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The final published paper can be found at DOI: 10.1109/LCSYS.2018.2851602

Stochastic Extremum Seeking for Dynamic Maps with Delays

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***Abstract*— We address a Newton-based extremum seeking algorithm for maximizing higher derivatives of unknown maps in the presence of known time delays. Different from all previous works on this topic, we employ stochastic instead of periodic perturbations in order to achieve better convergence rates, we allow arbitrarily long output delays as well as a dynamic map that is to be optimized. We incorporate a novel predictor feedback for delay compensation and show exponential stability and convergence to a small neighborhood of the unknown extremum point by using a backstepping transformation and averaging theory in infinite dimensions for stochastic systems. Moreover, simulations highlight the effectiveness of the proposed predictor-feedback scheme.**

I. INTRODUCTION

Extremum Seeking (ES) control is a nonmodel-based adaptive technique for real-time optimization that became very popular in recent years with Newton-like schemes being the most popular [1], [13]. However, there are two noticeable challenges in ES. First challenge is not the optimization of the map itself, but rather of the map's higher derivative [3], [7], [20]. Second, and an even more challenging issue, is the effect of time delays in implemented ES, leading to a degradation of the system's behavior or even its instability.

In order to solve the problem of maximizing map sensitivity, the generalization of Newton-based ES [6] was presented in [11] to optimize arbitrary higher derivatives of an unknown map employing sinusoidal perturbations. In addition, by using the mathematical foundation introduced in [16] to handle ES for functional differential equations (FDEs), the authors in [17] expanded the application of the results of [11] to the case of static maps with time delays.

Furthermore, it is a known issue that ES techniques based on periodic perturbations suffer with higher dimensionality [2]. Orthogonality requirements on the elements of the periodic perturbation vector pose an implementation challenge [10]. Limitations of the deterministic ES scheme also include the fact that learning using a periodic excitation signal is rare in probing-based learning and optimization approaches [10], which may lead to slower converge rates. Furthermore, ES algorithms inspired by biomimicry and others sensitive to deterministic perturbation signals suggest other perturbation techniques rather than periodic ones [10]. Hence, in a more general context, there is a huge demand in developing nonperiodic ES techniques such as stochastic methods [12], with all its remarkable advantages including faster convergence rate and the possibility of achieving global maximum/minimum in the presence of local extremum points [10].

The key contribution of this letter is the extension of the stochastic method for dynamic maps presented in [12] for the cases where constant output delays occur. We prove the local stability of the proposed predictor-feedback for delay compensation and show convergence to a small neighbourhood of the unknown extremum point. As in [15], we employ a semi-model-based approach due to the treatment of the delay. While assuming known delay, our predictor construction follows a model-free approach. Moreover, we show that the stochastic method outperforms the periodic one in terms of faster convergence rate.

II. PROBLEM FORMULATION

We are interested in maximizing (w.l.o.g.) the constantly delayed output of an arbitrary, unknown, nonlinear map

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{1a}$$
$$y = g(\mathbf{x}), \tag{1b}$$

where $\mathbf{x} \in \mathbb{R}^m$ is the m-dimensional state vector, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ represent the scalar input and output, respectively, and $\mathbf{f}\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ as well as $g\colon \mathbb{R}^m \to \mathbb{R}$ are smooth [9]. Establishing that a smooth control law $\alpha\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$

$$u = \alpha(\mathbf{x}, \theta) \tag{2}$$

is acting upon the plant, one obtains the closed-loop system

$$\dot{\mathbf{x}} = \mathbf{f}\big(\mathbf{x}, \alpha(\mathbf{x}, \theta)\big). \tag{3}$$

Its equilibria, characterized by the scalar parameter θ, are specified through the following assumptions [12].

Assumption A1: Let $\mathbf{l}\colon \mathbb{R} \to \mathbb{R}^m$ be an existing, smooth vector field, such that only when the state vector $\mathbf{x}$ follows the assignment

$$\mathbf{x} = \mathbf{l}(\theta), \tag{4}$$

the closed-loop system (3) is in equilibrium

$$\mathbf{f}\big(\mathbf{x}, \alpha(\mathbf{x}, \theta)\big) = \mathbf{0}. \tag{5}$$

Assumption A2: For every value of the parameter $\theta \in \mathbb{R}$, the equilibrium (5) is exponentially stable with decay and overshoot constants uniform in θ.

For $D \geq 0$, the delayed output y is

$$y(t-D) = g\Big(\mathbf{l}\big(\theta(t-D)\big)\Big), \tag{6}$$

*This work was partially supported by Bavaria California Technology Center (BaCaTech).

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The final published paper can be found at DOI: 10.1109/LCSYS.2018.2851602

which is only true in equilibrium (Assumption A1 - A2). Input delays are not considered since, in the case of dynamic maps, they cannot be simply transformed to output delays.

