{3}	λ	2.772	λ	3.106
η	-6.162×10^{-4}	ω	3.038×10^{-9}	ω	9.520×10^{-5}
w	3.768×10^{-10}	β	9.026×10^{-1}		
a	2.472×10^{7}	α	3.660×10^{-6}		
b	-1.933×10^{1}	ρ	1.284×10^{2}		
c	4.142×10^{-6}				

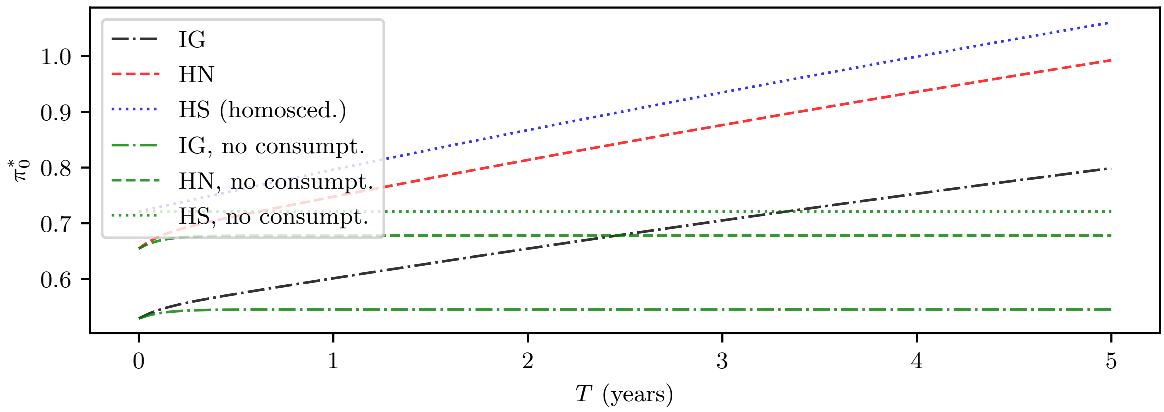
risky allocation among the strategies with consumption, followed by the HN-GARCH model. It is also interesting to observe that the numbers for π_0^* are significantly higher in general for this parameter set from Table C.1 than in the plot in Figure 4.2b created with the estimates from Table 4.1.

Given the similar behavior of the initial optimal relative consumption seen in Figure C.3a as compared to Figure 4.2a, it is not surprising to see that changes in risk preferences have a comparable impact on this value under the new parametric environment, too, see Figure C.4. As in Figure 4.3a, we use $\gamma \in [-6, -0.1]$ and $\bar{\gamma} \in \left[5 \times 10^2, 1 \times 10^4\right]$ to modify risk aversion for terminal wealth and consumption, respectively. If the risk aversion for utility derived from consumption is decreased, both relative consumption and the relative risky investment increase. Decreasing risk aversion regarding utility from terminal wealth leads to an increase in the risky allocation.

Figure C.5 once more shows that the impact of skewness and kurtosis on the optimal initial risky allocation and relative consumption, respectively, is relatively small. For this analysis, we again use the approach described in Section 4.4.1 and keep the first two moments of the IG-GARCH parameter set constant, while the parameter η is modified, to observe the isolated impact of skewness and excess kurtosis. In both cases, the optimal solution decreases as log asset-returns become more negatively skewed and leptokurtic, but the differences are very little. Again, as in Figure 4.4, we also plot dots for the optimal solutions to the original IG-GARCH parametrization as of Table C.1 as well as to the corresponding HN-GARCH and homoscedastic models with the same first two unconditional moments. Concerning the optimal initial risky allocation π_0^*, the homoscedastic solution suggests investing around 2% less in the risky asset than the heteroscedastic models, while the difference between HN-GARCH and IG-GARCH solution is much less. A similar pattern can be observed for the optimal initial relative consumption in the right plot.



(a) Optimal initial relative consumption, dependent on the time horizon.



(b) Optimal initial relative risky allocation, dependent on the time horizon.