Defining $\nu\colon \mathbb{R} \rightarrow \mathbb{R}$ as the composition of the SISO output function g in (1b) and the state vector function $\mathbf{l}$ in (4)

$$\nu(\cdot) \quad = \quad (g \circ \mathbf{l})(\cdot)\,, \tag{7}$$

with $n \in \mathbb{N}_0$, we formulate our optimization problem as

$$\max_{\theta\in\mathbb{R}} \nu^{(n)}(\theta(t-D)) \quad := \quad \max_{\theta\in\mathbb{R}} \frac{d^n\nu}{d\theta^n}(\theta(t-D))\,, \tag{8}$$

where the corresponding maximizing value is denoted by θ^*.

Assumption A3: Consider the set

$$\Theta_{max} \quad = \quad \Big\{\theta \mid \nu^{(n+1)}(\theta) = 0, \quad \nu^{(n+2)}(\theta) < 0\Big\} \;\neq\; \emptyset \tag{9}$$

including all stationary points which are local maxima, i.e. locally concave. We assume that $\exists \theta^* \in \Theta_{max}$.

III. SYSTEMS AND SIGNALS

We only require the map g (1b) itself to be measurable. Moreover, all constants, i.e. the perturbation amplitude a and frequency ω, the integrator gains

$$k_I \quad = \quad \varepsilon\omega k_I' \;=\; \mathcal{O}(\varepsilon\omega) \tag{10a}$$
$$k_R \quad = \quad \varepsilon\omega k_R' \;=\; \mathcal{O}(\varepsilon\omega)\,, \tag{10b}$$

the time scale separation ε, the entailing gains k_I' and k_R' are user-assignable, and have to be positive [12]. For the averaging theory it applies $0 < \varepsilon << 1$.

The analysis strategy leads to multiple time scales with the plant dynamics (3) being the fastest [14]. We employ a stochastic perturbation of the unknown map via the sinusoid of a Wiener process about the boundary of a circle [12],

$$\theta(t) \quad = \quad \hat{\theta}(t) + a\sin(\eta(t))\,, \tag{11}$$

where

$$\eta(t) \quad = \quad \omega\pi\big(1 + \sin(W_{\omega t})\big) \tag{12}$$

represents a homogenous ergodic Markov process. The special demodulation signals

$$\Upsilon_k(t) \quad = \quad C_k \sin\Big(k\eta(t) + \frac{\pi}{4}\big(1 + (-1)^k\big)\Big)\,, \tag{13}$$

with the normalizing gain

$$C_k \quad = \quad \frac{2^k k!}{a^k}(-1)^{\dfrac{k - \left|\sin\left(\frac{k\pi}{2}\right)\right|}{2}}\,. \tag{14}$$

allow a sufficiently precise estimate of the gradient and Hessian in an average sense.

Using the stochastic chain rule, one obtains

$$d\eta \quad = \quad -\omega\frac{\pi}{2}\sin(W_{\omega t})dt + \omega\pi\cos(W_{\omega t})dW_{\omega t} \tag{15}$$

as the perturbational middle time scale [12].

With $\hat{\theta}$ being the best estimate of the maximizing value θ^*, and from the block diagram in Fig. 1, we can write

$$\frac{d\hat{\theta}}{dt} \quad = \quad -\varepsilon\omega k_I' U(t)\,, \tag{16}$$

where the controlling signal $U(t)$ is generated through our proposed predictor in Sec. V.

Furthermore, we use a Riccati filter [6]

$$\frac{d\gamma}{dt} \quad = \quad \varepsilon\omega k_R'\gamma(t)\Big(1 - \gamma(t)\widehat{\nu^{(n+2)}}(t)\Big) \tag{17}$$

to dynamically estimate the inverse Hessian of $\nu^{(n)}(\theta)$. In addition, by setting the sign of the initial value $sgn(\gamma(0)) = sgn\big(\nu^{(n+2)}(\theta^*)\big)$ one can switch from a maximization to a minimization. Altogether, these differential parameter update equations (16), (17) follow the slowest time scale.

The demodulated signals are defined as

$$\widehat{\nu^{(k)}}(t) \quad = \quad \Upsilon_k(t-D)y(t-D). \tag{18}$$

The estimates for the Gradient (1st derivative) and Hessian (2nd derivative) of the n^{th} derivative map $\nu^{(n)}$ in (7)–(9) is obtained by means of (18) by setting $k = n+1$ and $k = n+2$, respectively. They are also delayed by D time units in order to cope with delayed output y. Additionally, dvancing the perturbation signal (12) by D time units would lead to the same solution [17].

Lastly, we define the measurable signal

$$z(t) \quad = \quad \gamma(t)\widehat{\nu^{(n+1)}}(t)\,, \tag{19}$$

where $\gamma(t)$, the inverse Hessian estimate, is updated according to (17), and $\widehat{\nu^{(n+1)}}(t)$ represents the gradient estimate (18) [17].

IV. AVERAGING ANALYSIS

Consider the error transformations

$$\tilde{\theta}(t) \quad = \quad \hat{\theta}(t) - \theta^* \tag{20a}$$
$$\tilde{\gamma}(t) \quad = \quad \gamma(t) - \gamma^*\,, \tag{20b}$$

with the inverse Hessian

$$\gamma^* \quad = \quad \frac{1}{\nu^{(n+2)}(\theta^*)} \tag{21}$$

being the corresponding optimum value of γ.