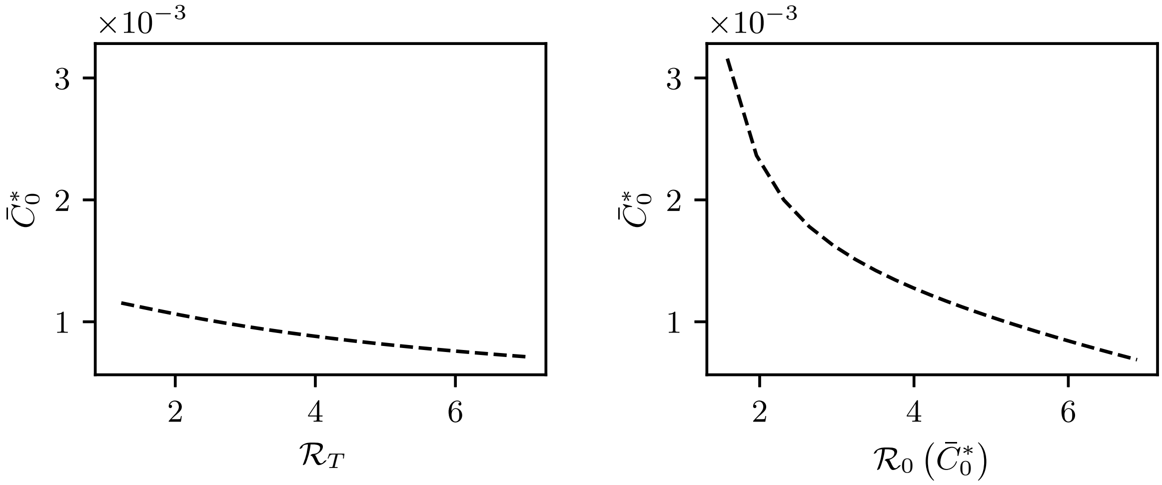
Figure C.3: Values of the optimal initial strategy at time $t = 0$, dependent on the time horizon. The plots include an IG-GARCH, an HN-GARCH and a homoscedastic Gaussian, all fitted to the same data set. The solutions without consumption are obtained via setting $\bar{\gamma} = 1 \times 10^8$ to approximate the case without consumption. One year is assumed to have 252 trading days. We used the parameter MLEs of Christoffersen et al. (2006).

C.2.3 Wealth-Equivalent Loss

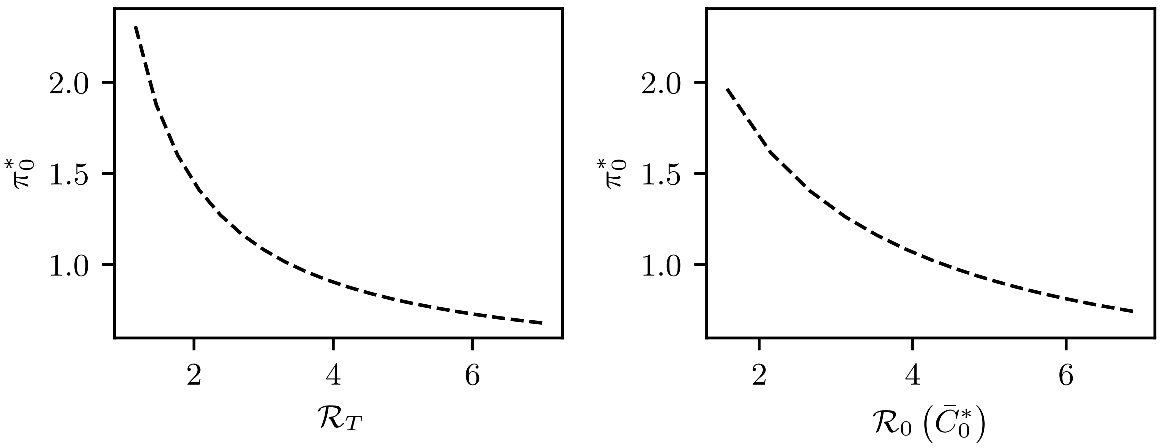
This section presents an extra analysis concerning the wealth-equivalent loss (WEL) arising from subobtimal strategies. We follow the method described in Remark 4.9 to measure losses in closed form via the expected utility and iterated expectations. This study complements the real-world analysis presented in Section 4.4.2, where the outcome was calculated explicitly for a five-year time horizon based on S&P 500 index data.

For the following plots, we assume that log asset-returns actually evolve according to an IG-GARCH model, while the investor implements the strategy originating from an HN-GARCH model or a homoscedastic variant. The parameters are taken from Tables 4.1 (for Figures C.8, C.9, C.10) and C.1 (for Figures C.6 and C.7).

Figure C.6, based on the parameter set displayed in Table C.1, illustrates that losses are highest for investors with very low risk aversion. We observe this pattern both for HN-GARCH and homoscedastic investors, and for risk aversion concerning both terminal wealth and consumption. The heteroscedastic HN-GARCH model clearly outperforms the homoscedastic variant.



(a) Optimal initial relative consumption, dependent on the investor's risk aversion concerning utility from terminal wealth (left plot) and from consumption (right plot).



(b) Optimal initial relative risky allocation, dependent on the investor's risk aversion concerning utility from terminal wealth (left plot) and from consumption (right plot).

Figure C.4: Values of the optimal initial relative consumption and optimal initial risky allocation, dependent on relative risk aversion. Relative risk aversion for terminal wealth in the left plots is governed by the parameter γ, relative risk aversion for consumption in the right plots is modified via $\bar{\gamma}$, see Remark 4.2. The time horizon is five years.