Lemma 1: For maximizing any higher derivative of the unknown, dynamic, nonlinear and time delayed map $y(t-D)$ satisfying Assumptions A1 - A3, the reduced, averaged and linearized version of the measurable signal $z(t)$ as stated in (19) can be expressed as

$$z_{\mathrm{r}}^{\mathrm{a}}(t) \quad = \quad \tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\,, \tag{22}$$

when using the stochastic perturbation $\eta(t)$, given by (11) - (12) and being a homogenous ergodic Markov process, as well as the estimative demodulated signals (18).

Proof of Lemma 1:

For the uncompensated case where $U(t)$=$z(t)$, using (16) and (20a) as well as (17) and (20b), we can express the error dynamics as

$$\frac{d\tilde{\theta}}{dt} \quad = \quad -\varepsilon\omega k_I'\big(\tilde{\gamma}(t) + \gamma^*\big)\Upsilon_{n+1}(t-D)g\big(\mathbf{x}(t-D)\big) \tag{23a}$$
$$\frac{d\tilde{\gamma}}{dt} \quad = \quad \varepsilon\omega k_R'\big(\tilde{\gamma}(t) + \gamma^*\big) \Big(1 - \big(\tilde{\gamma}(t) + \gamma^*\big)\Upsilon_{n+2}(t-D)g\big(\mathbf{x}(t-D)\big)\Big)\,. \tag{23b}$$

After employing a singular perturbation reduction [12] in order to freeze the delayed state vector $\mathbf{x}(t-D)$ in (3) at its quasi-steady state value

$$\mathbf{x}(t-D) \quad = \quad \mathbf{l}\Big(\tilde{\theta}(t-D) + a\sin\big(\eta(t-D)\big)\Big)\,, \tag{24}$$

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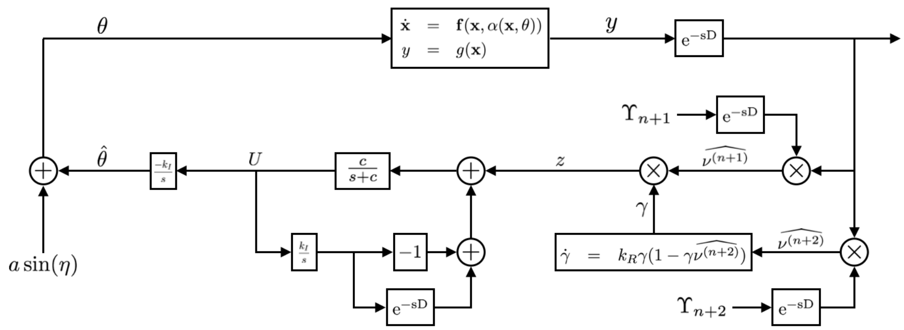

Fig. 1. Block diagram of the basic prediction scheme for time delay compensation in Newton-based extremum seeking with stochastic perturbation and an unknown, nonlinear, dynamic map for maximizing arbitrarily high derivatives.

we obtain the reduced error system of the form

$$\begin{aligned}
\frac{d\tilde{\theta}_{\mathrm{r}}}{dt} &= -\varepsilon\omega k_I'\big(\tilde{\gamma}_{\mathrm{a}}(t)+\gamma^*\big) \\
&\quad \Upsilon_{n+1}(t-D)\nu\Big(\tilde{\theta}_{\mathrm{r}}(t-D)+a\sin\big(\eta(t-D)\big)\Big) \qquad (25a)\\
\frac{d\tilde{\gamma}_r}{dt} &= \varepsilon\omega k_R'\big(\tilde{\gamma}_{\mathrm{r}}(t)+\gamma^*\big)\Bigg(1-\big(\tilde{\gamma}_{\mathrm{r}}(t)+\gamma^*\big) \\
&\quad \Upsilon_{n+2}(t-D)\nu\Big(\tilde{\theta}_{\mathrm{r}}(t-D)+a\sin\big(\eta(t-D)\big)\Big)\Bigg)\,. \qquad (25b)
\end{aligned}$$

We assume the maximizing value θ^* to remain constant in the reduced system, since the parameter differential equations follow the slowest time scale. For sufficiently small ε, the reduced parameter error system is applicable to the theory of averaging. Now, for the averaging method we replace the frozen quantities $\tilde{\theta}_{\mathrm{r}}$ and $\tilde{\gamma}_{\mathrm{r}}$ with autonomous values $\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}$ and $\tilde{\gamma}_{\mathrm{r}}^{\mathrm{a}}$, respectively. Then, comparing (25a) and (25b), we focus on the evaluation of the average

$$\xi_{\mathrm{r,k}}^{\mathrm{a}} = \mathrm{AVE}\Big\{\widehat{\nu^{(k)}}\big(\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\big)\Big\} \qquad (26)$$

exploiting the ergodicity and invariant distribution. The average of the reduced-demodulated error signal $\xi_{\mathrm{r,k}}^{\mathrm{a}}$ (26) can be solved for arbitrary $k\in\mathbb{N}$. and one obtains the expression

$$\xi_{\mathrm{r,k}}^{\mathrm{a}} = \nu^{(k)}\big(\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\big)+\frac{\nu^{(k+2)}\big(\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\big)}{4(k+1)}a^2+\mathcal{O}(a^4)\,. \qquad (27)$$