Furthermore, a comparison of the loss levels in Figure C.6 shows that the attitude towards consumption has a higher impact than risk-aversion concerning the utility derived from terminal wealth. This results in losses up to 4% for a homoscedastic investor with low risk aversion for consumption over a five-year horizon.

Figure C.7a confirms that conditional skewness and excess kurtosis have minimal impact on the investment strategy's performance. For modifications in the parameter η of the IG-GARCH model, the remaining parameters are adjusted such that the first two moments are kept constant, and the effect of changes in skewness and kurtosis is isolated. As seen above, the HN-GARCH model outperforms the homoscedastic variant. For both models, however, even for rather extreme values for the third and fourth moments (see the corresponding scale in Figures 4.4 and C.5), the losses show little difference to the original values on the right end of the horizontal axis. For the plot in Figure C.7b, we adjust the market price of risk (MPR) in all models to the

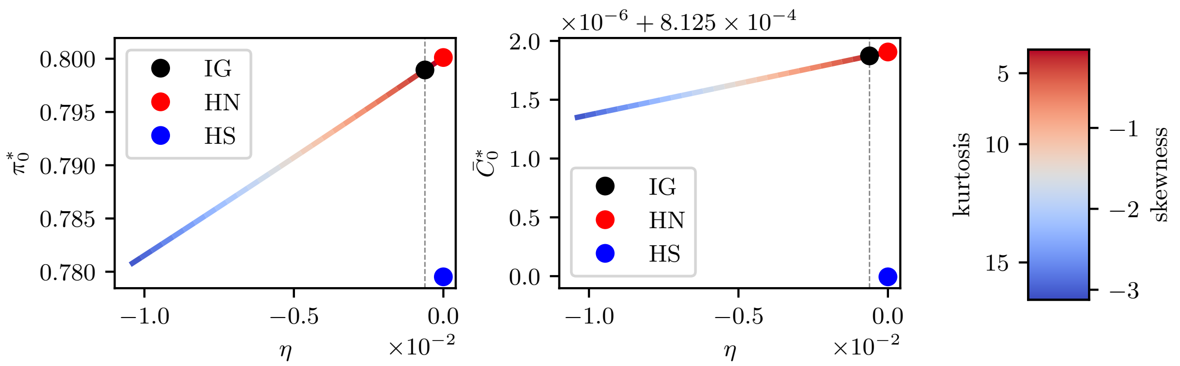
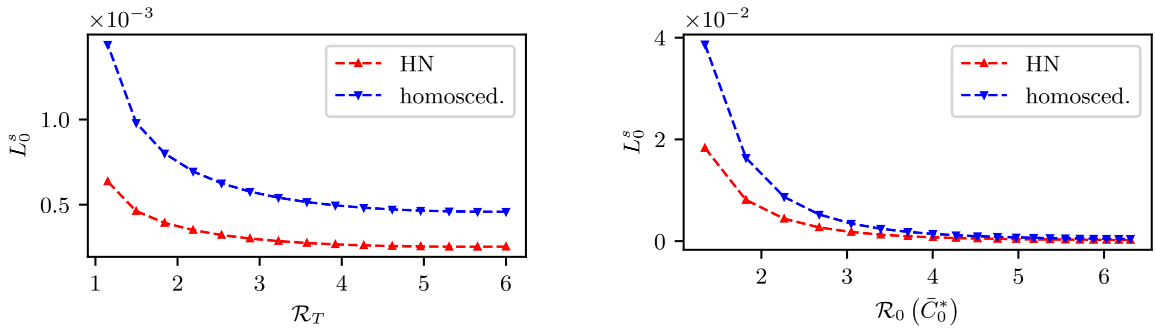


Figure C.5: Values of the optimal initial risky allocation and relative consumption dependent on the parameter η. The time horizon is five years. The color bar on the right-hand side indicates the corresponding values for skewness and excess kurtosis of log asset-returns, the vertical dashed line marks the original MLE for η in the considered parameter set. For this plot, the first two moments are kept constant under changes in η.



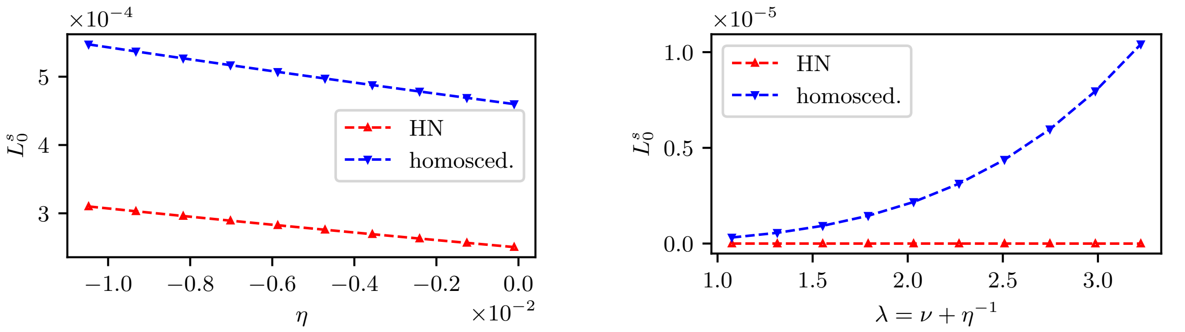
(a) WEL dependent on risk aversion concerning utility from terminal wealth.