Consequently, we can now express the averaged-reduced error system as

$$\begin{aligned}
\frac{d\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}}{dt} &= -\varepsilon\omega k_I'\big(\tilde{\gamma}_{\mathrm{r}}^{\mathrm{a}}(t)+\gamma^*\big)\xi_{\mathrm{r,n+1}}^{\mathrm{a}} \qquad (28a)\\
\frac{d\tilde{\gamma}_{\mathrm{r}}^{\mathrm{a}}}{dt} &= \varepsilon\omega k_R'\big(\tilde{\gamma}_{\mathrm{r}}^{\mathrm{a}}(t)+\gamma^*\big)\Big(1-\big(\tilde{\gamma}_{\mathrm{r}}^{\mathrm{a}}(t)+\gamma^*\big)\xi_{\mathrm{r,n+2}}^{\mathrm{a}}\Big)\,. \qquad (28b)
\end{aligned}$$

We use a local quadratic approximation of the objective function [17] at $\theta=\theta^*$, which yields

$$\nu^{(n)}(\theta) = Q^*+\frac{\nu^{(n+2)}(\theta^*)}{2}\big(\theta(t)-\theta^*\big)^2\,, \qquad (29)$$

with a positive constant $Q^*>0$ by linearization. Thus, the equilibria of (28a) and (28b) become

$$\begin{aligned}
\tilde{\theta}_r^{\mathrm{a,e}} &= 0 \qquad (30a)\\
\tilde{\gamma}_r^{\mathrm{a,e}} &= 0\,. \qquad (30b)
\end{aligned}$$

After plugging (29) into (27) for $k=n+1$, we obtain

$$\xi_{\mathrm{r,n+1}}^{\mathrm{a}} = \nu^{(n+2)}(\theta^*)\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\,. \qquad (31)$$

Finally, consider the averaged-reduced version of (19):

$$z_{\mathrm{r}}^{\mathrm{a}}(t) = \big(\tilde{\gamma}_{\mathrm{r}}^{\mathrm{a}}(t)+\gamma^*\big)\nu^{(n+2)}(\theta^*)\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\,. \qquad (32)$$

Performing linearization at the desired extremizing operating point corresponding to the error equilibrium (30a)–(30b) we get

$$z_{\mathrm{r}}^{\mathrm{a}}(t) = \tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)\,, \qquad (33)$$

which completes the proof of Lemma 1. □

V. PREDICTOR FEEDBACK DESIGN

The idea is to derive a controller which, taking (33) as the input, feeds back the future state into the equivalent averaged-reduced system

$$U_{\mathrm{r}}^{\mathrm{a}}(t) = z_{\mathrm{r}}^{\mathrm{a}}(t+D)=\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t+D-D)=\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t)\,. \qquad (34)$$

Using (16) and (20a), we consider

$$\dot{\hat{\theta}}(t-D) = -k_I U(t-D) \qquad (35)$$

as well as its shifted, averaged-reduced version

$$\dot{\hat{\theta}}_{\mathrm{r}}^{\mathrm{a}}(t) = -k_I U_{\mathrm{r}}^{\mathrm{a}}(t). \qquad (36)$$

Applying the known time delay D to (36) and using (33), we obtain

$$\dot{z}_{\mathrm{r}}^{\mathrm{a}}(t) = -k_I U_{\mathrm{r}}^{\mathrm{a}}(t-D)\,. \qquad (37)$$

Now, applying Laplace-Transformation to ((37)) we are able to express the future state as

$$z_{\mathrm{r}}^{\mathrm{a}}(t+D) = -k_I\int_0^t U_{\mathrm{r}}^{\mathrm{a}}(\tau)d\tau\,. \qquad (38)$$

Since we are only interested in predicting the change over the next D time units, we herefore transform (38) into

$$z_{\mathrm{r}}^{\mathrm{a}}(t+D) = z_{\mathrm{r}}^{\mathrm{a}}(t)-k_I\int_{t-D}^t U_{\mathrm{r}}^{\mathrm{a}}(\tau)d\tau\,. \qquad (39)$$

Thus, we arrive at the following expression for our predictor-based controller $\forall t\geq D$:

$$U_{\mathrm{r}}^{\mathrm{a}}(t) = z_{\mathrm{r}}^{\mathrm{a}}(t)-k_I\int_{t-D}^t U_{\mathrm{r}}^{\mathrm{a}}(\tau)d\tau\,. \qquad (40)$$

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Now, we propose the usage of the nonaverage and unreduced version of (40)

$$U(t) \;=\; \frac{c}{s+c}\left\{ z(t) - k_I \int\limits_{t-D}^{t} U(\tau)d\tau \right\}, \tag{41}$$

employing an additional low-pass filter with a sufficiently large $c > 0$. The braces $\{\cdot\}$ in (41) are used to indicate the effect of the frequency-domain transfer function of the filter as an operator on the according time-domain signal.