(b) WEL dependent on risk aversion concerning utility from consumption.

Figure C.6: WEL for the HN-GARCH and the homoscedastic model, dependent on risk aversion concerning utility from terminal wealth and from consumption. Relative risk aversion for terminal wealth in the left plot is governed by the parameter γ, and relative risk aversion for consumption in the right plot is modified via $\bar{\gamma}$, see Remark 4.2. The time horizon is five years. The parameters are given in Table C.1.

same value. This is implemented via changes in ν for the IG-GARCH model and via changes in λ in the suboptimal models. The original value for the MPR in the IG-GARCH model with the parameters from Table C.1 is $\lambda = \nu + \eta^{-1} \approx 2.15$. The first finding resulting from this plot is that matching the MPR decreases the overall level of losses for both suboptimal models. Secondly, the losses for the homoscedastic model exceed the losses originating form the HN-GARCH model significantly, leading to a flat line for the heteroscedastic model in a joint plot (see Figure C.7b). The WEL increases as the MPR increases.

The observations from Figures C.6 and C.7 are confirmed when performing the same analysis with the parameter set presented in Table 4.1 (Christoffersen et al., 2006) instead of the one in Table C.1. In this case, however, the difference between the losses for the heteroscedastic HN-GARCH model and the homoscedastic variant is so large that plotting the losses for both models at once would result in a seemingly flat line close to zero for the heteroscedastic GARCH model.



(a) WEL dependent on the parameter η.

(b) WEL dependent on the MPR. In this plot, all models are modified to share the same MPR.

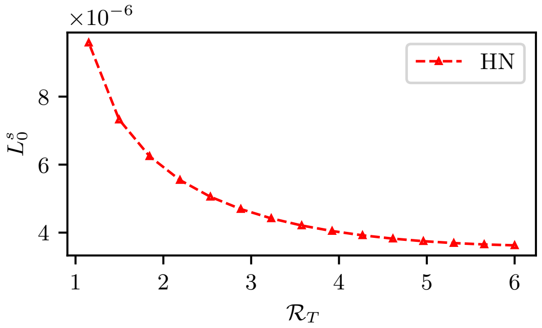
Figure C.7: WEL for the HN-GARCH and the homoscedastic model, dependent on the parameters η and ν. Modifications in η lead to different levels of conditional skewness and excess kurtosis, see Figures 4.4 and C.5. The parameter ν is used to modify the MPR. The time horizon is five years. The original parameter set is given in Table C.1.

In general, Figure C.8 again verifies that losses increase as the investor's level of risk aversion decreases. This trend is observed for both models and for both types of risk aversion. Furthermore, the impact of risk aversion concerning consumption, governed via the parameter $\bar{\gamma}$, is larger than the effect changes in the level of risk aversion concerning utility from terminal wealth. Overall, however, the losses resulting from the parameter sets in Table 4.1 are smaller than the ones presented in Figures C.6 for the parameter sets in Table C.1. In particular, even for very low risk aversion concerning consumption, the WEL for the homoscedastic model does not exceed 1% for a five-year horizon.

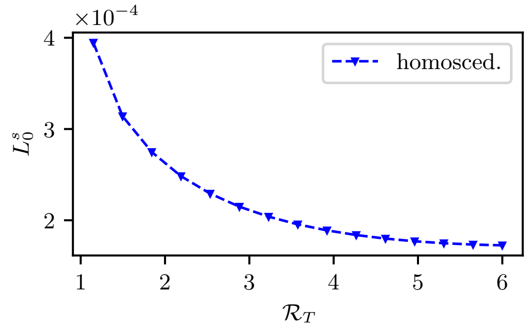
The plots in Figure C.9 seem surprising at first since more negative skewness and more excess kurtosis, both induced by a decrease in η, lead to lower WEL levels for both the HN-GARCH and the homoscedastic model. A closer look at Figures 4.2 and 4.4, however, reveals that a decrease in η also decreases the optimal risky allocation (left plots in Figures 4.4 and C.5, respectively), and furthermore that the IG-GARCH yields the highest fraction of wealth invested in the risky assets (Figure 4.2b). That is, the changes induced by conditional skewness and excess kurtosis are eclipsed by the original pattern stemming from the differences in the first and second moments. As opposed to that, the parameter set from Table C.1 leads to the inverse pattern regarding the risky allocation (with the IG-GARCH yielding the lowest fraction invested in the risky asset, see Figure C.3b), and thus a decrease in η also leads to higher WEL levels for the HN-GARCH and the homoscedastic model. We also note that significant changes in the levels of conditional skewness and excess kurtosis do not lead to large differences in the corresponding WEL – the effect is practically negligible for both models.