VI. Stability Analysis

In the following, utilizing the approaches of [16], [17], [18] we prove the stability of the closed-loop, stochastic perturbation-based ES system from Sec. III applying the predictor (41). Note that we only consider its mathematically reduced form via singular perturbation reduction in order to cope with the unknown nonlinear dynamics.

Theorem 1: There exists a low-pass filter constant $c^* > 0$ such that, $\forall c \geq c^*$, the reduced closed-loop system described by (16), (17) and (41) with error signals (20a)–(20b), and states $\tilde{\theta}_{\mathrm{r}}(t-D)$, $U_{\mathrm{r}}(\tau)$, $\tilde{\gamma}_{\mathrm{r}}(t)$ has a unique, locally exponentially stable solution, $\forall t \geq 0, \forall s, \forall \tau \in [t-D, t]$ which satisfies the following upper bound for the expectation:

$$\begin{aligned} \mathrm{E}^{(s)}\Bigg\{ \left|\tilde{\gamma}_{\mathrm{r}}(s+t)\right|^2 \;&+\; \left|\tilde{\theta}_{\mathrm{r}}(s+t-D)\right|^2 \\ &+\; U_{\mathrm{r}}^2(s+t) + \int_{s+t-D}^{s+t} U_{\mathrm{r}}^2(\tau)d\tau \Bigg\} \\ &\leq a_1 e^{-a_2 t\varepsilon}|\varphi|^2 + \Delta(\varepsilon)\,, \end{aligned} \tag{42}$$

with $\Delta(\varepsilon) > 0$ such that $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, some positive constants a_1, a_2 and any continuous three-dimensional vector function $\varphi \in \mathbf{C}_3$ as defined in [8] with initial condition

$$\varphi(\delta) \;=\; \begin{bmatrix} \tilde{\theta}_{\mathrm{r}}(s+\delta-D) \\ U_{\mathrm{r}}(s+\delta) \\ \tilde{\gamma}_{\mathrm{r}}(s+\delta) \end{bmatrix}, \quad -D \leq \delta \leq 0\,. \tag{43}$$

Moreover, there exist constants $r > 0$, $M > 0$, $\lambda > 0$, $\Delta_1 > 0$ and function $T(\varepsilon) : (0\,,1) \to \mathbb{N}$ such that for any initial condition $|\varphi| < r$ and any $\Delta_1 > 0$ the error vector norm $|\varphi(t)|$ converges below a residual value Δ_1 exponentially fast *almost surely* (a.s.) and *in probability*:

$$\liminf_{\varepsilon\to 0} \left\{ \forall t \geq 0 : |\varphi(t)| > M|\varphi(0)|e^{-\lambda t} + \Delta_1 \right\} = \infty\,, \text{a.s.}\,, \tag{44}$$

$$\lim_{\varepsilon\to 0} \mathrm{P}\, \left\{ |\varphi(t)| \leq M|\varphi(0)|e^{-\lambda t} + \Delta_1\,, \forall t \in [0\,, T(\varepsilon)] \right\} = 1\,, \tag{45}$$

with $\lim_{\varepsilon\to 0} T(\varepsilon) = \infty$. Finally, by choosing $\Delta_1(\varepsilon) = \mathcal{O}(\varepsilon)$, we can conclude

$$\lim_{\varepsilon\to 0} \mathrm{P}\, \left\{ \limsup_{t\to\infty} |\theta_r(t) - \theta^*| \right\} = \mathcal{O}(a+\varepsilon)\,. \tag{46}$$

⌟

The proof of Theorem 1 is organized in the following steps A-E.

A. Reduced ODE-PDE System

Using (16) and (20a), we consider

$$\dot{\tilde{\theta}}(t-D) \;=\; -k_I U(t-D). \tag{47}$$

We define

$$u(x,t) \;=\; U_{\mathrm{r}}(t+x-D)\,, \tag{48}$$

where t is time, D is the delay, and x the state enabling the following representation of (47) as an ODE-PDE system:

$$\dot{\tilde{\theta}}_{\mathrm{r}}(t-D) \;=\; -k_I u(0,t)\,, \tag{49a}$$
$$\partial_t u(x,t) \;=\; \partial_x u(x,t)\,, \quad x \in [0,D]\,, \tag{49b}$$
$$u(D,t) \;=\; U_{\mathrm{r}}(t)\,. \tag{49c}$$

Consequently, (48) represents the solution of the PDE subsystem (49b)–(49c) above, with (49a) being an ODE.