As in Figure C.7b, we adjust the MPR in all three models to obtain the WEL values plotted in Figures C.10a and C.10b. Again, matching the market price of risk leads to lower WEL levels in general, with all values remaining far below one basis point. As observed previously, the level of losses increases as the MPR increases.

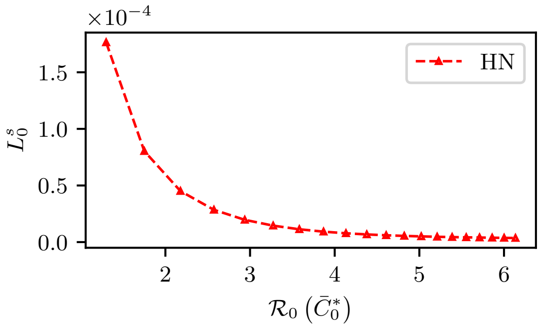
In summary, the key observation deduced from the plots in this section is that heteroscedasticity matters, which can be concluded from the fact that the HN-GARCH model outperforms the homoscedastic variant in all studies. An interpretation of the specific numbers quantifying the loss, and especially a comparison to the real-world study in Section 4.4.2, remains difficult since the approaches are fundamentally different.



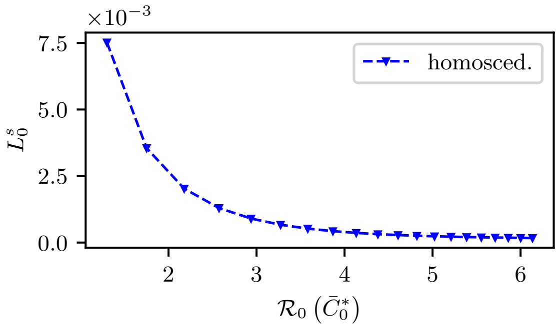
(a) WEL for the HN-GARCH model, dependent on the investor's risk aversion concerning utility from terminal wealth.



(b) WEL for the homoscedastic model, dependent on the investor's risk aversion concerning utility from terminal wealth.

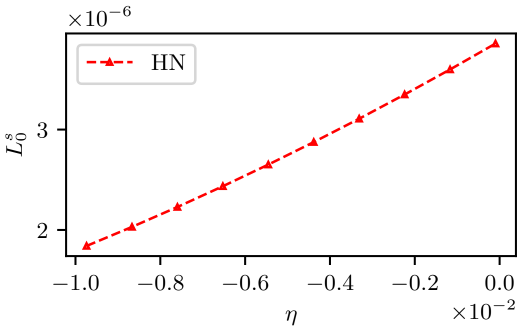


(c) WEL for the HN-GARCH model, dependent on the investor's risk aversion concerning utility from consumption.

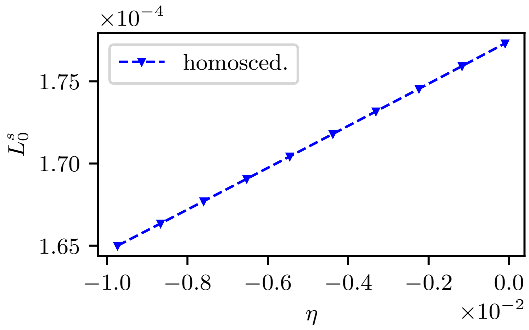


(d) WEL for the homoscedastic model, dependent on the investor's risk aversion concerning utility from consumption.

Figure C.8: WEL for the HN-GARCH and the homoscedastic model, dependent on risk aversion concerning utility from terminal wealth and from consumption. Relative risk aversion for terminal wealth in the left plot is governed by the parameter γ, and relative risk aversion for consumption in the right plot is modified via $\bar{\gamma}$, see Remark 4.2. The time horizon is five years. The model parameters are given in Table 4.1.

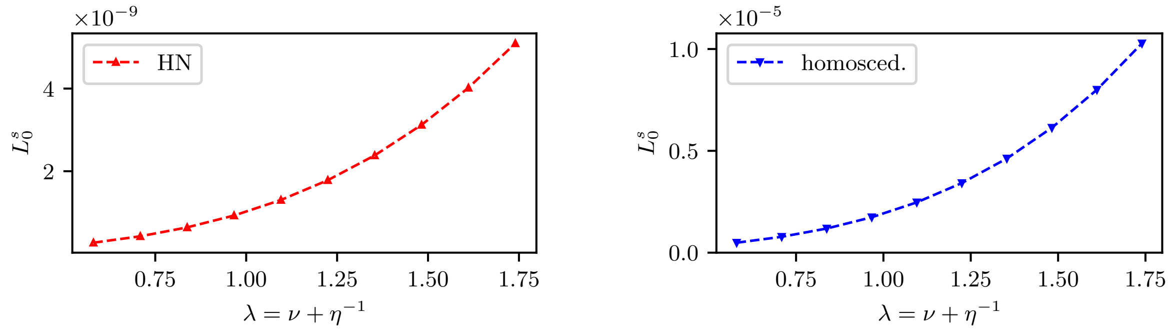


(a) WEL for the HN-GARCH model, dependent on the parameter η.



(b) WEL for the homoscedastic model, dependent on the parameter η.

Figure C.9: WEL for the HN-GARCH and the homoscedastic model, dependent on the parameter η. Modifications in η lead to different levels of conditional skewness and excess kurtosis, see Figures 4.4 and C.5. The time horizon is five years. The original parameter set is given in Table 4.1.



(a) WEL dependent on the MPR.

(b) WEL dependent on the MPR.

Figure C.10: WEL for the HN-GARCH and the homoscedastic model, dependent on the MPR. In this plot, all models are modified to share the same MPR. The time horizon is five years. The original parameter set is given in Table 4.1. The original MPR for the IG-GARCH model with the parameters as of Table 4.1 is 1.16.

D Appendix for Chapter 6

D.1 Proofs from Sections 6.2 and 6.3

Proof of Prop 6.1. According to the model setup presented in Section 6.2, the innovations z_{t+1} and y_{t+1} are contemporaneously independent. Furthermore, for $z_{t+1} \sim \mathcal{N}(0, h_{z,t+1})$, integrating by substitution and completing the square shows that (Badescu et al., 2019, p. 33)

$$\mathbb{E}_t\left[\exp\left\{\alpha z_{t+1} + \beta z_{t+1}^2\right\}\right] = \exp\left\{\frac{\alpha^2 h_{z,t+1}}{2\left(1 - 2\beta h_{z,t+1}\right)} - \frac{1}{2}\log\left(1 - 2\beta h_{z,t+1}\right)\right\}. \tag{D.1}$$

Using (D.1) in the last step, we prove Proposition 6.1:

$$\begin{aligned}
&\Psi_{(R_{t+1},\, h_{z,t+2},\, h_{y,t+2})}\left(u, v_z, v_y \,|\, \mathcal{F}_t\right) \\
&= \mathbb{E}_t\left[\exp\left\{u \cdot R_{t+1} + v_z \cdot h_{z,t+2} + v_y \cdot h_{y,t+2}\right\}\right] \\
&= \mathbb{E}_t\Bigg[\exp\Bigg\{u\Big[r_{t+1} + \left(\lambda_z - \xi_z(1)\right) h_{z,t+1} + \left(\lambda_y - \xi_y(1)\right) h_{y,t+1} + z_{t+1} + y_{t+1}\Big] \\
&\qquad\qquad + v_z \cdot h_{z,t+2} + v_y \cdot h_{y,t+2}\Bigg\}\Bigg] \\
&= \exp\left\{u\left[r_{t+1} + \left(\lambda_z - \xi_z(1)\right) h_{z,t+1} + \left(\lambda_y - \xi_y(1)\right) h_{y,t+1}\right]\right\} \\
&\quad \times \mathbb{E}_t\Bigg[\exp\Bigg\{u \cdot \left(z_{t+1} + y_{t+1}\right) \\
&\qquad\qquad + v_z\left[\omega_z + b_z h_{z,t+1} + \frac{a_z}{h_{z,t+1}}\left(z_{t+1} - c_z h_{z,t+1}\right)^2\right] \\
&\qquad\qquad + v_y\left[\omega_y + b_y h_{y,t+1} + \frac{a_y}{h_{z,t+1}}\left(z_{t+1} - c_y h_{z,t+1}\right)^2\right]\Bigg\}\Bigg] \\
&= \exp\Bigg\{u \cdot r_{t+1} + v_z \cdot \omega_z + v_y \cdot \omega_y \\
&\qquad + \left[u \cdot \left(\lambda_z - \xi_z(1)\right) + \ v_z \cdot \left(b_z + a_z c_z^2\right) + v_y a_y c_y^2\right] \cdot h_{z,t+1} \\
&\qquad + \left[u \cdot \left(\lambda_y - \xi_y(1)\right) + v_y \cdot b_y\right] \cdot h_{y,t+1}\Bigg\} \\
&\quad \times \mathbb{E}_t\Bigg[\exp\Bigg\{u \cdot y_{t+1} + \left[u - 2 v_z a_z c_z - 2 v_y a_y c_y\right] \cdot z_{t+1} \\
&\qquad\qquad + \left[v_z \cdot \frac{a_z}{h_{z,t+1}} + v_y \cdot \frac{a_y}{h_{z,t+1}}\right] \cdot z_{t+1}^2\Bigg\}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
= \exp\Bigg\{ & u r_{t+1} + v_z \omega_z + v_y \omega_y \\
& + \Big[u \cdot (\lambda_z - \xi_z(1)) + v_z \cdot \left(b_z + a_z c_z^2 \right) + v_y a_y c_y^2 \Big] \cdot h_{z,t+1} \\
& + \Big[u \cdot (\lambda_y - \xi_y(1)) + v_y b_y + \xi_y(u) \Big] \cdot h_{y,t+1} \\
& + \frac{(u - 2 v_z a_z c_z - 2 v_y a_y c_y)^2}{2\left[1 - 2\left(v_z a_z + v_y a_y\right)\right]} \cdot h_{z,t+1} - \frac{1}{2} \log\left(1 - 2\left(v_z a_z + v_y a_y\right)\right) \Bigg\},
\end{aligned}
$$