Employing a singular perturbation reduction [12] in order to freeze the delayed state vector $\mathbf{x}(t-D)$ in (3) at its quasi-steady state value

$$\mathbf{x}(t-D) \;=\; \mathbf{l}\Big(\tilde{\theta}(t-D) + a\sin\big(\eta(t-D)\big)\Big)\,, \tag{50}$$

we derive an expression for the delayed and reduced output $y_{\mathrm{r}}(t-D)$ in terms of $\tilde{\theta}_{\mathrm{r}}(t-D)$ by plugging (7), (4) and then (50) into (1b)

$$y_{\mathrm{r}}(t-D) \;=\; \nu\Big(\tilde{\theta}_{\mathrm{r}}(t-D) + a\sin\big(\eta(t-D)\big)\Big)\,. \tag{51}$$

Then, we rewrite the integrand of the reduced version of our predictor (41) using (48) to

$$U_{\mathrm{r}}(t) \;=\; \frac{c}{s+c}\left\{ z_{\mathrm{r}}(t) - k_I \int_0^D u(\tau,t)d\tau \right\}. \tag{52}$$

Following the same steps as in [18], using (52) we derive the averaged version of the reduced ODE-PDE system

$$\dot{\tilde{\theta}}^{\mathrm{a}}_{\mathrm{r}}(t-D) \;=\; -k_I u^{\mathrm{a}}(0,t)\,, \tag{53a}$$
$$\partial_t u^{\mathrm{a}}(x,t) \;=\; \partial_x u^{\mathrm{a}}(x,t), \quad x \in [0,D]\,, \tag{53b}$$
$$u^{\mathrm{a}}(D,t) \;=\; \frac{c}{s+c}\left\{ z^{\mathrm{a}}_{\mathrm{r}}(t) - k_I \int_0^D u^{\mathrm{a}}(\tau,t)d\tau \right\}. \tag{53c}$$

B. Exponential Stability in the Sense of the Full State Norm

Next, we consider the following infinite-dimensional backstepping transformation of the delay state

$$w(x,t) = u^{\mathrm{a}}(x,t) - \left(\tilde{\vartheta}^{\mathrm{a}}_{\mathrm{r}}(t) - k_I \int_0^x u^{\mathrm{a}}(\tau,t)d\tau\right), \tag{54}$$

mapping the local version the system (53a)–(53c) into

$$\dot{\tilde{\vartheta}}^{\mathrm{a}}_{\mathrm{r}}(t) \;=\; -k_I\big(\tilde{\vartheta}^{\mathrm{a}}_{\mathrm{r}}(t) + w(0,t)\big)\,, \tag{55a}$$
$$\partial_t w(x,t) \;=\; \partial_x w(x,t)\,, \quad x \in [0,D]\,, \tag{55b}$$
$$\partial_t u^{\mathrm{a}}(D,t) \;=\; -cw(D,t)\,. \tag{55c}$$

Using (53a), we partially derive the transformed state $w(x,t)$ (54) with respect to time t and for the delay state $x = D$

$$\partial_t w(D,t) = \partial_t u^{\mathrm{a}}(D,t) + k_I u^{\mathrm{a}}(D,t)\,. \tag{56}$$

Furthermore, consider the inverse transformation of (54)

$$u^{\mathrm{a}}(x,t) = w(x,t) + e^{-k_I x}\tilde{\vartheta}^{\mathrm{a}}_{\mathrm{r}}(t) - k_I \int_0^x e^{-k_I(x-\tau)}w(\tau,t)d\tau\,. \tag{57}$$

After plugging (55c) and (57) into (56), we obtain

$$\begin{aligned} \partial_t w(D,t) \;&=\; -cw(D,t) + k_I w(D,t) + k_I e^{-k_I D}\tilde{\vartheta}^{\mathrm{a}}_{\mathrm{r}}(t) \\ &-\; k_I^2 \int_0^D e^{-k_I(D-\tau)}w(\tau,t)d\tau\,. \end{aligned} \tag{58}$$

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Given the following Lyapunov-Krasovskii Functional

$$V(t)=\frac{\left(\tilde{\vartheta}_{\mathrm{r}}^{\mathrm{a}}(t)\right)^2}{2}+\frac{b}{2}\int_0^D(1+x)w^2(x,t)dx+\frac{1}{2}w^2(D,t), \tag{59}$$

as already done in [18], one can show that

$$\dot{V}(t)\leq-\mu V(t) \tag{60}$$

is guaranteed for some $\mu>0$. Thus, the closed-loop system is exponentially stable in the sense of the full state norm

$$\sqrt{|\tilde{\vartheta}_{\mathrm{r}}^{\mathrm{a}}(t)|^2+w^2(D,t)+\int_0^D w^2(x,t)dx}\,, \tag{61}$$

i.e. in the transformed variable $(\tilde{\vartheta}_{\mathrm{r}}^{\mathrm{a}}, w)$.

C. Exponential Stability Estimate of the Averaged-reduced System

To obtain exponential stability in the sense of the norm

$$\sqrt{|\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)|^2+\left(U_{\mathrm{r}}^{\mathrm{a}}(t)\right)^2+\int_{t-D}^t\left(U_{\mathrm{r}}^{\mathrm{a}}(\tau)\right)^2d\tau}\,, \tag{62}$$

we need to show

$$\alpha_1\Psi(t)\leq V(t)\leq\alpha_2\Psi(t), \tag{63}$$

where α_1 and α_2 are positive numbers and

$$\Psi(t)\triangleq|\tilde{\theta}_{\mathrm{r}}^{\mathrm{a}}(t-D)|^2+\left(U_{\mathrm{r}}^{\mathrm{a}}(t)\right)^2+\int_{t-D}^t\left(U_{\mathrm{r}}^{\mathrm{a}}(\tau)\right)^2d\tau\,. \tag{64}$$

This is done by utilizing a similar approach as in [17]. Ultimately, one obtains

$$\Psi(t)\leq\frac{\alpha_2}{\alpha_1}e^{-\mu t}\Psi(0)\,, \tag{65}$$

which completes the proof of exponential stability for the averaged-reduced system.

D. Invoking the Averaging Theorem

Using (47), (41), (23b) while plugging in (10a), (19), (10b) as well as (18) and (13), we obtain

$$\frac{d}{dt}\begin{bmatrix}\tilde{\theta}(t-D)\\ U(t)\\ \tilde{\gamma}(t)\end{bmatrix}=\begin{bmatrix}0\\ -cU(t)\\ 0\end{bmatrix}$$

$$+\varepsilon\begin{bmatrix}-\omega k_I' U(t-D)\\ -c\left((\tilde{\gamma}(t)+\gamma^*)\frac{1}{\varepsilon}\widehat{\nu^{(n+1)}}(t)-\omega k_I'\int\limits_{t-D}^{t}U(\tau)d\tau\right)\\ \omega k_R'(\tilde{\gamma}(t)+\gamma^*)\left(1-(\tilde{\gamma}(t)+\gamma^*)\widehat{\nu^{(n+2)}}(t)\right)\end{bmatrix} \tag{66}$$

with

$$\begin{aligned}\widehat{\nu^{(k)}}(t) &= \frac{2^k k!}{a^k}(-1)^{\lfloor\frac{k}{2}\rfloor}y(t-D)\\ &\quad\sin\left(k\eta(t-D)+\frac{\pi}{4}\left(1+(-1)^k\right)\right).\end{aligned} \tag{67}$$

We define the state vector

$$\mathbf{u}^{\varepsilon}(t)=\begin{bmatrix}\tilde{\theta}(t-D)\\ U(t)\\ \tilde{\gamma}(t)\end{bmatrix}, \tag{68}$$

which allows us to generally express (66) in form of the three-dimensional stochastic functional differential equation

$$\frac{d}{dt}\mathbf{u}^{\varepsilon}(t)=G(\mathbf{u}_t^{\varepsilon})+\varepsilon F(t,\mathbf{u}_t^{\varepsilon},\eta(t),\varepsilon)\,. \tag{69}$$

Therefore, since $\eta(t)$ is a homogenous ergodic Markov process with invariant measure $\mu(d\eta)$ and the property of exponential ergodicity, $\mathbf{u}_t^{\varepsilon}(\delta)=\mathbf{u}^{\varepsilon}(t+\delta)$ for $-D\leq\delta\leq0$ and $G\colon\mathbf{C}_3([-D,0])\to\mathbb{R}^3$ as well as the Lipschitz $F\colon\mathbb{R}_+\times\mathbf{C}_3([-D,0])\times Y\times[0,1)\to\mathbb{R}^3$ with $F(t,0,\eta,\varepsilon)=0$ are continuous mappings, we can apply the averaging theorem in [8] together with its exponential p-stability result for the initial random system using ε.

E. Asymptotic Convergence to the Extremum

By defining the stopping time [10]:

$$\tau_{\varepsilon}^{\Delta_1(\varepsilon)}:=\inf\left\{\forall t\geq0:|\varphi(t)|>M|\varphi(0)|e^{-\lambda t}+\mathcal{O}(\varepsilon)\right\}\,, \tag{70}$$

as the first time when the norm of the error vector does not satisfy the exponential decay property. The error vector norm $|\varphi(t)|$ converges below a residual value $\Delta_1(\varepsilon)=\mathcal{O}(\varepsilon)$ exponentially fast almost surely (44) and in probability (45). From (44) it is clear that $\tau_{\varepsilon}^{\Delta_1(\varepsilon)}$ most surely approaches infinity as ε goes to zero. Similarly in (45), the deterministic function $T(\varepsilon)$ tends to infinity as ε goes to zero. It follows from (44) and (45) that the exponential convergence is satisfied over an arbitrarily long time interval. Either component of the error vector converges to below $\Delta_1(\varepsilon)=\mathcal{O}(\varepsilon)$, particularly the $\tilde{\theta}_r(t)$ component. Then, we can write $\lim\limits_{\varepsilon\to0}\mathrm{P}\left\{\limsup\limits_{t\to\infty}|\tilde{\theta}_{\mathrm{r}}(t)|\right\}=\mathcal{O}(\varepsilon)$. From (11) and (20a), one has that $\theta(t)-\theta^*=\tilde{\theta}(t)+a\sin(\eta(t))$. Since the first term in the right-hand side for the reduced system is ultimately of order $\mathcal{O}(\varepsilon)$ and the second term is of order $\mathcal{O}(a)$, then we state (46). Hence, the proof of Theorem 1 is completed. □

Remark 1: In the stochastic ES for dynamical systems with output equilibrium map, we focus on the stability of the *reduced system* [10], where the the closed-loop system has stochastic perturbations and thus generally, there is no equilibrium solution or periodic solution so that the singular perturbation (SP) methods [9] are not applicable. Despite the lack of such a SP theorem for FDEs, for the reduced system we could analyze properties of the solution by the developed averaging theory in [8] to obtain the approximation to the maximum of the higher-derivative for the equilibrium map.