justifying (6.6) together with the definitions in (6.7). This completes the proof of Proposition 6.1. □

Proof of Theorem 6.2. We use Bellman's principle in order to maximize via backwards induction and use the optimal solution at time $t+1$ for our approach at time t. This results in the calculations below, where for the terminal step at time $t = T-1$ we just work with

$$
\phi_T\left(W_T\right) = U\left(V_T\right) = \frac{1}{\gamma} \exp\left\{\gamma W_T\right\},
$$

resulting in $D_{T,T} = E^z_{T,T} = E^y_{T,T} = 0$. We obtain:

$$
\begin{aligned}
& \max_{\pi_t} \ \mathbb{E}_t\left[\phi_{t+1}\left(W_{t+1}, h_{z,t+2}, h_{y,t+2}\right)\right] \\
= & \max_{\pi_t} \ \mathbb{E}_t\Bigg[\frac{1}{\gamma} \exp\Big\{D_{t+1,T}(\pi^*_{t+1}) + \gamma W_{t+1} \\
& \qquad + E^z_{t+1,T}(\pi^*_{t+1}) \cdot h_{z,t+2} + E^y_{t+1,T}(\pi^*_{t+1}) \cdot h_{y,t+2}\Big\}\Bigg] \\
= & \max_{\pi_t} \ \frac{1}{\gamma} \exp\left\{D_{t+1,T}(\pi^*_{t+1})\right\} \\
& \times \mathbb{E}_t\Bigg[\exp\Bigg\{\gamma \cdot \Bigg[W_t + \pi_t \cdot R_{t+1} \\
& \qquad + \frac{1}{2}\left(\pi_t - \pi_t^2\right) \cdot \left(h_{z,t+1} + h_{y,t+1} \cdot \xi_y''(0)\right) + (1 - \pi_t)\, r_{t+1}\Bigg] \\
& \qquad + E^z_{t+1,T}(\pi^*_{t+1}) \cdot h_{z,t+2} + E^y_{t+1,T}(\pi^*_{t+1}) \cdot h_{y,t+2}\Bigg\}\Bigg] \\
= & \max_{\pi_t} \ \frac{1}{\gamma} \exp\Big\{D_{t+1,T}(\pi^*_{t+1}) + (1 - \pi_t)\, \gamma r_{t+1} + \gamma W_t \\
& \qquad + \frac{\gamma}{2}\left(\pi_t - \pi_t^2\right) \cdot \left(h_{z,t+1} + h_{y,t+1} \cdot \xi_y''(0)\right)\Big\} \\
& \times \mathbb{E}_t\Big[\exp\Big\{\gamma \pi_t R_{t+1} + E^z_{t+1,T}(\pi^*_{t+1}) \cdot h_{z,t+2} + E^y_{t+1,T}(\pi^*_{t+1}) \cdot h_{y,t+2}\Big\}\Big] \\
= & \max_{\pi_t} \ \frac{1}{\gamma} \exp\Big\{D_{t+1,T}(\pi^*_{t+1}) + (1 - \pi_t)\, \gamma r_{t+1} + \gamma W_t \\
& \qquad + \frac{\gamma}{2}\left(\pi_t - \pi_t^2\right) \cdot \left(h_{z,t+1} + h_{y,t+1} \cdot \xi_y''(0)\right)\Big\} \\
& \times \Psi_{(R_{t+1}, h_{z,t+2}, h_{y,t+2})}\left(\gamma \pi_t;\ E^z_{t+1,T}(\pi^*_{t+1}),\ E^y_{t+1,T}(\pi^*_{t+1}) \,\Big|\, \mathcal{F}_t\right) \\
= & \max_{\pi_t} \ \frac{1}{\gamma} \exp\Big\{D_{t,T}(\pi_t) + \gamma W_t + E^z_{t,T}(\pi_t) \cdot h_{z,t+1} + E^y_{t,T}(\pi_t) \cdot h_{y,t+1}\Big\},
\end{aligned}
\tag{D.2}
$$