VII. SIMULATIONS

In order to show the effectiveness of the stochastic perturbation technique, we compare it with the periodic method [17] considering the following map

$$\dot{x} = -\frac{1}{\sqrt{100\pi}}\exp\left(-\frac{(t-40)^2}{100}\right)+\dot{\theta},\quad x(0)=\theta(0) \tag{71a}$$

$$y = g(x) = -\frac{1}{6}x^3+3x\,, \tag{71b}$$

where we are interested in the maximization of $\nu^{(1)}(\theta)$ (7).

We assume $D=5s$. For the periodic algorithm, we apply (30) from [17], and for the stochastic case, the predictor (41). Both scenarios have identical paramteres: $c=20$, $\varepsilon=1/1000$, $a=0.1$, $\omega=35$, $k_I'=1$, $k_R'=0.25$. The stochastic perturbation process' standard deviation is $\sigma=0.1$.

As shown in Fig. 2, all signals converge to the expected values, despite the delay. The convergence rate of the

This is a preprint version of the author's article published by IEEE in the IEEE Control Systems Letters.
The final published paper can be found at DOI: 10.1109/LCSYS.2018.2851602

stochastic method outperforms the periodic one when using identical simulation parameters. This confirms the theoretical train of thought in [10, Chapter 2].

In order to improve the performance of the deterministic method it is prohibited in theory to increase arbitrarily the perturbation frequency ω due to the application of the singular perturbation reduction [9] to dynamic maps.

Additionally, we state that the predictor is vital for the compensation of the delays, since the delays massively degrades the ES performance or leads to instability.

Remark 2: High-frequency switching leads to *chattering* or *limit cycles* in actuators. The inability to remove it and achieve equilibrium stabilization in ES may also be associated with actuator constraints, such as magnitude and rate saturation. Here, the best control requirement could be to enforce a stable, "smallest" limit cycle [21], [2, Chapter 5]. However, ES-based controllers whose control efforts vanish as the system approaches equilibrium have been proposed [22] . In [19] and [5], Lie bracket-basedES was introduced to obtain ES feedback with bounded update rates. Thus, limit cycle can be reduced or completely eliminated.

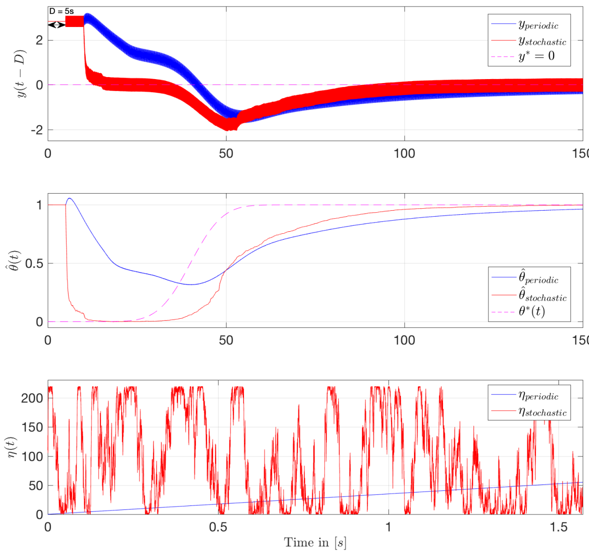

Fig. 2. Maximization of $\nu^{(1)}(\theta)$ with nonlinear time-variant dynamic (71a) and cubic output function (71b) using stochastic and periodic perturbations: (a) delayed output $y(t-D)$; (b) estimate of maximizing parameter $\hat{\theta}(t)$; (c) excerpts of the stochastic signal $\eta(t)$ and the deterministic ωt. Initial conditions are $\hat{\theta}_0 = 1$ and $\gamma_0 = -2$. The dashed lines indicate the target values $y^* = 0$ and $\theta^*(t) = 0.5\text{erf}((t-40)/10) + 0.5$.

VIII. CONCLUSIONS

In this letter, we presented the design of a Newton-based ES for higher derivatives of unknown dynamic maps using stochastic perturbations in the presence of known time delays. We proposed a predictor feedback for delay compensation and stated the application of the stochastic averaging theorem for the stability proof. Simulation shows that the stochastic version outperforms the periodic one in terms of faster convergence rates.In addition, for unknown constant delays the results would necessarily be local due to nonlinear parametritation of the delay (the initial estimate of the delay would have to be close to the actual delay). This would offer little advantage over the existing robustness of the predictor feedback to small perturbations in the delay (proved in [4]).

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