where we used the multivariate generating function from the proof of Proposition 6.1 in the Lévy GARCH setting and the definitions from Equations (6.9) in the last step. We seek to maximize (D.2) with respect to the single variable π_t. Thus, we approach this step by taking derivatives, setting the first derivative to zero while having a negative second derivative at this point (or globally in case of concave functions). Concerning the first of the above-mentioned steps, note that, given some $\gamma \neq 0$, the first derivative is equal to zero if and only if

$$D'_{t,T}(\pi_t) + \left(E^z_{t,T}\right)'(\pi_t) \cdot h_{z,t+1} + \left(E^y_{t,T}\right)'(\pi_t) \cdot h_{y,t+1} = 0, \tag{D.3}$$

which, according to (6.9), is equivalent to

$$\begin{aligned} &-\gamma r_{t+1} + \frac{\partial}{\partial u} A\left(\gamma\pi_t,\, E^{z,*}_{t+1,T},\, E^{y,*}_{t+1,T}; t, t+1\right) \cdot \gamma \\ &+ \left[\frac{\gamma}{2} - \gamma\pi_t + \frac{\partial}{\partial u} B\left(\gamma\pi_t,\, E^{z,*}_{t+1,T},\, E^{y,*}_{t+1,T}; t, t+1\right) \cdot \gamma\right] \cdot h_{z,t+1} \\ &+ \left[\xi''_y(0) \cdot \left(\frac{\gamma}{2} - \gamma\pi_t\right) + \frac{\partial}{\partial u} C\left(\gamma\pi_t,\, E^{y,*}_{t+1,T}; t, t+1\right) \cdot \gamma\right] \cdot h_{y,t+1} = 0, \end{aligned} \tag{D.4}$$

where $E^{x,*}_{t+1,T}$ is short for $E^x_{t+1,T}(\pi^*_{t+1})$ with $x \in \{y, z\}$. Now, using Definitions (6.7), we note that the first line reduces to zero, and, also dividing by γ, we arrive at

$$\begin{aligned} &\left[\frac{1}{2} - \pi_t + \lambda_z - \xi_z(1) + \frac{\gamma\pi_t - 2E^{z,*}_{t+1,T}a_z c_z - 2E^{y,*}_{t+1,T}a_y c_y}{1 - 2E^{z,*}_{t+1,T}a_z - 2E^{y,*}_{t+1,T}a_y}\right] \cdot h_{z,t+1} \\ &\qquad + \left[\xi''_y(0)\left(\frac{1}{2} - \pi_t\right) + (\lambda_y - \xi_y(1)) + \xi'_y(\gamma\pi_t)\right] \cdot h_{y,t+1} = 0. \end{aligned}$$

Exploiting the Gaussian distribution of z, we further use $\xi_z(1) = 1/2$ to obtain

$$\begin{aligned} &\left[\lambda_z - \pi_t + \frac{\gamma\pi_t - 2E^{z,*}_{t+1,T}a_z c_z - 2E^{y,*}_{t+1,T}a_y c_y}{1 - 2E^{z,*}_{t+1,T}a_z - 2E^{y,*}_{t+1,T}a_y}\right] \cdot h_{z,t+1} \\ &+ \left[\xi''_y(0)\left(\frac{1}{2} - \pi_t\right) + (\lambda_y - \xi_y(1)) + \xi'_y(\gamma\pi_t)\right] \cdot h_{y,t+1} = 0. \end{aligned} \tag{D.5}$$

Remember that the two time-homogeneous parameters are assumed to be linearly dependent, i.e.,

$$h_{y,t+1} = \Lambda \cdot h_{z,t+1} \tag{D.6}$$

holds for some $\Lambda \in \mathbb{R}$. This yields the single equation

$$\begin{aligned} &\left[\lambda_z - \pi_t + \frac{\gamma\pi_t - 2E^{z,*}_{t+1,T}a_z c_z - 2E^{y,*}_{t+1,T}a_y c_y}{1 - 2E^{z,*}_{t+1,T}a_z - 2E^{y,*}_{t+1,T}a_y}\right] \\ &+\Lambda \cdot \left[\xi''_y(0)\left(\frac{1}{2} - \pi_t\right) + (\lambda_y - \xi_y(1)) + \xi'_y(\gamma\pi_t)\right] = 0. \end{aligned} \tag{D.7}$$

How easily the resulting equation can be solved for π_t now largely depends on the shape of ξ_y, i.e. on the explicit distribution of the Lévy jump innovation y. In order to ensure that the solution to (D.7) is actually a maximum, we require (6.11). This concludes the proof. □

Proof of Corollary 6.4. If we require (6.18) with strict inequality, then the continuous function on the left-hand side in the optimality equation (6.20) is strictly decreasing in π_t. Furthermore, using that $\gamma < 0$, we obtain that the left-hand side goes to $\pm\infty$ if $\pi_t \to \mp\infty$. Using the intermediate value theorem, the existence of a unique solution follows. □